

Behavioral Foundations of Nested Stochastic Choice and Nested Logit *

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Abstract

We provide the first behavioral characterization of **nested logit**, a foundational and widely applied model in discrete choice, through the introduction of a non-parametric version of nested logit that we call **Nested Stochastic Choice** (NSC). NSC is characterized by a single axiom that weakens **Independence of Irrelevant Alternatives** based on revealed similarity to allow for the **similarity effect**. Nested logit is characterized by an additional menu-independence axiom. Our axiomatic characterization allows us to derive a practical, data-driven algorithm that identifies the true nest structure from choice data. We also discuss limitations of generalizing nested logit by studying testable implications of cross-nested logit.

Keywords: Nested Logit; Nested Stochastic Choice; Luce Model; IIA; Similarity Effect; Regularity; Revealed Similarity; Cross-Nested Logit; Nest Identification.

JEL: D01, D81, D9.

1 Introduction

Nested logit (Ben-Akiva (1973), McFadden (1978)) is the most widely applied generalization of multinomial logit (or Luce’s (1959) model) due to its ability to capture various substitution patterns.¹ In nested logit, each alternative belongs to a nest (or subset) of “similar” alternatives, and choice may be decomposed into two Luce procedures: the probability that a is chosen from menu A is the probability that a ’s nest is chosen from among nests available in A multiplied

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¹Nested logit has been used to study transportation demand (Anderson and De Palma (1992), Forinash and Koppelman (1993)), airline competition (Lurkin, Garrow, Higgins, Newman, and Schyns (2018)), automobile demand (Brownstone and Small (1989), Goldberg (1995)), telephone use (Train, McFadden, and Ben-Akiva (1987), Lee (1999)), and much more. See Chapter 4 of Train (2009) for an excellent discussion.

by the conditional probability that a is chosen from that nest. Despite its immense importance, nested logit has escaped behavioral characterization. In this paper, we provide the first behavioral characterization of nested logit through the introduction of a fully non-parametric version that we term **Nested Stochastic Choice** (NSC). This axiomatic characterization sheds light on the implicit assumptions behind nested logit and related models while also inspiring a tractable method to identify *subjective* nests from data.

Nested logit was developed to address the limitations of multinomial logit when dealing with “similar alternatives.” In the Luce model, choice probabilities are proportional to a utility index and hence satisfy **Independence of Irrelevant Alternatives** (IIA): probability ratios are menu-independent. The **similarity effect** (Debreu (1960), Tversky (1972)) is a violation of IIA in which the introduction of an alternative to a menu has a much larger effect on the choice probabilities of alternatives of a “similar type” than on those of a “different type.” Nested logit allows the similarity effect by assuming nested (similar) alternatives are more substitutable (e.g., because they receive a correlated utility shock). Our main result shows that nesting of similar alternatives, a key behavioral feature of nested logit (and also NSC), is captured by a weakening of IIA that allows for the similarity effect. This finding reveals a deep connection between nested logit and the similarity effect.

To illustrate the similarity effect, consider the red bus/blue bus example from Debreu (1960). A commuter making a choice between a red bus and a train may choose either with probability 0.5. If the option to take a blue bus is introduced, it is plausible that it will have no effect on the commuter’s likelihood of taking the train; the blue bus only affects the probability of selecting the red bus (e.g., by reducing it to 0.25). The intuition behind this is that the buses are similar to each other in a way that neither one is to the train. Nested logit handles this by placing the two buses into a bus nest (buses receive a correlated utility shock). While this example provides an extreme case of the similarity effect (the buses are perfect substitutes, or “duplicates”), the principle that “similar alternatives affect each other” readily extends to many situations of interest to economists, firms, and policy-makers, such as a consumer’s choice of vehicle or apartment.²

Our analysis of nested logit relies on the introduction of NSC, a non-parametric version of nested logit. Formally, a stochastic choice function p is an NSC if there exist nests X_1, \dots, X_K that partition the set of all alternatives X and functions v and u such that, for each choice set A , the probability of choosing $a \in A \cap X_i$ is given by

$$(1) \quad p(a, A) = \frac{v(A \cap X_i)}{\sum_{j=1}^K v(A \cap X_j)} \frac{u(a)}{\sum_{b \in A \cap X_i} u(b)}.$$

The NSC is defined by two Luce rules where the attractiveness of the nest $A \cap X_i$ is measured by $v(A \cap X_i)$, and the attractiveness of the alternative a is measured by $u(a)$ (or simply Luce’s utility of a). In terms of the red bus/blue bus example, v governs the choice between general modes, “a

²In these cases, the correct nest specification is not easily observed by the analyst. For instance, apartments in a city might be nested based on subjective neighborhoods, which may depend on a variety of factors, some of which may be observable and others may be subjective or difficult to observe.

bus” or “a train,” while u governs the choice between specific alternatives, the red or blue bus.

Notice that nested logit is the special case of NSC in which, for each $i \leq K$,

$$(2) \quad v(A \cap X_i) = \left(\sum_{a \in A \cap X_i} u(a) \right)^{\eta_i} \text{ for some } \eta_i > 0.$$

Hence, NSC is nested logit without any assumptions on the relationship between v and u . It is commonly assumed in the applied literature that $\eta_i \leq 1$, as this ensures that nested logit is a GEV (McFadden, 1978), and therefore this restriction is sufficient for consistency with the random utility model (RUM). However, this restriction ($\eta_i \leq 1$) is not necessary for Equation 2 to be a RUM and nested logit has been estimated without this restriction.³ Therefore, for simplicity of exposition we will refer to Equation 2 as a nested logit for any parameter value. To provide further clarity, we may sometimes refer to nested logit satisfying the restriction ($\eta_i \leq 1$) as the *random utility (RU) nested logit*. We provide behavioral foundations for NSC and nested logit, along with a characterization of random utility nested logit.

We utilize a revealed preference approach to identify the subjective/endogenous nest structure of the NSC. This is achieved by introducing a notion of revealed similarity. In nested logit, if two alternatives are in the same nest, then their probability ratio will always be independent of other alternatives. Consistent with nested logit, we use this insight to define a notion of similarity: alternatives a and b are **revealed categorically similar**, denoted $a \sim_p b$, if IIA holds between a and b at any menu. Otherwise, they are *revealed categorically dissimilar*. Therefore, the core notion of similarity underpinning nested logit is binary: alternatives are similar or not.

Equipped with this notion of revealed similarity, we can weaken IIA to allow for the similarity effect. To do so, we decompose IIA into two axioms. The first axiom, **Independence of Symmetric Alternatives** (ISA), imposes an IIA condition between a and b in the presence of a third alternative x , when x is symmetrically related to a and b in terms of revealed similarity (i.e., both are revealed categorically similar or dissimilar to x). In terms of the red bus/blue bus example, IIA should hold between the buses in the presence of the train. The second axiom, **Independence of Asymmetric Alternatives** (IAA), “completes” ISA by imposing an IIA condition when x is asymmetrically similar to a and b (i.e., x is similar to one and not the other). For example, IAA implies that the introduction of the blue bus impacts the red bus and the train equally, which directly rules out the similarity effect.

Our main result is that ISA characterizes NSC (Theorem 1). Further, since ISA is the minimal departure from IIA that allows for the similarity effect (i.e., violations of IAA), our analysis reveals

³This parameter restriction is sometimes referred to as the Daly-Zachary-McFadden condition (Daly and Zachary (1978), McFadden (1978)), as they showed that this is sufficient for consistency with RUM for arbitrary values of the other variables (e.g., utilities). However, this restriction is not always imposed. For instance, Train, McFadden, and Ben-Akiva (1987) provide estimates of a model for which $\eta_i > 1$, remarking that it represents greater substitutability between nests than within nests (see also Train, Ben-Akiva, and Atherton (1989), Lee (1999), Foubert and Gijsbrechts (2007)). Further, nested logit with $\eta_i > 1$ may still be consistent with RUM (see Börsch-Supan (1990) and Herriges and Kling (1996)).

that NSC is the model obtained when IIA is relaxed to allow for the similarity effect.

To see how our axiomatization provides a clearer picture of nested logit, recall its standard textbook description. For instance, Chapter 4 of [Train \(2009\)](#) states that for nested logit “IIA holds over alternatives in each nest and independence of irrelevant nests (IIN) holds over alternatives in different nests.” NSC also satisfies these properties. Indeed, ISA ensures the existence of endogenous nests and imposes exactly these properties on them. But this finding shows that “IIA and IIN” are not sufficient for a nested logit representation, as ISA is equivalent to NSC. Since there are missing behavioral assumptions behind nested logit, the textbook description is incomplete.

To provide a complete picture of nested logit, we establish two characterizations of nested logit as well as a characterization of random utility nested logit. Our first characterization shows that an NSC is a nested logit if and only if it satisfies **Log Ratio Invariance**. This axiom requires that the natural logarithm of certain probability ratios featuring collections of similar alternatives is menu-independent. The explicit use of a functional form in the axiom allows us to establish necessary and sufficient conditions for the functional form assumed in nested logit with finite data.

Our second characterization is based upon a novel monotonicity condition, **Relative Likelihood Independence**, that is necessary for nested logit and becomes sufficient under a mild richness assumption. To understand this axiom, note that nested logit ([Equation 2](#)) requires that the attractiveness of a nest is increasing in the sum of utilities. **Relative Likelihood Independence** implies that this feature must hold in a relative sense; the attractiveness of a nest *relative to another nest* is increasing in the sum of utilities, holding the alternatives in the other nest fixed. Finally, random utility nested logit is characterized by one additional axiom, **Regularity**, a well-known monotonicity property that all random utility models must satisfy (under the same richness assumption). We summarize all of our characterization results in [Table 1](#).

| | | |
|-----------------|-----------------------------|--|
| NSC | \Leftrightarrow ISA | |
| Luce | \Leftrightarrow ISA + IAA | \Leftrightarrow IIA |
| Nested Logit | \Leftrightarrow ISA + LRI | \Leftrightarrow ISA + RLI (Richness) |
| RU Nested Logit | \Leftrightarrow | Nested Logit + Regularity (Richness) |

Table 1: Summary Of Characterization Results.

Independence of Symmetric Alternatives implies that the revealed similarity relation \sim_p is transitive, which ensures that the nests form a partition. Hence, our axiomatic characterizations show that the notion of (categorical) similarity in nested logit, as well as in NSC, is quite structured. In some applications, an analyst may want to allow for more flexible forms of substitutability. For this reason, **cross-nested logit**, a generalization of nested logit, has been proposed and widely applied in empirical work (see [Vovsha \(1997\)](#), [Ben-Akiva and Bierlaire \(1999\)](#)). The main difference between nested logit and cross-nested logit is that alternatives may belong to several nests in cross-nested logit. Since we are focused on the problem of recovering the endogenous nesting structure, we consider a generalization of cross-nested logit that relaxes the typical parameter restriction, and refer to this as the *unrestricted cross-nested logit*. We show that the unrestricted cross-nested

logit does not have testable implications. Therefore, our results reveal the trade-off between nested logit and cross-nested models: relaxing the partition structure of nested logit results in an overly permissive model. In other words, the behavioral content of cross-nested logit is essentially driven by the analyst’s assumption of the nest structure and parameter restrictions.

In practice, p is estimated from observed choices and “IIA like” conditions never hold exactly. However, we show that the true nest structure can still be identified, for any NSC, by solving a minimization problem. In particular, our axiomatic characterization allows us to derive a “distance” function D that measures, for a given nest structure, the degree of violations of IIA within and across nests for a given set of observations. When the data are close to the true (or theoretical) p , the true nest structure will be the unique minimizer of D . In applied settings where the researcher has several nest structures in mind, D may also be useful as a selection criteria.

Because the number of possible nests grows so rapidly in comparison to the number of alternatives, the full minimization problem may become intractable quickly. However, this issue can be managed due to insights from our similarity relation; one only needs to check nest structures that are consistent with an empirical approximation of \sim_p . In fact, we show that there are at most $|X|$ potential nests that we need to check, where $|X|$ is the number of alternatives. We illustrate our theoretical finding and our data-driven algorithm to reduce the number number of candidate nests with a simulation exercise.

The rest of the paper is organized as follows. In [section 2](#), we discuss setup and notation as well as define NSC and nested logit. In [section 3](#), we define revealed similarity and the similarity effect (3.1) before characterizing NSC (3.2) and nested logit (3.3). We discuss ways of extending our notion of similarity and the testable implications of unrestricted cross-nested logit in [section 4](#). The identification of nest structure from choice data is presented in [section 5](#). We conclude with a discussion of related literature in [section 6](#). We also discuss the relationship between the similarity effect and regularity in [Appendix B.1](#).

2 Nested Stochastic Choice and Nested Logit

All of our models are developed in the standard stochastic choice setup. Accordingly, let X be a finite set of alternatives and \mathcal{A} be the collection of all nonempty subsets of X (menus). Let \mathbb{R}_+ (\mathbb{R}_{++}) denote the non-negative (positive) real numbers.

Definition 1. A function $p : X \times \mathcal{A} \rightarrow [0, 1]$ is a **stochastic choice function** if for any $A \in \mathcal{A}$, $\sum_{a \in A} p(a, A) = 1$ and $p(x, A) = 0$ when $x \notin A$. In some instances, we may write $p(B, A) = \sum_{b \in B} p(b, A)$ for $B \in \mathcal{A}$.

Throughout this paper, we assume that p is *positive*; i.e., $p(a, A) > 0$ for all $A \in \mathcal{A}$ and $a \in A$. For notational simplicity, we write $A \cup x$ instead of $A \cup \{x\}$.

The Luce model is the most widely-known and influential stochastic choice model. In this model, choice probabilities are proportional to a utility index. In his seminal paper, [Luce \(1959\)](#)

proves that a stochastic choice function can be represented by the Luce model if and only if it satisfies IIA for every pair of alternatives.

Definition 2 (IIA). A stochastic choice function p **satisfies IIA at** $a, b \in X$ if, for any $A \in \mathcal{A}$ with $a, b \in A$,

$$\frac{p(a, A)}{p(b, A)} = \frac{p(a, \{a, b\})}{p(b, \{a, b\})}.$$

Further, we say that p **satisfies IIA** if p satisfies IIA at any $a, b \in X$.

It is well known that IIA may fail when similar alternatives are added to the menu, as was illustrated by Debreu’s (1960) famous “red bus/blue bus” example. In this example a commuter is initially equally likely to take the (red) bus or the train. After the city introduces a second (blue) bus, the commuter’s likelihood for taking the train is unchanged and the two buses now split the remaining probability. This particular violation of IIA is an example of the similarity effect (Tversky, 1972) — introducing the blue bus takes more from the red bus than from the train since the busses are similar.

The nested logit is the most commonly applied generalization of Luce’s model and was developed to accommodate violations of IIA like the similarity effect. We now formally define nested logit and the NSC, which is a novel, non-parametric version of nested logit.

Definition 3. A stochastic choice function p is a **Nested Stochastic Choice** (NSC) if there exist a partition X_1, \dots, X_K of X , a utility function $u : X \rightarrow \mathbb{R}_{++}$, and a nest utility function $v : \bigcup_{i=1}^K 2^{X_i} \rightarrow \mathbb{R}_+$ with $v(\emptyset) = 0$ such that for any $A \in \mathcal{A}$ and $a \in A \cap X_i$,

$$(3) \quad p(a, A) = \frac{v(A \cap X_i)}{\sum_{j=1}^K v(A \cap X_j)} \frac{u(a)}{\sum_{b \in A \cap X_i} u(b)}.$$

Moreover, p is a **nested logit** if there exist real numbers $\eta_1, \dots, \eta_n > 0$ such that for any $A \in \mathcal{A}$ and $i \leq K$,

$$(4) \quad v(A \cap X_i) = \left(\sum_{x \in A \cap X_i} u(x) \right)^{\eta_i}.$$

Finally, we say that p is a **random utility (RU) nested logit** when $\eta_i \leq 1$ for each $i \leq K$.

The NSC is defined by two Luce procedures, where v governs the choice over nests (e.g., transportation modes or neighborhoods) and u governs the choice over the particular alternatives in the selected nest (e.g., the red/blue bus or a specific apartment). Note that the nest value function v is not necessarily related to alternative utilities u , which enables the NSC to capture rich behavior (see 3.4). Despite this generality, the NSC may be falsified with relatively few observations. This is because behavior is disciplined by u and the partition structure of the nests, both of which are menu-independent.⁴

⁴It is straightforward to derive from the representation that for any $A \subseteq X$ with $|A| = 3$, there is a distinct pair

The nested logit imposes a specific parametric relationship between v and u . Notice that the NSC, and consequently nested logit, reduces to the Luce model when there is a single nest. Additionally, it is simple to see from [Definition 3](#) that any NSC satisfies (i) IIA within a nest and (ii) an “IIA like” property across nests that is often referred to as **Independence of Irrelevant Nests** (IIN).⁵ These properties are often taken as the hallmark of nested logit, yet they apply to all NSC (with endogenous nests). Since NSC permits behavior that nested logit does not (three examples are discussed in [section 3.4](#)), this means that there are additional behavioral assumptions underpinning nested logit. We elucidate these assumptions in [section 3.3](#).

In nested logit, $1 - \eta_i$ is usually considered a measure of correlation or substitutability between alternatives in nest i . When $\eta_i < 1$ alternatives within the same nest are substitutes. Further, it is well known that when $\eta_i < 1$, the nested logit is always a RUM for any profile of utilities.

When $\eta_i > 1$, choice frequencies may (but do not always) violate *regularity*, a necessary property of every random utility model (RUM) which states that the probability of choosing some alternative must never increase as the menu expands.⁶ Behaviorally, we can interpret $\eta_i > 1$ as indicating complementarities among alternatives.⁷ In certain contexts, we may even anticipate $\eta_i > 1$. For instance, [Foubert and Gijsbrechts \(2007\)](#) study the effects of “product bundling” and find a parameter greater than one, consistent with the effectiveness of bundling.⁸ As NSC allows for violations of regularity, formally defined below, the NSC is not nested by RUM.

Definition 4 (Regularity). A stochastic choice function p satisfies **regularity** if $p(x, A \cup y) \leq p(x, A)$ for any $A \in \mathcal{A}$ and $x, y \in X$ with $x \in A$.

Finally, note that when $\eta_i = 1$ the nest value is exactly proportional to the sum of Luce utilities. If this proportionality happens for every nest i , the model reduces to Luce. In fact, in this case the Luce model has multiple NSC (and nested logit) representations with different partitions and identification of a unique nest structure is not possible. To rule this out, we say that an NSC p with $(v, u, \{X_i\}_{i=1}^K)$ is *nondegenerate* if there is at most one nest where this proportionality occurs: there is at most one $i \leq K$ such that for some $a \in X_i$,

$$\frac{\sum_{x \in A_i} u(x)}{v(A_i)} = \frac{u(a)}{v(a)} \text{ for any } A_i \subseteq X_i \text{ with } a \in A_i.$$

$a, b \in A$ such that $\frac{p(a, \{a, b\})}{p(b, \{a, b\})} = \frac{p(a, A)}{p(b, A)}$. This is because for any three alternatives, either (i) at least two belong to the same nest or (ii) all three belong to distinct nests. Hence there must exist some pair for which IIA holds and therefore NSC may be rejected with only three alternatives.

⁵If a and b are from distinct nests, then the addition of an alternative c from a third nest will not affect the relative probabilities of a and b .

⁶There is some experimental evidence that violations of regularity occur when similar alternatives are introduced, in-line with convex aggregation. This has been observed in humans ([Rieskamp, Busemeyer, and Mellers, 2006](#)) and animals ([Shafir, Waite, and Smith, 2002](#)). Recently, [Batley and Hess \(2016\)](#) estimated nested logit parameters to check for consistency with regularity (and various forms of stochastic transitivity) and found that parameter values consistent with violations of regularity provided the best fit.

⁷Relatedly, [Cerreia-Vioglio, Dillenberger, Ortoreva, and Riella \(2019\)](#) suggests that regularity may be violated due to deliberate randomization between complementary lotteries.

⁸Indeed, in regard to whether η_i should be less than or greater than one, [Train, McFadden, and Ben-Akiva \(1987\)](#) state that “...the value of $[\eta_i]$ indicates relative substitutability within and among nests, and neither possibility can be rule out *a priori*.”

This restriction rules out cases when v is always proportional to the sum of Luce utilities. Further, the Luce model has a unique nondegenerate NSC representation in which there is a single nest, $X_1 = X$.⁹ This nondegeneracy condition will be crucial for the unique identification of nests, but it is not required for the sufficiency part of our characterization (Theorem 1).

3 Behavioral Characterizations

3.1 The Similarity Effect and Revealed Similarity

Following the intuition behind the similarity effect, we introduce a notion of revealed similarity that will be essential to our analysis. Consider the effect of adding an alternative x on the probabilities of choosing a and b from some menu A . Adding x might decrease these probabilities as it competes with a and b . If x disproportionately affects one of them, say a relative to b , this reveals that a and b are dissimilar. Conversely, if x takes away from a and b proportionally, then this reveals that a and b are similar (symmetric) in menu A . We take a conservative approach and call two alternatives similar only if this is true for any menu A (they are symmetric to all other alternatives).

