Pivots and Prestige: Venture Capital Contracts with Experimentation*

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Abstract

We study a dynamic informed principal problem with learning. An entrepreneur contracts with an investor and has private information about a project, which requires costly experimentation by both parties to succeed. In equilibrium, investors learn about the project from the arrival of exogenous information and from the entrepreneur's contract offers. Pivots and prestige projects emerge as signaling devices. Technological progress, which lowers the cost of experimentation or which increases the rate of learning, makes entrepreneurs pivot more aggressively in equilibrium.

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"... if you're not in a state where you're rapidly changing ideas or assumptions over, and over, and over again in quick succession, you are likely doing it wrong."

— Dalton Caldwell, Y-Combinator[1]

"A reorientation is an implicit admission that the plan to which the founders were once deeply committed was flawed. This [...] can suggest a lack of consistency and competence."

— Harvard Business Review, September 2020[2]

1 Introduction

Entrepreneurial firms are key for innovation and growth. In recent years, technological progress has reduced the cost of starting firms by orders of magnitude, which has dramatically transformed how startups operate (e.g. Kerr et al. (2014)). Instead of developing a complete product and bringing it to market, startups increasingly focus on "minimum viable products" and on quick improvements in response to market feedback. Pivots are integral to this approach. If the initial response to a product is underwhelming, entrepreneurs quickly abandon it and start developing a new one. Examples of successful pivots abound: Groupon initially started as a social network, Twitter emerged from a failed podcasting platform, and Slack initially developed on online game.[3]

Yet, this "lean startup" approach has its detractors. A pivot reveals that the founders' initial idea is flawed, which may reflect negatively on their ability to generate ideas or to execute them. Many pivots fail because investors lose confidence following an admission of failure and refuse to fund another idea. Investors have also become wary about startups pivoting excessively for the sake of appearance, without ever making progress.

In this paper, we reconcile these conflicting views in a dynamic contracting model with adverse selection. When the cost of pivoting is intermediate, a high ability entrepreneur pivots to separate from a low ability entrepreneur. Then, a pivot is a positive signal about entrepreneurial ability. When the cost of pivoting is high, however, the low ability entrepreneurs pivots first, and a pivot is a bad signal about the entrepreneur's ability. Overall, our model suggests that seemingly excessive pivots serve a valuable social role, by helping investors distinguish entrepreneurial talent.

[1] See https://www.ycombinator.com/library/6p-all-about-pivoting.

[2] See https://hbr.org/2020/09/when-its-time-to-pivot-whats-your-story.

[3] See https://www.inc.com/jeff-haden/21-side-projects-that-became-million-dollar-startups-and-how-yours-can-too.html.

In our model, the entrepreneur (she) is privately, but imperfectly, informed about the quality of her project, which is either good or bad. She is either a high type, who knows that her project is likely to be good, or a low type. We can understand the type as the entrepreneur's ability, i.e. the high type generates better projects on average. We assume that in each period, the entrepreneur offers a contract to an investor (he), which consists of an equity share that grants the investor a stake in the project. This assumption closely follows Bergemann and Hege (1998) and Bergemann and Hege (2005). The project requires costly experimentation by both the entrepreneur and the investor to succeed.[4] When both experiment, the good project generates a breakthrough with positive probability, while the bad project generates no breakthrough.[5] Over time, both entrepreneur and investor learn from the absence of breakthroughs, and revise their beliefs about the project downwards.

The entrepreneur has a real option to abandon the initial project and start a new one, after paying a fixed cost. We call this a *pivot*. Why do pivots signal information? We model the entrepreneur's type as the ability to generate ideas. Since this ability exhibits persistence (see e.g. Gompers et al. (2010)), the high type's new project is more likely to succeed than the low type's. After the initial project fails to yield a breakthrough for some time, the high type prefers to pivot, since for her the value of starting a new project is higher. The low type, however, prefers to continue her initial project, since the value from starting the new project is lower. Then, the high type pivots, the low type does not, and pivoting signals that the type is high. As we show in Section 5.3, this equilibrium occurs whenever the cost of pivoting is in an intermediate range. If the cost is relatively large, the high type instead prefers to wait until everyone has grown sufficiently pessimistic about the project. Then, the low type randomizes between pivoting to a new project (and thereby revealing her type) and continuing the old one. Finally, if the cost of pivoting is relatively small, pivots cannot signal information and the equilibrium is pooling. That is, the low and high type pivot at the same time. Intuitively, when the cost of pivoting is too low, the low type can easily imitate the high type.

The pivoting decision is inherently a dynamic one. To see the tradeoffs, consider an equilibrium where investors expect only the high type to pivot at time t. If the entrepreneur pivots, she gives up her value from continuing the existing project, albeit at off-path beliefs which identify her as the low type. In return, she starts a new project and investors believe

[4]We can interpret the cost to the investor in two ways: (1) the investor must exert effort, as in the literature on double moral hazard (e.g. Schmidt (2003), Casamatta (2003), Repullo and Suarez (2004), and Hellmann (2006).) (2) Running the startup costs c each period and the investor must provide these funds to keep the startup going. Both interpretations are equivalent in the model.

[5]Thus, our model features an exponential bandit with good news, as in e.g. Bergemann and Hege (1998), Bergemann and Hege (2005), and Keller et al. (2005).

that she is the high type. The value of continuing the initial project depends on the entrepreneur's and the investor's past learning. Early on, pivoting is not valuable, since the initial project is likely to succeed, while late, the low type pivots as well, since her value from continuing the initial project is small. Thus, learning determines whether pivots are feasible.

A large literature starting with Myers and Majluf (1984) studies equity issuance under adverse selection. In line with that literature, why doesn't the high type separate by offering a higher equity share? In our model, signaling via a higher share is feasible, but not optimal. Instead, without pivots, the optimal contract is pooling. Intuitively, separating is feasible because the low type's project is less likely to succeed than the high type's and thus her continuation value from imitating the high type is lower.[6] There exists an equity share for which the high type prefers to separate, and the low type does not imitate. However, separating via equity is relatively costly for the high type, since her project is more likely to succeed. To reduce the low type's value from imitating by one, the high type incurs a cost that is strictly larger than one, because her shares are more likely to pay out to investors. In this sense, the classical *single crossing condition* is inverted in our model,[7], which renders separation suboptimal. Thus, without pivots, the optimal contract is pooling.

This pooling contract is of independent interest, since it generates vesting and dilution based entirely on adverse selection. Specifically, dilution occurs because investor becomes more pessimistic about the project over time. Then, the entrepreneur pledges successively larger shares to prevent the investor from abandoning the project. Eventually, however, the low type starts liquidating while the high type continues with certainty. Then, the investor updates his belief about the project upwards, because the likelihood that he faces the high type increases. In response, the entrepreneur optimally lowers the investor's share and increases her own. This feature resembles a delayed vesting schedule: the entrepreneur's share initially decreases, but it starts to increase after sufficient time has passed.[8]

The traditional understanding (e.g. Kaplan and Strömberg (2003)) is that vesting is an incentive device for the entrepreneur and dilution occurs because the firm has to raise additional financing. In particular, vesting schedules are usually agreed upon ex-ante, i.e. they are part of a long-term contract. By contrast, both vesting and dilution are the result of adverse selection in our model, and we do not need commitment to long-term contracts

[6] In this sense, we have single crossing in the entrepreneur's continuation values. See Equation (14).

[7] In much of the literature on signaling, sending a signal is more costly for the low type than for the high type, which ensures the existence of separating equilibria. In our model, the opposite is true.

[8] Our model mechanics are consistent with Bengtsson and Sensoy (2015), who document that if a firm's performance between financing rounds is poor, later rounds award higher cash-flow rights to investors. This is consistent with investors becoming pessimistic about the likelihood of success, just as our model predicts.

to generate these features. In other words, adverse selection and learning are enough to generate a contract that *resembles* a delayed vesting schedule.

As an alternative to pivots, we also consider signaling via prestige projects. Prestige projects are also common among early stage firms. Perhaps paradoxically, startups divert resources from their main project and use them to generate publicity and goodwill.[9] Prestige projects can serve as signaling devices, because they tempt low types to liquidate. Intuitively, since the low type's value from continuing her project is lower at all times, the low type prefers to liquidate her project once she has generated enough prestige, while the high type prefers to continue.

In Section 6, we deliver additional empirical predictions via comparative statics. When the cost of operating the startup decreases, entrepreneurs pivot earlier. Thus, shocks which decrease the cost of running startups should lead to more pivots in the cross section of a given cohort of startups. When rate of breakthroughs, or equivalently the rate of learning, increases, entrepreneurs also pivot earlier. Thus, we expect that information spillovers, e.g. from patent disclosures (Hegde et al. (2018)) lead to earlier pivots. Finally, as the ex-ante quality of the entrepreneur decreases, entrepreneurs again pivot earlier. Thus, we should expect more pivots in industries with lower entry barriers for new founders or those which require less skill.

Interestingly, our model suggest that the rise of pivots may be the consequence of recent technological progress. As Kerr et al. (2014) report, the cost of starting internet companies has decreased substantially.[10] Simultaneously, cohort-based accelerators (such as Y-Combinator) have increased entry by relatively inexperienced founders. Arguably, experimentation about startups has sped up, and investors discover more quickly whether a startup is going to be successful. Our comparative statics suggest that all these changes lead entrepreneurs to pivot earlier.

[9]For example, WeWork, a co-working platform, founded an elementary school (see `https://www.reuters.com/article/us-wework-wegrow/wework-to-close-its-wegrow-elementary-school-in-new-york-next-year-idUSKBN1WQ28V`, last accessed 10/13/19), Uber offered helicopter rides (see `https://www.theverge.com/2019/10/3/20897427/uber-helicopter-trips-manhattan-jfk-airport-price`, last accessed 10/13/19), and Tesla's Elon Musk has sold a device which closely resembles, but is not, a flamethrower (see `https://www.boringcompany.com/not-a-flamethrower`, last accessed 10/13/19).

[10]They write "firms in these sectors that would have cost $5 million to set up a decade ago can be done for under $50,000 today. For example, open-source software lowers the costs associated with hiring programmers. In addition, fixed investments in high-quality infrastructure, servers, and other hardware are no longer necessary [...] because they can be rented in tiny increments from cloud computing providers"

2 Literature

Our paper contributes to the literature on experimentation in venture capital financing. In seminal work, Bergemann and Hege (1998) and Bergemann and Hege (2005) study optimal contracts with moral hazard in an experimentation setting. We extend this literature by considering private information on the entrepreneur's side and by characterizing how adverse selection changes as information arrives over time. In Bergemann and Hege's papers, there is no information asymmetry on the equilibrium path and the questions about signaling and separation do not arise. Hence, our results on pivots and prestige projects cannot be obtained in their frameworks.

In our model, a pivot trades off the value of continuing the current project against the value of starting a new one. Both these values depend on the entrepreneur's type, the investor's beliefs, and the equilibrium contract. This distinguishes our model from existing papers on signaling with real options, which have no such notion of pivots. Specifically, in Morellec and Schürhoff (2011) and Bouvard (2012), the entrepreneur experiments and chooses when to start a project. These papers differ from ours as follows: (1) the entrepreneur is not endowed with a project and learning is about the value of the real option, not the value of the current project. Thus, pivots do not arise, since the entrepreneur has no project she can abandon for a new one. (2) There is no ongoing contracting. Instead, the entrepreneur chooses when to start a project and how to split payoffs *once the project is started*. By contrast, the entrepreneur splits proceeds with the investor both before and after the pivot in our paper. (3) There is no vesting or dilution, since there is no allocation of cash flows before the project is started. (4) The entrepreneur signals by her choice of cash flow rights, which is not optimal in our model.

A number of papers study signaling via the length of experimentation, i.e. Grenadier et al. (2014), Dong (2016), and Thomas (2019). The key contribution of our paper is to characterize the dynamics of the contract and to explore signaling via pivots and prestige projects. The above settings do not feature contracting or pivots, since the length of experimentation is the only choice variable.

Also related is Kaniel and Orlov (2018), which studies the relationship between a mutual fund family and a manager. As in our paper, there is experimentation about the manager's skill and information is revealed by both news arrival and retention/continuation decisions. In their paper, however, retention is the only signaling device, whereas in our paper, the terms of the contract also can be used to signal. Our results on vesting and dilution, pivots, and prestige projects do not appear in Kaniel and Orlov (2018).

To render our analysis tractable, we borrow from the literature on relational contracts with adverse selection (i.e. Halac (2012), Fahn and Klein (2017), and Kartal (2018)). The key difference is that in our model, all parties learn about the project by observing whether a success arrived, so that the degree of adverse selection changes over time. By contrast, in the above papers, there is no exogenous information about the principal's type and agents can learn only by observing the principal's choices. Indeed, the arrival of information is crucial for our results. Without it, the high type would separate either immediately or never and there would be no dynamics.

Finally, our paper is related to recent work on learning in dynamic contracts. Fong and Li (2016) and Li et al. (2021) respectively study information revelation and the agent's experimentation in relational contracts. Kuvalekar and Lipnowski (2020), Fudenberg et al. (2021), and Smolin (2021) study environments where effort reveals additional information.

3 Model

Environment An entrepreneur (she) has no wealth and seeks to finance a project. There is a competitive market of investors, which in our model is represented by a single representative investor. The entrepreneur's project is either good or bad. It requires costly experimentation by both the entrepreneur and the investor. When both experiment, the good project generates a single payoff V, which realizes with probability $\lambda \in (0,1)$ in each period $t \in \{1, 2, ...\}$. The bad project never generates a payoff.[11] Once either the entrepreneur or the investor stops experimenting, the project is irreversibly liquidated.

The entrepreneur is privately, but imperfectly, informed about the quality of the project. We denote the entrepreneur's type with $\theta \in \{l, h\}$. A high type entrepreneur knows that the project is good with ex-ante probability p_1^h, while a low type knows that the project is good with probability p_1^l, where $0 < p_1^l < p_1^h < 1$. The ex-ante likelihood of a high type is $q_0 \in (0,1)$.[12] Both parties are risk-neutral and have a common discount factor $\delta \in (0,1)$.

The entrepreneur is endowed with a (one-time) real option to abandon her project and start a new one at a fixed cost $F > 0$.[13] We call this the *pivot*. To focus on signaling considerations, we assume that the entrepreneur's type is fixed across projects, so that

[11] Thus, we have an exponential bandit with good news, as in Bergemann and Hege (1998), Bergemann and Hege (2005), and Keller et al. (2005).

[12] In the model, q_t is updated at the beginning of each period, while p_t is updated at the end. This is why our notation for the ex-ante probabilities, i.e., p_1^θ vs. q_0, differs.

[13] We interpret F as an effort cost which is incurred by the entrepreneur. Assuming that F is a cost that must be paid by the investor (e.g because the pivot requires additional capital) yields qualitatively similar results. We assume that the cost is the same for both types for simplicity. Assuming that the cost is e.g. higher for the low type yields similar results, provided that the difference in costs between low and high type is not too large. Finally, as is standard in the real options literature, the entrepreneur can exercise her option at most once, i.e. she loses the option to pivot again.

the new project has an ex-ante likelihood of success of p_1^l for the low type and p_1^h for the high type. Equivalently, the initial and the new projects are ex-ante identical, conditional on the entrepreneur's type. We relax this assumption in Section 7.1, where we allow the entrepreneur's type to change after the pivot, and provide conditions so that our results go through.

Contracts Our notion of contracting closely follows Bergemann and Hege (1998), Bergemann and Hege (2005), and Bouvard (2012). At the beginning of each period t, the entrepreneur chooses a liquidation probability $l_t^\theta \in [0,1]$. If she continues, the entrepreneur pays a cost $k > 0$ and offers the investor a contract $C_t^\theta = \left(I_t^\theta, \alpha_t^\theta\right)$. This contract consists of a decision to pivot $(I_t = 1)$ or to wait $(I_t = 0)$, and an equity share $\alpha_t^\theta \in [0,1]$, which is contingent on the project succeeding.[14] The contract, and in particular the decision to pivot, is observable by the investor. Given the contract, the investor chooses whether to continue $(e_t = 1)$ or whether to abandon the project $(e_t = 0)$. Continuing has cost $c > 0$ for the investor. This cost can represent a cash investment which is made each period, the opportunity cost of already committed funds, or advising effort (see Section 4 for a discussion). If the project is liquidated, the entrepreneur and investor each receive an outside option of zero.[15]

Beliefs The high type entrepreneur, the low type entrepreneur, and the investor each have different beliefs about the likelihood that the project is good. Figure 1 shows how beliefs are updated. Each entrepreneur type learns from the absence of successes. Consider a period t at which neither type has pivoted yet. The type-θ entrepreneur enters period t with belief p_t^θ. Without a success, she updates her belief to

$$p_{t+1}^\theta = \frac{(1-\lambda)\, p_t^\theta}{1-\lambda p_t^\theta}, \tag{1}$$

which is strictly decreasing over time. Equation (1) implies that $p_t^h > p_t^l$ for all t. That is, the high type believes that her project is more likely to succeed at all times.

[14]In our main analysis, the entrepreneur is cashless, which captures the fact that entrepreneurs rarely pay out investors prior to an IPO or acquisition. We study cash payments to investors in Section C.

Also, once the equity is pledged, it is enforceable. Thus, the entrepreneur cannot renege once the project succeeds, unlike in e.g. Halac (2012).

[15]The value of the outside option does not affect the qualitative properties of the contract. We set it to zero to simplify notation. We study the case when the entrepreneur can choose different projects, which have different outside options, in Section 5.4. We may alternatively assume the following timing structure within each period t: (1) the entrepreneur offers the contract, (2) the investor decides whether to accept or reject, (3) the entrepreneur and investor pay costs k and c, respectively, (4) the project succeeds or not. All results go through under this specification.

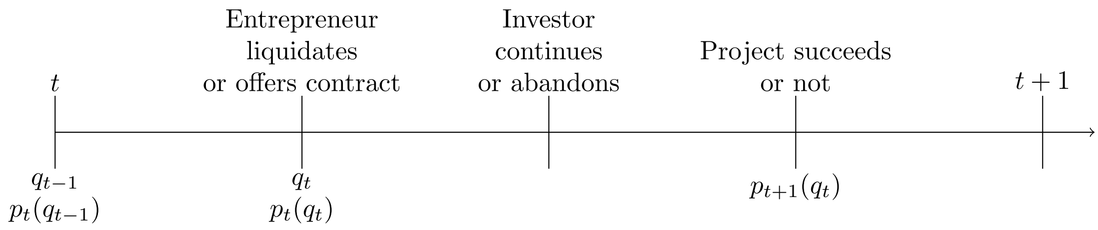


Figure 1: Timeline and Beliefs

The investor learns from two sources. First, he may learn about entrepreneur's type from the contract offered. At the beginning of period t, he believes he is facing the high type with probability q_{t-1}. Upon observing the contract, he updates this belief to q_t. From the investor's perspective, the likelihood that the project is good is then

$$p_t(q_t) = q_t p_t^h + (1 - q_t) p_t^l. \tag{2}$$

Second, if the project does not succeed, he updates this belief to $p_{t+1}(q_t)$, using Bayes' rule in Equation (1). If type θ pivots in period t, she changes her belief to $p_t^\theta = p_1^\theta$ and the investor updates his belief q_t as before.[16]

Payoffs Denote with τ the period in which the game ends, either because the project succeeds or because it is liquidated.[17] Denote with $\mathbf{1}_s$ the indicator function which is one if and only if the project succeeds in period s, and with τ_P^θ the time at which type θ pivots.[18]

In period t, the payoffs for the type-θ entrepreneur and the investor are

$$\Pi_t^\theta = E_t^\theta \left[\sum_{s=t}^{\tau} \delta^{s-t} \left(\mathbf{1}_s (1 - \alpha_s) V - k - I_s^\theta F \right) \right] \tag{3}$$

and

$$U_t = E_t \left[\sum_{s=t}^{\tau} \delta^{s-t} (\mathbf{1}_s \alpha_s V - c) \right]. \tag{4}$$

Here, $E_t^\theta[.]$ is the expectation given type θ's belief p_t^θ and $E_t[.]$ is the expectation given the investor's beliefs $p_t(q_t)$ and q_t.

[16] For example, if the high type pivots and the low type does not, we have $q_t = 1$ and $p_t(q_t) = p_t(1) = p_t^h = p_1^h$.

[17] Since liquidation is irreversible, $e_t = 1$ for all $t < \tau$.

[18] Formally, $\tau_P^\theta = \min\{t : I_t^\theta = 1\}$ and we set $I_t^\theta = 0$ for all $t > \tau_P^\theta$ without loss of generality.

For $t < \tau$, the entrepreneur's and investor's values can be written recursively as

$$\Pi_t^\theta = \left(1 - l_t^\theta\right) \left(\lambda p_t^\theta \left(1 - \alpha_t\right) V - k - I_t^\theta F + \delta \left(1 - \lambda p_t^\theta\right) \Pi_{t+1}^\theta\right) \tag{5}$$

and

$$U_t = \left(1 - l_t \left(q_{t-1}\right)\right) \left(\lambda p_t \left(q_t\right) \alpha_t V - c + \delta \left(1 - \lambda p_t \left(q_t\right)\right) U_{t+1}\right), \tag{6}$$

where

$$1 - l_t \left(q_{t-1}\right) = q_{t-1} \left(1 - l_t^h\right) + \left(1 - q_{t-1}\right) \left(1 - l_t^l\right)$$

is the investor's expectation about the entrepreneur's liquidation probability.

Equilibrium Concept We focus on Perfect Bayesian Equilibria. A Perfect Bayesian Equilibrium is a set of strategies and posterior beliefs, such that the strategies are sequentially rational at each history given the beliefs, and the beliefs are updated according to Bayes' rule whenever possible. We provide a formal equilibrium definition in Appendix A. Following Halac (2012), we require Bayesian updating both on and off the equilibrium path. Bayes' rule does not apply at histories at which the investor's belief about the entrepreneur is degenerate.[19] We follow the literature in making the following assumption.[20]

Assumption 1 *If, at any history, the investor believes he is facing type θ with certainty, he will continue to believe so no matter which contract is offered.*

Throughout the paper, we refer to a Perfect Bayesian Equilibrium as *equilibrium*. We consider pooling and separating equilibria. In a pooling equilibrium, both types offer the same contract each period, but the low type may liquidate the project earlier and thereby reveal her type. In a separating equilibrium, types *separate in period t* if they pool until period $t-1$ and offer different contracts in period t. An equilibrium contract is *optimal* if it maximizes a weighted average of the low and high type's ex-ante values, where $\gamma \in [0, 1]$ is the weight on the high type.[21]

Parametric Assumptions To avoid uninteresting cases, we maintain the following assumptions throughout the paper.

[19] That is, $q_t \in \{0, 1\}$. These histories arise after the high type successfully separates from the low type. See e.g. Section 5.2.

[20] This, or similar assumptions, are common in dynamic adverse selection models. See e.g. Osborne and Rubinstein (1990)'s "Never Dissuaded Once Convinced" condition.

[21] We focus on optimal PBE throughout the paper. As a robustness check, we adapt the D1 criterion of Cho and Kreps (1987) to our setting and show that our results are qualitatively unchanged. The extension is in Section 7.2.

Assumption 2 *In the first best, the good project is never liquidated, i.e.,*

$$\lambda V > k + c, \tag{7}$$

and in the pooling equilibrium, the low type does not immediately liquidate, i.e.,

$$\lambda p_1^l \left(1 - \frac{c}{\lambda p_1 (q_0) V}\right) V > k. \tag{8}$$

Without Equation (7), the entrepreneur would immediately liquidate the project in any equilibrium. Without Equation (8), there may exist a pooling equilibrium in which either the low type or both types immediately liquidate the project. Then, the investor's belief evolution is trivial. He either learns nothing (if both liquidate) or immediately learns he is facing the high type (if only the low type liquidates).

For tractability, we also impose the following parametric assumption.

Assumption 3 *We have* $\left(1 - \lambda p_1^h\right)\left(1 - \lambda p_1^l\right) > 1 - \lambda$.

Assumption 3 implies that the degree of adverse selection, as measured by the difference in the high and low type's beliefs, $p_t^h - p_t^l$, is decreasing over time.[22] It holds whenever the initial probabilities p_1^l and p_1^h are sufficiently small.

4 Discussion

We now discuss how our modeling assumptions map to observed patterns in venture capital and how they relate to existing literature.

Venture Capital We follow the literature on "double moral hazard" (see Schmidt (2003), Casamatta (2003), Repullo and Suarez (2004), and Hellmann (2006)). As in these papers, both entrepreneur and VC must exert costly effort, and we can interpret the VC's effort as monitoring or advising.[23] Alternatively, we can understand the VC's effort cost c as ongoing

[22]This is because

$$p_{t+1}^h - p_{t+1}^l = \frac{1-\lambda}{\left(1 - \lambda p_t^h\right)\left(1 - \lambda p_t^l\right)} \left(p_t^h - p_t^l\right).$$

Since $p_t^\theta < p_1^\theta$, it is sufficient to show this condition holds at $t = 1$. We use Assumption 3 to establish a single-crossing condition in Lemma 50. Note that $p_t^h - p_t^l$ is decreasing at time t whenever p_t^h and p_t^l are sufficiently small. Thus, even without the assumption, $p_t^h - p_t^l$ is decreasing when t is sufficiently large. Assumption 3 merely ensures that this is true for all t.

[23]Consistent with this view, the empirical literature documents that venture capital investors provide valuable services to entrepreneurs (see Sahlman (1990), Gorman and Sahlman (1989), Lerner (1995), and Hellmann and Puri (2000)), which include providing advice, helping determine strategy, or helping recruit talent (see e.g. Kortum and Lerner (2000) and Bernstein et al. (2016)).

investment that is necessary to keep the project going. Thus, our modeling is consistent with both the "double moral hazard" view and with the VC's role as a financier.

We can interpret the investor's exit as the firm shutting down or being bought out, or as the founder being replaced (see Wasserman (2003)). We can interpret the arrival of a success as an IPO.

Contracting Our model falls into the class of infinite-horizon informed principal problems. As is well-known, it is generally not feasible to characterize long-term contracts in such settings.[24] Therefore, we follow Halac (2012) and consider PBE, which implies that contracts are sequentially rational at each history. In Section 7.3, we study renegotiation-proof long-term contracts and provide sufficient conditions such that the optimal contract in our main model remains optimal.

From an applied perspective, our modeling choice is consistent with the fact that startups are subject to rapid changes and significant uncertainty, which often render contractual commitments moot (see Kaplan and Strömberg (2001), Kaplan and Strömberg (2003), Kaplan and Strömberg (2004) and Kerr et al. (2014)). For example, Kaplan and Strömberg (2003) and Bengtsson and Sensoy (2015) document that VC's contracts are frequently renegotiated based on the startup's interim performance.

The Role of Pivots In our model, a pivot implies starting a new project and abandoning the current one. For simplicity, we assume that the entrepreneur's type is the same across projects, i.e. the high (or low) type's new project has an ex-ante likelihood of success of p_1^h (or p_1^l). This assumption is not crucial for our results. We allow the entrepreneur's type to change in Section 7.1 and provide conditions so that our results go through. In applied terms, our assumption captures the idea that the founders' quality is most important for the startup's success. Indeed, many VC firms consider the founders skills more important than the actual project the startup is working on, since they anticipate the startup to change their project in the future (see e.g. Gompers and Lerner (2001), Kaplan et al. (2009), and Gompers et al. (2020)).

Experimentation and Adverse Selection We assume that entrepreneur and investor learn about the firm over time. This is consistent with Kerr et al. (2014), who document that even conditional on investing, VCs face significant residual uncertainty, and with Ewens

[24]Intuitively, with commitment, the principal can signal her type at time t by offering an inefficient contract at an arbitrary time $T > t$. This yields a large and intractable set of IC constraints. By contrast, in our setting we only have to consider one IC constraint for each time t. See e.g. Equations (IC_l) and (IC_h) below.

et al. (2018), who document that investors adjust their contracts as information becomes available. Our modeling of learning follows Bergemann and Hege (1998) and Bergemann and Hege (2005), who also assume that information arrives in the form of successes or their absence. Given the substantial skewness of returns in the venture capital industry, which features few startups with high profits and many startups with profits close to zero, this assumption is reasonable (see Scherer and Harhoff (2000) and Hall and Woodward (2010)).

