THE BAYESIAN THEORY OF GAMES: A STATISTICAL DECISION THEORETIC BASED ANALYSIS OF STRATEGIC INTERACTIONS

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Abstract

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Abstract

Bayesian rational prior equilibrium requires agent to make rational statistical predictions and decisions, starting with first order non informative prior and keeps updating with statistical decision theoretic and game theoretic reasoning until a convergence of conjectures is achieved.

The main difference between the Bayesian theory of games and the current games theory are:

I. It analyzes a larger set of games, including noisy games, games with unstable equilibrium and games with double or multiple sided incomplete information games which are not analyzed or hardly analyzed under the current games theory.

II. For the set of games analyzed by the current games theory, it generates far fewer equilibria and normally generates only a unique equilibrium and therefore functions as an equilibrium selection and deletion criterion and, selects the most common sensible and statistically sound equilibrium among equilibria and eliminates insensible and statistically unsound equilibria.

III. It differentiates between simultaneous move and imperfect information. The Bayesian theory of games treats sequential move with imperfect information as a special case of sequential move with observational noise term. When the variance of the noise term approaches its maximum such that the observation contains no informational value, there is imperfect information (with sequential move).

IV. It treats games with complete and perfect information as special cases of games with incomplete information and noisy observation whereby the variance of the prior distribution function on type and the variance of the observation noise term tend to zero. Consequently, there is the issue of indeterminacy in statistical inference and decision making in these games as the equilibrium solution depends on which variances tends to zero first. It therefore identifies equilibriums in these games that have so far eluded the classical theory of games.

1. Introduction

Current prevailing games theory solves by asking which combinations of strategies are equilibria. Agents are assumed to know the strategies adopted by the other agents and which equilibrium they are in. If there are incomplete information, then it is required that the beliefs about the type of the other agents be consistent with the equilibrium combination of strategy.

The Bayesian theory of games, in contrast, solves by assuming that the agents start with first order uninformative prior probability distribution functions on the strategy of the other agents and the agents have no idea which equilibrium they are in. The agents then keep updating their prior beliefs with game theoretic and statistical decision theoretic reasonings, until convergences of prior beliefs or conjectures are achieved.

Why starts with the first order noninformative prior? This is to let the game solves itself and selects its own equilibrium, rather than being imposed or affected by the informative first order priors. Selten and Harsanyi (1988) has a tracing precedure to select the most reasonable equilibrium among multiple Nash equilibria. The Bayesian solution concept for simultaneous games is quite similar to the tracing procedure of Selten and Harsanyi (1988). However, the Bayesian approach to simultaneous games does not start its tracing with only Nash equilibria but all possible actions or strategies of the playes. Another difference is that the equilibrium selection precedure of Selten and Harsanyi (1988) will always has an equilibrium if there is at least a Nash equilibrium. In contrast, the Bayesian might yields no equilibrium when there is a mixed strategy Nash equilibrium. This difference arises because when a player is indifferent among different actions, the Bayesian approach treats his response as stochastic. In contrast, the current games theory arbitrarily assumes that the player will selects a particular action.

Section 2 presents the Bayesian approach to noisy sequential games, section 3 presents the Bayesian approach to simultaneous games, section 4 deals with the difference between simultaneous move and imperfect information, section 5 presents sequential games with incomplete but perfect information and section 6 concludes the paper.

2. Noisy Sequential Games with Incomplete Information

Nnoisy sequential games are sequential games with inaccurate observation clouded by noises. In present modelling of incomplete information sequential games, there is the uncertainty about the type of one or more players. The uncertainty about the type of a player is modelled by a prior distribution function on the type of that player held by other players. This prior distribution function is normally assumed to be a common knowledge.[1] Other players normally either observe accurately the action of the player whose type is unknown or they do not observe his action at all. In simultaneous move Bayesian games, the action of the player whose type is unknown is not observed by the other players before they make their moves. They choose their optimal strategy given their prior beliefs about the type of the player with unknown type. The equibrium so obtained is termed the Bayesian Nash equilibrium. In most of the sequential Bayesian games, for instance, the signalling games, the action of the player whose type is unknown is accurately observed by the other players. After observing the action of the player with unknown type, the other players use game theoretic reasoning and the Bayes rule to update their prior beliefs about the type of the player with unknown type. They then choose their optimal strategy given their posterior beliefs about the type of the player with unknown type. The equilibrium so obtained is termed the perfect Bayesian Equilibrium.

The stipulation that the action of the player with unknown type is either accurately observed or completely unobserved is too restrictive. Given this restriction, there is no statistical inference involved in these Bayesian games though Bayes rule is used to update the beliefs on the type of the player with unknown type. This restriction will be relaxed. This section analyzes the game theoretic situation where the other player observes inaccurately the action of the player with unknown type. Inaccurate observation means that the other player observes the action of the player with unknown type with a noise term or there is a positive probability that they will make observational error due to the noise term. He therefore must makes statistical inference on the action of the player with unknown type. He does so bases upon two sources of information. One source of information is the inaccurate observations on the action of the player with unknown type. This is the sample data. The other source of information is the evidence which concerns the motive of the player with unknown type constructed through game theoretic reasoning, basing upon knowledge such as the prior distribution function on the type of the player with unknown type and the structure of the game. The information so constructed gives a belief about the probability of possible actions taken by the player with unknonw type. This belief is the prior distribution function on the action of the player with unknown type.

Given the need for statistical inference and decision, the player has to

[1] Refer to Harsanyi (1967, 1968a, 1968b).

decide which statistical decision rule to use. Since in games theory, the basic assumption is that the player is rational, the decision rule has to be a Bayes rule. A decision rule is a Bayes rule if it attains the infimum of the expected loss function or the supremum of the expected utility function.[2] On the other hand, the knowledge that the player has about the game will affect his belief about what action the other player is likely to take. That is to say, he will have a prior distribution on the action of the other player. Therefore, in a Bayesian game with inaccurate observation, there could be many possible equilibria given that there are many possible statistical decision rules and many possible Bayesian decision rules with their different prior beliefs.

To narrow down the number of equilibria, one has to further strengthen the concept of Bayesian Nash equilibrium and perfect Bayesian equilibrium. Presently in games theory there are no equilibrium concept for sequential Bayesian games with incomplete information and inaccurate observation of actions. This section uses the concept of Bayesian rational prior equilibrium to fill in this gap. Rational prior refers to the rational prior distribution function on the action of the player with unknown type. The rational prior distribution function is formed through iterative reasoning, starting with a first order uninformative prior distribution function on the action of the player with unknown type and keeps being updated by game theoretic and statistical decision theoretic reasoning until a convergence of the prior distribution function is achieved, thereby incorporating all available useful information such as the structure of strategic interaction, the prior distribution function on the type of the player with unknown type and the nature of the game theoretic equilibrium. A rational prior distribution function is consistent with the equilibrium it supported. Bayesian rational prior equilibrium rules out equilibria that are based on prior beliefs that are inconsistent with the equilibria they supported. The Bayesian decision rule starting with first order uninformative priors and ending with rational priors consistent with the equilibrium they supported is an undominated decision rule, attaining the supremum of the expected utility of the player making the inference.[3]

Sequential games with incomplete and perfect information where players accurately observe the actions of the other players and sequential games with incomplete and imperfect information where players completely do not

[2] A decision rule is a Bayes rule if it attains the infimum of the expected loss function or the supremum of the expected utility function. Other criteria for selecting decision rule include the minimax rule and admissibility. Refer to Berger (1980).

[3] Teng (2004).

observe the actions of the other players are both then special cases of noisy sequential games with incomplete information. When the inaccurate observation of action is so poor that no valuable information is contained in the observation, a noisy sequential game becomes an imperfect information sequential game. On the other hand, when the inaccurateness in observation is so small that the posterior distribution function on action depends only on the sample data and not on the prior distribution function at all, a noisy sequential game becomes a perfect information sequential game.

2.1. Example: Market Leadership

There are two players: Firm 1, the market leader and Firm 2, the market follower. Firm 1 moves first by setting its output level. Firm 2 observes imperfectly the output level of Firm 1 due to a confounding noise term. Firm 2 makes rational Bayesian inference on the output level of Firm 1 and then sets its output level.

The structure of the game is common knowledge. The cost efficiency of Firm 1 which determines the type of Firm 1, is choosen by Nature from a predetermined distribution function which is common knowledge. Once chosen, the type of Firm 1 is private knowledge. The type of Firm 2 is common knowledge. Firm 2 therefore must makes inference on the type and action of Firm 1. The distribution function of the noise term that confounds the observation by the Firm 2 on the actual output level of Firm 1 is common knowledge.

The Model

q_1, the output level of Firm 1, is the action of Firm 1. q_2, the output level of Firm 2, is the action of Firm 2. Total level of output in the market is Q. The price level is $P = D - Q$. The payoff function of Firm 1 is $\pi_1 = (D - q_1 - q_2 - c_1)\, q_1$. c_1 is the average and marginal cost of production of Firm 1. c_1 decides the type of Firm 1. Firm 1 knows c_1 but not Firm 2 does not know c_1. c_1 has a normal distribution which is common knowledge: $c_1\tilde{\ }N\left(\bar{c_1}, \zeta\right)$. The action of Firm 1 is imperfectly observed by Firm 2 with a noise term: $R = q_1 + \epsilon$. ϵ is the noise term. ϵ has a normal distribution: $\epsilon\tilde{\ }N\left(0, \kappa\right)$. The above leads to the following sampling distribution on R: $R \mid q_1\tilde{\ }N\left(q_1, \kappa\right)$ and the likelihood function: $q_1 \mid R\tilde{\ }N\left(R, \kappa\right)$.

The game is solved starting with a non-informative first order prior. Firm 2 solves

$q_2 \max E\left(\pi_2\right) = \int_{-\infty}^{\infty} \left(D - q_1 - q_2 - c_2\right) q_2 f\left(q_1 \mid R\right) dq_1$. The optimal solution is $q_2 = \frac{D-R-c_2}{2}$. Firm 1 being the first mover, anticipates the stochastic response of firm 2 and solves $q_1 \max E(\pi_1) = \int_{-\infty}^{\infty} \left(D - q_1 - \frac{D-R-c_2}{2} - c_1\right) q_1 f(\epsilon)\, d\epsilon$. The optimal solution is $q_1 = \frac{D+c_2-2c_1}{2}$. Therefore, $q_1\tilde{}N\left(\bar{q}_1, \zeta\right)$ and $\bar{q}_1 = \frac{D+c_2-2\bar{c}_1}{2}$, which is the second order prior. The second order posterior is $q_1 \mid R\tilde{}N\left(\hat{q}_1, \hat{\rho}\right), \hat{q}_1 = \frac{\zeta}{\kappa+\zeta}R + \frac{\kappa}{\kappa+\zeta}\bar{q}_1 = \theta R + (1-\theta)\bar{q}_1$ where $\theta = \frac{\zeta}{\kappa+\zeta}$ and $\hat{\rho} = \frac{\zeta\kappa}{\kappa+\zeta}$.