Definition 5. For any alternatives $a, b \in X$, we say that a and b are **revealed categorically similar**, denoted by $a \sim_p b$, if p satisfies IIA at a, b . We also say that a and b are **revealed categorically dissimilar** if $a \not\sim_p b$.

The similarity effect is often defined using an exogenously given similarity relation. With our formal notion of revealed similarity, we may establish a fully behavioral definition of the similarity effect given \sim_p . Recalling the red bus/blue bus example, adding the blue bus had a larger effect on the red bus than on the train. Hence the blue bus “takes more away” from similar alternatives than from dissimilar alternatives.

Definition 6. A stochastic choice function p exhibits the **similarity effect** if for all $A \in \mathcal{A}$, $a, b \in A$, and $x \notin A$,

$$\text{if } a \sim_p x \text{ and } b \not\sim_p x, \text{ then } \frac{p(a, A \cup x)}{p(b, A \cup x)} < \frac{p(a, A)}{p(b, A)}.$$

Intuitively, x hurts the revealed categorically similar alternative a more than a revealed categorically dissimilar alternative b . Since a and x are closer substitutes, x competes more with a than it does with b .

3.2 Characterization of NSC

In order to introduce our axiom, we consider the general effect of introducing an alternative x on the choice probabilities of two alternatives a and b . IIA requires that the relative probability between a

⁹Indeed, if there are i, j such that $\frac{\sum_{x \in A_i} u(x)}{v(A_i)} = \frac{u(a)}{v(a)}$ and $\frac{\sum_{y \in A_j} u(y)}{v(A_j)} = \frac{u(b)}{v(b)}$ for any $A_i \subseteq X_i, A_j \subseteq X_j, a \in A_i$, and $b \in A_j$, then $X_i \cup X_j$ should be treated as one nest. It is also not difficult to show that the set of degenerate NSC is measure zero with respect to the set of all NSC.

and b is *always* independent of x . However, as the similarity effect suggests, (asymmetric) similarity between x and a, b might affect the relative probabilities. We therefore divide IIA into two logically independent axioms based on the revealed similarity between x and a, b .

AXIOM 1 (Independence of Symmetric Alternatives). For any $A \in \mathcal{A}$, $a, b \in A$, and $x \notin A$,

$$\begin{array}{ccc} a \sim_p x \text{ and } b \sim_p x & & \\ \text{or} & \implies & \frac{p(a, A)}{p(b, A)} = \frac{p(a, A \cup x)}{p(b, A \cup x)}. \\ a \not\sim_p x \text{ and } b \not\sim_p x & & \end{array}$$

AXIOM 2 (Independence of Asymmetric Alternatives). For any $A \in \mathcal{A}$, $a, b \in A$, and $x \notin A$,

$$a \sim_p x \text{ and } b \not\sim_p x \implies \frac{p(a, A)}{p(b, A)} = \frac{p(a, A \cup x)}{p(b, A \cup x)}.$$

The first axiom requires that the relative probability between a and b is independent of x when a and b are revealed categorically (dis)similar to x . Intuitively, if a and b are symmetric from the perspective of x , then x should symmetrically influence a and b ; it does not affect the relative probability between a and b . The similarity effect directly contradicts the second axiom yet is unrelated to the first axiom.

Observation 1. *Luce's IIA is equivalent to the joint assumption of **Independence of Symmetric Alternatives** and **Independence of Asymmetric Alternatives**. Moreover, the two axioms are independent.*

We show in [Theorem 1](#) that NSC is characterized by **Independence of Symmetric Alternatives**, and thus NSC is precisely the generalization of Luce's model that accommodates the similarity effect.

Theorem 1. *Let p be a stochastic choice function with at least three alternatives that are dissimilar to each other. Then p satisfies **Independence of Symmetric Alternatives** if and only if p is a **nondegenerate NSC**.*

[Theorem 1](#) characterizes NSC when there are at least three nests; $a \not\sim_p b$, $b \not\sim_p c$, and $a \not\sim_p c$ for distinct alternatives.¹⁰ While the proof is in the appendix, we discuss briefly how our axiom characterizes NSC. It should be apparent from the definition that \sim_p is reflexive and symmetric. It turns out that the first part of **Independence of Symmetric Alternatives** ($a \sim_p x$ and $b \sim_p x$) implies that \sim_p is transitive.¹¹ Hence, transitivity of \sim_p immediately generates a partition X_1, \dots, X_k of X

¹⁰When the assumption is violated (i.e., there are only two nests), we can still obtain the characterization result by modifying Independence of Symmetric Alternatives. In particular, we can impose a modification of Luce's (1959) Product Rule instead of the second part of Independence of Symmetric Alternatives. It is well known that IIA is equivalent to the Product Rule for menus with two alternatives (see Luce (1959)).

¹¹More general notions of similarity may be intransitive (e.g., due to context dependence). Since we take a conservative definition of similarity, we find transitivity quite reasonable in our setting. That is, by requiring $\frac{p(a, A)}{p(b, A)} = \frac{p(a, \{a, b\})}{p(b, \{a, b\})}$ for any menu, we eliminate much of the context dependence. Further, transitivity of this revealed similarity relation is implicitly assumed in nested logit. See [section 4](#).

(or disjoint nests) such that any two alternatives in X_i are revealed categorically similar.¹² However, by itself it imposes no particular structure on choice, nor does it establish a relationship between the partition and choices (except that IIA is satisfied within each nest). The essential structure of NSC is captured by the second part of [Independence of Symmetric Alternatives](#) ($a \approx_p x$ and $b \approx_p x$). Therefore, almost all of the proof is devoted to showing that the second part of [Independence of Symmetric Alternatives](#) implies a nested choice structure consistent with this partition.

Lastly, we state the uniqueness properties of the NSC representation. The following proposition shows that the nest structure is unique, the nest utility v is unique up to a positive scalar, and Luce's utility u is unique up to a positive scalar at each nest.

Proposition 1 (Uniqueness). *Suppose p is a nondegenerate NSC with respect to $(v, u, \{X_i\}_{i=1}^K)$ as well as to $(v', u', \{X'_i\}_{i=1}^{K'})$. Then $K = K'$ and $\{X_i\}_{i=1}^K = \{X'_i\}_{i=1}^{K'}$. Moreover, there is $(\alpha_1, \dots, \alpha_K, \delta) \in \mathbb{R}_{++}^{K+1}$ such that $v' = \delta v$ and for any $x_i \in X_i$, $u'(x_i) = \alpha_i u(x_i)$.*

3.3 Characterizations of Nested Logit

The most well-known special case of NSC is nested logit, which was specifically created to handle the similarity effect. The difference between NSC and nested logit is that the latter imposes additional restrictions on the nest values: $v(A \cap X_i) = (\sum_{a \in A \cap X_i} u(a))^{\eta_i}$ with $\eta_i > 0$.

Despite its widespread use, nested logit has not been subject to careful axiomatic analysis in the way that other choice models have been. We provide two characterizations of nested logit that clarify the behavioral assumptions embedded in this model. The first characterization uses one additional axiom that imposes a menu independence condition on certain probability ratios.

AXIOM 3 (Log Ratio Invariance). For any $a, x \in X$ and $A, B \in \mathcal{A}$ such that $a \sim_p a'$ for all $a' \in A \cup B$,

$$\frac{\log \left(\frac{p(A, A \cup x)}{p(x, A \cup x)} / \frac{p(a, \{a, x\})}{p(x, \{a, x\})} \right)}{\log \left(\frac{p(A, A \cup a)}{p(a, A \cup a)} \right)} = \frac{\log \left(\frac{p(B, B \cup x)}{p(x, B \cup x)} / \frac{p(a, \{a, x\})}{p(x, \{a, x\})} \right)}{\log \left(\frac{p(B, B \cup a)}{p(a, B \cup a)} \right)}.$$

Log Ratio Invariance requires that the ratio $\log \left(\frac{p(A, A \cup x)}{p(x, A \cup x)} / \frac{p(a, \{a, x\})}{p(x, \{a, x\})} \right)$ and $\log \left(\frac{p(A, A \cup a)}{p(a, A \cup a)} \right)$ are proportional.

Theorem 2. *A nondegenerate NSC satisfies [Log Ratio Invariance](#) if and only if it is an **nested logit**.*

The explicit use of a functional form in Log Ratio Invariance allows us to establish testable implications for the functional form assumed in nested logit even with finite data.

In the rest of this section, we discuss an alternative axiom that captures the essential features of nested logit without an explicit functional form and shows that it characterizes nested logit under a richer domain assumption. To state this axiom, first notice that an important behavioral property

¹²Transitivity of \sim_p is imposed in [Li and Tang \(2016\)](#), which will be carefully discussed in [section 6](#). A weak version of transitivity of \sim_p is also used in [Echenique et al. \(2018\)](#).

of nested logit, beyond its treatment of similarity (ISA), is that the probability of choosing a nest depends on the total attractiveness of the nest: $v(A \cap X_i)$ is increasing in $\sum_{a \in A \cap X_i} u(a)$.

This behavior is characterized by a simple monotonicity property imposed among similar alternatives. Suppose that $A, A' \in \mathcal{A}$ are menus such that all alternatives in $A \cup A'$ are revealed similar. When A is more attractive than A' , then alternatives in A are always chosen more frequently than alternatives in A' when they are compared with any other alternative x . More formally, $p(A, A \cup A') \geq p(A', A \cup A')$ implies $p(A, A \cup x) \geq p(A', A' \cup x)$ for any $x \notin A \cup A'$. This can be viewed as an additional form of context independence, as it requires that there is no interaction between alternatives in $A \cup A'$ and x which might create a choice frequency reversal.

Because nested logit involves a power function, it satisfies a stronger version of the monotonicity property above. In particular, the monotonicity property holds even in relative terms: if A is relatively more attractive than A' when they are compared to any other menus, B and B' , then alternatives in A will be chosen relatively more frequently than A' when they are chosen against x .

AXIOM 4 (Relative Likelihood Independence). For any $x \in X$ and $A, B, A', B' \in \mathcal{A}$ such that $a \sim_p a'$ for any $a, a' \in A \cup B \cup A' \cup B'$,

$$\frac{p(A, A \cup B)}{p(B, A \cup B)} \geq \frac{p(A', A' \cup B')}{p(B', A' \cup B')} \implies \frac{p(A, A \cup x)}{p(x, A \cup x)} \bigg/ \frac{p(B, B \cup x)}{p(x, B \cup x)} \geq \frac{p(A', A' \cup x)}{p(x, A' \cup x)} \bigg/ \frac{p(B', B' \cup x)}{p(x, B' \cup x)}.$$

In our next result, we prove that Relative Likelihood Independence is a necessary condition for nested logit. Moreover, it implies that $v(A \cap X_i)$ is increasing in $\sum_{a \in A \cap X_i} u(a)$.

Proposition 2. *Any nested logit satisfies **Relative Likelihood Independence**. Conversely, if a nondegenerate NSC with $(v, u, \{X_i\}_{i=1}^K)$ satisfies Relative Likelihood Independence, then for each $i \leq K$ there is an increasing function $f_i : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ such that $v(A) = f_i(\sum_{x \in A} u(x))$ for any $A \subseteq X_i$.*

While **Relative Likelihood Independence** is not sufficient for nested logit, this is essentially due to the limitations of finite data. Indeed, we show that Relative Likelihood Independence characterizes nested logit when the following richness condition is satisfied.

AXIOM 5 (Richness). For any $a \in X$ and $\rho \in (0, 1)$, there is $b \in X$ such that $a \sim_p b$ and $p(a, \{a, b\}) = \rho$.

On its own, **Richness** is relatively mild and simply ensures that there are alternatives for each utility value. However, under this condition **Relative Likelihood Independence** yields the well-known functional form of nested logit. Consequently, **Relative Likelihood Independence** captures all remaining behavioral features of nested logit.

Theorem 3. *Any nondegenerate NSC that satisfies **Relative Likelihood Independence** and **Richness** is a nested logit.*

In applied settings, nested logit is often restricted to $\eta_i \in (0, 1]$, as this is sufficient for it to be a RUM. Since any RUM satisfies **Regularity**, a random utility nested logit must as well. It is

well known that [Regularity](#) is necessary but not sufficient for a model to be a RUM in general. However, we show that [Regularity](#) is sufficient for the nested logit to be a RUM under [Richness](#).

Proposition 3. *Any nested logit that satisfies [Regularity](#) and [Richness](#) is a random utility nested logit.*

If $\eta_i > 1$, nested logit will violate regularity for some specifications of u . Therefore [Richness](#) strengthens the bite of [Regularity](#) and $\eta_i \in (0, 1]$ is ensured.

3.4 Beyond Nested Logit

It is a matter of folk knowledge that the aforementioned “IIA within a nest and IIN” properties serve as the behavioral underpinnings of nested logit. However, our results show that “IIA within a nest and IIN” (with an endogenous nest structure) in fact characterize NSC, and nested logit requires an additional property ([Relative Likelihood Independence](#)). In this subsection, we present three special cases of NSC that are distinct from nested logit. These examples illustrate some natural choice behaviors that [Relative Likelihood Independence](#) rules out, further clarifying the behavioral assumptions behind nested logit.

3.4.1 Linear NSC

One interesting example of NSC that is distinct from nested logit is the Linear NSC. In this example, the nest value is linear in total nest utility, in contrast to the power function used in nested logit. For each nest i , there exist parameters $\lambda_i \geq 0$ and ν_i , so that

$$(5) \quad v(A \cap X_i) = \lambda_i \left(\sum_{x \in A \cap X_i} u(x) \right) + \nu_i.$$

In the Linear NSC, the attractiveness of a nest depends on both its instrumental utility, through λ_i , and an intrinsic “category” utility, through ν_i . It turns out that the Linear NSC is a special case of both Elimination-by-Aspects (EBA) of [Tversky \(1972\)](#) and the Attribute Rule (AR) of [Gul, Natenzon, and Pesendorfer \(2014\)](#). Consequently, the Linear NSC is also a RUM.

3.4.2 Menu-Dependent Substitutability

In nested logit, the nest parameter η_i captures substitutability of alternatives. While the standard nested logit only allows for a single substitution parameter for each nest, NSC can accommodate *menu-dependent substitutability*. For instance, consider the following example where substitutability depends on the size of the menu, capturing the idea that consumers may find it more difficult to distinguish between alternatives in larger option sets.

For each nest i , there exists a threshold $\tau_i \in \{1, \dots, |X_i|\}$, and nest parameters $\eta_i, \tilde{\eta}_i > 0$, so

that

$$(6) \quad v(A \cap X_i) = \begin{cases} (\sum_{x \in A \cap X_i} u(x))^{\eta_i} & \text{if } |A \cap X_i| > \tau_i \\ (\sum_{x \in A \cap X_i} u(x))^{\tilde{\eta}_i} & \text{if } |A \cap X_i| \leq \tau_i. \end{cases}$$

If $1 - \eta_i > 1 - \tilde{\eta}_i$, this means the agent perceives fewer differences among alternatives as the nest becomes “more represented.” That is, when $|A \cap X_i|$ exceeds τ_i , alternatives are “more substitutable.” In this example, τ_i has a natural interpretation as the consumer’s “distinction capacity.” Further, if $1 - \eta_i > 0 > 1 - \tilde{\eta}_i$, then whether the alternatives are complements or substitutes depends on the size of the nest. Lastly, when the change in substitutability is small (η_i and $\tilde{\eta}_i$ are sufficiently close), this example is also consistent with RUM.

3.4.3 Attention and Spillover Effects

The NSC also allow for cases where the nest value is not directly tied to alternative utility. We consider a particular example in which v is determined by the “salience” of alternatives. For some function $S : X \rightarrow \mathbb{R}_{++}$,

$$(7) \quad v(A \cap X_i) = \max_{x \in A \cap X_i} S(x).$$

In this specification, the value of a nest is determined the “attractiveness” or “noticeability” of its most salient alternative. To illustrate its behavioral implications, suppose there are three alternatives, $X = \{x, y, z\}$, with nests $X_1 = \{x, z\}$ and $X_2 = \{y\}$. When z is highly salient but low utility, $S(z)u(x) > S(x)[u(x) + u(z)]$, then $\frac{p(x, \{x, y, z\})}{p(y, \{x, y, z\})} > \frac{p(x, \{x, y\})}{p(y, \{x, y\})}$. Examples of such z include brands offering a high-end good with a high price to attract attention, expecting all consumers to purchase their “moderate” offering x . When the value of $S(z)$ is large enough relative to the value of $u(z)$, this may induce a violation of regularity. Similar examples can generate “spillover” effects. For example, one successful or attractive product may funnel attention to others, causing demand spillover. This is the traditional rationale behind the use of “loss-leaders” (Lal and Matutes (1994)) or “attention-grabbers” (Eliaz and Spiegler (2011)).

4 Revealed Similarity and its Extensions

We say that two alternatives are revealed categorically similar if IIA is satisfied between them at all menus. Requiring this eliminates the menu dependence of similarity, and so our notion of revealed similarity captures a form of absolute or fundamental similarity. Consequently, similarity is symmetric and, under [Independence of Symmetric Alternatives](#), transitive.¹³ One drawback is

¹³There are other ways to define similarity and other properties one might demand of a similarity relation. For instance, Rubinstein (1988) studies similarity and choice under risk. In his paper, the similarity relation is reflexive and symmetric, like ours, but also must satisfy a form of betweenness with respect to objective attributes and violates transitivity, unlike ours.

that this notion does not allow statements about comparative similarity; two alternatives are similar or not. Additionally, in some cases impressions of similarity may be context-dependent.¹⁴ Because of these apparent limitations, we consider two ways in which to extend NSC to accommodate more complex notions of similarity.

The first extension of NSC relaxes the requirement that an alternative must belong to a single nest. In section 4.1, we consider the (unrestricted) *cross-nested logit* (Vovsha, 1997) and the (more general) *generalized nested logit* (Wen and Koppelman, 2001), which allow for each alternative to be “allocated” across several nests. While overlapping nests allows for the most flexible notion of similarity, these models have no testable implications if the nests are not known *a priori*. Thus we demonstrate an important trade-off between nested and cross-nested models.

The second extension of NSC allows for “intermediate nests.” These intermediate nests are often visually represented through a multi-level decision tree. Within this structure, we can allow for a more nuanced notion of similarity through the introduction of a second similarity relation that is conceptually related to our core similarity notion. This secondary relation captures “context-dependent” similarity and allows for comparative statements. We provide an axiomatic characterization of this model (Theorem 6) in appendix B.3.¹⁵

4.1 Overlapping Nests

In NSC, each alternative belongs to one, and only one, nest. This feature of NSC places restrictions on the similarity relation. Because of these restrictions, in some settings, applied researchers have proposed allowing alternatives to exist in multiple nests. This leads to a class of models known as “cross-nested logits” (see Vovsha (1997), Ben-Akiva and Bierlaire (1999), Wen and Koppelman (2001), Papola (2004), and Bierlaire (2006)). In the cross-nested logit and the generalized nested logit, each alternative is allocated among the various nests.¹⁶ This allocation is specified with a vector of weights, one for each alternative, which describes to what extent an alternative belongs to each nest.

We show that any stochastic choice rule p can be rationalized by some unrestricted cross-nested logit. That is, for any p , there exist some nesting structure, X_1, \dots, X_K , allocations to these nests, $(\alpha_x^k)_{k=1}^K$, and utilities so that the resulting unrestricted cross-nested logit generates identical choice frequencies. Hence, unlike the nested logit and Luce models, there can be no

¹⁴There is a sense in which our notion is somewhat moderate. Consider Debreu’s red bus/blue bus example. In this case, the similar alternatives (buses) are in fact identical, often called duplicates (or in some cases replicas). Duplicates are not merely similar alternatives; they are similar and *provide the exact same utility value*. For example, in Gul et al. (2014) the use of duplicates is essential to their characterization of the Attribute Rule. Formally, x and y are duplicates if $p(a, A \cup x) = p(a, A \cup y)$ for any A and $a \in A$. However, our notion of similarity is not tied to utility. A commuter may regard all buses as similar (i.e., they belong to the same nest), yet nothing in our model restricts an agent from exhibiting a preference over different buses (e.g., because some bus routes may be faster or cheaper than others).

¹⁵Just as our similarity relation identifies endogenous nests, this secondary relation identifies endogenous, intermediate nests. Thus, Theorem 6 shows that we may identify an endogenous tree structure.

¹⁶The generalized nested logit was introduced by Wen and Koppelman (2001) and includes various “cross nested logits” as special cases.

behavioral characterization of the unrestricted cross-nested logit. More importantly, our result demonstrates the critical importance of the *a priori* specification of nests and model parameters typically utilized by applied researchers. The only testable implications of the model are due to the analysts' assumptions about alternative categorization and parameter restrictions.

