Labor Market Friction, Firm Heterogeneity, and Aggregate Employment and Productivity*

Rasmus Lentz
University of Wisconsin–Madison
D.T. Mortensen Center at University of Aarhus

Dale T. Mortensen

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Abstract

The paper is based on a synthesis of a "product variety" version of the firm life cycle model developed by Klette and Kortum (2004) and an equilibrium search model of the labor market with job to job flows introduced by Mortensen (2003). The model is particularly well suited for the measurement of labor misallocation in a frictional environment where job creation requires investment over and beyond that required to overcome the matching frictions. In the setting at hand, job creation requires investment into a broad notion of know-how capital.

In the model, a final consumption good is produced from continuum of intermediate product and service varieties that are themselves products of an aggregation of heterogenous labor inputs. Intermediate goods producers generally differ with respect to their productivity. New firms enter and continuing firms grow by developing new product varieties through a costly innovation process. The time required to match workers and jobs in the model depends on the total number of vacancies and possibly on the fraction of employed to unemployed workers. Workers receive job offers both while employed and unemployed. Wages are set through bargaining over marginal match surplus where the worker's bargaining position may improve with the arrival of outside job opportunities as in Dey and Flinn (2005) and Cahuc et al. (2006). A job separation occurs if either a worker quits or a job is destroyed. We show that a general equilibrium solution to the model exists and that the equilibrium is broadly consistent with observed dispersion in firm productivity, wages, and the relationship between them as well as patterns of worker flows. The model implies that frictions, both in the labor market and in the firm growth process, can be important determinants of aggregate productivity as well as aggregate employment. The model delivers a wage dispersion decomposition into worker and firm side contributions as well as frictional dispersion that can itself be decomposed into bargaining position dispersion and idiosyncratic dispersion due to frictional labor force fluctuations. The model furthermore

*Dale T. Mortensen died in 2014. I miss him dearly. It has taken me much too long to return to this paper, which is the last paper I did with him in a research agenda we started almost 20 years ago as an exploration of the demand side of frictional labor markets. The last version of the paper while he was still alive is dated 2012. It includes a full model equilibrium formulation, solution, and numerical simulation of a homogenous worker model that is otherwise identical to that in the current version of the paper. To the extent that the additions to the paper after his death would not have been to his liking, you now know what they are and need not hold them against him. With that said, I do believe the shape of the paper is consistent with our joint agenda. I have benefited from encouraging and supportive advice from Mike Elsby as well as the comments from seminar participants at Cornell, Carnegie-Mellon, SUNY Stony Brook, University of British Columbia, University College London, and Yale. Financial support of this research includes grants from the U.S. National Science Foundation and the Danish National Research Foundation.

displays rich sorting patterns both between worker and firm types as well as between worker and coworker types. Inference is drawn from Danish matched employer-employee data.

1 Introduction

Firm productivity differentials are large and persistent. Average wages paid by firms are positively correlated with firm productivity and more productive firms are larger and are more likely to export. Empirical evidence supports the view that workers move from lower to higher paying jobs. These differences imply that the reallocation process may be an important determinate of aggregate productivity as well as employment. The purpose of this paper is to develop a tractable equilibrium model that explains these and other stylized facts relating worker flows, wages, and productivity across firms. The equilibrium solution to the model also provides a framework for studying the determinants of the distribution of productivity across firms as well as the level of aggregate employment and productivity when frictions in the labor market are present.[1]

The paper is based on a synthesis of a "product variety" version of the firm life cycle model developed by Klette and Kortum (2004), the Melitz (2003) model of heterogeneous firms, and an equilibrium search model of the labor market in the spirit of that introduced in Mortensen (2003). Households value future streams of consumption and leisure. A continuum of intermediate product and service varieties are produced with labor input. These serve as inputs in the production of a final good which can be used either for consumption or investment.

Both potential and continuing firms invest in costly R&D. Potential firms that innovate enter the economy as operating firms. Continuing firms grow by creating and developing new product lines. The creator of a new variety is the sole supplier of the variety which through monopoly rents allows the needed surplus to motivate the investment required to innovate. The firm's derived demand for labor is limited by the demand for the firm's portfolio of products. Existing products are destroyed at an exogenous rate and a firm dies when all of its products lines are destroyed.

Time is required to match workers and jobs in the model. Workers search for job opportunities both while employed and unemployed. Wages are set as the outcome of a strategic bargaining problem over marginal match surplus where the worker's bargaining position may improve with the arrival of outside job opportunities similar to Dey and Flinn (2005) and Cahuc et al. (2006). A job separation occurs if either a worker quits or a job is destroyed. Firms with vacancies also invest in recruiting workers. The rates at which workers and vacancies meet are determined by a matching function that depends on aggregate worker search and recruiting effort.

[1] In this respect, the paper is closely related to Lagos (2006).

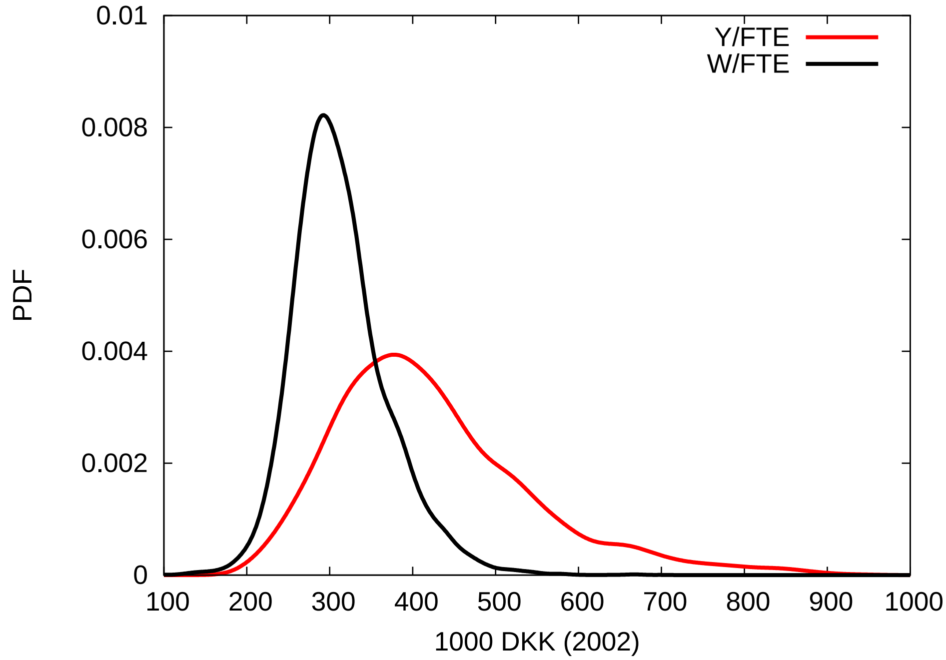
In general steady state equilibrium, consumption, the total number of product lines supplied by each firm type, the number of worker employed by each firm type, and the number of employed workers are all stationary. The steady state equilibrium distributions of both labor productivity and wages across firms and the employment rate reflect these steady state conditions as well as optimality requirements. A steady state equilibrium is defined as a consumption flow, a firm entry rate, a product creation rate and a measure of products supplied for each firm type, a firm recruiting strategy, and a worker search strategy that satisfy optimality and the steady state conditions. The existence of at least one steady state equilibrium solution is established. The equilibrium solution illustrates the importance of both labor market frictions and the cost of firm growth as determinants of aggregate productivity and employment.

2 Danish Wage and Productivity Dispersion

Both productivity and wage dispersion are large and persistent in the micro data. (Early work documenting productivity dispersion is reviewed in Bartelsman and Doms (2000). See Davis and Haltiwanger (1991) on wage dispersion.) Foster et al. (2001) present reduced form evidence that workers are reallocated from less to more productive firms as well. Finally, Lentz and Mortensen (2008) estimate a structural model of firm dynamics, closely related to that studied here, using Danish longitudinal firm data. It implies that about half of productivity growth can be attributed to reallocation within firm birth cohort as reflected in the higher relative growth rates of more productive firms.

Nagypál (2004) finds that over half of all U.S. prime age full time workers who separate in a month are reemployed with another employer in the next month and that 70% of those who don't leave the labor force experience such a job-to-job movement. That workers who quit to move to another employer typically receive an increase in earnings has long been know. (Bartel and Borjas (1981) and Mincer (1986) represent early work on the finding.) Recently structural models of job to job movements have added more details. For example, Jolivet et al. (2006) show that an estimated off-the-shelf search on-the-job model does a good job of explaining the observed extent to which the distribution of wages offered to new employees is stochastically dominated by the distribution of wages earned in 9 out of 11 OECD countries. Christensen et al. (2005) estimate a structural model that allows for an endogenous choice of search intensity using Danish matched worker-employer data.

Figure 1: Danish Firm Wage and Productivity Distributions



All of these studies supports the basic view that wage dispersion exits in the sense that different firms pay similar workers differently and that workers respond to these differences by moving from lower to higher paying employers.

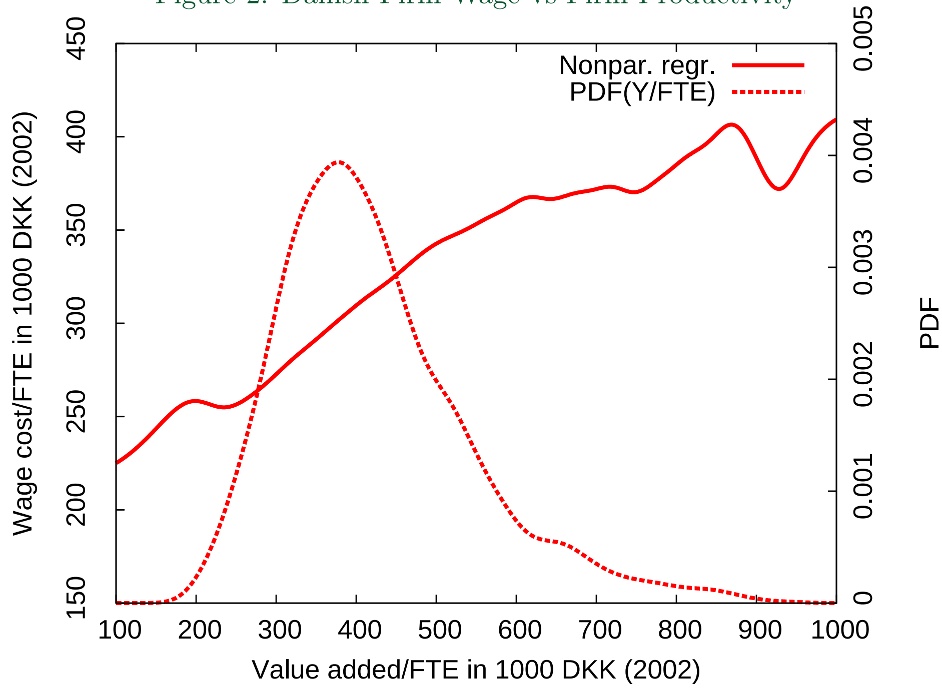
This paper is based on evidence that wage dispersion is induced by productive firm heterogeneity through "rent sharing." Given search friction, match rents are larger at more productive firms. Hence, if the wage is determined by some form of rent sharing, then we should see a positive cross section relationship between the average wage paid and firm productivity.

In this section, we present Danish evidence for wage and productivity dispersion as well as a positive association between them drawn from an longitudinal files on the value added (Y), full time equivalent (FTE) employment (N) and the wage bill (W) paid by privately owned Danish firms. These firm accounting data are collected annually in a survey conducted by Statistics Denmark and are supplemented from tax records. The survey is a rolling panel and the sampling of firms is based on firm size and revenue.[2]

In Figure 1, cross firm probability density functions for wage cost per worker, as reflected in the ratio of the wage bill to FTE employment (W/N), and for average productivity per worker,

[2]Only firms with 5 or more employees and revenue above 500M DKK are included. All firms with labor force at or above 50 are included in the survey. The sampling proportions for 20-49 workers is 50%, for 10-19 is 20%, and for 5-9 is 10%.

Figure 2: Danish Firm Wage vs Firm Productivity



as measured by the ratio of valued added to employment (Y/N), are plotted for the year 2002.[3] Note that dispersion in both the average wage paid and labor productivity are large. Specifically, four fold productivity differentials lie within the 5-95 percentile support of the data as well as over 100% differences in the average wages paid. The relative positions of the two distributions suggest that average labor productivity exceeds the average wage paid, as one should expect. In Figure 2, a non-parametric regression of firm labor productivity on average firm wage cost is plotted. This evidence suggests that the relationship is close to linear. The strong positive relationship between firm wage and productivity supports "rent-sharing" theories of the wage determination such as the one embodied in this paper.

3 The Model

3.1 Households

There is a continuum L of identical households each composed of a unit measure of individuals who are either employed or not. Individuals differ by type, $h \in \{1, 2, \ldots, H\}$ with measure L_h in the population, $\sum_h L_h = 1$. The type composition of individuals within a household is identical to that

[3] At the moment, we have data for 1999-2002 inclusive. All the properties characteristics of the data pointed out here are virtually identical for each of the other years as well.

of the overall population. There is a single final good used for either consumption or investment and produced by a competive market sector using a continuum of intermediate goods. Each intermediate good is supplied by a single firm. Employed workers produce intermediate goods and those not employed produce a substitute for the final good at rate b_h depending on the individual's type. Household utility is the expected present value of consumption where the fixed discount rate r is the equilibrium rate of interest.

3.2 Intermediate Product Demand and Supply

A single final good is supplied by a competitive sector and the aggregate quantity produced is a function of a continuum of intermediate inputs each supplied by a monopolist. Specifically, final output is determined by the following constant returns to scale Dixit-Stigliz production function,

$$Y = A\left[\frac{1}{K^{\sigma}}\int_0^K \alpha(j)^{\frac{1}{\sigma}} x(j)^{\frac{\sigma-1}{\sigma}} dj\right]^{\frac{\sigma}{\sigma-1}}, \tag{1}$$

where $x(j)$ is the quantity of intermediate product j available, K is the measure of intermediate goods, and $\sigma > 1$ is the elasticity of substitution between inputs, and $\alpha(j)$ is proportional to the product j share of demand expenditure. The demand for each input given final output is

$$x(j) = \alpha(j)\frac{Y}{K}A^{\sigma-1}p(j)^{-\sigma},$$

where $p(j)$ is the price of input j expressed in units of the final output.

Each intermediate good is produced with labor and supplied by a single firm according to the constant elasticity of substitution aggregation of labor inputs,

$$x_{\vec{n}}(j \mid \vec{q})^{\frac{\sigma-1}{\sigma}} = \sum_{h=1}^{H} q_h(j) n_h^{\frac{\sigma-1}{\sigma}},$$

where $q_h(j)$ captures the relative contribution of labor type h to production of good j and $n(j) = (n_1(j), \ldots, n_H(j))$ is product j employment of the different types of labor. The demand and productivity parameters pair $(\alpha, \vec{q})$ determine the revenue schedule of the supplier of product j in the model. Given this pair of parameters, the supplying firm's revenue from the product is

$$px = R_{\vec{n}}(\vec{\mu}) = \left(\frac{Y}{K}\right)^{1/\sigma} \sum_{h=1}^{H} \mu_h n_h^{\frac{\sigma-1}{\sigma}} = \sum_{h=1}^{H} \hat{R}_{n_h}(\mu_h), \tag{2}$$

where $\mu_h = \left(\alpha A^{\sigma-1}\right)^{1/\sigma} q_h$ and $\hat{R}_n(\mu) = (Y/K)^{1/\sigma} \mu n^{\frac{\sigma-1}{\sigma}}$. Equation (2) makes clear that in this model with monopolistic pricing, demand and productivity shocks work in the same way by shifting the revenue function. Absent direct observation of quantity produced, the model cannot distinguish value productivity from quantity productivity when σ is finite. That is, we generally cannot distinguish if a firm looks highly productive because it has a high demand realization α or a high productivity realization q. In the perfect substitutes case ($\sigma = \infty$), however, there are no demand constraints and we have $\mu_h = Aq_h$, that is the revenue function shifts across product lines only as a result of q dispersion.

The assumption that the substitution elasticity between labor types equals that of the final goods aggregator delivers that the revenue function is additively separable in each labor factor. Thus, the marginal revenue of a given labor factor is independent of the state of the other factors. While not without loss of generality, it allows the analysis a powerful simplification of the vacancy and wage determination while maintaining.

3.3 Bargaining and Search

The labor market is segmented by worker type h. Within each segment search is random subject to the segment specific equilibrium meeting rates and vacancy offer distributions.

When an individual worker meets an employer, the two bargain over their match surplus. Given that the value of the worker's current employment status is the threat point in the bilateral bargaining game, the generalized Nash solution allocates the share $\beta \in (0, 1)$ to the worker and the residual to the employer. A match is formed if its value exceeds the worker's outside option.

For example, if a type h worker is unemployed, then match surplus to be divided is $S = X - U_h$ where X is the match value and U_h is the worker's value of unemployment. Given the Nash solution to the bargaining problem, the worker's value of unemployment is

$$rU_h = b_h + \lambda_{0h}\beta \int_U^\infty (X - U_h) dF_h(X - U_h) = b_h + \lambda_{0h}\beta E_h [S], \tag{3}$$

where b_h is the flow value of home production, λ_{0h} is the segment conditional rate a which an unemployed worker meets vacancies, $F_h(S)$ is the offer c.d.f., the distribution of match surplus over vacancies, and $E_h [S] = \int_0^\infty S dF_h (S)$.

We adopt a bargaining model that resembles that of Dey and Flinn (2005) and Cahuc et al. (2006) where firms match outside offers, and the employment contract specifies that the worker receives the share $\beta \in (0, 1)$ of the surplus value of the match. It can be viewed as the outcome of a bilateral bargaining problem between worker and employer when they meet and the worker's outside option is full surplus extraction from the outside employer. In this case, a worker voluntarily moves from job-to-job if and only if the total value of the new match exceeds that of the old. Hence, the worker's value of employment at the beginning of a new job is $W = W_0 + \beta(X - W_0)$ where X is the value of the marginal match with the new employer when employed and W_0 is equal to the value of the previous match if the worker was hired voluntary and the value of unemployment if either the worker was previously unemployed or the move was involuntary.

