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Overcoming Free-Riding in Bandit Games

Johannes Hörner*
Nicolas Klein**
Sven Rady***

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*Yale University, 30 Hillhouse Ave., New Haven, CT 06520, USA, and TSE (CNRS), and CEPR, johannes.horner@yale.edu
**Université de Montréal, Département de Sciences Économiques, C.P. 6128 succursale Centre-ville; Montréal, H3C 3J7, Canada, and CIREQ, kleinnic@yahoo.com
***University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany, and CEPR, rady@hcm.uni-bonn.de

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Johannes Hörner† Nicolas Klein‡ Sven Rady§

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†Yale University, 30 Hillhouse Ave., New Haven, CT 06520, USA, and TSE (CNRS), and CEPR, johannes.horner@yale.edu.

‡Université de Montréal, Département de Sciences Économiques, C.P. 6128 succursale Centre-ville; Montréal, H3C 3J7, Canada, and CIREQ, kleinnic@yahoo.com.

§University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany, and CEPR, rady@hcm.uni-bonn.de.

Abstract

This paper considers a class of experimentation games with Lévy bandits encompassing those of Bolton and Harris (1999) and Keller, Rady and Cripps (2005). Its main result is that efficient (perfect Bayesian) equilibria exist whenever players' payoffs have a diffusion component. Hence, the trade-offs emphasized in the literature do not rely on the intrinsic nature of bandit models but on the commonly adopted solution concept (MPE). This is not an artifact of continuous time: we prove that such equilibria arise as limits of equilibria in the discrete-time game. Furthermore, it suffices to relax the solution concept to strongly symmetric equilibrium.

KEYWORDS: Two-Armed Bandit, Bayesian Learning, Strategic Experimentation, Strongly Symmetric Equilibrium.

JEL CLASSIFICATION NUMBERS: C73, D83.

1 Introduction

Bandit models involve trade-offs. The exploration vs. exploitation dilemma of the classic multi-armed bandit problem coined by Thompson (1933) and Robbins (1952) has been supplanted by free-riding vs. encouragement effects in a strategic context (Bolton and Harris, 1999). These two effects might be essential to our economic intuition, but the trade-off only arises because of the solution concept, as we show in this paper.

First, we provide some context. Typically, bandit games are modeled in continuous time, with Markov (perfect) equilibrium as the solution concept. This is a sensible choice. Continuous time provides elegant characterizations and even closed-form solutions. Markov equilibrium is the obvious counterpart to the criterion used in operations research, enabling meaningful comparisons with the team solution. It is also dictated by continuous time because standard game-theoretic notions raise conceptual problems (Simon and Stinchcombe, 1989). However, there is a price to pay. Markov equilibrium excludes rewards and punishments, the cornerstones of dynamic games. What Markov taketh away, continuous time giveth: equilibria arise with no discrete-time equivalent.[1]

We show that the main equilibrium prediction, namely inefficiently low experimentation, (mostly) disappears once the model is cast in the framework traditionally used in dynamic games. Relaxing Markov equilibrium and pruning out artifacts of continuous time requires discretizing the game. We then let the time interval between successive actions vanish to obtain a similarly clean characterization and a proper comparison with the literature. Asymptotically, efficiency obtains, as long as payoffs involve an informative diffusion (as opposed to a pure jump) component.

What efficient experimentation entails depends on the players' patience. Hence, this is not a folk theorem, which would not apply in any case: beliefs are not reversible. For example, players would never be convinced that the risky arm is good if it is not. In addition, efficiency does not always hold: with pure jumps, this depends on how good news impacts the common belief.

Efficiency only obtains if a selfish, lone player would not experiment given any resulting posterior belief, had he been on the brink of stopping as a selfless team player given the prior belief. Intuitively, having all other players stop experimenting is the worst punishment a deviating player can face.

This punishment is needed but not a given with pure jumps. Indeed, it fails with conclusive good news. However, when there is a diffusion component, the predominant

[1] For instance, the "infinite switching equilibria" of Keller et al. (2005). This is because there is no last time before a given date in continuous time.

effect of "good news" on the belief is not a jump but a slight uptick because of the Brownian term. Had the team player been on the brink of stopping, a slight uptick would definitely push his selfish version into stopping territory, given his newly discovered solitude. Hence, efficiency is obtained in the setting of Bolton and Harris (1999), independent of the parameters.

We show that efficiency is attainable with strongly symmetric equilibria (SSEs), that is, equilibria in which all players use the same continuation strategy for any given history, independent of their identity (e.g., regardless of whether they had been the sole deviator). There is no need to resort to more complicated perfect Bayesian equilibria (PBEs). Of course, PBE need not involve symmetric payoffs, but we show that in terms of total payoffs across players, there is no difference between SSE and PBE: the worst and best total equilibrium payoffs coincide.[2]

A few caveats are in order.

First, showing that the free-rider and encouragement effects do not determine the outcome of bandit games is akin to noting that efficiency is achievable in the repeated prisoner's dilemma even if defection is dominant in the stage game: it would not occur to us to demonstrate how cooperation arises in the repeated version without first remarking that free-riding is the problem we are addressing. In stochastic games such as bandits, discerning the underlying incentives is difficult and subtle: solving for the Markov equilibria is the advisable approach. Our point is that we must disentangle these incentives from the possible equilibrium outcomes.

Second, we have emphasized the importance of studying the discrete-time game to factor out equilibria that are continuous-time quirks. However, our results are asymptotic to the extent that they only hold when the time interval between rounds is small enough. There is no difference between an arbitrarily small uptick vs. a discrete jump when the interval length is bounded away from zero. Our results rely heavily on what is known about the continuous-time limits and hence on the analyses of Bolton and Harris (1999) and Keller et al. (2005), among others. To the extent that some of our proofs are involved, it is because they require careful comparison and convergence arguments.

Third, because we rely on discrete time, we must settle on a particular discretization. We consider our choice to be natural: players may revise their action choices at equally spaced time opportunities, while payoffs and information accrue in continuous time, independent of the duration of the intervals. That is, ours is the simplest version

[2]One appealing property of SSEs is that payoffs can be studied via a coupled pair of functional equations that extends the functional equation characterizing MPE payoffs (see Proposition 11).

of inertia strategies, as introduced by Bergin and MacLeod (1993). Other discretization choices may lead to different predictions.

Fourth, our results do not cover all bandit games. Because we build on existing results of the single-agent case, we cannot go beyond the framework used for them. In particular, we must make restrictions similar to Cohen and Solan (2013) in their analysis of the continuous-time bandit problem. In fact, our assumptions are stronger than theirs.[3] Our main restriction, just as theirs, is that bad-news jumps are not permitted, which means that our framework does not subsume Keller and Rady (2015), in particular.[4]

Our paper belongs to the growing literature on strategic bandits. We have already discussed the standard references in that literature. There is no need to review the large and growing literature on extensions, variations and applications. With few exceptions, these papers model the game in continuous time and focus on MPEs unless actions on at least one side are not observed (meaning that applying standard game-theoretic solution concepts raises no difficulty).

Second, our paper contributes to the literature on SSE. We hope that it illustrates how SSE can be usefully applied to games usually cast in continuous time, such as bandit games. SSEs have been studied in repeated games since Abreu (1986). They are known to be restrictive. First, they make no sense if the model itself fails to be symmetric. However, as Abreu (1986) notes for repeated games, they are (i) easily calculated, being completely characterized by two simultaneous scalar equations; (ii) more general than static Nash, or even Nash reversion; and even (iii) without loss in terms of total welfare, at least in some cases, as in ours. See also Abreu, Pearce and Stacchetti (1986) for the optimality of symmetric equilibria within a standard oligopoly framework and Abreu, Pearce and Stacchetti (1993) for a motivation of the solution concept based on a notion of equal bargaining power. Cronshaw and Luenberger (1994) conduct a more general analysis for repeated games with perfect monitoring, showing how the set of SSE payoffs can be obtained by solving for the largest scalar solving a certain equation. Hence, our paper shows that Properties (i)–(iii) extend to bandit games, with "Markov perfect" replacing "Nash" in statement (ii) and "functional" replacing "scalar" in (i): as mentioned above, a pair of functional equations replaces the usual Hamilton-Jacobi-Bellman (HJB) (or Isaacs) equation from optimal control.

[3] We do not a priori perceive a difficulty in adopting theirs, but we also do not perceive any benefits.

[4] The technical difficulty with bad-news jumps is that the value functions cannot be described explicitly. They are rather defined recursively, with the functional form depending on the number of bad news events triggering an end to all experimentation. Because of this complication, we leave the analysis of this case to future work.

Section 2 introduces the model. Section 3 characterizes the efficient solution when actions can be chosen in continuous time and shows that MPEs cannot achieve efficiency. Section 4 presents the game in which actions can only be adjusted at regularly spaced points in time, the discrete-time game or discrete game for short. Section 5 contains the main results regarding the set of equilibrium payoffs in the discrete game as the time between consecutive choices tends to zero. Section 6 is devoted to the construction of SSE in the discrete game. Section 7 studies functional equations that characterize SSE payoffs in both the discrete game and the continuous-time limit. Section 8 concludes the paper. Appendix A presents auxiliary results on the evolution of beliefs and on various payoff functions. The proofs of all other results are relegated to Appendix B.

2 The Model

Time $t \in [0, \infty)$ is continuous. There are $N \geq 2$ players, each facing the same two-armed bandit problem with one safe and one risky arm.