Definition 7 (Generalized Nested Logit). A stochastic choice function p is an unrestricted **generalized nested logit** if there is a collection of subsets X_1, \dots, X_K of X and a vector $(\alpha_x^k)_{k=1}^K \in \mathbb{R}_+^K$ with $\sum_{k=1}^K \alpha_x^k = 1$ for each $x \in X$ such that $x \notin X_k$ iff $\alpha_x^k = 0$, a utility function $u : X \rightarrow \mathbb{R}_{++}$, and parameters $(\lambda_k)_{k=1}^K \in \mathbb{R}_{++}^K$ such that for any $A \in \mathcal{A}$ and $x \in A$,

$$(8) \quad p(x, A) = \sum_{k: x \in A \cap X_k} \frac{(\alpha_x^k u(x))^{\frac{1}{\lambda_k}}}{\sum_{y \in A \cap X_k} (\alpha_y^k u(y))^{\frac{1}{\lambda_k}}} \cdot \frac{\left(\sum_{y \in A \cap X_k} (\alpha_y^k u(y))^{\frac{1}{\lambda_k}} \right)^{\lambda_k}}{\sum_{l: A \cap X_l \neq \emptyset} \left(\sum_{z \in A \cap X_l} (\alpha_z^l u(z))^{\frac{1}{\lambda_l}} \right)^{\lambda_l}}.$$

Moreover, we say that p is an **unrestricted cross-nested logit** if $\lambda_k = \lambda_{k'}$ for any $k, k' \leq K$.

Theorem 4. *Every stochastic choice function is an **unrestricted cross-nested logit**.*

Corollary 1. Every stochastic choice function is an **unrestricted generalized nested logit**.

Our result relies on the key insight that the cross-nested logit is behaviorally equivalent to a form of menu-dependent utility. We first prove this equivalence as [Lemma 1](#) and show how we can go from menu-dependent utility to weighted allocations and back. This equivalence between the cross-nested logit and menu-dependent utility allows us to reduce the problem of finding weights to the problem of finding menu-dependent utility values for each menu that satisfy the cross-nested logit equation. The bulk of the proof is dedicated to showing that the existence of these menu-dependent utilities is equivalent to the existence of a fixed point for some self-map. The proof is completed by applying Brouwer's fixed point theorem.

This result precisely shows the trade-offs between using nested logit and related models: either accept a restrictive form of similarity or impose assumptions on nest structure and model parameters. As we mentioned previously, further assumptions on parameters or nest structure may lead to testable restrictions. In the literature, similar to nested logit, it is commonly assumed that $\lambda \leq 1$, since this is sufficient for cross-nested logit to be RUM. As with our handling of nested logit, we refer to such specifications as the *random utility cross-nested logit*. Note that our result shows that this restriction is not necessary for consistency with RUM; By [Theorem 4](#), every RUM has an unrestricted cross-nested logit representation with $\lambda > 1$.

In any case, a random utility cross-nested logit must have, at least, the same testable restrictions as RUM. However, our result suggests that random utility cross-nested logit may not have any testable restrictions beyond RUM. In fact, although it does not prove our hypothesis, [Fosgerau, McFadden, and Bierlaire \(2013\)](#) prove that the set of random utility cross-nested logit models is dense in the set of RUMs. Note that our [Theorem 4](#) is quite different from the result of [Fosgerau et al. \(2013\)](#) for the following reasons: (i) we prove an exact result while they prove an approximation result, (ii) they focus on random utility cross-nested logit, and (iii) our proof techniques are

completely different because their proof relies on the properties of the CDF for GEV distributions while we use Brouwer’s fixed point theorem.

5 Identifying Nests

In most applications of nested logit to market data, researchers assume nests based on knowledge of alternative attributes. This is potentially problematic, as in many environments there are many plausible structures. When studying vehicle choice, the researcher might construct nests based on brand, body type (e.g., sedan vs. truck), or country of origin.¹⁷ In other environments, classification may be subjective. When studying choice over apartments, nests might depend on both observable attributes and a myriad of unobservables: subjective impressions of neighborhoods, proximity to landmarks, or a host of other features. If the nest structure is misspecified, this may lead to biased conclusions regarding substitutability of goods and systematically inaccurate predictions.¹⁸

We show in [subsection 5.1](#) that the true (subjective) nest structure can be identified from the data by solving a minimization problem. Any potential nest structure has implications for when IIA may and may not be violated between alternatives. For a hypothesized nest structure, \mathcal{Y} , we propose a measure of the total magnitude of IIA violations within and across the proposed nests, $D(\mathcal{Y})$. We show that the true nest is a minimizer of D and that it will be the unique minimizer of D under a mild identification assumption ([Proposition 4](#)). In cases where the researcher has several potential nest structures in mind, such as in vehicle choice, our procedure for nest identification could be useful for nest selection. The researcher can calculate D for the particular nests in mind and select the one that best fits.

Because the number of possible nests grows rapidly in comparison to the number of alternatives, the full minimization problem becomes intractable. However, this issue can be managed due to insights from our similarity relation; by [Proposition 5](#), one only needs to check nest structures that are consistent with an empirical approximation of \sim_p . Note that in finite data sets it is unlikely that IIA will hold between any alternatives (e.g., since we observe a finite sample from the true distribution). However, one can measure the magnitude of the the IIA violation between a, b across various menus in the data. If this magnitude is below some threshold ϵ , then we conclude that a and b are approximately similar: $a \sim_\epsilon b$. When \sim_ϵ is transitive, then there are at most $|X|$ potential nests that we need to check, as stated in [Proposition 6](#).

¹⁷A common approach to this type of problem is to utilize multiple levels of nesting (which we characterize in [appendix B.3](#)). Even under this approach, the order of the levels matters.

¹⁸[Greene \(2003\)](#) provides an excellent summary of this issue: “*To specify the nested logit model, it is necessary to partition the choice set into branches [nests]. Sometimes there will be a natural partition ... In other instances, however, the partitioning of the choice set is ad hoc and leads to the troubling possibility that the results might be dependent on the branches so defined. ... There is no well-defined testing procedure for discriminating among tree structures, which is a problematic aspect of the model.*”

5.1 Identifying Nests by Distance Minimization

To analyze the problem of nest identification, we consider a data set denoted $\mathcal{O} = \{A, \{p_t(\cdot, A)\}_{t=1}^{N_A}\}_{A \in \mathcal{A}}$, where N_A is the number of observations of menu A and $p_t(a, A) = 1$ means that a was chosen from A at observation $t \leq N_A$. We also require $\sum_{x \in A} p_t(x, A) = 1$ for each A , so that $p_t(a, A) = 1$ implies $p_t(b, A) = 0$ for any $b \in A \setminus \{a\}$. Note that we may always write

$$p_t(a, A) = \bar{p}(a, A) + \epsilon_{t,a,A},$$

where $\bar{p}(a, A)$ is the probability that a is chosen from A according to the NSC with $(v, u, \{X_i\}_{i=1}^K)$. Then, the observed choice frequency of a from A in \mathcal{O} is

$$p(a, A) \equiv \frac{\sum_{t=1}^{N_A} p_t(a, A)}{N_A} = \bar{p}(a, A) + \epsilon_{a,A} \text{ where } \epsilon_{a,A} \equiv \frac{\sum_{t=1}^{N_A} \epsilon_{t,a,A}}{N_A}.$$

We assume that $\{p_t(\cdot, A)\}_{t=1}^{N_A}$ are independently drawn according to $\bar{p}(\cdot, A)$. Then, by the classical Glivenko-Cantelli theorem, $\epsilon_{a,A} \xrightarrow{a.s.} 0$.¹⁹ For notational simplicity, we write

$$r_A(A', B') \equiv \frac{p(A', A)}{p(B', A)} \text{ and } \bar{r}_A(A', B') \equiv \frac{\bar{p}(A', A)}{\bar{p}(B', A)} \text{ for any } A, A', B' \in \mathcal{A}.$$

Finally, let \mathcal{X} denote the set of all partitions of X and \mathcal{X}^* denote the true partition $\{X_i\}_{i=1}^K$.

Consider the following minimization problem.

$$(\text{NMP}) \quad \min_{\mathcal{Y} \in \mathcal{X}} D(\mathcal{Y}) = D_1(\mathcal{Y}) + D_2(\mathcal{Y}),$$

$$(9) \quad D_1(\mathcal{Y}) = \frac{\sum_{Y \in \mathcal{Y}} \sum_{A, B \in \mathcal{A}, a, b \in A \cap B \cap Y} \left(\log(r_A(a, b)) - \log(r_B(a, b)) \right)^2}{\sum_{Y \in \mathcal{Y}} |\{(A, B, a, b) | a, b \in A \cap B \cap Y\}|},$$

$$(10) \quad D_2(\mathcal{Y}) = \frac{\sum_{Y, Y' \in \mathcal{Y}} \sum_{A, B \in \mathcal{A}: A \cap Y = B \cap Y, A \cap Y' = B \cap Y'} \left(\log(r_A(Y, Y')) - \log(r_B(Y, Y')) \right)^2}{\sum_{Y, Y' \in \mathcal{Y}} |\{(A, B) | A \cap Y = B \cap Y, A \cap Y' = B \cap Y'\}|}.$$

Intuitively, $D_1(\mathcal{Y})$ measures the degree to which the data violates IIA among elements in the same nest in \mathcal{Y} , while $D_2(\mathcal{Y})$ measures the degree to which the data violates IIA among different nests in \mathcal{Y} . These measures are motivated by our axiom [Independence of Symmetric Alternatives](#): D_1 measures the extent to which the first part of [Independence of Symmetric Alternatives](#) holds, and D_2 measures the extent to which the second part of [Independence of Symmetric Alternatives](#) holds.

Similarly, let us define loss functions D^*, D_1^*, D_2^* when there is no observational noise; these

¹⁹All convergence statements in this paper are with respect to $N^* \rightarrow \infty$ where $N^* = \min_{A \in \mathcal{A}} N_A$.

are defined by replacing p with \bar{p} in Equations 9-10. Moreover, let $\hat{\mathcal{X}} = \arg \min_{\mathcal{Y} \in \mathcal{X}} D(\mathcal{Y})$. Note that $\hat{\mathcal{X}}$ is an M-estimator (Amemiya (1985), Newey and McFadden (1994)). Hence, by standard results, $\hat{\mathcal{X}}$ is a strongly consistent estimator of \mathcal{X}^* if \mathcal{X}^* is the unique minimizer of D^* . Indeed, \mathcal{X}^* is a minimizer of D^* since $D^*(\mathcal{X}^*) = 0$. It turns out that it is the unique minimizer under the following identification assumption.

Assumption 1. For all subsets $A_i \subset X_i$ and $A_j \subseteq X_j$, there are menus $A, B \in \mathcal{A}$ such that $\bar{r}_A(A_i, A_j) \neq \bar{r}_B(A_i, A_j)$, $A \cap A_i = B \cap A_i$, and $A \cap Y_j = B \cap Y_j$.

We now can state our strong consistency result.

Proposition 4. $\hat{\mathcal{X}} \xrightarrow{a.s.} \mathcal{X}^*$ under Assumption 1.

Proposition 4 shows that the true nest structure can be found by solving NMP. The intuition behind the result is as follows. As we see in our axiomatization, $a \sim_p b$ if and only if $a, b \in X_i$ for some i . Hence, IIA is satisfied between a and b when $a, b \in X_i$ and IIA is violated at least once between a and b when $a \in X_i$ and $b \in X_j$. Hence, the distance $\sum_{A, B \in \mathcal{A}, a, b \in A \cap B \cap Y} \left(\log(r_A(a, b)) - \log(r_B(a, b)) \right)^2$ between a and b is smaller whenever $a, b \in X_i$. Hence, minimizing $D_1(\mathcal{Y})$ helps us to correctly identify that elements from different nests are in fact from different nests.

However, it is important to notice that $D_1(\mathcal{Y})$ alone is not sufficient to identify \mathcal{X}^* . For example, suppose $X = \{a_1, \dots, a_5\}$ and \mathcal{X}^* is given by $X_1 = \{a_1, a_2, a_3\}$ and $X_2 = \{a_4, a_5\}$. Since the data provide a noisy measure of \bar{p} , it is possible that D_1 is minimized at some finer partition, say $Y_1 = \{a_1\}$, $Y_2 = \{a_2, a_3\}$, and $Y_3 = \{a_4, a_5\}$. Note that \mathcal{Y} splits X_1 , and since D_1 measures IIA violations within nests, $D_1(\mathcal{Y}) \leq D_1(\mathcal{X}^*)$ because \mathcal{Y} never combines two elements from different nests into the same nest (i.e., it is finer than \mathcal{X}^*).

This example illustrates a potential problem. D_1 by itself tends to select finer partitions (it wants to create “too many nests”). The second term, D_2 , corrects this problem. If \mathcal{Y} were the true nest structure, our axiomatization (i.e., the second half of Independence of Symmetric Alternatives) requires that the relative likelihoods between alternatives in Y_2 (for instance, a_2) and alternatives in Y_3 (for instance, a_4) are unaffected by the presence of a_1 . Accordingly, \mathcal{Y} is penalized by D_2 if introducing a_1 changes the relative likelihoods between alternatives in Y_2 and Y_3 . Importantly, since the true nest structure, \mathcal{X}^* , groups a_1 with a_2 and a_3 , \mathcal{X}^* will not be penalized, and so when N^* becomes sufficiently large, $D_2(\mathcal{Y}) > D_2(\mathcal{X}^*) \rightarrow 0$ almost surely. Thus $D_2(\cdot)$ enables us to correctly conclude that a_1 and a_2 do in fact belong to the same nest.

Notice that solving NMP is quite different from the typical exercise of selecting a nest structure in the literature. In a typical nested logit estimation, a researcher assumes a nest structure and then runs a maximum likelihood (ML) estimation to identify model parameters. To compare different nest structures, the researcher has to run a full ML estimation for each nest structure. Hence, it is computationally expensive to compare many different nest structures. However, our NMP provides a data-driven way to compare different nest structures without estimating the full parametric model. Moreover, NMP does not rely on the functional form of nested logit, since it applies to any NSC.

Finally, note that \mathcal{X}^* is a minimizer of D without any further assumptions; our identifying [Assumption 1](#) is only required to ensure that \mathcal{X}^* is the *unique* minimizer. Consequently, when [Assumption 1](#) is violated, \mathcal{X}^* will always be contained in the set of minimizers of D . This suggests that D may still be used to eliminate nest structures and that solving [NMP](#) can facilitate identification of the true nest structure.

5.2 Revealed Similarity and Nest Selection

There is a practical concern with directly applying [Proposition 4](#) to identify the nest structure because $|\mathcal{X}|$ grows exponentially as $|X|$ increases.²⁰ Therefore, we further refine our result by showing that we only need to compare $|X|$ different partitions, rather than $|\mathcal{X}|$. This dramatically reduces the number of calculations; comparing $|X|$ different partitions is computationally inexpensive even when X contains hundreds of alternatives. To establish this result, we introduce the following measure of IIA violations. For any $a, b \in X$, let

$$(11) \quad d(a, b) \equiv \frac{\sum_{A, B \in \mathcal{A}: a, b \in A \cap B} \left(\log(r_A(a, b)) - \log(r_B(a, b)) \right)^2}{|\{(A, B, a, b) | a, b \in A \cap B\}|}.$$

The value of $d(a, b)$ captures the total “distance” between a and b , in terms of IIA violations in the data \mathcal{O} . Consistent with our axiomatization, and the intuition behind D_1 , the value of $d(a, b)$ is smaller when a and b are from the same nest than when they are from different nests. While conceptually similar to D_1 , note that it is defined over the alternatives, not on nest structures. This crucial distinction allows us to use d to narrow our candidate nests purely based on the data.

Proposition 5. *Under Assumption 1, there are $\epsilon^*, \bar{N} > 0$ such that for any $N^* > \bar{N}$,*

$$\max_i \max_{a, b \in X_i} d(a, b) < \epsilon^* < \min_{i < j} \min_{a' \in X_i, b' \in X_j} d(a', b') \text{ with probability one.}$$

[Proposition 5](#) shows that for large enough N^* , there exists a “separating” threshold that correctly identifies whether two alternatives belong, or do not belong, to the same nest. If the researcher knows ϵ^* , then identifying the nest structure is a straightforward task due to this result. But when ϵ^* is unknown, [Proposition 5](#) is not sufficient to identify the nest structure.²¹ However, the insights provided by [Proposition 5](#) allow us to show that in order to identify the correct nest structure for any NSC, only $|X|$ different partitions are worth considering. In fact, we will explicitly construct the set of partitions that need to be considered using d and prove that this set contains the true nest \mathcal{X}^* .

²⁰Unlike the standard method of estimating nested logit, it is not computationally expensive to solve [NMP](#) by going through all possible partitions of X when $|X|$ is small. For instance, there are 877 different partitions when $|X| = 7$. Indeed, many papers in the literature study situations with relatively few alternatives (e.g., transportation modes or cellphone providers), and [Proposition 4](#) can be applied to these situations directly.

²¹Suppose a researcher chooses ϵ and uses it to construct nests: she decides that a and b are in the same nest if $d(a, b) < \epsilon$. When ϵ is too small compared to ϵ^* , she may wrongly conclude that a and b belong to different nests due to noise. If ϵ is too large compared to ϵ^* , she may wrongly conclude that a and b belong to the same nest.

In order to construct the set of relevant partitions, we introduce the following “approximately similar” relation: for any $\epsilon \geq 0$ and $a, b \in X$, let $a \sim_\epsilon b$ if $d(a, b) < \epsilon$. When \sim_ϵ is transitive, let $\mathcal{X}_\epsilon \equiv X / \sim_\epsilon$, which is the partition of X such that for any $A \in \mathcal{X}_\epsilon$, $a \in A$, and $b \in X$, $a \sim_\epsilon b$ if and only if $b \in A$.

Since we have finite data, if ϵ and ϵ' are close enough, they will result in the same relation ($\sim_\epsilon = \sim_{\epsilon'}$), except for certain knife-edge cases (which happens at most $|X|$ times). Notice that for smaller ϵ , we are “more discriminating” in declaring similarity and this results in a finer partition. For larger ϵ , we are “less discriminating” in declaring similarity and this results in a coarser partition. Let $\bar{\epsilon} \equiv \max_{a, b \in X} d(a, b)$, the maximal distance calculated in the data. Then for any $\epsilon > \bar{\epsilon}$, the resulting relation \sim_ϵ is complete, which reduces to the Luce model ($\epsilon = 0$ also gives the Luce model). Consequently, we never need to use an ϵ above $\bar{\epsilon}$. Because of these two key features of \sim_ϵ , it turns out that the set $\mathcal{X}^* \equiv \{\mathcal{X}_\epsilon\}_{\epsilon \in [0, \bar{\epsilon}]}$ is the desired collection of partitions and contains at most $|X|$ different elements.

Proposition 6. $|\mathcal{X}^*| \leq |X|$ under Assumption 1.

Combining Propositions 4-6, we can immediately show that \mathcal{X}^* contains the true partition structure, and it can be found by solving **NMP**, as formalized below. Let $\hat{\mathcal{X}}^* = \arg \min_{\mathcal{Y} \in \mathcal{X}^*} D(\mathcal{Y})$.

Corollary 2. $\hat{\mathcal{X}}^* \xrightarrow{a.s.} \mathcal{X}^*$ under Assumption 1.

The minimization problem **NMP** is not computationally demanding since $|\mathcal{X}^*| \leq |X|$. Hence, we can find the true nest structure even if there are many products. In practice, computing \mathcal{X}^* from choice frequencies is quite simple. First note that any partition of X can be represented by an $|X| \times |X|$ matrix M such that $M_{x,y} = 1$ when x and y are from the same nest and 0 otherwise. Hence, to compute \mathcal{X}^* , we follow the following steps:

- 1) Calculate $\epsilon_{ab} := d(a, b)$ for each pair (a, b) in $X \times X$;
- 2) For each ϵ_{ab} , construct the $|X| \times |X|$ matrix M^{ab} such that $M_{x,y}^{ab} = 1$ if $d(x, y) < \epsilon_{ab}$ and 0 otherwise;
- 3) Include the matrix M^{ab} in \mathcal{X}^* if for any $x, y, z \in X$, $M_{x,z}^{ab} = 1$ whenever $M_{x,y}^{ab} = M_{y,z}^{ab} = 1$.

The first step determines the collection of relevant thresholds from the data to construct candidate relations \sim_ϵ . The second step generates $|X|(|X|-1)/2$ matrices, which represent the similarity thresholds found in the previous step. The third step reduces the number to $|X|$, as we prove in Proposition 6, since \sim_ϵ must be transitive.