An overwhelming part of the venture capital literature highlights the entrepreneur's information advantage as a source of frictions, going back at least to Gompers (1995). As investors learn about the firm, however, the information advantage disappears and the optimal contract changes (see Kaplan and Strömberg (2003) and Ewens et al. (2018) for evidence). This is exactly what happens in our model. As time passes, the projects of the good and bad type become indistinguishable. This evolution of the adverse selection friction is a key driver for our results.

Alternative Formulations In our model, the entrepreneur offers the contract to the investors. This assumption is commonly made in the VC literature, see e.g. Repullo and Suarez (2004), Hellmann (2006), and Bouvard (2012), and can easily be relaxed. If, as in Axelson (2007), we assume that the investor has private information and chooses the contract, our entire analysis goes through, except that the roles of the entrepreneur's and investor's share are reversed.

Alternatively, we could interpret the model as contracting between a founder and an early employee, who is compensated by a significant equity stake. Such arrangements are common in startups (see Hand (2008)) and other industries (see Eisfeldt et al. (2018)). In this interpretation, the employee learns about the firm's prospects the longer he is employed and prefers to leave if the prospects become sufficiently unfavorable. None of our results change.

Finally, instead of having a single investor, we could have a sequence of short-lived investors. That is, the entrepreneur is matched with a new investor each period, who observes the history of the contract. All results go through under this specification.

5 Analysis

We start with some notation. To distinguish whether to option to pivot has been exercised yet, we denote the value of type θ after exercise as Π_t^θ and before exercise at $\widehat{\Pi}_t^\theta$. We denote the payoffs given belief q_t and contract C_t as $\Pi_t^\theta(q_t, C_t)$ and $\widehat{\Pi}_t^\theta(q_t, C_t)$, irrespectively of whether this is on or off the equilibrium path. Finally, we denote with $\Pi_t^\theta(q_t)$ and $\widehat{\Pi}_t^\theta(q_t)$

the *equilibrium* payoffs given belief q_t. All proofs are in the Appendix.

5.1 Symmetric Information Benchmark

Suppose that the entrepreneur's type is public and that she offers a contract $\bar{C}_t^\theta = (\overline{I}_t^\theta, \bar{\alpha}_t^\theta)$. The investor's belief about the project is the same as the entrepreneur's, i.e., $p_t(q_t) = p_t^\theta$. The investor is willing to continue the project whenever

$$\lambda p_t^\theta \alpha_t^\theta V - c + \delta \left(1 - \lambda p_t^\theta\right) U_{t+1} \geq 0. \tag{9}$$

The left-hand side (LHS) is the investor's payoff from continuing, which must exceed his outside option. The optimal contract leaves the investor indifferent between continuing or abandoning the project. That is, $U_t = 0$ for all t, the optimal share is

$$\bar{\alpha}_t^\theta = \frac{c}{\lambda p_t^\theta V}. \tag{10}$$

Any other contract can be improved upon by the entrepreneur. If $U_t > 0$ for some t, lowering α_t^θ increases the entrepreneur's payoff without violating Equation (9). The optimal equity share is increasing in time. As time passes without a success, the investor becomes more pessimistic about the project, i.e. p_t^θ decreases. Then, the entrepreneur must pledge a larger share to ensure that the investor continues. Moreover, the low type pledges a larger share than the high type, i.e. $\bar{\alpha}_t^l > \bar{\alpha}_t^h$, because the likelihood that the low type's project succeeds is lower.

Given the optimal share $\bar{\alpha}_t^\theta$, the entrepreneur's payoff in period t (without a pivot) is

$$\widehat{\Pi}_t^\theta = \left(1 - l_t^\theta\right) \left(\lambda p_t^\theta V - c - k + \delta \left(1 - \lambda p_t^\theta\right) \widehat{\Pi}_{t+1}^\theta\right). \tag{11}$$

When t becomes large, the entrepreneur's period payoff becomes negative, because the project is unlikely to succeed. Then, she either pivots or liquidates. Specifically, type θ liquidates whenever $\Pi_t^\theta \leq 0$ and type θ pivots if $\Pi_1^\theta - F \geq \widehat{\Pi}_t^\theta$. Since the low type's project is less likely to succeed than the high type's, the low type liquidates earlier. By pivoting, the entrepreneur resets the belief to p_1^θ but she loses the option to pivot and incurs a cost F.[25] Her continuation value is then the same as her initial value without the option to pivot later. If she does not pivot, the entrepreneur continues her project with belief p_t^θ. We summarize these results in the following Lemma.

[25] Here, recall that the entrepreneur may exercise the option to pivot at most once.

Lemma 1 *The type-θ entrepreneur offers share*

$$\bar{\alpha}_t^\theta = \frac{c}{\lambda p_t^\theta V}$$

and liquidates the project whenever $\lambda p_t^\theta V - c - k \leq 0$. Let τ^θ be the period in which liquidation occurs under symmetric information. We have $\tau^l \leq \tau^h$ and $\bar{\alpha}_t^l > \bar{\alpha}_t^h$ for all $t < \tau^l$. Further, if $\Pi_1^\theta > F$, the type-θ entrepreneur pivots at time

$$\bar{\tau}_P^\theta = \min\left\{t : \Pi_1^\theta - F \geq \widehat{\Pi}_t^\theta\right\}.$$

The condition $\Pi_1^\theta > F$ is necessary for pivots to occur with symmetric information. Otherwise, the fixed cost F is so large that the entrepreneur prefers to liquidate instead of pivoting. In the following, we denote the high and low type's symmetric information payoffs as $\widehat{\Pi}_t^h(1)$ and $\widehat{\Pi}_t^l(0)$ (before the pivot) and $\Pi_t^h(1)$ and $\Pi_t^l(0)$ (after the pivot).[26]

With symmetric information, pivots occur because the entrepreneur learns that the initial project is unlikely to succeed. Our main contribution is to show that pivots can signal the entrepreneur's ability. Thus, with private information, there is a different and novel reason for why firms pivot. In other words, that pivots occur in this framework should not be surprising. Our goal is to provide a novel rationale for *why* pivots occur.

5.2 No Pivots

With private information, offering the symmetric information contracts is not incentive compatible. Since $\bar{\alpha}_t^h < \bar{\alpha}_t^l$, the low type prefers to imitate the high type, because then she can offer a lower equity share. This is the source of adverse selection in our model.

We first consider the optimal contract without pivots and show that it is pooling. Both types offer the same equity share, but the low type liquidates earlier than the high type and thereby reveals her type. The entrepreneur's share first decreases and then increases. This resembles dilution, i.e. as the project continues the entrepreneur's share becomes increasingly diluted, and vesting, i.e. after enough time has passed, the entrepreneur's shares vest and her stake in the firm increases.

Proposition 2 *The following pooling contract is optimal. There exist two periods $\underline{\tau}^l \leq \bar{\tau}^l$, such that both types continue for $t < \underline{\tau}^l$. The low type liquidates with positive probability for*

[26] That is, if $\bar{C}_t^\theta$ is the optimal symmetric information contract, then $\Pi_t^h(1) = \Pi_t^h\left(1, \bar{C}_t^h\right)$ and $\Pi_t^l(0) = \Pi_t^l\left(0, \bar{C}_t^l\right)$.

$t \geq \underline{\tau}^l$ and liquidates with certainty in period $\bar{\tau}^l$. For all $t < \bar{\tau}^l$, both types offer equity share

$$\alpha_t^P = \frac{c}{\lambda p_t\left(q_t\right) V}. \tag{12}$$

This share is increasing in time for $t < \underline{\tau}^l$ and decreasing for $t \geq \underline{\tau}^l$. After the low type liquidates, the high type offers $\bar{\alpha}_t^h$ and continues until period τ^h.

Our notion of vesting and dilution crucially differs from standard explanations (i.e. vesting is used to incentivize the entrepreneur or to increase retention, and dilution occurs mechanically by having to raise equity). Instead, vesting and dilution are driven by the interplay between learning and adverse selection. When breakthrough arrives, investors become more pessimistic about the project, which leads to dilution for the entrepreneur. Eventually, the low type starts liquidating with positive probability while the high type continues. Then, not liquidating is good news about the entrepreneur's type (and hence the value of the project), even in the absence of a breakthrough. This leads the entrepreneur's share to increase, which resembles vesting. Hence, our results provide a novel explanation for why vesting and dilution occur.

We now informally derive the main results of the proposition.[27] The low type knows that her project is less likely to succeed. Thus, when offering the same contract as the high type, her value from continuing is lower. After enough time without a success, the low type starts liquidating with positive probability, while the high type continues. Thus, even though both types offer the same contract, the investor learns the entrepreneur's type over time. Based on the low type's liquidation strategy, the investor updates his belief according to

$$q_t = \frac{q_{t-1}}{q_{t-1} + \left(1 - q_{t-1}\right)\left(1 - l_t^l\right)}. \tag{13}$$

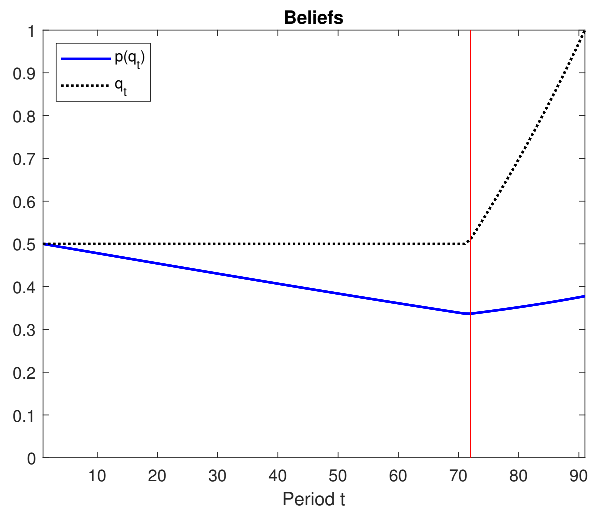
The belief is constant when the low type does not liquidate ($l_t^l = 0$) and increasing when she does ($l_t^l > 0$). Since the low type never liquidates before period $\underline{\tau}_l$, we have $q_t = q_0$ for any $t < \underline{\tau}^l$.

The investor continues the project whenever

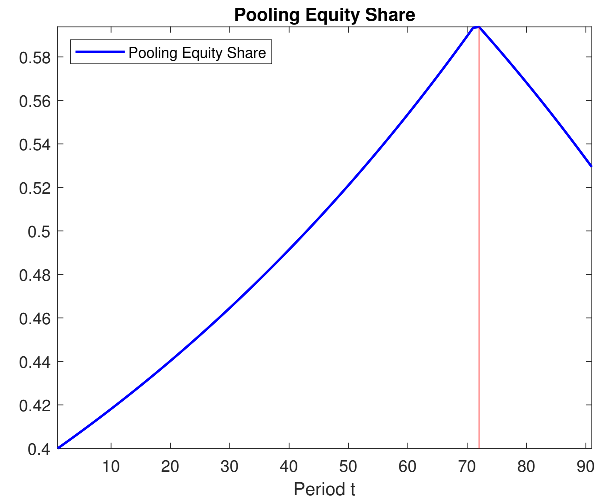
$$\lambda p_t\left(q_t\right) \alpha_t^P V - c + \delta\left(1 - \lambda p_t\left(q_t\right)\right) U_{t+1} \geq 0. \tag{IC_I}$$

The optimal pooling equity share (in Equation (12)) leaves the investor indifferent between continuing and abandoning the project. Any higher share is suboptimal, because both types can lower the share until the investor's incentive compatibility (IC) condition (IC_I) binds.

[27]The formal proof is in Appendix B.



(a) Pooling Equilibrium: Investor's Beliefs



(b) Pooling Equilibrium: Investor's Equity Share

Figure 2: On each panel, the vertical line indicates $\underline{\tau}^l$. Before $\underline{\tau}^l$, the investor does not change his beliefs about the entrepreneur's type, because neither type liquidates. Between $\underline{\tau}^l$ and $\bar{\tau}^l$, the low type liquidates with positive probability, so that q_t increases. This causes $p_t(q_t)$, the investor's belief about the project, to increase (left panel). Between $\underline{\tau}^l$ and $\bar{\tau}^l$, the optimal pooling share decreases (right panel), because the investor becomes more optimistic about the project, which resembles a vesting schedule for the entrepreneur.

The optimal equity share is increasing in time when the low type does not liquidate, i.e. before period $\underline{\tau}^l$, because the investor's belief $p_t(q_0)$ is decreasing.[28] To keep the investor indifferent, his share must increase. However, starting from period $\underline{\tau}^l$, the low type liquidates with positive probability and the equity share is decreasing. Intuitively, the low type must be indifferent between liquidating and continuing, i.e. $\Pi_t^l = \Pi_{t+1}^l = 0$, and Equation (5) reduces to

$$\lambda p_t^l \left(1 - \alpha_t^P\right) V = k.$$

The belief p_t^l decreases over time, so the equity share must also decrease to keep the low type indifferent. The equilibrium liquidation probability l_t^l, together with Bayes rule in Equation (13), ensure that the investor continues the project in any such period.[29] Intuitively, when the low type liquidates with positive probability, q_t increases, and thus the investor is willing to continue despite receiving a lower share.

[28] Recall that both p_t^h and p_t^l decrease without a success. Keeping q_t at q_0, $p_t(q_t)$ decreases as well.

[29] That is, we have for $\underline{\tau}^l \leq t < \bar{\tau}^l$,

$$\alpha_t^P = \frac{\lambda p_t^l V - k}{\lambda p_t^l V} = \frac{c}{\lambda p_t\left(q_t\right)} V.$$

In period $\bar{\tau}^l$, the low type liquidates with certainty and the investor learns whether he is facing the high type. The high type then offers the symmetric information contract $\bar{\alpha}_t^h$. Figure 2 illustrates the dynamics of the investor's equity share and beliefs.[30]

In addition to pooling equilibria, there exist equilibria in which the high type separates in period t. That is, both types offer a pooling contract before period t, and the high type separates in period t by offering an inefficiently large equity share. However, as we show next, all separating equilibria are suboptimal.

Proposition 3 *For any $t < \bar{\tau}^l$, there exists an equilibrium in which the high type separates in period t by offering a share $\alpha_t^h > \alpha_t^P$ and both types pool before period t. Any such equilibrium is suboptimal.*

In a separating equilibrium, the following IC conditions must hold. The low type prefers to reveal her type instead of offering the high type's equity share α_t^h, i.e.

$$\Pi_t^l(0) \geq \lambda p_t^l \left(1 - \alpha_t^h\right) V - k + \delta \left(1 - \lambda p_t^l\right) \Pi_{t+1}^l(1). \tag{IC_l}$$

Similarly, the high type prefers to offer α_t^h, so that

$$\Pi_t^h(0) \leq \lambda p_t^h \left(1 - \alpha_t^h\right) V - k + \delta \left(1 - \lambda p_t^h\right) \Pi_{t+1}^h(1). \tag{IC_h}$$

The continuation values following separation are the symmetric information values $\Pi_{t+1}^h(1)$ and $\Pi_t^l(0)$. If the low type imitates the high type, she optimally offers the high type's symmetric information share $\bar{\alpha}_{t+1}^h$ the next period, and we denote her value with $\Pi_{t+1}^l(1)$, while if the high type imitates the low type, she optimally offers $\bar{\alpha}_t^l$ and receives $\Pi_t^h(0)$.[31]

Intuitively, pledging a large share dissuades the low type from imitating, because she has to give up a larger portion of the project's value if it succeeds. Then, the low type prefers instead to be discovered. The high and low type's values satisfy a variant of single crossing in any period $t < \bar{\tau}^l$,[32]

$$\frac{\Pi_t^h(1) - \Pi_t^h(0)}{p_t^h} \geq \frac{\Pi_t^l(1) - \Pi_t^l(0)}{p_t^l}. \tag{14}$$

That is, the value of being perceived as the high type is larger for the high type than for the low type.[33] Because single crossing holds, the high type can separate in any period.

[30]The investor's share is α_t, which is depicted in Figure 2, and the entrepreneur's share is $1 - \alpha_t$.

[31]This is because of Assumption 1. After types separate, the investor never changes his belief, and he will accept any contract which promises a share of at least $\bar{\alpha}_t^h$ (if $q = 1$) or $\bar{\alpha}_t^l$ (if $q = 0$). The optimal contract for type θ is then the symmetric information contract of Section 5.1.

[32]See Lemma 25 in Appendix B.3.

[33]Intuitively, if type θ is being perceived as the high type in a future period $s > t$, she can offer a share

However, separating via a higher share is relatively costly. The high type's project is more likely to succeed, and thus she is more likely to pay the investor. If she increases the share by ε, she reduces the low type's value from imitating by $\lambda p_t^l V \varepsilon$ and her own value by $\lambda p_t^h V \varepsilon$. Thus, to reduce the low type's value by one, the high type has to give up value $p_t^h / p_t^l > 1$. Because of this, the high type's cost of deterring the low type exceeds her benefit from separating. Thus, the optimal contract is pooling.

5.3 Pivots as Signals

With symmetric information, pivots occur because the entrepreneur learns that the initial project is unlikely to succeed (Section 5.1). In this section, we show that pivots can also signal information. This is the main contribution of our paper and relies on incorporating both adverse selection and learning into the same model. Without learning, the entrepreneur would either pivot immediately or never so the decision would be trivial. Without adverse selection, no signaling can occur.[34] The trade off to pivot is inherently a dynamic one. When the pivot serves as a signal, both the high and low type trade off losing their continuation value from the current project against starting a new project, with a belief of $q = 1$. Thus, when, and whether, signaling occurs depends on the history of the contract and evolution of the continuation values. Indeed, these features are what distinguishes our dynamic framework from a static one.

To rule out uninteresting cases we make the following assumption, in addition to Assumptions 1-3. Equation (15) is not crucial, but allows us to reduce the number of cases we have to consider.[35]

Assumption 4 *Both types pivot rather than liquidate, i.e.*

$$\Pi_1^l(0) - F \geq 0. \tag{15}$$

We consider three different kinds of equilibria. (1) The *high type separates via a pivot* if the high type pivots in period τ_S, and the low type pivots in a later period. (2) The *low type separates via a pivot* if the low type pivots in period τ_S, and the high type pivots later. (3) The equilibrium is *pooling* if both the low and the high type pivot in the same period τ_P.

$\bar{\alpha}_s^h$, while when she is perceived as the low type, she offers share $\bar{\alpha}_s^l$. Type θ's value of being perceived as the high type is thus $\lambda p_s^\theta (\bar{\alpha}_s^l - \bar{\alpha}_s^h)$. This value is larger for the high type, whose project is more likely to succeed. After discounting and considering the high and low type's liquidation decisions, this leads to Equation (14).

[34] Again, the point of our paper is not to argue that pivots occur. The point of our paper is to show that pivots can signal information.

[35] Without the assumption, the high and/or the low type may prefer to liquidate rather than to pivot in the first best, which yields different value functions for the equilibrium.

Suppose that the high type separates via a pivot in some period t. Then, if the entrepreneur does not pivot, she retains her current project, but investors will infer that she is the low type. Thus, her continuation value is given by $\widehat{\Pi}_t^{\theta}(0)$, which depends on the entrepreneur's type, the investor's off-path belief, and the current period. If the entrepreneur pivots, investors infer that she is the high type, and she effectively gets to restart the likelihood of success at p_1^{θ}. Her continuation value is then given by $\Pi_1^{\theta}(1)$. Thus, to determine whether signaling can occur, we must compare across time, beliefs, and types.

Specifically, when the high type separates via a pivot, the following IC conditions must hold. First, the low type prefers not to imitate the high type by pivoting as well, i.e.

$$\Pi_1^l(1) - F \leq \widehat{\Pi}_t^l(0), \qquad (IC_l^{Piv})$$

where t is the time at which a pivot occurs.[36] The LHS is the low type's value from pivoting. She starts a new project, so that her belief is p_1^l, and the investor believes he is facing the high type, i.e. $q = 1$. When the low type does not pivot, she continues her initial project, which has a likelihood of p_t^l of being good, but the investor knows he is facing the low type. The high type's IC constraint is similarly given by

$$\Pi_1^h(1) - F \geq \widehat{\Pi}_t^h(0), \qquad (IC_h^{Piv})$$

i.e. the high type prefers to pivot rather than continuing her initial project and being perceived as the low type.

Overall, it is feasible for the high type to separate via a pivot whenever

$$F \in \left[\Pi_1^l(1) - \widehat{\Pi}_t^l(0), \Pi_1^h(1) - \widehat{\Pi}_t^h(0)\right]. \qquad (16)$$

The interval on the RHS is nonempty for all t, which is a variant of single crossing in the high and low type's continuation values.[37]

Suppose instead that the low type separates via a pivot. We show in Proposition 4 that in any equilibrium, the low type randomizes between pivoting or continuing. This is similar to the low type randomizing between liquidating and continuing in Proposition 2. In that case, the relevant IC constraints are

$$\Pi_1^l(0) - F = \widehat{\Pi}_t^l(q_t)$$

[36] Here, recall that the entrepreneur can pivot only once.

[37] See Corollary 34 in Appendix B.4. Specifically, the value of pivoting vs. continuing and being perceived as the low type is higher for the high type, i.e., $\Pi_1^l(1) - \widehat{\Pi}_t^l(0) \leq \Pi_1^h(1) - \widehat{\Pi}_t^h(0)$ because the high type's new project is more likely to succeed.

for the low type and

$$\Pi_1^h(0) - F \leq \widehat{\Pi}_t^h(q_t)$$

for the high type. That is, the low type is indifferent between pivoting and continuing to offer the pooling contract α_t^P, and the high type prefers to continue rather than pivot and imitate the low type.

Finally, if both types pool, they offer the same equity share α_t^P before the pivot and both pivot at time τ_P. In that case, investors do not learn about the entrepreneur's type, and we have $q_t = q_0$. The relevant IC conditions are

$$\Pi_1^l(q_0) - F \geq \widehat{\Pi}_t^l(0) \tag{17}$$

for the low type and

$$\Pi_1^h(q_0) - F \geq \widehat{\Pi}_t^h(0) \tag{18}$$

for the high type. That is, both types prefer pivoting to not pivoting, given an off-path belief of zero and given that the belief stays at q_0 after the pivot.

The proposition below characterizes the optimal contract.

Proposition 4 *There exist two periods $\underline{\tau}^l \leq \bar{\tau}^l$ such that the optimal contract takes the following form whenever δ is sufficiently small and γ is sufficiently large.*

1. *If $F < \Pi_1^l(1) - \widehat{\Pi}_1^l(0)$, the optimal contract is pooling. Both types pivot at time $\tau_P \leq \underline{\tau}^l$.*
2. *If $\Pi_1^l(1) - \widehat{\Pi}_1^l(0) \leq F \leq \Pi_1^h(1) - \widehat{\Pi}_{\underline{\tau}^l}^h(q_0)$, then in the optimal contract the high type separates via a pivot at time $\tau_S < \underline{\tau}^l$.*
3. *If $F > \Pi_1^h(1) - \widehat{\Pi}_{\underline{\tau}^l}^h(q_0)$, then in the optimal contract the low type separates via a pivot at time $\underline{\tau}^l \leq \tau_S \leq \bar{\tau}^l$.*

Whenever γ is sufficiently small, the optimal contract is pooling. In any of these cases, both types offer the equity share α_t^P in Equation (12) before a pivot occurs.

Thus, when the cost of pivoting F is intermediate, then a pivot is good news about the entrepreneur's type. Investors who observe a pivot conclude that they are facing the high type. When the cost of pivoting is large, however, a pivot is bad news, i.e. only the low type pivots. Finally, when the cost of pivoting is too low, the optimal contract is pooling and pivots are not informative.

Let us briefly comment on the parameter restrictions. The parameter condition δ is small helps us ensure that the high type's value from separating earlier vs. separating later is monotone. The assumption that γ is large ensures a sufficiently large Pareto weight on the high type, who generally prefers to separate rather than pool with the low type. A sufficient condition for $F \leq \Pi_1^h(1) - \widehat{\Pi}_{\underline{\mathcal{I}}^l}^h(q_0)$ is that q_0 is sufficiently small, and that F is sufficiently close to $\Pi_1^l(1) - \widehat{\Pi}_1^l(0)$. Intuitively, assuming that q_0 and F are small ensures that the high type prefers separating rather than pooling or waiting for the low type to separate. Then, the high type separates via a pivot in the optimal contract. When F is relatively large, the high type can separate earlier, but instead prefers to wait until the low type pivots. Finally, for γ small, the Pareto weight on the low type is high, and the low type prefers to pool with the high type rather than reveal his type, or have the high type separate.

In equilibrium, pivots are generally delayed. For example, suppose that the high type separates. Early on, pivoting is not incentive compatible for the high type, because the value $\widehat{\Pi}_t^h(0)$ is relatively large. Then, the high type prefers to continue her current project instead of pivoting, even if this leads investors to believe she is the low type. As time passes, $\widehat{\Pi}_t^h(0)$, the value from continuing the current project, decreases and the high type eventually prefers to pivot. As long as t is not too large, the low type's IC condition holds as well, because her value from pivoting is lower than the high type's.[38]

Interestingly, pivoting in our model may occur earlier than with symmetric information. Intuitively, the high type benefits from separating from the low type, which implies that she prefers to pivot earlier than with symmetric information.

Corollary 5 *When the high type separates via a pivot, the high type pivots weakly earlier than in the symmetric information case whenever γ is sufficiently large.*

5.4 Prestige Projects

Many early stage firms divert resources towards prestige projects in order to generate publicity or goodwill.[39] As we show next, such prestige projects can act as signaling devices.

The entrepreneur can now implement a publicly observable prestige project in each period. Doing so reduces the payoff of the original project and generates a higher outside option for the entrepreneur. Specifically, when implementing the prestige project, a breakthrough yields value $V - V_0$ and the outside option is $\pi > 0$. For example, generating prestige makes

[38]We can see this from the IC conditions (IC_l^{Piv}) and (IC_h^{Piv}), which satisfy a single crossing type condition.

[39]For example, DoorDash, a food delivery platform, has started donating drivers' time to deliver surplus food from restaurants to nonprofits (see https://thespoon.tech/with-project-dash-doordash-uses-logistics-to-rescue-over-1-million-pounds-of-surplus-food/, last accessed 10/13/19). SpaceX, an aerospace manufacturer, has a stated, if lofty, goal to colonize Mars with one million inhabitants (see https://www.businessinsider.com/elon-musk-mars-iac-2017-transcript-slides-2017-10, last accessed 10/13/19).

it more likely that the entrepreneur can fund another startup[40] or obtain outside employment, but to do so the entrepreneur has to divert resources from her main project. The entrepreneur decides whether to implement the prestige project at the same time she offers the contract to the investor.[41] For the sake of clarity, we assume that the prestige project is the only signaling device and that the entrepreneur cannot pivot.

The high type can use the prestige project to separate, but separation occurs only after some time. Here is the intuition. Early on, the low type is relatively optimistic about the likelihood of success. Her value from continuing to experiment is therefore relatively large. If the high type implements a prestige project, the low type will simply imitate and continue experimenting despite the higher outside option. However, after enough time has passed, the low type is pessimistic about the likelihood of success and her value from continuing is small. Then, she liquidates immediately upon implementing the prestige project. The prestige project can now be used to signal. Specifically, the high type always has a higher value of continuing, so that there exists a region of time in which the low type liquidates when implementing the prestige project while the high type does not.

Since the prestige project has a lower value, separating is costly and can be suboptimal. Intuitively, if both V_0 and π are very large, both types prefer to liquidate instead of continuing with the prestige project. As we show in the proposition below, for certain parameter values, separating via a prestige project is indeed optimal.[42]

Proposition 6 *Suppose that γ is sufficiently large. Then, there exists a pair (π, V_0) such that the optimal contract features pooling in all periods $t < \tau_S$, and in period τ_S, the high type separates by implementing a prestige project and the low type liquidates.*

Formally, the following IC constraints hold when the high type separates at time t. First, the low type prefers not to implement the prestige project, i.e.,

$$\Pi_t^l(1) - \lambda p_t^l V_0 \leq \Pi_t^l(0). \qquad (IC_l^{Pres})$$

The LHS is the low type's value from imitating the high type, which consists of her continuation value $\Pi_t^l(1)$ and the loss in value if the project succeeds, while the RHS is the low

[40]This is broadly consistent with Gompers et al. (2010), who find that an entrepreneur's who had successful projects in the past can more easily obtain future financing.

[41]The particular timing is irrelevant, as long as the decision to implement the prestige project occurs before the investor's continuation decision.