3.4 Vacancy Creation and Marginal Match Surplus

Firms are composed of product lines, each of which faces a destruction risk equal to δ_1. Each product line has its own labor force; direct reallocation across lines within the firm has no cost advantage relative to worker reallocation through the labor market. Hence, the employing unit in the model is the product line, a coalition composed of an employer and its labor force. The product line's hiring decision is a specification of how much to hire in each market segment, $\vec{v} = (v_1, \ldots, v_H)$ at a total hiring cost of $c(\vec{v}) = \sum_h c_{0h}(v_h)$. The decision is made in service of the maximization of the coalition surplus thereby eliminating the Stole and Zviebel (1996) over-hiring inefficiency.

The size of the type h labor force employed in a product line, denoted as n_h, is a discrete variable defined on the non-negative integers. As the model is formulated in continuous time and hires and separations are individual decisions, only a one unit change in n_h can occur in any instant if the line continues into the future. Only in the case of product destruction, an event which occurs infrequently at rate δ_1, is this rule violated. In this case, all workers are laid off. Hence, type h employment in a product line is a birth-death stochastic process with transition rate from n_h to $n_h + 1$ equal its market segment conditional hire frequency, from n_h to $n_h - 1$ equal to the frequency with which workers separate, and from any n_h to $n_h = 0$ with frequency δ_1.

Define $V_{\vec{n}}(\vec{\mu})$ as the expected present value of the future income of the members of a product line composed of the employer and the n employees. Furthermore, define $S_{\vec{n}}(\vec{\mu}) = V_{\vec{n}}(\vec{\mu}) - \sum_h n_h U_h$ as the value of the coalition net of the workers' combined value of unemployment. The additive nature of the product line revenue over worker type conditional employment implies that coalition value and surplus can in a similar way be written as $V_{\vec{n}}(\vec{\mu}) = \sum_h \hat{V}_{n_h}(\mu_h)$ and $S_{\vec{n}}(\vec{\mu}) = \sum_h \hat{S}_{n_h,h}(\mu_h)$. $\hat{V}_{n_h}(\mu_h)$ is the segment h coalition value of a product line's type h workers and the $\hat{R}_{n_h}(\mu_h)$ part of the firm's revenue. The h coalition surplus is defined as $\hat{S}_{n,h}(\mu) = \hat{V}_n(\mu) - nU_h$. The marginal segment h coalition surplus of a worker is $\Delta\hat{S}_{n,h}(\mu) = \hat{S}_{n,h}(\mu) - \hat{S}_{n-1,h}(\mu) = \Delta\hat{V}_n(\mu) - U_h$.

Circumflex is in the following used to signify a type h dependence of the variable in question and the h and μ_h dependence is suppressed in notation to ease exposition whenever it can be done without ambiguity. Thus, $\hat{S}_n$ will be used as a short hand for $\hat{S}_{n,h}(\mu_h)$ where n is the number of type h workers in the product line. $\hat{U}$ is used to reference U_h, and so forth.

If a type h worker receives a job offer from another firm where the marginal value of a worker X in the h segment exceeds $\Delta\hat{V}_n$, the worker quits and receives value $W = \Delta\hat{V}_n + \beta\left(X - \Delta\hat{V}_n\right)$ with the new firm. In this case, the total value of the remainder of the type h coalition moves from $\hat{V}_n$ to $\hat{V}_{n-1}$. The asset equation for the h coalition surplus given type h labor force size n can be written as,

$$\begin{aligned} r\hat{S}_n = {} & \hat{R}_n + \hat{\pi}\big(\Delta\hat{V}_{n+1} - \hat{U}\big) + \delta_1\big(n\hat{U} - \hat{V}_n\big) + \hat{\delta}_0 n\big(\hat{U} - \Delta\hat{V}_n\big) \\ & + \hat{\lambda}_1 n \int_{\Delta\hat{V}_n}^{\infty} \Big[\Delta\hat{V}_n + \beta\big(X - \Delta\hat{V}_n\big) + \hat{V}_{n-1} - \hat{V}_n\Big]\, d\hat{F}(X-U) \\ & + \hat{\lambda}_2 n \int_{\hat{U}}^{\infty} \Big[\hat{U} + \beta(X - \hat{U}) + \hat{V}_{n-1} - \hat{V}_n\Big]\, d\hat{F}(X-U) - nr\hat{U}, \end{aligned}$$

where $\hat{\lambda}_1$ is the rate at which employed type h workers generate outside offers, $\hat{\lambda}_2$ represents the quit rate of those moving to another job for other reasons with a bargaining position equal to that of unemployment, $\hat{\delta}_0$ is the transition rate to unemployment, δ_1 denotes the product destruction rate, $\hat{F}(S)$ is again the market h segment offer distribution, and finally $\hat{\pi}\left(\Delta\hat{V}_{n+1} - \hat{U}\right)$ is the value of the product line's type h worker recruiting operation. This simplifies to,

$$\begin{aligned} (r+\delta_1)\hat{S}_n = {} & \hat{R}_n - \hat{b}n + \hat{\pi}\big(\Delta\hat{S}_{n+1}\big) - n\big(\hat{\delta}_0 + \hat{\lambda}_2\big)\Delta\hat{S}_n \\ & + n\beta\left[\hat{\lambda}_1 \int_{\Delta\hat{S}_n}^{\infty}\Big[1 - \hat{F}(X)\Big]\, dX + \left(\hat{\lambda}_2 - \hat{\lambda}_0\right)\hat{E}\left[S\right]\right]. \end{aligned} \tag{4}$$

The firm's recruiting strategy determines the number of vacancies posted contingent on the current level of employment in a firm. An optimal strategy maximizes the value that accrues to the current coalition from recruitment. Upon meeting and hiring a type h worker with outside option of $W_0 \leq \Delta\hat{V}_{n_h+1}(\mu_h)$, the value of the h-coalition changes from $\hat{V}_{n_h}(\mu_h)$ to $\hat{V}_{n_h+1}(\mu_h)$ and there is no change to the rest of product line coalition. The outside worker receives value $W_0 + \beta\left(\Delta\hat{V}_{n_h+1}(\mu_h) - W_0\right)$. Hence, the gain to the coalition in this case is $\Delta\hat{V}_{n+1}(\mu_h) - W_0 - \beta\left(\Delta\hat{V}_{n+1}(\mu) - W_0\right) = (1-\beta)\left(\Delta\hat{V}_{n+1}(\mu) - W_0\right)$.

The value of the recruitment operation maximizes the expectation of the product of the coalition's share of the match surplus and the rates at which workers in different employment states are met. As all unemployed workers but only those employed at smaller match values accept, the number of segment h vacancies, v_h, posted by a product line with n_h type h employees is the solution to the optimization problem,

$$\pi_h(X) = \max_{v\geq 0}\left\{(1-\beta)v\left[(\eta_{0h}+\eta_{2h})\,X+\eta_{1h}\int_0^X (X-Y)dG_h(Y)\right]-c_{0h}(v)\right\}, \tag{5}$$

where $G_h(X)$ denotes the fraction of type h workers employed at firms with marginal match surplus X or less, and $c_{0h}(v)$ represents the cost of posting v vacancies in market segment h, an increasing and convex function. The parameters η_{0h}, η_{1h}, and η_{2h} are rates per vacancy posted at which employers meet unemployed workers, employed workers who have the option of turning down an outside offer, and employed workers who move for reasons exogenous to the model. Denote the optimal number of type h vacancies given segment h coalition surplus X by,

$$\hat{v}_h(X) = \arg\max_{v\geq 0}\left\{(1-\beta)v\left[(\eta_{0h}+\eta_{2h})\,X+\eta_{1h}\int_0^X G_h(Y)dY\right]-c_{0h}(v)\right\}, \tag{6}$$

Note that the optimal number of vacancies does not depend directly on either productivity or market size. In short, the value of the match is a sufficient statistic.

As all offers are acceptable to an unemployed worker and one who otherwise transitions to unemployment but only those employed worker in a match with a lower value that the match will voluntarily accept an offer, the employment size contingent hire frequency is

$$\hat{a}_{n,h}(\mu) = \left[\eta_{0h}+\eta_{1h}\hat{G}_h\big(\Delta\hat{S}_{n+1,h}(\mu)\big)+\eta_{2h}\right]\hat{v}_h\big(\Delta\hat{S}_{n+1,h}(\mu)\big). \tag{7}$$

By equation (4) one obtains the expression for the marginal h-coalition surplus,

$$\begin{aligned}\left(r+\delta_1+n\left(\hat{\delta}_0+\hat{\lambda}_2\right)\right)\Delta\hat{S}_n = {} & \Delta\hat{R}_n-\hat{b}+\hat{\pi}(\Delta\hat{S}_{n+1})-\hat{\pi}(\Delta\hat{S}_n)-\left(\hat{\lambda}_0-\hat{\lambda}_2\right)\beta\hat{E}\left[S\right] \\ & +\left(\hat{\delta}_0+\hat{\lambda}_2\right)(n-1)\,\Delta\hat{S}_{n-1}+\hat{\lambda}_1 n\beta\int_{\Delta\hat{S}_n}^{\infty}\left[1-\hat{F}(X)\right]dX \\ & -\hat{\lambda}_1(n-1)\beta\int_{\Delta\hat{S}_{n-1}}^{\infty}\left[1-\hat{F}(X)\right]dX,\end{aligned} \tag{8}$$

were $\Delta\hat{R}_n(\mu)=\hat{R}_n(\mu)-\hat{R}_{n-1}(\mu)$ is the revenue product of the marginal type h worker.

3.5 Recruiting and the Maximum Labor Force Size

Because marginal revenue product of a type h worker declines in type h labor force size n_h, marginal match surplus, $\Delta\hat{S}_{n_h}$, diminishes as well. As a consequence, a maximum labor force size exists that depends on type h value TFP, μ_h. Define $\hat{m}_h(\mu)$ as the upper product line size for a given μ realization. With the usual short hand, it is defined by $\Delta\hat{S}_{\hat{m}} = 0$. Since $\Delta\hat{S}_n$ is decreasing in n for any finite σ, there is no option value from hiring at the upper firm size bound. Hence, by equation (8) and integration by parts (while ignoring integer constraints on $\hat{m}$),

$$\begin{aligned}\Delta\hat{R}_{\hat{m}} = {} & \hat{b} + \left(\hat{\lambda}_0 - \hat{\lambda}_2 - \hat{\lambda}_1\right)\beta\hat{E}\left[S\right] \\ & - \lambda_1\left(\hat{m}-1\right)\beta\int_0^{\Delta S_{\hat{m}-1}}[1-\hat{F}(X)]dX - \left(\hat{\delta}_0 + \hat{\lambda}_2\right)\left(\hat{m}-1\right)\Delta\hat{S}_{\hat{m}-1}. \end{aligned} \tag{9}$$

Since $\Delta\hat{R}_{\hat{m}}$ is monotonically decreasing in $\Delta\hat{S}_{\hat{m}-1}$ one can bound the expression for $\Delta\hat{R}_{\hat{m}}$ by,

$$\Delta\hat{R}_{\hat{m}} \leq \Delta\hat{R}_{\hat{m}^*} = \hat{b} + \left(\hat{\lambda}_0 - \hat{\lambda}_1 - \hat{\lambda}_2\right)\beta\hat{E}\left[S\right]. \tag{10}$$

Equation (10) states that ignoring labor hoarding incentives, labor is hired to the point where the marginal revenue product of an additional worker equals an unemployed worker's income flow plus the value of the unemployed search technology advantage relative to that of employed search, $\hat{\lambda}_0 - \hat{\lambda}_1 - \hat{\lambda}_2$. This is independent of the product line's h-type value productivity parameter, μ_h. Denote this value of n as $\hat{m}^*$. Generally, $\hat{m} > \hat{m}^*$ reflecting that firms may choose to hoard labor in anticipation of the costly replacement of future quits. Of course in the perfect substitutes case ($\sigma = \infty$), there is no upper size bound and by definition no labor hoarding. In this case, firm size is finite purely as a result of labor market frictions.

3.6 Wages

A type h worker's wage is a result of the worker's history and the h-coalition marginal match surplus value. The latter depends on the number of workers employed in the product line's h-coalition and the value factor productivity μ_h, $\Delta\hat{S}_{n_h,h}(\mu_h)$. The surplus value offered to any new type h employee must equal the surplus value of the worker's previous employment status, represented as $W_0 - U$, plus the share β of the difference between it and the new match surplus. Consequently, the surplus value of employment given that the worker is hired when there are n_h type h employees in the product line is $\hat{W}_{n_h+1,h}\left(W_0 - \hat{U}_h, \mu_h\right) - \hat{U}_h = W_0 - \hat{U}_h + \beta\left(\Delta\hat{S}_{n_h+1,h}\left(\mu_h\right) - \left(W_0 - \hat{U}_h\right)\right)$, where $W_0 = \hat{U}_h$ if the worker is hired while unemployed or in the process of an involuntary job-to-job move, and the value of employment in the worker's previous match otherwise. Although there are a continuum of payment streams that are consistent with $\hat{W}_{n_h+1,h}\left(W_0 - \hat{U}_h, \mu_h\right)$, it is unique under the additional restriction that the flow wage paid must be the same for all current employees with the same market experience as embodied in the value of their employment status when hired. Both legal requirements and fairness considerations provide reasons for this equal pay condition.

By the fact that separation is efficient, $W_0 - \hat{U}_h$ is necessarily less than the surplus of the match at its inception, however as time passes in the match, the worker's share of the surplus of the marginal worker may drop and possibly below $W_0 - \hat{U}_h$. However, efficient separation requires the lifetime wage is revised downward if the marginal surplus is positive but is less than $W_0 - \hat{U}_h$. Hence, if n_h in the firm evolves so as $W_0 - \hat{U}_h > \Delta\hat{S}_{n_h,h}(\mu_h) \geq 0$, we assume that the employment contract awards the worker with the full surplus of the match, $\Delta\hat{S}_{n_h,h}(\mu_h)$. Hence, conditional on the worker's labor market history and the number of employees in the product line, the worker's continuation value is given by,

$$\hat{W}_{n,h}(X,\mu_h) = U_h + \min\left[\Delta\hat{S}_{n,h}(\mu_h), X + \beta\left(\Delta\hat{S}_{n,h}\left(\mu_h\right) - X\right)\right], \tag{11}$$

where X is a sufficient statistic for the worker's labor market history and represents the surplus value of the best outside offer given since the most recent layoff or exogenous reallocation shock. Given this specification, the worker only quits voluntarily if offered a job that offers surplus Y in excess of $\Delta\hat{S}_n$. If the worker receives an outside offer that exceeds, X but is less than $\Delta\hat{S}_n$, then X evolves to take on this new and higher value, $X = Y$. From the perspective of the product line's coalition surplus, this is a non-event, but the law of motion of X matters for the determination of the wage flow.

Denote by $\hat{w}_{n,h}\left(X,\mu\right)$ the wage flow of a type h worker with labor market history X in an h-coalition with n workers and value factor productivity μ. It satisfies,

$$\begin{aligned}\left(r + \delta_1 + \hat{d}_n\right)\left(\hat{W}_n(X) - \hat{U}\right) = {} & \hat{w}_n(X) - \hat{b} + \left(\hat{\lambda}_2 - \hat{\lambda}_0\right)\beta\hat{E}\left[S\right] + \\ & \hat{a}_n\Delta\hat{W}_{n+1}(X) - (n-1)\hat{d}_n\Delta\hat{W}_n(X) + \\ & \hat{\lambda}_1\int_X^{\Delta\hat{S}_n}\hat{W}_n(X')d\hat{F}(X') + \\ & \hat{\lambda}_1\int_{\Delta\hat{S}_n}^{\infty}\left(\Delta\hat{S}_n + \beta\left(X' - \Delta\hat{S}_n\right)\right)d\hat{F}(X'),\end{aligned} \tag{12}$$

where the separation rate of type h workers in a μ productivity h-coalition with n workers not including product line destruction is,

$$\hat{d}_{n,h}(\mu) = \hat{\delta}_{0h} + \hat{\lambda}_{2h} + \hat{\lambda}_{1h}\big[1 - \hat{F}_h\big(\Delta\hat{S}_{n,h}(\mu)\big)\big]. \tag{13}$$

The wage flow is readily calculated for a given solution of $\Delta\hat{S}_{n,h}(\mu)$ from equation (8).

As emphasized in both Dey and Flinn (2005) and Postel-Vinay and Robin (2002) one may observe both wage increases and decreases between jobs in this model. However, unlike these two previous examples, wages can both decrease and increase within the job as well. Wages will increase as a result of an arrival of an outside offer that increases W_0 but is below the marginal coalition surplus of the worker's current job. However, wages will also rise and fall as changes in the coalitions labor force size impact the coalition's marginal match surplus. In the early life of a product line, the expansion of its labor force size will tend to imply decreasing wage profiles which will be impacted in the opposite direction as workers improve their bargaining position as a result of outside offer arrivals.

3.7 Product Innovation and Entry.

Firm size as reflected in the number of products supplied is generated by a research and development process as introduced in Klette and Kortum (2004). At any point in time, a firm with k product lines can grow only by creating new product varieties. Investment in R&D is required to create new products. Specifically, the firm's R&D investment flow generates new product arrivals at frequency γk where γ represents the firm's innovation rate per product line. The total R&D investment cost expressed in terms of output is $c_1(\gamma)k$ where $c_1(\gamma)$ is assumed to be strictly increasing and convex function. The assumption that the total cost of R&D investment is linearly homogeneous in the new product arrival flow, γk, and the number of existing product, k, "captures the idea that a firm's knowledge capital facilitates innovation," in the words of Klette and Kortum (2004). The specification assumption also implies that Gibrat's law holds in the sense that innovation rates are size independent contingent on type, a property needed to match the data on firm growth. Finally, every product is subject to destruction risk with exogenous frequency δ_1. Given this specification, the number of products supplied by any firm is a stochastic birth-death process characterized by "birth rate" γ and "death rate" δ_1. Firms enter with one product and exit occurs when a single product firm has its last product destroyed.

The value productivity of product lines, μ, are realized independent from each other when a new product is created. The distribution of market size is assumed independent of type. Firm type heterogeneity enters the model through type dependence of the distribution of product line factor productivity, $\vec{q}$. Consequently, the distribution of the value of random variable $\mu = \left(\alpha A^{\sigma-1}\right)^{1/\sigma} \vec{q}$

is also type dependent. Let $\Upsilon_\tau(\mu), \tau \in \{1, 2...., T\}$, denote the cumulative distribution of μ for any firm of type τ. The magnitude of the type index reflects the value productivity rank of the type in the sense of the expected value of a new product line, $E_\tau\left[S_{\vec{0}}(\vec{\mu})\right]$.