5.3 Identification from Simulations

To illustrate our algorithm and our theoretical result on identification, we ran the following simulation with six alternatives. We assumed that the true nest structure is $X_1 = \{x_1, x_2, x_3\}$ and $X_2 = \{x_4, x_5, x_6\}$, with $X = X_1 \cup X_2$, and calculated the fraction of trials in which our procedure

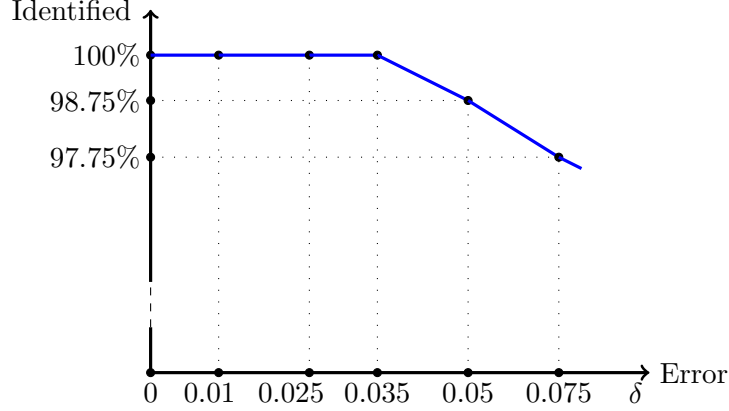


Figure 1: Percentage of trials with correctly identified nest structure with error δ .

correctly identified the nest structure. To do so, we randomly generated values for u and v and calculated \bar{p} , which is the NSC given by $(v, u, \{X_1, X_2\})$. To introduce sampling error, we drew independent errors from a uniform distribution $U[0, \delta]$ and perturbed \bar{p} .²² As shown by Proposition 4, when δ is small enough, the true nest structure will be identified correctly. This was confirmed by our simulation.

We considered six different values for δ ($\{0, 0.01, 0.025, 0.035, 0.05, 0.075\}$) and ran a total of 2400 trials, the results of which are summarized in Figure 1. For $\delta \in \{0, 0.01, 0.025, 0.035\}$, the nest structure was correctly identified in all trials. For $\delta = 0.05$ (0.075), the nest structure was correctly identified 395 (391) times out of 400 trials. In other words, in line with our theoretical result (Proposition 4), when error is relatively small (e.g., $\delta \leq 0.035$) the true nest is recovered 100% of the time. Even for relatively large errors (e.g., $\delta \geq 0.05$), we recover the true nests over 97% of the time.

6 Related Literature

NSC has a large overlap with RUM, which goes back to Block and Marschak (1960), Falmagne (1978), and Barbera and Pattanaik (1986). For example, both nested logit and Linear NSC are RUM. The nested logit goes back to Ben-Akiva (1973) and McFadden (1978) and has been widely applied in many fields, as mentioned in the introduction.

Aside from nested logit, the two most prominent models dealing with the similarity effect are Elimination-By-Aspects (EBA) of Tversky (1972) and the Attribute Rule (AR) of Gul et al. (2014). Both EBA and AR are RUMs and generalize the Luce model. In Tversky’s EBA, each alternative is a collection of *aspects*. The decision maker randomly selects one of these aspects from the aspects available in the menu, via a Luce rule, and eliminates alternatives that do not have the selected aspect. The decision maker repeats this procedure until a single alternative remains. EBA

²²Specifically, for each menu A and each simulation trial t , we independently draw errors $\{\zeta_{a,A,t}\}_{a \in A}$ from $U[0, \delta]$ and construct $\bar{p}^t(\cdot, A)$ as follows: $\bar{p}^t(a, A) = \frac{\bar{p}^t(a, A) + \zeta_{a,A,t}}{\sum_{b \in A} \bar{p}^t(b, A) + \zeta_{b,A,t}}$ for each $a \in A$.

is conceptually similar to an N -step nested logit, where N is the total number of alternatives. We can show that the Linear NSC is a special case of EBA, but EBA is disjoint from nested logit.

In the AR of Gul et al. (2014), each alternative has many *attributes*. A decision maker randomly selects one attribute from the attributes available in the menu via a Luce rule. When the selected attribute is ω , alternative x will be chosen with a probability that is proportional to γ_x^ω , where γ_x^ω is the intensity of ω in x . The AR is conceptually similar to cross-nested logit. In fact, one can show that the AR is a special case of a non-parametric version of cross-nested logit in which weights assigned to nests are menu-independent (i.e., γ_x^ω is menu-independent). Because of the menu independence of γ_x^ω , the AR is more restrictive than unrestricted cross-nested logit. We can also show that the Linear NSC is a special case of the AR but nested logit is not.

Another recent paper motivated by the similarity effect is Natenzon (2018), which introduces the Bayesian probit model. In the Bayesian probit model, the decision maker receives a signal about the true utility of each alternative, where signals follow a normal distribution. With regards to similarity, the author utilizes a notion of revealed similarity that is not directly related to IIA. His notion of similarity typically does not induce a partition structure on the set of alternatives.

In addition to EBA, AR, and nested logit, many special cases of RUM have been proposed, including: Gul and Pesendorfer (2006), in which each preference has an expected utility representation; Apesteguia et al. (2017), in which the collection of preferences satisfy the single-crossing property; and Manzini and Mariotti (2014), in which randomness occurs due to stochastic consideration. Our characterization of random utility nested logit contributes to this area of the stochastic choice literature.

One related RUM is the *Luce Model with Replicas* (LR) by Faro (2018). LR is a special case of Tversky’s EBA and corresponds to the special case of Linear NSC in which nest values and Luce utilities are constant: $v(A) = v_i$ for each $A \subseteq X_i$ and $u(a) = u(b)$ for any $a, b \in X_i$. In terms of behavior, Faro’s model only allows for restrictive forms of the similarity effect in which similar alternatives are replicas.

Another closely related model is the *Associationistic Luce Model* (AL) by Li and Tang (2016). AL is a special case of NSC with $v(A) = \sum_{a \in A} \gamma(a)$ for some function γ . Because of the additive structure of v , AL is significantly more restrictive than NSC. In fact, the Luce model is the only intersection between nested logit and AL. We note that the AL also allows for violations of regularity (e.g., the attraction effect). However, since v is increasing, the AL cannot simultaneously allow for violations of regularity and the similarity effect (see Appendix B.1). In terms of axiomatic foundations, they also use the revealed similarity relation \sim_p , and impose transitivity of \sim_p as one of their axioms.

Another model that has an interesting connection to NSC is by Ravid and Stevenson (2018).²³ They introduce the following stochastic choice model that captures a decision-making procedure

²³This model is a special case of Marley (1991).

that involves a sequence of binary comparisons:

$$p(x, A) = \frac{\prod_{y \in A \setminus \{x\}} \pi(x, y)}{\sum_{z \in A} \prod_{t \in A \setminus \{z\}} \pi(z, t)}.$$

This model is disjoint from nested logit, but when $\pi(x, y) = \frac{1}{u(y)}$ and $\pi(x, z) = \frac{1}{w(z)}$ for any $x, y \in X_i$ and $z \in X_j$, we obtain an NSC with $v(A \cap X_i) = \left(\sum_{y \in A \cap X_i} u(y) \right) \frac{\prod_{y \in A \cap X_i} w(y)}{\prod_{y \in A \cap X_i} u(y)}$.

Finally, we note that [Manzini and Mariotti \(2012\)](#) study a choice procedure similar to NSC in a deterministic choice environment.

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A Proofs

A.1 Proof of Theorem 1

Sufficiency. We first prove the sufficiency part of Theorem 1 by the following nine steps. Suppose p satisfies [Independence of Symmetric Alternatives](#).

Step 1: \sim_p is transitive.

Take any $x, y, z \in X$ such that $x \sim_p y$ and $y \sim_p z$. We shall prove that $\frac{p(x, \{x, z\})}{p(z, \{x, z\})} = \frac{p(x, A)}{p(z, A)}$ for any A with $x, z \in A$.

Take any A with $x, z \in A$ and $y \notin A$. Since $x \sim_p y$ and $y \sim_p z$, by [Independence of Symmetric Alternatives](#) we have $\frac{p(x, A \cup y)}{p(z, A \cup y)} = \frac{p(x, A)}{p(z, A)}$ and $\frac{p(x, \{x, y, z\})}{p(z, \{x, y, z\})} = \frac{p(x, \{x, z\})}{p(z, \{x, z\})}$. By the definition of \sim_p , $x \sim_p y$ implies $\frac{p(x, A \cup y)}{p(y, A \cup y)} = \frac{p(x, \{x, y, z\})}{p(y, \{x, y, z\})}$ and $y \sim_p z$ implies $\frac{p(y, A \cup y)}{p(z, A \cup y)} = \frac{p(y, \{x, y, z\})}{p(z, \{x, y, z\})}$. By combining all the

previous equalities,

$$\begin{aligned}\frac{p(x, A)}{p(z, A)} &= \frac{p(x, A \cup y)}{p(z, A \cup y)} = \frac{p(x, A \cup y)}{p(y, A \cup y)} \cdot \frac{p(y, A \cup y)}{p(z, A \cup y)} \\ &= \frac{p(x, \{x, y, z\})}{p(y, \{x, y, z\})} \cdot \frac{p(y, \{x, y, z\})}{p(z, \{x, y, z\})} = \frac{p(x, \{x, y, z\})}{p(z, \{x, y, z\})} = \frac{p(x, \{x, z\})}{p(z, \{x, z\})}.\end{aligned}$$

Hence, $x \sim_p z$.

Step 2: Let $X / \sim_p \equiv \{X_i\}_{i=1}^K$; that is, for any $x_i, x'_i \in X_i$ and $x_j \in X_j$, $x_i \sim_p x'_i$ and $x_i \not\sim_p x_j$. Since \sim_p is reflexive, transitive, and symmetric, we have a well-defined partition of X .

Step 3: The construction of u .

Notice that for each $i \leq K$, IIA is satisfied at all subsets of X_i . Therefore, for each i , there is a utility function $u_i : X_i \rightarrow \mathbb{R}_{++}$ such that $p(a, A) = \frac{u_i(a)}{\sum_{b \in A} u_i(b)}$ for any $A \subseteq X_i$ and $a \in A$ (as in the characterization of the Luce model). Since X_1, \dots, X_k are disjoint, we also have $u : X \rightarrow \mathbb{R}_{++}$ such that for any $A \subseteq X_i$ and $a \in A$, $p(a, A) = \frac{u(a)}{\sum_{b \in A} u(b)}$.

Step 4: For any $A \in \mathcal{A}$ and $a \in A \cap X_i$,

$$p(a, A) = \frac{u(a)}{\sum_{x \in A \cap X_i} u(x)} p(A \cap X_i, A).$$

Take any A and $a \in A \cap X_i$. By the definitions of \sim_p , u , and $\{X_j\}_{j=1}^K$, we have $\frac{p(a', A)}{p(a, A)} = \frac{p(a', \{a', a\})}{p(a, \{a', a\})} = \frac{u(a')}{u(a)}$ for any $a' \in A \cap X_i$. Then $\frac{p(A \cap X_i, A)}{p(a, A)} = \frac{\sum_{a' \in A \cap X_i} u(a')}{u(a)}$. Hence, $p(a, A) = \frac{u(a)}{\sum_{a' \in A \cap X_i} u(a')} p(A \cap X_i, A)$.

Step 5. Take alternatives a, b, x such that $a \in X_i, b \in X_j$, and $x \in X_k$. By [Independence of Symmetric Alternatives](#), for any $A \in \mathcal{A}$ with $a, b, x \in A$, we have $\frac{p(a, A)}{p(b, A)} = \frac{p(a, A \setminus \{x\})}{p(b, A \setminus \{x\})}$ since $a \not\sim_p x$ and $b \not\sim_p x$. Equivalently,

$$\frac{\frac{u(a)}{\sum_{y \in A \cap X_i} u(y)}}{\frac{u(b)}{\sum_{z \in A \cap X_j} u(z)}} \cdot \frac{p(A \cap X_i, A)}{p(A \cap X_j, A)} = \frac{\frac{u(a)}{\sum_{y \in A \cap X_i} u(y)}}{\frac{u(b)}{\sum_{z \in A \cap X_j} u(z)}} \cdot \frac{p(A \cap X_i, A \setminus \{x\})}{p(A \cap X_j, A \setminus \{x\})}.$$

Therefore, for any $A \in \mathcal{A}$ and $a, b, x \in A$ such that $a \in X_i, b \in X_j$, and $x \in X_k$,

$$(12) \quad \frac{p(A \cap X_i, A)}{p(A \cap X_j, A)} = \frac{p(A \cap X_i, A \setminus \{x\})}{p(A \cap X_j, A \setminus \{x\})}.$$

Step 6. We will construct the nest utility function $v : \bigcup_{i=1}^K 2^{X_i} \rightarrow \mathbb{R}_+$ for subsets of X_2, \dots, X_k in the following way.

First, let us take $\alpha_i \subseteq X_i$ where $i \geq 2$. Let

$$v(\alpha_i) \equiv \frac{p(\alpha_i, \alpha_i \cup X_1)}{p(X_1, \alpha_i \cup X_1)}.$$

Fact 1. For any $\alpha_i \subseteq X_i, \alpha_j \subseteq X_j$ with $i, j \geq 2$,

$$\frac{p(\alpha_i, \alpha_i \cup \alpha_j)}{p(\alpha_j, \alpha_i \cup \alpha_j)} = \frac{v(\alpha_i)}{v(\alpha_j)}.$$

Proof of Fact 1. Notice that from Equation (12) we can obtain the following by repeatedly eliminating $x \in X_1$ from $\alpha_i \cup \alpha_j \cup X_1$:

$$\frac{p(\alpha_i, \alpha_i \cup \alpha_j \cup X_1)}{p(\alpha_j, \alpha_i \cup \alpha_j \cup X_1)} = \frac{p(\alpha_i, \alpha_i \cup \alpha_j)}{p(\alpha_j, \alpha_i \cup \alpha_j)}.$$

Similarly, from Equation (12) we obtain

$$\frac{p(\alpha_i, \alpha_i \cup \alpha_j \cup X_1)}{p(X_1, \alpha_i \cup \alpha_j \cup X_1)} = \frac{p(\alpha_i, \alpha_i \cup X_1)}{p(X_1, \alpha_i \cup X_1)} = v(\alpha_i)$$

and

$$\frac{p(\alpha_j, \alpha_i \cup \alpha_j \cup X_1)}{p(X_1, \alpha_i \cup \alpha_j \cup X_1)} = \frac{p(\alpha_j, \alpha_j \cup X_1)}{p(X_1, \alpha_j \cup X_1)} = v(\alpha_j).$$

Combining the above three equalities, we obtain

$$\frac{v(\alpha_i)}{v(\alpha_j)} = \frac{\frac{p(\alpha_i, \alpha_i \cup \alpha_j \cup X_1)}{p(X_1, \alpha_i \cup \alpha_j \cup X_1)}}{\frac{p(\alpha_j, \alpha_i \cup \alpha_j \cup X_1)}{p(X_1, \alpha_i \cup \alpha_j \cup X_1)}} = \frac{p(\alpha_i, \alpha_i \cup \alpha_j \cup X_1)}{p(\alpha_j, \alpha_i \cup \alpha_j \cup X_1)} = \frac{p(\alpha_i, \alpha_i \cup \alpha_j)}{p(\alpha_j, \alpha_i \cup \alpha_j)}.$$

Fact 2. For any $\alpha_i \subseteq X_i, \alpha_j \subseteq X_j$ with $i, j \geq 2$ and $A \subseteq \cup_{s \neq i, j} X_s$,

$$\frac{p(\alpha_i, \alpha_i \cup \alpha_j \cup A)}{p(\alpha_j, \alpha_i \cup \alpha_j \cup A)} = \frac{v(\alpha_i)}{v(\alpha_j)}.$$

Proof of Fact 2. By Equation (12), we obtain the following equality by repeatedly eliminating $x \in A$ from $\alpha_i \cup \alpha_j \cup A$:

$$\frac{p(\alpha_i, \alpha_i \cup \alpha_j \cup A)}{p(\alpha_j, \alpha_i \cup \alpha_j \cup A)} = \frac{p(\alpha_i, \alpha_i \cup \alpha_j)}{p(\alpha_j, \alpha_i \cup \alpha_j)} = \frac{v(\alpha_i)}{v(\alpha_j)}.$$

Step 7. We will construct the nest utility function $v : \bigcup_{i=1}^K 2^{X_i} \rightarrow \mathbb{R}_+$ for subsets of X_1 in the following way.

First, let us take $\alpha_1 \subseteq X_1$. Let

$$v(\alpha_1) \equiv \frac{p(\alpha_1, \alpha_1 \cup X_2)}{p(X_2, \alpha_1 \cup X_2)} \cdot \frac{p(X_2, X_1 \cup X_2)}{p(X_1, X_1 \cup X_2)}.$$

Fact 3. For any $\alpha_2 \subseteq X_2$,

$$\frac{p(\alpha_1, \alpha_1 \cup \alpha_2)}{p(\alpha_2, \alpha_1 \cup \alpha_2)} = \frac{v(\alpha_1)}{v(\alpha_2)}.$$

Proof of Fact 3. Since $v(\alpha_2) = \frac{p(\alpha_2, X_1 \cup \alpha_2)}{p(X_1, X_1 \cup \alpha_2)}$, we shall prove that

$$\frac{p(\alpha_1, \alpha_1 \cup X_2)}{p(X_2, \alpha_1 \cup X_2)} \cdot \frac{p(X_2, X_1 \cup X_2)}{p(X_1, X_1 \cup X_2)} = \frac{p(\alpha_1, \alpha_1 \cup \alpha_2)}{p(\alpha_2, \alpha_1 \cup \alpha_2)} \cdot \frac{p(\alpha_2, X_1 \cup \alpha_2)}{p(X_1, X_1 \cup \alpha_2)}.$$

Notice that from Equation (12) we can obtain the following equalities by repeatedly eliminating $x \in X_3$ from $\alpha_1 \cup X_2 \cup X_3$ and $\alpha_1 \cup \alpha_2 \cup X_3$,

$$\frac{p(\alpha_1, \alpha_1 \cup X_2)}{p(X_2, \alpha_1 \cup X_2)} \cdot \frac{p(X_2, X_1 \cup X_2)}{p(X_1, X_1 \cup X_2)} = \frac{p(\alpha_1, \alpha_1 \cup X_2 \cup X_3)}{p(X_2, \alpha_1 \cup X_2 \cup X_3)} \cdot \frac{p(X_2, X_1 \cup X_2)}{p(X_1, X_1 \cup X_2)}$$

and

$$\frac{p(\alpha_1, \alpha_1 \cup \alpha_2)}{p(\alpha_2, \alpha_1 \cup \alpha_2)} \cdot \frac{p(\alpha_2, X_1 \cup \alpha_2)}{p(X_1, X_1 \cup \alpha_2)} = \frac{p(\alpha_1, \alpha_1 \cup \alpha_2 \cup X_3)}{p(\alpha_2, \alpha_1 \cup \alpha_2 \cup X_3)} \cdot \frac{p(\alpha_2, X_1 \cup \alpha_2)}{p(X_1, X_1 \cup \alpha_2)}.$$

Therefore, we shall prove that

$$\frac{p(\alpha_1, \alpha_1 \cup X_2 \cup X_3)}{p(X_2, \alpha_1 \cup X_2 \cup X_3)} \cdot \frac{p(X_2, X_1 \cup X_2)}{p(X_1, X_1 \cup X_2)} = \frac{p(\alpha_1, \alpha_1 \cup \alpha_2 \cup X_3)}{p(\alpha_2, \alpha_1 \cup \alpha_2 \cup X_3)} \cdot \frac{p(\alpha_2, X_1 \cup \alpha_2)}{p(X_1, X_1 \cup \alpha_2)}.$$

Moreover,

$$\begin{aligned} \frac{p(\alpha_1, \alpha_1 \cup X_2 \cup X_3)}{p(X_2, \alpha_1 \cup X_2 \cup X_3)} \cdot \frac{p(X_2, X_1 \cup X_2)}{p(X_1, X_1 \cup X_2)} &= \frac{p(\alpha_1, \alpha_1 \cup X_2 \cup X_3)}{p(X_3, \alpha_1 \cup X_2 \cup X_3)} \cdot \frac{p(X_3, \alpha_1 \cup X_2 \cup X_3)}{p(X_2, \alpha_1 \cup X_2 \cup X_3)} \cdot \frac{p(X_2, X_1 \cup X_2)}{p(X_1, X_1 \cup X_2)} \\ &= \frac{p(\alpha_1, \alpha_1 \cup X_2 \cup X_3)}{p(X_3, \alpha_1 \cup X_2 \cup X_3)} \cdot \frac{v(X_3)}{v(X_2)} \cdot \frac{p(X_2, X_1 \cup X_2)}{p(X_1, X_1 \cup X_2)}, \text{ by Fact 2,} \\ &= \frac{p(\alpha_1, \alpha_1 \cup X_2 \cup X_3)}{p(X_3, \alpha_1 \cup X_2 \cup X_3)} \cdot \frac{v(X_3)}{v(X_2)} \cdot v(X_2), \text{ by the definition of } v, \end{aligned}$$

and

$$\begin{aligned} \frac{p(\alpha_1, \alpha_1 \cup \alpha_2 \cup X_3)}{p(\alpha_2, \alpha_1 \cup \alpha_2 \cup X_3)} \cdot \frac{p(\alpha_2, X_1 \cup \alpha_2)}{p(X_1, X_1 \cup \alpha_2)} &= \frac{p(\alpha_1, \alpha_1 \cup \alpha_2 \cup X_3)}{p(X_3, \alpha_1 \cup \alpha_2 \cup X_3)} \cdot \frac{p(X_3, \alpha_1 \cup \alpha_2 \cup X_3)}{p(\alpha_2, \alpha_1 \cup \alpha_2 \cup X_3)} \cdot \frac{p(\alpha_2, X_1 \cup \alpha_2)}{p(X_1, X_1 \cup \alpha_2)} \\ &= \frac{p(\alpha_1, \alpha_1 \cup \alpha_2 \cup X_3)}{p(X_3, \alpha_1 \cup \alpha_2 \cup X_3)} \cdot \frac{v(X_3)}{v(\alpha_2)} \cdot \frac{p(\alpha_2, X_1 \cup \alpha_2)}{p(X_1, X_1 \cup \alpha_2)}, \text{ by Fact 2,} \\ &= \frac{p(\alpha_1, \alpha_1 \cup \alpha_2 \cup X_3)}{p(X_3, \alpha_1 \cup \alpha_2 \cup X_3)} \cdot \frac{v(X_3)}{v(\alpha_2)} \cdot v(\alpha_2), \text{ by the definition of } v. \end{aligned}$$

Finally, we shall prove that

$$(13) \quad \frac{p(\alpha_1, \alpha_1 \cup X_2 \cup X_3)}{p(X_3, \alpha_1 \cup X_2 \cup X_3)} = \frac{p(\alpha_1, \alpha_1 \cup \alpha_2 \cup X_3)}{p(X_3, \alpha_1 \cup \alpha_2 \cup X_3)},$$

which immediately follows from Equation (12) by repeatedly eliminating $x \in X_2 \setminus \alpha_2$ from $\alpha_1 \cup$

$X_2 \cup X_3$.