Firm 2 solves $q_2 \max E(\pi_2) = \int_{-\infty}^{\infty} (D - q_1 - q_2 - c_2)\, q_2 f(q_1 \mid R)\, dq_1$. The optimal solution is $q_2 = \frac{D-\hat{q}_1-c_2}{2}$ and $q_2 \mid q_1\tilde{}N\left(\frac{D-c_2-\left(\theta q_1+(1-\theta)\bar{q}_1\right)}{2}, \frac{\theta^2}{4}\kappa\right)$. Firm 1 solves $q_1 \max E(\pi_1) = \int_{-\infty}^{\infty}\left(D - q_1 - \frac{D-c_2}{2} + \frac{\hat{q}_1}{2} - c_1\right) q_1 f(\epsilon)\, d\epsilon$. The optimal solution is $q_1 = \frac{D+c_2+(1-\theta)\bar{q}_1-2c_1}{2(2-\theta)}$. Therefore, the third order prior is $q_1\tilde{}N\left(\bar{q}_1, \rho\right), \bar{q}_1 = \frac{D+c_2-2\bar{c}_1}{3-\theta}, \rho = \left(\frac{1}{2-\theta}\right)^2\zeta$. The third order posterior is $q_1 \mid R\tilde{}N\left(\hat{q}_1, \hat{\rho}\right), \hat{q}_1 = \frac{\rho}{\kappa+\rho}R + \frac{\kappa}{\kappa+\rho}\bar{q}_1 = \theta R + (1-\theta)\bar{q}_1$ where $\theta = \frac{\rho}{\kappa+\rho}$ and $\hat{\rho} = \frac{\rho\kappa}{\kappa+\rho}$

Firm 2 solves $q_2 \max E(\pi_2) = \int_{-\infty}^{\infty} (D - q_1 - q_2 - c_2)\, q_2 f(q_1 \mid R)\, dq_1$. The optimal solution is $q_2 = \frac{D-\hat{q}_1-c_2}{2}$ and therefore $q_2 \mid q_1\tilde{}N\left(\frac{D-c_2-\left(\theta q_1+(1-\theta)\bar{q}_1\right)}{2}, \frac{\theta^2}{4}\kappa\right)$. Firm 1 solves

$q_1 \max E(\pi_1) = \int_{-\infty}^{\infty}\left(D - q_1 - \frac{D-c_2}{2} + \frac{\hat{q}_1}{2} - c_1\right) q_1 f(\epsilon)\, d\epsilon$. The optimal solution is $q_1 = \frac{D+c_2+(1-\theta)\bar{q}_1-2c_1}{2(2-\theta)}$ and therefore the fourth order prior is $q_1\tilde{}N\left(\bar{q}_1, \rho\right), \bar{q}_1 = \frac{D+c_2-2\bar{c}_1}{3-\theta}, \rho = \left(\frac{1}{2-\theta}\right)^2\zeta$. At this point, the process converges.

The rational prior density function of Firm 2 on the output level of Firm 1 is:

$q_1\tilde{}N\left(\bar{q}_1, \rho\right), \bar{q}_1 = \frac{D+c_2-2\bar{c}_1}{3-\theta}, \rho = \left(\frac{1}{2-\theta}\right)^2\zeta$ and therefore $q_1 = \frac{D+c_2}{3-\theta} - \frac{1}{2-\theta}\left(c_1 + \frac{1-\theta}{3-\theta}\bar{c}_1\right)$

Ex ante q_1 and q_2 have the following joint distribution function:

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \tilde{} N \left(\begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \end{pmatrix} cov\left(q_1, q_2\right) \right)$$

$$\begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \end{pmatrix} = \begin{pmatrix} \frac{D+c_2-2\bar{c_1}}{3-\theta} \\ \frac{D(2-\theta)-c_2(4-\theta)-2\bar{c_1}}{2(3-\theta)} \end{pmatrix}$$

$$cov\left(q_1, q_2\right) = \begin{pmatrix} \left(\frac{1}{2-\theta}\right)^2 \zeta & \frac{-\theta}{2}\left(\frac{1}{2-\theta}\right)^2 \zeta \\ \frac{-\theta}{2}\left(\frac{1}{2-\theta}\right)^2 \zeta & \frac{\theta^2}{4}\left(\left(\frac{1}{2-\theta}\right)^2 \zeta + \kappa\right) \end{pmatrix}$$

2.2. Perfect and Complete Information and Indeterminacy.

The next few paragraphs let the variance of the type distribution function (ζ) tends to zero and let the variance of the noice term (κ) tends to either its zero or positive infinity. The variance of the prior distribution function on action (ρ) therefore tends to zero as well. Now let $\rho \to 0 \lim \kappa \to 0 \lim \theta = 1$[4]. In this case, at BRPE,

$q_1 = \frac{D+c_2+(1-\theta)\bar{q_1}-2c_1}{2(2-\theta)} = \frac{D+c_2+(1-\theta)q_1-2c_1}{2(2-\theta)} = \frac{D+c_2}{2} - c_1$

and

$q_2 = \frac{D-c_2}{2} - \frac{\theta(q_1+\epsilon)+(1-\theta)\bar{q_1}}{2} = \frac{D-c_2}{2} - \frac{\theta(q_1+\epsilon)+(1-\theta)q_1}{2} = \frac{D-c_2-q_1}{2} = \frac{D}{4} - \frac{3c_2}{4} + \frac{c_1}{2}$

This is the Stackelberg solution for the complete and perfect information game. To verify, note that in choosing q_2, Firm 2 solves $q_2 \max \pi_2 = \left(D - q_1 - q_2 - c_2\right) q$. The optimal solution is $q_2 = \frac{D-c_2-q_1}{2}$. Firm 1, being the first mover, anticipates the reaction of Firm 2. In determining the optimal level of output, Firm 1 solves $q_1 \max \pi_1 = \left(D - q_1 - \frac{D-q_1-c_2}{2} - c_1\right) q_1$. The optimal solution is $q_1 = \frac{D+c_2}{2} - c_1$.

Next let $\rho \to 0 \lim \kappa \to \infty \lim \theta = 0$. In this case, at BRPE,

$q_1 = \frac{D+c_2+(1-\theta)\bar{q_1}-2c_1}{2(2-\theta)} = \frac{D+c_2-2c_1}{3}$

$q_2 = \frac{D-c_2}{2} - \frac{\theta(q_1+\epsilon)+(1-\theta)\bar{q_1}}{2} = \frac{D-2C_2+C_1}{3}$

This is the Cournot solution for the complete and imperfect information (or simultaneous) game. To verify, note that in choosing q_2, Firm 2 solves $q_2 \max \pi_2 = \left(D - q_1 - q_2 - c_2\right) q$. The optimal solution is $q_2 = \frac{D-c_2-q_1}{2}$. In determining the optimal level of output, Firm 1 solves $q_1 \max \pi_1 = \left(D - q_1 - q_2 - c_1\right) q_1$. The optimal solution is $q_1 = \frac{D-q_2-c_1}{2}$. Solving the two first order conditions simultaneously gives

$q_1 = \frac{D+c_2-2c_1}{3}$

[4]The case presented here assumes that the inferring agent bases his statistical inference and decision entirely on his observation and not prior reasoning.

$q_2 = \frac{D-c_2-q_1}{2} = \frac{D-2C_2+C_1}{3}$

At the Bayesian rational prior equilibrium, firm 2 produces $q_2 = \frac{D-c_2}{2} - \frac{\theta(q_1+\epsilon)+(1-\theta)\bar{q_1}}{2}$.

The rational prior density function of Firm 2 on the output level of Firm 1 is: $q_1 \tilde{} N\left(\bar{q_1}, \rho\right), \bar{q_1} = \frac{D+c_2-2\bar{c_1}}{3-\theta}, \rho = \left(\frac{1}{2-\theta}\right)^2 \zeta$ and therefore $q_1 = \frac{D+c_2}{3-\theta} - \frac{1}{2-\theta}\left(c_1 + \frac{1-\theta}{3-\theta}\bar{c_1}\right)$. The slope of the reaction function is $\frac{\partial q_2}{\partial q_1} = -\frac{\theta}{2}$

Now let the variance of the type distribution function (ζ) and variance of the noice term (κ) both tend to zero. The variance of the rational prior distribution functino on action (ρ) therefore tends to zero as well. The equilibrium q_1 and q_2 when all the three variances tend to zero depend upon the value of $\kappa \to 0 \lim \rho \to 0 \lim \theta$. If

$\rho \to 0 \lim \kappa \to 0 \lim \theta = 0$, then $q_1 = \frac{D+c_2+(1-\theta)\bar{q_1}-2c_1}{2(2-\theta)} = \frac{D+c_2+(1-\theta)q_1-2c_1}{2(2-\theta)} = \frac{D+c_2-2c_1}{4} + \frac{1}{4}q_1 = \frac{D+c_2-2c_1}{3}$ and $q_2 = \frac{D-c_2}{2} - \frac{\theta(q_1+\epsilon)+(1-\theta)\bar{q_1}}{2} = \frac{D-c_2}{2} - \frac{\theta(q_1+\epsilon)+(1-\theta)q_1}{2} = \frac{D-c_2-q_1}{2} = \frac{D-2C_2+C_1}{3}$. This is the same as the Cournot solution for the complete and imperfect information game.

$\kappa \to 0 \lim \rho \to 0 \lim \theta$ could take on any value from 0 to 1. For instance, it could be that $\kappa \to 0 \lim \rho \to 0 \lim \theta = 0.5$. In this case, $q_1 = \frac{D+c_2+(1-\theta)\bar{q_1}-2c_1}{2(2-\theta)} = \frac{2(D+c_2-2c_1)}{5}$ and $q_2 = \frac{D-c_2}{2} - \frac{\theta(q_1+\epsilon)+(1-\theta)\bar{q_1}}{2} = \frac{2}{5}D - \frac{7}{10}C_2 + \frac{2}{5}C_1$

The example is illustrated in the diagram below:

In the above disgram, C is the solution when $\theta = 0$, S is the solution when $\theta = 1$, G is the solution when $\theta = \frac{1}{2}$.