[42]Broadly, V_0 cannot be too large, because then the high type never prefers to separate, but it also cannot be too small, because otherwise the low type is never dissuaded from mimicking. The range of feasible V_0 is affected by the outside option π. It is larger whenever π is smaller.

type's value from separating. The high type's IC constraint is similarly given by

$$\Pi_t^h(1) - \lambda p_t^h V_0 \geq \Pi_t^h(0). \qquad (IC_h^{Pres})$$

Separating is feasible whenever

$$\frac{\Pi_t^h(1) - \Pi_t^h(0)}{p_t^h} \geq \lambda V_0 \geq \frac{\Pi_t^l(1) - \Pi_t^l(0)}{p_t^l}. \qquad (19)$$

A variant of the single-crossing condition in Equation (14) holds in this extension. Thus, the interval in Equation (19) is nonempty, and for each period, there exists a V_0 that separates types. When the low type liquidates upon implementing the prestige project, her IC constraint becomes

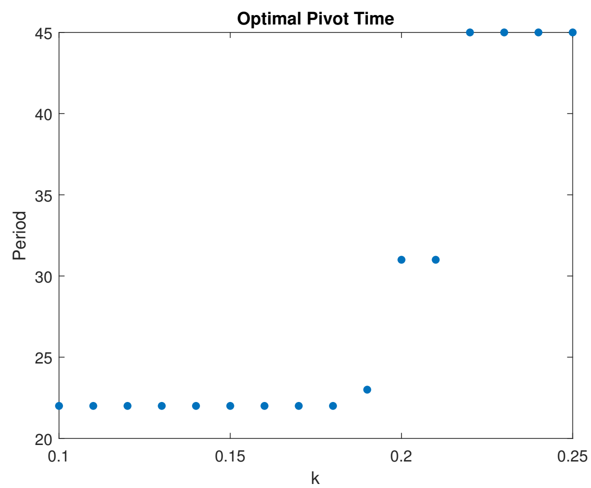
$$\Pi_t^l(1) - \lambda p_t^l V_0 \leq \pi.$$

Whenever the outside option π is sufficiently close to $\Pi_t^l(1)$, separating can be achieved relatively cheaply. That is, the loss in value V_0 which is necessary for the low type to separate is relatively small. This shows, intuitively, that there exists a pair (π, V_0) which makes separation optimal for the high type. As in the main model, the low type prefers to never separate. Thus, separating is ex-ante optimal whenever γ, the weight on the high type's value, is sufficiently large.

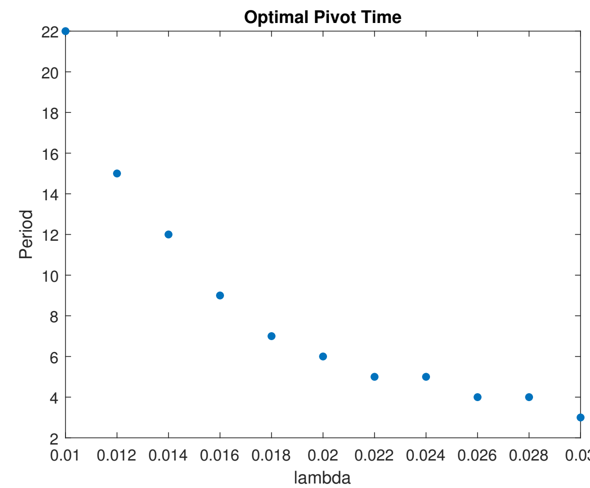
6 Implications and Empirical Predictions

We now discuss empirical implications of our model. We do this in the context of the equilibrium in which the high type separates (Proposition 4, Part 2). In particular, we predict that technological progress, which reduces the cost of experimentation or increases the speed of learning, and emerging patterns in the VC industry, which generate higher entry by inexperienced founders, can both explain the rise of pivots. We give detailed examples below.

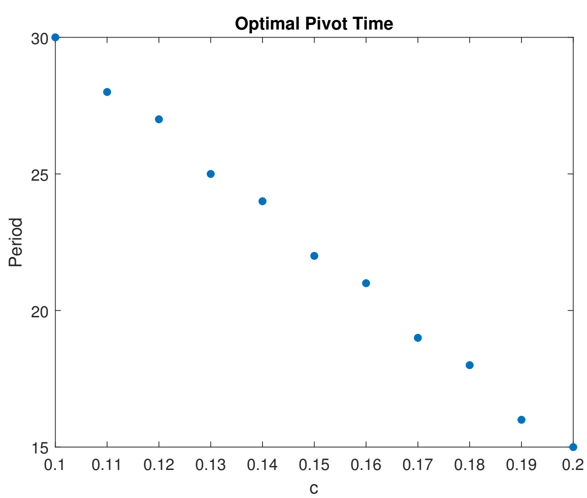
Timing of Pivots and Success Rates In our separating equilibrium, early pivots signal that the entrepreneur's ability is high. Thus, earlier pivots (by the high type) are more likely to succeed than later pivots (by the low type). This implies that in a given group of startups, the timing of pivoting is negatively correlated with future firm valuation and the probability of a successful exit, such as an IPO.



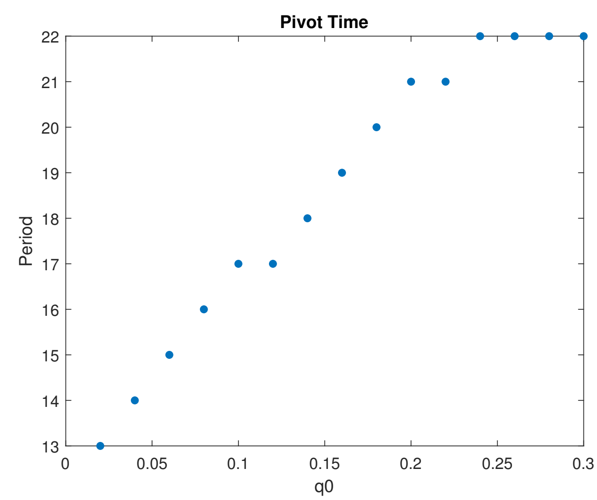
(a) As the cost of experimenting k increases, the high type pivots later.



(b) As the likelihood of breakthrough increases, the high type pivots earlier.



(c) As experimentation becomes costlier for the investor, i.e. c increases, the high type pivots earlier.



(d) As the expected quality of the entrepreneur q_0 increases, the high type pivots later.

Figure 3: Comparative Statics. Baseline parameters are $\lambda = 0.01$, $V = 100$, $c = k = 0.15$, $\gamma = 1$, $\delta = 0.7$, $q_0 = 0.5$, $p_1^l = 0.4$, $p_1^h = 0.6$, and $F = 0.1$.

Cost of Experimenting When the entrepreneur's cost of experimenting k decreases, the high type pivots earlier. Intuitively, as k decreases, the value of separating form the low type increases, since the high type expects to continue the new project for a longer period of time. We illustrate this result in Figure 3a, which is obtained by solving the model numerically. The empirical implication is that industries with low operational costs will feature more pivots by startups. Exogenous cross-sectional variation in operational costs can stem from exposures to IT infrastructure innovations, such as the development of cloud computing services discussed in Ewens et al. (2018), or government subsidies for R&D.[43]

Faster Learning As the likelihood of breakthroughs λ increases, learning about the entrepreneur's type speeds up (see Figure 3b). Then, the high type pivots earlier. Intuitively, a higher λ implies that in any period t, the belief in the pooling equilibrium $p_t(q_0)$ is lower and the pooling share α_t^P is higher. Then, the high type prefers to pivot earlier, since pooling is more costly. Recently, several papers have explored the effects of learning rates on innovation. Hegde et al. (2018) considers public disclosure of patents after passage of the American Investor's Protection Act, which increased the rate of follow-up inventions, while Budish et al. (2015) use cancer survival times as a measure of the speed of learning in drug development. Our model can relate such measures to the rate and timing of pivots.

Cost of VC Involvement As the VC's cost decreases, firms separate later (see Figure 3c). Intuitively, a decrease in c increases the high type's value from both pooling and separating. However, because of the adverse selection discount, the effect of c on the pooling share is amplified, so that as c decreases, the pooling share decreases more than the high type's full information share.[44] Thus, as c decreases, pooling becomes relatively more appealing for the high type and the high type pivots later. In the literature, the VC's monitoring costs have been proxied by physical distance or travel time (Bernstein et al. (2016)). Using this measure, our model predicts that startups far away from their lead VCs will pivot earlier.

Entrepreneur Quality As q_0, the ex-ante quality of the entrepreneur, decreases, pooling becomes more costly for the high type, since the pooling share α_t^P increases. Then, the high type prefers to pivot earlier (see Figure 3d). Thus, in industries with lower entry barriers, which attract less qualified entrepreneurs, we should expect earlier pivots. Alternatively,

[43] One example is in the biotech area, where the government provides tax benefits for companies developing drugs for orphan diseases. This subsidy has spurred the growth of "repurposing," i.e. firms use a drug originally developed for a different disease to treat orphan diseases (?). In our model, repurposing can be interpreted as a pivot.

[44] Formally, we have $\alpha_t^P = c/\lambda p_t(q_0)V$ and $\bar{\alpha}_t^h = c/\lambda p_t^h V$. As c decreases, the pooling share α_t^P decreases more, because $p_t(q_0) < p_t^h$.

decade-long quantitative easing has decreased the cost of capital for VCs, which can lead to VCs to overinvest and fund lower ability entrepreneurs.

Technological Progress and the Rise of Pivots Technological advances in recent decades have dramatically changed how VCs finance startups (see Kerr et al. (2014) and Ewens et al. (2018)). Three aspects are particularly relevant to our model. First, the cost of experimentation has declined for entrepreneurs. For example, cloud computing services have lowered the cost of operating IT startups by orders of magnitude. Second, cohort-based accelerators (such as Y-Combinator) have increased entry by relatively inexperienced founders, potentially driving down the average quality of entrepreneurs. Third, existing advances have made follow-up innovations easier, so that learning about startups has sped up. In our model, all these changes lead firms to pivot earlier, or, equivalently, to more pivots in the cross section. Thus, our model suggests that the rising popularity of pivots is a consequence of recent technological changes.

7 Extensions and Robustness

7.1 Entrepreneur's Type Changes after Pivot

We now allow the entrepreneur's type to change after the pivot, i.e. a good entrepreneur may become bad and vice versa. This captures the notion that the entrepreneur may be less well suited to the new project than to the old one. Our main results (i.e. Proposition 4) go through as long as the types are sufficiently persistent.

Suppose that after a pivot, the entrepreneur has probability $\eta > 1/2$ to remain the initial type and becomes the other type with probability $1-\eta$.[45] In a separating equilibrium where only the high type pivots, the investor's initial belief after the pivot is thus $q = \eta$. Following the analysis of Section 5.2, the optimal contract in the second stage is the optimal contract of Proposition 2. For the IC conditions to hold, the low type must prefer not to imitate

$$(1-\eta)\,\Pi_1^h(\eta) + \eta\Pi_1^l(\eta) - F \leq \widehat{\Pi}_t^l(0)\,, \tag{20}$$

where t is the time at which a pivot occurs. The entrepreneur becomes the high type with probability $1-\eta$ and stays as the low type with probability η. In both cases, the entrepreneur

[45]That is, the high type remains the high type with probability η and becomes the low type with probability $1-\eta$ and vice versa.

believes $q = \eta$ so the payoff is $\Pi_1^\theta(\eta)$. The high type's IC constraint is similarly given as

$$\eta\Pi_1^h(\eta) + (1-\eta)\Pi_1^l(\eta) - F \geq \widehat{\Pi}_t^h(0). \tag{21}$$

In this case, separating via a pivot is feasible whenever

$$F \in \left[(1-\eta)\Pi_1^h(\eta) + \eta\Pi_1^l(\eta) - \widehat{\Pi}_t^l(0), \eta\Pi_1^h(\eta) + (1-\eta)\Pi_1^l(\eta) - \widehat{\Pi}_t^h(0)\right]. \tag{22}$$

Whenever η is sufficiently large, the IC conditions become sufficiently similar to the IC conditions of our baseline model (i.e. Equations (IC_h^{Piv})) and (IC_l^{Piv})). Then, the optimal contract features signaling via pivots, just as in the baseline model.

Proposition 7 *Suppose that δ, F, q_0, and γ satisfy the conditions such that the optimal contract is the high type will pivot in period τ_S as in Proposition 4. There exists $\bar{\eta} > 1/2$ such that this contract is still optimal for all $\eta \geq \bar{\eta}$.*

7.2 Equilibrium Refinements

Multiple equilibria exist in this game. The analysis in Section 5 focuses on Pareto optimality. Instead, in this section we adopt the D1 criterion of Cho and Kreps (1987). It specifies that the belief of the VC upon observing an off-equilibrium contract should place no weight on type-θ entrepreneur if the other type θ' has a strict incentive to deviate whenever the first type has a strict or weak incentive to deviate. We first need to modify the definition of the D1 criterion as we are working in a dynamic setup different from Cho and Kreps (1987). This requires specifying the continuation game after deviating.

To be precise, consider the entrepreneur offers a deviated contract $C_t' = (I_t', \alpha_t')$ and the VC's off-equilibrium belief is q_t'. We assume that following the deviation at t, the entrepreneur will offer an optimal contract that maximizes her continuation value given q_t'. Precisely, if the belief is degenerate ($q_t' = 1$ or $q_t' = 0$), then the continuation game follows Lemma 1. Otherwise, the low type always offers the pooling contract in Proposition 2, and the high type will either offer a pooling contract or a separating contract as in Proposition 4, if pivots are still feasible. The choice of continuation contracts therefore is decided by q_t'. Define $M^\theta(C_t') \in [0,1]$ as the set of off-equilibrium beliefs that will make the θ type weakly better off after paying C_t'. If $M^{\theta'}(C_t') \subset M^\theta(C_t')$, then the D1 criterion assumes that the VC believes facing θ with certainty upon seeing C_t'. We further define a tie-breaking rule that if $M^h(C_t') = M^l(C_t')$ then $q_t' = q_t$. In this definition, we focus on optimal contracts in continuation. Doing so is sufficient for characterizing the largest set of beliefs that make deviations weakly profitable.[46]

Proposition 8 *The unique equilibrium satisfying the D1 criterion is the separating contract in Proposition 4 if the high type pivots. Otherwise, the pooling contract in Proposition 2 is the unique equilibrium.*

Consider the case when pivots are available. Multiple separating equilibria exist for all the periods such that Equation (16) hold. The pivoting time τ_S in Proposition 4 is the optimal stopping time for the high type, i.e. it is the first period when separation at t dominates doing so at $t+1$. This implies that for equilibria separating strictly later than τ_S (including the pooling equilibrium), the following condition holds at a period (i.e. τ_S) before the separation:

$$\Pi_1^l(1) - \widehat{\Pi}_{\tau_S}^l(q_{\tau_S}) < F < \Pi_1^h(1) - \widehat{\Pi}_{\tau_S}^h(q_{\tau_S}). \tag{23}$$

Equation (23) shows that relative to the equilibrium payoff, the incentive for the high type to separate is strictly larger than the pivoting cost, which is even larger than the incentive for the low type to imitate. Therefore, if the high type deviates by pivoting, $M^l\left(I_t' = 1\right) \subset M^h\left(I_t' = 1\right)$ holds and the off-equilibrium belief is $q_t' = 1$ under the D1 criterion. Deviation is profitable for the high type given this off-equilibrium belief. Similarly, separating strictly earlier than τ_S is dominated by delaying the separation since both types are strictly better-off. Therefore, the D1 criterion picks the optimal separating equilibrium at τ_S in Proposition 4.

If only equity payments are feasible, we need to normalize the deviation incentives by the probability of paying the deviating contract, which leads to a variant of Equation (23):

$$\frac{\Pi_t^l(1) - \Pi_t^l(q_t)}{p_t^l} < \frac{\Pi_t^h(1) - \Pi_t^h(q_t)}{p_t^h}. \tag{24}$$

This never holds for the optimal pooling contract since the marginal cost of separation increases to $p_t^h/p_t^l > 1$. Therefore, it survives the D1 criterion. For the same reason, the separating contract in Proposition 3 is pruned. At the separating period, the entrepreneur will deviate to a smaller equity contract. Since the high type has a strictly larger expected payment reduction, the off-equilibrium belief is $q_t' = 1$, which makes the deviation profitable for both types.

[46]If the entrepreneur is weakly better off with some suboptimal continuation contract, then she must be strictly better with an optimal contract given the same q_t'. However, if she is just indifferent with an optimal continuation contract given q_t', then she will become strictly worse off with suboptimal contacts.

Lastly, we discuss the necessity of using the D1 criterion instead of the intuitive criterion in Cho and Kreps (1987). The definition of the intuitive criterion in our setting is as follows. Suppose there exists a deviating contract C_t' such that (i) the low is strictly worse off even if she is regarded as the high type, i.e. $M^l(C_t') = \emptyset$, and (ii) the high type is strictly better off if the VC knows she is not the low type, i.e $\{1\} \subset M^h(C_t')$, then the VC believes that the deviating entrepreneur is the high type.[47] Following the intuitive criterion is equivalent to using the D1 criterion when pivots are feasible since Equation (23) leads to both (i) and (ii). The intuitive criterion is sufficient in this case because the action of pivoting is binary, and we only need to consider *whether* paying the fixed cost and deviating are profitable. However, when pivots are not available, deviations are achieved by offering a different equity share, and there exist a continuum of such off-equilibrium contracts. Consider a deviating equity contract that pays slightly larger than the equilibrium one. Given the small cost, the low type will become strictly better-off if the off-equilibrium is $q' = 1$, i.e. $M^l(C_t') \neq \emptyset$. This violates (i) and the off-equilibrium belief is not defined under the intuitive criterion. This is why we have to rely on the D1 criterion.

7.3 Renegotiation

We now study whether our main result (Prop. 4) can be obtained in a setting with long-term contracts and renegotiation. Our results go through if pivoting decisions are not contracted ex-ante, i.e. the long-term contract consists of a sequence of equity shares, but the entrepreneur chooses when to pivot optimally at each period. Otherwise, if the entrepreneur also contracts on pivoting decisions, no renegotiation-proof contract can induce separation. We start with the latter case.

Consider the following signaling game in long-term contracts. At time 1, the entrepreneur chooses a long-term contract C^θ, which is observable to the investor. Given these contracts, the investor updates his beliefs about the entrepreneur's type and then chooses whether to continue or abandon the project in each period. A long-term contract consists of a sequence of equity shares $\left\{\alpha_t^\theta\right\}_{t>0}$ and pivoting decisions $\left\{I_t^\theta\right\}_{t>0}$, i.e. $C^\theta := \left\{C_t^\theta\right\}_{t>0}$ where $C_t^\theta = \left\{\alpha_t^\theta, I_t^\theta\right\}$. As in the main model, we do not consider the liquidation decisions $\left\{l_t^\theta\right\}_{t>0}$ to be part of this contract, i.e. they are unobservable to the investor, unless liquidation actually occurs.[48] Given a long-term contract, the history for the investor consists of $h^t = \{C, 1\{t \leq \tau_B\}\} \in H^t$, i.e. the investor observes the long-term contract that is offered at

[47] The case when the VC believes facing the low type is interchangeably defined by switching the above types.

[48] If the contract includes liquidation decisions, i.e. $C^\theta = \left\{\alpha_t^\theta, I_t^\theta, l_t^\theta\right\}_{0\leq t\leq\tau}$, the result below survives, i.e. there is still no separating equilibrium in renegotiation-proof long-term contracts.

time 1 and then observes whether a breakthrough has occurred. Here, τ_B is the time of the breakthrough and $1\{.\}$ is the indicator function.

A PBE with long-term contracts consists of contracts C^h and C^l, liquidation decisions by the investor $\{e_t\}_{t>0}$, and beliefs $\left\{p_t^\theta, p_t(q_t), q_t(h^t)\right\}_{t>0, h^t\in H^t}$, such that given beliefs, the contracts and liquidation decisions are optimal and the beliefs follow Bayes' rule whenever possible. Note that in a separating PBE with long-term contracts, i.e. $C^h \neq C^l$, we must have $q_1 \in \{0,1\}$, i.e. the entrepreneur's type is revealed immediately.

We now define renegotiation-proof long-term contracts. First, define with $C^{\theta|t} := \left\{C_s^\theta\right\}_{t\leq s}$ the continuation contract at time t. Consider the following renegotiation protocol, which is adapted from Maskin and Tirole (1992). At each time $0 < t \leq \tau$, the entrepreneur offers the investor a continuation contract $\widehat{C}^{\theta|t}$. Based on this contract, the investor updates his belief q_t and then either accepts or rejects the contract. In case of rejection, the original continuation contract $C^{\theta|t}$ is played.[49] A long-term contract is renegotiation-proof, if given the investor's beliefs (which potentially depend on the continuation contract offered), there exists no continuation contract $\widehat{C}^{\theta|t}$ which yields a higher continuation value for either entrepreneur type θ at any time t. A PBE in renegotiation-proof long-term contracts consists of renegotiation-proof contract, liquidation decisions, and beliefs, such that the contracts and liquidation decisions are optimal given beliefs, and beliefs follow Bayes' rule whenever possible.

Proposition 9 *There exists no separating PBE in renegotiation-proof long-term contracts.*

This result shows that whenever the entrepreneur can commit to pivoting decisions ex-ante, a renegotiation-proof long-term contract cannot feature separation. Thus, it cannot generate our results on pivots as signaling devices. Intuitively, if a long-term contract is separating, then the investor learns immediately which type he is facing, so that $q_1 = 1$ whenever the high type's contract is offered. The only renegotiation-proof contract given $q_1 = 1$ is the high type's first-best contract. But then the low type prefers to offer the same contract.

Alternatively, we may consider long-term contracts in which the entrepreneur commits to an equity share, but does not commit to pivoting decisions. Define a history for the entrepreneur as $h^{\theta t} = \left\{\theta, I_1^\theta, ..., I_{t-1}^\theta\right\} \in H^{\theta^t}$, which includes past pivoting decisions, and a history for the investor as $h^t = \left\{C, I_1^\theta, ..., I_{t-1}^\theta\right\} \in H^t$. Suppose that at time 1, the entrepreneur offers a long-term contract $C = \{\alpha_t\}_{t>0}$ where $\alpha_t : H^{\theta t} \to [0,1]$, but chooses

[49] Alternatively, we may assume that in case of rejection the game ends and the entrepreneur and investor receiver zero. Doing so does not alter the results, since on the path of the optimal renegotiation-proof contract, the investor has zero continuation value in each period. Thus, he rejects a continuation contract if and only if his value form doing so is strictly negative.

whether to pivot at each time $t > 0$ and history $h^{\theta t}$. In particular, the investor does not observe the pivoting decision until the pivot actually occurs. This implies that at time t, the investor cannot alter his belief about the entrepreneur q_t based on a pivoting decision that will occur at some future time $s > t$.[50] A PBE now includes pivoting decisions which are sequentially rational at each history. With this definition of long-term contracts, the optimal contract in Prop. 4 is also the optimal renegotiation-proof long-term contract.

Proposition 10 *If the entrepreneur cannot commit to pivoting decisions in advance, then the optimal renegotiation-proof long-term contract is the contract in Prop. 4.*

Intuitively, when pivoting decisions are not part of the contract, both types can offer the same contract, which implements the pooling equity share before a pivot occurs, and the low and high type's first-best shares, respectively, after a pivot occurs. Then, the investor only learns which type he is facing when the pivot happens. The optimal contract of Prop. 4 is then sustained by imposing off-path beliefs $q_t = 0$ whenever a different contract is offered in renegotiation.

8 Discussion and Conclusion

Comparison to Static Setting Our main results rely on dynamic considerations and cannot be replicated in a static model. Specifically:

1. Vesting and dilution (Proposition 2) rely on having multiple periods in which the investor's belief changes.

2. In deciding whether to pivot (Proposition 4), the entrepreneur trades off continuing the current project versus starting a new one. In a static model, there is no notion of abandoning a project or of starting a new one.

3. The comparative statics in Section 6 also require a multi-period model, otherwise there is no notion of earlier or later liquidation.

4. Signaling via prestige projects (Proposition 6) entails a timing component, since the entrepreneur needs to wait until the beliefs become sufficiently pessimistic for signaling to be feasible.

We hope this illustrates why we consider a dynamic model, despite the apparent complexity.

[50]Crucially, this distinguishes this model variant from a setup in which the entrepreneur commits to the pivoting decision ex-ante.

Conclusion In this paper, we model startup financing a contracting problem with private information, in which both the entrepreneur and investor learn about the startup over time. We recover common features of VC contracts, such as dilution, vesting, and pivots, as equilibrium outcomes.

The dominant explanation for pivots is that entrepreneurs realize that their current project is unlikely to succeed and start a new project. Our model nests this explanation and provides a new one: pivots can act as signaling devices. This, to our knowledge, has not been recognized in the literature on venture capital.

Our comparative statics provide testable predictions on how pivot timing relates the cost of experimentation, the speed of learning, and entrepreneur quality. In reality, the rising popularity of pivots coincided with sweeping technological changes, which made running startups cheaper, sped up learning, and encouraged entry by inexperienced founders. As our model suggests, this timing is not a mere coincidence. In our framework, these technological changes indeed lead firms to pivot more.

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A Equilibrium Definition

Since the game ends as soon as $e_t = 0$, the only relevant history is the one in which the investor has chosen $e_s = 1$ for all $s < t$. We thus do not need to keep track of e_t. Let $\mathcal{C} = \{0,1\} \times [0,1]$ be the space of possible contracts in any given period, where each contract is given by a pair $C_t = (I_t, \alpha_t) \in \mathcal{C}$.

A history for entrepreneur type θ in period t is given by $h^{\theta t} = \{\theta, C_1, C_2, ..., C_{t-1}\} \in H^{\theta t}$. A history for the investor is given by $h^t = \{C_1, C_2, ..., C_{t-1}, C_t\} \in H^t$. A strategy for a type θ entrepreneur is pair $\sigma_t^\theta = \left(g_t^\theta\left(h^t, C\right), l_t^\theta\left(h^t\right)\right)$, where $g_t^\theta : H^{\theta t} \to \Delta(\mathcal{C})$ is the probability she offers contract C given history h^t and $l_t^\theta : H^{\theta t} \to [0,1]$ is the probability she liquidates the project. A strategy for the investor is $\sigma_t : H^t \to [0,1]$, which maps $\left(h^t, C_t\right)$ into a distribution over $e_t \in \{0,1\}$.

A Perfect Bayesian Equilibrium consists of strategies σ^θ and σ, and beliefs p_t^θ, $p_t(q_t)$, and q_t, such that for all $t \leq \tau$, σ_t^h, σ_t^l, and σ_t are sequentially rational at all histories and beliefs satisfy Bayes' rule whenever possible.

This definition is consistent with the following extensive form stage game in each period t:

- Stage 1: The entrepreneur chooses l_t^θ.
- Stage 2: If the entrepreneur has not liquidated, she chooses C_t conditional on $h^{\theta t}$.
- Stage 3: The investor observes C_t (and the fact that the game has not ended yet) and chooses e_t conditional on h^t (which includes C_t).
- Stage 4: The project succeeds or not. If not, the game proceeds to period $t+1$.

B Proofs

B.1 Proof of Lemma 1

Since $\Pi_1^\theta > F$ for $\theta = L, H$, either type prefers to pivot rather than liquidate. Thus, liquidation can only occur after the option to pivot has been exercised. To establish that liquidation is optimal whenever $\lambda p_t^\theta V - c - k \leq 0$, suppose by way of contradiction that the period payoff is strictly positive when the entrepreneur liquidates. Then $\Pi_t^\theta > 0$, because the entrepreneur always has the option of liquidating in the next period, so that $\Pi_{t+1}^\theta \geq 0$. Thus, liquidating in period t cannot be optimal. Conversely, if the period payoff is non-positive, it will be strictly negative in all future periods, because the belief p_t^θ is strictly decreasing. Thus, it must be the case that $\Pi_{t+1}^\theta < 0$. If the entrepreneur continues the project, she earns a negative value. Thus, liquidating is optimal.

Period τ^θ is defined as the first period in which $\lambda p_t^\theta V - c - k$ becomes negative, or, equivalently, the first period for which

$$p_t^\theta \leq \frac{c+k}{\lambda V}.$$

Since $p_t^l < p_t^h$ for all t, we have $\tau^l < \tau^h$.

That the optimal equity share is given $\bar{\alpha}_t^\theta$ has been established in the text. Finally, that type θ pivots at the first time when

$$\Pi_1^\theta - F \geq \widehat{\Pi}_t^\theta$$

is an immediate consequence of optimality.