Fact 4. For any $\alpha_i \subseteq X_i$ with $i \geq 3$,

$$\frac{p(\alpha_1, \alpha_1 \cup \alpha_i)}{p(\alpha_i, \alpha_1 \cup \alpha_i)} = \frac{v(\alpha_1)}{v(\alpha_i)}.$$

Proof of Fact 4. By Equation (12) and Facts 2-3,

$$\begin{aligned} \frac{p(\alpha_1, \alpha_1 \cup \alpha_i)}{p(\alpha_i, \alpha_1 \cup \alpha_i)} &= \frac{p(\alpha_1, \alpha_1 \cup \alpha_i \cup X_2)}{p(\alpha_i, \alpha_1 \cup \alpha_i \cup X_2)} = \frac{p(\alpha_1, \alpha_1 \cup \alpha_i \cup X_2)}{p(X_2, \alpha_1 \cup \alpha_i \cup X_2)} \cdot \frac{p(X_2, \alpha_1 \cup \alpha_i \cup X_2)}{p(\alpha_i, \alpha_1 \cup \alpha_i \cup X_2)} \\ &= \frac{p(\alpha_1, \alpha_1 \cup X_2)}{p(X_2, \alpha_1 \cup X_2)} \cdot \frac{v(X_2)}{v(\alpha_i)} = \frac{v(\alpha_1)}{v(X_2)} \cdot \frac{v(X_2)}{v(\alpha_i)} = \frac{v(\alpha_1)}{v(\alpha_i)}. \end{aligned}$$

Fact 5. For any $\alpha_i \subseteq X_i$ with $i \geq 2$ and $A \subseteq \cup_{s \neq i, 1} X_s$,

$$\frac{p(\alpha_1, \alpha_1 \cup \alpha_i \cup A)}{p(\alpha_i, \alpha_1 \cup \alpha_i \cup A)} = \frac{v(\alpha_1)}{v(\alpha_i)}.$$

Proof of Fact 5. From Equation (12), we have

$$\frac{p(\alpha_1, \alpha_1 \cup \alpha_i \cup A)}{p(\alpha_i, \alpha_1 \cup \alpha_i \cup A)} = \frac{p(\alpha_1, \alpha_1 \cup \alpha_i)}{p(\alpha_i, \alpha_1 \cup \alpha_i)} = \frac{v(\alpha_1)}{v(\alpha_i)}.$$

Step 8. By Facts 2, 4, and 5, for any $A \in \mathcal{A}$ and $i, j \leq K$,

$$\frac{p(A \cap X_i, A)}{p(A \cap X_j, A)} = \frac{v(A \cap X_i)}{v(A \cap X_j)}.$$

Since $\sum_{i=1}^K p(A \cap X_i, A) = 1$, we have $p(A \cap X_i, A) = \frac{v(A \cap X_i)}{\sum_{j=1}^K v(A \cap X_j)}$. Since $\frac{p(a, A)}{p(b, A)} = \frac{u(a)}{u(b)}$ for any $a, b \in X_i$, we have

$$p(a_i, A) = \frac{u(a_i)}{\sum_{x \in A \cap X_i} u(x)} \cdot \frac{v(A \cap X_i)}{\sum_{j=1}^K v(A \cap X_j)} \text{ for each } a_i \in A \cap X_i.$$

Step 9. An NSC p is nondegenerate.

By way of contradiction, suppose there are $i, j \leq K$ such that for some $a \in X_i$ and $b \in X_j$,

$$\frac{\sum_{x \in A_i} u(x)}{v(A_i)} = \frac{u(a)}{v(a)} \text{ and } \frac{\sum_{y \in A_j} u(y)}{v(A_j)} = \frac{u(b)}{v(b)} \text{ for any } A_i \subseteq X_i \text{ and } A_j \subseteq X_j.$$

In other words,

$$\frac{\sum_{x \in A_i} u(x)}{v(A_i)} \Big/ \frac{u(a)}{v(a)} = \frac{\sum_{y \in A_j} u(y)}{v(A_j)} \Big/ \frac{u(b)}{v(b)} \text{ for any } A_i \subseteq X_i \text{ and } A_j \subseteq X_j.$$

Then by NSC representation, we have $\frac{p(a, \{a, b\})}{p(b, \{a, b\})} = \frac{p(a, A)}{p(b, A)}$ for any $A \in \mathcal{A}$; i.e., $a \sim_p b$, which contradicts the construction of $\{X_k\}_{k=1}^K$.

Necessity. Suppose p is a nondegenerate NSC with $(v, u, \{X_i\}_{i=1}^K)$.

Step 1. For any $a, b \in X$, $a \sim_p b$ if and only if either $a, b \in X_i$ for some i .

Take any $a, b \in X$. We consider two cases.

Case 1. Suppose $a, b \in X_i$.

In this case, by NSC representation, $\frac{p(a, A)}{p(b, A)} = \frac{u(a)}{u(b)} = \frac{p(a, \{a, b\})}{p(b, \{a, b\})}$ for any $A \in \mathcal{A}$. Therefore, $a \sim_p b$.

Case 2. Suppose $a \in X_i$ and $b \in X_j$ with $i \neq j$.

We shall prove that $a \not\sim_p b$. By nondegeneracy of p , either

$$\frac{\sum_{x \in A_i} u(x)}{v(A_i)} \neq \frac{u(a)}{v(a)} \text{ for some } A_i \subseteq X_i \text{ with } a \in A_i$$

or

$$\frac{\sum_{y \in A_j} u(y)}{v(A_j)} \neq \frac{u(b)}{v(b)} \text{ for some } A_j \subseteq X_j \text{ with } b \in A_j.$$

Without loss of generality, suppose the former is true. Then we have

$$\frac{p(a, \{a, b\})}{p(b, \{a, b\})} = \frac{v(a)}{v(b)} \neq \frac{p(a, A_i \cup b)}{p(b, A_i \cup b)} = \frac{u(a)}{\sum_{x \in A_i} u(x)} \frac{v(A_i)}{v(b)}.$$

Therefore, $a \not\sim_p b$.

Step 2. The first part of [Independence of Symmetric Alternatives](#) is satisfied.

Take any $A \in \mathcal{A}$, $a, b \in A$, and $x \notin A$ such that $a \sim_p x$ and $b \sim_p x$. By Step 1, we have $a, b, x \in X_i$ for some i . Therefore, $a \sim_p b$ implies $\frac{p(a, A)}{p(b, A)} = \frac{p(a, \{a, b\})}{p(b, \{a, b\})} = \frac{p(a, A \cup x)}{p(b, A \cup x)}$.

Step 3. The second part of [Independence of Symmetric Alternatives](#) is satisfied.

Take any $A \in \mathcal{A}$, $a, b \in A$, and $x \notin A$ such that $a \not\sim_p x$ and $b \not\sim_p x$. By Step 1, $a \not\sim_p x$ and $b \not\sim_p x$ imply $x \in X_i$ and $a, b \notin X_i$ for some i .

Case 1. $a, b \in X_j$ for some j .

Since $a \sim_p b$,

$$\frac{p(a, A)}{p(b, A)} = \frac{p(a, \{a, b\})}{p(b, \{a, b\})} = \frac{p(a, A \cup x)}{p(b, A \cup x)}.$$

Case 2. $a \in X_j$ and $b \in X_k$ for some j, k with $j \neq k$.

In this case, we have

$$\frac{p(a, A)}{p(b, A)} = \frac{\frac{u(a)}{\sum_{y \in A_j} u(y)} v(A_j)}{\frac{u(b)}{\sum_{z \in A_k} u(z)} v(A_k)} = \frac{p(a, A \cup x)}{p(b, A \cup x)}$$

since $A_j = (A \cup x) \cap X_j = A \cap X_j$ and $A_k = (A \cup x) \cap X_k = A \cap X_k$.

A.2 Proof of Proposition 1

Since the uniqueness of Luce utilities and nest utilities follows standard arguments, we only prove that the nest structure is unique. By way of contradiction, suppose p is a nondegenerate NSC with respect to both of $(v, u, \{X_i\}_{i=1}^K)$ and $(v', u', \{X'_i\}_{i=1}^{K'})$, and $\{X'_i\}_{i=1}^{K'}$ is not a permutation of $\{X_i\}_{i=1}^K$. Without loss of generality, suppose $K' \geq K$. Then there are $x_i, x'_i \in X_i$ such that $x_i \in X'_j$ and $x'_i \in X'_k$. Since $\frac{p(x_i, \{x_i, x'_i\})}{p(x'_i, \{x_i, x'_i\})} = \frac{p(x_i, A)}{p(x'_i, A)}$ for any A , by NSC representation with $\{v', u', \{X'_i\}_{i=1}^{K'}\}$, we have

$$\frac{v'(x_i)}{v'(x'_i)} = \frac{\frac{u'(x_i)}{\sum_{x_j \in A_j} u'(x_j)} v'(A_j)}{\frac{u'(x'_i)}{\sum_{x_k \in A_k} u'(x_k)} v'(A_k)} \text{ for any } A_j \subseteq X'_j \text{ and } A_k \subseteq X'_k.$$

Let us first set $A_j = \{x_i\}$. Then we have $\frac{u'(x'_i)}{v'(x'_i)} = \frac{\sum_{x_k \in A_k} u'(x_k)}{v'(A_k)}$ for any $A_k \subseteq X'_k$. Similarly, by setting $A_k = \{x'_i\}$, we also obtain $\frac{u'(x_i)}{v'(x_i)} = \frac{\sum_{x_j \in A_j} u'(x_j)}{v'(A_j)}$ for any $A_j \subseteq X'_j$. Therefore, we obtain a contradiction since the above two equalities contradict the assumption that NSC p with $(v', u', \{X'_i\}_{i=1}^{K'})$ is nondegenerate.

A.3 Proof of Theorem 2

Since the necessity part is straightforward, we only prove the sufficiency part. Suppose p is an NSC with $(v, u, \{X_i\}_{i=1}^K)$ and that it satisfies [Log Ratio Invariance](#). If $K = 1$, then we immediately have a Luce model. Suppose now $K \geq 2$. For each $i \leq K$, fix some $a_i^* \in X_i$. Then define

$$\eta_i \equiv \frac{\log(v(X_i)/v(\{a_i^*\}))}{\log(\sum_{a_i \in X_i} u(a_i)/u(a_i^*))}.$$

For any $A \subset X_i$ and any $x \in X \setminus X_i$, by Log Ratio Invariance, we have

$$\frac{\log\left(\frac{p(X_i, X_i \cup x)}{p(x, X_i \cup x)} / \frac{p(a_i^*, \{a_i^*, x\})}{p(x, \{a_i^*, x\})}\right)}{\log\left(\frac{p(X_i, X_i \cup a_i^*)}{p(a_i^*, X \cup a_i^*)}\right)} = \frac{\log\left(\frac{p(A, A \cup x)}{p(x, A \cup x)} / \frac{p(a_i^*, \{a_i^*, x\})}{p(x, \{a_i^*, x\})}\right)}{\log\left(\frac{p(A, A \cup a_i^*)}{p(a_i^*, A \cup a_i^*)}\right)}.$$

Since

$$\frac{\log\left(\frac{p(X_i, X_i \cup x)}{p(x, X_i \cup x)} / \frac{p(a_i^*, \{a_i^*, x\})}{p(x, \{a_i^*, x\})}\right)}{\log\left(\frac{p(X_i, X_i \cup a_i^*)}{p(a_i^*, X \cup a_i^*)}\right)} = \frac{\log\left(\frac{v(X_i)}{v(\{a_i^*\})}\right)}{\log\left(\frac{\sum_{b \in X_i} u(b)}{u(a_i^*)}\right)} = \eta_i,$$

we have

$$\eta_i = \frac{\log\left(\frac{p(A, A \cup x)}{p(x, A \cup x)} / \frac{p(a_i^*, \{a_i^*, x\})}{p(x, \{a_i^*, x\})}\right)}{\log\left(\frac{p(A, A \cup a_i^*)}{p(a_i^*, A \cup a_i^*)}\right)} = \frac{\log\left(\frac{v(A)}{v(\{a_i^*\})}\right)}{\log\left(\frac{\sum_{b \in A} u(b)}{u(a_i^*)}\right)};$$

equivalently, $\frac{v(A)}{v(\{a_i^*\})} = \left(\frac{\sum_{b \in A} u(b)}{u(a_i^*)}\right)^{\eta_i}$. Let $\delta_i = \frac{v(\{a_i^*\})}{(u(a_i^*))^{\eta_i}}$. Then $v(A) = \delta_i (\sum_{b \in A} u(b))^{\eta_i}$. Therefore, p is the nested logit with $(\eta_1, \dots, \eta_K, u', \{X_i\}_{i=1}^K)$ such that $u'(x) = \delta_i^{\frac{1}{\eta_i}} u(x)$ when $x \in X_i$.

A.4 Proof of Proposition 2

Suppose p is a nested logit with $(\eta_1, \dots, \eta_K, u, \{X_i\}_{i=1}^K)$. Take any $A, A' \in \mathcal{A}$ such that IIA is satisfied at $A \cup A'$. We shall prove that Relative Likelihood Independence is satisfied. Since p is a nondegenerate NSC, by Step 1 of the necessity part proof of Theorem 1, $x \sim_p y$ if and only if $x, y \in X_i$ for some $i \leq K$. Take any $A, B, A', B' \in \mathcal{A}$ and $x \in X$ such that $a \sim_p a'$ for any $a, a' \in A \cup B \cup A' \cup B'$. By the previous argument, $A \cup B \cup A' \cup B' \subseteq X_i$. By the nested logit representation,

$$\frac{p(A, A \cup B)}{p(B, A \cup B)} = \frac{\sum_{a \in A} u(a)}{\sum_{b \in B} u(b)} \geq \frac{p(A', A' \cup B')}{p(B', A' \cup B')} = \frac{\sum_{a \in A'} u(a)}{\sum_{b \in B'} u(b)}.$$

When $x \in X_i$,

$$\frac{p(A, A \cup x)}{p(x, A \cup x)} \bigg/ \frac{p(B, B \cup x)}{p(x, B \cup x)} = \frac{\sum_{a \in A} u(a)}{\sum_{b \in B} u(b)} \geq \frac{p(A', A' \cup x)}{p(x, A' \cup x)} \bigg/ \frac{p(B', B' \cup x)}{p(x, B' \cup x)} = \frac{\sum_{a \in A'} u(a)}{\sum_{b \in B'} u(b)}.$$

When $x \in X_j$ and $i \neq j$,

$$\frac{p(A, A \cup x)}{p(x, A \cup x)} \bigg/ \frac{p(B, B \cup x)}{p(x, B \cup x)} = \frac{(\sum_{a \in A} u(a))^{\eta_i}}{(\sum_{b \in B} u(b))^{\eta_i}} \geq \frac{p(A', A' \cup x)}{p(x, A' \cup x)} \bigg/ \frac{p(B', B' \cup x)}{p(x, B' \cup x)} = \frac{(\sum_{a \in A'} u(a))^{\eta_i}}{(\sum_{b \in B'} u(b))^{\eta_i}}.$$

Hence,

$$\frac{p(A, A \cup B)}{p(B, A \cup B)} \geq \frac{p(A', A' \cup B')}{p(B', A' \cup B')} \text{ implies } \frac{p(A, A \cup x)}{p(x, A \cup x)} \bigg/ \frac{p(B, B \cup x)}{p(x, B \cup x)} \geq \frac{p(A', A' \cup x)}{p(x, A' \cup x)} \bigg/ \frac{p(B', B' \cup x)}{p(x, B' \cup x)}.$$

Suppose p is the nondegenerate NSC with $(v, u, \{X_i\}_{i=1}^K)$ and satisfies Relative Likelihood Independence. If $K = 1$, we trivially obtain the desired result. Suppose $K \geq 2$. Take any $i, j \leq K$ with $i \neq j$. Take any $A, B, A' \subseteq X_i$ and $x \in X_j$. By Relative Likelihood Independence,

$$\frac{p(A, A \cup B)}{p(B, A \cup B)} \geq 1 \text{ implies } \frac{p(A, A \cup x)}{p(x, A \cup x)} \bigg/ \frac{p(B, B \cup x)}{p(x, B \cup x)} \geq 1;$$

equivalently,

$$p(A, A \cup B) \geq p(B, A \cup B) \text{ implies } \frac{p(A, A \cup x)}{p(x, A \cup x)} \geq \frac{p(B, B \cup x)}{p(x, B \cup x)}.$$

By the NSC representation, $\sum_{a \in A} u(a) \geq \sum_{b \in B} u(b)$ implies

$$\frac{p(A, A \cup x)}{p(x, A \cup x)} = \frac{v(A)}{v(x)} \geq \frac{p(B, B \cup x)}{p(x, B \cup x)} = \frac{v(B)}{v(x)}.$$

Then $\sum_{a \in A} u(a) \geq \sum_{b \in B} u(b)$ implies $v(A) \geq v(B)$. Therefore, there is an increasing function $f_i : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ such that $v(A) = f_i(\sum_{a \in A} u(a))$ for any $A \subseteq X_i$.

A.5 Proof of Theorem 3

Suppose p is the nondegenerate NSC with $(v, u, \{X_i\}_{i=1}^K)$ and satisfies Relative Likelihood Independence and Richness. We shall show that p is a nested logit. If $K = 1$, we trivially obtain the desired result. Suppose $K \geq 2$. Let us fix $i \leq K$.

Take any $j \leq K$ with $i \neq j$. Take any $A, B, A', B' \subseteq X_i$ and $x \in X_j$. By Relative Likelihood Independence and the NSC representation,

$$\begin{aligned} \frac{p(A, A \cup B)}{p(B, A \cup B)} &= \frac{\sum_{a \in A} u(a)}{\sum_{b \in B} u(b)} \geq \frac{p(A', A' \cup B')}{p(B', A' \cup B')} = \frac{\sum_{a \in A'} u(a)}{\sum_{b \in B'} u(b)} \\ \text{implies } \frac{p(A, A \cup x)}{p(x, A \cup x)} / \frac{p(B, B \cup x)}{p(x, B \cup x)} &= \frac{v(A)}{v(B)} \geq \frac{p(A', A' \cup x)}{p(x, A' \cup x)} / \frac{p(B', B' \cup x)}{p(x, B' \cup x)} = \frac{v(A')}{v(B')}. \end{aligned}$$

Equivalently,

$$\frac{\sum_{a \in A} u(a)}{\sum_{b \in B} u(b)} \geq \frac{\sum_{a \in A'} u(a)}{\sum_{b \in B'} u(b)} \text{ implies } \frac{v(A)}{v(B)} \geq \frac{v(A')}{v(B')}.$$

When $A' = B'$, $\sum_{a \in A} u(a) \geq \sum_{b \in B} u(b)$ implies $v(A) \geq v(B)$. Therefore, there is an increasing function $f_i : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ such that $v(A) = f_i(\sum_{a \in A} u(a))$ for any $A \subseteq X_i$. Let $R_i = \{x \in \mathbb{R}_{++} | x = \sum_{a \in A} u(a) \text{ for some } A \in \mathcal{A}\}$. Then we have for any $v, v', w, w' \in R_i$,

$$\frac{v}{v'} \geq \frac{w}{w'} \text{ implies } \frac{f_i(v)}{f_i(v')} \geq \frac{f_i(w)}{f_i(w')}.$$

Take any $a \in X_i$. By Richness, for any $\rho \in (0, 1)$, there is $b \in X_i$ such that $u(b) = \frac{1-\rho}{\rho} u(a)$. Therefore, for any $\alpha > 0$, there is $b \in X_i$ such that $u(b) = \alpha$. Hence, $u(X_i) = \mathbb{R}_{++}$.