3. Sequential Games with Incomplete but Perfect Information

In current games theory, the solution algorithm of sequential games with incomplete and perfect information starts with the assumption of equilibrium strategy of the first moving player with uncertain type. Given the equilibrium strategy of the first moving player with uncertain type, upon perfectly observing the action of the first moving player, the second moving player infers about the type of the first moving player and decides his own strategy. Finally, the decisions of the first and second moving players are checked if they accord with the initially assumed equilibrium. If so, the initially assumed equilibrium is the perfect Bayesian equilibrium or any of its many refinements. The assumption that players know the equilibrium of the game and strategy of the other player removes much of the inherent uncertainty about the strategy of the other player in games of incomplete information.

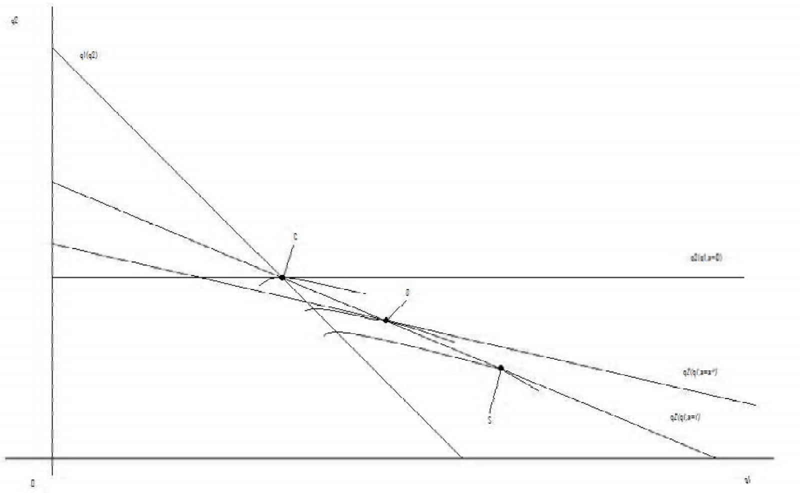

Figure 0.1:

The Bayesian approach to games investigates how the conjectures of players about the strategies of the other players and their conjectures converges to an equilibrium, the Bayesian rational prior equilibrium. In solving sequential games with incomplete and perfect information, the Bayesian approach assumes that players do not know the other player's strategy nor the equilibrium of the game, though the second moving player observes perfectly the action of the first moving player. The conjectures of players about the other player's strategy and his conjectures are conditional upon the fact that the action of the player who moves earlier are perfectly observed by the player who moved later and that this is common knowledge. Both the first moving player and the player who moves later conjecture about the strategy and conjectures of the other player. The conjectures about the strategies and actions of the other player start with first order uninformative priors. The point of convergence of such conjectures constitute the Bayesian raitonal prior equilibrium for the incomplete and perfect information game.

In a sequential game of incomplete and perfect information, the second moving player does not know the type of the first mover. Therefore, despite the fact that he observes the action of the first moving player perfectly, he must still infer about the strategy of each type of the first moving player through game theoretic and statistical decision theoretic reasonings. The first mover of course must also conjecture about the strategy and conjectures of the second moving player when selecting his strategy. That means the agent cannot condition his strategy upon the other player's strartegy: Player 1 cannot connot do so for he moves first and player 2 cannot do so for player 1 has more than one type and player 2 observes player 1's action but not strategy.

3.1. Example.

The following example shows the Bayesian approach to solving a game of incomplete and perfect information. It also illustrates the relationship between incomplete and perfect information sequential games and complete and perfect information sequential games. When the variance of type tends to zero, a sequential game with incomplete and perefect information becomes a sequential game with complete and perfect information where the player relies upon the observation totally for his statistical inference and decision and not the prior. The equilibrium of the latter must equal to the equilibrium of the former in the limiting case. The Bayesian approach passes this test while the PBE approach fails to.

Example: coordination game.

This example is important for the intuitive criterion cannot eliminate the unreasonable equilibrium but Bayesian Rational Prior Equilibrium approach could.

There are two equilibria by the classical PBE approach, both separating:

i. (L, R; u(L), d(R)). This equilibrium is socially optimal.

ii. (R, L; d(L), u(R)). This equilibrium is socially suboptimal.

Refinements by determining the off equilibrium beliefs would not work here since there is no off equilibrium belief.

Solving by the Bayesian Rational Prior Equilibrium approach:

Let the probability that the receiver plays U when observed L be a and the probability that the receiver plays U when observed R be b. Let the probability that the type 1 sender plays L be x and the probability that the type 2 sender plays L be y.

When L is observed, the receiver plays U if

$2x\,(0.5) > x\,(0.5) + 5y\,(0.5)$

$x > 5y$

$E\,(a) = \int_0^1 \int_0^1 a\,(x, y)\,dxdy = \frac{1}{10}$

When R is observed, the receiver plays U if

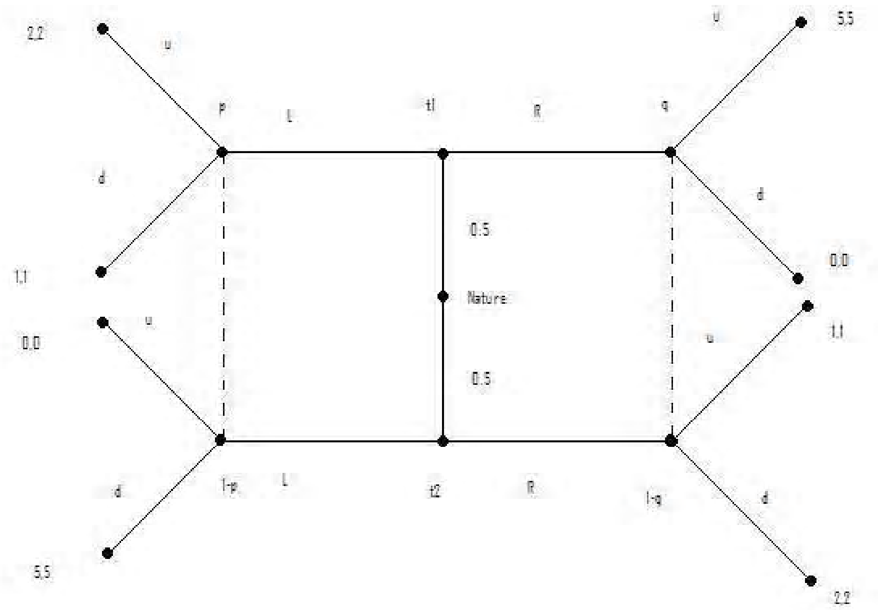

Figure 0.2:

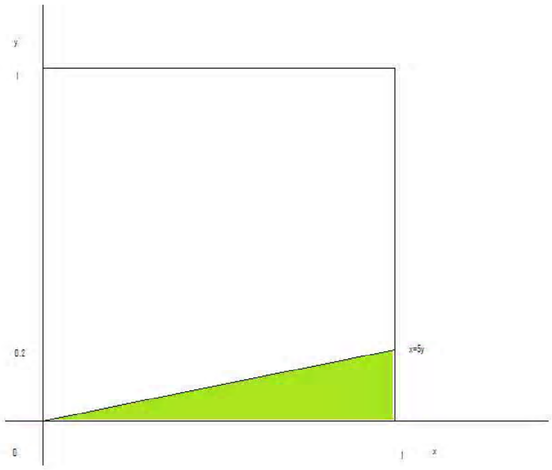

Figure 0.3:

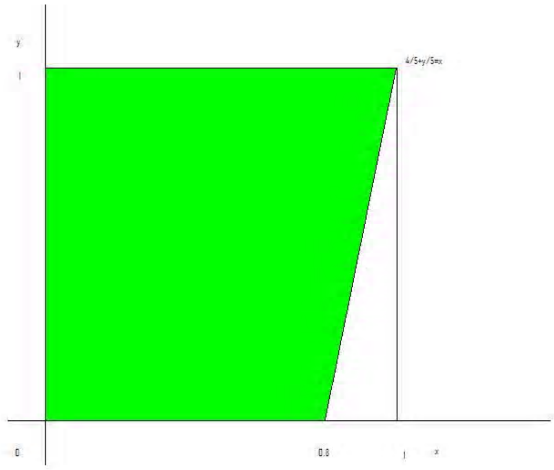

Figure 0.4:

$5\left(1-x\right)\left(0.5\right)+\left(1-y\right)\left(0.5\right)>2\left(1-y\right)\left(0.5\right)$

$\frac{4}{5}+\frac{y}{5}>x$

$E\left(b\right)=\int_0^1\int_0^1 b\left(x,y\right)dxdy=\frac{9}{10}$

Type 1 sender plays L if

$2a+\left(1-a\right)>5b$

$\frac{1}{5}+\frac{a}{5}>b$

$E\left(x\right)=\int_0^1\int_0^1 x\left(a,b\right)dadb=\frac{3}{10}$

Type 2 sender plays L if

$5\left(1-a\right)>b+2\left(1-b\right)$

$\frac{3}{5}+\frac{b}{5}>a$

$E\left(y\right)=\int_0^1\int_0^1 y\left(a,b\right)dadb=\frac{7}{10}$

Starting with $E\left(x\right)$ and $E\left(y\right)$, $p=\frac{3}{10}$ and $q=\frac{7}{10}$. So when L is observed, the receiver plays D since

$2\left(\frac{3}{10}\right)<\frac{3}{10}+5\left(\frac{7}{10}\right)$

When R is observed, the receiver plays U since

$5\left(\frac{7}{10}\right)+\frac{3}{10}>2\left(\frac{3}{10}\right)$

Anticipating the receiver's responses, type 1 sender plays R and type 2 sender plays L. The equilibrium is (R, L; D, U).