The safe arm generates a known constant payoff $s > 0$ per unit of time. The distribution of the payoffs generated by the risky arm depends on the state of the world, $\theta \in \{0, 1\}$, which nature draws at the outset with $\mathbb{P}[\theta = 1] = p$. Players do not observe θ, but they know p. They also understand that the evolution of the risky payoffs depends on θ. Specifically, the payoff process X^n associated with player n's risky arm evolves according to

$$dX_t^n = \alpha_\theta \, dt + \sigma \, dZ_t^n + h \, dN_t^n,$$

where Z^n is a standard Wiener process, N^n is a Poisson process with intensity λ_θ, and the scalar parameters $\alpha_0, \alpha_1, \sigma, h, \lambda_0, \lambda_1$ are known to all players. Conditional on θ, the processes $Z^1, \ldots, Z^N, N^1, \ldots, N^N$ are independent. As Z^n and $N^n - \lambda_\theta t$ are martingales, the expected payoff increment from using the risky arm over an interval of time $[t, t + dt)$ is $m_\theta \, dt$ with $m_\theta = \alpha_\theta + \lambda_\theta h$.

Players share a common discount rate $r > 0$. We write $k_{n,t} = 0$ if player n uses the safe arm at time t and $k_{n,t} = 1$ if the player uses the risky arm at time t.[5] Given actions $(k_{n,t})_{t \geq 0}$ such that $k_{n,t} \in \{0, 1\}$ is measurable with respect to the information

[5]Bolton and Harris (1999), Keller et al. (2005) and Keller and Rady (2010) allow the players to allocate one unit of a perfectly divisible resource freely across the two arms at each point in time, so the fraction allocated to the risky arm can be $k_{n,t} \in [0, 1]$. As the efficient solution in continuous time does not require such interior allocations, we do not consider them here.

available at time t, player n's total expected discounted payoff, expressed in per-period units, is

$$\mathbb{E}\left[\int_0^\infty re^{-rt}\left[(1-k_{n,t})s+k_{n,t}m_\theta\right]\,dt\right],$$

where the expectation is over both the random variable θ and the stochastic process $(k_{n,t})$.[6]

We make the following assumptions: (i) $m_0 < s < m_1$, so each player prefers the risky arm to the safe arm in state $\theta = 1$ and prefers the safe arm to the risky arm in state $\theta = 0$. (ii) $\sigma > 0$ and $h > 0$, so the Brownian payoff component is always present and jumps of the Poisson component entail positive lump-sum payoffs;[7] (iii) $\lambda_1 \geq \lambda_0 \geq 0$, so jumps are at least as frequent in state $\theta = 1$ as in state $\theta = 0$.

Players begin with a common prior belief about θ, given by the probability p with which nature draws state $\theta = 1$. Thereafter, they learn about this state in a Bayesian fashion by observing one another's actions and payoffs; in particular, they hold common posterior beliefs throughout time. A detailed description of the evolution of beliefs is presented in Appendix A.1. When $\lambda_1 = \lambda_0$ (and hence $\alpha_1 > \alpha_0$), the arrival of a lump-sum payoff contains no information about the state of the world, and our setup is equivalent to that in Bolton and Harris (1999), with the learning being driven entirely by the Brownian payoff component. When $\alpha_1 = \alpha_0$ (and hence $\lambda_1 > \lambda_0$), the Brownian payoff component contains no information, and our setup is equivalent to that in Keller et al. (2005) or Keller and Rady (2010), depending on whether $\lambda_0 = 0$ or $\lambda_0 > 0$, with the learning being driven entirely by the arrival of lump-sum payoffs.[8]

3 Efficiency and Markov Perfect Equilibria in Continuous Time

The authors cited in the previous paragraph assume that players use continuous-time Markov strategies with the posterior belief as the state variable, so that $k_{n,t}$ is a time-

[6]Note that we have not yet defined the set of strategies available to each player and hence are silent at this point on how the players' strategy profile actually induces a stochastic process of actions $(k_{n,t})_{t\geq 0}$ for each of them. We will close this gap in two different ways in Sections 3 and 4: by imposing Markov perfection in the former and a discrete time grid of revision opportunities in the latter.

[7]This rules out "breakdowns" as in Keller and Rady (2015).

[8]Keller et al. (2005) and Keller and Rady (2010) consider compound Poisson processes where the distribution of lump-sum payoffs (and their mean h) at the time of a Poisson jump is independent of, and hence uninformative about, the state of the world. By contrast, Cohen and Solan (2013) allow for Lévy processes where the size of lump-sum payoffs contains information about the state, but a lump sum of any given size arrives weakly more frequently in state $\theta = 1$.

invariant deterministic function of the probability p_t assigned to state $\theta = 1$ at time t.[9] In this section, we show how some of their main insights generalize to the present setting. First, we present the efficient benchmark. Second, we show that efficient behavior cannot be sustained as an MPE.

Consider a planner who maximizes the *average* of the players' expected payoffs in continuous time by selecting an entire action profile $(k_{1,t}, \ldots, k_{N,t})$ at each time t. The corresponding average expected payoff increment is

$$\left[\left(1 - \frac{K_t}{N}\right) s + \frac{K_t}{N} m_\theta\right] dt \qquad \text{with} \qquad K_t = \sum_{n=1}^{N} k_{n,t}.$$

A straightforward extension of the main results of Cohen and Solan (2013) shows that the evolution of beliefs also depends on K_t only[10] and that the planner's value function, denoted by V_N^*, has the following properties.

First, V_N^* is the unique once-continuously differentiable solution of the HJB equation

$$v(p) = s + \max_{K \in \{0,1,\ldots,N\}} K \left[b(p, v) - \frac{c(p)}{N}\right]$$

on the open unit interval subject to the boundary conditions $v(0) = m_0$ and $v(1) = m_1$. Here,

$$b(p, v) = \frac{\rho}{2r} p^2 (1-p)^2 v''(p) - \frac{\lambda_1 - \lambda_0}{r}\, p(1-p)\, v'(p) + \frac{\lambda(p)}{r}\, [v(j(p)) - v(p)]$$

can be interpreted as the expected informational benefit of using the risky arm when continuation payoffs are given by a (sufficiently regular) function v.[11] Its first term reflects Brownian learning. Its second term captures the downward drift in the belief when no Poisson lump sum arrives. Its third term expresses the discrete change in the overall payoff once such a lump sum arrives, with the belief jumping up from p to

$$j(p) = \frac{\lambda_1 p}{\lambda(p)};$$

[9] In the presence of discrete payoff increments, one actually has to take the left limit p_{t-} as the state variable, owing to the informational constraint that the action chosen at time t cannot depend on the arrival of a lump sum at t. In the following, we simply write p_t with the understanding that the left limit is meant whenever this distinction is relevant. Note that $p_{0-} = p_0$ by convention.

[10] Cf. Appendix A.1.

[11] Up to division by r, this is the infinitesimal generator of the process of posterior beliefs for $K = 1$, applied to the function v; cf. Appendix A.1 for details.

this occurs at the expected rate

$$\lambda(p) = p\lambda_1 + (1-p)\lambda_0.$$

The function

$$c(p) = s - m(p)$$

captures the opportunity cost of playing the risky arm in terms of expected current payoff forgone; here,

$$m(p) = pm_1 + (1-p)m_0$$

denotes the risky arm's expected flow payoff given the belief p. Thus, the planner weighs the shared opportunity cost of each experiment on the risky arm against the learning benefit, which accrues fully to each agent because of the perfect informational spillover.

Second, there exists a cutoff p_N^* such that all agents using the safe arm ($K = 0$) is optimal for the planner when $p \leq p_N^*$, and all agents using the risky arm ($K = N$) is optimal when $p > p_N^*$. This cutoff is given by

$$p_N^* = \frac{\mu_N(s - m_0)}{(\mu_N + 1)(m_1 - s) + \mu_N(s - m_0)},$$

where μ_N is the unique positive solution of the equation

$$\frac{\rho}{2}\mu(\mu+1) + (\lambda_1 - \lambda_0)\mu + \lambda_0\left(\frac{\lambda_0}{\lambda_1}\right)^{\mu} - \lambda_0 - \frac{r}{N} = 0.$$

Both μ_N and p_N^* increase in r/N. Thus, the interval of beliefs for which all agents using the risky arm is efficient widens with the number of agents and their patience.

Third, the value function satisfies $V_N^*(p) = s$ for $p \leq p_N^*$, and

$$V_N^*(p) = m(p) + \frac{c(p_N^*)}{u(p_N^*; \mu_N)}\, u(p; \mu_N) > s, \tag{1}$$

for $p > p_N^*$, where

$$u(p; \mu) = (1-p)\left(\frac{1-p}{p}\right)^{\mu}$$

is strictly decreasing and strictly convex for $\mu > 0$. The function V_N^* is strictly increasing and strictly convex on $[p_N^*, 1]$.

By setting $N = 1$, one obtains the single-agent value function V_1^* and corresponding cutoff $p_1^* > p_N^*$.

Now consider $N \geq 2$ players acting noncooperatively. Suppose that each of them uses a Markov strategy with the common belief as the state variable. As in Bolton and Harris (1999), Keller et al. (2005) and Keller and Rady (2010), the HJB equation for player n when he or she faces opponents who use Markov strategies is given by

$$v_n(p) = s + K_{\neg n}(p)b(p, v_n) + \max_{k_n \in \{0,1\}} k_n \left[b(p, v_n) - c(p) \right],$$

where $K_{\neg n}(p)$ is the number of n's opponents that use the risky arm. That is, when playing a best response, each player weighs the opportunity cost of playing risky against his or her own informational benefit only. Consequently, V_N^* does not solve the above HJB equation when player n's opponents use the efficient strategy. Efficient behavior therefore cannot be sustained in MPE.

4 The Discrete Game

Henceforth, we restrict players to changing their actions only at the times $t = 0, \Delta, 2\Delta, \ldots$ for some fixed $\Delta > 0$. This yields a discrete-time game evolving in a continuous-time framework; in particular, the payoff processes are observed continuously.[12] Moreover, we allow for non-Markovian strategies.

The expected discounted payoff increment from using the safe arm for the length of time Δ is $\int_0^\Delta r\, e^{-r\,t}\, s\, dt = (1-\delta)s$ with $\delta = e^{-r\,\Delta}$. Conditional on θ, the expected discounted payoff increment from using the risky arm is $\int_0^\Delta r\, e^{-r\,t}\, m_\theta\, dt = (1-\delta)m_\theta$. Given the probability p assigned to $\theta = 1$, the expected discounted payoff increment from the risky arm conditional on all available information is $(1-\delta)m(p)$.