B.2 Proof of Proposition 2

We start with some preliminaries. First, no pooling equilibrium with inefficient liquidation can be optimal.

Lemma 11 *There exists no optimal pooling equilibrium in which* $l_t^l = l_t^h = 1$ *, but* $\Pi_t^h(1) > 0$.

Proof. To characterize alternative equilibria with higher payoffs, we must consider a number of cases. Let $\Pi_t^l(q_{t-1})$ be the low type's value in period t under the strategies $l_t^{l\prime} = 0$, $l_t^{h\prime} = 0$, $\alpha_t' = \alpha_t^P(q_{t-1})$, $l_{t+1}^{l\prime} = l_t^l$, etc. Note that we have $\Pi_t^h(1) \geq \Pi_t^l(1) \geq \Pi_t^l(q_{t-1})$. Thus, if $\Pi_t^l(q_{t-1}) \geq 0$, both types simply continue and offer contract $\alpha_t' = \alpha_{t-1}^P$. Then, the belief in the alternative equilibrium is $q' = q_{t-1}$, which ensures that the investor's IC condition holds. The payoffs are then larger: $\Pi_t^{h\prime} > \Pi_t^{l\prime} = \Pi_t^l(q_{t-1}) \geq 0$.[51]

Suppose instead that $\Pi_t^l(q_{t-1}) < 0 \leq \Pi_t^l(1) \leq \Pi_t^h(1)$. Then, by continuity, there exists a belief q' such that $\Pi_t^l(q') = 0$. Consider the following alternative equilibrium: $l_t^{l\prime}, l_t^{h\prime} \in (0,1)$ such that

$$1 - l_t^{l\prime} = \left(1 - l_t^{h\prime}\right) q_{t-1} \left(\frac{1-q'}{q'}\right)$$

and

$$\alpha_t' = \frac{c}{\lambda p_t(q') V}.$$

The liquidation probabilities induce the belief q'. This yields the same payoff as in equilibrium for the low type, but a strictly larger payoff for the high type. Finally, suppose that $\Pi_t^h(1) > 0 = \Pi_t^l(1)$. Then, picking $l_t^{h\prime} = 0$ and $l_t^{l\prime} = 1$ is a Pareto improvement. ■

In equilibrium, the high type receives a larger payoff.

Lemma 12 *In any pooling equilibrium, we have* $\Pi_t^h \geq \Pi_t^l$*. If* $l_t^l < 1$*, then* $\Pi_t^h > \Pi_t^l$.

Proof. Since choosing l_t^l is (weakly) suboptimal for the high type, his value satisfies

$$\Pi_t^h \geq \sum_{s=t}^{\infty} \delta^{s-t} \left[\prod_{t \leq u \leq s-1} \left(1 - \lambda p_u^h\right)\left(1 - l_u^l\right)\right] \left(1 - l_s^l\right) \left(\lambda p_s^h \left(1 - \alpha_s^P\right) V - k\right). \tag{25}$$

Using the updating rule in Equation (1) repeatedly yields

$$\prod_{t \leq u \leq s-1} \left(1 - \lambda p_u^h\right) \lambda p_s^h = \lambda p_t^h (1-\lambda)^{s-t},$$

so that the RHS of Equation (25) equals

$$p_t^h \sum_{s=t}^{\infty} (\delta(1-\lambda))^{s-t} \left[\prod_{t \leq u \leq s} \left(1 - l_u^l\right)\right] \lambda \left(1 - \alpha_s^P\right) V$$
$$- \sum_{s=t}^{\infty} \delta^{s-t} \left[\prod_{t \leq u \leq s-1} \left(1 - \lambda p_u^h\right)\left(1 - l_u^l\right)\right] \left(1 - l_s^l\right) k.$$

[51] Recall that in any pooling equilibrium $\Pi_t^h > \Pi_t^l$, by Lemma 12.

We similarly obtain for the low type

$$\begin{aligned}\Pi_t^l &= p_t^l \sum_{s=t}^{\infty} (\delta (1-\lambda))^{s-t} \left[\prod_{t \le u \le s} \left(1 - l_u^l\right) \right] \lambda \left(1 - \alpha_s^P\right) V \\ &\quad - \sum_{s=t}^{\infty} \delta^{s-t} \left[\prod_{t \le u \le s-1} \left(1 - \lambda p_u^l\right) \left(1 - l_u^l\right) \right] \left(1 - l_s^l\right) k.\end{aligned}$$

Since $p_t^h > p_t^l$ for all t, combining the two expressions yields $\Pi_t^h \ge \Pi_t^l$ and the inequality is strict if $l_t^l < 1$. The lemma implies that if the low type does not liquidate, the high type will not liquidate either. We will exploit this fact throughout. ■

Since the high type receives a higher payoff, she liquidates later.

Corollary 13 *Whenever $l_t^l = 0$, we have $l_t^h = 0$. Whenever $l_t^h > 0$, we have $l_t^l = 1$. There exists no equilibrium in which $l_t^l, l_t^h \in (0,1)$.*

Proof. Liquidating with probability $l_t^l \in (0,1)$ is optimal for the low type if and only if

$$\Pi_t^l = 0,$$

where Π_t^l is the equilibrium value of the low type. Similarly, liquidating with probability $l_t^h \in (0,1)$ is optimal for the high type if and only if

$$\Pi_t^h = 0.$$

Lemma 12 then implies the results. ■

Moreover, the constraint $\Pi_t^\theta \ge \Pi_t^\theta(0)$ does not bind in equilibrium whenever both types continue.

Lemma 14 *For all $t < \bar{\tau}^l$, we have $\Pi_t^\theta > \Pi_t^\theta(0)$.*

Proof. We have, using a similar calculation as in Lemma 12,

$$\Pi_t^h - \Pi_t^h(0) = \lambda p_t^h V \sum_{s=t}^{\bar{\tau}^l - 1} (\delta (1-\lambda))^{s-t} \left(\bar{\alpha}_t^l - \alpha_s^P\right) + \delta^{\bar{\tau}^l - t} \Pi_{t \le u < \bar{\tau}^l - 1} \left(1 - \lambda p_u^h\right) \left(\Pi_{\bar{\tau}^l}^h(1) - \Pi_{\bar{\tau}^l}^h(0)\right)$$

and[52]

$$\Pi_t^l - \Pi_t^l(0) \ge \lambda p_t^l V \sum_{s=t}^{\bar{\tau}^l} (\delta (1-\lambda))^{s-t} \Pi_{t \le u < s-1} \left(1 - l_u^l\right) \left(\bar{\alpha}_s^l - \alpha_s^P\right).$$

For all $t < \bar{\tau}^l$, we have $q_0 \le q_t$ and therefore $\alpha_t^P < \bar{\alpha}_t^l$. We also have $l_t^l < 1$ and $\Pi_t^h(1) \ge \Pi_t^h(0)$ for all t. Thus, both expressions are strictly positive. ■

We are now done with preliminaries and ready to prove the proposition. We first show that the contract in Proposition 2 is optimal among all pooling contracts.

[52] Note that by construction, $\Pi_{\bar{\tau}^l}^l = \Pi_{\bar{\tau}^l}^l(0) = 0$.

Proposition 15 *Any other pooling equilibrium yields weakly lower payoffs for both types than the equilibrium in Proposition 2.*

Specifically, the next series of Lemmas establishes that $U_t = 0$, that

$$\alpha_t^P = \frac{c}{\lambda p_t(q_t) V}$$

for all $t < \bar{\tau}^l$, and that no equilibrium with different payoffs can be optimal.

Lemma 16 *Suppose that $1 = l_t^l \geq l_t^h$. Then, $U_t = 0$ and if $l_t^h < 1$, the high type offers the optimal contract $\bar{\alpha}_s^h$ for all $s \geq t$.*

Proof. That $U_t = 0$ follows directly from Equation (26). Since $l_t^l = 1$, we have $q_t = 1$. Then, any equilibrium must have the high type offer $\bar{\alpha}_s^h$ for $s \geq t$. Overpaying the investor, i.e. $\alpha_s' > \bar{\alpha}_s^h$, is not optimal for the high type and liquidating for $t < s < \tau^h$ cannot be optimal either. ■

Thus, if $l_t^l = 1$, the contract is uniquely pinned down for $s \geq t$. We can therefore restrict attention to periods in which $l_t^l < 1$ and, by Corollary 13, $l_t^h = 0$. We keep this restriction throughout the remainder of the section.

Lemma 17 *Any optimal pooling contract features $U_t = 0$ and*

$$\alpha_t^P = \frac{c - \delta(1 - \lambda p_t(q_t)) U_{t+1}}{\lambda p_t(q_t) V}$$

whenever $\Pi_t^l > 0$.

Proof. The investor experiments whenever

$$U_t = (1 - l_t(q_{t-1})) \left(\lambda p_t(q_t) \alpha_t^P V - c + \delta(1 - \lambda p_t(q_t)) U_{t+1} \right) \geq 0. \tag{26}$$

Suppose that $U_t > 0$, $\Pi_t^l > 0$, and $l_t^l = l_t^h = 0$ for some t.[53] We can generate an improvement for the entrepreneur by picking equity share $\alpha_t' = \alpha_t^P - \varepsilon$, where, ε is chosen sufficiently small to ensure that the investor's value remains positive. This clearly increases both types' payoffs. ■

We next show that for any pooling equilibrium in which $\Pi_t^l = 0$ and $U_t > 0$, there exists another equilibrium in which $U_t = 0$ and which at least weakly increases the payoffs to both types in period t. We distinguish two cases, when $\Pi_{t+1}^l > 0$ and when $\Pi_{t+1}^l = 0$.

Lemma 18 *Suppose that $\Pi_t^l = \Pi_{t+1}^l = 0$. Then, any pooling equilibrium in which*

$$\alpha_t^P > \frac{c}{\lambda p_t(q_{t-1}) V}$$

is not optimal.

[53] Recall that $l_t^l = 0$ implies $l_t^h = 0$ by Corollary 13.

Proof. Consider the following alternative equilibrium

$$\begin{aligned}
\alpha_t' &= \frac{c}{\lambda p_t\left(q_{t-1}\right)V} \\
l_t^{l\prime} &= l_t^{h\prime} = 0 \\
q_t' &= q_{t-1} \\
\alpha_{t+1}^{'} &= \alpha_t^P \\
l_{t+1}^{\theta\prime} &= l_t^{\theta} \text{ for } \theta = l, h \\
q_{t+1}' &= q_t \\
&\ldots
\end{aligned}$$

We now verify that the alternative contract is indeed an equilibrium and improves the entrepreneur's payoffs. First, in any equilibrium, we have $U_{t+1} \geq 0$. Thus, the investor experiments whenever his share exceeds $c/\left(\lambda p_t\left(q_t'\right)V\right)$. In particular, he experiments at α_t' given belief $q_t' = q_{t-1}$. Second, since $\Pi_{t+1}^{l\prime} = \Pi_t^l = 0$ and $\alpha_t' < \alpha_t^P$, it must be the case that $\Pi_t^{l\prime} > \Pi_t^l = 0$ and therefore $l_t^{l\prime} = 0$ is optimal. A similar argument holds for type h, which implies that $\Pi_t^{h\prime} > \Pi_t^h \geq 0$ and that $l_t^{h\prime} = 0$ is optimal.[54] Third, we have $l_t^{l\prime} = l_t^{h\prime} = 0$ and Bayesian updating implies that $q_t' = q_{t-1}$. Finally, since the continuation game in period $t+1$ in the alternative equilibrium is the same as the continuation game in period t under the original equilibrium, all conditions are satisfied from period $t+1$ onward. Thus, we have constructed an equilibrium which improves the entrepreneur's payoffs. ■

Lemma 19 *If* $\Pi_t^l = \Pi_{t+1}^l = 0$ *and*

$$\alpha_t^P \leq \frac{c}{\lambda p_t\left(q_{t-1}\right)V},$$

then there exists another pooling equilibrium which yields the same payoffs to both types in period t and which satisfies

$$\alpha_t^P = \frac{c}{\lambda p_t\left(q_t\right)V}.$$

Proof. Consider the following alternative equilibrium. We pick

$$\begin{aligned}
\alpha_t' &= \alpha_t^P \\
q_t' &: \alpha_t^P = \frac{c}{\lambda p_t\left(q_t'\right)V} \\
l_t^{h\prime} &= l_t^h \\
1 - l_t^{l\prime} &= \left(1 - l_t^h\right) q_{t-1}\left(\frac{1 - q_t'}{q_t'}\right) \\
\alpha_{t+1}' &= \alpha_{t+1}^P \\
q_{t+1}' &= q_{t+1} \\
l_{t+1}^{h\prime} &= l_{t+1}^h \\
1 - l_{t+1}^{l\prime} &= \left(1 - l_{t+1}^h\right) q_t'\left(\frac{1 - q_{t+1}'}{q_{t+1}'}\right).
\end{aligned}$$

That is, we keep the equity share the same, i.e. $\alpha_t^{'} = \alpha_t^P$. However, we change the likelihood of

[54] Of course, it is possible that $l_t^{h\prime} = l_t^h = 0$, which happens whenever $\Pi_t^h > 0$.

termination $l_t^{l\prime}$ so that the belief $q_t^{\prime}$ satisfies

$$\alpha_t^P = \frac{c}{\lambda p_t\left(q_t^{\prime}\right) V}$$

under Bayes' rule.[55] Note that this implies $q_t^{\prime} \geq q_{t-1}$. In period $t+1$, we keep the equity share and beliefs the same as in the original equilibrium, but we again adjust type l's likelihood of liquidation so that $q_{t+1}^{\prime} = q_{t+1}$.[56] From period $t+2$ onward, the strategies and beliefs in the alternative equilibrium are the same as in the original one.

Let us confirm that the alternative equilibrium exists. First, since $q_t^{\prime} \geq q_{t-1}$, the investor's IC condition in Equation (26) holds in period t given equity share $\alpha_t^{\prime}$ and belief $q_t^{\prime}$. Similarly, his IC condition in period $t+1$ holds because $q_{t+1}^{\prime} = q_{t+1}$ and $\alpha_{t+1}^{\prime} = \alpha_{t+1}^P$. Second, we have $\Pi_{t+1}^{l\prime} = \Pi_{t+1}^l = 0$, which holds because $\alpha_{t+1}^{\prime} = \alpha_{t+1}^P$ and because the continuation strategies after time $t+1$ are the same as in the original equilibrium. We also have $\Pi_t^{l\prime} = \Pi_t^l = 0$, because $\alpha_t^{\prime} = \alpha_t^P$ and $\Pi_{t+1}^{l\prime} = \Pi_{t+1}^l$.[57] Since the low type is indifferent in period t, we can freely pick $l_t^{l\prime}$ to ensure that the investor's belief is indeed $q_t^{\prime}$. Similarly, we can pick $l_{t+1}^{l\prime}$ such that $q_{t+1}^{\prime} = q_{t+1}$. ∎

Now, we consider the case when $\Pi_t^l = 0$ and $\Pi_{t+1}^l > 0$. We will show that either (i) this case is equivalent to the previous one, where $\Pi_t^l = \Pi_{t+1}^l = 0$, or (ii) we can pick an alternative equilibrium in which $U_t = 0$.

Lemma 20 *Suppose that $\Pi_t^l = 0$ and $\Pi_{t+1}^l > 0$. Then, there exists another pooling equilibrium which yields the same payoffs to both types and in which either $\Pi_{t+1}^l = 0$ or $U_t = 0$.*

Proof. Consider the following alternative equilibrium

$$\begin{aligned}
\alpha_t^{\prime} &= \alpha_t^P - \varepsilon \\
\alpha_{t+1}^{\prime} &= \alpha_{t+1}^P + \varepsilon / \left(\delta\left(1-\lambda\right)\right) \\
q_t^{\prime} &= q_t \\
q_{t+1}^{\prime} &= q_{t+1} \\
l_s^{\theta\prime} &= l_s^{\theta} \text{ for } \theta = l, h \text{ and } s = t, t+1.
\end{aligned}$$

Let us confirm that this is indeed an equilibrium. Type l's payoff in the alternative equilibrium is

$$\begin{aligned}
\Pi_t^l\left(q_t, \alpha_t^{\prime}\right) &= \left(1 - l_t^{l\prime}\right)\left(\lambda p_t^l\left(1 - \alpha_t^p + \varepsilon\right) V - k + \delta\left(1 - \lambda p_t^l\right) \Pi_{t+1}^l\left(q_{t+1}, \alpha_{t+1}^{\prime}\right)\right) \\
&= \left(1 - l_t^{l\prime}\right)\left(\lambda p_t^l\left(1 - \alpha_t^p + \varepsilon\right) V - k + \delta\left(1 - \lambda p_t^l\right) \Pi_{t+1}^l\left(q_{t+1}, \alpha_{t+1}^P\right)\right. \\
&\quad \left. - \delta\left(1 - \lambda p_t^l\right) \lambda p_{t+1}^l \frac{\varepsilon V}{\delta\left(1-\lambda\right)}\right).
\end{aligned}$$

[55] This is always possible, since we are considering the case when $l_t^h < 1$.

[56] That is, the investor's beliefs in the alternative and original equilibrium coincide.

[57] A similar argument for the high type yields $\Pi_t^{h\prime} = \Pi_t^h$ and $\Pi_{t+1}^{h\prime} = \Pi_{t+1}^h$. Thus, both types' payoffs are unchanged.

Using the updating rule in Equation (1) yields

$$\begin{aligned}\Pi_t^l\left(q_t, \alpha_t'\right) &= \left(1-l_t^l\right)\left(\lambda p_t^l\left(1-\alpha_t^p+\varepsilon\right)V - k + \delta\left(1-\lambda p_t^l\right)\Pi_{t+1}^l\left(q_{t+1}, \alpha_{t+1}^P\right)\right. \\ &\quad \left. -\delta\left(1-\lambda\right)\lambda p_t^l \frac{\varepsilon V}{\delta\left(1-\lambda\right)}\right) \\ &= \Pi_t^l.\end{aligned}$$

Thus, type l receives the same payoff as in equilibrium. A similar calculation for type h yields

$$\Pi_t^h\left(q_t, \alpha_t'\right) = \Pi_t^h.$$

Thus, $l_t^{\theta\prime} = l_t^\theta$ for $\theta = l, h$ is optimal. Since the liquidation probabilities are the same, the beliefs are the same as well, i.e. $q_t' = q_t$. Notice that $\Pi_{t+1}^{\theta\prime}$ is decreasing in ε. Thus, if we pick ε sufficiently large, we have $\Pi_{t+1}^{l\prime} = 0$ and we can then pick $l_{t+1}^{l\prime}$ to ensure that $q_{t+1}' = q_{t+1}$.

Finally, it remains to check whether the investor's incentive compatibility constraint holds in period t in the alternative equilibrium. If this is not true, i.e. for the ε for which $\Pi_{t+1}^l = 0$, we have $U_t < 0$, then, since U_t is continuous in ε, there exists another ε' such that $U_t = 0$. We have thus established the result in the statement of the lemma. ■

So far, we have shown that for each t, any optimal pooling contract must feature either $U_t = 0$ or $\alpha_t^p = \frac{c}{\lambda p_t(q_t)V}$. The following Lemma shows concludes this part of our argument by showing that $U_t = 0$ for all t.

Lemma 21 *Suppose that either $U_t = 0$ or $\alpha_t^p = c/\left(\lambda p_t\left(q_t\right)V\right)$ for all t. Then, for all t, we have $U_t = 0$ and*

$$\alpha_t^p = \frac{c}{\lambda p_t\left(q_t\right)V}.$$

Proof. Suppose $U_t > 0$. Then, we have

$$U_t = \delta\left(1-\lambda p_t\left(q_t\right)\right)U_{t+1}$$

and thus $U_{t+1} > 0$. Proceeding inductively, we must have $U_s > 0$ for all $s \geq t$. But this is impossible. Under the contract α_t^P, the low type will eventually liquidate with probability one, which leaves the investor with a continuation value of zero, since either the game ends or the high type offers his optimal symmetric information contract. Therefore, we must have $U_t = 0$ for all t. But this immediately implies that

$$\alpha_t^p = \frac{c}{\lambda p_t\left(q_t\right)V}$$

for all t. ■

We have now shown that there is no pooling equilibrium which yields a strictly higher payoff to any type than the equilibrium of Proposition 2. We next show that the equilibrium of Proposition 2 exists. A necessary condition is that given

$$\alpha_t^P = \frac{c}{\lambda p_t\left(q_t\right)V},$$

the following conditions are satisfied. For all t and $\theta \in \{l, h\}$,

$$\begin{aligned} \Pi_t^\theta &\geq \Pi_t^\theta(0) \\ \Pi_t^l &> 0 \Rightarrow l_t^h = l_t^l = 0 \\ l_t^\theta &> 0 \Rightarrow \Pi_t^\theta = 0 \end{aligned} \tag{27}$$

and q_t satisfies Equation (13).[58] The first equation says that deviating to any other contract (in which case we can set the off-path belief to zero) makes each type worse off than staying in equilibrium. The two following equations ensure that the liquidation decisions are optimal for both types. Note that we do not have to consider the investor's incentives, since in this equilibrium, we have $U_t = 0$ for all t.

Lemma 22 *Let $\underline{\tau}^l$ be the first period for which*

$$\lambda p_t^l \left(1 - \frac{c}{\lambda p_t(q_0) V}\right) V - k \leq 0$$

and let $\bar{\tau}^l$ be the first period for which

$$\lambda p_t^l \left(1 - \bar{\alpha}_t^h\right) V - k \leq 0.$$

We have $1 < \underline{\tau}^l \leq \bar{\tau}^l$. Consider the following strategies and beliefs. For any $t < \underline{\tau}^l$, we have $q_t = q_0$ and $l_t^l = l_t^h = 0$. If $\underline{\tau}^l < \bar{\tau}^l$, then for any $\underline{\tau}^l \leq t < \bar{\tau}^l$, we have $l_t^h = 0$, l_t^l satisfies

$$q_t = \frac{q_{t-1}}{q_{t-1} + (1 - q_{t-1})\left(1 - l_t^l\right)},$$

and q_t satisfies

$$\lambda p_t^l \left(1 - \frac{c}{\lambda p_t(q_t) V}\right) V = k. \tag{28}$$

Finally, we have $l_{\bar{\tau}^l}^l = 1$. Then, the belief q_t is strictly increasing for all $\underline{\tau}^l \leq t < \bar{\tau}^l$ and the conditions in Equation (27) are satisfied.

Intuitively, we adjust liquidation probabilities so that the low type remains indifferent between continuing and liquidating, given that q_t is updated using Bayes rule. As we show, this is possible and satisfies all relevant incentive constraints.

Proof. Condition (8) implies that

$$\lambda p_1^l \left(1 - \alpha_1^P\right) V - k > 0.$$

Thus, $l_1^l = l_1^h = 0$, $q_1 = q_0$, and $\Pi_1^l > 0$. For any t such that $l_s^l = 0$ for all $s \leq t$, we have $q_t = q_0$

[58] Recall that we are restricting attention to times at which $l_t^l < 1$, since by Lemma 16, the contract is pinned down if $l_t^l = 1$. Thus, Bayes' rule applies.

and the low type's period payoff is

$$\lambda p_t^l \left(1 - \frac{c}{\lambda p_t (q_0) V}\right) V - k.$$

This expression crosses zero exactly once from above, since $p_t (q_0)$ is strictly decreasing in t and vanishes as t becomes large. Thus, $\underline{\tau}^l$ exists and we have $\underline{\tau}^{\,l} > 1$. We have $\Pi_t^l > 0$ for $t < \underline{\tau}^l$ and thus $l_t^l = l_t^h = 0$ is optimal for any such t.

A similar argument implies that $\bar{\tau}^l$ exists. That $\underline{\tau}^l \leq \bar{\tau}^l$ is straightforward, because $\alpha_t^P \geq \alpha_t^h$ for all t.

In period $\underline{\tau}^l$, the low type liquidates with strictly positive probability. Here is the argument. If she continues with certainty, then the equilibrium must feature a belief $q_{t+1} = q_0$ and a contract $\alpha_{t+1}^P = c/ (\lambda p_{t+1} (q_0) V)$, etc. But then, the low type's period payoff for any period $s > t$ in which she continues is strictly negative, so that $\Pi_{t+1}^l \leq 0$. This, in turn implies that

$$\Pi_t^l = \left(1 - l_t^l\right) \left(\lambda p_t^l \left(1 - \alpha_t^P\right) V - k + \delta \left(1 - \lambda p_t^l\right) \Pi_{t+1}^l\right) < 0,$$

so that the low type's decision to continue in period t must be suboptimal. Thus, no such equilibrium can exist, and we have $\Pi_{\underline{\tau}^l}^l = 0$.

Now, consider the case when $\bar{\tau}^l > \underline{\tau}^l$[59] and recall that by construction of $\bar{\tau}^l$, we have

$$\lambda p_{\bar{\tau}^l}^l \left(1 - \alpha_{\bar{\tau}^l}^h\right) V \leq k. \tag{29}$$

Then, for any $\underline{\tau}^l \leq t < \bar{\tau}^l$, there exists a unique belief $q_t \in (q_0, 1)$ such that

$$\lambda p_t^l \left(1 - \frac{c}{\lambda p_t (q_t) V}\right) V = k. \tag{30}$$

Using the updating rule in Equation (13), we can find a unique l_t^l which induces belief q_t given q_{t-1}. We can now inductively construct a sequence $\left\{l_t^l, q_t\right\}$ such that Equation (30) holds in each period $\underline{\tau}^l \leq t < \bar{\tau}^l$. In period $\bar{\tau}^l$, we pick $l_{\bar{\tau}^l}^l = 1$. If $\underline{\tau}^l = \bar{\tau}^l$, we also pick $l_{\bar{\tau}^l}^l = 1$.

In periods $\underline{\tau}^l \leq t < \bar{\tau}^l$, we have $\Pi_t^h > 0$ and therefore it is optimal for the high type to continue. For the low type, the indifference condition $\Pi_t^l = 0$ must hold. This is true. In period $\bar{\tau}^l$, the low type receives zero value, i.e. $\Pi_{\bar{\tau}^l}^l = 0$, and in any period $\underline{\tau}^l \leq t < \bar{\tau}^l$, Equation (30) implies that her period payoff is zero. Backwards induction then implies that $\Pi_t^l = 0$.

In period $\bar{\tau}^l$, it is optimal for the low type to liquidate with certainty. We must distinguish two cases. Suppose that $l_{\bar{\tau}^l}^h < 1$. Then, Equation (13) implies that $q_{\bar{\tau}^l} = 1$. If the low type continues instead, she receives a payoff of

$$\lambda p_{\bar{\tau}^l}^l \left(1 - \alpha_{\bar{\tau}^l}^h\right) V - k + \delta \left(1 - \lambda p_{\bar{\tau}^l}^l\right) \Pi_{\bar{\tau}^l+1}^l.$$

Equation (29) implies that the deviation payoff is negative. Thus, the low type indeed prefers to liquidate. Now, suppose that $l_{\bar{\tau}^l}^h = 1$, so that the game ends with certainty in period $\bar{\tau}^l$. In that case, Equation (29) guarantees that the low type's deviation payoff is negative for any off-path

[59] This implies that $\lambda p_t^l \left(1 - \bar{\alpha}_t^h\right) V > k$ at $t = \underline{\tau}^l$.

belief. ■

We have now established that the proposed strategies satisfy the necessary conditions in Equation (27).[60] It remains to show that the pooling equilibrium yields at least a weakly higher payoff to both types than any separating equilibrium. To prove the result, we must first characterize separating equilibria. Therefore, we defer this proof. It can be found in Corollary 28 below.

We conclude by showing that q_t is strictly increasing for $\underline{\tau}^l \leq t < \bar{\tau}^l$. This follows, because $p_t(q)$ is strictly decreasing in t for any fixed q and strictly increasing in q for any fixed t. Thus, to satisfy the indifference condition in Equation (30) in consecutive periods, q_t must be strictly increasing.

Finally, the equilibrium in Lemma 22 is unique, provided we fix the equity share offered. To establish this, we need the following auxiliary result.