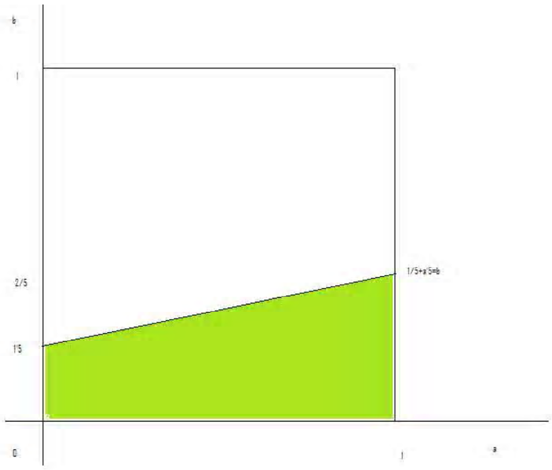

Figure 0.5:

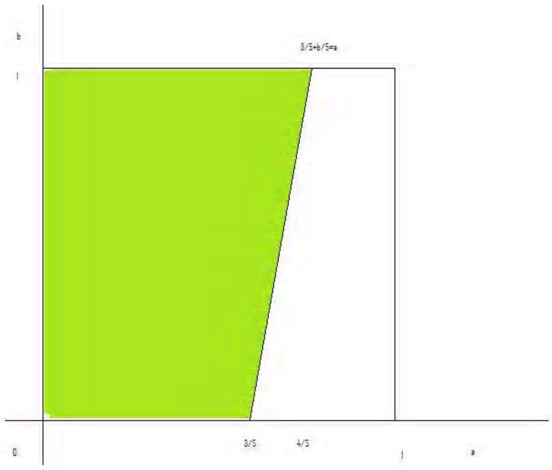

Figure 0.6:

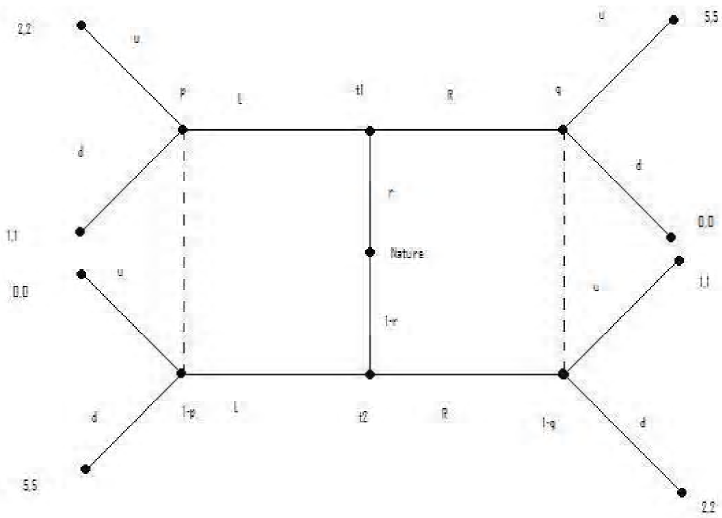

Figure 0.7:

Consider the following game case where the probability of player 1 being type 1 and type 2 is r and $1-r$.

There are three equilibria by the classical PBE approach: i. Separating equilibrium (L, R; u(L), d(R)). ii. Separating equilibrium (R, L; d(L), u(R)). iii. Pooling equilibrium (R, R; u(L), u(R); p>$\frac{5}{6}$). Equilibrium iii is ruled out by intuitive criterion. Since only type 2 sender has incentive to deviate from the equilibrium and switch from R to L, $p = 0$.

Solving by the Bayesian Rational Prior Equilibrium approach: Let the probability that the receiver plays U when observed L be a and the probability that the receiver plays U when observed R be b. Let the probability that the type 1 sender plays L be x and the probability that the type 2 sender plays L be y.

When L is observed, the receiver plays U if

$2x\,(r) > x\,(r) + 5y\,(1-r)$ or $x\,(r) > 5y\,(1-r)$

When R is observed, the receiver plays U if

$5\,(1-x)\,(r) + (1-y)\,(1-r) > 2\,(1-y)\,(1-r)$ or $5\,(1-x)\,(r) > (1-y)\,(1-r)$

Type 1 sender plays L if

$2a + (1-a) > 5b$ or $\frac{1}{5} + \frac{a}{5} > b$

$E\,(x) = \int_0^1 \int_0^1 x\,(a,b)\,da\,db = \frac{3}{10}$

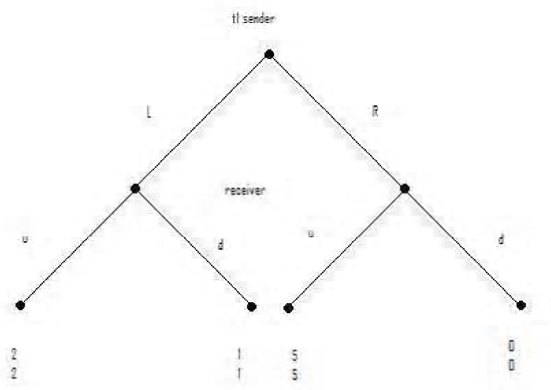

Figure 0.8:

Type 2 sender plays L if
$5(1-a) > b + 2(1-b)$ or $\frac{3}{5} + \frac{b}{5} > a$
$E(y) = \int_0^1 \int_0^1 y(a,b)\, da db = \frac{7}{10}$
Starting with $E(x)$ and $E(y)$, $r \rightarrow 1 \lim p = \frac{\frac{3}{10}(r)}{\frac{3}{10}(r)+\frac{7}{10}(1-r)} = 1$ and $r \rightarrow 1 \lim q = \frac{\frac{7}{10}(r)}{\frac{7}{10}(r)+\frac{3}{10}(1-r)} = 1$. So when L is observed, the receiver plays U since $2(p) > p + 5(1-p)$. When R is observed, the receiver plays U since $5(q) + (1-q) > 2(1-q)$. Anticipating the receiver's responses, type 1 sender plays R and type 2 sender plays R. The equilibrium is (R, R; U, U). It is the equilibrium iii of the PBE approach which is ruled out by intuitive criterion.

Note that the BRPE for the limiting case sequential incomplete and perfect information game agrees with the equilibrium for the sequential complete and perfect information game which is represented by the diagram below:

The equilibrium is (R; u(L), u(R)). This is derived through backward induction. The above examples casts doubt about the validity of the intuitive criterion as a refinement of PBE. The intuitive criterion eliminates the most logical equilibrium and is no help in selecting among the remaining two equilibria.

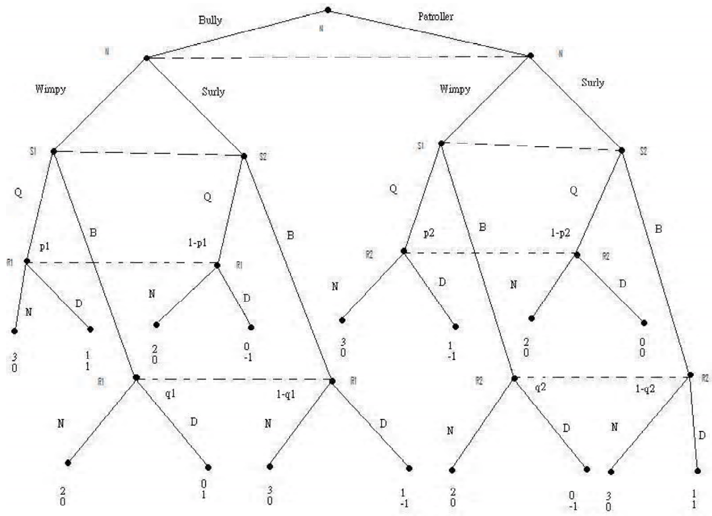

Figure 0.9:

Example: Two Sided Incomplete Information Sequential Game with Perfect Information

In the above game, there are two types of senders, wimpy and surly, and two types of receivers, bully and patrollers. The probability of wimpy is 0.1 and the probability of surly is 0.9. The bully type enjoys picking on the wimpy type. The left hand side of the game is therefore the Beer-Quiche game. The patroller type, on the other hand, has the duty of challenging the surly type when the surly orders beer and only if the surly orders beer. However, if the patroller challenges the wimpy, the patroller is humiliated. The probability of bully is 0.1 and the probability of patroller is 0.9.

Let the probability that the bully plays duels when Beer is observed be u, the probability that the bully plays duels when Quiche is observed be v, the probability that the patroller plays duels when Beer is observed be s and the probability that the patroller plays duels when Quiche is observed be t. Please note that when Quiche is observed, the patroller has the dominant strategy of choosing No Duel and therefore $t = 0$. Let the probability that the surly plays Beer be x and the probability that the wimpy plays Beer be y.

Given u, v, s and t, the surly chooses beer if

$(0.1)[u+3(1-u)]+(0.9)[s+3(1-s)] >$
$(0.1)2(1-v)+(0.9)2(1-t)$
or $1 > (0.2)(u-v)+(1.8)(s-t)$

The above is the combination of $[u+3(1-u)-2(1-v)] > 0$ and $[s+3(1-s)-2(1-t)] > 0$ and weighted by 0.1 and 0.9. The expected value of x is therefore $E(x) = \frac{7}{8}$

Given u, v, s and t, the wimpy chooses beer if
$(0.1)[2(1-u)]+(0.9)[2(1-s)] >$
$(0.1)[v+3(1-v)]+(0.9)[t+3(1-t)]$
or $(0.2)(v-u)+(1.8)(t-s) > 1$

The above is the combination of $[2(1-u)-v-3(1-v)] > 0$ and $[2(1-s)-t-3(1-t)] > 0$ and weighted by 0.1 and 0.9. The expected value of y is therefore $E(y) = \frac{1}{8}$

Given x and y, when observed Beer the bully reasons that
$(-1)\left(\frac{7}{8}\right)\left(\frac{9}{10}\right)+(1)\left(\frac{1}{8}\right)\left(\frac{1}{10}\right) < 0$
The bully therefore chooses No Duel when observed Beer and $u = 0$.

Given x and y, when observed Quiche the bully reasons that
$(-1)\left(\frac{1}{8}\right)\left(\frac{9}{10}\right)+(1)\left(\frac{7}{8}\right)\left(\frac{1}{10}\right) < 0$
The bully therefore chooses No Duel when observed Quiche and $v = 0$.

Given x and y, when observed Beer the patroller reasons that
$(1)\left(\frac{7}{8}\right)\left(\frac{9}{10}\right)+(-1)\left(\frac{1}{8}\right)\left(\frac{1}{10}\right) > 0$
The patroller therefore chooses Duel when observed Beer and $s = 1$.

Therefore at this stage of reasoning, $u = 0, v = 0, s = 1$ and $t = 0$. For the Surly, the payoff of playing Beer is
$(0.1)[u+3(1-u)]+(0.9)[s+3(1-s)] = 1.2$
For the Surly, the payoff of playing Quiche is
$(0.1)2(1-v)+(0.9)2(1-t) = 2$
The Surly therefore chooses Quiche.

For the Wimpy, the payoff of playing Beer is
$(0.1)[2(1-u)]+(0.9)[2(1-s)] = 0.2$
For the Wimpy, the payoff of playing Quiche is
$(0.1)[v+3(1-v)]+(0.9)[t+3(1-t)] = 3$
The Wimpy therefore chooses Quiche.

At this stage of reasoing, $x = 0$ and $y = 0$.

Given x and y, when observed Quiche the bully reasons that
$(-1)(0.9)+(1)(0.1) < 0$
The bully therefore chooses No Duel when observed Quiche and $v = 0$.

At this stage of reasoing, $v = 0$ and $t = 0$. From last stage, $u = 0$ and $s = 1$.

For the Surly, the payoff of playing Beer is

$(0.1)[u+3(1-u)]+(0.9)[s+3(1-s)]=1.2$

For the Surly, the payoff of playing Quiche is

$(0.1)\,2(1-v)+(0.9)\,2(1-t)=2$

The Surly therefore chooses Quiche and $x=0$.