A history of length $t = \Delta, 2\Delta, \ldots$ is a sequence

$$h_t = \left(\left(k_{n,0}, \widetilde{Y}^n_{[0,\Delta)}\right)_{n=1}^N, \left(k_{n,\Delta}, \widetilde{Y}^n_{[\Delta,2\Delta)}\right)_{n=1}^N, \ldots, \left(k_{n,t-\Delta}, \widetilde{Y}^n_{[t-\Delta,t)}\right)_{n=1}^N \right),$$

where $k_{n,\ell\Delta} = 1$ if player n uses the risky arm on the time interval $[\ell\Delta, (\ell+1)\Delta)$;

[12] While arguably natural, our discretization remains nonetheless *ad hoc*, and other discretizations might yield other results. Not only is it well known that the limits of the discrete-time models might differ from the continuous-time solutions, but the particular discrete structure might also matter; see, among others, Müller (2000), Fudenberg and Levine (2009), Hörner and Samuelson (2013), and Sadzik and Stacchetti (2015). In Hörner and Samuelson (2013), for instance, there are multiple solutions to the optimality equations, corresponding to different boundary conditions, and to select among them, it is necessary to investigate in detail the discrete-time game (see their Lemma 3). However, the role of the discretization goes well beyond selecting the "right" boundary condition; see Sadzik and Stacchetti (2015).

$k_{n,\ell\Delta} = 0$ if player n uses the safe arm on this interval; $\widetilde{Y}^n_{[\ell\Delta,(\ell+1)\Delta)}$ is the observed sample path $Y^n_{[\ell\Delta,(\ell+1)\Delta)}$ on the interval $[\ell\Delta,(\ell+1)\Delta)$ of the payoff process associated with player n's risky arm if $k_{n,\ell\Delta} = 1$; and $\widetilde{Y}^n_{[\ell\Delta,(\ell+1)\Delta)}$ equals the empty set if $k_{n,\ell\Delta} = 0$. We write H_t for the set of all histories of length t, set $H_0 = \{\emptyset\}$, and let $H = \bigcup_{t=0,\Delta,2\Delta,\ldots}^{\infty} H_t$.

In addition, we assume that players have access to a public randomization device in every period, namely, a draw from the uniform distribution on $[0,1]$, which is assumed to be independent of θ and across periods. Following standard practice, we omit its realizations from the description of histories.

A behavioral strategy σ_n for player n is a sequence $(\sigma_{n,t})_{t=0,\Delta,2\Delta,\ldots}$, where $\sigma_{n,t}$ is a measurable map from H_t to the set of probability distributions on $\{0,1\}$; a pure strategy takes values in the set of degenerate distributions only.

Along with the prior probability p_0 assigned to $\theta = 1$, each profile of strategies induces a distribution over H. Given his or her opponents' strategies σ_{-n}, player n seeks to maximize

$$(1-\delta)\,\mathbb{E}^{\sigma_{-n},\sigma_n}\left[\sum_{\ell=0}^{\infty}\delta^{\ell}\left\{[1-\sigma_{n,\ell\Delta}(h_{\ell\Delta})]s+\sigma_{n,\ell\Delta}(h_{\ell\Delta})m_{\theta}\right\}\right].$$

By the law of iterated expectations, this equals

$$(1-\delta)\,\mathbb{E}^{\sigma_{-n},\sigma_n}\left[\sum_{\ell=0}^{\infty}\delta^{\ell}\left\{[1-\sigma_{n,\ell\Delta}(h_{\ell\Delta})]s+\sigma_{n,\ell\Delta}(h_{\ell\Delta})m(p_{\ell\Delta})]\right\}\right].$$

Nash equilibrium, PBE and MPE, with actions after history h_t depending only on the associated posterior belief p_t, are defined in the usual way. Imposing the standard "no signaling what you don't know" refinement, beliefs are pinned down after all histories, on and off path.[13]

An SSE is a PBE in which all players use the same strategy: $\sigma_n(h_t) = \sigma_{n'}(h_t)$ for all n, n' and $h_t \in H$. This implies symmetry of behavior after *any* history, not just on the equilibrium path of play. By definition, any symmetric MPE is an SSE, and any SSE is a PBE.

[13]While we could equivalently define this Bayesian game as a stochastic game with the common posterior belief as a state variable, no characterization or folk theorem applies to our setup, as the Markov chain (over consecutive states) does not satisfy the sufficient ergodicity assumptions; see Dutta (1995) and Hörner, Sugaya, Takahashi and Vieille (2011).

5 Main Results

Fix $\Delta > 0$. For $p \in [0,1]$, let $\overline{W}^{\Delta}_{\text{PBE}}(p)$ and $\underline{W}^{\Delta}_{\text{PBE}}(p)$ denote the supremum and infimum, respectively, of the set of average payoffs (per player) over all PBE, given prior belief p. Let $\overline{W}^{\Delta}_{\text{SSE}}(p)$ and $\underline{W}^{\Delta}_{\text{SSE}}(p)$ be the corresponding supremum and infimum over all SSE. If such equilibria exist,

$$\overline{W}^{\Delta}_{\text{PBE}}(p) \geq \overline{W}^{\Delta}_{\text{SSE}}(p) \geq \underline{W}^{\Delta}_{\text{SSE}}(p) \geq \underline{W}^{\Delta}_{\text{PBE}}(p). \tag{2}$$

Given that we assume a public randomization device, these upper and lower bounds define the corresponding equilibrium average payoff sets.

As any player can choose to ignore the information contained in the other players' experimentation results, the value function W_1^{Δ} of a single agent experimenting in isolation constitutes a lower bound on a player's payoff in any PBE. Lemma A.2 establishes that this lower bound converges to V_1^* as $\Delta \to 0$. Hence, we obtain a lower bound to the limits of all terms in (2), namely $\liminf_{\Delta\to 0} \underline{W}^{\Delta}_{\text{PBE}} \geq V_1^*$.

An upper bound is also easily found. As any discrete-time strategy profile is feasible for the continuous-time planner from the previous section, it holds that $\overline{W}^{\Delta}_{\text{PBE}} \leq V_N^*$.

The main theorem provides an exact characterization of the limits of all four functions. It requires introducing a new family of payoffs. Namely, we define the players' common payoff in continuous time when they all use the risky arm if, and only if, the belief exceeds a given threshold $\hat{p}$. This function admits a closed form that generalizes the first-best payoff V_N^* (cf. (1)). It is equal to

$$V_{N,\hat{p}}(p) = m(p) + \frac{c(\hat{p})}{u(\hat{p};\mu_N)}\, u(p;\mu_N),$$

for $p > \hat{p}$, and $V_{N,\hat{p}}(p) = s$ otherwise.[14]

Theorem 1 (i) *There exists $\hat{p} \in [p_N^*, p_1^*]$ such that*

$$\lim_{\Delta\to 0} \overline{W}^{\Delta}_{\text{PBE}} = \lim_{\Delta\to 0} \overline{W}^{\Delta}_{\text{SSE}} = V_{N,\hat{p}},$$

and

$$\lim_{\Delta\to 0} \underline{W}^{\Delta}_{\text{PBE}} = \lim_{\Delta\to 0} \underline{W}^{\Delta}_{\text{SSE}} = V_1^*,$$

[14] This function is continuous, strictly increasing and strictly convex on $[\hat{p}, 1]$, and continuously differentiable except for a convex kink at $\hat{p}$. For $\hat{p} = p_N^*$, $V_{N,\hat{p}}$ coincides with the cooperative value function V_N^*. For $\hat{p} > p_N^*$, we have $V_{N,\hat{p}} < V_N^*$ on $(p_N^*, 1)$.

uniformly on $[0,1]$.

(ii) *If* $\rho > 0$, *then* $\hat{p} = p_N^*$ *(and hence* $V_{N,\hat{p}} = V_N^*$*).*

(iii) *If* $\rho = 0$, *then* $\hat{p}$ *is the unique belief in* $[p_N^*, p_1^*]$ *satisfying*

$$N\lambda(\hat{p})\left[V_{N,\hat{p}}(j(\hat{p})) - s\right] - (N-1)\lambda(\hat{p})\left[V_1^*(j(\hat{p})) - s\right] = rc(\hat{p}); \tag{3}$$

moreover, $\hat{p} = p_N^*$ *if, and only if,* $j(p_N^*) \leq p_1^*$, *and* $\hat{p} = p_1^*$ *if, and only if,* $\lambda_0 = 0$.

To understand this result, let us begin with SSEs and the characterization of the cutoff $\hat{p}$ in the last item, when learning is entirely driven by the jump process. The players' temptation to deviate to the safe arm is strongest when the belief is so low that, absent good news, the belief drops into the region where safe prevails in *any* SSE, whether a single player has deviated or not. The cost of such a deviation, captured by the left-hand side of (3), thus arises only if good news arrives. Starting out from $\hat{p}$, in expectation, this happens at the rate $N\lambda(\hat{p})$ if no player deviates; a deviation reduces this rate to $(N-1)\lambda(\hat{p})$. Without a deviation, a player's continuation payoff then amounts at most to the cooperative payoff given that the use of the risky arm is disallowed below $\hat{p}$; in the event of a deviation, it is at least the single-player payoff (both evaluated at the revised belief $j(\hat{p})$ and net of the value of the safe arm). The right-hand side of (3) represents the benefit of a deviation, that is, the saved opportunity cost of playing risky. The cutoff belief $\hat{p}$ thus solves the familiar trade-off between the benefit from deviating and the cost of the worst punishment that may follow the deviation.