Lemma 23 *For $\underline{\tau}^l \leq t < \bar{\tau}^l$, we have*

$$\alpha_t^P = \frac{\lambda p_t^l V - k}{\lambda p_t^l V}$$

and $\Pi_t^l = 0$. For $t < \underline{\tau}^l$, we have

$$\alpha_t^P < \frac{\lambda p_t^l V - k}{\lambda p_t^l V}.$$

Proof. For $t < \underline{\tau}^l$, the low type continues with certainty and $q_t = q_0$, so that

$$\alpha_t^P = \frac{c}{\lambda p_t(q_0) V}.$$

The inequality

$$\frac{c}{\lambda p_t(q_0) V} \geq \frac{\lambda p_t^l V - k}{\lambda p_t^l V},$$

is equivalent to

$$c \geq \lambda p_t(q_0) V - k \left(q_0 \frac{p_t^h}{p_t^l} + (1 - q_0) \right). \tag{31}$$

Since $p_t(q_0)$ is decreasing in t and p_t^h / p_t^l is increasing, the RHS is decreasing. We will exploit this fact throughout the proof.

First, we show that there exists no $\underline{\tau}^l \leq t < \bar{\tau}^l$ for which $\Pi_t^l > 0$. Assume towards a contradiction that there exists such a period and let $\hat{t}$ be the largest one. This implies that $\Pi_{\hat{t}+1}^l = 0$, $l_{\hat{t}}^l = 0$, $q_{\hat{t}} = q_{\hat{t}-1}$, and

$$\Pi_{\hat{t}}^l = \lambda p_{\hat{t}}^l \left(1 - \frac{c}{\lambda p_{\hat{t}}(q_{\hat{t}-1}) V} \right) V - k > 0.$$

By construction of $\underline{\tau}^l$, we have

$$\frac{c}{\lambda p_{\underline{\tau}^l}^l (q_0) V} \geq \frac{\lambda p_{\underline{\tau}^l}^l V - k}{\lambda p_{\underline{\tau}^l}^l V}.$$

[60] Recall that $\alpha_t^P = \frac{c}{\lambda p_t(q_t) V}$ guarantees that the investor is always willing to experiment. Thus, we do not need to consider the investor's incentive compatibility constraints. Similarly, we do not need to consider deviations in the contract offered, i.e. $\alpha_t' \neq \alpha_t^P$, since the off-path belief $q' = 0$ renders such deviations unprofitable for either type.

If $\Pi^l_t > 0$ for all $\underline{\tau}^l \le t < \hat{t}$, then we have $q_{\hat{t}} = q_0$. But since the RHS in Equation (31) is decreasing in time, this implies that

$$\frac{c}{\lambda p_{\hat{t}}\left(q_0\right) V} > \frac{\lambda p_{\hat{t}} V - k}{\lambda p_{\hat{t}} V} \tag{32}$$

and therefore $\Pi^l_{\hat{t}} < 0$, a contradiction.

Otherwise, there exists a $\underline{\tau}^l \le \tilde{t} < \hat{t}$ such that $\Pi^l_{\tilde{t}} = 0$ and $\Pi^l_t > 0$ for all $\tilde{t} < t \le \hat{t}$. Then, we have $l^l_t = 0$ for any such t and $q_{\hat{t}} = q_{\tilde{t}}$. Moreover,

$$\Pi^l_{\tilde{t}} = \lambda p^l_{\tilde{t}}\left(1 - \alpha^P_{\tilde{t}}\right) V - k + \delta\left(1 - \lambda p^l_{\tilde{t}}\right)\Pi^l_{\tilde{t}+1} = 0$$

implies that

$$\frac{c}{\lambda p^l_{\tilde{t}}\left(q_{\tilde{t}}\right) V} \ge \frac{\lambda p^l_{\tilde{t}} V - k}{\lambda p^l_{\tilde{t}} V},$$

since $\Pi^l_{\tilde{t}+1} \ge 0$. Because the belief does not change between $\tilde{t}$ and $\hat{t}$, a variant of Equation (31) implies that Inequality (32) holds again, and we get a contradiction.

That

$$\alpha^P_t = \frac{\lambda p^l_t V - k}{\lambda p^l_t V}$$

for $\underline{\tau}^l \le t < \bar{\tau}^l$ follows because $\Pi^l_t = \Pi^l_{t+1} = 0$ for any such t, so that

$$\lambda p^l_t\left(1 - \alpha^P_t\right) V - k + \delta\left(1 - \lambda p^l_t\right) \cdot 0 = 0.$$

To show the second part of the lemma, note simply that by construction, $\underline{\tau}^l$ is the first period in which Inequality (31) holds. ■

Corollary 24 *Suppose that*

$$\alpha^P_t = \frac{c}{\lambda p_t\left(q_t\right) V}.$$

Then, the equilibrium of Lemma 22 is unique, i.e. there does not exist another pooling equilibrium with the same equity share but different liquidation probabilities.

Proof. Any equilibrium which features liquidation before period $\underline{\tau}^l$ violates the entrepreneur's incentive constraints. By Lemma 23, we have for any $\underline{\tau}^l \le t < \bar{\tau}^l$,

$$\frac{c}{\lambda p_t\left(q_t\right)} = \frac{\lambda p^l_t V - k}{\lambda p^l_t V}.$$

Thus, the sequence of beliefs $\{q_t\}$ is unique and so is the sequence of liquidation probabilities $\left\{l^l_t\right\}$. Any other choice of liquidation probabilities will either violate the low type's incentive constraint for some t or violate Bayesian updating in Equation (13). Finally, there is no equilibrium in which the low type continues past $\bar{\tau}^l$, because even if $q_t = 1$, her value from continuing is negative. ■

B.3 Proof of Proposition 3

We now construct separating equilibria and show that any separating equilibrium is suboptimal. To show existence, we must ensure that the low type does not mimic the high type and vice versa. For this, we need to consider the continuation payoff of the high type when $q = 0$, i.e. the investor believes he is facing the low type, and the low type's continuation payoff when $q = 1$. If $q = 0$, the high type's continuation contract is $\bar{\alpha}^l_{t+1}$. Any lower share leads to the investor abandoning the project while any higher share is suboptimal.[61] We denote the high type's continuation value in that case as $\Pi^h_{t+1}(0) = \Pi^h_{t+1}\left(0, \bar{\alpha}^l_{t+1}\right)$. Similarly, if the low type succeeds in mimicking the high type, he optimally offers $\bar{\alpha}^h_{t+1}$ and receives a value of $\Pi^l_{t+1}(1) = \Pi^l_{t+1}\left(1, \bar{\alpha}^h_{t+1}\right)$. When the investor's beliefs are degenerate, the project is liquidated at a deterministic time. We denote with $\tau^{\theta\prime}$ the liquidation times after a deviation. That is $\tau^{l\prime}$ is the low type's liquidation time if $q' = 1$ and $\tau^{h\prime}$ is the high type's liquidation time when $q' = 0$. We have $\tau^{l\prime} \geq \tau^l$ and $\tau^{h\prime} \leq \tau^h$.

If $t < \tau^l$, combining the two incentive constraints yields the necessary condition

$$\alpha^h_t \in \left[\bar{\alpha}^l_t + \delta(1-\lambda)\frac{\Pi^l_{t+1}(1) - \Pi^l_{t+1}(0)}{\lambda p^l_{t+1} V}, \bar{\alpha}^l_t + \delta(1-\lambda)\frac{\Pi^h_{t+1}(1) - \Pi^h_{t+1}(0)}{\lambda p^h_{t+1} V}\right], \tag{33}$$

while if $\tau^l \leq t < \tau^{l\prime}$, we have

$$\alpha^h_t \in \left[\frac{\lambda p^l_t V - k}{\lambda p^l_t V} + \delta(1-\lambda)\frac{\Pi^l_{t+1}(1) - \Pi^l_{t+1}(0)}{\lambda p^l_{t+1} V}, \bar{\alpha}^l_t + \delta(1-\lambda)\frac{\Pi^h_{t+1}(1) - \Pi^h_{t+1}(0)}{\lambda p^h_{t+1} V}\right], \tag{34}$$

because the low type liquidates if his type is revealed. If $t = \tau^{l\prime}$, the low type liquidates even if she successfully imitates the high type and therefore the high type simply offers the symmetric information contract, i.e. $\alpha^h_t = \bar{\alpha}^h_t$.

Finally, there is no equilibrium in which the high type separates in period $t > \tau^{l\prime}$. Any such equilibrium requires pooling in period $\tau^{l\prime}$. But even if the belief under pooling were $q_{\tau^{l\prime}} = 1$, the low type would liquidate with certainty. Thus, period $t > \tau^{l\prime}$ cannot be reached.

The intervals in Equation (33) and (34) are nonempty, because the high and low type's values satisfy a variant of single crossing. We prove this in Lemma 25 below.[62]

Lemma 25 *We have*

$$\frac{\Pi^h_t(1) - \Pi^h_t(0)}{p^h_t} \geq \frac{\Pi^l_t(1) - \Pi^l_t(0)}{p^l_t}$$

for all $t < \tau^h$.

Proof. The low type's gain from imitating the high type vs. revealing her type in a given period

[61] If $q = 0$, then the investor experiments whenever $\lambda p^l_t(\alpha_t V_t - c) + \delta\left(1 - \lambda p^l_t\right) U_{t+1} \geq 0$. The optimal contract for type h induces $U_t = 0$ for all t, just as in the symmetric information benchmark.

[62] Specifically, in Equation (34), we have $\lambda p^l_t\left(1 - \bar{\alpha}^l_t\right) V \leq k$, which implies that

$$\frac{\lambda p^l_t V - k}{\lambda p^l_t V} \leq \bar{\alpha}^l_t.$$

Together with Lemma 25, this ensures that the interval in Equation (34) is nonempty.

is

$$\Delta_t^l = \begin{cases} \lambda p_t^l(\bar{\alpha}_t^l - \bar{\alpha}_t^h)V & \text{if } t < \tau^l, \\ \lambda p_t^l\left(1 - \bar{\alpha}_t^h\right)V - k & \text{if } \tau^l \le t < \tau^{l\prime}, \\ 0 & \text{if } \tau^{l'} \le t. \end{cases} \tag{35}$$

If the project succeeds before τ_l, the low type pays the investor $\bar{\alpha}_t^h V$ instead of $\bar{\alpha}_t^l V$. If the project succeeds after τ_l, she receives an additional continuation value since she would have liquidated the project otherwise.

Similarly, the gain for the high type from being indeed perceived as the high type is

$$\Delta_t^h = \begin{cases} \lambda p_t^h(\bar{\alpha}_t^l - \bar{\alpha}_t^h)V & \text{if } t < \tau^{h\prime}, \\ \lambda p_t^h\left(1 - \bar{\alpha}_t^h\right)V - k & \text{if } \tau^{h\prime} \le t < \tau^h, \\ 0 & \text{if } \tau^h \le t. \end{cases} \tag{36}$$

Thus, we have

$$\Pi_t^l(1) - \Pi_t^l(0) = E_t^l\left[\sum_{s=t}^{\tau^{l\prime}-1} \delta^{s-t}\Delta_s^l\right]$$

and

$$\Pi_t^h(1) - \Pi_t^h(0) = E_t^h\left[\sum_{s=t}^{\tau^h-1} \delta^{s-t}\Delta_s^h\right].$$

To prove the result, we distinguish two cases. Suppose first that $\tau^l \le \tau^{l\prime} \le \tau^{h\prime} \le \tau^h$. Then, we have

$$\begin{aligned} \Pi_t^h(1) - \Pi_t^h(0) &\ge E_t^h\left[\sum_{s=t}^{\tau^{h\prime}-1} \delta^{s-t}\Delta_s^h\right] \\ &= \lambda p_t^h \sum_{s=t}^{\tau^{h\prime}-1} (\delta(1-\lambda))^{s-t}(\bar{\alpha}_s^l - \bar{\alpha}_s^h)V, \end{aligned}$$

and

$$\begin{aligned} \Pi_t^l(1) - \Pi_t^l(0) &\le E_t^l\left[\sum_{s=t}^{\tau^{l\prime}-1} \delta^{s-t}\lambda p_s^l\left(\bar{\alpha}_s^l - \bar{\alpha}_s^h\right)V\right] \\ &= \lambda p_t^l \sum_{s=t}^{\tau^{l\prime}-1} (\delta(1-\lambda))^{s-t}\left(\bar{\alpha}_s^l - \bar{\alpha}_s^h\right)V. \end{aligned}$$

Since $\tau_h' \ge \tau_l'$ and $\bar{\alpha}_t^l > \bar{\alpha}_t^h$ for all t, the above expressions imply

$$\frac{\Pi_t^h(1) - \Pi_t^l(0)}{p_t^h} \ge \frac{\Pi_t^l(1) - \Pi_t^l(0)}{p_t^l},$$

which is what we set out to prove.

Suppose now that $\tau^l \leq \tau^{h\prime} < \tau^{l\prime} \leq \tau^h$. We have

$$\begin{aligned}\Pi_t^h(1) - \Pi_t^h(0) &\geq E_t^h\left[\sum_{s=t}^{\tau^{l\prime}-1} \delta^{s-t}\Delta_s^h\right] \\ &= E_t^h\left[\sum_{s=t}^{\tau^{h\prime}-1} \delta^{s-t}\lambda p_s^h\left(\bar{\alpha}_s^l - \bar{\alpha}_s^h\right)V + \sum_{s=\tau^{h\prime}}^{\tau^{l\prime}-1} \delta^{s-t}\left(\lambda p_s^h\left(1-\bar{\alpha}_s^h\right)V - k\right)\right]\end{aligned}$$

and

$$\begin{aligned}\Pi_t^l(1) - \Pi_t^l(0) &= E_t^l\left[\sum_{s=t}^{\tau^{l\prime}-1} \delta^{s-t}\Delta_s^l\right] \\ &= E_t^l\left[\sum_{s=t}^{\tau^{l}-1} \delta^{s-t}\lambda p_s^h\left(\bar{\alpha}_s^l - \bar{\alpha}_s^h\right)V + \sum_{s=\tau^{l}}^{\tau^{l\prime}-1} \delta^{s-t}\left(\lambda p_s^h\left(1-\bar{\alpha}_s^h\right)V - k\right)\right].\end{aligned}$$

If $t \leq s < \tau^l$, both types continue. Then, we have

$$\frac{E_t^h\left[\Delta_s^h\right]}{p_t^h} = (1-\lambda)^{s-t}\lambda\left(\bar{\alpha}_s^l - \bar{\alpha}_s^h\right)V = \frac{E_t^l\left[\Delta_s^l\right]}{p_t^l}.$$

If $\tau^l \leq s < \tau^{h'}$, then the high type always continues, while the low type liquidates if her type is known. Therefore, we have

$$\lambda p_s^l\left(1-\bar{\alpha}_s^l\right)V \leq k$$

and

$$\lambda p_s^l\left(1-\bar{\alpha}_s^h\right)V \geq k,$$

which together imply

$$\lambda p_s^l\left(\bar{\alpha}_s^l - \bar{\alpha}_s^h\right)V \geq \lambda p_s^l\left(1-\bar{\alpha}_s^h\right)V - k.$$

This inequality, in turn, implies that

$$\begin{aligned}\frac{E_t^h\left[\Delta_s^h\right]}{p_t^h} &= \frac{1}{p_t^h}\left[\prod_{t\leq u<s-1}\left(1-\lambda p_u^h\right)\right]\lambda p_s^h\left(\bar{\alpha}_s^l - \bar{\alpha}_s^h\right)V \\ &= (1-\lambda)^{s-t}\lambda\left(\bar{\alpha}_s^l - \bar{\alpha}_s^h\right)V \\ &= \frac{1}{p_t^l}\left[\prod_{t\leq u<s-1}\left(1-\lambda p_u^l\right)\right]\lambda p_s^l\left(\bar{\alpha}_s^l - \bar{\alpha}_s^h\right)V \\ &\geq \frac{1}{p_t^l}E_t^l\left[\lambda p_s^l\left(1-\bar{\alpha}_s^h\right)V - k\right] \\ &= \frac{E_t^l\left[\Delta_s^l\right]}{p_t^l}.\end{aligned}$$

If $\tau^{h'} \leq s < \tau^{l'}$, then both types liquidate if $q_s = 0$ and continue if $q_s = 1$. We have

$$\begin{aligned}\frac{E_t^h\left[\Delta_s^h\right]}{p_t^h} &= \frac{1}{p_t^h}\left[\prod_{t\leq u<s-1}\left(1-\lambda p_u^h\right)\right]\left(\lambda p_s^h\left(1-\bar{\alpha}_s^h\right)V-k\right) \\ &= (1-\lambda)^{s-t}\lambda\left(1-\bar{\alpha}_s^h\right)V-\frac{1}{p_t^h}\left[\prod_{t\leq u<s-1}\left(1-\lambda p_u^h\right)\right]k.\end{aligned}$$

An analog expression holds for the low type. Since $p_t^h > p_t^l$ for all t, we have

$$\frac{1}{p_t^h}\prod_{t\leq u<s-1}\left(1-\lambda p_u^h\right) < \frac{1}{p_t^l}\prod_{t\leq u<s-1}\left(1-\lambda p_u^l\right)$$

and therefore

$$\frac{E_t^h\left[\Delta_s^h\right]}{p_t^h} \geq \frac{E_t^l\left[\Delta_s^l\right]}{p_t^l}.$$

Combining the three cases yields the result.

Finally, for $\tau^{l\prime} \leq t < \tau^h$, the result is obvious. The low type always liquidates and receives zero, while the high type continues if his type is known and receives a strictly positive payoff. ∎

The above Lemma establishes that for each $t < \tau^h$, there is an equilibrium in which the high type separates in period t. In the optimal separating equilibrium, the low type's IC constraint binds, i.e.,

$$\alpha_t^h = \bar{\alpha}_t^l + \delta\frac{1-\lambda}{\lambda V}\frac{\Pi_{t+1}^l(1)-\Pi_{t+1}^l(0)}{p_{t+1}^l} \tag{37}$$

if $t < \tau^l$ and[63]

$$\alpha_t^h = \frac{\lambda p_t^l V - k}{\lambda p_t^l V} + \delta\frac{1-\lambda}{\lambda V}\frac{\Pi_{t+1}^l(1)}{p_{t+1}^l} \tag{38}$$

if $\tau^l \leq t < \tau^h$.

We now show that in any separating equilibrium, both types would at least weakly prefer to offer the pooling contract. We split the argument into two cases, depending on whether the low type continues once her type is revealed. The low type is at least weakly better off in the pooling equilibrium compared to the separating equilibrium. Thus, we only need to show that the high type is better off.

Lemma 26 *Any equilibrium in which the high type separates in period* $t < \min\left\{\tau^l, \tau^h - 1\right\}$ *is suboptimal. The entrepreneur can strictly improve by pooling in period* t *and separating in period* $t+1$.

Proof. The high type's payoff from separating in period t is

$$\Pi_t^h = \lambda p_t^h\left(1-\alpha_t^h\right)V - k + \delta\left(1-\lambda p_t^h\right)\Pi_{t+1}^h(1),$$

where $\Pi_{t+1}^h(1)$ is the symmetric information continuation payoff, which we defined in Section 5.1,

[63]Note that $\Pi_{t+1}^l(0) = 0$ in this case.

while her payoff from pooling in period t and separating in period $t+1$ is

$$\Pi_t^{h\prime} = \lambda p_t^h \left(1-\alpha_t^P\right) V - k + \delta \left(1-\lambda p_t^h\right) \Pi_{t+1}^{h\prime},$$

where $\Pi_{t+1}^{h\prime}$ is her payoff from offering the separating contract in period $t+1$.[64]

Suppose first that $t < \tau^l - 1 \leq \tau^{l\prime}$. Then, the low type continues after her type is revealed, both in the initial separating contract and in the alternative one. Using Equation (37), we can write

$$\Pi_t^h = \lambda p_t^h \left(V - \frac{c}{\lambda p_t^l}\right) - k - \delta \left(1-\lambda\right) p_t^h \frac{\Pi_{t+1}^l\left(1\right) - \Pi_{t+1}^l\left(0\right)}{p_{t+1}^l} + \delta \left(1-\lambda p_t^h\right) \Pi_{t+1}^h\left(1\right)$$

and

$$\begin{aligned}\Pi_t^{h\prime} \quad = \quad & \lambda p_t^h \left(V - \frac{c}{\lambda p_t\left(q_t\right)}\right) - k \\ & + \delta \left(1-\lambda p_t^h\right) \left(\lambda p_{t+1}^h \left(V - \frac{c}{\lambda p_{t+1}^l}\right) - k \right. \\ & \left. - \delta\left(1-\lambda\right) p_{t+1}^h \frac{\Pi_{t+2}^l\left(1\right) - \Pi_{t+2}^l\left(0\right)}{p_{t+2}^l} + \delta \left(1-\lambda p_{t+1}^h\right) \Pi_{t+2}^h\left(1\right)\right).\end{aligned}$$

We can now plug in the expression

$$\frac{\Pi_{t+1}^l\left(1\right) - \Pi_{t+1}^l\left(0\right)}{p_{t+1}^l} = \frac{1}{p_{t+1}^l} \left(\lambda p_{t+1}^l \left(\frac{c}{\lambda p_{t+1}^l} - \frac{c}{\lambda p_{t+1}^h}\right) + \delta \left(1-\lambda p_{t+1}^l\right) \left(\Pi_{t+2}^l(1) - \Pi_{t+2}^l(0)\right)\right),$$

plug in the symmetric information value

$$\Pi_{t+1}^h\left(1\right) = \lambda p_{t+1}^h V - c - k + \delta \left(1 - \lambda p_{t+2}^h \Pi_{t+2}^h\left(1\right)\right),$$

and use the Bayesian updating rule in Equation (1) . This yields, after some algebra,

$$\Pi_t^{h\prime} - \Pi_t^h = \lambda p_t^h \left(\frac{c}{\lambda p_t\left(q_t\right)} - \frac{c}{\lambda p_t^l}\right) > 0.$$

Thus, pooling in period t and separating in period $t+1$ yields a strictly larger payoff for the high type.

If $t = \tau^l - 1 < \tau^{l\prime}$, the low type liquidates in period $t+1$ if her type is revealed. This changes the high type's payoff from separating later. The separating contract in period $t+1$ is given by Equation (38) and we have

$$\frac{\Pi_{t+1}^l\left(1\right) - \Pi_{t+1}^l\left(0\right)}{p_{t+1}^l} = \frac{1}{p_{t+1}^l} \left(\lambda p_{t+1}^l \left(V - \frac{c}{\lambda p_{t+1}^h}\right) - k + \delta \left(1-\lambda p_{t+1}^l\right) \Pi_{t+2}^l(1)\right).$$

[64] For $t < \tau^{l\prime}$, the high type does not liquidate when offering the pooling contract. See Proposition 2.

A similar argument as in the previous case yields

$$\Pi_t^{h\prime} - \Pi_t^h = \lambda p_t^h \left(\frac{c}{\lambda p_t^l} - \frac{c}{\lambda p_t(q_t)} \right) > 0.$$

■

Note that the alternative equilibrium we construct is only meaningful if the high type does not liquidate in period $t+1$. This complication occurs when $t = \tau^h - 1$. Then, the pooling and separating contracts coincide, i.e.

$$\alpha_t^P = \alpha^h = \frac{\lambda p_t^l V - k}{\lambda p_t^l V}.$$

This is because we have $\tau^h \geq \tau^{l\prime}$, so the low type will liquidate with certainty in period $t+1$ under both the pooling and separating contracts. In this case, pooling and separating contracts yields the same payoffs to both types, and they both induce liquidation. The distinction in that case is thus purely notational.

Now, we consider the case when the low type liquidates if her type is known and the separating contract is given by Equation (38).

Lemma 27 *Any equilibrium in which the high type separates in period $\tau^l \leq t < \tau^{l\prime}$ is suboptimal. If instead the entrepreneur pools in period t and separates in period $t+1$, her payoff is at least weakly larger.*

Proof. Suppose first that $t < \tau^{l\prime} - 1$. Using Equation (38), the high type's payoff from separating is

$$\Pi_t^h = \lambda p_t^h \left(1 - \frac{\lambda p_t^l V - k}{\lambda p_t^l V} \right) V - k - \delta \left(1 - \lambda \right) p_t^h \frac{\Pi_{t+1}^l \left(1 \right)}{p_{t+1}^l} + \delta \left(1 - \lambda p_t^h \right) \Pi_{t+1}^h \left(1 \right),$$

and her payoff from pooling in period t and separating in period $t+1$ is

$$\begin{aligned} \Pi_t^{h\prime} &= \lambda p_t^h \left(1 - \alpha_t^P \right) V - k + \delta \left(1 - \lambda p_t^h \right) \Pi_{t+1}^{h\prime} \\ &= \lambda p_t^h \left(1 - \alpha_t^P \right) V - k \\ &\quad + \delta \left(1 - \lambda p_t^h \right) \left(k \left(\frac{p_{t+1}^h}{p_{t+1}^l} - 1 \right) \right. \\ &\quad \left. - \delta \left(1 - \lambda \right) p_{t+1}^h \frac{\Pi_{t+2}^l \left(1 \right)}{p_{t+2}^l} + \delta \left(1 - \lambda p_{t+1}^h \right) \Pi_{t+2}^h \left(1 \right) \right). \end{aligned}$$

Using

$$\Pi_{t+1}^l \left(1 \right) = \lambda p_{t+1}^l \left(V - \frac{c}{\lambda p_{t+1}^h} \right) - k + \delta \left(1 - \lambda p_{t+1}^l \right) \Pi_{t+2}^l \left(1 \right)$$

and substituting the high type's symmetric information value $\Pi^h_{t+1}(1)$, we can write

$$\begin{aligned}\Pi^h_t &= \lambda p^h_t \left(1-\frac{\lambda p^l_t V-k}{\lambda p^l_t V}\right)V-k \\ &\quad -\delta\left(1-\lambda\right)p^h_t\left(\lambda V-\frac{c}{p^h_{t+1}}-\frac{k}{p^l_{t+1}}+\delta\left(1-\lambda\right)\frac{\Pi^l_{t+2}(1)}{p^l_{t+2}}\right) \\ &\quad +\delta\left(1-\lambda p^h_t\right)\left(\lambda p^h_{t+1}V-c-k+\delta\left(1-\lambda p^h_{t+1}\right)\Pi^h_{t+2}(1)\right),\end{aligned}$$

which yields

$$\Pi^{h\prime}_t-\Pi^h_t=\lambda p^h_t\left(\frac{\lambda p^l_t V-k}{\lambda p^l_t V}-\alpha^P_t\right)V$$

after some algebra. Lemma 23 then implies that[65] $\Pi^{h\prime}_t \geq \Pi^h_t$.

Finally, if $t=\tau^{l\prime}-1$, the low type liquidates in period $t+1$ even after mimicking the high type. In that case, we have $\Pi^l_{t+1}(1)=0$. A similar calculation as above yields

$$\Pi^{h\prime}_t-\Pi^h_t=\lambda p^h_t\left(\frac{\lambda p^l_t V-k}{\lambda p^l_t V}-\alpha^P_t\right)V-\delta\left(1-\lambda\right)\frac{p^h_t}{p^l_{t+1}}\left(\lambda p^l_{t+1}\left(V-\frac{c}{\lambda p^h_{t+1}}\right)-k\right).$$

The first term is positive because of Lemma 23. The second term is positive because the low type prefers to liquidate even if $q_{t+1}=1$, which implies that

$$\lambda p^l_{t+1}\left(V-\frac{c}{\lambda p^h_{t+1}}\right)\leq k.$$

Thus, we again have $\Pi^{h\prime}_t-\Pi^h_t \geq 0$. ■

Corollary 28 *The pooling equilibrium yields an at least weakly higher payoff for both types than any separating equilibrium.*

Proof. We can apply the two previous Lemmas inductively. Separating in period t is Pareto dominated by separating in period $t+1$, which is Pareto dominated by separating in period $t+2$, etc. Thus, separating in period t is Pareto dominated by pooling in period $\tau^{l\prime}-1$. In period $\tau^{l\prime}$, the low type liquidates with certainty in the pooling equilibrium, so pooling and separating contracts are identical. ■

This concludes our proof of Proposition 3.

B.4 Proof of Proposition 4

We first characterize the optimal equilibrium when the low type separates via a pivot, i.e. the low type pivots earlier than the high type. In the baseline model, the low type liquidates probabilistically after enough time has passed. With pivots, the low type instead pivots probabilistically.

[65] The inequality binds for $t>\underline{\tau}^l$.