Take any $\alpha, \beta \in \mathbb{R}_{++}$. Since $u(X_i) = \mathbb{R}_{++}$, there are alternatives $a, b, a', b' \in X_i$ such that $u(a) = \alpha\beta, u(b) = \beta, u(a') = \alpha$, and $u(b') = 1$. By the above implication of Relative Likelihood Consistency, we have

$$\frac{f_i(\alpha\beta)}{f_i(\beta)} = \frac{f_i(\alpha)}{f_i(1)} \text{ for any } \alpha, \beta > 0.$$

Let $g(t) \equiv \frac{f_i(t)}{f_i(1)}$. Then $g(1) = 1$ and $g(\alpha\beta) = g(\alpha)g(\beta)$ for any $\alpha, \beta > 0$. Finally, we can prove that g is a power function. Since $g > 0$, let $h(t) = \log(g(\exp(t)))$ for any $t \in \mathbb{R}$. Then for any $t, t' \in \mathbb{R}$, we have $h(t+t') = \log(g(\exp(t+t'))) = \log(g(\exp(t) \exp(t'))) = \log(g(\exp(t))g(\exp(t'))) = \log(g(\exp(t))) + \log(g(\exp(t'))) = h(t) + h(t')$. We have obtained a typical Cauchy functional equation for h . Hence, there is $\eta_i \geq 0$ such that $h(t) = \eta_i t$. In other words, $g(a) = a^{\eta_i}$. Therefore,

$$\begin{aligned} p(a, A) &= \frac{f_i(1) \left(\sum_{x \in A \cap X_i} u(x) \right)^{\eta_i}}{\sum_{j: A \cap X_j \neq \emptyset} f_j(1) \left(\sum_{y \in A \cap X_j} u(y) \right)^{\eta_j}} \frac{u(a)}{\sum_{b \in A \cap X_i} u(b)} \\ &= \frac{\left(\sum_{x \in A \cap X_i} \bar{u}(x) \right)^{\eta_i}}{\sum_{j: A \cap X_j \neq \emptyset} \left(\sum_{y \in A \cap X_j} \bar{u}(y) \right)^{\eta_j}} \frac{\bar{u}(a)}{\sum_{b \in A \cap X_i} \bar{u}(b)}, \end{aligned}$$

where $\bar{u}(x) = (f_i(1))^{\frac{1}{\eta_i}} u(x)$ for each $x \in X_i$. That is, p is a nested logit.

A.6 Proof of Proposition 3

Suppose p is the nested logit with $(\eta_1, \dots, \eta_K, u, \{X_i\}_{i=1}^K)$ and satisfies Regularity and Richness. If $K = 1$, we obtain the desired result since the Luce model is a nested logit. Suppose $K \geq 2$. Let us fix $i \leq K$. We shall prove that $\eta_i \leq 1$. Take any $a, a' \in X_i$ and $b' \in X_j$ with $i \neq j$. By Richness, for any $\rho \in (0, 1)$ there is $b \in X_j$ such that $u(b) = \frac{1-\rho}{\rho} u(b')$. By Regularity, $p(a, \{a, b\}) \leq p(a, \{a, b, a'\})$; i.e.,

$$p(a, \{a, b\}) = \frac{(u(a))^{\eta_i}}{(u(a))^{\eta_i} + (u(b))^{\eta_j}} \geq p(a, \{a, b, a'\}) = \frac{u(a)}{u(a) + u(a')} \cdot \frac{(u(a) + u(a'))^{\eta_i}}{(u(a) + u(a'))^{\eta_i} + (u(b))^{\eta_j}}.$$

After simplifying the above inequality, we obtain

$$\frac{(u(a) + u(a'))^{\eta_i} + (u(b))^{\eta_j}}{(u(a))^{\eta_i} + (u(b))^{\eta_j}} = 1 + \frac{(u(a) + u(a'))^{\eta_i} - (u(a))^{\eta_i}}{(u(a))^{\eta_i} + (u(b))^{\eta_j}} \geq \frac{(u(a) + u(a'))^{\eta_i-1}}{(u(a))^{\eta_i-1}};$$

equivalently,

$$\frac{(u(a))^{\eta_i-1}}{(u(a))^{\eta_i} + (u(b))^{\eta_j}} \geq \frac{(u(a) + u(a'))^{\eta_i-1} - (u(a))^{\eta_i-1}}{(u(a) + u(a'))^{\eta_i} - (u(a))^{\eta_i}}.$$

Notice that when ρ is close to 0, we can obtain arbitrary large $u(b)$. Then the left-hand side of above inequality can be arbitrary close to zero. Therefore, the right-hand side must be negative. Hence, since $\eta_i > 0$ implies $(u(a) + u(a'))^{\eta_i} - (u(a))^{\eta_i} > 0$, we have $\eta_i \leq 1$.

A.7 Proof of Theorem 4

Before we proceed to the proof of Theorem 4, it is useful to consider the following generalization of unrestricted cross-nested logit:

$$(14) \quad p(x, A) = \sum_{k: x \in A \cap X_k} \frac{u_x^k}{\sum_{y \in A \cap X_k} u_y^k} \cdot \frac{\left(\sum_{y \in A \cap X_k} u_y^k \right)^\lambda}{\sum_{l: A \cap X_l \neq \emptyset} \left(\sum_{z \in A \cap X_l} u_z^l \right)^\lambda}.$$

Note that the representation (14) reduces to unrestricted cross-nested logit by setting $u_x^k = (\alpha_x^k u(x))^{\frac{1}{\lambda}}$. It turns out that the representation (14) is behaviorally equivalent to the unrestricted cross-nested logit.

Lemma 1. *Any stochastic choice p that admits the representation (14) is an unrestricted cross-nested logit.*

Proof. Suppose that p admits the representation (14) with $\{X_k\}_{k=1}^K$, $\{u_x^k\}_{k \leq K, x \in X}$, and λ . Let us define $u : X \rightarrow \mathbb{R}_{++}$ and α_x^k as follows: for each $x \in X$ and $k \leq K$,

$$u(x) \equiv \sum_{l=1}^K (u_x^l)^\lambda \text{ and } \alpha_x^k \equiv \frac{(u_x^k)^\lambda}{\sum_{l=1}^K (u_x^l)^\lambda}.$$

Then we have $u_x^k = (\alpha_x^k u(x))^{\frac{1}{\lambda}}$ and $\sum_{k=1}^K \alpha_x^k = 1$. Hence, we obtain an unrestricted cross-nested logit. \square

By Lemma 1, we shall prove that any stochastic choice function p admits the representation (14) with some $\{X_k\}_{k=1}^K$, $\{u_x^k\}_{k \leq K, x \in X}$, and λ . We first set the collection of subsets X_1, \dots, X_K to be equivalent to \mathcal{A} . That is, for any $A \in \mathcal{A}$, there $i \leq K$ such that $A = X_k$. Moreover, $X_i \neq X_j$ whenever $i \neq j$. Let us now write u_x^A rather than u_x^k when $A = X_k$. Then we shall find $U = \{u_x^A\}_{A \in \mathcal{A}, x \in A} \in \mathbb{R}_{++}^N$ where $N \equiv \sum_{A \in \mathcal{A}} |A|$ and $\lambda > 0$ such that

$$(15) \quad p(x, A) = \sum_{B: x \in B} \frac{u_x^B}{\sum_{y \in A \cap B} u_y^B} \cdot \frac{\left(\sum_{y \in A \cap B} u_y^B \right)^\lambda}{\sum_{C: A \cap C \neq \emptyset} \left(\sum_{z \in A \cap C} u_z^C \right)^\lambda}.$$

We prove the above by two steps.

Step 1. There is some function $\sigma : \mathbb{R}_{++}^N \rightarrow \mathbb{R}^N$ such that $U \in \mathbb{R}_{++}^N$ is a fixed point of σ iff p admits the representation (16) with respect to U .

Let $\tilde{U} = \{\tilde{u}_x^A\}_{A \in \mathcal{A}, x \in A}$. Let us define the following mapping for each U, \tilde{U} , and λ :

$$\begin{aligned} q(x, A|U, \tilde{U}, \lambda) &\equiv \frac{u_x^A}{\sum_{y \in A} u_y^A} \cdot \frac{\left(\sum_{y \in A} \tilde{u}_y^A \right)^\lambda}{\sum_{C: A \cap C \neq \emptyset} \left(\sum_{z \in A \cap C} \tilde{u}_z^C \right)^\lambda} \\ &+ \sum_{B: x \in B, A \neq B} \frac{\tilde{u}_x^B}{\sum_{y \in A \cap B} \tilde{u}_y^B} \cdot \frac{\left(\sum_{y \in A \cap B} \tilde{u}_y^B \right)^\lambda}{\sum_{C: A \cap C \neq \emptyset} \left(\sum_{z \in A \cap C} \tilde{u}_z^C \right)^\lambda}. \end{aligned}$$

Now note that it is enough for us to find U and λ such that $p(x, A) = q(x, A|U, U, \lambda)$. For notational simplicity, let

$$f^A(\tilde{U}, \lambda) \equiv \frac{\left(\sum_{y \in A} \tilde{u}_y^A \right)^\lambda}{\sum_{C: A \cap C \neq \emptyset} \left(\sum_{z \in A \cap C} \tilde{u}_z^C \right)^\lambda}$$

and

$$g_x^A(\tilde{U}, \lambda) \equiv \sum_{B: x \in B, A \neq B} \frac{\tilde{u}_x^B}{\sum_{y \in A \cap B} \tilde{u}_y^B} \cdot \frac{\left(\sum_{y \in A \cap B} \tilde{u}_y^B \right)^\lambda}{\sum_{C: A \cap C \neq \emptyset} \left(\sum_{z \in A \cap C} \tilde{u}_z^C \right)^\lambda}.$$

Then we have $q(x, A|U, \tilde{U}, \lambda) = \frac{u_x^A}{\sum_{y \in A} u_y^A} \cdot f^A(\tilde{U}, \lambda) + g_x^A(\tilde{U}, \lambda)$. For any $M \equiv \{m^A\}_{A \in \mathcal{A}} \in \mathbb{R}_{++}^{|\mathcal{A}|}$, let

$$\sigma_x^A(\tilde{U}, \lambda, M) \equiv m^A \frac{p(x, A) - g_x^A(\tilde{U}, \lambda)}{f^A(\tilde{U}, \lambda)} \text{ and } \sigma(\tilde{U}, \lambda, M) \equiv \{\sigma_x^A(\tilde{U}, \lambda, M)\}_{A \in \mathcal{A}, x \in A}.$$

Note that $f^A(\cdot, \lambda, M)$ and $g_x^A(\cdot, \lambda, M)$ are strictly positive and continuous functions on \mathbb{R}_{++}^N . Hence,

$\sigma(\cdot, \lambda, M) : \mathbb{R}_{++}^N \rightarrow \mathbb{R}^N$ is continuous for each (λ, M) . Moreover,

$$\begin{aligned}
q(x, A | \sigma(\tilde{U}, \lambda, M), \tilde{U}, \lambda) &= \frac{\sigma_x^A(\tilde{U}, \lambda, M)}{\sum_{y \in A} \sigma_y^A(\tilde{U}, \lambda, M)} \cdot f^A(\tilde{U}, \lambda) + g_x^A(\tilde{U}, \lambda) \\
&= \frac{m^A \frac{p(x, A) - g_x^A(\tilde{U}, \lambda)}{f^A(\tilde{U}, \lambda)}}{\sum_{y \in A} m^A \frac{p(y, A) - g_y^A(\tilde{U}, \lambda)}{f^A(\tilde{U}, \lambda)}} \cdot f^A(\tilde{U}, \lambda) + g_x^A(\tilde{U}, \lambda) \\
&= p(x, A) \text{ since } f^A(\tilde{U}, \lambda) = 1 - \sum_{x \in A} g_x^A(\tilde{U}, \lambda).
\end{aligned}$$

Therefore, it is enough to find $U \in \mathbb{R}_{++}^N$ such that $\sigma(U, \lambda, M) = U$; i.e., a fixed point of $\sigma(\cdot, \lambda, M)$ in \mathbb{R}_{++}^N . To apply Brouwer's fixed point theorem,²⁴ we shall show that for some (λ, M) there is a non-empty, convex, compact set $S \subset \mathbb{R}_{++}^N$ such that $\sigma(\cdot, \lambda, M)$ is a self-map on S ; that is, $\sigma(\cdot, \lambda, M) : S \rightarrow S$.

Step 2. For some λ and M , there is a non-empty, closed, convex set $S \subset \mathbb{R}_{++}^N$ such that $\sigma(U, \lambda, M) \in S$ for any $U \in S$. Let $p^* \equiv \min\{\min_{B \in \mathcal{A}, y \in B} p(y, B), \frac{1}{|X|}\} > 0$ and

$$S \equiv \{U \in \mathbb{R}_{++}^N \mid \sum_{x \in A} u_x^A = 1 + \frac{|A|}{|X|^2} p^* \text{ and } \sum_{x \in B} u_x^A \leq 1 - \frac{(p^*)^2}{4} \text{ for any } A, B \text{ with } B \subset A\}.$$

Step 2.1. S is non-empty.

We will show that $U \in S$ when $u_x^A = \frac{1}{|A|} + \frac{p^*}{|X|^2}$ for each $A \in \mathcal{A}$ and $x \in X$. First, $\sum_{x \in A} u_x^A = 1 + \frac{|A|}{|X|^2} p^*$. Second, for any $B \subset A$,

$$\begin{aligned}
\sum_{x \in B} u_x^A &\leq (|A| - 1) \left(\frac{1}{|A|} + \frac{p^*}{|X|^2} \right) = 1 - \left(\frac{1}{|A|} + \frac{p^*}{|X|^2} - \frac{|A|}{|X|^2} p^* \right) < 1 - \left(\frac{1}{|A|} - \frac{|A|}{|X|^2} p^* \right) \\
&\leq 1 - \left(\frac{1}{|X|} - \frac{|X|}{|X|^2} p^* \right) = 1 - \frac{(1 - p^*)}{|X|} \leq 1 - \frac{1}{2|X|} \leq 1 - \frac{p^*}{2} \leq 1 - \frac{(p^*)^2}{4}.
\end{aligned}$$

Step 2.2. S is convex.

Let

$$S_1 \equiv \{U \in \mathbb{R}_{++}^N \mid \sum_{x \in A} u_x^A = 1 + \frac{|A|}{|X|^2} p^*\}$$

and

$$S_2 \equiv \{U \in \mathbb{R}_{++}^N \mid \sum_{x \in B} u_x^A \leq 1 - \frac{(p^*)^2}{4} \text{ for any } A, B \text{ with } B \subset A\}.$$

Note that S_1, S_2 are convex sets. Hence, $S = S_1 \cap S_2$ is convex.

²⁴**Brouwer's fixed point theorem:** Let $S \subset \mathbb{R}^m$ be convex and compact and let $f : S \rightarrow S$ be continuous. Then f has a fixed point; that is, there is $s \in S$ such that $f(s) = s$. For example, see [Ok \(2007\)](#), p.279.

Step 2.3. S is compact.

Note that $S \subset [0, 2]^N$. Hence, S is bounded. Moreover, note that S_1, S_2 are closed sets. Hence, $S = S_1 \cap S_2$ is closed.

Step 2.4. For some λ , $f^A(U, \lambda) \in (1 - \frac{p^*}{2}, 1)$ for any $U \in S$.

First, it is immediate that

$$f^A(U, \lambda) = \frac{\left(\sum_{y \in A} u_y^A\right)^\lambda}{\sum_{C: A \cap C \neq \emptyset} \left(\sum_{z \in A \cap C} u_z^C\right)^\lambda} < 1 \text{ when } U \gg 0.$$

Second, in order to show that

$$f^A(U, \lambda) = \frac{\left(\sum_{y \in A} u_y^A\right)^\lambda}{\sum_{C: A \cap C \neq \emptyset} \left(\sum_{z \in A \cap C} u_z^C\right)^\lambda} > 1 - \frac{p^*}{2},$$

it is enough to prove that

$$\frac{1}{f^A(U, \lambda)} - 1 = \frac{\sum_{C: A \cap C \neq \emptyset, C \neq A} \left(\sum_{z \in A \cap C} u_z^C\right)^\lambda}{\left(\sum_{y \in A} u_y^A\right)^\lambda} < \frac{p^*}{2}.$$

By the construction of S , we have $\left(\sum_{y \in A} u_y^A\right)^\lambda = \left(1 + \frac{|A|}{|X|^2} p^*\right)^\lambda$. Moreover,

$$\sum_{C: A \cap C \neq \emptyset, C \neq A} \left(\sum_{z \in A \cap C} u_z^C\right)^\lambda = \sum_{B \subset A} \left(\sum_{y \in B} u_y^B\right)^\lambda + \sum_{C: A \cap C \neq \emptyset, C \not\subseteq A} \left(\sum_{z \in A \cap C} u_z^C\right)^\lambda$$

and by the construction of S ,

$$\sum_{B \subset A} \left(\sum_{y \in B} u_y^B\right)^\lambda = \sum_{B \subset A} \left(1 + \frac{|B|}{|X|^2} p^*\right)^\lambda < 2^{|X|} \left(1 + \frac{|A| - 1}{|X|^2} p^*\right)^\lambda.$$

Moreover, since $C \cap A \neq C$ whenever $C \not\subseteq A$, by the construction of S ,

$$\sum_{C: A \cap C \neq \emptyset, C \not\subseteq A} \left(\sum_{z \in A \cap C} u_z^C\right)^\lambda \leq \sum_{C: A \cap C \neq \emptyset, C \not\subseteq A} \left(1 - \frac{(p^*)^2}{4}\right)^\lambda < 2^{|X|}.$$

Combining the last two inequalities, we have

$$\frac{1}{f^A(U, \lambda)} - 1 = \frac{\sum_{C: A \cap C \neq \emptyset, C \neq A} \left(\sum_{z \in A \cap C} u_z^C\right)^\lambda}{\left(\sum_{y \in A} u_y^A\right)^\lambda} < \frac{2^{|X|} \left(1 + \frac{|A| - 1}{|X|^2} p^*\right)^\lambda + 2^{|X|}}{\left(1 + \frac{|A|}{|X|^2} p^*\right)^\lambda}.$$

Let $\lambda^* \equiv \max_{A \in \mathcal{A}} \frac{\log \left(\frac{2^{|X|+2}}{p^*} \right)}{\log \left(\frac{1 + \frac{|A|}{|X|^2} p^*}{1 + \frac{|A|-1}{|X|^2} p^*} \right)}$ and $\lambda > \lambda^*$. Then we have $\frac{2^{|X|+2}}{p^*} < \left(\frac{1 + \frac{|A|}{|X|^2} p^*}{1 + \frac{|A|-1}{|X|^2} p^*} \right)^\lambda$. Consequently,

$$\frac{p^*}{2} > \frac{2^{|X|+1} \left(1 + \frac{|A|-1}{|X|^2} p^* \right)^\lambda}{\left(1 + \frac{|A|}{|X|^2} p^* \right)^\lambda} > \frac{2^{|X|} \left(1 + \frac{|A|-1}{|X|^2} p^* \right)^\lambda + 2^{|X|}}{\left(1 + \frac{|A|}{|X|^2} p^* \right)^\lambda} > \frac{1}{f^A(U, \lambda)} - 1.$$

Step 2.5. When $\lambda > \lambda^*$, $g_x^A(U, \lambda) \in [0, \frac{p^*}{2})$ for any $U \in S$.

It is immediate that $g_x^A(U, \lambda) \geq 0$. Moreover, since $\sum_{x \in A} g_x^A(U, \lambda) = 1 - f^A(U, \lambda) \in (0, \frac{p^*}{2})$ by Step 2.4, $g_x^A(U, \lambda) < \frac{p^*}{2}$.

Step 2.6. Let $m^A = 1 + \frac{|A|}{|X|^2} p^*$ and $\lambda > \lambda^*$. Then $\sigma(U, \lambda, M) \in S$ for any $U \in S$.

To show that $\sigma(U, \lambda, M) \in S$, we shall prove that for any $A \in \mathcal{A}$, (i) $\sigma_x^A(U, \lambda, M) > 0$, (ii) $\sum_{x \in A} \sigma_x^A(U, \lambda, M) = 1 + \frac{|A|}{|X|^2} p^*$, and (iii) $\sum_{y \in B} \sigma_y^A(U, \lambda, M) \leq 1 - \frac{(p^*)^2}{4}$ for any $B \subset A$.

Step 2.6.(i). $\sigma_x^A(U, \lambda, M) > \frac{m^A p^*}{2}$.

By Step 2.5 and the definition of p^* , we have $p(x, A) \geq p^*$ and $\frac{p^*}{2} > g_x^A(U, \lambda)$. Therefore, since $f^A(U, \lambda) < 1$, $\sigma_x^A(U, \lambda, M) = m^A \frac{p(x, A) - g_x^A(U, \lambda)}{f^A(U, \lambda)} > m^A \frac{\frac{p^*}{2}}{f^A(U, \lambda)} > \frac{m^A p^*}{2}$.