Let Ψ_τ represent the value of research embodied in each existing product in a firm of type τ. The innovation frequency per product line is the choice variable γ. If an innovation arrives in the next instant, the employer will have a product line with no workers and some value productivity realization which has value. In addition, the option to create still another, which again has value Ψ, arrives with each product created. Because the option to create a new product is lost if the existing product line is destroyed, the research value of a new product for firm of type τ per worker solves

$$r\Psi_\tau = \max_\gamma \left\{\gamma\left(E_\tau\left[S_{\vec{0}}(\vec{\mu})\right] + \Psi_\tau\right) - c_1(\gamma) - \delta_1\Psi_\tau\right\}$$

$$\Updownarrow$$

$$\Psi_\tau = \max_\gamma \frac{\gamma\int_\mu \sum_{h=1}^H \frac{\hat{\pi}_h\left(\Delta\hat{S}_{1,h}(\mu_h)\right)}{r+\delta_1} d\Upsilon_\tau(\mu) - c_1(\gamma)}{r+\delta_1-\gamma}, \tag{14}$$

where the second line follows from the h-coalition surplus equation (4). The additive nature of the problem implies that this can also be formulated using the τ conditional marginal distributions for the h-coalition value productivities, $\hat{\Upsilon}_{\tau h}(\cdot)$,

$$\Psi_\tau = \max_\gamma \frac{\gamma\sum_{h=1}^H \int_{\mu_h} \hat{S}_0(\mu_h)\, d\hat{\Upsilon}_{\tau,h}(\mu_h) - c_1(\gamma)}{r+\delta_1-\gamma}.$$

By equation (14), a type τ firm's product creation decision is given by,

$$\gamma_\tau = \arg\max_\gamma \frac{\gamma\int_\mu \sum_{h=1}^H \frac{\hat{\pi}_h\left(\Delta\hat{S}_{1,h}(\mu_h)\right)}{r+\delta_1} d\Upsilon_\tau(\mu) - c_1(\gamma)}{r+\delta_1-\gamma},$$

which is characterized by the first order necessary condition,

$$c_1'(\gamma_\tau) = \int_{\tilde{\mu}} \sum_{h=1}^H \frac{\hat{\pi}_h\left(\Delta\hat{S}_{1,h}(\mu_h)\right)}{r+\delta_1} d\Upsilon_\tau(\mu) + \Psi_\tau. \tag{15}$$

Thus, a firm's product creation rate, γ_τ, is independent of its number of product lines, as well as their labor force and value productivity realizations. The product creation rate is firm type dependent in such a way as to be increasing in the expected value of a new production line, $E_\tau\left[S_{\vec{0}}(\vec{\mu})\right]$.

Finally, the entry rate is the product $\nu = \kappa\gamma_0$ where κ is a given measure of entrepreneurs and γ_0 is the frequency with which any one of them creates new product. As the optimal innovation rate by a potential entrant is chosen before value TFP is realized,

$$\nu = \kappa\gamma_0 = \kappa \arg\max_{\gamma\geq 0} \left\{ \frac{\gamma \sum_{\tau=1}^{T} E_\tau \left[S_{\vec{0}}\left(\vec{\mu}\right)\right] \phi_\tau - c_1(\gamma)}{r + \delta_1 - \gamma} \right\}, \tag{16}$$

where ϕ_τ is the probability that a new entrant is realized to be type τ.

The innovation choice is characterized in the following proposition.

Proposition 1. *If the cost of R&D, $c_1(\gamma)$, is increasing, strictly concave and $c_1(0) = c_1'(0) = 0$, and the measure of products supplied, K, is sufficiently large, then the optimal product creation rate is positive and less than the product destruction rate, δ_1. Furthermore, more productive firms innovate more frequently (γ_τ is increasing in τ).*

Proof. If a solution to (46) exists, then the value of an additional product line is defined by

$$\Psi_\tau = \max_{\gamma\geq 0} \left\{ \frac{\gamma E_\tau \left[S_{\vec{0}}\left(\vec{\mu}\right)\right] - c_1(\gamma)}{r + \delta_1 - \gamma} \right\}. \tag{17}$$

Hence, the first order condition for the optimal innovation rate can be written as

$$f(\gamma) = (r + \delta_1 - \gamma)\left(E_\tau \left[S_{\vec{0}}\left(\vec{\mu}\right)\right] - c_1'(\gamma)\right) + \gamma E_\tau \left[S_{\vec{0}}\left(\vec{\mu}\right)\right] - c_1(\gamma) = 0 \tag{18}$$

and the second order condition requires $f'(\gamma) = -\left(r + \delta_1 - \gamma\right) c"(\gamma) \leq 0$ at a maximal solution. As $f(0) = (r+\delta_1)E_\tau S_0\left(\tilde{q}\right) > 0$, the first order condition has a unique solution satisfying $0 < \gamma < \delta_1$ and the sufficient second order condition is satisfied if $f(\delta_1) = (r+\delta_1)E_\tau \left[S_{\vec{0}}\left(\vec{\mu}\right)\right] - rc_1'(\delta_1) - c_1(\delta_1) < 0$. As $E_\tau \left[S_{\vec{0}}\left(\vec{\mu}\right)\right]$ is bounded above by the largest value of a new product line and that bound converges to zero as $K \to \infty$ by equations (2) and (4), the claim follows. ☐

3.8 The value of the firm and its workers

Denote by $\Delta_\tau\left(\vec{n}^k, \vec{\mu}^k\right)$ the coalition value of a type τ firm and its workers given current state. The firm's labor force $\vec{n}^k = (\vec{n}_1, \ldots, \vec{n}_k)$ is the labor force of each of the firm's k product lines, where $\vec{n}_k = (n_{1k}, \ldots, n_{Hk})$ is product line k's employment size by worker type. Analogously for the value productivity state, $\vec{\mu}^k = (\vec{\mu}_1, \ldots, \vec{\mu}_k)$. The law of motion of the firm's labor force state is governed by a collection of independent Poisson arrival processes fully described in the previous sections. The value productivity state $\vec{\mu}^k$ changes only through additions or subtractions from the vector as governed by the birth-death process of product lines in the firm. For the purpose of further dividing the coalition value into separate firm and worker values, the state must expand to include each worker's employment history as captured by the worker's best outside value option to date in that worker's current employment spell. The evolution of worker employment histories is described in section 3.3. Importantly, the law of motion of $\left(\vec{n}^k, \vec{\mu}^k\right)$ does not depend on the workers' bargaining positions.

In the following, to capture the subtraction of a product line from a firm's state, denote by $\left(\vec{n}^k, \vec{\mu}^k\right)_{<j>}$ the state where product line j has been removed from the vectors. The asset flow equation for the coalition value is,

$$
\begin{aligned}
r\Lambda_\tau\left(\vec{n}^k, \vec{\mu}^k\right) = \; & (r+\delta_1)\sum_{j=1}^{k} V_{\vec{n}_j}(\vec{\mu}_j) + \\
& k \max_{\gamma \geq 0}\left[\gamma \int \left[\Lambda_\tau\left((\vec{n}^k, \vec{0}), (\vec{\mu}^k, \tilde{\mu})\right) - \Lambda_\tau\left(\vec{n}^k, \vec{\mu}^k\right)\right] d\Upsilon_\tau(\tilde{\mu}) - c_1(\gamma)\right] + \\
& \delta_1 \sum_{j=1}^{k}\left(\Lambda_\tau\left(\vec{n}^k, \vec{\mu}^k\right)_{<j>} - \Lambda_\tau\left(\vec{n}^k, \vec{\mu}^k\right)\right),
\end{aligned} \tag{19}
$$

where the first term on the right hand side is flow value of the product line operations including the flow value to workers in case of product line destruction. The second term is value of the innovation operation and the third term is change in coalition due to product line destruction.

The value of the firm-worker coalition takes the form,

$$
\Lambda_\tau\left(\vec{n}^k, \vec{\mu}^k\right) = \sum_{j=1}^{k} V_{\vec{n}_j}(\vec{\mu}_j) + k\Psi_\tau, \tag{20}
$$

where to summarize product line operation values are given by,

$$
V_{\vec{n}}(\vec{\mu}) = \sum_{h=1}^{H} \hat{V}_{n_h,h}(\mu_h),
$$

and the $h-$coalition value follows from equation (4)

$$
\begin{aligned}
(r+\delta_1)\hat{V}_n = \; & \hat{R}_n + \hat{\pi}\left(\Delta\hat{S}_{n+1}\right) - n\left(\hat{\delta}_0 + \hat{\lambda}_2\right)\Delta\hat{S}_n + n\delta_1\hat{U} \\
& + n\beta\left[\hat{\lambda}_1 \int_{\Delta\hat{S}_n}^{\infty}\left[1 - \hat{F}(X)\right] dX + \hat{\lambda}_2 \hat{E}\left[S\right]\right],
\end{aligned}
$$

and innovation value embodied in a product line is given in equation (14).

4 Steady State Market equilibrium

This section presents the labor market meeting technology as well as the equilibrium and steady state conditions.

4.1 The Meeting Process

The labor market is segmented by worker type h into H separate segments. In each segment, the aggregate rate at which workers and vacancies meet is determined by increasing concave and homogeneous of degree one matching function of aggregate vacancies and search effort, $\hat{m}\left(\hat{\theta}\right)$, where $\hat{\theta}$ is market segment tightness, vacancies relative to aggregate search effort. Maintaining that variables with circumflex are h segment specific, aggregate search effort in a market segment is equal to $\hat{\lambda}_0\hat{u} + (\hat{\lambda}_1 + \hat{\lambda}_2)(1-\hat{u})$ where the parameters $\hat{\lambda}_0$ and $\hat{\lambda}_1 + \hat{\lambda}_2$ reflect the search intensities of unemployed and employed workers in the segment, respectively. Hence, the aggregate segment meeting rate is $\hat{m}(\hat{\theta})\left(\hat{u} + (\hat{z}_1 + \hat{z}_2)(1-\hat{u})\right)\hat{L}$ where by an appropriate normalization

$$\hat{\lambda}_0 = \hat{m}(\hat{\theta}) \text{ and } \hat{\lambda}_i = \hat{z}_i\hat{\lambda}_0, : i \in \{1,2\} \tag{21}$$

represent the meeting rates, $\hat{u}$ is the unemployment rate, $\hat{z}_1$ and $\hat{z}_2$ are the relative search intensity of voluntary and involuntary search while employed, assumed to be constants. Because unemployed workers find jobs at rate $\hat{\lambda}_0 = \hat{m}(\hat{\theta})$ and lose them at rate $\hat{\delta} = \hat{\delta}_0 + \hat{\delta}_1$, the steady state unemployment rate is

$$\hat{u} = \frac{\hat{\delta}_0 + \hat{\delta}_1}{\hat{\delta}_0 + \hat{\delta}_1 + \hat{m}(\hat{\theta})} = \frac{\hat{\delta}}{\hat{\delta} + \hat{m}(\hat{\theta})}. \tag{22}$$

By assumption the function $\hat{m}(\cdot)$ is increasing and concave. The rate a vacancy in the segment meets some worker is $\hat{\eta} = \hat{m}(\hat{\theta})/\hat{\theta}$. The assumption that workers of each type are met at rates proportional to their relative search intensities implies that the vacancy meeting rates by search state (unemployed, employed, exogenous reallocation) are functions of tightness:

$$\begin{aligned}
\hat{\eta}_0 &= \left(\frac{\hat{u}}{\hat{u} + (1-\hat{u})(\hat{z}_1 + \hat{z}_2)}\right)\frac{\hat{m}\left(\hat{\theta}\right)}{\hat{\theta}} = \left(\frac{\hat{\delta}}{\hat{\delta} + (\hat{\delta} + \hat{m}(\hat{\theta}))\hat{z}}\right)\frac{\hat{m}\left(\hat{\theta}\right)}{\hat{\theta}} \\
\hat{\eta}_1 &= \left(\frac{(1-\hat{u})\,\hat{z}_1}{\hat{u} + (1-\hat{u})(\hat{z}_1 + \hat{z}_2)}\right)\frac{\hat{m}\left(\hat{\theta}\right)}{\hat{\theta}} = \left(\frac{\left(\hat{\delta} + \hat{m}(\hat{\theta})\right)\hat{z}_1}{\hat{\delta} + \left(\hat{\delta} + \hat{m}(\hat{\theta})\right)\hat{z}}\right)\frac{\hat{m}\left(\hat{\theta}\right)}{\hat{\theta}} \\
\hat{\eta}_2 &= \left(\frac{(1-\hat{u})\,\hat{z}_2}{\hat{u} + (1-\hat{u})(\hat{z}_1 + \hat{z}_2)}\right)\frac{\hat{m}\left(\hat{\theta}\right)}{\hat{\theta}} = \left(\frac{\left(\hat{\delta} + \hat{m}(\hat{\theta})\right)\hat{z}_2}{\hat{\delta} + \left(\hat{\delta} + \hat{m}(\hat{\theta})\right)\hat{z}}\right)\frac{\hat{m}\left(\hat{\theta}\right)}{\hat{\theta}}
\end{aligned} \tag{23}$$

where $\hat{z} = \hat{z}_1 + \hat{z}_2$.

4.2 Product Line Size, Product Distribution, and Average Firm Size

Denote by $\hat{\Upsilon}_{\tau,h}(\mu)$ the firm type τ marginal value productivity CDF for h-type workers. The offer CDF $\hat{F}_h(X)$ is the fraction of vacancies posted by product line h-coalitions with marginal match values less than or equal to X. As the number of vacancies posted by each product line h-coalition depends on its own labor force and the value productivity of the line, one needs to calculate the distribution of employment over product line h-coalitions of each type. Let $\hat{P}_{n,h}(\mu)$ represent the fraction of productivity μ product line h-coalitions with employment equal to n, where $\sum_{n=0}^{\hat{m}_h(\mu)} \hat{P}_{n,h}(\mu) = 1$. In steady state, the flow out of the state $n = 0$ which equal hires plus product destruction, $[\delta_1 + \hat{a}_{0,h}(\mu)]\, \hat{P}_{0,h}(\mu)$, balances the flow into the state of newly created products $\sum_\tau (\nu\phi_\tau + \gamma_\tau K_\tau)\, \hat{\Upsilon}'_{\tau,h}(\mu)$, in addition to, the mass of state $n = 1$ product lines that lose a worker, $\hat{d}_{1,h}(\mu)\, \hat{P}_{1,h}(\mu)$. The normalization that $\sum_{n=0}^{\infty} \hat{P}_{n,h}(\mu) = 1$, implies that the inflow of newly created product is rescaled to equal δ_1, which balances the overall outflow rate of mass from, $\sum_{n=0}^{\infty} \hat{P}_{n,h}(\mu)$. Thus, in steady state,

$$\delta_1 + \hat{d}_{1,h}(\mu)\, \hat{P}_{1,h}(\mu) = [\delta_1 + \hat{a}_{0,h}(\mu)]\, \hat{P}_{0,h}(\mu).$$

Since only transition from $n-1$ to n, $n+1$ to n, and n to zero can occur in an instant, all the other measures satisfy the difference equation

$$(n+1)\hat{d}_{n+1,h}(\mu)\hat{P}_{n+1,h}(\mu) + \hat{a}_{n-1,h}(\mu)\hat{P}_{n-1,h}(\mu) = \left[\delta_1 + \hat{a}_{n,h}(\mu) + \hat{d}_{n,h}(\mu)n\right] P_n(\mu), \tag{24}$$

where the functions $\hat{d}_{n,h}(\mu)$ and $\hat{a}_{n,h}(\mu)$ are those defined by equations (39) and (42). Solving for $\hat{P}_{n,h}(\mu)$ reduces to a simple tridiagonal equation system bounded above by the maximum labor force size $\hat{m}_h(\mu)$ as defined in equation (9).

The measure of products supplied by the set of type τ firms evolves according to the law of motion,

$$\dot{K}_\tau = (\nu\phi_\tau + \gamma_\tau K_\tau) - \delta_1 K_\tau$$

where ν is the innovation rate of new entrants, ϕ_τ is the fraction of entrant who are of type τ, γ_τ is the innovation rate of type τ firms per product line, and δ_1 is the product destruction rate. In other words, the net rate of change in the measure is equal to the sum of the flows of products supplied by new entrants and continuing firms respectively less the flow of product lines currently supplied by the type that are destroyed. Hence, in steady state, the measure of products supplied by type τ firms and the aggregate measure of products are

$$K_\tau = \frac{\nu\phi_\tau}{\delta_1 - \gamma_\tau} \tag{25}$$

and

$$K = \sum_{\tau} K_{\tau} = \sum_{\tau} \frac{\nu \phi_{\tau}}{\delta_1 - \gamma_{\tau}}. \tag{26}$$

Hence, more value productive firms supply more products in steady state relative to their share at entry, as emphasized in Lentz and Mortensen (2008).

Each firm's number of product lines k evolves according to a birth-death process where new product lines are added at rate $k\gamma_{\tau}$ and product lines are destroyed at rate $k\delta_1$. When a firm is born, it enters with one product. The product line dynamic is identical to the one in Klette and Kortum (2004). Denote by $M_{\tau,k}$ the mass of type τ incumbent firms that have k product lines. It evolves according to,

$$\dot{M}_{\tau,k} = (k-1)\gamma_{\tau} M_{\tau,k-1} + (k+1)\delta_1 M_{\tau,k+1} - k(\gamma_{\tau} + \delta_1) M_{\tau,k}, \; k = 2, \ldots$$
$$\dot{M}_{\tau,1} = \upsilon \phi_{\tau} + 2\delta_1 M_{\tau,2} - (\gamma_{\tau} + \delta_1) M_{\tau,1}.$$

Define $\sum_{k=1}^{\infty} M_{\tau,k} = M_{\tau}$ and denote by $m_{\tau,k} \equiv M_{\tau,k}/M_{\tau}$ the probability that a type τ incumbent has k product lines. As shown in Klette and Kortum (2004), in steady state $m_{\tau,k}$ is distributed according the logarithmic distribution with parameter γ_{τ}/δ_1,

$$m_{\tau,k} = \frac{\left(\frac{\gamma_{\tau}}{\delta_1}\right)^k}{\ln\left(\frac{\delta_1}{\delta_1 - \gamma_{\tau}}\right) k}. \tag{27}$$

By implication, the average type conditional firm size is,

$$E_{\tau}[k] = \frac{\gamma_{\tau}}{\delta_1} \frac{\frac{\delta_1}{\delta_1 - \gamma_{\tau}}}{\ln\left(\frac{\delta_1}{\delta_1 - \gamma_{\tau}}\right)}. \tag{28}$$

The steady state mass of firms M_{τ} is given by,

$$M_{\tau} = \frac{\nu \phi_{\tau}}{\gamma_{\tau}} \ln\left(\frac{\delta_1}{\delta_1 - \gamma_{\tau}}\right). \tag{29}$$

In sum, more productive firms create more products, represent a larger fraction by equations (28), and (29) and Proposition 1.