Lemma 29 *There exist two periods $\underline{\tau}^l \leq \bar{\tau}^l$, with the following property. If the high type does not separate via a pivot before time $\underline{\tau}^l$, then in any optimal equilibrium, the low type pivots with positive probability whenever $\underline{\tau}^l \leq t \leq \bar{\tau}^l$, and the high type pivots after the low type. Before a pivot occurs, the equity share is given by*

$$\alpha_t^P = \frac{c}{\lambda p_t\left(q_t\right) V}.$$

The period $\underline{\tau}^l$ is the first period in which

$$\widehat{\Pi}_t^l = \Pi_1^l\left(0\right) - F.$$

The proof is analogous to the proof of Proposition 2 and hence omitted. The only difference is that when the low type pivots with positive probability, her value satisfies

$$\widehat{\Pi}_t^l = \lambda p_t^l\left(1 - \alpha_t^P\right)V - k + \delta\left(1 - \lambda p_t^l\right)\left(\Pi_1^l\left(0\right) - F\right) = \Pi_1^l\left(0\right) - F$$

and the pooling share α_t^P is now given by

$$\alpha_t^P = \frac{\lambda p_t^l V - k}{\lambda p_t^l V} - \left(1 - \delta\left(1 - \lambda p_t^l\right)\right)\frac{\Pi_1^l\left(0\right) - F}{\lambda p_t^l V}$$

for $\underline{\tau}^l \leq t \leq \bar{\tau}^l$.

The above lemma shows that there exists an equilibrium in which the low type separates after period $\underline{\tau}^l$. The next lemma shows that there is no equilibrium in which the low type separates earlier.

Lemma 30 *There exists no equilibrium in which the low type separates via a pivot at time $t < \underline{\tau}^l$.*

Proof. Suppose that the investor expects the low type to pivot in period t and expects the high type not to pivot. Then, conditional on a pivot the belief is $q_t = 0$ and conditional on no pivot, the belief is $q_t = 1$. The low type prefers to pivot at time $t < \underline{\tau}^l$ if

$$\Pi_1^l(0) - \widehat{\Pi}_t^l(1) \geq F.$$

The LHS is increasing in t, since $\widehat{\Pi}_t^l(1)$ is decreasing in t. By construction, period $\underline{\tau}^l$ is the first period such that

$$\widehat{\Pi}_t^l(q_0) = \Pi_1^l\left(0\right) - F,$$

and for all $t < \underline{\tau}^l$, we have

$$\widehat{\Pi}_t^l(q_0) \geq \Pi_1^l\left(0\right) - F.$$

But then, for all $t < \underline{\tau}^l$

$$F \geq \Pi_1^l(0) - \widehat{\Pi}_t^l(q_0) > \Pi_1^l(0) - \widehat{\Pi}_t^l(1),$$

which implies that the low type's IC constraint cannot hold. ■

We now characterize equilibria in which the high type separates via a pivot, i.e. the high type pivots before the low type.

The following Lemma allows us to restrict attention to contracts which separate before period $\underline{\tau}^l$. Thus, if separation occurs in period t, the low type cannot be indifferent between pivoting and continuing.

Lemma 31 *Separating via pivots by the high type is not incentive compatible in any period $t \geq \underline{\tau}^l$.*

Proof. We have for $t \geq \underline{\tau}^l$, $\Pi_1^l(1) - F \geq \Pi_1^l(0) - F = \widehat{\Pi}_t^l$, which implies that separating in period $t \geq \tau^l$ is not incentive compatible because the low type prefers to imitate. ■

We next show that the set $\left[\Pi_1^l(1) - \widehat{\Pi}_t^l(0), \Pi_1^h(1) - \widehat{\Pi}_t^h(0)\right]$ in Equation (16) is nonempty. To start, we characterize the pivoting timing for both types when they are regarded as the low type (i.e $\widehat{\Pi}_t^\theta(0)$) first.

Lemma 32 *There exists a $\underline{\delta}$ such that for any $\delta < \underline{\delta}$, the high type pivots weakly earlier than the low type when investors believe that the entrepreneur is the low type.*

Proof. Consider the high type's pivoting decision first. Since the value of continuing the current project at a constant belief $q_t = 0$ decreases over time, the high type pivots in the first period when

$$\Pi_1^h(0) - F \geq \lambda p_t^h\left(1 - \bar{\alpha}_t^l\right)V - k + \delta\left(1 - \lambda p_t^h\right)\left(\Pi_1^h(0) - F\right).$$

The above equation implies that the high type finds it optimal to pivot in period t instead of delaying to period $t+1$. We can transform this equation into

$$\begin{aligned}
F &\leq \Pi_1^h(0) - \frac{\lambda p_t^h\left(1 - \bar{\alpha}_t^l\right)V - k}{1 - \delta\left(1 - \lambda p_t^h\right)} \\
&= \Pi_1^h(0) - \sum_{s=0}^{\infty}\left(\delta\left(1 - \lambda p_t^h\right)\right)^s\left(\lambda p_t^h\left(1 - \bar{\alpha}_t^l\right)V - k\right) \\
&= \sum_{s=1}^{\tau^{h\prime}}\left(\prod_{1 \leq u \leq s-1}\delta\left(1 - \lambda p_u^h\right)\right)\left(\lambda p_s^h\left(1 - \bar{\alpha}_s^l\right)V - k\right) - \sum_{s=0}^{\infty}\left(\delta\left(1 - \lambda p_t^h\right)\right)^s\left(\lambda p_t^h\left(1 - \bar{\alpha}_t^l\right)V - k\right) \\
&= \lambda p_1^h\left(1 - \bar{\alpha}_1^l\right)V - \lambda p_t^h\left(1 - \bar{\alpha}_t^l\right)V - \delta K^h,
\end{aligned}$$

where K^h represents higher order terms in the summation.[66] K^h can be negative or positive depending on the parameters. Similarly, it is optimal for the low type to pivot in the first period t in which

$$F \leq \lambda p_1^l\left(1 - \bar{\alpha}_1^l\right)V - \lambda p_t^l\left(1 - \bar{\alpha}_t^l\right)V - \delta K^l.$$

The high type pivots weakly earlier than the low type if

$$\begin{aligned}
& \lambda p_1^h\left(1 - \bar{\alpha}_1^l\right)V - \lambda p_t^h\left(1 - \bar{\alpha}_t^l\right)V - \delta K^h \geq \lambda p_1^l\left(1 - \bar{\alpha}_1^l\right)V - \lambda p_t^l\left(1 - \bar{\alpha}_t^l\right)V - \delta K^l \\
\Leftrightarrow\ & \lambda\left(p_1^h - p_1^l\right)\left(1 - \bar{\alpha}_1^l\right)V - \lambda\left(p_t^h - p_t^l\right)\left(1 - \bar{\alpha}_t^l\right)V \geq \delta\left(K^h - K^l\right).
\end{aligned}$$

[66] Here, recall that we use the convention $\prod_\emptyset = 1$.

In the above equation, we have $p_1^h - p_1^l > p_t^h - p_t^l$ by Assumption 3, and $\bar{\alpha}_1^l < \bar{\alpha}_t^l$ by the symmetric information contract (Lemma 1). Therefore, the left-hand side is positive. If $K^h < K^l$, then we can set $\underline{\delta} = 1$. Otherwise, we have

$$\underline{\delta} = \min \left\{ 1, \frac{\lambda \left(p_1^h - p_1^l\right) \left(1 - \bar{\alpha}_1^l\right) V - \lambda \left(p_t^h - p_t^l\right) \left(1 - \bar{\alpha}_t^l\right) V}{K^h - K^l} \right\}.$$

■

The above Lemma implies that the high type always pivots weakly earlier than the low type for sufficiently small δ, given the off-equilibrium belief of $q_t = 0$. We use this fact in the next Lemma.

Lemma 33 *For δ sufficiently small, we have for all t*

$$\Pi_1^l(1) - \widehat{\Pi}_t^l(0) \leq \Pi_1^h(1) - \widehat{\Pi}_t^h(0).$$

Proof. We prove that

$$\widehat{\Pi}_t^h(0) - \widehat{\Pi}_t^l(0) \leq \Pi_1^h(1) - \Pi_1^l(1), \tag{39}$$

which is equivalent to the expression in the Lemma statement. We have

$$\widehat{\Pi}_t^h(0) = \max \left\{ \Pi_1^h(0) - F, \lambda p_t^h \left(1 - \bar{\alpha}_t^l\right) V - k + \delta \left(1 - \lambda p_t^h\right) \widehat{\Pi}_{t+1}^h(0) \right\}$$

and

$$\widehat{\Pi}_t^l(0) = \max \left\{ \Pi_1^l(0) - F, \lambda p_t^l \left(1 - \bar{\alpha}_t^l\right) V - k + \delta \left(1 - \lambda p_t^l\right) \widehat{\Pi}_{t+1}^l(0) \right\}.$$

By Lemma 32, we need to consider the following three cases for a given period t and investor belief $q_t = 0$: (1) both types pivot, (2) the high type pivots and the low type continues the project, and (3) both types continue the project.

If both types pivot in period t, then

$$\widehat{\Pi}_t^h(0) - \widehat{\Pi}_t^l(0) = \Pi_1^h(0) - \Pi_1^l(0).$$

By the proof of Proposition 3, we know that $\Pi_1^h(0) - \Pi_1^l(0) < \Pi_1^h(1) - \Pi_1^l(1)$, which implies that Inequality (39) holds.

Suppose now that the high type pivots in period t and the low type continues the current project. Then, since pivoting is suboptimal for the low type, we have

$$\Pi_1^l(0) - F < \widehat{\Pi}_t^l(0),$$

which implies that

$$\begin{aligned} \widehat{\Pi}_t^h(0) - \widehat{\Pi}_t^l(0) &< \widehat{\Pi}_t^h(0) - \left(\Pi_1^l(0) - F\right) \\ &= \left(\Pi_1^h(0) - F\right) - \left(\Pi_1^l(0) - F\right) \\ &= \Pi_1^h(0) - \Pi_1^l(0). \end{aligned}$$

Again, the proof of Proposition 3 implies that the last term is strictly smaller than $\Pi_1^h(1) - \Pi_1^l(1)$, which implies that Inequality (39) holds.

Finally, suppose that both types continue the project in period t. Then,

$$\widehat{\Pi}_t^h(0) - \widehat{\Pi}_t^l(0) = \lambda\left(p_t^h - p_t^l\right)\left(1 - \bar{\alpha}_t^l\right)V + \delta\left(\left(1 - \lambda p_t^h\right)\widehat{\Pi}_{t+1}^h(0) - \left(1 - \lambda p_t^l\right)\widehat{\Pi}_{t+1}^l(0)\right).$$

We use the same method as in Lemma 32. We denote

$$\widehat{K} = \left(1 - \lambda p_t^h\right)\widehat{\Pi}_{t+1}^h(0) - \left(1 - \lambda p_t^l\right)\widehat{\Pi}_{t+1}^l(0)$$

and write

$$\Pi_1^h(1) - \Pi_1^l(1) = \lambda\left(p_1^h - p_1^l\right)\left(1 - \bar{\alpha}_1^h\right)V + \delta K.$$

If $K > \widehat{K}$, then $\widehat{\Pi}_t^h(0) - \widehat{\Pi}_t^l(0) < \Pi_1^h(1) - \Pi_1^l(1)$ for all $\delta \leq 1$. Otherwise, the Inequality (39) holds if

$$\delta \leq \min\left\{1, \frac{\lambda\left(p_1^h - p_1^l\right)\left(1 - \bar{\alpha}_1^h\right)V - \lambda\left(p_t^h - p_t^l\right)\left(1 - \bar{\alpha}_t^l\right)V}{\widehat{K} - K}\right\}.$$

This concludes the proof. ■

Lemma 31 implies that separation by the high type is not incentive compatible after period $\underline{\tau}^l$ so we can focus on $t < \underline{\tau}^l$. If $F \leq \Pi_1^h(1) - \widehat{\Pi}_{\underline{\tau}^l}^h(0)$, given that $\widehat{\Pi}_t^h(0)$ strictly decreases with t, there exists $\underline{\tau}_{Piv} \geq 1$ such that IC constraint (IC_h^{Piv}) holds only if $t \geq \underline{\tau}_{Piv}$. For $t < \underline{\tau}_{Piv}$, separation by the high type is not feasible, because it is too costly for the high type. Similarly, as t increases beyond $\bar{\tau}_{Piv}$, we eventually have $F < \Pi_1^l(1) - \widehat{\Pi}_t^l(0)$, so that separation is not incentive-compatible, because the low type would pivot together with the high type. The following Corollary summarizes when separation through pivoting is feasible.

Corollary 34 *Suppose that $F \geq \Pi_1^l(1) - \widehat{\Pi}_1^l(0)$ and that $F \leq \Pi_1^h(1) - \widehat{\Pi}_{\underline{\tau}^l}^h(0)$. Then, there exist two periods $1 \leq \underline{\tau}_{Piv} \leq \bar{\tau}_{Piv}$, such that for $\underline{\tau}_{Piv} \leq t \leq \bar{\tau}_{Piv}$*

$$F \in \left[\Pi_1^l(1) - \widehat{\Pi}_t^l(0), \Pi_1^h(1) - \widehat{\Pi}_t^h(0)\right].$$

Now we can characterize the equilibrium when separation via pivots is feasible.

Lemma 35 *For δ sufficiently small and $\Pi_1^l(1) - \widehat{\Pi}_1^l(0) \leq F \leq \Pi_1^h(1) - \widehat{\Pi}_{\underline{\tau}^l}^h(q_0)$, there exists a period $\underline{\tau}_{Piv} \leq \tau_S \leq \bar{\tau}_{Piv}$, such that the high type prefers to separate by pivoting in period τ_S and prefers to wait for another period for all $t < \tau_S$.*

Proof. We can wlog restrict attention to the case when $\bar{\tau}_{Piv} < \underline{\tau}^l$. This implies that at any time at which we consider separation, we have $q_t = q_0$.

First, consider a period $t < \bar{\tau}_{Piv}$. If the high type separates in period t, her payoff is $\Pi_1^h(1) - F$, while if she pools in period t and separates in period $t+1$, her payoff is

$$\lambda p_t^h\left(1 - \alpha_t^P\right)V - k + \delta\left(1 - \lambda p_t^h\right)\left(\Pi_1^h(1) - F\right).$$

The high type prefers separating in period t rather than in period $t+1$ whenever

$$f_t := \Pi_1^h(1) - F - \left(\lambda p_t^h\left(1 - \alpha_t^P\right)V - \delta\lambda p_t^h\left(\Pi_1^h(1) - F\right) - k\right) \geq 0.$$

Since we have $t <\underline{\tau}^l$, we can write

$$\begin{aligned} \lambda p_t^h\left(1-\alpha_t^P\right)V-\delta\lambda p_t^h\left(\Pi_1^h\left(1\right)-F\right) &= \lambda p_t^h\left(V-\delta\left(\Pi_1^h\left(1\right)-F\right)\right)-c\frac{p_t^h}{p_t\left(q_0\right)} \\ &= \lambda p_t^h\left(V-\delta\left(\Pi_1^h\left(1\right)-F\right)\right)-c\frac{1}{q_0+\left(1-q_0\right)\frac{p_t^l}{p_t^h}}, \end{aligned}$$

which is strictly decreasing in t, because both p_t^h and p_t^l/p_t^h are strictly decreasing and $V > \Pi_1^h(1)$. Then, f_t is strictly increasing in t and crosses zero at most once.

Since no separation is feasible after period $\bar{\tau}_{Piv}$, we denote

$$f_{\bar{\tau}_{Piv}} := \Pi_1^h\left(1\right)-F-\widehat{\Pi}_{\bar{\tau}_{Piv}}^h(q_0).$$

Define τ_S as the period in which f_t crosses zero for the first time. For any $t > \tau_S$, the high type prefers to separate earlier (since $f_t > 0$) and for any $t < \tau_S$, the high type prefers to separate later (since $f_t < 0$). Thus, the high type prefers to separate in period τ_S over separating in any other period.

Note that when $F \leq \Pi_1^h(1) - \widehat{\Pi}_{\underline{\tau}^l}^h(q_0)$, $f_{\bar{\tau}_{Piv}}$ is non-negative as $\bar{\tau}_{Piv}$ approaches $\underline{\tau}^l$. This establishes the existence of $\tau_S \in [\underline{\tau}_{Piv}, \bar{\tau}_{Piv}]$.

■

When $F > \Pi_1^h(1) - \widehat{\Pi}_{\underline{\tau}^l}^h(q_0)$, the high type finds it optimal to wait until $\underline{\tau}^l$. We show in the Lemma below, that the high type does not prefer to pool and pivot at the same time $\tau_P \geq \underline{\tau}^l$ as the low type.

Lemma 36 *For $F > \Pi_1^h(1) - \widehat{\Pi}_{\underline{\tau}^l}^h(q_0)$ and for any $t \geq \underline{\tau}^l$, the high type prefers to wait until the low type separates rather than pool and pivot at the same time as the low type.*

Proof. The condition $F > \Pi_1^h(1) - \widehat{\Pi}_{\underline{\tau}^l}^h(q_0)$ implies that

$$\Pi_1^h(q_0) - F < \widehat{\Pi}_{\underline{\tau}^l}^h(q_0)$$

so that the high type does not prefer to pool at time $t = \underline{\tau}^l$. For $t > \underline{\tau}^l$, the high type prefers not to pool whenever

$$\Pi_1^h(q_t) - F < \widehat{\Pi}_t^h.$$

Using the condition on F, a sufficient condition is that

$$\widehat{\Pi}_t^h > \widehat{\Pi}_{\underline{\tau}^l}^h(q_0).$$

We now establish that this condition holds for any $t > \underline{\tau}^l$, because $\widehat{\Pi}_t^h$ is strictly increasing in t for $t \geq \underline{\tau}^l$.

Since the low type is indifferent between pivoting and not pivoting, we have

$$\lambda p_t^l\left(1-\alpha_t^P\right)V = \left(1-\delta\left(1-\lambda p_t^l\right)\right)\left(\Pi_1^l\left(0\right)-F\right)+k,$$

which implies that

$$\lambda p_t^h \left(1-\alpha_t^P\right) V = \frac{p_t^h}{p_t^l}\left(1-\delta\left(1-\lambda p_t^l\right)\right)\left(\Pi_1^l(0)-F\right)+\frac{p_t^h}{p_t^l}k.$$

Thus,

$$\widehat{\Pi}_t^h - \widehat{\Pi}_{t+1}^h = \lambda p_t^h \left(1-\alpha_t^P\right) V - k - \left(1-\delta\left(1-\lambda p_t^h\right)\right)\widehat{\Pi}_{t+1}^h,$$

or equivalently

$$\widehat{\Pi}_t^h - \widehat{\Pi}_{t+1}^h = \frac{p_t^h}{p_t^l}\left(\Pi_1^l(0)-F\right)+\frac{p_t^h-p_t^l}{p_t^l}k-\widehat{\Pi}_{t+1}^h+\delta\left(\left(1-\lambda p_t^h\right)\widehat{\Pi}_{t+1}^h-\left(1-\lambda p_t^l\right)\left(\Pi_1^l(0)-F\right)\right).$$

The last term can be either positive or negative so the above equation will monotonically increase or decrease with δ. As a result, it is sufficient to show $\widehat{\Pi}_t^h < \widehat{\Pi}_{t+1}^h$ given both $\delta = 0$ and $\delta = 1$.

If $\delta = 0$, then

$$\widehat{\Pi}_t^h = \frac{p_t^h}{p_t^l}\left(\Pi_1^l(0)-F\right)+\frac{p_t^h}{p_t^l}k-k.$$

Bayes' rule implies that p_t^h/p_t^l is strictly increasing in t, so $\widehat{\Pi}_t^h$ monotonically increases.

If $\delta = 1$, then

$$\widehat{\Pi}_t^h = \lambda p_t^h\left(\Pi_1^l(0)-F\right)+\frac{p_t^h}{p_t^l}k-k+\left(1-\lambda p_t^h\right)\widehat{\Pi}_{t+1}^h.$$

To show that $\widehat{\Pi}_t^h$ strictly increases in this case, consider the alternative continuation value below

$$\check{\Pi}_t^h = \lambda p_t^h\left(\Pi_1^l(0)-F\right)+\frac{p_t^h}{p_t^l}k-k+\left(1-\lambda p_t^h\right)\left(\Pi_1^h(0)-F\right).$$

This continuation value is based on the strategy that if the high type does not succeed in t, then she will pivot (thereby being regarded as a low type) in the next period. Given that p_t^h strictly decreases and $\Pi_1^h(0) > \Pi_1^l(0)$, $\check{\Pi}_t^h$ strictly increases over time. After some algebra, we can confirm that

$$\check{\Pi}_t^h < \lambda p_t^h\left(\Pi_1^l(0)-F\right)+\frac{p_t^h}{p_t^l}k-k+\left(1-\lambda p_t^h\right)\check{\Pi}_{t+1}^h < \check{\Pi}_{t+1}^h.$$

But by construction, $\check{\Pi}_t^h < \widehat{\Pi}_t^h$ for any t since the high type can optimally choose not to pivot. Therefore,

$$\begin{aligned}\widehat{\Pi}_t^h &= \lambda p_t^h\left(\Pi_1^l(0)-F\right)+\frac{p_t^h}{p_t^l}k-k+\left(1-\lambda p_t^h\right)\check{\Pi}_{t+1}^h+\left(1-\lambda p_t^h\right)\left(\widehat{\Pi}_{t+1}^h-\check{\Pi}_{t+1}^h\right)\\ &< \check{\Pi}_{t+1}^h+\left(1-\lambda p_t^h\right)\left(\widehat{\Pi}_{t+1}^h-\check{\Pi}_{t+1}^h\right)\\ &< \widehat{\Pi}_{t+1}^h.\end{aligned}$$

■

The following Lemma summarizes our results and characterizes when the high type separates via a pivot in the optimal contract, depending on γ and F.

Lemma 37 *For δ sufficiently small, γ sufficiently large, and $\Pi_1^l(1) - \widehat{\Pi}_1^l(0) \leq F \leq \Pi_1^h(1) - \widehat{\Pi}_{\underline{\tau}^l}^h(q_0)$, the high type separates via a pivot in the optimal contract. If instead $F > \Pi_1^h(1) - \widehat{\Pi}_{\underline{\tau}^l}^h(q_0)$, then in the optimal contract the low type separates via a pivot.*

Proof. If $\Pi_1^l(1) - \widehat{\Pi}_1^l(0) \leq F \leq \Pi_1^h(1) - \widehat{\Pi}_{\underline{\tau}^l}^h(q_0)$, the high type prefers separating via a pivot in period $\underline{\tau}_{Piv} \leq \tau_S \leq \bar{\tau}_{Piv}$ Note that the high type strictly prefers separating in period τ_S instead of pooling (and pivoting together with the low type). Hence, for γ sufficiently large, there exists a period τ_S such that $\underline{\tau}_{Piv} \leq \tau_S \leq \bar{\tau}_{Piv}$, so that pivoting in period τ_S is optimal. If $F > \Pi_1^h(1) - \widehat{\Pi}_{\underline{\tau}^l}^h(q_0)$, the high type prefers to wait until at least period $\underline{\tau}^l$ instead of separating earlier. Since the value of separating strictly exceeds the value of pooling, the high type also prefers to wait until at least period $\underline{\tau}^l$ instead of pooling earlier. For any $t \geq \underline{\tau}^l$, the low type pivots before the high type in any equilibrium. Then, in the optimal contract, the low type separates via a pivot. ■

We now show that the equilibrium is pooling whenever

$$F < \Pi_1^l(1) - \widehat{\Pi}_1^l(0),$$

i.e. there exists $\tau_P \leq \underline{\tau}^l$ such that both types pivot at τ_P with certainty. Following Bayes' Rule, the posterior after pivots is $q_t = q_0$. First, we prove a Lemma mirroring Lemma 32 that shows the high type pivots weakly earlier in this case.

Lemma 38 *For δ sufficiently small, the high type pivots weakly earlier than the low type when investors have a constant belief $q_t = q_0$.*

Proof. The algebra of this proof is close to Lemma 32 so we omit the details. The high type pivots in the fist period when

$$F \leq \lambda p_1^h \left(1 - \alpha_1^P\right) V - \lambda p_t^h \left(1 - \alpha_t^P\right) V - \delta K^h,$$

where K^h represents higher order terms in the summation and

$$\alpha_t^P = \frac{c}{\lambda p_t(q_0) V}.$$

The low type pivots when

$$F \leq \lambda p_1^l \left(1 - \alpha_1^P\right) V - \lambda p_t^l \left(1 - \alpha_t^P\right) V - \delta K^L.$$

So the high type pivots weakly earlier than the low type if

$$\lambda \left(p_1^h - p_1^l\right) \left(1 - \alpha_1^P\right) V - \lambda \left(p_t^h - p_t^l\right) \left(1 - \alpha_t^P\right) V \geq \delta \left(K^h - K^l\right).$$

If $K^h < K^l$, then we can set $\underline{\delta} = 1$. Otherwise, the inequality holds when

$$\delta \leq \min\left\{1, \frac{\lambda \left(p_1^h - p_1^l\right) \left(1 - \alpha_1^P\right) V - \lambda \left(p_t^h - p_t^l\right) \left(1 - \alpha_t^P\right) V}{K^h - K^l}\right\}.$$

■

The above Lemma, along with the fact that the low type will mimic the pivots, implies that in equilibrium both types will pivot at the same time. The following Lemma shows that $\tau_P \leq \underline{\tau}^l$.

Lemma 39 *For δ sufficiently small, both types will pivot at $\tau_P \leq \underline{\tau}^l$. After pivots, investors have the same belief $q_t = q_0$.*

Proof. Recall that $\widehat{\Pi}_t^l = \Pi_1^l(0) - F$ at $t = \underline{\tau}^l$ whereas the pivoting payoff is $\Pi_1^l(q_0) - F$ for the low type. Thus, the she strictly prefers to pivot at $t = \underline{\tau}^l$. Then by Lemma 38, the high type prefers to pivot weakly earlier (while forecasting the low type will mimic at the same time). This establishes that $\tau_P \leq \underline{\tau}^l$. Investors' belief follows from Bayes' Rule. ■

Finally, it remains to show that for γ sufficiently large, the optimal contract is pooling, i.e. both the high and low type pivot at some time τ_P. For this, it is sufficient to show that the low type prefers pooling with the high type at some period τ_P.

For any t where separation by the high type is feasible, the low type strictly prefers the high type to not separate, because $\widehat{\Pi}_t^l(q_0) > \widehat{\Pi}_t^l(0)$. For any $t \geq \underline{\tau}^l$, separation by the low type is feasible and pooling is feasible as well. For any such t, the low type strictly prefers to pool. Thus, the contract which maximizes the low type's value is a pooling contract.

B.5 Proof of Corollary 5

For γ sufficiently large, pivoting occurs in the separating equilibrium in first period that $f_t \geq 0$, i.e.

$$F \leq \Pi_1^h(1) - \frac{\lambda p_t^h (1 - \bar{\alpha}_t^p) V - k}{1 - \delta\left(1 - \lambda p_t^h\right)}.$$

In the symmetric information case, the high type pivots in the first period that

$$F \leq \Pi_1^h(1) - \frac{\lambda p_t^h \left(1 - \bar{\alpha}_t^h\right) V - k}{1 - \delta\left(1 - \lambda p_t^h\right)}.$$

The statement is true since $\bar{\alpha}_t^h < \bar{\alpha}_t^p$ (the inequality is strict if separation occurs earlier than τ^l) implies

$$\Pi_1^h(1) - \frac{\lambda p_t^h (1 - \bar{\alpha}_t^p) V - k}{1 - \delta\left(1 - \lambda p_t^h\right)} > \Pi_1^h(1) - \frac{\lambda p_t^h \left(1 - \bar{\alpha}_t^h\right) V - k}{1 - \delta\left(1 - \lambda p_t^h\right)}.$$

B.6 Proof of Proposition 6

The proof of Proposition 6 is similar to the proof of Proposition 4. We hence only provide a sketch.