When $\lambda_0 = 0$, the arrival of good news freezes the belief at 1, and the resulting cooperative and single-player payoffs both equal $\lambda_1 h$. Starting out from $\hat{p}$, therefore, a player's continuation payoffs coincide with those of a single agent in all circumstances, so that it is impossible to sustain experimentation below the single-agent cutoff. Hence, $\hat{p} = p_1^*$.

If the second term on the left-hand side of (3) were zero, that is, if $j(p_N^*) \leq p_1^*$, so that a player left to his or her own devices would stop experimenting at the revised belief after the arrival of good news, and hence obtain a zero payoff (net of the value of the safe arm), the solution to this equation is the first-best cutoff p_N^*. To see this, note that the first term on the left-hand side can equivalently be interpreted as the social value of experimentation by a single player. Indeed, a player contributes to the arrival of news at rate $\lambda(\hat{p})$, but all N players then reap the gain $V_{N,\hat{p}}(j(\hat{p})) - s$. The right-hand side is the cost of such experimentation. Hence, $\hat{p} = p_N^*$ follows immediately from the equation.

The same logic immediately implies that first-best efficiency obtains when $\rho > 0$. Indeed, for small Δ, the leading term in the updating of beliefs is driven by the diffusion component of observed payoffs. Since this term involves no jumps, it will definitely keep the belief in a region where a player left to his or her own devices would stop experimenting.[15]

First-best efficiency not only depends on the cutoff but also requires play to be exclusively risky at all higher beliefs. Hence, the best equilibrium must involve a pure strategy, at least asymptotically. This is not straightforward. Indeed, symmetric pure-strategy PBE fail to exist with conclusive good news ($\rho = \lambda_0 = 0$) in discrete time. If all others play risky for certain, the posterior belief also declines for certain, unless good news arrives. If players randomized, there would be the added opportunity to punish if the posterior belief remained the same. When good news is conclusive, our proof relies on the existence of two symmetric mixed-strategy equilibria for beliefs close to the cutoff. It is then possible to choose continuation play as a function of history to incentivize players to experiment at beliefs that are sufficiently many rounds away from the cutoff (a negligible difference in beliefs once the time interval is small enough). Matters are simpler when news is inconclusive or a diffusion term is present.

Turning to point (i) of the theorem, there is no difference between the set of SSE and PBE payoffs, at least on average across players. This is shown in Sections 6.1–6.2. Regarding the highest equilibrium payoff, this may seem plausible (though not obvious) because efficiency requires symmetric play. Regarding the lowest equilibrium payoff, either playing safe forever is an equilibrium of the game given the current belief, or best-responding to being minmaxed provides a higher payoff to the punished player than also playing the minmaxing action (using the safe arm). In the latter case, one can incentivize the punished player to play safe by promising that all players will revert to risky (cooperative) play at a later time, thereby compensating the punished player for the flow payoff deficit that playing safe involves in the meantime. This eventual reversion also motivates the punishing players to play safe.

Figure 1 shows the cooperative continuous-time payoff V_N^* as well as the supremum $V_{N,\hat{p}}$ and infimum V_1^* of the limit average PBE payoffs for a parameter configuration that implies $p_N^* < \hat{p} < p_1^*$.

[15] A more technical intuition can be given in the spirit of smooth pasting in stopping problems for diffusion processes; see Dixit and Pindyck (1994). If all SSE experimentation stopped at a belief $\hat{p} > p_N^*$, the limiting payoff function $V_{N,\hat{p}}$ would exhibit a convex kink at $\hat{p}$. Given the diffusion component of the posterior-belief process, this kink could be used to provide all players incentives to use the risky arm at beliefs slightly below $\hat{p}$. Indeed, the informational benefit of experimentation in the presence of a kink is of lower order in Δ than its opportunity cost and hence dominates for small Δ.

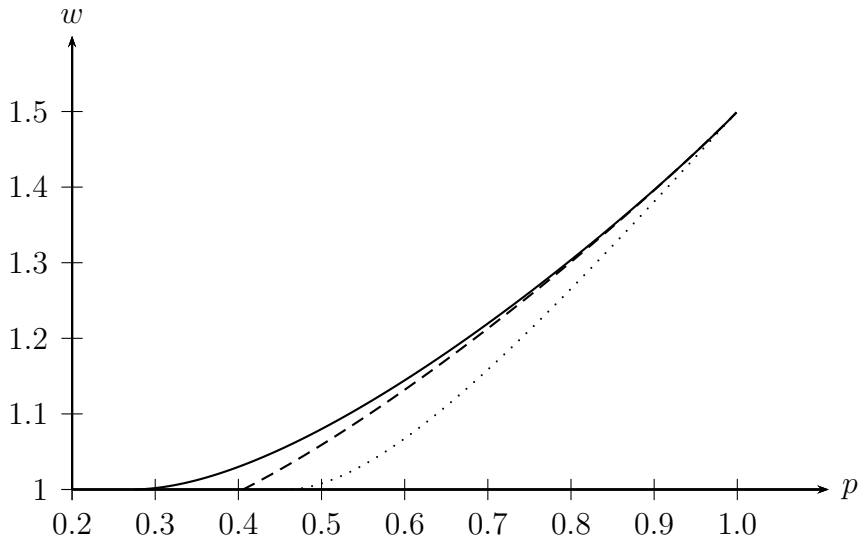

Figure 1: Payoffs V_N^* (solid), $V_{N,\hat{p}}$ (dashed) and V_1^* (dotted) for $\rho = 0$ and $(r, s, h, \lambda_1, \lambda_0, N) = (1, 1, 1.5, 1, 0.2, 5)$, implying $(p_N^*, \hat{p}, p_1^*) \simeq (.27, .40, .45)$.

We conclude this section with a few comparisons and comparative statics. They pertain to the case of pure jump processes, since when there is a diffusion component, Theorem 1(ii) implies that these are trivial corollaries of the first-best results given in Section 3.

In terms of comparisons, we find that, even when the first-best is not achievable, the best SSE performs strictly better than the symmetric MPE along two dimensions: the MPE not only involves a higher cutoff (hence, a lower *amount of experimentation*) but also entails too low a *speed* of experimentation, as it involves an interior level of experimentation for a range of beliefs.

Proposition 1 *For $\rho = 0$ and $\lambda_0 > 0$, the cutoff $\hat{p}$ is strictly lower than the belief at which all experimentation stops in the symmetric MPE of the continuous-time game.*

Turning to comparative statics, when is the first-best achievable with jump processes? The next proposition characterizes the area (in the (λ_1, λ_0)-plane) where asymptotic efficiency obtains. As is intuitive, having more players, or more patience, increases the scope for the first-best.

Proposition 2 *Let $\rho = 0$. Then, $j(p_N^*) > p_1^*$ whenever $\lambda_0 \leq \lambda_1/N$. On any ray in $\mathbb{R}^2_+$ emanating from the origin $(0,0)$ with a slope strictly between $1/N$ and 1, there is a unique critical point $(\lambda_1^*, \lambda_0^*)$ at which $j(p_N^*) = p_1^*$; moreover, $j(p_N^*) > p_1^*$ at all points of the ray that are closer to the origin than $(\lambda_1^*, \lambda_0^*)$, and $j(p_N^*) < p_1^*$ at all points that are farther from the origin than $(\lambda_1^*, \lambda_0^*)$. These critical points form a continuous curve that is bounded away from the origin and asymptotes to the ray of slope $1/N$. The curve shifts downward as r falls or N rises.*

This result is illustrated in Figure 2. Furthermore, in the case of $\lambda_0 > 0$, the more players participate in the game, the more experimentation can be sustained. (Recall that for $\lambda_0 = 0$, the threshold belief $\hat{p}$ is independent of N.) Hence, the comparative statics of the best SSE with respect to the number of players mirrors that for symmetric MPE (see Keller and Rady (2010)).

Proposition 3 *For $\rho = 0$ and $\lambda_0 > 0$, $\hat{p}$ is decreasing in N.*

It is instructive to consider what happens when the players become arbitrarily impatient or patient. If players are myopic, they do not react to future rewards and punishments. It is therefore no surprise that the cooperative solution cannot be attained in the limit. By contrast, if players are very patient, asymptotic efficiency is achieved if the number of players is large.

Proposition 4 *For $\rho = 0$ and $\lambda_0 > 0$,*

$$\lim_{r\to\infty} \frac{j(p_N^*)}{p_1^*} = \frac{\lambda_1 h}{s},$$

and

$$\lim_{r\to 0} \frac{j(p_N^*)}{p_1^*} = \frac{\lambda_1}{N\lambda_0}.$$

The next section is devoted to the construction of SSEs that underlies the proof of Theorem 1. Missing details are provided in the appendix.

6 Construction of Equilibria

We first consider the case of a diffusion component (Section 6.1) and then turn to the case of pure jump processes (Section 6.2).

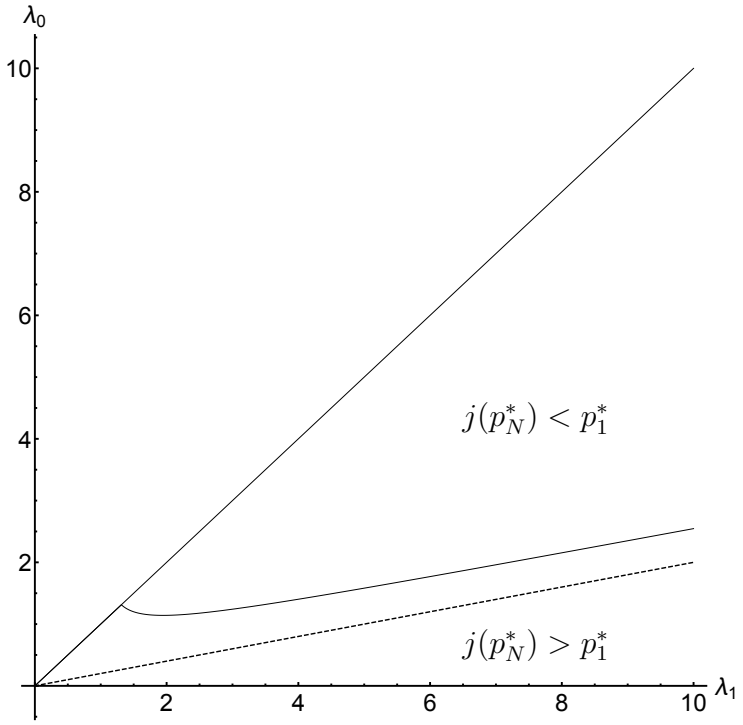

Figure 2: Asymptotic efficiency is achieved for parameter combinations (λ_1, λ_0) between the diagonal and the curve but not below the curve. The dashed line is the ray of slope $1/N$. Parameter values: $r = 1$, $N = 5$.