Step 2.6.(ii). $\sum_{x \in A} \sigma_x^A(U, \lambda, M) = 1 + \frac{|A|}{|X|^2} p^*$.

Since $f^A(U, \lambda) = 1 - \sum_{x \in A} g_x^A(U, \lambda)$ and by the definition of m^A , $\sum_{x \in A} \sigma_x^A(U, \lambda, M) = \sum_{x \in A} m^A \frac{p(x, A) - g_x^A(U, \lambda)}{f^A(U, \lambda)} = m^A \frac{1 - \sum_{x \in A} g_x^A(U, \lambda)}{f^A(U, \lambda)} = m^A = 1 + \frac{|A|}{|X|^2} p^*$.

Step 2.6.(iii). $\sum_{y \in B} \sigma_y^A(U, \lambda, M) \leq 1 - \frac{(p^*)^2}{4}$ for any $B \subset A$.

Suppose $x \in A \setminus B$. Then by Step 2.6.(i),

$$\sum_{y \in B} \sigma_y^A(U, \lambda, M) \leq \sum_{y \in A} \sigma_y^A(U, \lambda, M) - \sigma_x^A(U, \lambda, M) = m^A - \sigma_x^A(U, \lambda, M) < m^A - \frac{m^A p^*}{2}.$$

Finally,

$$m^A - \frac{m^A p^*}{2} = \left(1 - \frac{p^*}{2} \right) \left(1 + \frac{|A|}{|X|^2} p^* \right) \leq \left(1 - \frac{p^*}{2} \right) \left(1 + \frac{|X|}{|X|^2} p^* \right) \leq \left(1 - \frac{p^*}{2} \right) \left(1 + \frac{1}{2} p^* \right) = 1 - \frac{(p^*)^2}{4}.$$

To sum up, Step 1 shows that p admits the representation (16) with respect to U if and only if U is a fixed point of σ . Step 2 shows that σ has a fixed point by Brouwer's fixed point theorem. Therefore, by Steps 1-2 and Lemma 1, any p is an unrestricted cross-nested logit.

A.8 Proof of Proposition 4

We use standard strong consistency results for M-estimators (e.g., see p. 2121-2 of Newey and McFadden (1994)). To obtain $\hat{\mathcal{X}} \xrightarrow{a.s.} \mathcal{X}^*$, since the set of all nest structures is finite, we only need to prove that \mathcal{X}^* is the unique minimizer of D^* and $D \xrightarrow{a.s.} D^*$.

We first simplify the calculation of $D_1(\mathcal{Y})$. Let $\bar{\epsilon}_{B,A} \equiv \frac{\sum_{a \in A \cap B} \epsilon_{a,A}}{\bar{p}(A \cap B, A)}$. Then $r_A(a, b) = \frac{\bar{p}(a, A) + \epsilon_{a,A}}{\bar{p}(b, A) + \epsilon_{b,A}} = \bar{r}_A(a, b) \frac{1 + \bar{\epsilon}_{a,A}}{1 + \bar{\epsilon}_{b,A}}$. Let

$$\zeta_{a,b,A,B} \equiv \log \left(\frac{1 + \bar{\epsilon}_{a,A}}{1 + \bar{\epsilon}_{b,A}} \right) \text{ and } \delta_{a,b,A,B} \equiv \log \left(\frac{\bar{r}_A(a, b)}{\bar{r}_B(a, b)} \right).$$

Then

$$\log \left(\frac{r_A(a, b)}{r_B(a, b)} \right) = \delta_{a,b,A,B} + \zeta_{a,b,A,B}.$$

We now simplify the calculation of $D_2(\mathcal{Y})$. Note that

$$r_A(Y, Y') = \frac{p(A \cap Y, A)}{p(A \cap Y', A)} = \frac{\bar{p}(A \cap Y, A) + \sum_{a \in A \cap Y} \epsilon_{a,A}}{\bar{p}(A \cap Y', A) + \sum_{a \in A \cap Y'} \epsilon_{a,A}} = \bar{r}_A(Y, Y') \frac{1 + \bar{\epsilon}_{Y,A}}{1 + \bar{\epsilon}_{Y',A}}.$$

Similarly, let

$$\zeta_{Y,Y',A,B} \equiv \log \left(\frac{1 + \bar{\epsilon}_{Y,A}}{1 + \bar{\epsilon}_{Y',A}} \right) \text{ and } \delta_{Y,Y',A,B} \equiv \log \left(\frac{\bar{r}_A(Y, Y')}{\bar{r}_B(Y, Y')} \right).$$

Then

$$\log \left(\frac{r_A(Y, Y')}{r_B(Y, Y')} \right) = \delta_{Y,Y',A,B} + \zeta_{Y,Y',A,B}.$$

Let $N_1(\mathcal{Y}) = \sum_{Y \in \mathcal{Y}} |\{(A, B, a, b) | a, b \in A \cap B \cap Y\}|$ and $N_2(\mathcal{Y}) = \sum_{Y, Y' \in \mathcal{Y}} |\{(A, B) | A \cap Y = B \cap Y, A \cap Y' = B \cap Y'\}|$ and take any $M > \max_{\mathcal{Y}} N_1(\mathcal{Y}), \max_{\mathcal{Y}} N_2(\mathcal{Y})$. Hence,

$$D(\mathcal{Y}) = \frac{\sum_{Y \in \mathcal{Y}} \sum_{A, B \in \mathcal{A}, a, b \in A \cap B \cap Y} (\delta_{a,b,A,B} + \zeta_{a,b,A,B})^2}{N_1(\mathcal{Y})} + \frac{\sum_{Y, Y' \in \mathcal{Y}} \sum_{A, B \in \mathcal{A}: A \cap Y = B \cap Y, A \cap Y' = B \cap Y'} (\delta_{Y,Y',A,B} + \zeta_{Y,Y',A,B})^2}{N_2(\mathcal{Y})}.$$

We then show that \mathcal{X}^* is the unique minimizer of D^* . Since $D^*(\mathcal{X}^*) = 0$, we shall show that $D^*(\mathcal{Y}) > 0$ for any nest structure $\mathcal{Y} \neq \mathcal{X}^*$. It is enough to consider the following two cases.

Case 1. \mathcal{Y} is a partition of X such that there are $a \in X_i$ and $b \in X_j$ such that $a, b \in Y$ for some $Y \in \mathcal{Y}$.

By Assumption 1, there are A, B with $\bar{r}_A(a, b) \neq \bar{r}_B(a, b)$; i.e., $\delta_{a,b,A,B} \neq 0$. Then we have $D^*(\mathcal{Y}) \geq D_1^*(\mathcal{Y}) > (\delta_{a,b,A,B})^2/M > 0$.

Case 2. \mathcal{Y} is a partition of X such that for any $Y \in \mathcal{Y}$, $Y \subseteq X_i$ for some i , and $Y' \subset X_j$ for some $Y' \in \mathcal{Y}$ and j .

Take any Y, Y' such that $Y \subset X_i$ and $Y' \subseteq X_j$. By Assumption 1, there are A, B such such that $\bar{r}_A(Y, Y') \neq \bar{r}_B(Y, Y')$, $A \cap Y = B \cap Y$, and $A \cap Y' = B \cap Y'$. That is, $\delta_{Y, Y', A, B} \neq 0$. Then we have $D^*(\mathcal{Y}) \geq D_2^*(\mathcal{Y}) \geq (\delta_{Y, Y', A, B})^2 / M > 0$.

We finally show that $D(\mathcal{Y}) \xrightarrow{a.s.} D^*(\mathcal{Y})$ for every \mathcal{Y} . We have

$$\begin{aligned} D(\mathcal{Y}) - D^*(\mathcal{Y}) &= \sum \frac{((\delta_{a,b,A,B} + \zeta_{a,b,A,B})^2 - (\delta_{a,b,A,B})^2)}{N_1(\mathcal{Y})} + \sum \frac{((\delta_{Y,Y',A,B} + \zeta_{Y,Y',A,B})^2 - (\delta_{Y,Y',A,B})^2)}{N_2(\mathcal{Y})} \\ &= \sum \frac{\zeta_{a,b,A,B} (2\delta_{a,b,A,B} + \zeta_{a,b,A,B})}{N_1(\mathcal{Y})} + \sum \frac{\zeta_{Y,Y',A,B} (2\delta_{Y,Y',A,B} + \zeta_{Y,Y',A,B})}{N_2(\mathcal{Y})} \xrightarrow{a.s.} 0 \end{aligned}$$

since $\zeta_{a,b,A,B} \xrightarrow{a.s.} 0$, $\zeta_{Y,Y',A,B} \xrightarrow{a.s.} 0$, and $\delta_{a,b,A,B}$ and $\delta_{Y,Y',A,B}$ are constants.

A.9 Proof of Proposition 5

Take any $a, b \in X$. As we showed in the proof of Proposition 6, we have $\log(r_A(a, b)/r_B(a, b)) = \delta_{a,b,A,B} + \zeta_{a,b,A,B}$. If $a, b \in X_i$ for some i , then

$$d(a, b) = \frac{\sum_{A, B \in \mathcal{A}: a, b \in A \cap B} \zeta_{a,b,A,B}^2}{|\{(A, B, a, b) | a, b \in A \cap B\}|} \xrightarrow{a.s.} 0.$$

If $a \in X_i$ and $b \in X_j$ for some i, j with $i \neq j$, then by Assumption 1, there are A^*, B^* with $\bar{r}_{A^*}(a, b) \neq \bar{r}_{B^*}(a, b)$; i.e., $\delta_{a,b,A^*,B^*} \neq 0$. Hence,

$$d(a, b) > \frac{(\delta_{a,b,A^*,B^*} + \zeta_{a,b,A^*,B^*})^2}{M} \geq \frac{(\delta_{a,b,A^*,B^*})^2}{2M} \text{ almost surely.}$$

Let

$$\epsilon^* \equiv \min_{a', b', A', B': \delta_{a', b', A', B'} \neq 0} \frac{(\delta_{a', b', A', B'})^2}{2M}.$$

Then by the previous inequality, $d(a, b) > \epsilon^*$ almost surely. Since $\epsilon^* > 0$, there is \bar{N} such that for any $N^* > \bar{N}$,

$$\max_i \max_{a, b \in X_i} d(a, b) < \epsilon^* < \min_{i < j} \min_{a' \in X_i, b' \in X_j} d(a', b') \text{ with probability one.}$$

A.10 Proofs of Proposition 6 and Corollary 2

Proof of Proposition 6. Since X and \mathcal{A} are finite, there is a set $\{d_1, \dots, d_m\}$ of positive real numbers such that $\max_{a,b} d(a, b) = d_m > \dots > d_2 > d_1 = \min_{a,b} d(a, b) \geq 0$ and for any $a', b' \in X$, $d(a', b') = d_s$ for some $s \leq m$. Hence, it is immediate that $|\mathcal{X}^*| \leq |X|^2$.

Let us prove $|\mathcal{X}^*| \leq |X|$ by induction on the number of alternatives $|X|$. When $|X| = 2$, we have $|\mathcal{X}^*| \leq |\mathcal{X}| = 2 = |X|$. Suppose that the hypothesis is true for any set X with $|X| = k$. We shall prove that this also holds for all sets X with $|X| = k + 1$.

Take a set X and suppose $|X| = k + 1$. Let a^*, b^* be elements of X such that $d_1 = d(a^*, b^*)$.

Hence, for any $\epsilon \in (d_1, d_m]$, $a^* \sim_\epsilon b^*$. In other words, a^* and b^* belong to the same nest for any partition \mathcal{X}_ϵ with $\epsilon \in (d_1, d_m]$. If $d_1 = d(a^*, b^*) = d_m = \max_{a,b \in X} d(a, b)$, then we obtain the desired result since $|\mathcal{X}| = 1$. Suppose that $d(a^*, b^*) < d_m$. Then, without loss of generality, we can assume that $d_m = \max_{a,b \in X'} d(a, b)$ where $X' = X \setminus \{b^*\}$. Because $|X'| \leq k$, it follows from the induction assumption that there are at most k different partitions in $\mathcal{X}'^* = \{\mathcal{X}'_\epsilon\}_{\epsilon \in [0, d_m]}$.

Now let us consider \mathcal{X}^* . When $\epsilon \in (d_1, d_m]$, adding b^* to X' does not increase the number of distinct partitions in $\{\mathcal{X}'_\epsilon\}_{\epsilon \in [0, d_m]}$ since a^* and b^* must belong to the same nest. Note that if \sim_ϵ is transitive on X , then it is also transitive on X' . Hence, adding b^* to X' does not extend the set of ϵ such that \sim_ϵ is transitive. Therefore, there is at most one new partition when b^* is added to X' . Therefore, by induction, $|\mathcal{X}^*| \leq |X|$.

Proof of Corollary 2. In the proof of [Proposition 6](#), we show that there is $\epsilon^* > 0$ such that

$$\max_i \max_{a,b \in X_i} d(a, b) < \epsilon^* < \min_{i < j} \min_{a' \in X_i, b' \in X_j} d(a', b') \text{ almost surely.}$$

Hence $\mathcal{X}_{\epsilon^*} = \mathcal{X}^*$ almost surely. Therefore, $\mathcal{X}^* \in \mathcal{X}^*$ almost surely. By [Proposition 6](#), $\hat{\mathcal{X}} \xrightarrow{a.s.} \mathcal{X}^*$. Therefore, since $\mathcal{X}^* \in \mathcal{X}^*$ almost surely and $\mathcal{X}^* \subset \mathcal{X}$, we have $\hat{\mathcal{X}}^* \xrightarrow{a.s.} \mathcal{X}^*$.

B Additional Results

B.1 Regularity, Increasing NSC, and The Similarity Effect

[Proposition 3](#) shows that regularity has important behavioral implications for nested logit. In this section, we study the implications of regularity for general NSC. We show that there is a deep connection between regularity, increasing NSCs, and the similarity effect. To clarify the implications of regularity, we divide regularity into two logically independent axioms (as we did for IIA).

AXIOM 6 (Dissimilar Regularity). For any $A \in \mathcal{A}$, $x \in A$, and $y \in X$,

$$p(x, A \cup y) \leq p(x, A) \text{ when } x \approx_p y.$$

AXIOM 7 (Similar Regularity). For any $A \in \mathcal{A}$, $x \in A$, and $y \in X$,

$$p(x, A \cup y) \leq p(x, A) \text{ when } x \sim_p y.$$

The first axiom, [Dissimilar Regularity](#), says that regularity should hold when x and y are revealed dissimilar, while the second axiom, [Similar Regularity](#), requires regularity when x and y are revealed similar. It is immediate that the joint assumption of Dissimilar Regularity and Similar Regularity is equivalent to regularity.

The first result shows that Similar Regularity is closely related to the similarity effect. In fact, the similarity effect implies Similar Regularity in a setting that is more general than NSC.

Proposition 7. *For any stochastic choice function p , if \sim_p is transitive, then the **similarity effect** implies **Similar Regularity**.*

The intuition behind this result is quite simple. If an alternative y is introduced and it is similar to some existing alternative x , the similarity effect requires that y hurts x more than it hurts anything that it is not similar to. Thus, the probability of x must decrease. Note that this result only requires that \sim_p is transitive; it does not rely on the structure of NSC. Consequently, it is difficult to explain both the similarity effect and violations of regularity. Further, we can show that under some richness condition, **Similar Regularity** will imply a weak version of the similarity effect, and thus the similarity effect is essentially equivalent to **Similar Regularity**.

The second result shows that **Dissimilar Regularity** is equivalent to a mild but behaviorally important restriction on v : monotonicity in the size of the nest. We say p is an **increasing NSC** if $v(A) \geq v(B)$ for any $i \leq K$ and nonempty sets $A, B \subseteq X_i$ with $B \subseteq A$.

Proposition 8. *A nondegenerate NSC p satisfies **Dissimilar Regularity** if and only if it is an **increasing NSC**.*

Increasing NSC are interesting because they subsume many of the models in the literature, including nested logit. However, increasing NSC are incompatible with certain violations of regularity, such as choice overload.²⁵ In fact, Propositions 7 and 8 imply that increasing NSC cannot allow the similarity effect and violations of regularity simultaneously. Thus, non-increasing NSC (e.g., the menu-dependent substitutability example from section 3.4) are of independent interest.

B.2 Alternative Axiomatization

In this section, we provide an alternative axiomatic characterization of NSC in which characterizing axioms do not rely on our revealed similarity relation \sim_p . To characterize NSC, we “divide” ISA into two axioms.

AXIOM 8 (ISA-1). For any $A \in \mathcal{A}$, $a, b \in A$, and $x \notin A$,

$$\frac{p(a, \{a, x\})}{p(x, \{a, x\})} = \frac{p(a, \{a, b, x\})}{p(x, \{a, b, x\})} \text{ and } \frac{p(b, \{b, x\})}{p(x, \{b, x\})} = \frac{p(b, \{a, b, x\})}{p(x, \{a, b, x\})} \implies \frac{p(a, A)}{p(b, A)} = \frac{p(a, A \cup x)}{p(b, A \cup x)}.$$

Note that the above axiom is essentially identical to the first part of ISA with $a \sim_p x$ and $b \sim_p x$.

AXIOM 9 (ISA-2). For any $A, B, C \in \mathcal{A}$, $a \in A \cap B$, $b \in A \cap C$, and $x \in B \cap C$,

$$\frac{p(a, \{a, x\})}{p(x, \{a, x\})} \neq \frac{p(a, B)}{p(x, B)} \text{ and } \frac{p(b, \{b, x\})}{p(x, \{b, x\})} \neq \frac{p(b, C)}{p(x, C)} \implies \frac{p(a, A)}{p(b, A)} = \frac{p(a, A \cup x)}{p(b, A \cup x)}.$$

²⁵Moreover, in an increasing NSC, the likelihood of choosing a nest can never decrease after expansion. This rules out “aversion to large choice sets” or “preference for commitment” type behavior which may be accommodated by more general NSC.

Similarly, the above axiom is essentially identical to the second part of ISA with $a \not\sim_p x$ and $b \not\sim_p x$.

We also need to strengthen our notion of nondegeneracy as follows. The NSC p with $(v, u, \{X_i\}_{i=1}^N)$ is **strict** if for any $A_i \subset X_i$ with $a \in A_i$ and $x \in X_i \setminus A_i$,

$$\text{if } \frac{u(a) + u(x)}{u(x)} = \frac{v(\{a, x\})}{v(x)}, \text{ then } \frac{\sum_{a' \in A_i} u(a') + u(x)}{\sum_{a' \in A_i} u(a')} = \frac{v(A_i \cup x)}{v(A_i)}.$$

Theorem 5. *Let p be a stochastic choice function with at least three alternatives that are dissimilar to each other. Then p satisfies [AXIOM 8](#) and [AXIOM 9](#) if and only if it is a strict nondegenerate NSC.*

B.3 Three-Step Nested Stochastic Choice

[Theorem 1](#) shows that all two-level nested logit models are in fact special cases of NSC and are characterized by a strong notion of categorical similarity. A natural question is, can we capture more complex substitution patterns through a more general notion of similarity? Put another way, can we allow for contextual or comparative similarity?

Consider a decision maker who is choosing between wines and beers. As in NSC, it is natural to think that there are three nests: one for white wines, one for red wines, and one for beers. Intuitively, the wines are “more similar” to each other than they are to the beers. This can be captured through an intermediate step in which, before deciding between red or white wines, the decision maker decides between wines and beers. After deciding between wine and beer, the consumer decides between different styles of wine (red vs. white), and then selects a specific one to consume. This can be represented through a three-level nested structure, which we refer to as a 3-step NSC. In this section, we show that we can capture such complex relationships through the introduction of a second similarity relation and a generalization of our main axiom.

Formally, any 3-step NSC consists of a nesting structure (tree) and conditional Luce rules.

Definition 8 (3-step NSC). A stochastic choice function p is a **3-Step Nested Stochastic Choice** if there exist a partition X_1, \dots, X_K of X , a partition $X_k^1, \dots, X_k^{q_k}$ of X_k for each $k \leq K$, and functions $u : X \rightarrow \mathbb{R}_{++}$, $w : \bigcup_{i=1}^K 2^{X_i} \rightarrow \mathbb{R}_+$, and $v : \bigcup_{k=1}^K \bigcup_{l=1}^{q_k} 2^{X_k^l} \rightarrow \mathbb{R}_+$ with $w(\emptyset) = v(\emptyset) = 0$ such that for any $A \in \mathcal{A}$ and $x \in A \cap X_k^j$,

$$p(x, A) = \frac{u(x)}{\sum_{y \in A \cap X_k^j} u(y)} \cdot \frac{v(A \cap X_k^j)}{\sum_{l=1}^{q_k} v(A \cap X_k^l)} \cdot \frac{w(A \cap X_k)}{\sum_{i=1}^K w(A \cap X_i)}.$$

We now introduce a secondary notion of similarity which applies to alternatives that are not categorically similar, but satisfy IIA in the presence of mutually dissimilar alternatives.