By the steady state condition, the mass of firms is constant over time which implies that the aggregate exit rate must equal the entry rate. To verify, the aggregate exit rate of firms is given the combined mass of firms with a single product line multiplied by the product creation rate. By equations (27) and (29) it follows that aggregate entry and exit rates are equal,

$$\sum_{\tau} \delta_1 M_{\tau,1} = \upsilon.$$

4.3 Offer and Employment Distributions

Given the constructs in the previous section and the fact that the optimal number of vacancies posted depends only on the marginal value, the market steady state distribution of vacancies over match surplus value offers is

$$\hat{F}_h(X) = \frac{\sum_\tau K_\tau \int_\mu \sum_{n=0}^\infty \mathbb{I}\left[\Delta\hat{S}_{n+1,h}(\mu) \leq X\right] \hat{v}_h\left(\Delta\hat{S}_{n+1,h}(\mu)\right) \hat{P}_{n,h}(\mu)\, d\hat{\Upsilon}_{\tau,h}(\mu)}{\sum_\tau K_\tau \int_\mu \sum_{n=0}^\infty \hat{v}_h\left(\Delta\hat{S}_{n+1}(\mu)\right) \hat{P}_{n,h}(\mu)\, d\hat{\Upsilon}_{\tau,h}(\mu)}. \tag{30}$$

where $\hat{v}_h\left(\Delta\hat{S}_{n+1}(\mu)\right)$ is the number of vacancies posted by μ productivity h-coalitions when employment is n, $\mathbb{I}(\cdot)$ is the indicator function equal to unity if the argument is true and zero otherwise, K_τ is the number of products supplied by firms of type τ. Similarly, the distribution of employment over match surplus values is given by

$$\hat{G}_h(X) = \frac{\sum_\tau K_\tau \int_\mu \sum_{n=0}^\infty \mathbb{I}\left[\Delta\hat{S}_{n,h}(\mu) \leq X\right] n\hat{P}_{n,h}(\mu)\, d\hat{\Upsilon}_{\tau,h}(\mu)}{\sum_\tau K_\tau \int_\mu \sum_{n=0}^\infty n\hat{P}_{n,h}(\mu)\, d\hat{\Upsilon}_{\tau,h}(\mu)}, \tag{31}$$

4.4 Market Tightness and Unemployment

The ratio of the aggregate vacancies to aggregate search effort, market tightness in segment h, is

$$\hat{\theta}_h = \frac{\sum_\tau K_\tau \int_\mu \sum_{n=0}^\infty \hat{v}_h\left(\Delta\hat{S}_{n+1,h}(\mu)\right) \hat{P}_{n,h}(\mu)\, d\hat{\Upsilon}_{\tau,h}(\mu)}{\left(\hat{u}_h + (\hat{z}_{1h} + \hat{z}_{2h})(1-\hat{u}_h)\right)\hat{L}_h}. \tag{32}$$

Finally, because unemployed type h workers find jobs at the rate $\hat{m}_h(\hat{\theta}_h)$ and lose them at rate equal to $\hat{\delta}_{0h} + \delta_1$, the steady state unemployment rate, that which balances inflow and outflow, is the solution to

$$\frac{\hat{u}_h}{1-\hat{u}_h} = \frac{\hat{\delta}_{0h} + \delta_1}{\hat{\delta}_{0h} + \delta_1 + \hat{m}_h(\hat{\theta}_h)}. \tag{33}$$

4.5 Aggregate Output

Given the production function specified in equation (1) and the fact that input j is supplied in quantity $x(j) = \left(\sum_{h=1}^H q_h(j) n_h(j)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$ where $q(j)$ is the productivity and $\vec{n}(j)$ is employment of product j, final market output in steady state is produced at rate

$$
\begin{aligned}
Y &= A\left[\frac{1}{K^{\sigma}}\int_0^K\left(\sum_{h=1}^{H}\alpha(j)^{\frac{1}{\sigma}}q_h(j)n_h(j)^{\frac{\sigma-1}{\sigma}}\right)dj\right]^{\frac{\sigma}{\sigma-1}} \\
&= \left[\sum_{\tau}\frac{K_\tau}{K^{\sigma}}\int_{\mu}\left(\sum_{h=1}^{H}\mu_h\sum_{n=1}^{\infty}n^{\frac{\sigma-1}{\sigma}}\hat{P}_{n,h}\left(\mu_h\right)\right)d\Upsilon_\tau\left(\mu\right)\right]^{\frac{\sigma}{\sigma-1}}, \\
&= \left[\sum_{\tau}\frac{K_\tau}{K^{\sigma}}\sum_{h=1}^{H}\int_{\mu}\mu\sum_{n=1}^{\infty}n^{\frac{\sigma-1}{\sigma}}\hat{P}_{n,h}\left(\mu\right)d\hat{\Upsilon}_{\tau,h}\left(\mu\right)\right]^{\frac{\sigma}{\sigma-1}} \qquad (34)
\end{aligned}
$$

by the law of large numbers and where the linearity of the problem allows use of marginal distributions in the last line.

The final good is demanded both as consumption by households and in the hiring and innovation processes by firms. Aggregate consumption is in equilibrium given by $C = Y+\sum_h \hat{L}_h\hat{u}_h\hat{b}_h-\kappa c_1(\gamma_0)-\sum_\tau K_\tau\left[c_1(\gamma_\tau)+\sum_h\int_\mu\left(\sum_{n=0}^{\infty}\hat{c}_{0h}\left(\hat{\nu}_{n,h}(\mu_h)\right)\hat{P}_{n,h}\left(\tilde{\mu}_h\right)d\right)\Upsilon_\tau\left(\mu\right)\right]$

4.6 Steady state equilibrium

The market equilibrium is defined by,

Definition. A steady state equilibrium is a collection

$$
\left\{Y,\upsilon,K_\tau,\gamma_\tau,\Psi_\tau,\left\{\hat{\theta},\hat{F},\hat{G},\hat{u},\hat{w}_n(X,\mu),\hat{a}_n(\mu),\hat{d}_n(\mu),\hat{P}_n(\mu),\hat{S}_n(\mu)\right\}_h\right\},
$$

where for given market segment h equilibrium $(\hat{\theta},\hat{F},\hat{G})$, h-coalition size and productivity conditional surplus, hiring, separations, wages, and labor force size distribution $\left(\hat{S}_n(\mu),\hat{a}_n(\mu),\hat{d}_n(\mu),\hat{w}_n(X,\mu),\hat{P}_n(\mu)\right)_h$ solve equations (4), (7), (13), (12), and (24). The labor market h segment steady state equilibrium objects $(\hat{\theta},\hat{F},\hat{G},\hat{u})$ solve equations (32), (30), (31), and (33). Firm type conditional innovation value, rates, and steady state product mass $(\Psi_\tau,\gamma_\tau,K_\tau)$ solve equations (14), (15), and (25). Finally, entry and output solve equations (16) and (34).

Existence of equilibrium is provided in the Appendix for the perfect substitutes case.

4.7 Characterization of equilibrium

4.7.1 Dispersion

While wages are the real source of interest, they are at their core tied to marginal coalition surplus, which is the statistic that guides labor reallocation in the model. The competitive pressures in the labor market seek to equalize marginal surplus across matches within a market segment. Labor market frictions manifest in within segment dispersion. Within a market segment there are two sources of dispersion: Within and across coalition type dispersion. Within coalition type dispersion is in part driven by coalition churn whereby new coalitions are born unstaffed and in steady state replace the exit of incumbent coalitions. Second, the stochastic nature of hirings and quits induces variation around a coalition's long term average size with resulting marginal surplus variation. Across coalition type dispersion is a result of the convex hiring cost which in the absence of integer constraints in worker hiring drives a positive relationship between coalition productivity and average marginal surplus. Across market segment marginal surplus variation is in the model tied to worker heterogeneity but is also affected by the level of labor market friction through their impact on innovation and hiring.

Marginal surplus variation across matches can by the law of total variance be decomposed as follows,

$$V(\Delta S) = V_h(E[\Delta S \mid h]) + E_h[V_\mu(E[\Delta S \mid h, \mu])] + E_{h,\mu}[V(\Delta S \mid h, \mu)]. \tag{35}$$

The first term is variance going across market segments. Market segment is in th analysis a permanent effect to workers and so it naturally correponds to a worker side variance contribution. The second term is the average within market segment variation in marginal surplus across (h, μ)-coalitions. It has a kinship with firm effects, but with the important distinction that firms are collections of (h, μ)-coalitions, and at this stage there is no particular discipline in the analysis on how coalition types are correlated within firms. Finally, the third term is the economy wide average marginal surplus dispersion within (h, μ)-coalitions, which represents pure frictional dispersion. Relative to a wage variance decomposition, the marginal surplus decomposition excludes the variance that is attributable to bargaining position variation, which in the seminal Postel-Vinay and Robin (2002) decomposition is the single source of frictional wage dispersion. The current analysis involves additional sources of frictional dispersion.

4.7.2 The frictionless counterfactual

The model allows a meaningful frictionless counterfactual and therefore an alternative method of measuring the impact of frictions on dispersion as well as other outcomes like productivity and welfare. In the counterfactual where there are no labor market frictions, (μ, h)-coalitions hire to the point where marginal surplus values are equalized across μ coalitions within market segments and so that aggregate employment within a market segment given marginal surplus equalization equals L_h. Product creation and destruction in this counterfactual follows the same process, but now subject to a valuation of a new product line in a frictionless labor market. The frictionless counterfactual has marginal surplus dispersion across segments, but no other sources of dispersion. The difference in productivity and welfare can be attributed to the presence of labor market frictions.

In the frictionless counterfactual, (μ, h) teams immediately adjust to their desired equilibrium size, $n^*(\mu, h)$. Within an h-type market all teams have the same marginal surplus. In a competitive labor market, the value of unemployment equals that of employment and jobs are found instantly. The serial nature of the competition in the model means that it is characterized by a Diamond (1971) type paradox. Thus, the competitive limit in the model would involve taking $\beta \to 1$, to ensure that for each h-type market we have that $\hat{U}_h = \Delta\hat{S}^*$, and that $\Delta\hat{S}_{n^*(\mu,h),h}(\mu) = \Delta\hat{S}^*$. To simplify matters, drop integer constraints. The frictionless surpluss is, $(r + \delta_1)\,\hat{S}_n = \hat{R}_n - \hat{b}n = (Y/K)^{1/\sigma}\,\mu n^{\frac{\sigma-1}{\sigma}} - \hat{b}n$. The marginal surplus is, $(r+\delta_1)\hat{S}_n' = \frac{\sigma-1}{\sigma}\,(Y/K)^{1/\sigma}\,\mu n^{-1/\sigma} - \hat{b}$. Thus, in the frictionless equilibrium, the labor force size solves,

$$n^*(\mu, h) = \frac{Y}{K}\left(\frac{\frac{\sigma-1}{\sigma}\mu}{(r + \delta_1)\Delta\hat{S}^* + \hat{b}}\right)^{\sigma}.$$

And the h-market equilibrium marginal surplus balances labor demand and supply, where for $\Delta\hat{S}^* > 0$,

$$\sum_{\tau} K_\tau \int_\mu n^*(\mu, h) d\hat{\Upsilon}_{\tau,h}\,(\mu) = L_h,$$

and if $\Delta\hat{S}^* = 0$,non-employment solves,

$$1 - \hat{u}_h = \frac{\sum_\tau K_\tau \int_\mu n^*(\mu, h) d\hat{\Upsilon}_{\tau,h}\,(\mu)}{L_h}.$$

The frictionless wage is for any $\Delta\hat{S}^* \geq 0$,

$$\hat{w}^* = (r + \delta_1)\,\Delta\hat{S}^* + \hat{b}.$$

The firm value attached to a product line is,

$$
\begin{aligned}
(r+\delta_1)\,\hat{V}_h^*(\mu) &= \hat{R}_{n^*(\mu,h),h}(\mu) - n^*(\mu,h)\hat{w}^* \\
&= \left(\frac{Y}{K}\right)^{1/\sigma} \mu \left(\frac{Y}{K}\left(\frac{\frac{\sigma-1}{\sigma}\mu}{\hat{w}^*}\right)^{\sigma}\right)^{\frac{\sigma-1}{\sigma}} - \frac{Y}{K}\left(\frac{\frac{\sigma-1}{\sigma}\mu}{\hat{w}^*}\right)^{\sigma}\hat{w}^* \\
&= \frac{Y}{K}\left(\frac{\frac{\sigma-1}{\sigma}\mu^{\frac{\sigma}{\sigma-1}}}{\hat{w}^*}\right)^{\sigma-1}\frac{1}{\sigma} = \frac{n^*(\mu,h)}{(\sigma-1)\,\hat{w}^*} > 0.
\end{aligned}
$$

With this, innovation is characterized by,

$$
\Psi_\tau^* = \max_\gamma \frac{\gamma\sum_{h=1}^{H}\int_{\mu_h}\hat{V}_h^*(\mu)\,d\hat{\Upsilon}_{\tau,h}(\mu_h) - c_1(\gamma)}{r+\delta_1-\gamma},
$$

and the rest of the equilibrium is as described in section 4.6 with degenerate match and offer distributions massed at the equilibrium marginal surplus for each h-market. Notice, that as $\sigma\to\infty$ (the perfect substitutes case), the firm value of a product line limits to zero in the competitive case and there are no ex post rents to reward product innovation in the frictionless model.

4.7.3 Allocation (sorting) and reallocation.

Sorting between workers and firms across matches in the economy are dictated by how μ realizations differ across firm types. The discussion requires restrictions on Υ_τ to first of all allow a sensible ranking of firm types. One such restriction is to assume that Υ_τ is stochastically increasing in τ, where we adopt a stochastic dominance concept that applies to the multidimensional setting. Supermodularity is then roughly speaking going to be a condition that says that as τ increases, productivity realizations increase faster for higher type worker segments than for lower type worker segments. Such a pattern would then result in a compositional variation across firm types whereby higher firm types also have relatively more higher type workers, and they would be paid relatively more.

[Lopes de Melo co-worker correlation deserves mention in this context. As pointed to in Bagger and Lentz (2019), it remains something of a mystery how strong the co-worker correlation is for the following exercise: Take all firms with similar firm effects and bin them together. Pool all their worker and create random worker coalitions to produce the same size distribution as before, but now with synthetic firms. The co-worker correlations that result across these synthetic firms drops dramatically relative to the actual co-worker correlations. This demonstrates that workers of similar types sort into firms on something more than the estimated firm wage fixed effect. In the context of the current analysis, this analysis has a real shot at understanding this: Some firms may consists primarily of particular market segment workers because they have poor mu realizations in other segments. Such firms will have high co-worker correlations and the sorting into the firm is closely tied to the firm's μ realization in that occupation. If anything, it is a manifestation of relatively low μ realizations in market segments that the firm is therefore not hiring in. With a nod to Flinn and Heckman (1984), while the analysis will struggle to identify the distribution of these low μ's, the co-worker correlation will speak to the relative frequency of low μ realizations that make certain market segment workers close to non-existent in some firms.]

[The analysis allows the calculation of transition matrix of worker reallocation between different product line types within a market segment. Product line types have a clear ranking and will be associated with observable differences like value added per worker where higher product types will have higher observables. The fact that ΔS varies substantially more within than across product lines makes it reasonable to suspect significant worker flows between all product types even in the absence of Godfather like shocks. There is a clear link to recent analyses like Hall et al (????), Sorkin (2018), and Lentz, Piyapromdee and Robin (2021) where the first two references argue significant amenity effects through revealed preference. This analysis argues that there are natural allocation and reallocation patterns resulting from a model in this paper that follow natural marginal productivity chasing arguments that are difficult to measure and result in conclusions that suggest that mobility is weakly linked to compensation types. This is a measurement point, though, more so than evidence of amenities.]

The model allows a study of how worker h types are allocated to firm types τ. This serves as a useful complement to the wage type focused analysis of AKM, BSS, BLM, and LPR that is done on simulated data that contain wages. For the allocation patterns on latent types, a useful statistic is that of mutual information (MI), which for the model can be written as,

$$MI = \sum_{h,\tau} e(h,\tau) \ln \left(\frac{e(h,\tau)}{e(h)e(\tau)} \right),$$

where $e(h,\tau)$ is the joint pdf of worker and firm types over matches, and $e(h)$ and $e(\tau)$ are the respective marginal distributions, which in this context reduce to employment mass by worker type, $e\left(h\right) = (1-u_h)L_h / \left(\sum_{h'}(1-u_{h'})L_{h'}\right)$, and firm type. The joint distribution of worker and firm types over matches is,

$$e(h,\tau)=\frac{K_\tau\sum_\mu\sum_{n=0}^\infty n\hat{P}_{n,h}\left(\mu\right)d\Upsilon_{h,\tau}(\mu)}{\sum_{\tau'}K_{\tau'}\sum_{h'}\sum_\mu\sum_{n=0}^\infty n\hat{P}_{n,h'}\left(\mu\right)d\Upsilon_{h',\tau'}(\mu)}.$$

The MI measure shows the distance of the joint distribution of employment mass over firm and worker types from the that of independent allocation. Unlike that of the correlation, there is in this measure no concept of ordering of types.

Sorting can also be quantified in terms of marginal coalition surplus. Denote the average marginal marginal surplus conditional on firm and worker type by,

$$E_\mu\left[\Delta S\mid h,\tau\right]=\frac{\int_\mu E\left[\Delta S\mid h,\mu\right]E\left[n\mid h,\mu\right]d\hat{\Upsilon}_{\tau,h}\left(\mu\right)}{\int_\mu E\left[n\mid h,\mu\right]d\hat{\Upsilon}_{\tau,h}\left(\mu\right)}.$$

For the purpose of a single dimensional ordering of firms and workers, apply a linear projection,

$$E_\mu\left[\Delta S\mid h,\tau\right]=\alpha_h+\varphi_\tau+\varepsilon_{h\tau}. \tag{36}$$

With this, one can label worker types by α_h and firm types by φ_τ and quantify sorting in these ordered dimensions by for example the correlation coefficient over the match distribution, $Corr(\alpha,\varphi)$.