Under symmetric information, each type liquidates at time τ^θ. We have $\tau^l \leq \tau^h$. Each type implements the prestige project at the time of liquidation, since by doing so she receives a higher outside option and does not need to suffer the decrease in project value. Thus, each type liquidates at the first time at which

$$\Pi_t^\theta \leq \pi.$$

A pooling equilibrium exists and has the same features as the equilibrium in Proposition 2. Specifically, the equity share is still given by

$$\alpha_t^P = \frac{c}{\lambda p_t(q_t)V},$$

and there exist two times $\underline{\tau}^l \leq \bar{\tau}^l$, such that in any period between $\underline{\tau}^l$ and $\bar{\tau}^l$, the low type randomizes between implementing the prestige project and liquidating or continuing. In any such period, the following indifference condition holds:

$$\lambda p_t^l \left(1 - \alpha_t^P\right) V - k + \delta \left(1 - \lambda p_t^l\right) \pi = \pi. \tag{40}$$

This equation implies that the pooling equity share satisfies

$$\alpha_t^P = \frac{\lambda p_t^l V - k}{\lambda p_t^l V} - \pi \frac{1 - \delta \left(1 - \lambda p_t^l\right)}{\lambda p_t^l}. \tag{41}$$

We now consider a separating equilibrium in period t. Using the IC conditions for the low and high type, Equations (IC_l^{Pres}) and (IC_h^{Pres}), we can see that separating is incentive compatible whenever

$$\frac{\Pi_t^l(1) - \Pi_t^l(0)}{p_t^l} \leq \lambda V_0 \leq \frac{\Pi_t^h(1) - \Pi_t^h(0)}{p_t^h}.$$

A similar argument as in Lemma 25 implies that

$$\frac{\Pi_t^l(1) - \Pi_t^l(0)}{p_t^l} \leq \frac{\Pi_t^h(1) - \Pi_t^h(0)}{p_t^h}.$$

Thus, for any period t, there exists a V_0, such that we can achieve separation in that period. It only remains to find conditions such that separation is optimal.

Consider the high type's value from separating in period t versus separating in period $t+1$. If the high type separates in period $t < \bar{\tau}^l - 1$, her value is

$$\Pi_t^h(1) - \lambda p_t^h V_0,$$

while if she separates in period $t+1$, her value in period t is

$$\lambda p_t^h \left(1 - \alpha_t^P\right) V - k + \delta \left(1 - \lambda p_t^h\right) \left(\Pi_{t+1}^h(1) - (1-\delta) \lambda p_{t+1}^h V_0\right).$$

After some algebra, the difference between these two values is

$$f_t = c \left(\frac{p_t^h}{p_t(q_t)} - 1\right) - \lambda p_t^h V_0 \left(1 - \delta(1-\lambda)\right).$$

Whenever this expression is positive, the high type prefers separating in period t rather than in period $t+1$.

For $t = \bar{\tau}^l - 1$, the high type knows that the low type will liquidate in period $\bar{\tau}^l$. Thus, separating in period $\bar{\tau}^l$ does not require costly signaling, and we have

$$f_{\bar{\tau}^l-1} = c\left(\frac{p^h_{\bar{\tau}^l-1}}{p_{\bar{\tau}^l-1}\left(q_{\bar{\tau}^l-1}\right)} - 1\right) - \lambda p^h_{\bar{\tau}^l-1} V_0.$$

Next, we find a sufficient condition, such that $f_t > 0$. Consider the region $\underline{\tau}^l \leq t \leq \bar{\tau}^l$. On this region, the low type will liquidate if her type is discovered. Thus, the low type's IC condition becomes

$$\Pi^l_t\left(1\right) - \lambda p^l_t V_0 \leq \pi,$$

since we have $\Pi^l_t\left(0\right) = \pi$. Let us pick V_0 such that the above inequality binds. Then, the high type prefers separating in period t over separating in the next period whenever

$$\hat{f}_t = c\left(\frac{p^h_t}{p_t\left(q_t\right)} - 1\right) - \frac{p^h_t}{p^l_t}\left(\Pi^l_t\left(1\right) - \pi\right)\left(1 - \delta\left(1 - \lambda\right)\right) \geq 0$$

for $t < \bar{\tau}^l - 1$ or

$$\hat{f}_{\bar{\tau}^l-1} = c\left(\frac{p^h_{\bar{\tau}^l-1}}{p_{\bar{\tau}^l-1}\left(q_{\bar{\tau}^l-1}\right)} - 1\right) - \frac{p^h_{\bar{\tau}^l-1}}{p^l_{\bar{\tau}^l-1}}\left(\Pi^l_{\bar{\tau}^l-1}\left(1\right) - \pi\right) \geq 0.$$

Here is the significance of $\hat{f}_t$. Whenever $\hat{f}_t$ is positive, there exists a V_0, such that f_t is positive. Thus, $\hat{f}_t > 0$ is a necessary and sufficient condition for the existence of a V_0 such that the high type prefers to separate in period t rather than in period $t+1$.

Pick period $t = \bar{\tau}^l - 1$. The low type liquidates in period $t + 1 = \bar{\tau}^l$, even if the investor's belief is $q_{t+1} = 1$. Thus, we have

$$\Pi^l_{\bar{\tau}^l-1}\left(1\right) = \lambda p^l_{\bar{\tau}^l-1} V - c\frac{p^l_{\bar{\tau}^l-1}}{p^h_{\bar{\tau}^l-1}} - k + \delta\left(1 - \lambda p^l_{\bar{\tau}^l-1}\right)\pi.$$

Plugging this expression into $\hat{f}_t$, we get,

$$\begin{aligned}
\hat{f}_{\bar{\tau}^l-1} &= -\frac{p^h_{\bar{\tau}^l-1}}{p^l_{\bar{\tau}^l-1}}\left(\lambda p^l_{\bar{\tau}^l-1} V - c\frac{p^l_{\bar{\tau}^l-1}}{p_{\bar{\tau}^l-1}\left(q_{\bar{\tau}^l-1}\right)} - k + \delta\left(1 - \lambda p^l_{\bar{\tau}^l-1}\right)\pi - \pi\right) \\
&= -\frac{p^h_{\bar{\tau}^l-1}}{p^l_{\bar{\tau}^l-1}}\left(\Pi^l_{\bar{\tau}^l-1} - \pi\right) = 0,
\end{aligned}$$

where $\Pi^l_{\bar{\tau}^l-1}$ is the low type's value in period $\bar{\tau}^l - 1$ if there is pooling. Since $\bar{\tau}^l - 1 \geq \underline{\tau}^l$, we have $\Pi^l_{\bar{\tau}^l-1} = \pi$.

For any $\underline{\tau}^l \leq t < \bar{\tau}^l - 1$, we can write

$$\begin{aligned}
\hat{f}_t &= c\left(\frac{p_t^h}{p_t(q_t)} - 1\right) - \frac{p_t^h}{p_t^l}\left(\Pi_t^l(1) - \pi\right)(1 - \delta(1-\lambda)) \\
&= c\left(\frac{p_t^h}{p_t(q_t)} - 1\right) - \frac{p_t^h}{p_t^l}\left(\lambda\left(p_t^l V - c\frac{p_t^l}{p_t^h} - k\right) + \delta\left(1 - \lambda p_t^l\right)\Pi_{t+1}^l(1) - \pi\right) \\
&\quad + \delta(1-\lambda)\frac{p_t^h}{p_t^l}\left(\Pi_t^l(1) - \pi\right) \\
&= -\frac{p_t^h}{p_t^l}\left(\lambda p_t^l V - c\frac{p_t^l}{p_t(q_t)} - k + \delta\left(1 - \lambda p_t^l\right)\Pi_{t+1}^l(1) - \pi\right) \\
&\quad + \delta(1-\lambda)\frac{p_t^h}{p_t^l}\left(\Pi_t^l(1) - \pi\right) \\
&= \delta\frac{p_t^h}{p_t^l}\left((1-\lambda)\left(\Pi_t^l(1) - \pi\right) - \left(1 - \lambda p_t^l\right)\left(\Pi_{t+1}^l(1) - \pi\right)\right),
\end{aligned}$$

since Equation (40) implies that

$$\lambda p_t^l V - c\frac{p_t^l}{p_t(q_t)} - k - \pi = -\delta\left(1 - \lambda p_t^l\right)\pi.$$

Now, we pick $t = \bar{\tau}^l - 2$ and we pick $\pi \leq \Pi_{\bar{\tau}^l-1}^l(1)$, arbitrarily close to $\Pi_{\bar{\tau}^l-1}^l(1)$. Then, we have $\hat{f}_{\bar{\tau}^l-2} > 0$ and $\hat{f}_{\bar{\tau}^l-1} = 0$. Thus, there exists a V_0 such that the two IC conditions in Equation (IC_l^{Pres}) and Equation (IC_h^{Pres}) hold in period $\bar{\tau}^l - 2$ and such that $f_{\bar{\tau}^l-2} > 0$. This implies that the high type prefers separating in period $\bar{\tau}^l - 2$ over separating in any later period. This, in turn, implies that there exists a period $\tau_S^h \leq \bar{\tau}^l - 2$ at which the high type prefers to separate.

As before, the low type strictly prefers to pool until period $\bar{\tau}^l$. Thus, for γ sufficiently large, there exists an optimal separation period $\tau_S < \bar{\tau}^l - 1$ which maximizes the entrepreneur's ex-ante value.

B.7 Proof of Proposition 7

First, we show that there exists an η such that the RHS of Equation (22) is non-empty, which is equivalent to

$$\widehat{\Pi}_t^h(0) - \widehat{\Pi}_t^l(0) \leq (2\eta - 1)\left(\Pi_1^h(\eta) - \Pi_1^l(\eta)\right). \tag{42}$$

Recall that in the proof of Lemma 32, we establish that, given the parameters,

$$\widehat{\Pi}_t^h(0) - \widehat{\Pi}_t^l(0) \leq \Pi_1^h(1) - \Pi_1^l(1).$$

We first confirm that $\Pi_1^h(\eta) - \Pi_1^l(\eta)$ strictly increases with η. This is because

$$\begin{aligned}\Pi_1^h(\eta) - \Pi_1^l(\eta) &= \lambda\left(p_1^h - p_1^l\right)\sum_{t=1}^{\bar{\tau}^l-1}\left(\delta(1-\lambda)\right)^{t-1}\left(\left(1-\alpha_t^P\right)V - k\right) \\ &\quad +\lambda p_1^h \sum_{t=\bar{\tau}^l}^{\tau^h}\left(\delta(1-\lambda)\right)^{t-1}\left(\left(1-\bar{\alpha}_t^h\right)V - k\right).\end{aligned}$$

In the above equation, the second line of the RHS is not a function of η. In the first line, for $\underline{\tau}_l \leq t < \bar{\tau}^l$,

$$\alpha_t^P = \frac{\lambda p_t^l - k}{\lambda p_t^l V},$$

which is not a function η. If $t < \underline{\tau}_l$,

$$\alpha_t^P = \frac{c}{\lambda p_t(\eta) V},$$

which will strictly decrease with η. As a result, $\Pi_1^h(\eta) - \Pi_1^l(\eta)$ strictly increases with η and so does the RHS of Equation (42).

Next we show that

$$\lim_{\eta\to 1}(2\eta-1)\left(\Pi_1^h(\eta) - \Pi_1^l(\eta)\right) = \Pi_1^h(1) - \Pi_1^l(1).$$

This is because in the optimal pooling contract, the sequence of belief q_t weakly increases over time and is bounded by 1. If $q_0 = \eta$ goes to 1 in the limit, then all q_t should do so as well, which implies

$$\lim_{q_t\to 1}\alpha_t^P = \frac{c}{\lambda p_t(q_t) V} = \bar{\alpha}_t^h,$$

and thus $\lim_{\eta\to 1}\Pi_1^\theta(\eta) = \Pi_1^\theta(1)$. By the monotone convergence, there exists $\eta_1 > 1/2$ such that Equation (42) holds for all $\eta > \eta_1$.

Next, we show that there exists η such that the high type will still pivot at the same τ_S. In Proposition 4, τ_S is defined as the first time that

$$\Pi_1^h(1) - F - \left(\lambda p_t^h\left(1-\alpha_t^P\right)V - \delta\lambda p_t^h\left(\Pi_1^h(1) - F\right) - k\right) \geq 0$$

holds. With non-persistent pivots, τ_S is defined as the first time that

$$\eta\Pi_1^h(\eta) + (1-\eta)\Pi_1^l(\eta) - F - \left(\lambda p_t^h\left(1-\alpha_t^P\right)V - \delta\lambda p_t^h\left(\eta\Pi_1^h(\eta) + (1-\eta)\Pi_1^l(\eta) - F\right) - k\right) \geq 0$$

holds. Notice that

$$\eta\Pi_1^h(\eta) + (1-\eta)\Pi_1^l(\eta) = \Pi_1^l(\eta) + \eta\left(\Pi_1^h(\eta) - \Pi_1^l(\eta)\right).$$

Following the previous proof, the RHS strictly increases with η and converges to $\Pi_1^h(1)$. Therefore, there exists $\eta_2 > 1/2$ such that the above two inequalities hold at the same time for all $\eta > \eta_2$, and thus τ_S becomes identical in the two cases. Define $\bar{\eta} = \max\{\eta_1, \eta_2\}$ and the proposition statement is proved.

B.8 Proof of Proposition 8

We start with a monotonicity lemma so that we can characterize $M^\theta(C'_t)$ by a lower bound belief.

Lemma 40 *In the optimal pooling equilibrium, if $1 > q_1 > q_2 > 0$, then $\Pi_t^\theta(q_1) > \Pi_t^\theta(q_2)$ holds.*

Proof. We start with the low type,

$$\begin{aligned} & \Pi_t^l(q_1) - \Pi_t^l(q_2) \\ = & \lambda p_t^l \left(\sum_{s=t}^{\underline{\tau}_2^l - 1} (\delta(1-\lambda))^{s-t} \left(\frac{c}{\lambda p_s(q_2)} - \frac{c}{\lambda p_s(q_1)} \right) \right. \\ & \left. + \sum_{s=\underline{\tau}_2^l}^{\underline{\tau}_1^l - 1} (\delta(1-\lambda))^{s-t} \left(V - \frac{c}{\lambda p_s(q_1)} - \frac{k}{\lambda p_s^l} \right) \right) \\ > & 0. \end{aligned}$$

Recall that $\underline{\tau}_i^l$ is the first period when the low type starts to drop out with positive probability if $q_t = q_i$. Notice for $t \geq \underline{\tau}_1^l$, both contracts offer the same expected payoff 0 for the low type. In other words, these two contracts coincide in such periods. The definition of $\underline{\tau}_i^l$ implies $\underline{\tau}_1^l \geq \underline{\tau}_2^l$.[67] Similarly, for the high type,

$$\begin{aligned} & \Pi_t^h(q_1) - \Pi_t^h(q_2) \\ = & \lambda p_t^h \left(\sum_{s=t}^{\underline{\tau}_2^l - 1} (\delta(1-\lambda))^{s-t} \left(\frac{c}{\lambda p_s(q_2)} - \frac{c}{\lambda p_s(q_1)} \right) \right. \\ & \left. + \sum_{s=\underline{\tau}_2^l}^{\underline{\tau}_1^l - 1} (\delta(1-\lambda))^{s-t} \left(\alpha_s^p - \frac{c}{\lambda p_s(q_1)} \right) \right) \\ > & 0, \end{aligned}$$

where $\alpha_s^p = \left(\lambda p_s^l V - k\right) / \lambda p_s^l$ is the pooling contract that generates 0 payoff for the low type in expectation. ■

Corollary 41 *In the optimal pooling equilibrium, $\Pi_t^l(q) > \Pi_t^l(0)$ holds for any $0 < q \leq 1$.*

[67] If $\underline{\tau}_1^l = \underline{\tau}_2^l$, then we follow the convention that $\sum_{s=\underline{\tau}_2^l}^{\underline{\tau}_1^l - 1} x_s = 0$.

Proof. After some algebra, we have

$$
\begin{aligned}
& \Pi_t^l(q) - \Pi_t^l(0) \\
= \; & \lambda p_t^l \left(\sum_{s=t}^{\tau_l - 1} (\delta(1-\lambda))^{s-t} \left(\frac{c}{\lambda p_s^l} - \frac{c}{\lambda p_s(q)} \right) \right. \\
& \left. + \sum_{s=\tau_l}^{\underline{\tau_l}-1} (\delta(1-\lambda))^{s-t} \left(V - \frac{c}{\lambda p_s(q)} - \frac{k}{\lambda p_s^l} \right) \right) \\
> \; & 0,
\end{aligned}
$$

where $\underline{\tau_l}$ is the first period when the low type starts to drop out with positive probability with the optimal pooling contract.[68] ■

Lemma 40 implies $M^\theta(C_t')$ must be a closed interval $\left[q^\theta(C_t'), 1\right]$. It is sufficient to find the off-equilibrium belief $q^\theta(C_t')$ such that the θ type is indifferent with deviating and offering the optimal contract in continuation. Therefore, if $q^\theta(C_t') < q^{\theta'}(C_t')$, then $M^{\theta'}(C_t') \subset M^\theta(C_t')$ and the VC should believe facing θ with certainty.

We first focus on the case when pivots are feasible. Lemma 42 shows if both the optimal pooling equilibrium and separating equilibrium exist, the D1 criterion will pick the separating equilibrium.

Lemma 42 *The optimal pooling equilibrium does not satisfy the D1 criterion if there exists an optimal separating equilibrium that pivots at* τ_S*.*

Proof. Consider period $t = \tau_s$, by the construction and definition of τ_s, the following equation must be true according to the proof of Proposition 4:

$$\widehat{\Pi}_t^h(q_t) \leq \Pi_1^h(1) - F.$$

Also notice

$$\widehat{\Pi}_t^l(q_t) > \widehat{\Pi}_t^l(0) \geq \Pi_1^l(1) - F,$$

These inequalities imply that $M^l(I_t' = 1) = \emptyset$ and $\{1\} \subset M^h(I_t' = 1)$. Therefore, there exists the following deviation for the pooling equilibrium: the high type deviates by pivoting. Upon observing the pivots, the investor believes the deviating type is the high type. ■

Next we lay out a necessary condition that all separating equilibria have to satisfy in order to survive the D1 criterion.

Lemma 43 *For any equilibrium separating with pivoting* I_t *at* t*, the equilibrium does not satisfy the D1 criterion if there exists* $t' < t$ *such that Equation (23) holds in* t'*.*

Proof. The proof is similar to the previous lemma. The high type can pivot earlier at t'. Equation (23) implies that $M^l(I_{t'}' = 1) = \emptyset$ and $\{1\} \subset M^h(I_{t'}' = 1)$. In other words, once the high type pivots, the off-equilibrium is $q_{t'} = 1$. ■

Recall that in the proof of Proposition 4, we show that separating equilibria exist in all periods $\underline{\tau}_{Piv} \leq t \leq \bar{\tau}_{Piv}$ and construct the separating time τ_S as the first period such that separation at t is strictly more profitable than doing at $t+1$. The above lemma implies that all the separating

[68] In the case of $q = 1$, this is the first period when the low type drops out if she is regarded as the high type.

equilibria that pivot strictly later than τ_S do not survive the D1 criterion. This is because for such equilibria, the following equation holds

$$\Pi_1^l(1) - \widehat{\Pi}_{\tau_S}^l(q_{\tau_S}) < F < \Pi_1^h(1) - \widehat{\Pi}_{\tau_S}^h(q_{\tau_S}).$$

Then the high type will deviate at τ_S. Next we show that all equilibria separating at $t < \tau_S$ do not survive the D1 criterion as well.

Lemma 44 *All equilibria separating at $t < \tau_S$ by pivots do not survive the D1 criterion.*

Proof. Consider the following deviation: not pivot at t, offer equity share α_t^P at t, and pivot at $t+1$. In other words, the entrepreneur deviates to separate at $t+1$. Notice the payoff of this deviation does not depend on the off equilibrium belief at t as long as the investor is willing to accept this offer. This is because the payment at t is fixed and the belief at $t+1$ will jump to either 1 or 0, decided by whether pivots occur. Since separation is delayed, the low type is strictly better off. Since $t < \tau_S$, delaying separation to $t+1$ increases the high type's payoff as well. Therefore, both type will deviate. ■

Next, we need to confirm that separating by pivots at τ_S survives the criterion. Deviation by separating later than τ_S does not make sense since the separation has occurred by then. Deviation by separating earlier than τ_S is not profitable since for all such periods,

$$\Pi_1^l(1) - \widehat{\Pi}_t^l(q_t) < \Pi_1^h(1) - \widehat{\Pi}_t^h(q_t) < F.$$

This implies $M^l\left(I_t' = 1\right) = M^h\left(I_t' = 1\right) = \emptyset$, i.e. regardless of the belief, both types will not deviate. So far we have shown if pivots are feasible, there exists a separating equilibrium surviving the D1 criterion. The optimal pooling equilibrium and all other separating equilibria do not satisfy it. To prove the uniqueness, it remains to show all suboptimal pooling equilibria are also pruned. We delay this after Corollary 47.]We now prove the second statement of the Proposition, which applies to the cashless entrepreneurs. The following Lemma is useful as an extension of Lemma 25.

Lemma 45 *For any $0 < q < 1$, the following equation holds for all t*

$$\frac{\Pi_t^h\left(q\right) - \Pi_t^h\left(0\right)}{\lambda p_t^h} > \frac{\Pi_t^l\left(q\right) - \Pi_t^l\left(0\right)}{\lambda p_t^l}.$$

Proof.

$$\begin{aligned}
&\Pi_t^h\left(q\right) - \Pi_t^h\left(0\right) \\
= \;& \lambda p_t^h \left(\sum_{s=t}^{\tau_h'-1} \left(\delta\left(1-\lambda\right)\right)^{s-t} \left(\frac{c}{\lambda p_s^l} - \frac{c}{\lambda p_s(q_1)}\right) \right. \\
& + \sum_{s=\tau_h'}^{\underline{\tau_l}-1} (\delta(1-\lambda))^{s-t} \left(V - \frac{c}{\lambda p_s(q_1)} - \frac{k}{\lambda p_s^h}\right) \\
& \left. + \sum_{s=\max\{\tau_h', \underline{\tau_l}\}}^{\tau^h-1} (\delta(1-\lambda))^{s-t} \left(V - \alpha_s^p - \frac{k}{\lambda p_s^h}\right) \right).
\end{aligned}$$

Recall that τ'_h is the optimal stopping time if the high type is regarded as the low type. We use $\max\{\tau'_h, \underline{\tau_l}\}$ in the last line because τ'_h could be strictly larger than $\underline{\tau_l}$, in which case we set the second term 0. Also notice that,

$$\begin{aligned} & V - \frac{c}{\lambda p_s(q_1)} - \frac{k}{\lambda p^l_s} \geq \frac{c}{\lambda p^l_s} - \frac{c}{\lambda p_s(q_1)} \\ \Leftrightarrow \quad & \lambda p^l_s \left(V - \frac{c}{\lambda p^l_s} \right) - k \geq 0 \\ \Leftrightarrow \quad & s \leq \tau_l. \end{aligned}$$

In the last line we use weak inequality because we follow the tie-breaker that players stop with certainty if they are indifferent. This implies

$$\begin{aligned} & \frac{\Pi^h_t(q) - \Pi^h_t(0)}{\lambda p^h_t} \\ > \quad & \sum_{s=t}^{\tau'_h - 1} (\delta(1-\lambda))^{s-t} \left(\frac{c}{\lambda p^l_s} - \frac{c}{\lambda p_s(q_1)} \right) + \sum_{s=\tau'_h}^{\underline{\tau_l}-1} (\delta(1-\lambda))^{s-t} \left(V - \frac{c}{\lambda p_s(q_1)} - \frac{k}{\lambda p^l_s} \right) \\ \geq \quad & \sum_{s=t}^{\tau^l - 1} (\delta(1-\lambda))^{s-t} \left(\frac{c}{\lambda p^l_s} - \frac{c}{\lambda p_s(q_1)} \right) + \sum_{s=\tau_l}^{\underline{\tau_l}-1} (\delta(1-\lambda))^{s-t} \left(V - \frac{c}{\lambda p_s(q_1)} - \frac{k}{\lambda p^l_s} \right) \\ = \quad & \frac{\Pi^l_t(q) - \Pi^l_t(0)}{\lambda p^l_t}. \end{aligned}$$

The first inequality follows from $k/p^h_s < k/p^l_s$. The second inequality is strict unless $\tau'_h = \tau_l$. ■

We start with eliminating all suboptimal equilibria.

Lemma 46 *No suboptimal separating equilibrium survives the D1 criterion if pivots are not feasible.*

Proof. Rewrite the separating contract α^h_t as $c/\left(\lambda p^h_t\right) + \Delta$, $\Delta > 0$. By definition, the following equation is true, i.e. the low type is indifferent between mimicking or not,

$$\Pi^l_t(0) = \Pi^l_t(1) - \lambda p^l_t \Delta.$$

By Lemma 45 and the fact this is suboptimal,

$$\Pi^h_t(0) < \Pi^h_t(1) - \lambda p^h_t \Delta < \Pi^h_t(q_t).$$

By continuity and Lemma 40, there exists $0 < q' < q_t$ such that $\Pi^h_t(1) - \lambda p^h_t \Delta = \Pi^h_t(q')$. Define

$\Delta' < \Delta$ such that [69]

$$\Pi_t^l(1) - \lambda p_t^l \Delta = \lambda p_t^l \left(V - \frac{c}{\lambda p_t^h} - \Delta' \right) - k + \delta \left(1 - \lambda p_t^l \right) \Pi_{t+1}^l(q').$$

In other words, if the deviating contract is $\alpha_t' = c / \left(\lambda p_t^h\right) + \Delta'$, then $M_t^l\left(\alpha_t'\right) = [q', 1]$. However, the high type is strictly better off after paying additional Δ' instead of Δ. Therefore, $[q', 1] \subset M^h\left(\alpha_t'\right)$. This implies the off-equilibrium belief is $q' = 1$ after observing α_t', then both types will deviate. ■

Corollary 47 *No suboptimal pooling contracts survive D1 criterion.*

Proof. Rewrite the suboptimal pooling contract as $\alpha_t^p + \Delta$, $\Delta > 0$. By definition, the following equation is true:

$$\lambda p_t^l \left(V - \alpha_t^p - \Delta\right) - k + \delta \left(1 - \lambda p_t^l\right) \Pi_{t+1}^l(q_t') = 0.$$

This is because for $q_t' > q_t$ to be true, the low type drops out with non-zero probability. If

$$\Pi_t^h(0) < \lambda p_t^h \left(V - \alpha_t^p - \Delta\right) - k + \delta \left(1 - \lambda p_t^h\right) \Pi_{t+1}^h(q_t') < \Pi_t^h(q_t),$$

then define q' and Δ' similarly as the proof of Lemma 46, and the rest of proof is essentially the same. Alternatively if,

$$\lambda p_t^h \left(V - \alpha_t^p - \Delta\right) - k + \delta \left(1 - \lambda p_t^h\right) \Pi_{t+1}^h(q_t') < \Pi_t^h(0),$$

then both types strictly want to deviate by offering a_t^l even though they are regarded as the low type. Lastly, if

$$\lambda p_t^h \left(V - \alpha_t^p - \Delta\right) - k + \delta \left(1 - \lambda p_t^h\right) \Pi_{t+1}^h(q_t') = \Pi_t^h(0),$$

then players still want to deviate unless $\Pi_t^l(0) = 0$. In the latter case, we have $M^h\left(\alpha_t^l\right) = M^l\left(\alpha_t^l\right) = [0, 1]$. Our tie-breaking rule stipulates the off-equilibrium belief is $q_t' = q_t$ so both types will deviate. ■

Notice the proof of Corollary 47 shows deviating with equity payment exists for suboptimal pooling equilibria. So we can eliminate them regardless of whether pivots are feasible. This concludes that the first statement of Proposition 8. For the second statement, we next show the optimal pooling equilibrium survives the D1 criterion. We express any deviation in the format as $\alpha_t' = a_t^p + \Delta$, satisfying

$$\lambda p_t^l \Delta = \delta \left(1 - \lambda p_t^l\right) \left(\Pi_{t+1}^l(q) - \Pi_{t+1}^l(q_{t+1})\right)$$
$$\implies \frac{\lambda}{1-\lambda} \Delta = \delta \frac{\Pi_{t+1}^l(q) - \Pi_{t+1}^l(q_{t+1})}{p_{t+1}^l}.$$

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$$\frac{\lambda}{1-\lambda}\left(\Delta - \Delta'\right) = \delta \frac{\Pi_{t+1}^l(1) - \Pi_{t+1}^l(q')}{p_{t+1}^l}.$$

The difference is not 0 because suboptimal separation happens before $\underline{\tau_l}$. See Lemma 25.