We need the following notation. Let $F_K^\Delta(\cdot|p)$ denote the cumulative distribution function of the posterior belief p_Δ when $p_0 = p$ and K players use the risky arm on the time interval $[0, \Delta)$. For any measurable function w on $[0, 1]$ and $p \in [0, 1]$, we write

$$\mathcal{E}_K^\Delta w(p) = \int_0^1 w(p')\, F_K^\Delta(dp'|p),$$

whenever this integral exists. Thus, $\mathcal{E}_K^\Delta w(p)$ is the expectation of $w(p_\Delta)$ given the prior p and K experimenting players.

6.1 Learning with a Brownian Component ($\rho > 0$)

For a sufficiently small $\Delta > 0$, we specify an SSE that can be summarized by two functions, $\overline{\kappa}$ and $\underline{\kappa}$, which do not depend on Δ. The equilibrium strategy is characterized by a two-state automaton. In the "good" state, play proceeds according to $\overline{\kappa}$, and the

equilibrium payoff satisfies

$$\overline{w}^{\Delta}(p) = (1-\delta)[(1-\overline{\kappa}(p))s + \overline{\kappa}(p)m(p)] + \delta\mathcal{E}^{\Delta}_{N\overline{\kappa}(p)}\overline{w}^{\Delta}(p), \tag{4}$$

while in the "bad" state, play proceeds according to $\underline{\kappa}$, and the payoff satisfies

$$\underline{w}^{\Delta}(p) = \max_{k}\left\{(1-\delta)[(1-k)s + km(p)] + \delta\mathcal{E}^{\Delta}_{(N-1)\underline{\kappa}(p)+k}\underline{w}^{\Delta}(p)\right\}. \tag{5}$$

That is, $\underline{w}^{\Delta}$ is the value from the best response to all other players following $\underline{\kappa}$.

A unilateral deviation from $\overline{\kappa}$ in the good state is punished by a transition to the bad state in the following period; otherwise, we remain in the good state. If there is a unilateral deviation from $\underline{\kappa}$ in the bad state, we remain in the bad state. Otherwise, a draw of the public randomization device determines whether the state next period is good or bad; this probability is chosen such that the expected payoff is indeed given by $\underline{w}^{\Delta}$ (see below).

With continuation payoffs given by $\overline{w}^{\Delta}$ and $\underline{w}^{\Delta}$, the common action $\kappa \in \{0,1\}$ is incentive compatible at a belief p if, and only if,

$$\begin{aligned}(1-\delta)[(1-\kappa)s + \kappa m(p)] + \delta\mathcal{E}^{\Delta}_{N\kappa}\overline{w}^{\Delta}(p) \\ \geq \quad (1-\delta)[\kappa s + (1-\kappa)m(p)] + \delta\mathcal{E}^{\Delta}_{(N-1)\kappa+1-\kappa}\underline{w}^{\Delta}(p).\end{aligned} \tag{6}$$

Therefore, the functions $\overline{\kappa}$ and $\underline{\kappa}$ define an SSE if, and only if, (6) holds for $\kappa = \overline{\kappa}(p)$ and $\kappa = \underline{\kappa}(p)$ at all p.

The probability $\eta^{\Delta}(p)$ of a transition from the bad to the good state in the absence of a unilateral deviation from $\underline{\kappa}(p)$ is pinned down by the requirement that

$$\begin{aligned}\underline{w}^{\Delta}(p) \quad = \quad & (1-\delta)[(1-\underline{\kappa}(p))s + \underline{\kappa}(p)m(p)] \\ & + \delta\left\{\eta^{\Delta}(p)\,\mathcal{E}^{\Delta}_{N\underline{\kappa}(p)}\overline{w}^{\Delta}(p) + [1-\eta^{\Delta}(p)]\,\mathcal{E}^{\Delta}_{N\underline{\kappa}(p)}\underline{w}^{\Delta}(p)\right\}.\end{aligned} \tag{7}$$

If $k = \underline{\kappa}(p)$ is optimal in (5), we simply set $\eta^{\Delta}(p) = 0$. Otherwise, (5) and (6) imply

$$\delta\mathcal{E}^{\Delta}_{N\overline{\kappa}(p)}\overline{w}^{\Delta}(p) \geq \underline{w}^{\Delta}(p) - (1-\delta)[(1-\underline{\kappa}(p))s + \underline{\kappa}(p)m(p)] > \delta\mathcal{E}^{\Delta}_{N\underline{\kappa}(p)}\underline{w}^{\Delta}(p),$$

so (7) holds with

$$\eta^{\Delta}(p) = \frac{\underline{w}^{\Delta}(p) - (1-\delta)[(1-\underline{\kappa}(p))s + \underline{\kappa}(p)m(p)] - \delta\mathcal{E}^{\Delta}_{N\underline{\kappa}(p)}\underline{w}^{\Delta}(p)}{\delta\mathcal{E}^{\Delta}_{N\underline{\kappa}(p)}\overline{w}^{\Delta}(p) - \delta\mathcal{E}^{\Delta}_{N\underline{\kappa}(p)}\underline{w}^{\Delta}(p)} \in (0,1].$$

It remains to specify $\overline{\kappa}$ and $\underline{\kappa}$. Let

$$p^m = \frac{s - m_0}{m_1 - m_0}\,.$$

As $m(p^m) = s$, this is the belief at which a myopic agent is indifferent between the two arms. It is straightforward to verify that $p_1^* < p^m$. Fixing $\underline{p} \in (p_N^*, p_1^*)$ and $\bar{p} \in (p^m, 1)$, we let $\overline{\kappa}(p) = \mathbb{1}_{p>\underline{p}}$ and $\underline{\kappa}(p) = \mathbb{1}_{p>\bar{p}}$.[16] Note that punishment and reward strategies coincide outside of $(\underline{p}, \bar{p})$.

Proposition 5 *For $\rho > 0$, there are beliefs $p^\flat \in (p_N^*, p_1^*)$ and $p^\sharp \in (p^m, 1)$ such that for all $\underline{p} \in (p_N^*, p^\flat)$ and $\bar{p} \in (p^\sharp, 1)$, there exists $\bar{\Delta} > 0$ such that for all $\Delta \in (0, \bar{\Delta})$, the two-state automaton with functions $\overline{\kappa}$ and $\underline{\kappa}$ defines an SSE of the experimentation game with period length Δ.*

The proof consists of verifying that, for a sufficiently small Δ, the actions $\overline{\kappa}(p)$ and $\underline{\kappa}(p)$ satisfy the incentive-compatibility constraint (6) at all p. First, we find $\varepsilon > 0$ small enough that $\underline{w}^\Delta = s$ in a neighborhood of $\underline{p}+\varepsilon$. The payoff functions $\overline{w}^\Delta$ and $\underline{w}^\Delta$ resulting from the two-state automaton are then bounded away from one another on $[\underline{p} + \varepsilon, \bar{p}]$ for small Δ. In this range, therefore, the difference in expected continuation values across states does not vanish as Δ tends to 0, whereas the difference in current expected payoffs across actions is of order Δ, rendering deviations unattractive for small enough Δ. On $(\bar{p}, 1]$ and $[0, \underline{p}]$, $\overline{\kappa}$ and $\underline{\kappa}$ both prescribe the myopically optimal action. Given that continuation payoffs are weakly higher in the good state, it is easy to show that there are no incentives to deviate on these intervals. For beliefs in $(\underline{p}, \underline{p}+\varepsilon)$, $\underline{\kappa}$ again prescribes the myopically optimal action. The proof of incentive compatibility of $\overline{\kappa}$ on this interval crucially relies on the fact that, for small Δ, $\overline{w}^\Delta$ is bounded below by $V_{N,\underline{p}}$, which has a convex kink at $\underline{p}$. This, together with the fact that, conditional on no lump sum arriving, the log-likelihood ratio of posterior beliefs is Gaussian, allows us to demonstrate the existence of some constant $C_1 > 0$ such that, for Δ small enough, $\mathcal{E}_N^\Delta \overline{w}^\Delta(p) \geq s + C_1 \Delta^{\frac{3}{4}}$ to the immediate right of $\underline{p}$, whereas $\mathcal{E}_{N-1}^\Delta \underline{w}^\Delta(p) \leq s + C_0 \Delta$ with some constant $C_0 > 0$. For small Δ, therefore, the linearly vanishing current-payoff advantage of the safe over the risky arm is dominated by the incentives provided through continuation payoffs.

The next result essentially follows from letting $\underline{p} \to p_N^*$ and $\bar{p} \to 1$ in Proposition 5.

[16] $\mathbb{1}_A$ denotes the indicator function of the event A.

Proposition 6 *For $\rho > 0$, $\lim_{\Delta\to 0} \overline{W}^{\Delta}_{\text{SSE}} = V_N^*$ and $\lim_{\Delta\to 0} \underline{W}^{\Delta}_{\text{SSE}} = V_1^*$, uniformly on $[0,1]$.*

6.2 Pure Poisson Learning ($\rho = 0$)

Let $\rho = 0$, and take $\hat{p}$ as in part (iii) of Theorem 1.