4.7.4 Firm size, productivity, and wages.

The model has rich implications for firm related observables such as value added per worker, number of workers, and wage bills. The value added distribution of an h-coalition in steady state is given by the CDF,

$$\Pr\left(Y_h\leq\tilde{Y}\mid\mu\right)=\sum_{n=0}^{\infty}\mathbb{I}\left[\hat{R}_{h,n}\left(\mu\right)-c_{0h}\big(\hat{v}_h\big(\Delta\hat{S}_{n+1,h}(\mu)\big)\big)\leq\tilde{Y}\right]\hat{P}_{n,h}\left(\mu\right),$$

where only the hiring cost of the coalition have been subtracted. The determination of the output distribution of a product line of a given firm type is a somewhat more complicated question because the h-coalitions in that product line are initiated at zero at the same time and therefore have non-zero labor force size covariance despite their independent laws of motion. This is easily resolved in simulations, but makes for a complicated analytical question.

4.7.5 Productivity and type selection.

[TBC].

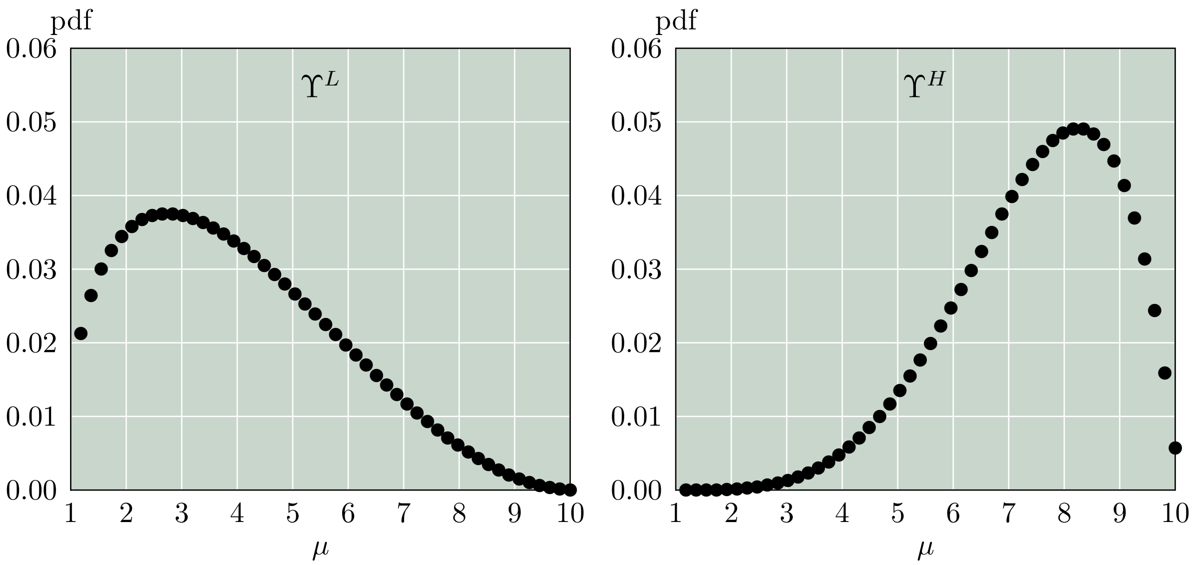
5 Illustrative scenarios

To illustrate the mechanisms in the model consider an example with 2 firm types and 2 worker types (market segments). At entry, firm type 1 is over represented relative to firm type 2, $\phi_1 = 2/3$ and $\phi_2 = 1/3$. In the examples to follow, firm type 2 will draw from a better productivity distribution and the equilibrium will be characterized by a more even representation as type 2 firms as the economy selects into both more type 2 product lines and those product lines also individually tend to employ more workers. The strength of decreasing returns is governed by the elasticity of substitution, $\sigma = 2.5$. Bargaining over rents is parameterized by $\beta_h = 0.5$, and the churn of product lines and firms is dictated by $\delta_1 = 0.07$. The mass of potential entrants is set at $\kappa = 0.1$. Hiring costs are the same in both market segments and are specified as $c_{0h}(v) = 0.1v^4$. Exogenous reallocation shocks are eliminated, $z_{2h} = 0$, and employed search is assumed to be substantially less efficient than unemployed search, $z_{1h} = 0.5$. The unemployed income flow in each market segment is set to be half of the revenue of the first worker in the worst type of team in a given market, $b_h = \frac{1}{2}\hat{R}_{1h}(1)$. Exogenous job destruction is in each market segment set at $\delta_{0h} = 0.05$. The matching function is assumed Cobb-Douglas and the same in each market segment, $m_h(\theta) = \sqrt{\theta}$, where the constant term is normalized at unity. Finally, the innovation cost function is for both firm types parameterized at $c_{1\tau}(\gamma) = (75\gamma)^3$.

The 3 scenarios in this section all work with 2 basic marginal value productivity distributions shown in Figure 3, where Υ^L is dominated by Υ^H. For the initial scenario, the labor force is split equally over the two market segments, $L_1 = L_2 = 1/2$. Furthermore, to enforce a design where the equilibrium exhibits heterogeneity only on the firm side, a given firm type draws μ_h realizations from a distribution that is invariant across h. Specifically, for $h = 1, 2$, $\hat{\Upsilon}_{1,h} = \Upsilon^L$ and $\hat{\Upsilon}_{2,h} = \Upsilon^H$. That is, firm type 1 draws from the low marginal productivity distribution for both market segments and type 2 draws from the high productivity distribution for both market segments. Scenario 1 is designed to illustratre the model's selection on more productive firm types and the model's capacity to generate marginal surplus dispersion within and across h-coalitions. The model parameters and specifications for the 3 scenarios in this section are summarized in Table 1.

The second scenario makes a simple change to the labor force composition so that $L_1 = 2/3$ and $L_2 = 1/3$. That is, $h = 1$ is now abundant relative to $h = 2$ which results in marginal surplus heterogeneity across worker types (market segments). In addition to the scenario 2 change

Figure 3: μ distributions



Note: TBC

Table 1: Model parameters and specifications

			$h = 1$	$h = 2$
$c_{0h}(v) = 0.1v^4,\ h = 1, 2$		δ_{0h}	0.05	0.05
$c_{1\tau}(\gamma) = (75\gamma)^3,\ \tau = 1, 2$		z_{1h}	0.50	0.50
$m_h(\theta) = \sqrt{\theta},\ h = 1, 2$		z_{2h}	0.00	0.00
$\sigma = 2.50$		β_h	0.50	0.50
$\kappa = 0.10$		L_h	1/2	1/2
$\delta_1 = 0.07$	Scenario 1	$\hat{\Upsilon}_{1,h}$	Υ^L	Υ^L
$\phi = (2/3, 1/3)$		$\hat{\Upsilon}_{2,h}$	Υ^H	Υ^H
$r = 0.05$		L_h	2/3	1/3
$\hat{b}_h = \frac{1}{2}\hat{R}_{1h}(1)$	Scenario 2	$\hat{\Upsilon}_{1,h}$	Υ^L	Υ^L
		$\hat{\Upsilon}_{2,h}$	Υ^H	Υ^H
		L_h	2/3	1/3
	Scenario 3	$\hat{\Upsilon}_{1,h}$	Υ^L	Υ^L
		$\hat{\Upsilon}_{2,h}$	Υ^L	Υ^H

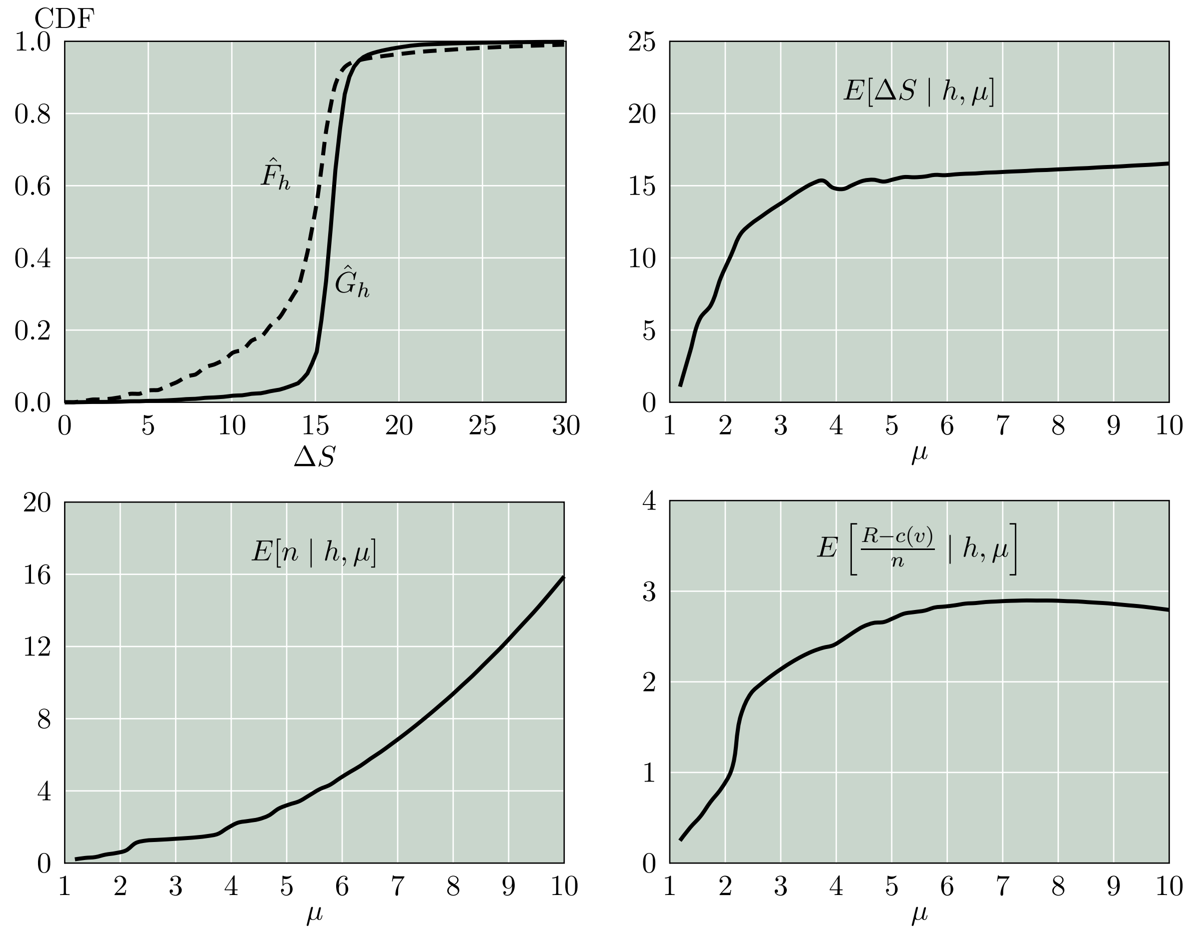
Table 2: Steady state equilibrium

	Scenario 1			Scenario 2			Scenario 3		
		$h=1$	$h=2$		$h=1$	$h=2$		$h=1$	$h=2$
Y	32.671			31.660			22.056		
C	29.816			28.911			20.484		
K	0.087			0.086			0.075		
K_1	0.045			0.044			0.045		
K_2	0.043			0.042			0.029		
ν	0.003			0.003			0.003		
γ_0	0.032			0.032			0.030		
γ_1	0.022			0.022			0.018		
γ_2	0.045			0.045			0.030		
Ψ_1	78.658			76.918			125.120		
Ψ_2	647.999			633.180			329.884		
$\hat{b}_h$		5.349	5.349		5.316	5.316		4.869	4.869
θ_h		3.404	3.404		2.642	4.662		2.034	4.021
u_h		0.061	0.061		0.069	0.053		0.078	0.057
λ_{0h}		1.845	1.845		1.625	2.159		1.426	2.005
λ_{1h}		0.923	0.923		0.813	1.080		0.713	1.003
$E[\Delta S \mid h]$		15.900	15.900		14.657	17.491		8.413	14.979
$E_\mu[V(\Delta S \mid h, \mu)]$		2.896	2.896		2.233	4.494		1.573	3.358
$V_\mu(E[\Delta S \mid h, \mu])$		0.945	0.945		1.007	0.986		0.668	0.927
$V(\Delta S)$	0.227			0.347			1.100		
- $V_h(E[\Delta S \mid h])$	0.0%			31.0%			76.7%		
- $E_h[V_\mu(E[\Delta S \mid h, \mu])]$	24.6%			17.3%			6.0%		
- $E_{h,\mu}[V(\Delta S \mid h, \mu)]$	75.4%			51.7%			17.3%		
MI	0.000			0.000			0.037		
$Cor(\alpha, \varphi)$	0.000			-0.003			0.270		

to the labor force composition, the third scenario changes firm type 2's productivity realization distribution. Specifically, $\hat{\Upsilon}_{2,1} = \Upsilon^L$ and $\hat{\Upsilon}_{2,2} = \Upsilon^H$. Thus, firm type 2's productivity draws for type $h = 1$ workers are now degraded to match that of the firm type 1 distribution. Thus, firm type 2 now has a comparative and absolute advantage in its use of type $h = 2$ workers. The third scenario is designed to illustrate the model's implications for sorting outcomes between workers and firm types.

The steady state equilibria in the three different scenarios are summarized in Table 2. In this scenario, firm type 2 draws from a better distribution than firm type 1 in both market segments. The value of a new product line is as a result higher for type 2 firms that choose greater innovation rates than type 1 firms. As a result type 2 product lines are over-represented in equilibrium relative

Figure 4: Scenario 1 Equilibrium Outcomes



Note: TBC

to their frequency at birth. Specifically, firm type 2 product lines make up the fraction $K_2/K = 0.49$ of the economy which compared to the birth distribution $\phi_2 = 1/3$ represents the economy's positive selection into firm type 2. The associated productivity gain is the central focus of the Lentz and Mortensen (2008) selection effect.

Figure 4 illustrates the positive association between coalition μ and average marginal surplus that results from the convex hiring cost technology. Both h market segments are identical in this scenario, so the graphs do not distinguish between h markets. In the non-frictional setting coalitions will staff up to the point where marginal surplus is equalized across coalitions. For greater productivity coalitions this means higher employment levels. In the frictional setting, the convex hiring cost implies a higher marginal cost of replacing workers lost to regular churn for larger coalitions than smaller ones. This effect is also emphasized in Kass and Kircher (2015). As a result under frictions, marginal surplus equalization is not achieved and higher productivity coalitions have on average greater marginal surplus and greater labor force sizes. As also illustrated in the figure, the integer nature of hiring workers in the model can locally upset this relationship at low employment levels.

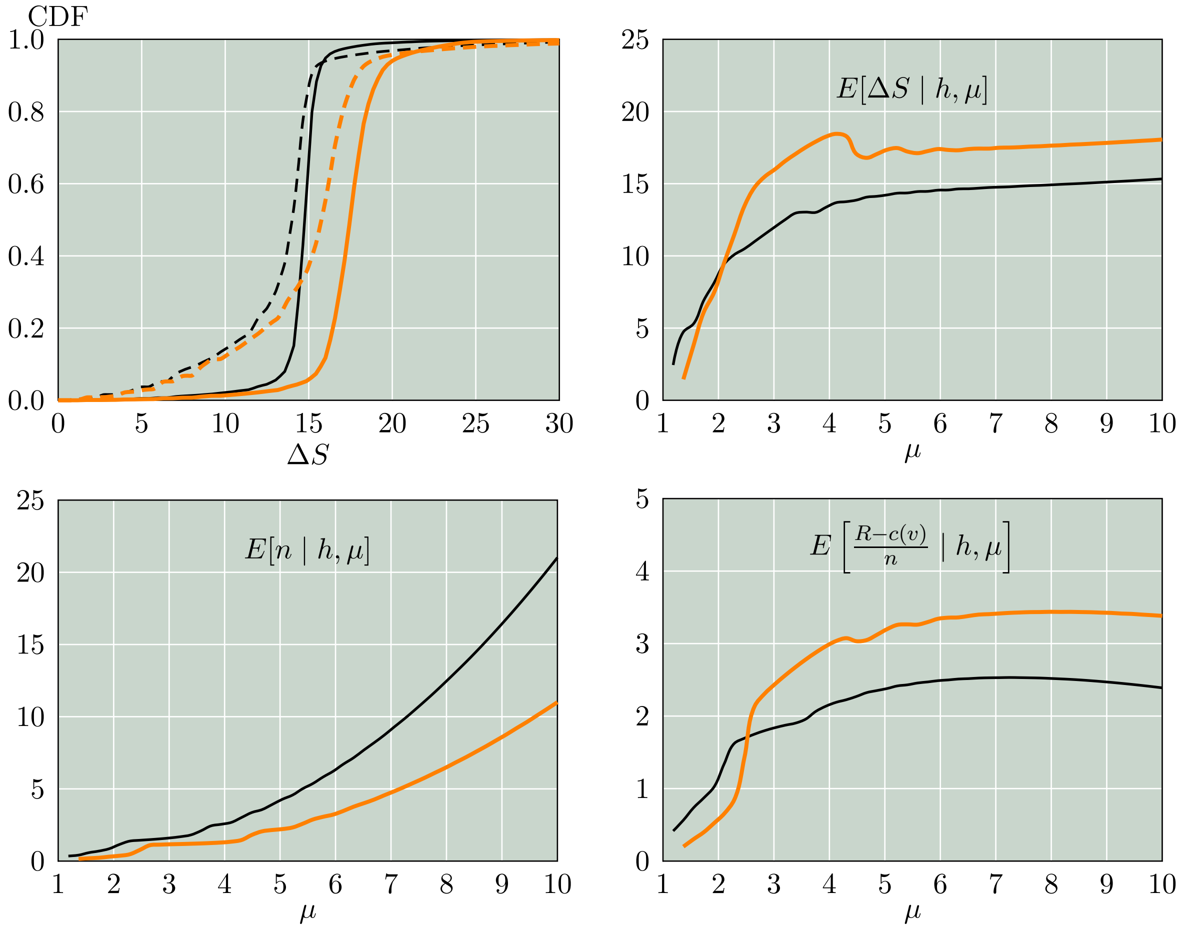
Thus, marginal surplus systematically varies across product lines within market segments. It also varies within such coalitions as a result of product line destruction and creation, and due to the stochastic nature of worker separations and hires. As stated in Table 2, in scenario 1, the within h coalition variation (75%) vastly dominates the across variation (25%). The distribution of marginal surplus in offers and matches is illustrated in the upper left panel in Figure 4. As expected, the match distribution overall dominates the offer distribution, but compared to a standard on-the-job search model like Burdett and Mortensen (1998) the dynamically evolving marginal surplus process within matches manifests itself in the crossing of the offer and match CDFs at high ΔS levels. As in Kass and Kircher (2015), the convex hiring cost associates on average higher ΔS offers with higher coalition μ, which is a permanent characteristic of the job. However, high ΔS offers also stem from relatively understaffed low μ coalition and such positions do not remain high marginal surplus positions as the coalition staffs up. It is this latter effect that results a heavier right tail offer distribution than that of the match distribution.