For the Wimpy, the payoff of playing Beer is

$(0.1)[2(1-u)]+(0.9)[2(1-s)]=0.2$

For the Wimpy, the payoff of playing Quiche is

$(0.1)[v+3(1-v)]+(0.9)[t+3(1-t)]=3$

The Wimpy therefore chooses Quiche and $y=0$.

The process therefore converges here with $u=0, v=0, s=1, t=0, x=0$ and $y=0$.

4. Simultaneous Games

This section illustrates the Bayesian algorithm of solving simultaneous games. Starting with first order uninformativce priors, probability density distribution functions are constructed from reaction functions and second order priors are derived. The process of constructing higher order priors keep reinterated until convergence of conjectures is achieved, if there is convergence. The convergent prior distribution functions and the equilibrium they supported is the Bayesian rational prior equilibrium.

In a simultaneous move game, none of the players observed what the other players are doing and they all makes their decisions simultaneously and all these are common knowledge. By the very definition of simultaneous move, even if one of players will play a particular equilibrium strategy prescribed by the concept of Nash equilibrium, be it either a pure strategy or a mixed strategy, the other players still do not observe the action of the other players. They therefore have to conjecture about the move. Since what the players think or conjecture will affect their decisions, it therefore follows that the players must conjecture about the other player' conjectures, besides conjecturing what the other players are doing or will do.

There is a strong need for a new solution concept for simultaneous games. The existing concept, Nash Equilibrium, generates too many equilibria. Many of these equilibria do not make sense. Mixed strategy equilibrium poses particular problem for not only in many cases it does not make sense, there are also many different interpretations of mixed strategy and mixed strategy equilibrium and games theorists are at a loss here. For instance, it is said that mixed strategy is adopted by a player who randomizes to confuse opponent. A good example is the matching pennies games which has only mixed strategy equilibrium and no pure strategy equilibrium. Another interpretation is that the player are unsure of the other player's strategy and the probability of mixed strategy reflects the subjective probability interpretation about the other player's possible strategy. A good example of this interpretation is the battle of sexes game. None of the interpretations is totally satisfactory and game theorists have an ambivalent attitude towards mixed strategies equilibrium.[5]

The conjectural variations literature is one of the response. Conjectures play an important role in the Bayesian rational prior equilibrium approach. The Bayesian rational prior equilibrium solution of a simultaneous game traces out the whole statistical decision process starting with non informative first order priors and updates the priors by game theoretic reasoning to achieve convergence in conjectures, if there is any. The equilibrium conjectures are consistent with the equilibrium they supported. On this point, the Bayesian rational prior equilibrium approach agrees with the equilibrium consistent conjectural variations literature.[6]

The way the Nash equilibrium approach solves a simultaneous move game is to get the interaction points of the reaction functions. Implicit in this solution algorithm is that there is perfect information and the moves are sequential. That is what a reaction function means: the reaction of one player to the action of the other player. That implies perfect information for you must observe the action of the other party before you could react to his aciton. If there are simultaneity in moves and the players do not observe the moves of the other players, then they could not react to the actions of the other players. In this situation, a player would react to the expected values or predictive distribution functions of the actions of the other players. It is clear that conjecture plays a central role here. The reaction functions of a simultaneous game are therefore not really reaction functions as those of a perfect information sequential game and are best named as virtual reaction functions for differentiation from the real reaction functions of a perfect information sequential game.

By the Bayesian rational prior equilibrium approach, players forms rational prior or expectation of the actions of the other players and update the priors or expecations rationally using game theoretic reasoning. The Bayesian Rational Prior Equilibrium, if it exists, is the point of convergence of these rational conjectures. In the situation where there are more than one

[5] Aumann (1985) and Harsanyi and Selten (1988).

[6] Check references here.

stable Nash equilibria, the Bayesian Rational Prior Equilibrium approach selects the most sensible stable Nash equilibrium as the Bayesian Rational Prior Equilibrium.

Note that by the present classical games theory approach in which the players take no expectation or make no prediction on the other players's strategies, the Nash equilibrium or equilibria could be reached only by chance in a simulataneous game. This is so for in a simultaneous game, you cannot condition your strategy or move on your opponent's strategy or move. In contrast, in the Bayesian Rational Prior Equilibrium approach, players make predictions or expectations and the resultant equilibrium is not arrived haphazardly .

4.1. 2X2 Simultaneous Game Examples

Four types of equilibria could arise in the BRPE approach to 2X2 simultaneous games. The first is a unique pure strategy equiuilibrium where all conjectures that start from the different first order priors on each player's strategy converge to the same pure strategy equilibrium. This is in fact the focus point or Schelling point brought up by Schelling (1960). Example 1 Coordination Game is such a case. The Bayesian approach selects the most sensible of the Nash equilibria in this case, if there are multpile pure strategy Nash equilibria. The second scenario is multiple pure strategy equilibria where conjectures starting from different first order priors on each player's strategy converge to different pure strategy equilibria. Example 2 Battle of Sexes game is such a case. The third is a stochastic response equilibrium in which players are indifferent between a range of possible strategies. Example 3 Matching Pennies is such a case. The Bayesian rational prior equilibrium though looks very much like the mixed strategy equilibrium, has very different interpretative meaning from the latter. The fourth situation is where no convergence of conjectures is achieved. Example 4 Conflict Game is such a case. The unique mixed strategy Nash Equilibrium of current games theory is ruled out by the Bayesian approach in this case.

Example 1: Coordination Game

player 1\player 2	$L\,(y)$	$R\,(1-y)$
$U\,(x)$	a, a	$0, 0$
$D\,(1-x)$	$0, 0$	$1, 1$

$a > 1$. The game has two pure strategy Nash equilibria: (U,L) (or $x = 1, y = 1$) and (D,R) (or $x = 0, y = 0$) and a mixed strategy Nash equilibrium:

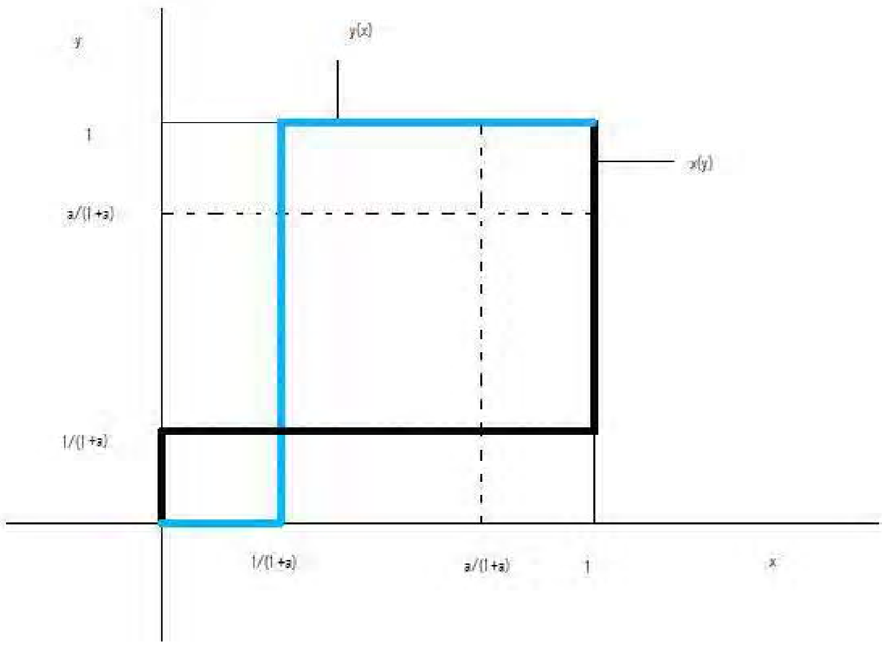

Figure 0.10:

$x = \frac{1}{1+a}$, $y = \frac{1}{1+a}$ where x is the probability that Player 1 plays U and y is the probability that Player 2 plays L.

The reaction functions are: $x(y) = 1$ if $y > \frac{1}{1+a}, x(y) \in [0,1]$ if $y = \frac{1}{1+a}, x(y) = 0$ if $y < \frac{1}{1+a}$ and $y(x) = 1$ if $x > \frac{1}{1+a}, y(x) \in [0,1]$ if $x = \frac{1}{1+a}, y(x) = 0$ if $x < \frac{1}{1+a}$.

Note that at the mixed strategy Nash equilibrium, as a increases, the probability that Player 1 plays U and Player 2 plays L decreases. This is not sensible.

Starting with first order uninformative priors and given $x(y)$ and $y(x)$, the expected value of x and y are: $E(x) = 0^1 x(y)\, dy = \frac{a}{1+a}$ and $E(y) = 0^1 y(x)\, dy = \frac{a}{1+a}$. Starting with $E(y) = \frac{a}{1+a}$, the outcome is $x\left(\frac{a}{1+a}\right) = 1$ and $y(1) = 1$. Starting with $E(x) = \frac{a}{1+a}$, the outcome is $y\left(\frac{a}{1+a}\right) = 1$ and $x(1) = 1$. Consequently, the unique Bayesian Rational Prior Equilibrium is $x = 1, y = 1$.

Example 2: Battle of Sexes

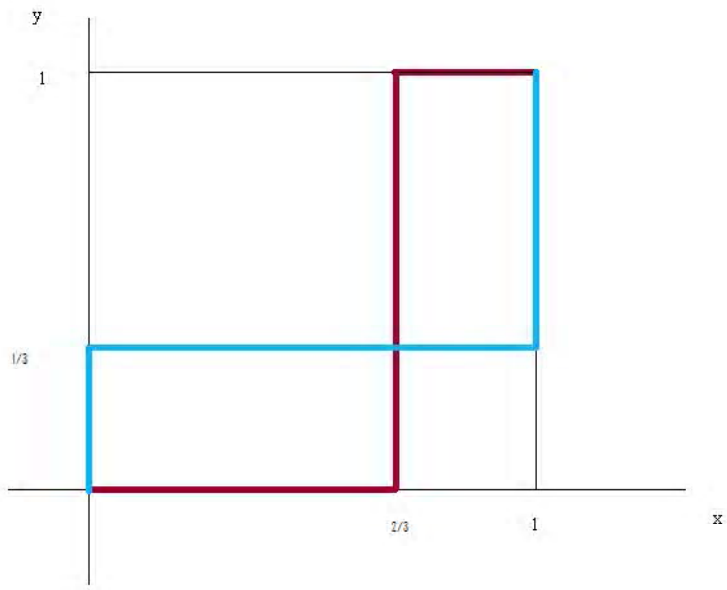

Figure 0.11:

Tom\Mary	Boxing (y)	Opera $(1-y)$
Boxing (x)	$10,5$	$0,0$
Opera $(1-x)$	$0,0$	$5,10$

The reaction functions are: $x(y) = 1$ if $y > \frac{1}{3}, x(y) \in [0,1]$ if $y = \frac{1}{3}, x(y) = 0$ if $y < \frac{1}{3}$ and $y(x) = 1$ if $x > \frac{2}{3}, y(x) \in [0,1]$ if $x = \frac{2}{3}, y(x) = 0$ if $x < \frac{2}{3}$.