Proposition 7 *Let $\rho = 0$. For any $\varepsilon > 0$, there is a $\Delta_\varepsilon > 0$ such that for all $\Delta \in (0, \Delta_\varepsilon)$, the set of beliefs at which experimentation can be sustained in a PBE of the discrete game with period length Δ is contained in the interval $(\hat{p} - \varepsilon, 1]$. In particular, $\limsup_{\Delta\to 0} \overline{W}^{\Delta}_{\text{PBE}}(p) \leq V_{N,\hat{p}}(p)$.*

For a heuristic explanation of the logic behind this result, consider a sequence of pure-strategy PBEs for vanishing Δ such that the infimum of the set of beliefs at which at least one player experiments converges to some limit $\tilde{p}$. Selecting a subsequence of Δs and relabeling players, if necessary, we can assume without loss of generality that players $1, \ldots, L$ play R immediately to the right of $\tilde{p}$, while players $L+1, \ldots, N$ play S. In the limit, players' individual continuation payoffs are bounded below by the single-agent value function V_1^* and cannot sum to more than $NV_{N,\tilde{p}}$, so the sum of the continuation payoffs of players $1, \ldots, L$ is bounded above by $NV_{N,\tilde{p}} - (N-L)V_1^*$. Averaging these players' incentive-compatibility constraints thus yields

$$L\lambda(\tilde{p}) \left[\frac{NV_{N,\tilde{p}}(j(\tilde{p})) - (N-L)V_1^*(j(\tilde{p}))}{L} - s \right] - rc(\tilde{p}) \geq (L-1)\lambda(\tilde{p}) \left[V_1^*(j(\tilde{p})) - s \right].$$

Simplifying the left-hand side, adding $(N-L)\lambda(\tilde{p}) \left[V_1^*(j(\tilde{p})) - s \right]$ to both sides and re-arranging, we obtain

$$N\lambda(\tilde{p}) \left[V_{N,\tilde{p}}(j(\tilde{p})) - s \right] - rc(\tilde{p}) \geq (N-1)\lambda(\tilde{p}) \left[V_1^*(j(\tilde{p})) - s \right],$$

which in turn implies $\tilde{p} \geq \hat{p}$, as we show in Lemma A.9 in the appendix. The proof of Proposition 7 makes this heuristic argument rigorous and extends it to mixed equilibria.

For non-revealing jumps ($\lambda_0 > 0$), the construction of SSEs that achieve the above bounds in the limit relies on the same two-state automaton as in Proposition 5, the only difference being that the threshold $\underline{p}$ is now restricted to exceed $\hat{p}$.

Proposition 8 *Let $\rho = 0$ and $\lambda_0 > 0$. There are beliefs $p^\flat \in (\hat{p}, p_1^*)$ and $p^\sharp \in (p^m, 1)$ such that for all $\underline{p} \in (\hat{p}, p^\flat)$ and $\bar{p} \in (p^\sharp, 1)$, there exists $\bar{\Delta} > 0$ such that for all*

$\Delta \in (0, \bar{\Delta})$, *the two-state automaton with functions* $\overline{\kappa}$ *and* $\underline{\kappa}$ *defines an SSE of the experimentation game with period length* Δ.

The strategy for the proof of this proposition is the same as that of Proposition 5, except for the belief region to the immediate right of $\underline{p}$, where incentives are now provided through terms of first order in Δ, akin to those in equation (3).

In the case $\lambda_0 > 0$, we are able to provide incentives in the potentially last round of experimentation by threatening punishment *conditional on there being a success* (that is, a successful experiment). This option is no longer available in the case of $\lambda_0 = 0$. Indeed, any success now takes us to a posterior of one, so that everyone plays risky forever after. This means that, irrespective of whether a success occurs in that round, continuation strategies are independent of past behavior, conditional on the players' belief. This raises the possibility of unravelling. If incentives just above the candidate threshold at which players give up on the risky arm cannot be provided, can this threshold be lower than in the MPE?

To settle whether unravelling occurs requires us to study the discrete game in considerable detail.[17] We start by noting that for $\lambda_0 = 0$, we can strengthen Proposition 7 as follows: there is no PBE with any experimentation at beliefs below the discrete-time single-agent cutoff $p_1^\Delta = \inf\{p \colon W_1^\Delta(p) > s\}$ (see Heidhues et al. (2015)).[18] The highest average payoff that can be hoped for, then, involves all players experimenting above p_1^Δ.

Unlike in the case of $\lambda_0 > 0$ (see Proposition 8), an explicit description of a two-state automaton implementing SSEs whose payoffs converge to the obvious upper and lower bounds appears elusive. This is partly because equilibrium strategies are, as it turns out, necessarily mixed for beliefs that are arbitrarily close to (but above) p_1^Δ. The proof of the next proposition establishes that the length of the interval of beliefs for which this is the case vanishes as $\Delta \to 0$. In particular, for higher beliefs (except for beliefs arbitrarily close to 1, when playing R is strictly dominant), both pure actions can be enforced in some equilibrium.

[17] The study of symmetric MPEs is difficult in discrete time. Unlike in continuous time, in which the explicit solution is known (see Keller et al. (2005)), they do not seem to admit an easy characterization. For some open sets of beliefs, there are multiple symmetric MPEs in discrete time, regardless of how small Δ is. It is not known whether any or all of these converge (in some sense) to the symmetric MPE in continuous time.

[18] In particular, this excludes the possibility that the asymmetric MPE of Keller et al. (2005) with an infinite number of switches between the two arms below p_1^* can be approximated in the discrete game.

Proposition 9 *Let $\rho = 0$ and $\lambda_0 = 0$. For any beliefs $\underline{p}$ and $\bar{p}$ such that $p_1^* < \underline{p} < p^m < \bar{p} < 1$, there exists a $\bar{\Delta} > 0$ such that for all $\Delta \in (0, \bar{\Delta})$, there exists*

- *an SSE in which, starting from a prior above $\underline{p}$, all players use the risky arm on the path of play as long as the belief remains above $\underline{p}$ and use the safe arm for beliefs below p_1^*; and*
- *an SSE in which, given a prior between $\underline{p}$ and $\bar{p}$, the players' payoff is no larger than their best-reply payoff against opponents who use the risky arm if, and only if, the belief lies in $[p_1^*, \underline{p}] \cup [\bar{p}, 1]$.*

While this is somewhat weaker than Proposition 8, its implications for limit payoffs as $\Delta \to 0$ are the same. Intuitively, given that the interval $[p_1^*, \underline{p}]$ can be chosen arbitrarily small (actually, of the order Δ, as the proof establishes), its impact on equilibrium payoffs starting from priors above $\underline{p}$ is of order Δ. This suggests that for the equilibria whose existence is stated in Proposition 9, the payoff converges to the payoff from all players experimenting above p_1^* and to the best-reply payoff against none of the opponents experimenting. Indeed, we have the following result, covering both inconclusive and conclusive jumps.

Proposition 10 *For $\rho = 0$, $\lim_{\Delta \to 0} \overline{W}^{\Delta}_{\mathrm{SSE}} = V_{N,\hat{p}}$ and $\lim_{\Delta \to 0} \underline{W}^{\Delta}_{\mathrm{SSE}} = V_1^*$, uniformly on $[0, 1]$.*

7 Functional Equations for SSE Payoffs

While it is possible to derive explicit solutions to the equilibrium payoff sets of interest, at least asymptotically, note that, already in the discrete game, a characterization in terms of optimality equations can be obtained, which defines the correspondence of SSE payoffs. As discussed in the introduction, these generalize the familiar equation characterizing the value function of the symmetric MPE. Instead of a single (HJB) equation, the characterization of SSE payoffs involves two coupled functional equations, whose solution delivers the highest and lowest equilibrium payoff. Proposition 11 states this in the discrete game, while Proposition 12 gives the continuous-time limit. As these propositions do not heavily rely on the specific structure of our game, we believe that they might be useful for analyzing SSE payoffs for more general processes or other stochastic games.

Fix $\Delta > 0$. For $p \in [0, 1]$, let $\overline{W}^{\Delta}(p)$ and $\underline{W}^{\Delta}(p)$ denote the supremum and infimum, respectively, of the set of payoffs over *pure-strategy* SSEs, given prior belief

p.[19] If such an equilibrium exists, these extrema are achieved, and $\overline{W}^{\Delta}(p) \geq \underline{W}^{\Delta}(p)$. For $\rho > 0$ or $\lambda_0 > 0$, we have shown in Sections 6.1–6.2 that in the limit as $\Delta \to 0$, the best and worst average payoffs (per player) over all PBEs are achieved by SSE in pure strategies. The following result characterizes $\overline{W}^{\Delta}$ and $\underline{W}^{\Delta}$ via a pair of coupled functional equations.

Proposition 11 *Suppose that the discrete game with time increment $\Delta > 0$ admits a pure-strategy SSE for any prior belief. Then, the pair of functions $(\overline{w}, \underline{w}) = (\overline{W}^{\Delta}, \underline{W}^{\Delta})$ solves the functional equations*

$$\overline{w}(p) = \max_{\kappa \in \mathcal{K}(p;\overline{w},\underline{w})} \left\{ (1-\delta)[(1-\kappa)s + \kappa m(p)] + \delta \mathcal{E}^{\Delta}_{N\kappa} \overline{w}(p) \right\}, \tag{8}$$

$$\underline{w}(p) = \min_{\kappa \in \mathcal{K}(p;\overline{w},\underline{w})} \max_{k \in \{0,1\}} \left\{ (1-\delta)[(1-k)s + km(p)] + \delta \mathcal{E}^{\Delta}_{(N-1)\kappa+k} \underline{w}(p) \right\}, \tag{9}$$

where $\mathcal{K}(p; \overline{w}, \underline{w}) \subseteq \{0, 1\}$ denotes the set of all κ such that

$$\begin{aligned} &(1-\delta)[(1-\kappa)s + \kappa m(p)] + \delta \mathcal{E}^{\Delta}_{N\kappa} \overline{w}(p) \\ &\quad \geq \max_{k \in \{0,1\}} \left\{ (1-\delta)[(1-k)s + km(p)] + \delta \mathcal{E}^{\Delta}_{(N-1)\kappa+k} \underline{w}(p) \right\}. \end{aligned} \tag{10}$$

Moreover, $\underline{W}^{\Delta} \leq \underline{w} \leq \overline{w} \leq \overline{W}^{\Delta}$ for any solution $(\overline{w}, \underline{w})$ of (8)–(10).