To understand the first line, the LHS is the expected additional payment, and the RHS is the jump in continuation value if the off-equilibrium belief is $q > q_{t+1}$. In other words, q is the off-equilibrium belief that makes the low type just indifferent and $M^l(\alpha_t') = [q, 1]$. With the same algebra of Lemma 40, for any $q > q_{t+1}$, we have

$$\delta \frac{\Pi_{t+1}^l(q) - \Pi_{t+1}^l(q_{t+1})}{p_{t+1}^l} = \delta \frac{\Pi_{t+1}^h(q) - \Pi_{t+1}^h(q_{t+1})}{p_{t+1}^h}.$$

This implies $M^l(\alpha_t') = M^h(\alpha_t')$ and out tie-breaking rule implies the off-equilibrium belief stays q_t, which is weakly lower than the q_{t+1}. But then both types will become strictly worse off by deviating.

B.9 Proof of Proposition 9

Suppose that such an equilibrium exists, i.e. $C^h \neq C^l$, where both contracts are renegotiation-proof. Then, we have either $q_1 = 1$ (if C^h is offered) or $q_1 = 0$ (if C^l is offered). Given Assumption 1 Ass 1, the only renegotiation-proof contract following belief $q_1 = 1$ is the high type's first-best contract $\bar{C}^h = \left\{\bar{\alpha}_t^h, \bar{I}_t^h\right\}_{t>0}$ and the only renegotiation-proof contract following belief $q_1 = 0$ is the low type's first-best contract $\bar{C}^l = \left\{\bar{\alpha}_t^l, \bar{I}_t^l\right\}_{t>0}$. But then we have

$$\widehat{\Pi}_1^l\left(1, \bar{C}^h\right) \geq \widehat{\Pi}_1^l\left(0, \bar{C}^l\right)$$

and the low type prefers to offer the high type's contract instead. Thus, no such equilibrium can exist.

B.10 Proof of Proposition 10

Fix contracts C^h and C^l. As in Proposition 9, if $C^h \neq C^l$, then we have $q_1 = 1$ if C^h is offered and $q_0 = 0$ if C^l is offered, which by Assumption 1 implies that such a contract cannot be renegotiation proof. Thus, as before, any renegotiation-proof contract must be pooling, i.e. $C^h = C^l = C$. Consider the optimal contract in Proposition 4, i.e. the high type pivots in period τ_P and both types offer α_t^P for $t < \tau_P$ and $\bar{\alpha}_t^h$ (for the high type) and $\bar{\alpha}_t^l$ (for the low type) for $t \geq \tau_P$. Now, define the following long-term contract. For all $t < \tau_P$ and $h^{\theta t}$, offer α_t^P. For $t \geq \tau_P$, offer $\bar{\alpha}_t^h$ if and only if $h^{\theta t}$ is such that $I_s^\theta = 0$ for all $s < t, s \neq \tau_P$ and $I_{\tau_P}^\theta = 1$ and offer $\bar{\alpha}_t^l$ if and only if $h^{\theta t}$ is such that $I_{\tau_P}^\theta = 0$. Assume that both types offer the same contract, and label it C^P.The contract C^P achieves the following. If a pivot occurred at time τ_P, then the contract offers the high type's first-best share. If no pivot occurred at that time, the contract offers the low type's first-best share. Before time τ_P, the contract offers the optimal pooling share. In the equilibrium of Prop. 4, the high type indeed pivots at time τ_P and the low type does not. Thus, the contract C^P implements exactly the same outcomes as the optimal contract in Prop. 4 as a long-term contract.

We now show that there exists a PBE in which offering contract C^P is optimal for both types (given beliefs) and that this contract is renegotiation-proof. Optimality can be achieved by setting $q_1 = 0$ whenever some contract $C \neq C^P$ is offered. Consider an arbitrary time t, history $h^{\theta t}$, and renegotiated contract $\widehat{C}^{\theta|t}$. Set $q_t = 0$ whenever $\widehat{C}^{\theta|t} \neq C^{P\theta|t}$. Now, offering a different contract in renegotiation is equivalent to deviating in the equilibrium in Prop. 4. Since the optimal contract in Prop. 4 is robust to such deviations, it is also robust to renegotiation given our assumption for off-path beliefs. Thus, the contract C^P is renegotiation-proof.

Finally, there exists no other contract which leads to a higher ex-ante utility (given Pareto weight γ) for the entrepreneur, which follows again from the fact that renegotiating the contract C^P is equivalent to deviating in the contract of Prop. 4. If such a contract were to exist, then the optimal contract of Prop. 4 is not optimal, a contradiction.

C Signaling via Cash Burning

We now consider the case with payouts. In reality, many startups are cash constrained and cannot pay investors before an IPO or acquisition occurs. However, the following analysis provides a useful benchmark, by providing another setting in which the optimal contract may be separating. For simplicity, payouts are the only signaling device in this section, i.e. we set $I_t^\theta = 0$ for all t. A contract is now given by $C_t^\theta = (d_t^\theta, \alpha_t^\theta)$, where $d_t^\theta \geq 0$ denotes the payout to investors in period t and α_t^θ denotes the equity share. Investors receive d_t^θ immediately and the share α_t^θ only if the project succeeds in period t.

In period t, the payoffs for the type-θ entrepreneur and the investor are

$$\Pi_t^\theta = E_t^\theta \left[\sum_{s=t}^{\tau} \delta^{s-t} \left(\mathbf{1}_s \left(1 - \alpha_s\right) V - k - d_s \right) \right] \tag{43}$$

and

$$U_t = E_t \left[\sum_{s=t}^{\tau} \delta^{s-t} \left(\mathbf{1}_s \alpha_s V - c + d_s \right) \right]. \tag{44}$$

For $t < \tau$, the entrepreneur's and investor's values can be written recursively as

$$\Pi_t^\theta = \left(1 - l_t^\theta\right) \left(\lambda p_t^\theta \left(1 - \alpha_t\right) V - k - d_t^\theta + \delta \left(1 - \lambda p_t^\theta\right) \Pi_{t+1}^\theta \right) \tag{45}$$

and

$$U_t = \left(1 - l_t\left(q_{t-1}\right)\right) \left(\lambda p_t\left(q_t\right) \alpha_t V - c + q_t d_t^h + (1 - q_t) d_t^l + \delta \left(1 - \lambda p_t\left(q_t\right)\right) U_{t+1} \right), \tag{46}$$

where

$$1 - l_t\left(q_{t-1}\right) = q_{t-1} \left(1 - l_t^h\right) + \left(1 - q_{t-1}\right) \left(1 - l_t^l\right)$$

is the investor's expectation about the entrepreneur's liquidation probability.

In the optimal contract, the high type separates by offering a payout in period τ_S, but separation is inefficiently delayed. Intuitively, the high type prefers to wait until the degree of adverse selection has decreased, since separating earlier is too costly.

Proposition 48 *If q_0 is sufficiently small and γ is sufficiently large, the optimal contract is separating in period τ_S. It consists of a payment $d_{\tau_S}^h$ and the high type's symmetric information share $\bar{\alpha}_{\tau_S}^h$. Before period τ_S, both types offer the pooling contract of Proposition 2. If either q_0 is sufficiently large, or if γ is sufficiently small, pooling is optimal.*

That cash payments can be used to signal follows from the familiar "cash-burning" intuition. After separating, the high type expects to continue the relationship longer than the low type. Thus, her value of separating is higher (see Equation (14)). Therefore, there exists a cash payment which deters the low type but not the high type. In the optimal contract, this cash payment is delayed, since early on the cost of separating is too high.

Here is the intuition for this result. With payouts, the pooling equilibrium of Proposition 2 still exists. Moreover, it is optimal among all pooling equilibria.[70] Suppose that the high type separates in period t by offering a contract $C_t^h = \left(d_t^h, \alpha_t^h\right)$. The relevant IC conditions are

$$\Pi_t^l(0) \geq \lambda p_t^l \left(1 - \alpha_t^h\right) V - d_t^h - k + \delta \left(1 - \lambda p_t^l\right) \Pi_{t+1}^l (1) \qquad (\tilde{IC}_l)$$

for the low type and

$$\Pi_t^h(0) \leq \lambda p_t^h \left(1 - \alpha_t^h\right) V - d_t^h - k + \delta \left(1 - \lambda p_t^h\right) \Pi_{t+1}^h (1) \qquad (\tilde{IC}_h)$$

for the high type. The first condition states that the low type's value from offering C_t^h and imitating the high type must be lower than the value from revealing her type. Specifically, once her type is discovered, the low type offers her symmetric information contract $\bar{C}_t^l = (0, \bar{\alpha}_t^l)$.[71] The second condition states that offering C_t^h must indeed be optimal for the high type. As before, once separated, the continuation values are the symmetric information values $\Pi_{t+1}^h (1)$ and $\Pi_t^l (0)$.

Separating via the equity share is costly for the high type. As we described in Section 5.2, to reduce the low type's payoff by one, the high type gives up a payoff of $p_t^h / p_t^l > 1$. By contrast, if she separates via the payout d_t^h, her cost of reducing the low type's payoff is one. Thus, separating via a payout is cheaper and the equity share is not distorted, i.e. $\alpha_t^h = \bar{\alpha}_t^h$. The low type's IC constraint binds in an optimal contract and the payout reduces to[72]

$$d_t^h = \Pi_t^l(1) - \Pi_t^l(0). \qquad (47)$$

That is, the payment equals the value of imitating for the low type, which is given by the difference in continuation values at beliefs $q_t = 1$ and $q_t = 0$.

Because the cost of separating is smaller, the high type may prefer to separate rather than pooling forever. This is true when pooling is relatively costly, i.e. when q_0 is small. Intuitively, when the investor believes he is unlikely to be facing the high type, i.e. q_0 is small, the pooling equity share α_t^P is large. Then, the high type must give up a large portion of the project when she continues pooling and her value of separating is relatively large. For sufficiently small q_0, there exists periods in which the value of separating value outweighs the cost.

In equilibrium, separation is inefficiently delayed. The high type prefers to separate whenever the loss from pooling exceeds the cost of separating d_t^h. As time passes, the pooling equilibrium becomes progressively worse, and, compared to separating, the high type must pledge successively larger shares.[73] Simultaneously, the cost of separating d_t^h decreases, because the low type's project

[70]Intuitively, any pooling equilibrium in which there are positive payouts $d_t^P > 0$ leaves rents to the investor and can be improved upon by setting the payouts to zero.

[71]This is straightforward. If the low type were to offer any other contract with $\alpha_t^l > \bar{\alpha}_t^l$ or $d_t^l > 0$ which reveals her type, that contract would be suboptimal.

[72]This follows from plugging $\alpha_t^h = \bar{\alpha}_t^h$ into Equation ($\tilde{IC}_l$).

[73]The ratio $\alpha_t^P / \bar{\alpha}_t^h$ is monotonically increasing, which implies that the high type's "adverse selection

becomes less likely to succeed. After sufficient time has passed, the high type prefers to separate.

The low type, by contrast, always prefers to pool until the project is liquidated. Whenever γ, the weight on the high type's payoff, is small, the optimal contract is pooling, while when the weight is large, it is separating.

When the high type separates, she pays the investor, and simultaneously reduces the investor's equity share and increases her own. This most closely resembles a management buyout. That is, the entrepreneur pays the investor to repurchase a fraction of his shares, so that her own share increases and the investor's share declines. While such buyouts occur in reality, they do so for a minority of startups, most often in later stages.[74]

C.1 Proof of Proposition 48

Combining the incentive constraints above yields the necessary condition

$$d_t^h \in \left[\Pi_t^l(1) - \Pi_t^l(0), \Pi_t^h(1) - \Pi_t^h(0)\right]. \tag{48}$$

In Lemma 25, we have shown that the entrepreneur's values satisfy the single-crossing condition

$$\frac{\Pi_t^h(1) - \Pi_t^h(0)}{p_t^h} \geq \frac{\Pi_t^l(1) - \Pi_t^l(0)}{p_t^l}.$$

Since $p_t^h > p_t^l$, Equation (14) implies the set in Equation (48) is non-empty. Thus, for any $t < \underline{\tau}^l$, there exists an equilibrium in which the high type separates in period t.

As mentioned in the text, the optimal separating contract must be the one with the lowest cost, i.e.,

$$d_t^h = \Pi_t^l(1) - \Pi_t^l(0).$$

Next, we establish that the pooling equilibrium constructed in Proposition 2 is suboptimal compared to separating in some period τ_S. For $t < \underline{\tau}^l$, any optimal pooling equilibrium must feature $d_t^P = 0$, otherwise, both types could improve by offering no payouts.[75] Thus, the equilibrium of Proposition 2 remains the optimal pooling equilibria. We first show that the high type prefers to separate rather than continue pooling for any $t \geq \tau_S$.

Lemma 49 *For any $t < \underline{\tau}^l$, type h prefers to separate with payouts in period t instead of pooling in period t and separating in period $t+1$ if and only if*

$$f_t(q_0) = c\left(\frac{(1-q_0)\,p_t^h}{q_0 p_t^h + (1-q_0)\,p_t^l} - 1\right) - \delta\lambda p_t^h \left(\Pi_{t+1}^l(1) - \Pi_{t+1}^l(0)\right) > 0.$$

Proof. Using Equation (47) and the fact that $\alpha_t^h = \bar{\alpha}_t^h$, if the high type separates in period t, her

discount" becomes progressively worse.

[74]See e.g. https://www.wsj.com/articles/ditch-the-venture-model-say-founders-who-buy-out-early-investors-to-make-a-clear-break-1531827001 for a prominent case.

[75]Recall that the equity share α_t^P provides all incentives to the investor while d_t^P simply serves as a transfer.

value is

$$\Pi_t^h = \lambda p_t^h \left(V - \frac{c}{\lambda p_t^h}\right) - k + \delta\left(1 - \lambda p_t^h\right)\Pi_{t+1}^h(1) - \left(\Pi_t^l(1) - \Pi_t^l(0)\right),$$

while if she pools in period t and separates in period $t+1$, her value is

$$\Pi_t^{h\prime} = \lambda p_t^h \left(V - \frac{c}{\lambda p_t(q_0)}\right) - k + \delta\left(1 - \lambda p_t^h\right)\left(\Pi_{t+1}^h(1) - \left(\Pi_{t+1}^l(1) - \Pi_{t+1}^l(0)\right)\right).$$

In the second case, type h offers the optimal pooling contract a_t^p before separation. By construction, no liquidation occurs before τ^l in the pooling equilibrium and therefore $q_t = q_0$. She prefers to separate earlier if and only if

$$\Pi_t^h - \Pi_t^{h\prime} = \lambda p_t^h \left(\frac{c}{\lambda p_t(q_0)} - \frac{c}{\lambda p_t^h}\right) - \left(\Pi_t^l(1) - \Pi_t^l(0)\right) + \delta\left(1 - \lambda p_t^h\right)\left(\Pi_{t+1}^l(1) - \Pi_{t+1}^l(0)\right)$$

is positive. Using the fact that

$$\Pi_t^l(1) - \Pi_t^l(0) = \lambda p_t^l \left(\frac{c}{\lambda p_t^l} - \frac{c}{\lambda p_t^h}\right) + \delta\left(1 - \lambda p_t^l\right)\left(\Pi_{t+1}^l(1) - \Pi_{t+1}^l(0)\right),$$

we have, after some algebra,

$$\frac{p_t^h}{p_t^h - p_t^l}\left(\Pi_t^h - \Pi_t^{h\prime}\right) = c\left(\frac{(1-q_0)\,p_t^h}{q_0 p_t^h + (1-q_0)\,p_t^l} - 1\right) - \delta\lambda p_t^h\left(\Pi_{t+1}^l(1) - \Pi_{t+1}^l(0)\right) = f_t(q_0). \quad (49)$$

Since $p_t^h > p_t^l$, $\Pi_t^h - \Pi_t^{h\prime}$ is positive if and only if $f_t(q_0)$ is positive. ■

The next lemma establishes the monotonicity of $f_t(\cdot)$ given the initial belief q_0.

Lemma 50 *Given q_0, f_t strictly increases in $1 \le t < \underline{\tau}^l$, i.e.,*

$$f_1(q_0) < f_2(q_0) < \ldots < f_{\underline{\tau}^l - 1}(q_0).$$

Proof. We first show that p_t^l/p_t^h decreases in t. This is because

$$\frac{p_{t+1}^l}{p_{t+1}^h} = \frac{p_t^l}{p_t^h}\frac{1-\lambda p_t^h}{1-\lambda p_t^l} < \frac{p_t^l}{p_t^h}.$$

This implies

$$\frac{1-q_0}{q_0 + (1-q_0)\frac{p_t^l}{p_t^h}} < \frac{1-q_0}{q_0 + (1-q_0)\frac{p_{t+1}^l}{p_{t+1}^h}},$$

which is equivalent to

$$\frac{(1-q_0)\,p_t^h}{q_0 p_t^h + (1-q_0)\,p_t^l} < \frac{(1-q_0)\,p_{t+1}^h}{q_0 p_{t+1}^h + (1-q_0)\,p_{t+1}^l}.$$

For the second part, we first generate an upper bound of $\delta\left(\Pi_t^l(1) - \Pi_t^l(0)\right)$. To start, notice that for $t < \underline{\tau}^l$,

$$\frac{c}{\lambda p_{t+1}^l} - \frac{c}{\lambda p_{t+1}^h} = \frac{c\left(1-\lambda p_t^h\right)}{\left(1-\lambda\right)p_t^h} - \frac{c\left(1-\lambda p_t^l\right)}{\left(1-\lambda\right)p_t^l} = \frac{1}{1-\lambda}\left(\frac{c}{\lambda p_t^l} - \frac{c}{\lambda p_t^h}\right),$$

and for $t \geq \underline{\tau}^l$, we have

$$\lambda p_t^l\left(V - \frac{c}{\lambda p_t^l}\right) - k \leq 0 = \lambda p_t^l\left(\frac{c}{\lambda p_t^h} - \frac{c}{\lambda p_t^h}\right),$$

since type l liquidates when she reveals her type, so that

$$V - \frac{c}{\lambda p_t^h} - \frac{k}{\lambda p_t^l} \leq \frac{c}{\lambda p_t^l} - \frac{c}{\lambda p_t^h}.$$

Therefore,[76]

$$\begin{aligned}
& \delta\left(\Pi_t^l(1) - \Pi_t^l(0)\right) \\
= \;& \delta\lambda p_t^l\left(\sum_{s=t}^{\underline{\tau}^l-1}\left(\delta\left(1-\lambda\right)\right)^{s-t}\left(\frac{c}{\lambda p_s^l} - \frac{c}{\lambda p_s^h}\right)\right. \\
& \left. + \sum_{s=\underline{\tau}^l}^{\tau^{l\prime}-1}\left(\delta(1-\lambda)\right)^{s-t}\left(V - \frac{c}{\lambda p_s^h} - \frac{k}{\lambda p_s^l}\right)\right) \\
\leq \;& \delta\lambda p_t^l\left(\frac{c}{\lambda p_t^l} - \frac{c}{\lambda p_t^h}\right)\left(\sum_{s=t}^{\tau^{l\prime}-1}\delta^{s-t}\right) \\
\leq \;& \frac{\delta}{1-\delta}\lambda p_t^l\left(\frac{c}{\lambda p_t^l} - \frac{c}{\lambda p_t^h}\right).
\end{aligned}$$

Second, using a similar derivation, if $\underline{\tau}^l < \tau^{l\prime}$ or if $t < \underline{\tau}^l - 1$, we have

$$\begin{aligned}
& \delta p_t^h\left(\Pi_t^l(1) - \Pi_t^l(0)\right) - \delta p_{t+1}^h\left(\Pi_{t+1}^l(1) - \Pi_{t+1}^l(0)\right) \\
= \;& \delta\lambda p_t^h p_t^l\left(\frac{c}{\lambda p_t^l} - \frac{c}{\lambda p_t^h}\right) - \left(1 - \delta\frac{\left(1-\lambda p_t^h\right)\left(1-\lambda p_t^l\right)}{1-\lambda}\right)\delta p_{t+1}^h\left(\Pi_{t+1}^l(1) - \Pi_{t+1}^l(0)\right) \\
\geq \;& \delta\lambda p_t^h p_t^l\left(\frac{c}{\lambda p_t^l} - \frac{c}{\lambda p_t^h}\right) - \left(1 - \delta\frac{\left(1-\lambda p_t^h\right)\left(1-\lambda p_t^l\right)}{1-\lambda}\right)\delta\lambda p_{t+1}^h p_{t+1}^l\left(\frac{c}{\lambda p_{t+1}^l} - \frac{c}{\lambda p_{t+1}^h}\right) \\
= \;& \frac{\delta}{1-\delta}\lambda p_t^h p_t^l\left(\frac{c}{\lambda p_t^l} - \frac{c}{\lambda p_t^h}\right)\frac{1-\lambda}{\left(1-\lambda p_t^h\right)\left(1-\lambda p_t^l\right)}\left(\frac{\left(1-\lambda p_t^h\right)\left(1-\lambda p_t^l\right)}{1-\lambda} - 1\right) \\
> \;& 0.
\end{aligned}$$

The inequality comes from Assumption 3. If $\underline{\tau}^l = \tau^{l\prime}$ and $t = \underline{\tau}^l - 1$, then $\Pi_{t+1}^l(1) - \Pi_{t+1}^l(0) = 0$

[76] If $\underline{\tau}^l > \tau^{l\prime} - 1$, we use the convention $\sum_{s=\underline{\tau}^l}^{\tau^{l\prime}-1}(...) = 0$.

and the expression is still positive. Together with the previous result, this generates the Lemma statement. ■

Next, we will characterize τ_S for two cases: $\underline{\tau}^l = \tau^{l\prime}$ and $\underline{\tau}^l < \tau^{l\prime}$.

Lemma 51 *If $\underline{\tau}^l = \tau^{l\prime}$, there exists a threshold $\bar{q}$ such that the high type always prefers to pool if and only if $q_0 \geq \bar{q}$.*

Proof. If $\underline{\tau}^l = \tau^{l\prime}$, then

$$f_{\underline{\tau}^l-1}(q_0) = c\left(\frac{(1-q_0)\,p^h_{\underline{\tau}^l-1}}{q_0 p^h_{\underline{\tau}^l-1} + (1-q_0)\,p^l_{\underline{\tau}^l-1}} - 1\right).$$

If $q_0 > 0.5$, the above expression is strictly negative. By continuity, there exists a $\bar{q} < 0.5$ such that $f_{\underline{\tau}^l-1}(q_0) \leq 0$ if and only if $q_0 \geq \bar{q}$. Since $f_t(q_0)$ is strictly increasing in t, we have $f_t(q_0) < 0$ for all $t < \underline{\tau}^l - 1$. Thus, for any equilibrium with separation in period $t \leq \underline{\tau}^l - 1$, we can increase the high type's payoff by separating in period $t+1$ instead. Applying the argument inductively implies that the high type prefers to never separate. ■

Corollary 52 *If $\underline{\tau}^l = \tau^{l\prime}$ and $q_0 \geq \bar{q}$, the optimal contract is pooling.*

Lemma 53 *If $\underline{\tau}^l = \tau^{l\prime}$ and $q_0 < \bar{q}$, we have $1 \leq \tau_S \leq \underline{\tau}^l - 1$.*

Proof. Define $\tau_S = \min\{t | f_t(q_0) \geq 0\}$. Since $q_0 < \bar{q}$, this set is non-empty, and we have $\tau_S \leq \underline{\tau}^l - 1$. ■

Now, consider the case $\underline{\tau}^l < \tau^{l\prime}$. In Section B.2, we have shown that separating through α^h_t has the same payoff as the pooling equilibrium for the high type when $\underline{\tau}^l \leq t < \tau^{l\prime}$. Since separating through payouts is cheaper, it is now preferred by the high type.

Lemma 54 *If $\underline{\tau}^l < \tau^{l\prime}$, the high type strictly prefers separating in period $\underline{\tau}^l$ and we have $\tau_S \leq \underline{\tau}^l$.*

Proof. For all $t \geq \underline{\tau}^l$, we can replicate the argument of Lemma 49. The difference is that separating in period $t+1$ has the following payoff structure, due to the change of α^p_t:

$$\lambda p^h_t\left(V - \frac{\lambda p^l_t V - k}{\lambda p^l_t}\right) - k + \delta\left(1 - \lambda p^h_t\right)\left(\Pi^h_{t+1}(1) - \left(\Pi^l_{t+1}(1) - \Pi^l_{t+1}(0)\right)\right).$$

Besides,

$$\Pi^l_t(1) - \Pi^l_t(0) = \lambda p^l_t\left(V - \frac{c}{\lambda p^h_t}\right) - k + \delta\left(1 - \lambda p^l_t\right)\left(\Pi^l_{t+1}(1) - \Pi^l_{t+1}(0)\right).$$

The difference in value for type h from separating in period t vs. period $t+1$ is

$$\Pi^{h\prime}_t - \Pi^h_t = \frac{\lambda\left(p^h_t - p^l_t\right)}{\lambda p^l_t}\left(\lambda p^l_t\left(V - \frac{c}{\lambda p^h_t}\right) - k\right) - \delta\lambda\left(p^h_t - p^l_t\right)\left(\Pi^l_{t+1}(1) - \Pi^l_{t+1}(0)\right).$$

Using the same argument as in Lemma 50 , we can show

$$\Pi_{t+1}^{l}(1)-\Pi_{t+1}^{l}(0)<\frac{1}{\lambda}\left(\lambda p_{t+1}^{l}\left(V-\frac{c}{\lambda p_{t+1}^{h}}\right)-k\right)<\frac{1}{\lambda p_{t}^{l}}\left(\lambda p_{t}^{l}\left(V-\frac{c}{\lambda p_{t}^{h}}\right)-k\right).$$

Therefore, $\Pi_t^{h\prime}-\Pi_t^h$ is strictly positive for all $t \geq \underline{\tau}^l$. Define $\tau_S = \min\{t|f_t(q_0) \geq 0, t \leq \underline{\tau}^l-1\} \cup \{\underline{\tau}^l\}$, then the high type will optimal separate at τ_S. ■

Next, we provide a sufficient condition for when $\underline{\tau}^l = \tau^{l\prime}$.

Lemma 55 *We have $\underline{\tau}^l = \tau^{l\prime}$ if and only if q_0 exceeds a threshold $\bar{q}'$.*

Proof. Given p_1^l and p_1^h, by Proposition 2, $\underline{\tau}^l$ is the first period when

$$\lambda p_t^l\left(V-\frac{c}{\lambda p_t(q_0)}\right)-k \leq 0.$$

Since $p_t(q_0)$ increases in q_0 and $\lim_{q_0 \to 1} p_t(q_0) = p_t^h$, $\underline{\tau}^l$ increases in q_0 and converges to $\tau^{l\prime}$. By continuity, there exists $\bar{q}'$ such that $\underline{\tau}^l = \tau^{l\prime}$ for all $q_0 > \bar{q}'$. ■

Combining Lemma 51 and 55, if $q_0 > \max\{\bar{q}, \bar{q}'\}$, the high type optimally chooses to pool. The low type, of course, prefers to never separate. The optimal equilibrium is pooling regardless of γ. This equilibrium can also be interpreted as "separating" at $\bar{\tau}^l$. In that period, the low type liquidates with certainty in the pooling equilibrium, so pooling and separating are equivalent. If $q_0 < \max\{\bar{q}, \bar{q}'\}$, there exists an optimal timing τ_S when the high type prefers to separate. Now we are optimizing over the finite set $\left\{\tau_S, \bar{\tau}^l\right\}$, an optimal separating time exists. Clearly, whenever γ is sufficiently small, pooling is optimal while whenever γ is sufficiently large, the optimal separating time satisfies either $f_{\tau_S}(q_0) > 0$ or $\tau_S = \underline{\tau}^l$.

Corollary 56 *The separation time in equilibrium weakly increases in q_0.*

Proof. Denote $\underline{\tau}^l$ as a function of q_0 by $\underline{\tau}^l(q_0)$. Then $\underline{\tau}^l(q_0)$ is a weakly increasing step function. We start from $q_0 < \min\{\bar{q}, \bar{q}'\}$. Notice $f_t(q_0)$ strictly decreases in q_0 for all $t \leq \tau^{l\prime} - 1$, therefore the first time when $f_t(q_0)$ crosses 0 from below weakly delays. If $f_t(q_0) < 0$ for all $t \leq \underline{\tau}^l(q_0) - 1$, then the separating time is $\underline{\tau}^l(q_0)$, which still weakly increases in q_0. This process continues until $q_0 = \bar{q}'$ and $\underline{\tau}^l(q_0) = \tau^{l\prime}$, then the separation timing weakly increases from $\tau^{l\prime} - 1$ to $\tau^{l\prime}$, depending on the relationship between $\bar{q}$ and $\bar{q}'$. ■