Figure 5 shows the Scenario 2 equilibrium outcomes. Here, type 1 workers are in abundant supply relative to type 2 workers, $L_1 = 2/3$ and $L_2 = 1/3$. This is the only difference between scenarios 1 and 2. The obvious impact on the equilibrium is an across worker type heterogeneity in labor market outcomes. Table 2 shows that on average type 1 workers are employed at a marginal surplus of 14.7 whereas type 2 workers are employed at a marginal surplus of 17.5. Worker type 2 market tightness is greater due to the firms' higher vacancy intensities. This results in a lower worker type 2 unemployment rate of 5.3% relative to that of the type 1 workers of 6.9%.

The worker type conditional offer and match distributions in Figure 5 tell the same story of marginal surplus heterogeneity in employment across the two worker types. For a given value productivity μ, a type 1 coalition on average has more workers than a type 2 coalition, reflecting the greater type 2 factor cost in equilibrium.

Table 5 documents the marginal surplus variance decomposition where worker type heterogeneity now accounts for 31% of the overall variation. Only 17% of the variation is attributed to systematic variation across h-coalitions for given worker types, and 52% is a result of within h-coalition

Figure 5: Scenario 2 Equilibrium Outcomes

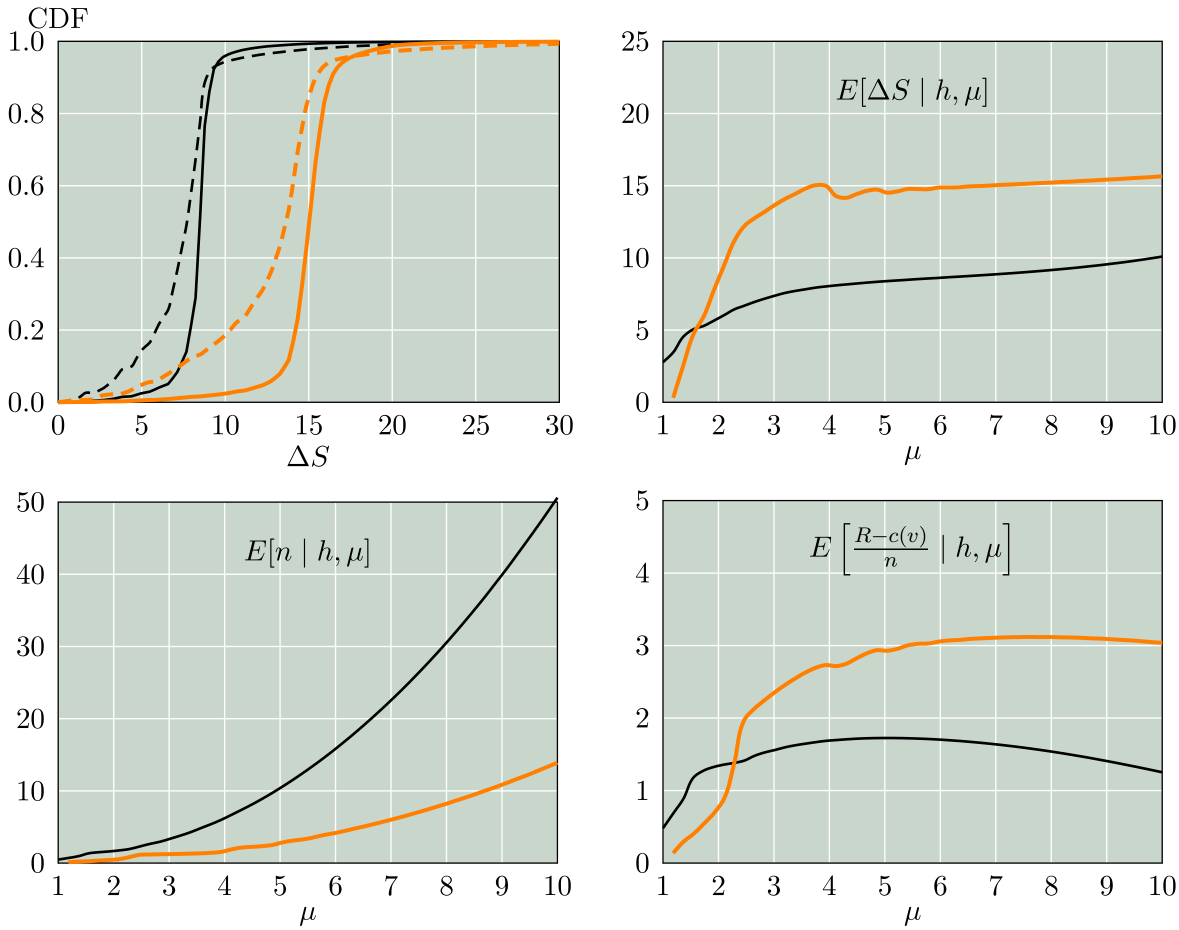


Note: Type $h = 1$ market outcomes in black and type $h = 2$ market outcomes in orange. The upper left panel has the offer distribution F in dashed line and the match distribution G in solid.

variation. In terms of the classic, firm-worker-friction decomposition, this is a decomposition result that attributes more contribution to worker side variation and a smaller role to the firm side where firms will be an aggregation of variation across h-teams. In this scenario, frictional noise remains the dominant contribution.

The final scenario outcomes are presented in Figure 6. Relative to scenario 2, scenario 3 considers a reduction in firm type 2's value productivity draws associated with type 1 workers. Specifically, it is assumed that $\hat{\Upsilon}_{2,1} = \Upsilon^L$. This results in a reduction in average marginal surplus for both worker types in equilibrium relative to scenario 2. Type 1 workers are now on average employed at a marginal surplus of 8.4 and type 2 workers are employed at 15. Output decreases and unemployment of both types increases. Firm type 2 now employ type 1 workers at lower productivity which directly

Figure 6: Scenario 3 Equilibrium Outcomes



Note: Type $h = 1$ market outcomes in black and type $h = 2$ market outcomes in orange. The upper left panel has the offer distribution F in dashed line and the match distribution G in solid.

lowers the marginal surplus distribution in worker type 1 employment. Type 2 workers are also adversely affected due to less product creation by type 2 firms whose product lines are now less valuable due to the lower productivity of the type 1 coalitions.

The variance decomposition in scenario 3 now swings towards increased contribution across market segments with a corresponding lower contribution from within product line dispersion. In addition to an illustration of a negative productivity shock, scenario 3 is designed to illustrate sorting. As shown in Table 2, the equilibrium now involves a positive correlation coefficient between firm and worker marginal surplus fixed effects as defined in equation (36). The mutual information measure also indicates sorting. The employment distribution over worker and firm types is shown in Table 3 compared to that of independent assignment according to the marginal distributions.

Table 3: Scenario 3 Employment distribution

Actual allocation

$h \diagdown \tau$	1	2	$e(h)$
1	0.40	0.26	0.66
2	0.11	0.23	0.34
$e(\tau)$	0.51	0.49	

Independent allocation

$h \diagdown \tau$	1	2	$e(h)$
1	0.33	0.32	0.66
2	0.18	0.17	0.34
$e(\tau)$	0.51	0.49	

The scenario 3 employment distribution puts more weight on the diagonal relative to that of type independent assignment. The mutual information measure quantifies this distance. The linear projection in equation (36) assigns fixed marginal surplus effects to firm and worker types which then allows a measurement of the kind of sorting along a cardinal measure. In this case, it demonstrates that sorting is positive. High marginal surplus type workers (type 2) match more frequently with high surplus type firms, which in this case are type 2 firms.

6 Estimation

[In progress]

7 Conclusion

TBC

Appendix I

Perfect Substitutes ($\sigma = \infty$)

Solving the model for the value of the marginal match and the associated optimal recruiting strategy is complicated in the general case in which intermediate good are imperfect substitutes, but doing so in the limiting case of perfect substitutes is quite straight forward. Since the conceptual issues are the same but the structure is much simpler, we study the special case in the main exposition. Later, we reintoduce the general case for the purpose of empirical estimation.

The Surplus Value of a Job-Worker Match

As noted earlier, the critical property of the special case is the proportionality of product line revenue to employment. That is $R_n = Aqn/K$ in the limit from equation (2). Therefore, $\Delta R = Aq/K$ is a constant which together with equation (8) implies that the surplus value of the marginal worker is also constant. Indeed, its value, denoted as $\Delta S(q)$ where the dependence on productivity q is emphasized, is the unique solution to

$$(r + \delta_0 + \delta_1)\Delta S(q) = Aq/K - b - \lambda_0 \beta \int_0^{\overline{X}} X dF(X) + \\ \lambda_1 \beta \int_{\Delta S(q)}^{\overline{X}} (X - \Delta S(q)) dF(X) + \lambda_2 \beta \int_0^{\overline{X}} (X - \Delta S(q)) dF(X)$$

An integration by part yields

$$(r + \delta_0 + \delta_1 + \lambda_2)\Delta S(q) = Aq/K - b \\ + \lambda_1 \beta \int_{\Delta S(q)}^{\overline{X}} [1 - F(X)] dX - (\lambda_0 - \lambda_2)\beta \int_0^{\overline{X}} [1 - F(X)] dX. \quad (37)$$

It is equal to the present value of marginal product, Aq/K, minus the worker's flow value if not matched, plus the expected gain in the coalitions joint value attributable to on-the-job search. It is straight forward to verify that $\Delta S(q)$ is differentiable and increasing in q. Indeed,

$$\Delta S'(q) = \frac{A/K}{r + d(q)} > 0 \quad (38)$$

if $F(X)$ is continuous where

$$d(q) = \delta_0 + \delta_1 + \lambda_1 [1 - F(\Delta S(q))] + \lambda_2. \quad (39)$$

is the separation rate from a product line of productivity q. Note that $\Delta S(q)$ is unique solution to the ODE (38) consistent with the boundary condition $\Delta S(\underline{q}) = 0$ where $\underline{q}$, the productivity of the marginal product line, is the unique solution to

$$A\underline{q}/K = b + (\lambda_0 - \lambda_1 - \lambda_2)\beta \int_0^{\overline{X}} [1 - F(X)]dX \tag{40}$$

Equation (5) implies that firms optimal choice of vacancies, denoted $v\,(q)$, is the solution to the problem on the right side of

$$\pi\,(q) = \max_{v \geq 0} \left\{ (1-\beta) \left[(\eta_0 + \eta_2)\,\Delta S\,(q) + \eta_1 \int_0^{\Delta S(q)} G(X)dX \right] v - c(v) \right\} \tag{41}$$

where $\pi\,(q)$ represents the net return to recruiting activity per worker. As the value of a match $\Delta S\,(q)$ is increasing in productivity and the FONC for an optimal choice of vacancies, more productive firms post more vacancies,

$$c_0'(v(q)) = (1-\beta) \left[(\eta_0 + \eta_2)\,\Delta S\,(q) + \eta_1 \int_0^{\Delta S(q)} G(X)dX \right]$$

more productive product lines engage in more recruiting effort. Indeed, the hire frequency

$$h(q) = [\eta_0 u + (\eta_1 G(\Delta S\,(q)) + \eta_2)\,(1-u)]\,v(q) \tag{42}$$

is increasing in worker productivity while the separation rate $d\,(q)$, defined in (39) is decreasing in q. As the employment process is mean reverting and tends toward the value of n that equates the flow of hires $h(q)$ to the quit flow $d(q)n$, more productive product lines will grow to become larger on average.

The expected present value of future income for any employee of a product line of productivity q is

$$W(q) = W_0 - U + \beta(\Delta S(q) + U - W_0) \tag{43}$$

where W_0 is reflects the worker's employment history. Namely, W_0 is value of an outside offer that resulted in a raise in lifetime wage relative to that earned at the time of employment. In the case of no such raise, then W_0 is value of employment in the worker's previous job if the worker transferred voluntarily to the firm or the value of unemployment if either hired from unemployed or an involuntary transfer from another firm. As W is also the solution to the following forward looking Bellman equation

$$\begin{aligned}
rW &= w + \lambda_1 \int_{\Delta S}^{\overline{X}} [U + \Delta S + \beta (X - \Delta S) - W] \, dF(X) \\
&\quad + \lambda_2 \int_0^{\overline{X}} [U + \beta (X - U) - W] \, dF(X) - (\delta_0 + \delta_1)(U - W) \\
&\Updownarrow \\
(r + d(q)) (W - U) &= w - b - (\lambda_1 - \lambda_2) \beta \int_0^{\overline{X}} [1 - F(X)] dX \\
&\quad + \lambda_1 \int_{\Delta S}^{\overline{X}} [\Delta S + \beta (X - \Delta S)] \, dF(X),
\end{aligned}$$

the wage flow satisfies,

$$w(q, A) = \Delta R - (1 - \beta) \left\{ [r + d(q)] (\Delta S(q) - A) + \lambda_1 \int_A^{\Delta S(q)} (S - A) \, dF(S) \right\}, \tag{44}$$

where $W_0 - U = A$.

Product Innovation and Entry

Firm size as reflected in the number of products supplied is generated by a research and development process as introduced in Klette and Kortum (2004). At any point in time, a firm has k product lines can grow only by creating new product varieties. Investment in R&D is required to create new products. Specifically, the firm's R&D investment flow generates new product arrivals at frequency γk where γ represents the firm's innovation rate per product line. The total R&D investment cost expressed in terms of output is $c_1(\gamma)k$ where $c_1(\gamma)$ is assumed to be strictly increasing and convex function. The assumption that the total cost of R&D investment is linearly homogeneous in the new product arrival flow, γk, and the number of existing product, k, "captures the idea that a firm's knowledge capital facilitates innovation," in the words of Klette and Kortum (2004). The specification assumption also implies that Gibrat's law holds in the sense that innovation rates are size independent contingent on type, a property needed to match the data on firm growth. Finally, every product is subject to destruction risk with exogenous frequency δ_1. Given this specification, the number of products supplied by any firm is a stochastic birth-death process characterized by "birth rate" γ and "death rate" δ_1.

The productivity of a product line, q, is realized when a new product is created. For a particular productivity realization, the product line faces a revenue function $R(q) = Aqn/K$. Firm type heterogeneity enters the model through type dependence of the distribution of product line productivity, q. Let $\Gamma_\tau(q), \tau \in \{1, 2...., T\}$, denote the cumulative distribution of q for any firm of type τ. Finally, we assume that the magnitude of the type index reflects the productivity rank of the type in the sense of first order stochastic dominance. That is, $\tau > \tau'$ implies that product lines of firms of τ are likely to have greater employment conditional revenue than firms of type τ', i.e., $\Gamma_\tau(q) \leq \Gamma_{\tau'}(q)$ for all $q \in \left[0, \underline{q}\right]$.

Let Ψ represent the value of research per existing product line. The innovation frequency per product line is the choice variable γ. If an innovation arrives in next instant, the employer will have a product line with no workers, which generates a revenue flow equal to the net profit associated with attempting to fill the first job, which is $\pi(\Delta S(q))$. As every product line faces destruction risk δ_1, its present value at the moment of discovery is $\pi(\Delta S(q))/(r + \delta_1)$. In addition, the option to create still another, which again has value Ψ, arrives with each product created. Because the option to create a new product is lost if the existing product line is destroyed, the productivity of the line is realized only after the product is created, and there are no employees at the date of creation, the value of a new product for firm of type τ per worker solves

$$(r + \delta_1)\Psi_\tau = \max_\gamma \left\{ \gamma \left(\frac{\int_{\underline{q}}^{\overline{q}} \pi(x) d\Gamma_\tau(x)}{r + \delta_1} + \Psi_\tau \right) - c_1(\gamma) \right\}. \tag{45}$$

As $\Gamma_\tau(\underline{q})$ may be positive, this formulation accounts for the possibility that an innovation will be marketed only if it yields a positive surplus.

From equation (45), the innovation frequency satisfies the FONC

$$c_1'(\gamma_\tau) = \int_{\underline{q}}^{\overline{q}} \frac{\pi(x)}{r + \delta_1} d\Gamma_\tau(x) + \Psi_\tau \tag{46}$$

Note that the choice is independent of both the number of products currently supplied and of the number of workers employed to supply each line.

Proposition 2. *If the cost of R&D, $c_1(\gamma)$, is increasing, strictly concave and $c_1(0) = c_1'(0) = 0$, and the measure of products supplied, K, is sufficiently large, then the optimal product creation rate is positive and less than the product destruction rate, δ_1. Furthermore, more productive firms innovate more frequently (γ_τ is increasing in τ).*

Proof. If a solution to (46) exists, then the value of an additional product line is defined by

$$\Psi_\tau = \max_{\gamma \geq 0} \left\{ \frac{\gamma E_\tau \left[S_0\left(\tilde{\mu}\right)\right] - c_1(\gamma)}{r + \delta_1 - \gamma} \right\}. \tag{47}$$

Hence, the first order condition for the optimal innovation rate can be written as

$$f(\gamma) = (r + \delta_1 - \gamma)\left(E_\tau S_0\left(\tilde{q}\right) - c_1'(\gamma)\right) + \gamma E_\tau S_0\left(\tilde{q}\right) - c_1(\gamma) = 0 \tag{48}$$

and the second order condition requires $f'(\gamma) = -\left(r + \delta_1 - \gamma\right) c"(\gamma) \leq 0$ at a maximal solution. As $f'(0) = (r + \delta_1) E_\tau S_0\left(\tilde{q}\right) > 0$, the first order condition has a unique solution satisfying $0 < \gamma < \delta_0$ and the sufficient second order condition is satisfied if $f'(\delta_1) = (r+\delta_1)E_\tau S_0\left(\tilde{q}\right) - r c_\gamma(\delta_1) - c_\gamma(\delta_1) < 0$. As $S_0\left(\tilde{\mu}\right)$ is bounded above by the largest value of a new product line and that bound converges to zero as $K \to \infty$ by equations (2) and (**??**) where $\sigma > 1$, the claim follows. $\square$

Assume that entry requires a successful innovation, that the cost of innovation activity by a potential entrant is $c_1(\gamma)$, and that firm type is unknown to any entrepreneur prior to entry. The entry rate is the product $\upsilon = \kappa\gamma_0$ where κ is a given measure of entrepreneurs and γ_0 is the frequency with which any one of them creates a new product. As the optimal innovation rate by a potential entrant maximizes the unconditional expected value of a job-worker match, the optimal choice is

$$\upsilon = \kappa\gamma_0 = \kappa \arg\max_{\gamma \geq 0} \left\{ \sum_\tau \left[\frac{\int_{\underline{q}}^{\overline{q}} \pi(x) d\Gamma_\tau(x)}{r + \delta_1} + \Psi_\tau \right] \phi_\tau \gamma - c_1(\gamma) \right\}. \tag{49}$$

where ϕ_τ is the exogenous probability of becoming a type τ firm.