This result relies on arguments that are familiar from Cronshaw and Luenberger (1994). We briefly sketch them here.

The above equations can be understood as follows. The ideal condition for a given (symmetric) action profile to be incentive compatible is that if each player conforms to it, the continuation payoff is the highest possible, while a deviation triggers the lowest possible continuation payoff. These actions are precisely the elements of $\mathcal{K}(p; \overline{w}, \underline{w})$, as defined by equation (10). Given this set of actions, equation (9) provides the recursion that characterizes the constrained minmax payoff under the assumption that if a player were to deviate to his myopic best reply to the constrained minmax action profile, the punishment would be restarted next period, resulting in a minimum continuation payoff. Similarly, equation (8) yields the highest payoff under this constraint, but here, playing the best action (within the set) is on the equilibrium path.

Note that in *any* SSE, given p, the action $\kappa(p)$ must be an element of $\mathcal{K}(p; \overline{W}^{\Delta}, \underline{W}^{\Delta})$. This is because the left-hand side of (10) with $\overline{w} = \overline{W}^{\Delta}$ is an upper bound on the

[19] For the existence of various types of equilibria in discrete-time stochastic games, see Mertens, Sorin and Zamir (2015), Chapter 7.

continuation payoff if no player deviates, and the right-hand side with $\underline{w} = \underline{W}^\Delta$ a lower bound on the continuation payoff after a unilateral deviation. Consider the equilibrium that achieves $\overline{W}^\Delta$. Then,

$$\overline{W}^\Delta(p) \leq \max_{\kappa \in \mathcal{K}(p;\overline{W}^\Delta,\underline{W}^\Delta)} \left\{ (1-\delta)[(1-\kappa)s + \kappa m(p)] + \delta \mathcal{E}^\Delta_{N\kappa} \overline{W}^\Delta(p) \right\},$$

as the action played must be in $\mathcal{K}(p;\overline{W}^\Delta,\underline{W}^\Delta)$, and the continuation payoff is at most given by $\overline{W}^\Delta$. Similarly, $\underline{W}^\Delta$ must satisfy (9) with "$\geq$" instead of "=." Suppose now that the "$\leq$" were strict. Then, we can define a strategy profile given prior p that (i) in period 0, plays the maximizer of the right-hand side, and (ii) from $t = \Delta$ onward, abides by the continuation strategy achieving $\overline{W}^\Delta(p_\Delta)$. Because the initial action is in $\mathcal{K}(p;\overline{W}^\Delta,\underline{W}^\Delta)$, this constitutes an equilibrium, and it achieves a payoff strictly larger than $\overline{W}^\Delta(p)$, a contradiction. Hence, (8) must hold with equality for $\overline{W}^\Delta$. The same reasoning applies to $\underline{W}^\Delta$ and (9).

Fix a pair $(\overline{w}, \underline{w})$ that satisfies (8)–(10). Note that this implies $\underline{w} \leq \overline{w}$. Given such a pair and any prior p, we specify two SSEs whose payoffs are $\overline{w}$ and $\underline{w}$, respectively. It then follows that $\underline{W}^\Delta \leq \underline{w} \leq \overline{w} \leq \overline{W}^\Delta$. Let $\overline{\kappa}$ and $\underline{\kappa}$ denote a selection of the maximum and minimum of (8)–(9). The equilibrium strategies are described by a two-state automaton, whose states are referred to as "good" or "bad." The difference between the two equilibria lies in the initial state: $\overline{w}$ is achieved when the initial state is good, $\underline{w}$ is achieved when it is bad. In the good state, play proceeds according to $\overline{\kappa}$; in the bad state, it proceeds according to $\underline{\kappa}$. Transitions are exactly as in the equilibria described in Sections 6.1–6.2. This structure precludes profitable one-shot deviations in either state, so that the automaton describes equilibrium strategies, and the desired payoffs are obtained.

As Δ tends to 0, equations (8)–(9) transform into differential-difference equations involving terms that are familiar from the continuous-time analysis in Section 3. A formal Taylor approximation shows that for any $\kappa \in \{0,1\}$, $K \in \{0,1,\ldots,N\}$ and a sufficiently regular function w on the unit interval,

$$\begin{aligned} &(1-\delta)[(1-\kappa)s + \kappa m(p)] + \delta \mathcal{E}^\Delta_K w(p) \\ &\quad = \quad w(p) + r \left\{ (1-\kappa)s + \kappa m(p) + K\, b(p,w) - w(p) \right\} \Delta + o(\Delta). \end{aligned}$$

Applying this approximation to (8)–(9), cancelling the terms of order 0 in Δ, dividing through by Δ, letting $\Delta \to 0$ and recalling the notation $c(p) = s - m(p)$ for the opportunity cost of playing risky, we obtain the coupled differential-difference equa-

tions that appear in the following result.

Proposition 12 *Let $\rho > 0$ or $\lambda_0 > 0$. As $\Delta \to 0$, the pair of functions $(\overline{W}^{\Delta}, \underline{W}^{\Delta})$ converges uniformly (in p) to a pair of functions $(\overline{w}, \underline{w})$ solving*

$$\overline{w}(p) = s + \max_{\kappa \in \overline{\mathcal{K}}(p)} \kappa \left[Nb(p, \overline{w}) - c(p) \right], \tag{11}$$

$$\underline{w}(p) = s + \min_{\kappa \in \overline{\mathcal{K}}(p)} (N-1)\kappa\, b(p, \underline{w}) + \max_{k \in \{0,1\}} k \left[b(p, \underline{w}) - c(p) \right], \tag{12}$$

where

$$\overline{\mathcal{K}}(p) = \begin{cases} \{0\} & \text{for } p \leq \hat{p}, \\ \{0,1\} & \text{for } \hat{p} < p < 1, \\ \{1\} & \text{for } p = 1, \end{cases} \tag{13}$$

and $\hat{p}$ is as in parts (ii) and (iii) of Theorem 1.

This result is an immediate consequence of the previous results. It follows from Sections 6.1–6.2 that, except when $\rho = \lambda_0 = 0$, there exist pure-strategy SSEs and the pair $(\overline{W}^{\Delta}, \underline{W}^{\Delta})$ converges uniformly to $(V_{N,\hat{p}}, V_1^*)$. It is straightforward to verify that $(\overline{w}, \underline{w}) = (V_{N,\hat{p}}, V_1^*)$ solves (11)–(13). First, as V_N^* satisfies[20]

$$V_N^*(p) = s + \max_{\kappa \in \{0,1\}} \kappa \left[Nb(p, V_N^*) - c(p) \right],$$

with $Nb(p, V_N^*) - c(p) > 0$ to the right of p_N^*, (11) is trivially solved by V_N^* whenever $\hat{p} = p_N^*$. Second, for $\hat{p} > p_N^*$, the function $V_{N,\hat{p}}$ satisfies

$$V_{N,\hat{p}}(p) = s + \mathbb{1}_{p > \hat{p}} \left[Nb(p, V_{N,\hat{p}}) - c(p) \right],$$

with $Nb(p; V_{N,\hat{p}}) - c(p) > 0$ on $(\hat{p}, 1)$. This implies that $V_{N,\hat{p}}$ solves (11) when $\hat{p} > p_N^*$. Third, V_1^* always solves (12). In fact, as $b(p; V_1^*) \geq 0$ everywhere, we have $\min_{\kappa \in \{0,1\}} (N-1)\kappa\, b(p, V_1^*) = 0$, and (12) with this minimum set to zero is just the HJB equation for V_1^*.

Note that the continuous-time functional equations (11)–(12) would be equally easy to solve for any *arbitrary* $\hat{p}$ in (13). However, only the solution with $\hat{p}$ as in Theorem 1 captures the asymptotics of our discretization of the experimentation game.

[20] This equation follows from the HJB equation in Section 3: because the maximand is linear in K, the continuous-time planner finds it optimal to set $K = 0$ or $K = N$ at any given belief.

8 Concluding Comments

We have shown that the inefficiencies arising in strategic bandit problems are driven by the solution concept, MPE. Inefficiencies entirely disappear when news has a Brownian component or good news events are not too informative. The best PBE can be achieved with an SSE, specifying a simple rule of conduct (unlike in an MPE): on-path play is of the cutoff type, with players using the risky (safe) arm exclusively if, and only if, the belief is above (below) a certain cutoff.

Of course, we do not expect the finding that SSE and PBE payoffs coincide to generalize to all symmetric stochastic games. For instance, SSE can be restrictive when actions are imperfectly monitored, as shown by Fudenberg, Levine and Takahashi (2007). Nonetheless, SSE is a class of equilibria that both allows for "stick-and-carrot" incentives, as in standard discrete-time repeated (or stochastic) games, but is also amenable to continuous-time optimal control techniques, as illustrated by Proposition 12 (for a given transversality condition that must be derived from independent considerations, such as a discretized version of the game).

The information/payoff processes we consider are a subset of those in Cohen and Solan (2013), which allows lump-sum sizes to be informative (assuming that lump sums of any size arrive more frequently in state $\theta = 1$). For processes with a Brownian component, our proof that risky play is incentive compatible immediately to the right of the threshold p_N^* only exploits the properties of the posterior belief process *conditional on no lump sum arriving*. As these properties are the same whether lump sums are informative or not, asymptotic efficiency when a Brownian component is present obtains more generally. When learning is driven by lump-sum payoffs only, inspection of equation (3) suggests that efficiency requires that a lump sum *of any size* arriving at the initial belief p_N^* lead to a posterior belief no higher than p_1^*. Therefore, the condition for asymptotic efficiency has a straightforward generalization.