Definition 9. For any $a, b \in X$, we say a and b are **approximately revealed similar**, denoted

by $a \bowtie_p b$, if $a \not\sim_p b$ and

$$\frac{p(a, A)}{p(b, A)} = \frac{p(a, A \cup x)}{p(b, A \cup x)} \text{ for any } A \in \mathcal{A} \text{ and } x \notin A \text{ with } x \not\sim_p a \text{ and } x \not\sim_p b.$$

We write $a \simeq_p b$ if either $a \sim_p b$ and $a \bowtie_p b$.

It is crucial to note that our approximately revealed similar relation requires that a and b are not categorically similar. Thus we have two distinct “layers” of similarity; \bowtie_p does not include \sim_p as a sub-relation. In terms of our drink example, all the red wines are categorically similar (related through \sim_p), while red and white wines are approximately similar (related through \bowtie_p), as IIA will hold between them when a beer is introduced but not if another wine were introduced. Hence this second layer delineates the “intermediate” nests in the tree and captures aspects of context-dependent similarity. Consequently, this approach distinguishes between fundamental and contextual similarity.

We now introduce a generalization of our main axiom, to characterize 3-step NSC.

AXIOM 10 (Generalized Independence of Symmetric Alternatives). For any $A \in \mathcal{A}$, $a, b \in A$, and $x \notin A$,

$$\begin{array}{c} a \sim_p x \text{ and } b \sim_p x, \\ a \bowtie_p x \text{ and } b \bowtie_p x, \\ \text{or} \\ a \not\sim_p x \text{ and } b \not\sim_p x \end{array} \implies \frac{p(a, A)}{p(b, A)} = \frac{p(a, A \cup x)}{p(b, A \cup x)}.$$

Recall that the second part of Independence of Symmetric Alternatives requires that $\frac{p(a, A)}{p(b, A)} = \frac{p(a, A \cup x)}{p(b, A \cup x)}$ when $a \not\sim_p x$ and $b \not\sim_p x$. However, the second part of Generalized Independence of Symmetric Alternatives requires that $\frac{p(a, A)}{p(b, A)} = \frac{p(a, A \cup x)}{p(b, A \cup x)}$ when either $a \bowtie_p x$ and $b \bowtie_p x$ or $a \not\sim_p x$ and $b \not\sim_p x$. Hence, Generalized Independence of Symmetric Alternatives relaxes the second part of Independence of Symmetric Alternatives.

Lastly, we need a consistency condition to hold between the similarity relations.

AXIOM 11 (Consistency of Revealed Similarities). For any $x, y, x' \in X$, if $x \sim_p x'$, then

$$x \bowtie_p y \text{ if and only if } x' \bowtie_p y.$$

In the language of our drink example, y is a white wine and x, x' are two red wines. Since $x \sim_p x'$, it must be the case that if a white wine y is approximately similar to some red wine x , then it is approximately similar to any other red wine x' .

Theorem 6. Consider a stochastic choice function p such that there are a, b, c with $a \not\sim_p b$, $b \not\sim_p c$, and $a \not\sim_p c$. Suppose, for any $x \in X$, there are $y, z \in X$ such that $x \bowtie_p y$, $y \bowtie_p z$, and $x \bowtie_p z$. If p satisfies **Generalized Independence of Symmetric Alternatives and Consistency of**

Revealed Similarities, then it is a **3-step NSC**.²⁶

The major insight from this result is that multi-step NSC is characterized by revealing multiple, layered similarity relations, and then imposing a generalization of our key axiom. Just as our similarity relation identifies endogenous nests, this secondary relation identifies endogenous, intermediate nests. Thus, [Theorem 6](#) shows that we may identify an endogenous tree structure.

Multi-level nested logit models have been applied to many situations. Most famously, [Goldberg \(1995\)](#) uses a multi-level nested logit to study automobile demand. It is well known that the “order” in which the tree-structure of nests is specified matters for estimates. Our approach reveals the entire, endogenous tree, and so the “order” is also recovered: \bowtie_p captures upper nests and \sim_p captures lower nests. It is straightforward to see how our approach could be extended to characterize an N -Step NSC.

B.4 Remaining Proofs

B.4.1 Proof of [Proposition 7](#)

As discussed in the proof of [Theorem 1](#), when \sim_p is transitive, there is a partition $\{X_i\}_{i=1}^K$ such that for any $x, y \in X$, $x \sim_p y$ if and only if $x, y \in X_i$ for some $i \leq K$. Suppose that p satisfies the similarity effect; that is, for any $A \in \mathcal{A}$ and $a, a' \in X_i$ and $b \in X_j$ with $a, b \in A$ and $a' \notin A$,

$$\frac{p(a, A)}{p(b, A)} > \frac{p(a, A \cup a')}{p(b, A \cup a')}.$$

Step 1. For any $A \in \mathcal{A}$ and $a' \in X_i \setminus A$ with $A \cap X_i \neq \emptyset$, $p(A \cap X_i, A) > p(A \cap X_i, A \cup a')$.

The similarity effect implies that for any $b \in A \setminus X_i$ and $a \in A \cap X_i$,

$$(16) \quad p(b, A \cup a') p(a, A) > p(b, A) p(a, A \cup a').$$

Let us first add Inequality (16) across all $a \in A \cap X_i$. Then we have

$$p(b, A \cup a') p(A \cap X_i, A) > p(b, A) p(A \cap X_i, A \cup a').$$

Let us add again the above inequality across all $b \in A \setminus X_i$. Then we obtain

$$(1 - p(A \cap X_i, A \cup a') - p(a', A \cup a')) p(A \cap X_i, A) > (1 - p(A \cap X_i, A)) p(A \cap X_i, A \cup a').$$

The above inequality implies

$$(1 - p(A \cap X_i, A \cup a')) p(A \cap X_i, A) > (1 - p(A \cap X_i, A)) p(A \cap X_i, A \cup a');$$

equivalently, $p(A \cap X_i, A) > p(A \cap X_i, A \cup a')$.

²⁶Indeed, just as in [Theorem 1](#), the necessity direction also holds when the appropriate nondegeneracy condition is imposed on p .

Step 2. For any $A \in \mathcal{A}$, $a \in A \cap X_i$, and $a' \in X_i \setminus A$, $p(a, A) > p(a, A \cup a')$.

For any $\tilde{a} \in A \cap X_i$, we have $\frac{p(a, A \cup a')}{p(\tilde{a}, A \cup a')} = \frac{p(a, A)}{p(\tilde{a}, A)}$. Let us add the equality $p(a, A \cup a') p(\tilde{a}, A) = p(a, A) p(\tilde{a}, A \cup a')$ for all $\tilde{a} \in A \cap X_i$. Then we obtain $p(a, A \cup a') p(A \cap X_i, A) = p(a, A) p(A \cap X_i, A \cup a')$. Finally, since $p(A \cap X_i, A \cup a') < p(A \cap X_i, A)$, we need to have $p(a, A) > p(a, A \cup a')$.

B.4.2 Proof of Proposition 8

Let p be a nondegenerate NSC with $(v, u, \{X_i\}_{i=1}^K)$.

Sufficiency. Take any $A_j \subset X_j$ and $y \in X_j \setminus A_j$. Take any $x \in X_i$ with $i \neq j$. Since $x \not\sim_p y$, by Dissimilar Regularity, we have

$$p(x, A_j \cup x \cup y) = \frac{v(x)}{v(x) + v(A_j \cup y)} \leq p(x, A_j \cup x) = \frac{v(x)}{v(x) + v(A_j)} \text{ iff } v(A_j) \leq v(A_j \cup y).$$

Necessity. Suppose v is increasing. Take any $A \in \mathcal{A}$, $x \in A$, and $y \notin A$ with $x \not\sim_p y$. Therefore, $x \in X_i$ and $y \in X_j$ for some i, j with $i \neq j$. Since $v(A \cap X_j \cup y) \geq v(A \cap X_j)$, we have

$$p(x, A \cup y) = \frac{p(x, A \cap X_i) v(A \cap X_i)}{v(A_j \cup y) + \sum_{k \neq j} v(A \cap X_k)} \leq p(x, A) = \frac{p(x, A \cap X_i) v(A \cap X_i)}{v(A_j) + \sum_{k \neq j} v(A \cap X_k)}.$$

B.4.3 Proof of Theorems 5

Sufficiency. Take any $A \in \mathcal{A}$, $a, b \in A$, and $x \notin A$. Suppose $a \sim_p x$ and $b \sim_p x$. Then we have $\frac{p(a, \{a, x\})}{p(x, \{a, x\})} = \frac{p(a, \{a, b, x\})}{p(x, \{a, b, x\})}$ and $\frac{p(b, \{b, x\})}{p(x, \{b, x\})} = \frac{p(b, \{a, b, x\})}{p(x, \{a, b, x\})}$. Hence, by Axiom 8, $\frac{p(a, A)}{p(b, A)} = \frac{p(a, A \cup x)}{p(b, A \cup x)}$. Suppose $a \not\sim_p x$ and $b \not\sim_p x$. Then there are B, C such that $\frac{p(a, \{a, x\})}{p(x, \{a, x\})} \neq \frac{p(a, B)}{p(x, B)}$ and $\frac{p(b, \{b, x\})}{p(x, \{b, x\})} \neq \frac{p(b, C)}{p(x, C)}$. Hence, by Axiom 9, $\frac{p(a, A)}{p(b, A)} = \frac{p(a, A \cup x)}{p(b, A \cup x)}$. Therefore, ISA is satisfied. Hence, by Theorem 1, p is a nondegenerate NSC with some $(v, u, \{X_i\}_{i=1}^K)$ where $X / \sim_p = \{X_i\}_{i=1}^K$.

To show the strictness, take any $a, x \in X_i$ such that $\frac{u(a) + u(x)}{u(x)} = \frac{v(\{a, x\})}{v(x)}$. Then for any $b \in X_j$, $\frac{p(b, \{b, x\})}{p(x, \{b, x\})} = \frac{p(b, \{a, b, x\})}{p(x, \{a, b, x\})}$. Since $\frac{p(a, \{a, x\})}{p(x, \{a, x\})} = \frac{p(a, \{a, b, x\})}{p(x, \{a, b, x\})}$, by Axiom 8, we have $\frac{p(a, A)}{p(b, A)} = \frac{p(a, A \cup x)}{p(b, A \cup x)}$. By NSC, we have

$$\frac{\frac{u(a)}{\sum_{a' \in A \cap X_i} u(a')} v(A \cap X_i)}{\frac{u(b)}{\sum_{b' \in A \cap X_j} u(b')} v(A \cap X_j)} = \frac{\frac{u(a)}{\sum_{a' \in A \cap X_i} u(a') + u(x)} v(A \cap X_i \cup x)}{\frac{u(b)}{\sum_{b' \in A \cap X_j} u(b')} v(A \cap X_j)}$$

equivalently,

$$\frac{\sum_{a' \in A \cap X_i} u(a') + u(x)}{\sum_{a' \in A \cap X_i} u(a')} = \frac{v(A \cap X_i \cup x)}{v(A \cap X_i)}.$$

Necessity. Suppose p is the strict nondegenerate NSC with $(v, u, \{X_i\}_{i=1}^K)$. By the necessity part of Theorem 1, p satisfies ISA and $a \sim_p b$ if and only if $a, b \in X_i$.

To prove Axiom 8, take any $A \in \mathcal{A}$, $a, b \in A$, and $x \notin A$ with $\frac{p(a, \{a, x\})}{p(x, \{a, x\})} = \frac{p(a, \{a, b, x\})}{p(x, \{a, b, x\})}$ and $\frac{p(b, \{b, x\})}{p(x, \{b, x\})} = \frac{p(b, \{a, b, x\})}{p(x, \{a, b, x\})}$. We shall prove that $\frac{p(a, A)}{p(b, A)} = \frac{p(a, A \cup x)}{p(b, A \cup x)}$. It is immediate when $a, b \in X_i$. Hence, suppose $a \in X_i$ and $b \in X_j$. If $x \in X_k$, then we have $a \not\sim_p x$ and $b \not\sim_p x$. Consequently, by ISA, we have $\frac{p(b, \{b, x\})}{p(x, \{b, x\})} = \frac{p(b, \{a, b, x\})}{p(x, \{a, b, x\})}$. Hence, suppose now either $x \in X_i$ or $x \in X_j$. Since

the role of a and b are symmetric, suppose $a \in X_i$ without loss of generality. By the NSC, $\frac{p(b, \{b, x\})}{p(x, \{b, x\})} = \frac{p(b, \{a, b, x\})}{p(x, \{a, b, x\})}$ implies $\frac{v(b)}{v(x)} = \frac{v(b)}{\frac{u(x)}{u(a)+u(x)}v(\{a, x\})}$; equivalently, $\frac{u(a)+u(x)}{u(x)} = \frac{v(\{a, x\})}{v(x)}$. Since by the strictness, $\frac{\sum_{a' \in A_i} u(a') + u(x)}{\sum_{a' \in A_i} u(a')} = \frac{v(A_i \cup x)}{v(A_i)}$ for any $A_i \subset X_i$.

By the NSC, $\frac{p(a, A)}{p(b, A)} = \frac{p(a, A \cup x)}{p(b, A \cup x)}$ is equivalent to

$$\frac{\frac{u(a)}{\sum_{a' \in A \cap X_i} u(a')} v(A \cap X_i)}{\frac{u(b)}{\sum_{b' \in A \cap X_j} u(b')} v(A \cap X_j)} = \frac{\frac{u(a)}{\sum_{a' \in A \cap X_i} u(a') + u(x)} v(A \cap X_i \cup x)}{\frac{u(b)}{\sum_{b' \in A \cap X_j} u(b')} v(A \cap X_j)}.$$

The above equality holds by the strictness since it is equivalent to

$$\frac{\sum_{a' \in A \cap X_i} u(a') + u(x)}{\sum_{a' \in A \cap X_i} u(a')} = \frac{v(A \cap X_i \cup x)}{v(A \cap X_i)}.$$

To prove Axiom 9, take any $A, B, C \in \mathcal{A}$, $a \in A \cap B$, $b \in A \cap C$, and $x \in B \cap C$ with $\frac{p(a, \{a, x\})}{p(x, \{a, x\})} \neq \frac{p(a, B)}{p(x, B)}$ and $\frac{p(b, \{b, x\})}{p(x, \{b, x\})} \neq \frac{p(b, C)}{p(x, C)}$. Then $a \not\sim_p x$ and $b \not\sim_p x$. Hence, by ISA, $\frac{p(a, A)}{p(b, A)} = \frac{p(a, A \cup x)}{p(b, A \cup x)}$.

B.4.4 Proof of Theorems 6

We prove Theorem 6 by four steps.

Step 1. Note that the first part of Generalized Independence of Symmetric Alternatives is identical to the first part of Independence of Symmetric Alternatives. Hence, by Steps 1-2 of the proof of Theorem 1, \sim_p is reflexive, transitive, and symmetric, we have a partition $X / \sim_p \equiv \{E_i\}_{i=1}^K$ of X such that for any $x_i, x'_i \in E_i$ and $x_j \in E_j$, $x_i \sim_p x'_i$ and $x_i \not\sim_p x_j$.

Step 2. \simeq_p is transitive.

Take any $x, y, z \in X$ such that $x \simeq_p y$ and $y \simeq_p z$. If $x \sim_p y$ and $y \sim_p z$, then by Step 1, $x \sim_p z$. If $x \sim_p y$ and $y \bowtie_p z$, then by Consistency of Revealed Similarities, $x \bowtie_p z$. If $x \bowtie_p y$ and $y \sim_p z$, then by Consistency of Revealed Similarities, $x \bowtie_p z$. Finally, we consider the case where $x \bowtie_p y$ and $y \bowtie_p z$.

Suppose $x \not\sim_p z$. Then we shall prove that $x \bowtie_p z$; i.e., for any A and $t \notin A$ such that $x \not\sim_p t$ and $z \not\sim_p t$, $\frac{p(x, A)}{p(z, A)} = \frac{p(x, A \cup t)}{p(z, A \cup t)}$.

Case 1. $y \in A$. In this case, we can write $\frac{p(x, A)}{p(z, A)} = \frac{p(x, A)}{p(y, A)} / \frac{p(z, A)}{p(y, A)}$ and $\frac{p(x, A \cup t)}{p(z, A \cup t)} = \frac{p(x, A \cup t)}{p(y, A \cup t)} / \frac{p(z, A \cup t)}{p(y, A \cup t)}$. If $y \not\sim_p t$, $x \bowtie_p y$ implies $\frac{p(x, A)}{p(y, A)} = \frac{p(x, A \cup t)}{p(y, A \cup t)}$ and $y \bowtie_p z$ implies $\frac{p(z, A)}{p(y, A)} = \frac{p(z, A \cup t)}{p(y, A \cup t)}$. Therefore, $\frac{p(x, A)}{p(z, A)} = \frac{p(x, A)}{p(y, A)} / \frac{p(z, A)}{p(y, A)} = \frac{p(x, A \cup t)}{p(z, A \cup t)} = \frac{p(x, A \cup t)}{p(y, A \cup t)} / \frac{p(z, A \cup t)}{p(y, A \cup t)}$. Instead, if $y \sim_p t$, then by Consistency of Revealed Preferences $x \bowtie_p y$ and $y \bowtie_p z$ imply $x \bowtie_p t$ and $z \bowtie_p t$. Then by Generalized Independence of Symmetric Alternatives, $\frac{p(x, A)}{p(z, A)} = \frac{p(x, A \cup t)}{p(z, A \cup t)}$.

Case 2. $y \notin A$. By Generalized Independence of Symmetric Alternatives, $x \bowtie_p y$ and $z \bowtie_p y$ imply $\frac{p(x, A)}{p(z, A)} = \frac{p(x, A \cup y)}{p(z, A \cup y)}$ and $\frac{p(x, A \cup t)}{p(z, A \cup t)} = \frac{p(x, A \cup y \cup t)}{p(z, A \cup y \cup t)}$. Now by Case 1, $\frac{p(x, A)}{p(z, A)} = \frac{p(x, A \cup y)}{p(z, A \cup y)} = \frac{p(x, A \cup y \cup t)}{p(z, A \cup y \cup t)} = \frac{p(x, A \cup t)}{p(z, A \cup t)}$.

Step 3. Let $X/\simeq_p \equiv \{X_i\}_{i=1}^n$. Since \simeq_p is reflexive, transitive, and symmetric, $\{X_i\}_{i=1}^n$ is a partition of X such that for any $x_i, x'_i \in X_i$ and $x_j \in X_j$, $x_i \simeq_p x'_i$ and $x_i \not\simeq_p x_j$. Moreover, by the definition of \simeq_p , for any $i \leq K$, there is some $j \leq n$ such that $E_i \subseteq X_j$. Hence, without loss of generality, let $X_i = \bigcup_{s=1}^{t_i} X_i^s$ such that for any $s \leq t_i$, $X_i^s = E_l$ for some $l \leq K$.

Step 4. For any $x, y \in X_i$, $x \bowtie_p y$ if and only if $x \not\sim y$. Hence, the first two parts of Generalized Independence of Symmetric Alternatives are equivalent to Independence of Symmetric Alternatives when p is restricted on X_i . Hence, by Theorem 1, p is an NSC on X_i with some $(u_i, v_i, \{X_i^s\}_{s=1}^{t_i})$. Since X_i and X_j are disjoint for each i, j with $j \neq i$, without loss of generality, we can say that p is an NSC on X_i with the same (v, u) .

Step 5. Since there are $a, b, c \in X$ such that $a \not\sim_p b$, $b \not\sim_p c$, and $a \not\sim_p c$, we have $n \geq 3$. Take any $A \in \mathcal{A}$. Let $A_i = A \cap X_i$ for each $i \leq n$. Take any $a \in A_i, b \in A_j$, and $x \in A_k$. Note that $a \not\sim_p x$ and $b \not\sim_p x$. Then by Generalized Independence of Symmetric Alternatives, we have

$$\frac{p(a, A)}{p(b, A)} = \frac{p(a, A \setminus \{x\})}{p(b, A \setminus \{x\})}.$$

Then by Steps 5-8 of Theorem 1 (also recall Equation (12)), there is a function $w : 2^X \rightarrow \mathbb{R}_+$ such that $\frac{p(A_i, A)}{p(A_j, A)} = \frac{w(A_i)}{w(A_j)}$. In other words, $p(A_i, A) = \frac{w(A_i)}{\sum_j w(A_j)}$. Since p is an NSC on X_i , we also have

$$p(a, A_i) = \frac{v(A_i \cap X_i^s)}{\sum_l v(A_i \cap X_i^l)} \frac{u(a)}{\sum_{b \in A_i \cap X_i^s} u(b)}$$

when $a \in A_i \cap X_i^s$. Finally,

$$p(a, A) = p(A_i, A) p(a, A_i) = \frac{w(A_i)}{\sum_j w(A_j)} \frac{v(A_i \cap X_i^s)}{\sum_l v(A_i \cap X_i^l)} \frac{u(a)}{\sum_{b \in A_i \cap X_i^s} u(b)}.$$