Steady State Market Equilibrium

The Meeting Process

The aggregate rate at which workers and vacancies meet is determined by increasing concave and homogeneous of degree one matching function of aggregate vacancies and search effort. Aggregate search effort is equal to $\lambda_0 u + (\lambda_1 + \lambda_2)(1 - u)$ where the parameters λ_0 and $\lambda_1 + \lambda_2$ reflect the search intensities of unemployed and employed workers respectively. Hence, the aggregate meeting rate is $m(\theta)\left(u + (a_1 + a_2)(1 - u)\right) L$ where by an appropriate normalization

$$\lambda_0 = m(\theta) \text{ and } \lambda_i = a_i \lambda_0, : i \in \{1, 2\} \tag{50}$$

represent the meeting rates, u is the unemployment rate, a_1 and a_2 are the relative search intensity of voluntary and involuntary search while employed, assumed to be constants. Because unemployed workers find jobs at rate $\lambda_0 = m(\theta)$ and lose them at rate $\delta = \delta_0 + \delta_1$, the steady state unemployment rate is

$$u = \frac{\delta_0 + \delta_1}{\delta_0 + \delta_1 + m(\theta)} = \frac{\delta}{\delta + m(\theta)}. \tag{51}$$

By assumption the function $m(\theta)$ is increasing and concave. The rate a vacancy meets some worker is $\eta = m(\theta)/\theta$. The assumption that workers of each type are met at rates proportional to their relative search intensities implies that the vacancy meeting rates by worker type are functions of tightness:

$$\begin{aligned}
\eta_0 &= \left(\frac{u}{u+(1-u)(a_1+a_2)}\right)\frac{m(\theta)}{\theta} = \left(\frac{\delta}{\delta+(\delta+m(\theta))a}\right)\frac{m(\theta)}{\theta} \\
\eta_1 &= \left(\frac{(1-u)\,a_1}{u+(1-u)(a_1+a_2)}\right)\frac{m(\theta)}{\theta} = \left(\frac{(\delta+m(\theta))\,a_1}{\delta+(\delta+m(\theta))\,a}\right)\frac{m(\theta)}{\theta} \\
\eta_2 &= \left(\frac{(1-u)\,a_2}{u+(1-u)(a_1+a_2)}\right)\frac{m(\theta)}{\theta} = \left(\frac{(\delta+m(\theta))\,a_2}{\delta+(\delta+m(\theta))\,a}\right)\frac{m(\theta)}{\theta}
\end{aligned} \tag{52}$$

where $a = a_1 + a_2$.

Product Distribution and Average Firm Size

The measure of products supplied by the set of type τ firms evolves according to the law of motion,

$$\dot{K}_\tau = (\upsilon\phi_\tau + \gamma_\tau K_\tau)\,[1-\Gamma_\tau(\underline{q})] - \delta_1 K_\tau$$

where υ is the innovation rate of new entrants, ϕ_τ is the fraction of entrant who are of type τ, γ_τ is the innovation rate of type τ firms per product line, δ_1 is the product destruction rate, and $\underline{q}$ is the reservation productivity of a new product. In other words, the net rate of change in the measure is equal to the sum of the flows of products supplied by new entrants and continuing firms respectively less the flow of product lines currently supplied by the type that are destroyed. Hence, in steady state, the measure of products supplied by type τ firms and the aggregate measure of products are

$$K_\tau = \frac{\upsilon\phi_\tau[1-\Gamma_\tau(\underline{q})]}{\delta_1 - \gamma_\tau[1-\Gamma_\tau(\underline{q})]} \tag{53}$$

and

$$K = \sum_\tau K_\tau = \sum_\tau \frac{\upsilon\phi_\tau[1-\Gamma_\tau(\underline{q})]}{\delta_1 - \gamma_\tau[1-\Gamma_\tau(\underline{q})]} \tag{54}$$

Hence, more productive firms supply more products in steady state relative to their share at entry by Proposition 1.

Each firm's number of product lines k evolves according to a birth-death process where new product lines are added at rate $k\gamma_\tau$ and product lines are destroyed at rate $k\delta_1$. When a firm is born, it enters with one product. The product line dynamic is identical to the one in Klette and

Kortum (2004). Denote by $M_{\tau,k}$ the mass of type τ incumbent firms that have k product lines. It evolves according to,

$$\begin{aligned}\dot{M}_{\tau,k} &= (k-1)\,\gamma_\tau M_{\tau,k-1} + (k+1)\,\delta_1 M_{\tau,k+1} - k\,(\gamma_\tau + \delta_1)\,M_{\tau,k},\ k = 2,\ldots\\ \dot{M}_{\tau,1} &= \upsilon\phi_\tau + 2\delta_1 M_{\tau,2} - (\gamma_\tau + \delta_1)\,M_{\tau,1}.\end{aligned}$$

Define $\sum_{k=1}^{\infty} M_{\tau,k} = M_\tau$ and denote by $m_{\tau,k} \equiv M_{\tau,k}/M_\tau$ the probability that a type τ incumbent has k product lines. As shown in Klette and Kortum (2004), in steady state $m_{\tau,k}$ is distributed according the logarithmic distribution with parameter γ_τ/δ_1,

$$m_{\tau,k} = \frac{\left(\frac{\gamma_\tau}{\delta_1}\right)^k}{\ln\left(\frac{\delta_1}{\delta_1-\gamma_\tau}\right)k}. \tag{55}$$

By implication, the average type conditional firm size is,

$$E_\tau\left[k\right] = \frac{\gamma_\tau}{\delta_1}\frac{\frac{\delta_1}{\delta_1-\gamma_\tau}}{\ln\left(\frac{\delta_1}{\delta_1-\gamma_\tau}\right)}. \tag{56}$$

The steady state mass of firms M_τ is given by,

$$M_\tau = \frac{\upsilon\phi_\tau}{\gamma_\tau}\ln\left(\frac{\delta_1}{\delta_1-\gamma_\tau}\right). \tag{57}$$

In sum, more productive firms create more products, represent a larger fraction by equations (56), and (57) and Proposition 1.

Wage Distributions and Market Tightness

Given the constructs in the previous section and the fact that the optimal number of vacancies posted depends only on the marginal value, the market steady state distribution of vacancies over match values is

$$\begin{aligned}\digamma(q) \quad &\equiv \quad F\left(\Delta S(q)\right) = \frac{\sum_\tau K_\tau \int_{\widehat{q}}^{q} v(x)d\Gamma_\tau(x)}{\sum_\tau K_\tau \int_{\underline{q}}^{\overline{q}} v(x)d\Gamma_\tau(x)}\\ &= \quad \frac{\sum_{\tau=1}^{T}\frac{\upsilon\phi_\tau[1-\Gamma_\tau(\underline{q})]}{\delta_1-\gamma_\tau[1-\Gamma_\tau(\underline{q})]}\int_{\widehat{q}}^{q} v(x)d\Gamma_\tau(x)}{\sum_{\tau=1}^{T}\frac{\upsilon\phi_\tau[1-\Gamma_\tau(\underline{q})]}{\delta_1-\gamma_\tau[1-\Gamma_\tau(\underline{q})]}\int_{\underline{q}}^{\overline{q}} v(x)d\Gamma_\tau(x)}\end{aligned} \tag{58}$$

where $\digamma(q)$ is the fraction of vacancies posted by product lines that are no more productive than q. Because the flows in and out of the set of workers employed in product line of productivity q or less are equal in steady state,

$$\lambda_0 F(\Delta S(q))u = (\delta + (\lambda_1+\lambda_2)\left[1 - F(\Delta S(q))\right)]\,G(\Delta S(q))(1-u). \tag{59}$$

where the odds of being unemployed are

$$\frac{u}{1-u} = \frac{\delta}{m(\theta)} \tag{60}$$

for (51) by $F(U) = G(U) = 0$. Hence, the steady state distribution of employment is

$$G\left(\Delta S(q)\right) = \frac{\delta F(q)}{\delta + am(\theta)[1 - F(q)]} \tag{61}$$

by equations (51) and (50) where $a = a_1 + a_2$. The ratio of the aggregate vacancies to aggregate search effort, market tightness, is

$$\theta = \frac{\sum_\tau K_\tau \int_{\underline{q}}^{\overline{q}} v(x) d\Gamma_\tau(x)}{\left(u + (a_1 + a_2)(1-u)\right) L} = \frac{(\delta + m(\theta)) \sum_{\tau=1}^{T} \frac{\upsilon \phi_\tau [1-\Gamma_\tau(\underline{q})]}{\delta_1 - \gamma_\tau[1-\Gamma_\tau(\underline{q})]} \int_{\underline{q}}^{\overline{q}} v(x) d\Gamma_\tau(x)}{(\delta + am(\theta)) L}. \tag{62}$$

Definition and Existence

Define

$$R(q) = (1-\beta)\frac{m(\theta)}{\theta}\left[\left(\frac{\delta + m(\theta)a_2}{\delta + m(\theta)a}\right)\Delta S(q) + \left(\frac{m(\theta)a_1}{\delta + m(\theta)a}\right)\int_{\underline{q}}^{q} \frac{\delta F(x)\Delta S'(x)dx}{\delta + am(\theta)[1-F(x)]}\right]. \tag{63}$$

Definition 1. A steady state labor market equilibrium is an optimal vacancy posting strategy

$$v(q) = \arg\max_{v} \left\{R(q)v - c_0(v)\right\}, \tag{64}$$

an optimal creation rate for each firm type,

$$\gamma_\tau = \arg\max_{\gamma} \left\{ \frac{\left(\int_{\underline{q}}^{\overline{q}} \frac{\max_v \{R(q)v - c_0(v)\} d\Gamma_\tau(q)}{r+\delta}\right)\gamma - c_1(\gamma)}{r + \delta_1 - \gamma} \right\}, \tau = 1, ..., T, \tag{65}$$

and entry rate

$$\upsilon = \kappa \arg\max_{\gamma \geq 0} \left\{ \frac{\left(\sum_\tau \phi_\tau \left(\int_{\underline{q}}^{\overline{q}} \frac{\pi(x) d\Gamma_\tau(x)}{r+\delta}\right)\right)\gamma - c_1(\gamma)}{r + \delta_1 - \gamma} \right\}, \tag{66}$$

where $R(q)$ and $F(q)$ are the solutions to the ODE system

$$R'(q) = \frac{\left(\frac{\delta + m(\theta)a_2}{\delta + m(\theta)a} + \frac{m(\theta)a_1}{\delta + m(\theta)a} \cdot \frac{\delta F'(q)}{\delta + am(\theta)[1-F(q)]}\right)(1-\beta)\frac{m(\theta)}{\theta}\frac{A}{K}}{r + \delta + a_1 m(\theta)\left[1 - F(q)\right] + a_2 m(\theta)} \tag{67}$$

$$F'(q) = \frac{\sum_\tau K_\tau v(q) \Gamma_\tau'(q)}{\sum_\tau K_\tau \int_{\underline{q}}^{\overline{q}} v(x) d\Gamma_\tau(x)} \tag{68}$$

associated with the boundary conditions $R(\overline{q}) = F(\overline{q}) = 0$ where

$$\underline{q} = \frac{1}{A}\left(b + (1 - a_1 - a_2)\beta m(\theta)\int_0^{\overline{X}} [1 - F(X)] dX\right)\sum_{\tau=1}^{T} K_\tau \tag{69}$$

is the reservation productivity together a distribution of products over firm type K_τ, $\tau = 1, ..., T$, and a market tightness parameter θ that satisfy equations (53) and (62).

The following two assumptions are standard. The third places reasonable restrictions on parameter values.

Assumption 1: The job finding rate $m(\theta)$ is strictly increasing and concave, and $\lim_{\theta\to 0}\{\theta/m(\theta)\} = 0$.

Assumption 2: The cost functions, $c_i(\cdot)$, $i \in \{0,1\}$, are both continuous, strictly convex, and satisfy $c_i(0) = c_i'(0) = 0$.

Assumption 3: The value of home production is strictly positive ($b > 0$) and the sum of the offer arrival rates when employed are less than the rate when unemployed ($a = a_1 + a_2 \leq 1$).

Proposition 3. *If the conditional productivity distributions $\Gamma_\tau, \tau = 1, ..., T$, are Lipschitz continuous for all $\tau = 1, 2, ..., T$ and the union of their supports is bounded above by $\overline{q} < \infty$, a steady state labor market equilibrium with strictly positive tightness exists.*

Proof: See Appendix II.

Appendix II

Proof of Existence

Let the triple $(\overline{\pi}, \overline{R}, \overline{\gamma})$ represent the unique solutions to

$$\frac{\kappa\overline{\gamma}}{\delta_1 - \overline{\gamma}} = \frac{A\overline{q}}{b}, \tag{70}$$

$$\overline{\gamma} = \max_{\gamma}\left\{\frac{\overline{\pi}\gamma - c_1(\gamma)}{r + \delta_1 - \gamma}\right\} \tag{71}$$

$$\overline{\pi} = \max_{v}\left\{\overline{R}v - c_0(v)\right\}. \tag{72}$$

Given $\overrightarrow{\gamma} = (\gamma_0, \gamma_1, ..., \gamma_T) \in \mathbb{R}_+^{T+1}$, $\overrightarrow{K} = (K_1, K_2, ..., K_T) \in \mathbb{R}_+^T$, and $(\underline{q}, \theta, K,) \in \mathbb{R}_+^3$, let $z = (\overrightarrow{\gamma}, \overrightarrow{K}, \theta, K, \underline{q}) \in \mathbb{Z}$ where

$$\mathbb{Z} = \left\{ \begin{array}{c} z \in \mathbb{R}_+^{2T+4} | \gamma_\tau \leq \overline{\gamma} \text{ for } \tau = 0, 1, ...T, K_\tau \leq \frac{A\overline{q}}{Tb} \text{ for } \tau = 1, ...T, \\ \theta \leq \frac{A\overline{q}}{bL} v(\overline{R}) \text{ and } (1-\beta)\, 2A(r+\delta) \leq \overline{R}\frac{\theta}{m(\theta)} \sum_{\tau=1}^{T} K_\tau \end{array} \right\}. \tag{73}$$

Note that $\mathbb{Z}$ is a bounded real vector space which is closed and convex given that $m(\theta)$ is increasing and concave.

The unique solution to the ODE system composed of (67) and (68), represented by $R(q, z)$ and $F(q, z)$, consistent with the boundary conditions $R(\underline{q}, 0) = F(\underline{q}, 0) = 0$ is continous and satisfy

$0 \leq R(q) \leq \overline{R}$ and $0 \leq F(q) \leq 1$ on $[\underline{q}, \overline{q}] \times \mathbb{Z}$. Continuity follows from the continuity of the RHS of both (67) and (68) in z. Furthermore,

$$\begin{aligned} R(q,z) &\leq R(\overline{q},z) = \int_{\underline{q}}^{q} \left(\frac{\left(\begin{array}{c} \left(\frac{\delta+m(\theta)a_2}{\delta+m(\theta)a}\right)\frac{m(\theta)}{\theta} \\ +\left(\frac{m(\theta)a_1}{\delta+m(\theta)a}\right)\frac{m(\theta)}{\theta}\frac{\delta F'(q)}{\delta+am(\theta)[1-F(x)]} \end{array} \right)(1-\beta)\frac{A}{\sum_{\tau=1}^{T}K_\tau}}{r+\delta+a_1m(\theta)\left[1-F(q)\right]+a_2m(\theta)} \right) dq \\ &\leq \frac{2A(1-\beta)\frac{m(\theta)}{\theta}}{(r+\delta)\sum_{\tau=1}^{T}K_\tau} = \overline{R}. \end{aligned}$$

and

$$F(q) \leq F(\overline{q}) = \frac{\int_{\underline{q}}^{\overline{q}}\sum_\tau K_\tau v(q)d\Gamma_\tau(q)}{\sum_\tau K_\tau \int_{\underline{q}}^{\overline{q}} v(x)d\Gamma_\tau(x)} = 1$$

Equations (64), (65), (66), (53), (62), and (69) define the map $M : \mathbb{Z} \to \mathbb{Z}$ where

$$\begin{aligned} (M\gamma_0)(z) &= \kappa \arg\max_{\gamma} \left\{ \frac{\left(\sum_{\tau=1}^{T}\phi_\tau\left(\int_{\underline{q}}^{\overline{q}}\frac{\pi(R(q,z))d\Gamma_\tau(q)}{r+\delta}\right)dz\right)\gamma - c_1(\gamma)}{r+\delta_1-\gamma} \right\} \leq \kappa\overline{\gamma} \\ (M\gamma_\tau)z &= \arg\max_{\gamma} \left\{ \frac{\left(\int_{\underline{q}}^{\overline{q}}\frac{\pi(R(q,z))d\Gamma_\tau(q)}{r+\delta}dz\right)\gamma - c_1(\gamma)}{r+\delta_1-\gamma} \right\} \leq \overline{\gamma}, \tau = 1, ..., T \end{aligned}$$

$$\begin{aligned} (MK_\tau)(z) &= \frac{\kappa\gamma_0\phi_\tau[1-\Gamma_\tau(\underline{q})]}{\delta_1-\gamma_\tau[1-\Gamma_\tau(\underline{q})]} \leq \frac{\kappa\gamma_T}{\delta_1-\gamma_T} \leq \frac{\kappa\overline{\gamma}}{\delta_1-\overline{\gamma}} = \frac{A\overline{q}}{Tb}, \tau = 1, ..., T \\ (M\theta)(z) &= \frac{(\delta+m(\theta))}{(\delta+am(\theta))L}\sum_{\tau=1}^{T}K_\tau\int_{\underline{q}}^{\overline{q}} v(R(q,z))d\Gamma_\tau(q) \leq \frac{A\overline{q}}{bL}v(\overline{R}) \\ (M\underline{q})(z) &= \left(\begin{array}{c} b+(1-a_1-a_2)\beta m(\theta) \\ \times\int_0^{\overline{q}}\frac{[1-F(x)]dx}{r+\delta+a_1m(\theta)[1-F(x)]+a_2m(\theta)} \end{array} \right)\frac{\sum_{\tau=1}^{T}K_\tau}{A} \leq b\frac{A\overline{q}}{Ab} = \overline{q}. \end{aligned}$$

As $\mathbb{Z}$ is compact and convex real vector space, M is continuous, and any fixed point is an equilibrium existence follows by Brouwer's fixed point theorem.