Given $x(y)$ and $y(x)$, the expected value of x and y are: $E(x) = 0^1 x(y)\,dy = \frac{2}{3}$ and $E(y) = 0^1 y(x)\,dy = \frac{1}{3}$. Starting with $E(x)$, $y\left(\frac{2}{3}\right) \in [0,1], E\left[y \mid x = \frac{2}{3}\right] = 0\overset{1}{\int} ydy = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}, x\left(\frac{1}{2}\right) = 1, y(1) = 1, x(1) = 1$. So starting with $E(x)$ and the point of convergence of conjectures is $x = 1, y = 1$. Starting with $E(y)$, $x\left(\frac{1}{3}\right) \in [0,1], E\left[x \mid y = \frac{1}{3}\right] = 0\overset{1}{\int} xdx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}, y\left(\frac{1}{2}\right) = 0, x(0) = 0$. So starting with $E(y)$ and the point of convergence of conjectures is $x = 0, y = 0$. There are two Bayesian rational prior equilibria:

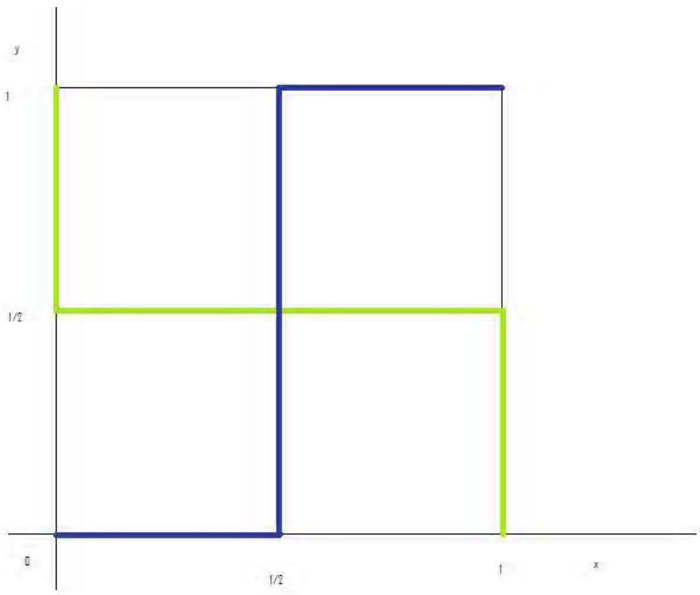

Figure 0.12:

$(x=1, y=1)$and $(x=0, y=0)$.

Example 3. Matching Pennies

$1\backslash 2$	$H(y)$	$T(1-y)$
$H(x)$	$-1,1$	$1,-1$
$T(1-x)$	$1,-1$	$-1,1$

There is no pure strategy Nash equilibrium but only mixed strategy Nash equilibrium, $\left(x=\frac{1}{2}, y=\frac{1}{2}\right)$. The reaction functions are $x(y)=0$ if $y>\frac{1}{2}, x(y)\in[0,1]$ if $y=\frac{1}{2}, x(y)=1$ if $y<\frac{1}{2}$ and $y(x)=1$ if $x>\frac{1}{2}, y(x)\in[0,1]$ if $x=\frac{1}{2}, y(x)=0$ if $x<\frac{1}{2}$.

The expected values of x and y are: $E(x(y))=\frac{1}{2}$ and $E(y(x))=\frac{1}{2}$. Starting with $E(x), y\left(\frac{1}{2}\right)\in[0,1], E\left[y \mid x=\frac{1}{2}\right] = \int_0^1 y dy=\frac{1}{2}, x\left(\frac{1}{2}\right)\in[0,1], E\left[x \mid y=\frac{1}{2}\right] = \int_0^1 x dx=\frac{1}{2}$ and so on. Starting with $E(y), x\left(\frac{1}{2}\right)\in[0,1], E\left[x \mid y=\frac{1}{2}\right] = \int_0^1 x dx=\frac{1}{2}, y\left(\frac{1}{2}\right)\in[0,1], E\left[y \mid x=\frac{1}{2}\right] = \int_0^1 y dy=\frac{1}{2}$

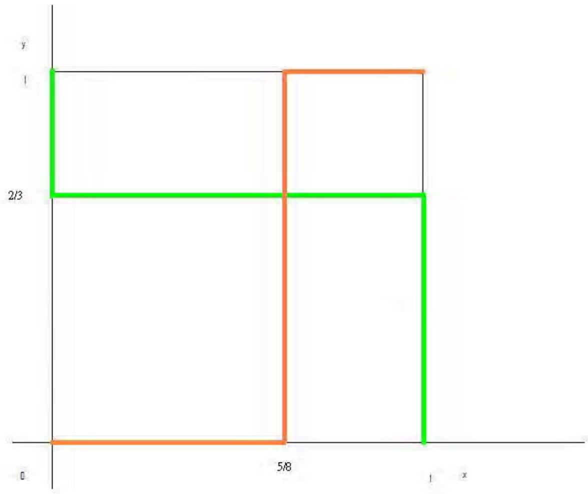

Figure 0.13:

and so on. So the conjectures stay at $E(x) = \frac{1}{2}$ and $E(y) = \frac{1}{2}$ and do not converge to a pure strategy equilibrium and that is the stochastic response Bayesian Rational Prior Equilibrium where $x \in [0,1]$ and $y\left(\frac{1}{2}\right) \in [0,1]$.

Example 4: Conflict Game

$1\backslash 2$	$H(y)$	$T(1-y)$
$H(x)$	$-2, 2$	$3, -1$
$T(1-x)$	$1, -2$	$-3, 3$

There is no pure strategy Nash equilibrium but only mixed strategy Nash equilibrium, $(x = \frac{5}{8}, y = \frac{2}{3})$. The mixed strategy Nash equilibrium is hard to explain. The equilibrium value of x is decided by the reaction function of y and the equilibrium value of y is decided by the x reaction function.

The reaction functions are $x(y) = 0$ if $y > \frac{2}{3}, x(y) \in [0,1]$ if $y = \frac{2}{3}, x(y) = 1$ if $y < \frac{2}{3}$ and $y(x) = 1$ if $x > \frac{5}{8}, y(x) \in [0,1]$ if $x = \frac{5}{8}, y(x) = 0$ if $x < \frac{5}{8}$. Starting with first order uninformative priors, the expected values of x and y are $E(x(y)) = \frac{2}{3}$ and $E(y(x)) = \frac{3}{8}$. Starting with $E(x), y\left(\frac{2}{3}\right) = 1, x(1) = 0, y(0) = 0, x(0) = 1, y(1) = 1, x(1) = 0$ and so on. Starting with $E(y), x\left(\frac{3}{8}\right) = 1, y(1) = 1, x(1) = 0, y(0) = 0, x(0) = 1, y(1) = 1$ and so on. There is no convergence of conjectures and therefore no Bayesian Rational

Prior Equilibrium.

Note that if $x = \frac{5}{8}$, then $y\tilde{}U(0,1)$, and then $E(y) = \frac{1}{2}$, and $x\left(\frac{1}{2}\right) = 1, y(1) = 1, x(1) = 0, y(0) = 0, x(0) = 1, y(1) = 1$ and so on. And if

$y = \frac{2}{3}$, then $x\tilde{}U(0,1)$, and then $E(x) = \frac{1}{2}$, and $y\left(\frac{1}{2}\right) = 0, x(0) = 1, y(1) = 1, x(1) = 0$ and so on. So $(x = \frac{5}{8}, y = \frac{2}{3})$ is not an equilibrium point but just a random point.

Example 5. Hybrid Equilibrium

$1\backslash 2$	$L(y)$	$R(1-y)$
$U(x)$	$2,2$	$0,2$
$D(1-x)$	$0,1$	$1,1$

There are two pure strategy Nash equilibria, (U,L) and (D,R), and a whole range of mixed strategy Nash equilibria: $x = 0, y \in \left[0, \frac{1}{3}\right]; x \in [0,1], y = \frac{1}{3}; x = 1, y \in \left[\frac{1}{3}, 1\right]$.

In this game, player 2 is always indifferent between L and R whatever the action of player 1. The game therefore degenerates into an economics of uncertainty exercise. The optimal action for player 1 is U or $x = 1$. The answer of economics of uncertainty exercise is $x = 1, y \in [0,1]$ which is not a subset of Nash equilibria.

The example shows that Nash equilibrium has no decision theoretic foundation.

Solving by the Bayesian approach:

Given the first order uninformative priors, $E(x) = \frac{2}{3}; y \in [0,1]$ and $E(y) = \frac{1}{2}; x = 1; y \in [0,1]$ and the process converges here.

Starting with $y(x)$, $y \in [0,1]$ and $E(y) = \frac{1}{2}; x = 1; y \in [0,1]$ and the process converges here.

Both processes converge to a hybrid Bayesian rational prior equilibrium: $x = 1, y \in [0,1]$ which is the same as the answer of the economics of uncertainty exercise.

Example 6. Equilibrium by Iterative Elimination of (Weakly) Dominated Strategies.

Very frequently one comes across examples of 2X2 simultaneous game the equilibrium of which could be attained by iterative elimination of weakly dominated strategies. The following game is an example.

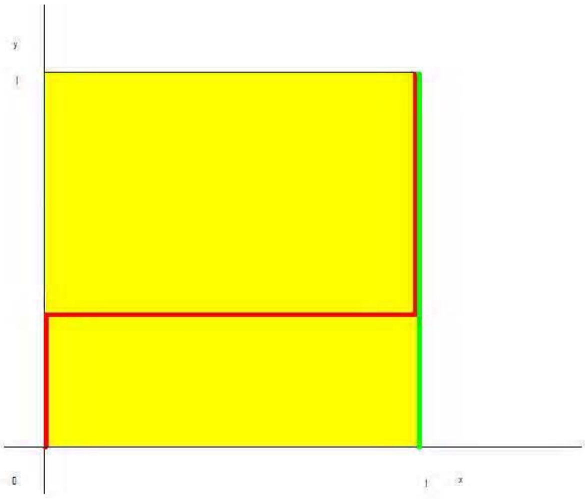

Figure 0.14:

1\2	L	R
U	0, 1	1, 1
D	1, 1	1, 0

There are two Nash equilibria: (D, L) and (U, R). Howecer, only one of them makes sense: (D, L). (U, R) would be weeded out by the elimination of (weakly) dominated strategy.