Simulation Algorithms

Simulating the match wage distribution

The sufficient statistic in order to understand the evolution of the worker's wage is (μ, n, A), that is the type and state of the worker's current product line along with his outside option. $\Delta S(\mu, n)$ is sufficient in order to determine the worker's acceptance rejection decisions of outside offers. However, the worker current match surplus also evolves according to the hiring and separation rates of the product line as defined in equations (39) and (42) which require (μ, n) for evaluation. In order to simulate the evolution of the worker's state, we combine the following independent Poisson processes:

- The product line may hire an additional worker, $h_n(\mu)$. In this case, the state transitions to $(\mu, n+1, A)$.
- The product line may lose a worker that is not the worker himself, $(n-1)(\delta_0 + s_n(\mu))$. In this case, the state transitions to $(\mu, n-1, A)$.
- The worker may be laid off, $\delta_0 + \delta_1$. In this case, the state transitions to unemployment.
- The worker may receive an outside offer, λ_1. Denote the outside offer by (μ', n'). If $\Delta S(\mu', n') > \Delta S(\mu, n)$ then the worker transitions to $(\mu', n', \Delta S(\mu, n))$. If $A < \Delta S(\mu', n') < \Delta S(\mu, n)$, then the worker transitions to $(\mu, n, \Delta S(\mu', n'))$. Otherwise the state does not change.
- The worker may be reallocated on the ladder at rate λ_2. In this case, the state transitions to $(\mu', n', 0)$.

An unemployed worker receives a job offer at rate λ_0. When that happens, the state transitions to $(\mu', n', 0)$.

For a given state (μ, n, A) determine the sum of all the hazards, $\lambda(\mu, n)$. The minimum of the independent poisson arrival times is exponentially distributed with parameter $\lambda(\mu, n)$. For a given monte carlo draw, $x \sim U[0,1]$, the duration is then found simply by using the inverse of the exponential CDF, $dur = -ln(1-x)/\lambda(\mu, n)$. The probability that poisson process i has the minimum duration is simply $\Pr(X_i = \min(X_1, \ldots, X_n)) = \lambda_i / (\lambda_1 + \cdots + \lambda_n)$. Hence, one monte carlo draw is used to determine the duration until one of the five events happen and another is used to determine which one of the events took place. If either an outside offer is drawn or a job-to-job reallocation happens, a third monte carlo draw is required to determine the new state.

In order to draw an offer from the offer distribution of (μ, n) simply run through all the product line types and states and determine the vacancy intensity associated with that particular (μ, n) pair. A vector is put together and the the vacancy intensities added up. This is a CDF and a monte carlo draw picks a (μ, n) pair by inverting this CDF.

Simulating firm dynamics

A firm's state and type is given by the tuple $(\vec{\mu}, \vec{n}, k, \tau)$ where k is the number of product lines and also the dimension of the $\vec{\mu}$ and $\vec{n}$ vectors which describe the type and state of each of the firm's product lines. Finally, τ describes the type of the firm which governs the creation rate of new products as well as the distribution of μ realizations. The formulation of the firm's state disregards information necessary to compute wages. In order to compute wages, one must keep track of each worker's outside option. In first cut, we avoid this complication and consequently, the firm dynamics statistics disregard wages.

In steady state, the number of product lines of a type τ incumbent is given by $m_{\tau,k}$ as defined in equation (**??**). It is consequently a straightforward matter to simulate the type conditional number of product lines given an initialization by steady state. The type distribution of each of the product lines is drawn from $\Gamma_\tau(\mu)$. In possible violation of the steady state initialization, we initialize the number of workers in each product line by the steady state distribution of $P_n(\mu)$ as defined in equation (**??**). The age distribution of the firm's product lines may differ from that of the overall product line population since it depends on the particular realization of product lines of that firm. To correct for this, we let the simulation run for a number periods prior to recording firm dynamics in order to let it run into steady state should the initialization be off in any significant way.

Thus, initialization is done by,

1. Firm type is determined by weighted sampling from the distribution of steady state firm types, M_τ defined in equation (**??**).

2. Conditional on the firm type draw, the number of products is sampled from the steady state product line distribution in equation (**??**). Specifically, this is done by inverting the CDF of the logarithmic distribution. Hence, for a particular monte carlo draw from the standardized uniform distribution $\tilde{x}$, the type conditional product realization draw is given by, $\tilde{k} = \min_k k \in N_+ | B(\gamma_\tau/\delta_1 | k+1, 0) \leq \ln(1-\gamma_\tau/\delta_1)(\tilde{x}-1)$, where $B(\cdot)$ is the incomplete beta distribution.

3. For each product line, the type of the product line is determined by weighted sampling from the $\Gamma_\tau(\mu)$ distribution.

4. Finally, for each product line, the labor force size is initialized by weighted sampling from the

steady state distribution $P_n(\mu)$ defined in equation (**??**).

Given the initialization, a firm is simulated forward in a manner similar to how simulation is performed for workers. Specifically, monte carlo draws determine durations until the next event occurs as a result of the many independent Poisson processes that the firm faces:

- At rate $k\delta_1$ a product line is destroyed. If this event occurs, a subsequent monte carlo draw determines which product line is destroyed.
- At rate $k\gamma_\tau$ a product line is created. If this event occurs, a subsequent monte carlo draw determines the μ realization of the new product line which starts with zero workers.
- Each of the product lines gain and lose workers according to the hire and separation poisson processes,
  - at rate $n_j\left(\delta_0 + s_{n_j}(\mu_j)\right)$ product line j loses a worker. In this case, product line j moves from n_j workers to $n_j - 1$.
  - at rate $h_{n_j}(\mu_j)$ product line j gains a worker. In this case, product line j moves from n_j workers to $n_j + 1$.

Hence, define the combined arrival rate of events by,

$$\lambda(\vec{\mu}, \vec{n}, k, \tau) = k(\delta_1 + \gamma_\tau) + \sum_{j=1}^{k} \left[h_{n_j}(\mu_j) + n_j\left(\delta_0 + s_{n_j}(\mu_j)\right)\right].$$

The time until next event is exponentially distributed with parameter $\lambda(\vec{\mu}, \vec{n}, k, \tau)$. Conditional on arrival of an event, the probability of any one event is proportional to the the event's contribution to $\lambda(\vec{\mu}, \vec{n}, k, \tau)$.

Coding notes

It is convenient for the solver to take the equilibrium product mass K as a given and treat δ as the endogenous variable whereas in truth, it's the other way around. For the purpose of equilibrium estimation, this is not an important distinction. For the purpose of counterfactuals care must be taken to remember to find the K that holds δ to its value in the counterfactual.

The solver algorithms iterates towards a fixed point in $(Y, \theta, \delta, F, G)$. With a given guess of this tuple, the algorithm proceeds as follows:

1. For the given guess of market tightness, θ, use matching function to determine,

$$\lambda_0 = L\theta^{\frac{1}{2}}$$
$$\lambda_1 = a_1\lambda_0$$
$$\lambda_2 = a_2\lambda_0.$$

and associated steady state unemployment rate,

$$u = \frac{\delta_0 + \delta_1}{\delta_0 + \delta_1 + \lambda_0}.$$

The firm side arrival rates are,

$$\begin{aligned}
\eta &= \frac{1}{\theta}\lambda_0 \\
\eta_0 &= \frac{\eta u}{u + (1-u)(a_1 + a_2)} \\
\eta_1 &= \frac{\eta\,(1-u)\,a_1}{u + (1-u)(a_1 + a_2)} \\
\eta_2 &= \frac{\eta\,(1-u)\,a_2}{u + (1-u)(a_1 + a_2)}.
\end{aligned}$$

Done in solvers.f90[fric_set] which uses only θ.

2. Proceed to solve for optimal vacancy behavior by solving for $S_n(\mu)$. This is done in solvers.f90[v_sol_ser]. A few numerically important notes are in order. To temper problems with convergence related to granularity in the μ and n that feeds through to F and G, optimal firm behavior (v, γ) is determined subject to smoothed versions of F and G. If you will, assume that firms are doing a kernel density estimation over vacancies and matches (which is how the smoothing is done). This is done using the solvers.f90[FG_sol_smooth] routine which is called just before v_sol_ser. As the suffix suggests, v_sol_ser is a solver that runs in seriel over the the different μ types. If there are many μ types, one can consider parallelization over this dimension. It is complicated by heterogenous solution times across μ. v_sol_ser is an iterative scheme that iterates on S toward fixed point in,

$$\begin{aligned}
(r + \delta + n(\delta_0 + \lambda_2))\,\Delta S_n = {} & R_n - R_{n-1} - b + \pi(\Delta S_{n+1}) - \pi(\Delta S_n) + (n-1)\,(\delta_0 + \lambda_2)\,\Delta S_{n-1} \\
& - (\lambda_0 - \lambda_2)\,\Omega(0) + \lambda_1 n\Omega(\Delta S_n) - \lambda_1(n-1)\Omega(\Delta S_{n-1}),
\end{aligned}$$

where $\Omega(\Delta S) = \beta \int_{\Delta S}^{\infty}(X - \Delta S)dF\left(X\right) = \beta \int_{\Delta S}^{\infty} \hat{F}\left(X\right)dX$ and where the value of the hiring operation is given by,

$$\pi(\Delta S) = \max_{v \geq 0} \left\{ (1-\beta)v \left[(\eta_0 + \eta_2)\,\Delta S + \eta_1 \int_0^{\Delta S} (\Delta S - X)dG(X) \right] - c_0(v) \right\}.$$

The revenue function is given by,

$$R_n\left(\mu\right) = \mu Y^{\frac{1}{\sigma}} \left(\frac{n}{K}\right)^{\frac{\sigma-1}{\sigma}}.$$

The first step is to determine the relevant support for the product line's labor force size. Denote by $m\left(\mu\right)$ the upper bound on the labor force size defined as the point where the firm optimally chooses to set hiring to zero, implying $\Delta S_{m(\mu)+1}(\mu) \leq 0 \leq \Delta S_{m(\mu)}\left(\mu\right)$. To make an educated guess, the upper labor force size bound absent labor market frictions, $\hat{m}(\mu)$, is given by,

$$\hat{m}\left(\mu\right) = \arg\max_n \left[R_n\left(\mu\right) - bn\right],$$

where b is the worker reservation wage. It follows that,

$$\left(\frac{\hat{m}\left(\mu\right)}{K}\right)^{\frac{-1}{\sigma}} = \frac{\sigma}{\sigma-1}\frac{b}{\mu}KY^{\frac{-1}{\sigma}}$$

$$\Updownarrow$$

$$\hat{m}\left(\mu\right) = YK^{1-\sigma}\left(\frac{\sigma-1}{\sigma}\frac{\mu}{b}\right)^{\sigma}.$$

The solutions algorithm proceeds to make a guess $m(\mu) = 1.3\hat{m}\left(\mu\right)$. If the solver has been invoked previously, the solver will invoke the old solution to $m\left(\mu\right)$ as a guess if smaller than $1.3\hat{m}\left(\mu\right)$. Another complication: It can happen that $m\left(\mu\right)$ is very large, yet vacancy choices are such that an artificial upper bound on n has negligable impact on labor force dynamics. Thus, the solver uses an upper bound on product line labor force size which is set in a parameter in model.f90[dim_n].

3. With product line vacancies in hand, proceed to solve for $(\gamma, k, \delta, \Psi)$ using,

$$\Psi_\tau = \max_{\gamma \geq 0} \left\{ \frac{\gamma E_\tau V_0\left(\tilde{\mu}\right) - c_\gamma(\gamma)}{r + \delta - \gamma} \right\}.$$

$$c_1'(\gamma_\tau) = E_\tau V_0\left(\tilde{\mu}\right) + \Psi_\tau.$$

Furthermore, product and creation rates are related to product mass by,

$$K_\tau = \frac{\upsilon\phi_\tau}{\delta - \gamma_\tau} \text{ and } K = \sum_\tau K_\tau = \frac{\nu + \sum_\tau K_\tau\gamma_\tau}{\delta}.$$

For the algorithm ν is taken as an exogenous parameter (which will later be justified by parameter in entry innovation cost function). Thus, the product destruction rate is,

$$\delta = \frac{\nu + \sum_\tau K_\tau\gamma_\tau}{K},$$

where

$$K_\tau = \frac{\upsilon\phi_\tau}{\delta - \gamma_\tau}$$

and $K = \sum_\tau K_\tau$. This follows from,

$$\sum K_\tau\left(\delta - \gamma_\tau\right) = \sum \nu\phi_\tau \Longleftrightarrow \delta K = \nu + \sum_\tau K_\tau\gamma_\tau$$

The solver finds K_τ by the implicit equation,

$$0 = K_\tau - \frac{\nu K\phi_\tau + K_\tau(K - K_\tau)\gamma_\tau}{\nu + \sum_{\tau'\neq\tau} K_{\tau'}\gamma_{\tau'}}$$

This combined process is done as an iteration towards convergence by the subroutines solvers.f90[gam_sol],[k_
The solver uses the endogeneity of δ trick, holding K as a constant. This implies solving,

$$K_\tau = \frac{\upsilon\phi_\tau}{\delta - \gamma_\tau} \Leftrightarrow K_\tau\left[\frac{\nu + \sum_{\tau'} K_{\tau'}\gamma_{\tau'}}{K} - \gamma_\tau\right] = \nu\phi_\tau \Leftrightarrow K_\tau\left[\nu + \sum_{\tau'} K_{\tau'}\gamma_{\tau'} - K\gamma_\tau\right] = K\nu\phi_\tau.$$

This can be formulated as,

$$K_\tau\left[\nu + \sum_{\tau'\neq\tau} K_{\tau'}\gamma_{\tau'}\right] + K_\tau\left(K_\tau\gamma_\tau - K\gamma_\tau\right) = K\nu\phi_\tau \Leftrightarrow K_\tau\left[\nu + \sum_{\tau'\neq\tau} K_{\tau'}\gamma_{\tau'}\right] = K\nu\phi_\tau + K_\tau\gamma_\tau\left(K - K_\tau\right) \Leftrightarrow K$$

Why is this a useful expression to solve for K_τ? First of all, the denominator is necessarily positive. The right hand side is continuous in K_τ. Furthermore, $RHS(0) > 0$ and $RHS(K) = \frac{K\nu\phi_\tau}{\nu+\sum_{\tau'\neq\tau} K_{\tau'}\gamma_{\tau'}} = \frac{K\nu\phi_\tau}{\nu+\sum_{\tau'\neq\tau} K_{\tau'}\gamma_{\tau'}} = K\frac{\phi_\tau}{1+\frac{1}{\nu}\sum_{\tau'\neq\tau} K_{\tau'}\gamma_{\tau'}} < K\phi_\tau < K$. It has derivative $RHS'(K_\tau) = \frac{\gamma_\tau(K-K_\tau)-\gamma_\tau K_\gamma}{\nu+\sum_{\tau'\neq\tau} K_{\tau'}\gamma_{\tau'}} = \frac{\gamma_\tau K - 2\gamma_\tau K_\tau}{\nu+\sum_{\tau'\neq\tau} K_{\tau'}\gamma_{\tau'}}$, where $RHS'(0) > 0$ as well as $RHS(0) > 0$. The second derivative of the right hand side is negative, $RHS''(K_\tau) = \frac{-2\gamma_\tau}{\nu+\sum_{\tau'\neq\tau} K_{\tau'}\gamma_{\tau'}} < 0$. Therefore, there exists a unique fixed point $RHS(K_\tau) = K_\tau$ where $K_\tau < K$. Alternatively, there is an explicit solution to the equation,

$$K_\tau\left[\nu + \sum_{\tau'} K_{\tau'}\gamma_{\tau'} - \gamma_\tau\sum_{\tau'} K_{\tau'}\right] = K\nu\phi_\tau \Leftrightarrow K_\tau\left[\nu + \sum_{\tau'\neq\tau} K_{\tau'}\left(\gamma_{\tau'} - \gamma_\tau\right)\right] = K\nu\phi_\tau \Leftrightarrow K_\tau = \frac{K\nu\phi_\tau}{\nu + \sum_{\tau'\neq\tau} K_{\tau'}}$$

This expression has implicitly used that $K = \sum K_\tau$ and does not treat it as a constant. There is not a guarantee that K_τ solves inside $[0, K]$ in the latter expression. Using the implicit expression, the solution implies,

$$K = \sum K_\tau = \sum_\tau \frac{K\nu\phi_\tau + K_\tau\gamma_\tau (K - K_\tau)}{\nu + \sum_{\tau' \neq \tau} K_{\tau'}\gamma_{\tau'}}$$

4. Finally, the algorithm proceeds to a solution for (P, F, G, θ) iterating over a solution to the difference equation,

$$\big(\delta_0 + d_{n+1}(\mu)\big)(n+1)P_{n+1}(\mu) + h_{n-1}(\mu)P_{n-1}(\mu) = \big[\delta_1 + h_n(\mu) + \big(\delta_0 + d_n(\mu)\big)n\big]\, P_n(\mu),$$

the offer and match distributions,

$$F(X) = \frac{\sum_\tau K_\tau E_\tau \left[\sum_{n=0}^{\infty} \mathbb{I}\left[\Delta V_n(\tilde{\mu}) \leq X\right] v_n(\tilde{\mu})\, P_n(\tilde{\mu})\right]}{\sum_\tau K_\tau E_\tau \left[\sum_{n=0}^{\infty} v_n(\tilde{\mu})\, P_n(\tilde{\mu})\right]}$$

$$G(X) = \frac{\sum_\tau K_\tau E_\tau \left[\sum_{n=0}^{\infty} \mathbb{I}\left[\Delta V_n(\tilde{\mu}) \leq X\right] n P_n(\tilde{\mu})\right]}{(1-u)\, L}$$

and market tightness,

$$\theta = \frac{\sum_\tau E_\tau \left\{\sum_{n=0}^{\infty} v_n(\tilde{\mu})\, P_n(\tilde{\mu})\right\} K_\tau}{\left(u + (a_1 + a_2)(1-u)\right) L},$$

where the iteration scheme iterates to an equilibrium holding firm behavior constant and allows meeting rates to update to the new θ value according to step 1.

5. loop over 2-4 until convergence in $(Y, \theta, \delta, F, G)$.

The key threats to breakdown in the solver is granularity in the problem that results in cycles as well as product lines that have very large upper bounds on labor. The problems can become very large.

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