Note that the Nash equilibrium $(x = 1, y = 0)$ is eliminated for $\Pr[x \in (0, 1)] = 0$ and $\Pr[y \in (0, 1)] = 0$. Therefore $(x = 0, y = 1)$ is the unique Bayesian Rational Prior Equilibrium. This example, as well as example 8a, shows that the Nash equilibrium approach is fundamental flawed.

Example 7. Cournot Competition

Production costs are $10q_1$ and $10q_2$.

$\pi_1 = \left(210 - \frac{q_1}{2} - \frac{6}{5}q_2 - 10\right) q_1$

$\pi_2 = \left(210 - \frac{q_2}{2} - \frac{11}{10}q_1 - 10\right) q_2$

$\frac{\partial \pi_1}{\partial q_1} = 200 - q_1 - \frac{6}{5}q_2 = 0$

$\frac{\partial^2 \pi_1}{\partial (q_1)^2} = -1$

$\frac{\partial^2 \pi_1}{\partial q_1 \partial q_2} = -\frac{6}{5}$

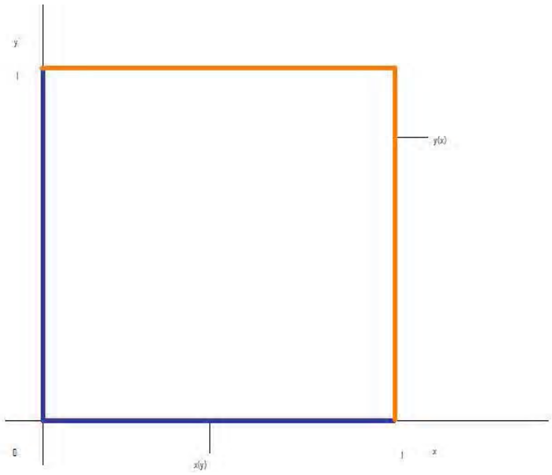

Figure 0.15:

$\frac{\partial \pi_2}{\partial q_2} = 200 - q_2 - \frac{11}{10} q_1 = 0$
$\frac{\partial^2 \pi_2}{\partial (q_2)^2} = -1$
$\frac{\partial^2 \pi_2}{\partial q_2 \partial q_1} = -\frac{11}{10}$
$\frac{\partial^2 \pi_1}{\partial (q_1)^2} \frac{\partial^2 \pi_2}{\partial (q_2)^2} - \frac{\partial^2 \pi_1}{\partial q_1 \partial q_2} \frac{\partial^2 \pi_2}{\partial q_2 \partial q_1} = -\frac{32}{100} < 0$
$q_1 (q_2) = 200 - \frac{6}{5} q_2$
$q_1 (0) = 200$
$q_1 \left(\frac{500}{3}\right) = 0$
$q_2 (q_1) = 200 - \frac{11}{10} q_1$
$q_2 (0) = 200$
$q_2 \left(\frac{2{,}000}{11}\right) = 0$
Nash equilibrium (interior and unstable):
$q_1 = 125, \pi_1 = 62.5$
$q_2 = \frac{125}{2}, \pi_2 = 31.25$
(There are two other Nash equilibria: $q_1 = 200, q_2 = 0; q_1 = 0, q_2 = 200$. These are corner solutions and they are stable.)

Strangely, though q_2 is a better substitute for q_1 than the reverse, firm 2 has only half of firm one's market share and profits. A convenient example for comparison is the symmetrical case, that is, $\pi_1 = \left(210 - \frac{q_1}{2} - \frac{6}{5} q_2 - 10\right) q_1$ and $\pi_2 = \left(210 - \frac{q_2}{2} - \frac{6}{5} q_1 - 10\right) q_2$. The symmetrical Nash equilibrium is

$q_1 = q_2 = \frac{1{,}000}{11}$ and $\pi_1 = \pi_2 = 45.4545$. By having a product that cuts deeper into the rival's demand, firm 2 ended up with a smaller market share, output level and profits while the rival that is supposedly disadvantaged ended up with a larger market share, output level and profits. The Nash equilibrium therefore does not make any sense. Please take note that the interior Nash equilibrium is strategically unstable while the corner Nash equilibria are strategically stable. Classical games theory does not provide guidelines on how to analyze situations such as this. The Bayesian rational prior equilibrium approach solution is presented below:

$E\left[q_1\left(q_2\right)\right] = \frac{3}{500} \int_0^{\frac{500}{3}} \left(200 - \frac{6}{5}q_2\right) dq_2 = 100$

$E\left[q_2\left(q_1\right)\right] = \frac{11}{2{,}000} \int_0^{\frac{500}{3}} \left(200 - \frac{11}{10}q_1\right) dq_1 = 100$

Through reiterations

$E\left(q_2\right) = 100;$

$q_1\left(100\right) = 80; q_2\left(80\right) = 112;$

$q_1\left(112\right) = 65.6; q_2\left(65.6\right) = 127.84;$

$q_1\left(127.84\right) = 46.592; q_2\left(46.592\right) = 148.7488...$

Starting from the other expectation,

$E\left(q_1\right) = 100;$

$q_2\left(100\right) = 90; q_1\left(90\right) = 92;$

$q_2\left(92\right) = 98.8; q_1\left(98.8\right) = 81.44;$

$q_2\left(81.44\right) = 110.416; q_1\left(110.416\right) = 67.5008...$

Both processes converge at $q_1 = 0; q_2 = 200$. This is therefore the Bayesian Rational Prior Equilibrium. At the Bayesian Rational Prior Equilibrium, $\pi_1 = 0, \pi_2 = 20{,}000$.

Example 8: Investment Entry Game.

Double and multiple sided incomplete information games with simultaneous moves or sequential moves are hardly analyzed under current classical theory of games. The probable existence of multiple equilibria makes the solution of such games a daunting task. This example is a double sided investment entry game. Both incumbent firm 1 and the potential entrant firm 2 have two types, high cost or low cost.

Now let the high investment cost firm 1 when facing the high cost firm 2 has the following payoff matrix:

High investment cost firm 1 (probability $\frac{1}{4}$); low cost firm 2 (probability $\frac{9}{10}$).

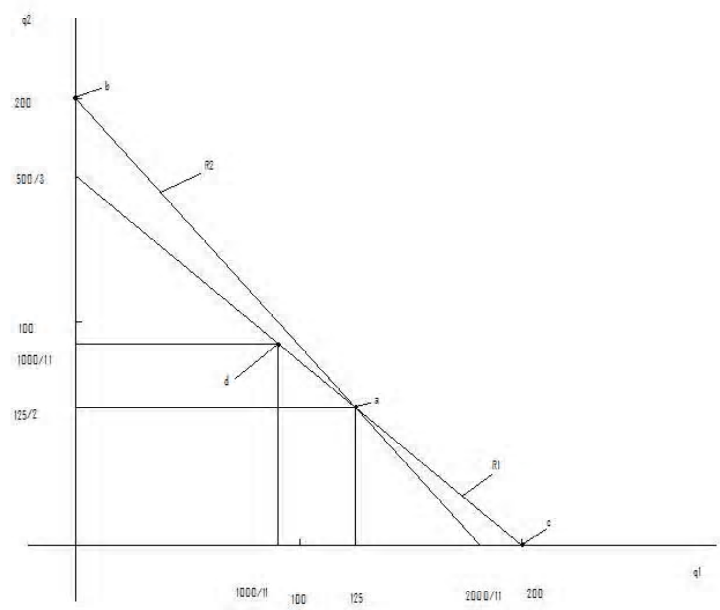
q2
b
200
R2
500/3
100
1000/11
d
a
125/2
R1
c
q1
1000/11
100
125
2000/11
200

Figure 0.16:

1\2	Enter (y)	Refrain (1-y)
Modern (w)	$0, -2$	$7, 0$
Antique (1-w)	$4, 2$	$6, 0$

(Three Nash equilibria: (Antique, Enter), (Modern, Refrain), ($w = 1$ for $y < 1/5, w \in [0, 1]$ for $y = 1/5$ and $w = 0$ for $y > 1/5; y = 1$ for $w < 1/2$, $y \in [0, 1]$ for $w = 1/2$, $y = 0$ for $w > 1/2$).)

When facing the low cost firm 2, the high investment cost firm 1 has the following payoff matrix:

High investment cost firm 1 (probability $\frac{1}{4}$); high cost firm 2 (probability $\frac{1}{10}$).

1\2	Enter (z)	Refrain (1-z)
Modern (w)	$0, -5$	$7, 0$
Antique (1-w)	$4, 1$	$6, 0$

(Three Nash equilibria: (Antique, Enter), (Modern, Refrain), ($w = 0$ for $z > \frac{1}{5}, w \in [0, 1]$ for $z = \frac{1}{5}$ and $w = 1$ for $z < \frac{1}{5}; z = 0$ for $w > \frac{1}{6}, z \in [0, 1]$ for $w = \frac{1}{6}, z = 1$ for $w < \frac{1}{6}$.)

Low investment cost (probability $\frac{3}{4}$); low cost firm 2 (probability $\frac{9}{10}$).

1\2	Enter (y)	Refrain (1-y)
Modern (x)	$3, -2$	$7, 0$
Antique (1-x)	$4, 2$	$6, 0$

(Three Nash equilibria: (Antique, Enter), (Modern, Refrain), ($x = 1$ for $y < 1/2, x \in [0, 1]$ for $y = 1/2$ and $x = 0$ for $y > 1/2; y = 1$ for $x < 1/2$, $y \in [0, 1]$ for $x = 1/2$, $y = 0$ for $x > 1/2$).)

Low investment cost (probability $\frac{3}{4}$); high cost firm 2 (probability $\frac{1}{10}$).

1\2	Enter (z)	Refrain (1-z)
Modern (x)	$3, -5$	$7, 0$
Antique (1-x)	$4, 1$	$6, 0$

(Three Nash equilibria: (Antique, Enter), (Modern, Refrain), ($x = 1$ for $z < 1/2, x \in [0, 1]$ for $z = 1/2$ and $x = 0$ for $z > 1/2; z = 1$ for $x < 1/6$, $z \in [0, 1]$ for $x = 1/6$, $z = 0$ for $x > 1/6$).)

The reaction functions are:

I.

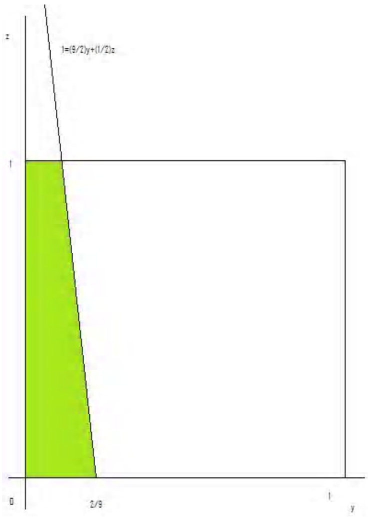

Figure 0.17:

$w(y,z) = 1$ if $\left(\frac{9}{10}\right)(7-7y)+\left(\frac{1}{10}\right)(7-7z) > \left(\frac{9}{10}\right)(6-2y)+\left(\frac{1}{10}\right)(6-2z)$
$w(y,z) \in [0,1]$ if $\left(\frac{9}{10}\right)(7-7y)+\left(\frac{1}{10}\right)(7-7z) = \left(\frac{9}{10}\right)(6-2y)+\left(\frac{1}{10}\right)(6-2z)$
$w(y,z) = 0$ if $\left(\frac{9}{10}\right)(7-7y)+\left(\frac{1}{10}\right)(7-7z) < \left(\frac{9}{10}\right)(6-2y)+\left(\frac{1}{10}\right)(6-2z)$
II.
$x(y,z) = 1$ if $\left(\frac{9}{10}\right)(7-4y)+\left(\frac{1}{10}\right)(7-4z) > \left(\frac{9}{10}\right)(6-2y)+\left(\frac{1}{10}\right)(6-2z)$
$x(y,z) \in [0,1]$ if $\left(\frac{9}{10}\right)(7-4y)+\left(\frac{1}{10}\right)(7-4z) = \left(\frac{9}{10}\right)(6-2y)+\left(\frac{1}{10}\right)(6-2z)$
$x(y,z) = 0$ if $\left(\frac{9}{10}\right)(7-4y)+\left(\frac{1}{10}\right)(7-4z) < \left(\frac{9}{10}\right)(6-2y)+\left(\frac{1}{10}\right)(6-2z)$
III.
$y(w,x) = 1$ if $\left(\frac{3}{4}\right)(2-4w)+\left(\frac{1}{4}\right)(2-4x) > 0$
$y(w,x) \in [0,1]$ if $\left(\frac{3}{4}\right)(2-4w)+\left(\frac{1}{4}\right)(2-4x) = 0$
$y(w,x) = 0$ if $\left(\frac{3}{4}\right)(2-4w)+\left(\frac{1}{4}\right)(2-4x) < 0$
IV.
$z(w,x) = 1$ if $\left(\frac{3}{4}\right)(1-6w)+\left(\frac{1}{4}\right)(1-6x) > 0$
$z(w,x) \in [0,1]$ if $\left(\frac{3}{4}\right)(1-6w)+\left(\frac{1}{4}\right)(1-6x) = 0$
$z(w,x) = 0$ if $\left(\frac{3}{4}\right)(1-6w)+\left(\frac{1}{4}\right)(1-6x) < 0$

The Bayesian theory of games solution is presented below. The expected values of w, x, y and z are: $E(w(y,z)) = \frac{1}{6}$
$E(x(y,z)) = \frac{1}{2}$
$E(y(w,x)) = \frac{1}{2}$

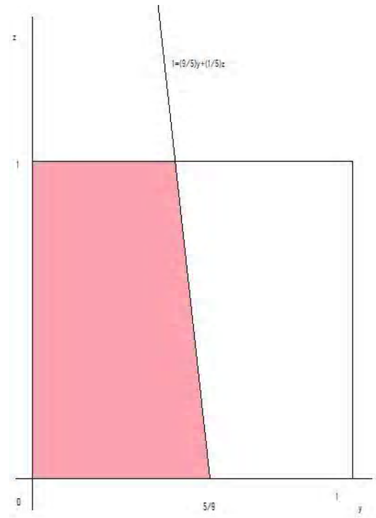

Figure 0.18:

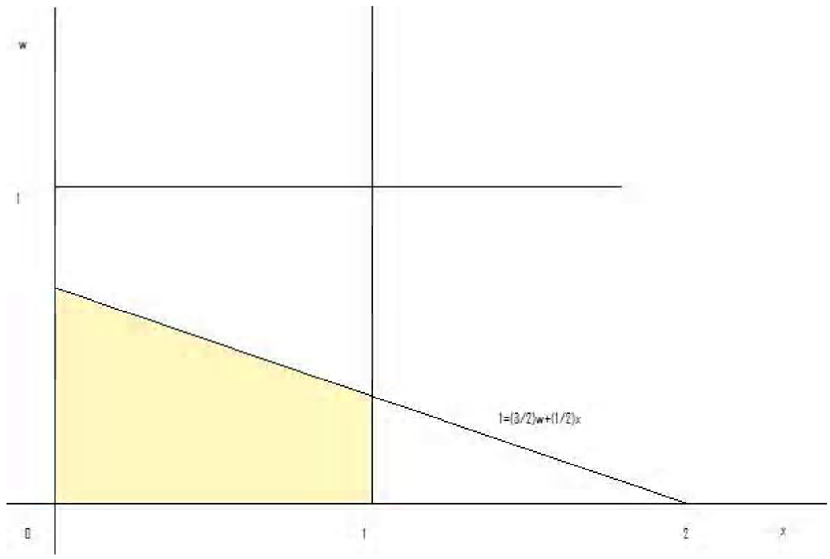

Figure 0.19:

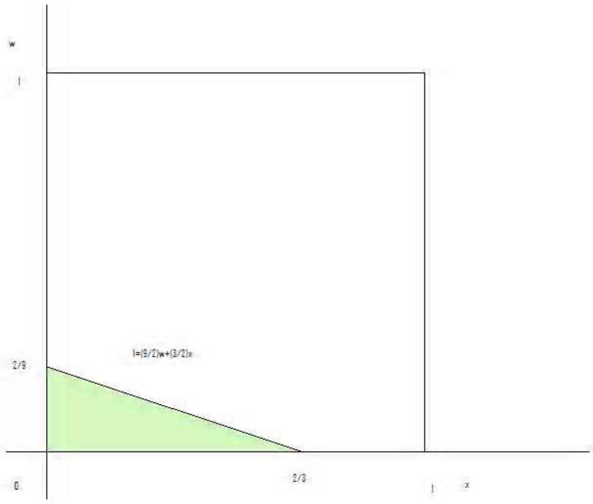

Figure 0.20:

$E(z(w,x)) = \frac{2}{27}$

Starting with $E(w)$ and $E(x)$,

$y\left(\frac{1}{6},\frac{1}{2}\right) = 1, z\left(\frac{1}{6},\frac{1}{2}\right) = 0$ and $w(1,0) = 0, x(1,0) = 0$ and $y(0,0) = 1, z(0,0) = 1$ and $w(1,1) = 0, x(1,1) = 0$ and the point of convergence is $(0,0,1,1)$.

Starting with $E(y)$ and $E(z)$ the process converges to $(0,0,1,1)$ as well. So the unique Bayesian Rational Prior Equilibrium is $w = 0, x = 0, y = 1, z = 1$.

5. Difference Between Simultaneous Move and Imperfect Information.

For illustration, an example of imperfect information sequential market competition game is presented below. The simultaneous game version has been solved in example 7 of section 3. In this imperfect information sequential game, there are two players: Firm 1, the market leader and Firm 2, the market follower. Firm 1 moves first by setting its output level. Firm 2 observes imperfectly the output level of Firm 1 due to a confounding noise term. Firm 2 makes Bayesian rational prior inference on the output level of Firm 1 and then sets its output level. The structure of the game is common knowledge. The cost parameter of Firm 1 which determines the type of Firm 1 is choosen by Nature from a predetermined distribution function which is common knowledge. Once chosen, the type of Firm 1 is private knowledge. The type of Firm 2 is common knowledge. Firm 2 must makes inference on the type and action of Firm 1. The distribution function of the noise term that confounds the observation by Firm 2 on the actual output level of Firm 1 is common knowledge.

q_1, the output level of Firm 1, is the action of Firm 1. q_2, the output level of Firm 2, is the action of Firm 2. The price level faced by firm 1 is $P_1 = 210 - \frac{1}{2}q_1 - \frac{6}{5}q_2$. The price level faced by firm 2 is $P_2 = 210 - \frac{1}{2}q_2 - \frac{11}{10}q_1$. q_2 is a better substitute for q_1 than the other way round. The average production cost of firm 2 is 10. There is no fixed cost. c_1 is the average cost of production of Firm 1. The is no fixed cost. c_1 decides the type of Firm 1. Firm 1 knows c_1 but not Firm 2 does not know c_1. c_1 has a normal distribution which is common knowledge. Firm 2 is almost absolutely sure that c_1 is 10, that is, $c_1\tilde{}N\left(\bar{c_1}, \zeta\right)$ and $\zeta \to 0 \text{lim} c_1 = 10$. The action of Firm 1 is imperfectly observed by Firm 2 with a noise term: $R = q_1 + \epsilon$. ϵ is the noise term. ϵ has a normal distribution: $\epsilon\tilde{}N(0, \kappa)$. The noise term variance approaches positive infinity, that is, $\kappa \to \infty$. The above leads to the following likelihood function: $q_1 \mid R\tilde{}N(q_1, \kappa)$

In solving the game for its Bayesian rational prior equilibrium, the first round of reasoning starts with an uninformative prior. In choosing q_2, Firm 2 solves $q_2 \text{max} E(\pi_2) = \int_{-\infty}^{\infty}\left(210 - \frac{1}{2}q_2 - \frac{11}{10}q_1 - 10\right) q_2 f(q_1 \mid R)\, dq_1$. The optimal solution is $q_2 = 200 - \frac{11}{10}q_1$. Firm 1, being the first mover leader, anticipates the reaction of Firm 2. In determining the optimal level of output, Firm 1 solves: $q_1 \text{max} E(\pi_1) = \int_{-\infty}^{\infty}\left(210 - \frac{1}{2}q_1 - \frac{6}{5}\left(200 - \frac{11}{10}q_1\right) - 10\right) q_1 f(\epsilon)\, d\epsilon$. The optimal solution is the corner solution of $q_1 = 200$. The second order prior distribution function of Firm 2 on the output level of Firm 1 is therefore $q_1\tilde{}N(200, 0)$

Given that, Firm 2 sets $q_2 = 0$. Firm 1 anticipates this and sets $q_1 = 200$. The process of conjectures converges here. The BRPE of the imperfect information sequential game of market competition is therefore $q_1 = 200$ and $q_2 = 0$. The profit levels are $\pi_1 = 20,000$ and $\pi_2 = 0$.

6. Conclusions.

The examples above illustrated the strengths of the Bayesian theory of games. They are all games of two stages. A natural agenda is to extend the framework to repeated games, finite and infinite.

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