As mentioned above, our model rules out lumpy bad news. Hence, it rules out models in which Poisson events are "breakdowns," as in the model of Keller and Rady (2015), for instance. Bad news amounts to assuming that the safe flow payoff and the average size of lump-sum payoffs are both negative with $\lambda_1 h < s < \lambda_0 h \leq 0$. Now, $\theta = 1$ is the *bad* state of the world, and the efficient and single-player solution cutoffs in continuous time satisfy $p_N^* > p_1^*$, with the stopping region lying to the *right* of the cutoff in either case. The associated value functions V_1^* and V_N^* solve the same HJB equations as in Section 3. In this model, $j(p_N^*) > p_N^* > p_1^*$, *i.e.*, starting from p_N^*, the belief remains in the single-agent stopping region for small Δ, whether a breakdown occurs or not. Hence, the harshest possible punishment, consisting of all other players

playing safe forever, can be meted out to any potential deviator, whether there is a breakdown or not. Thus, we conjecture that asymptotic efficiency also obtains in this framework.

Appendix

A Auxiliary Results

A.1 Evolution of Beliefs

For the description of the evolution of beliefs, it is convenient to work with the log odds ratio

$$\ell_t = \ln \frac{p_t}{1-p_t}\,.$$

Suppose that starting from $\ell_0 = \ell$, the players use the fixed action profile $(k_1, \ldots, k_N) \in \{0,1\}^N$. By Peskir and Shiryayev (2006, pp. 287–289 and 334–338), the log odds ratio at time $t > 0$ is then

$$\ell_t = \ell + \sum_{\{n:k_n=1\}} \left\{ \frac{\alpha_1 - \alpha_0}{\sigma^2} \left(X_t^n - \alpha_0 t - h N_t^n\right) - \left[\frac{(\alpha_1-\alpha_0)^2}{2\sigma^2} + \lambda_1 - \lambda_0\right] t + \ln \frac{\lambda_1}{\lambda_0} N_t^n \right\},$$

where X^n and N^n are the payoff and Poisson processes, respectively, associated with player n's risky arm. The terms involving α_1, α_0 and σ capture learning from the continuous component, $X_t^n - hN_t^n$, of the payoff process, with higher realizations making the players more optimistic. The terms involving λ_1 and λ_0 capture learning from lump-sum payoffs, with the players becoming more pessimistic on average as long as no lump-sum arrives, and each arrival increasing the log odds ratio by the fixed increment $\ln(\lambda_1/\lambda_0)$.[21]

Under the probability measure $\mathbb{P}_\theta$ associated with state $\theta \in \{0,1\}$, $X_t^n - \alpha_0 t - hN_t^n$ is Gaussian with mean $(\alpha_\theta - \alpha_0)t$ and variance $\sigma^2 t$, so that $\sum_{\{n:k_n=1\}} (\alpha_1 - \alpha_0)\sigma^{-2}\left(X_t^n - \alpha_0 t - hN_t^n\right)$ is Gaussian with mean $K(\alpha_1-\alpha_0)(\alpha_\theta-\alpha_0)\sigma^{-2}t$ and variance $K\rho t$, where $K = \sum_{n=1}^N k_n$ and $\rho = (\alpha_1-\alpha_0)^2\sigma^{-2}$. Conditional on the event that $\sum_{\{n:k_n=1\}} N_t^n = J$, therefore, ℓ_t is normally distributed with mean $\ell - K\left(\lambda_1 - \lambda_0 - \frac{\rho}{2}\right)t + J\ln(\lambda_1/\lambda_0)$ and variance $K\rho t$ under $\mathbb{P}_1$, and normally distributed with mean $\ell - K\left(\lambda_1 - \lambda_0 + \frac{\rho}{2}\right)t + J\ln(\lambda_1/\lambda_0)$ and variance $K\rho t$ under $\mathbb{P}_0$. Finally, the probability under measure $\mathbb{P}_\theta$ that $\sum_{\{n:k_n=1\}} N_t^n = J$ equals $\frac{(K\lambda_\theta t)^J}{J!} e^{-K\lambda_\theta t}$ by the sum property of the Poisson distribution.

Taken together, these facts make it possible to explicitly compute the distribution of

$$p_t = \frac{e^{\ell_t}}{1+e^{\ell_t}}$$

under the players' measure $\mathbb{P}_p = p\mathbb{P}_1 + (1-p)\mathbb{P}_0$. As this explicit representation is not needed in what follows, we omit it here.

Instead, we turn to the characterization of infinitesimal changes of p_t, once more assuming a fixed action profile with K players using the risky arm. Arguing as in Cohen and Solan (2013, Section 3.3), one shows that, with respect to the players' information filtration, the

[21] Here, λ_1/λ_0 is understood to be 1 when $\lambda_1 = \lambda_0 = 0$. When $\lambda_1 > \lambda_0 = 0$, we have $\ell_t = \infty$ and $p_t = 1$ from the arrival time of the first lump-sum on.

process of posterior beliefs is a Markov process whose infinitesimal generator $\mathcal{L}^K$ acts as follows on real-valued functions v of class C^2 on the open unit interval:

$$\mathcal{L}^K v(p) = K\left\{\frac{\rho}{2}p^2(1-p)^2 v''(p) - (\lambda_1 - \lambda_0)p(1-p)v'(p) + \lambda(p)\left[v(j(p)) - v(p)\right]\right\}.$$

In particular, instantaneous changes in beliefs exhibit linearity in K in the sense that $\mathcal{L}^K = K\mathcal{L}^1$.

By the very nature of Bayesian updating, finally, the process of posterior beliefs is a martingale with respect to the players' information filtration.

A.2 Payoff Functions

Our first auxiliary result concerns the function $u(\cdot;\mu_N)$ defined in Section 3.

Lemma A.1 *$\delta\mathcal{E}_K^\Delta u(\cdot;\mu_N)(p) = \delta^{1-\frac{K}{N}}u(p;\mu_N)$ for all $\Delta > 0$, $K \in \{1,\dots,N\}$ and $p \in (0,1]$.*

Proof: We simplify notation by writing u for $u(\cdot;\mu_N)$. Consider the process (p_t) of posterior beliefs in continuous time when $p_0 = p > 0$ and K players use the risky arm. By Dynkin's formula,

$$\begin{aligned}
\mathbb{E}\left[e^{-rK\Delta/N}u(p_\Delta)\right] &= u(p) + \mathbb{E}\left[\int_0^\Delta e^{-rKt/N}\left\{\mathcal{L}^K u(p_t) - \frac{rK}{N}u(p_t)\right\}dt\right] \\
&= u(p) + K\,\mathbb{E}\left[\int_0^\Delta e^{-rKt/N}\left\{\mathcal{L}^1 u(p_t) - \frac{r}{N}u(p_t)\right\}dt\right] \\
&= u(p),
\end{aligned}$$

where the last equality follows from the fact that $\mathcal{L}^1 u = ru/N$ on $(0,1]$.[22] Thus, $\delta^{K/N}\mathcal{E}_K^\Delta u(p) = u(p)$. ∎

We further note that $\mathcal{E}_K^\Delta m(p) = m(p)$ for all K by the martingale property of beliefs and the linearity of m in p.

These properties are used repeatedly in what follows. Their first application is in the proof of uniform convergence of the discrete-time single-agent value function to its continuous-time counterpart.

Let $(\mathcal{W}, \|\cdot\|)$ be the Banach space of bounded real-valued functions on $[0,1]$ equipped with the supremum norm. Given $\Delta > 0$, and any $w \in \mathcal{W}$, define a function $T_1^\Delta w \in \mathcal{W}$ by

$$T_1^\Delta w(p) = \max\left\{(1-\delta)m(p) + \delta\mathcal{E}_1^\Delta w(p),\ (1-\delta)s + \delta w(p)\right\}.$$

[22] To verify this identity, note that

$$u'(p) = -\frac{\mu_N + p}{p(1-p)}u(p), \quad u''(p) = \frac{\mu_N(\mu_N+1)}{p^2(1-p)^2}u(p), \quad u(j(p)) = \frac{\lambda_0}{\lambda(p)}\left(\frac{\lambda_0}{\lambda_1}\right)^{\mu_N}u(p),$$

and use the equation defining μ_N.

The operator T_1^Δ satisfies Blackwell's sufficient conditions for being a contraction mapping with modulus δ on $(\mathcal{W}, \|\cdot\|)$: monotonicity ($v \leq w$ implies $T_1^\Delta v \leq T_1^\Delta w$) and discounting ($T_1^\Delta(w+c) = T_1^\Delta w + \delta c$ for any real number c). By the contraction mapping theorem, T_1^Δ has a unique fixed point in $\mathcal{W}$; this is the value function W_1^Δ of an agent experimenting in isolation.

The corresponding continuous-time value function is V_1^* as introduced in Section 3. As any discrete-time strategy is feasible in continuous time, we trivially have $W_1^\Delta \leq V_1^*$.

Lemma A.2 *$W_1^\Delta \to V_1^*$ uniformly as $\Delta \to 0$.*

PROOF: A lower bound for W_1^Δ is given by the payoff function W_*^Δ of a single agent who uses the cutoff p_1^* in discrete time; this function is the unique fixed point in $\mathcal{W}$ of the contraction mapping T_*^Δ defined by

$$T_*^\Delta w(p) = \begin{cases} (1-\delta)m(p) + \delta\mathcal{E}_1^\Delta w(p) & \text{if} \quad p > p_1^*, \\ (1-\delta)s + \delta w(p) & \text{if} \quad p \leq p_1^*. \end{cases}$$

Next, choose $\check{p} < p_1^*$, and define $p^\natural = \frac{\check{p}+p_1^*}{2}$ and the function $v = m + Cu(\cdot;\mu_1) + \mathbb{1}_{[0,p^\natural]}(s - m - Cu(\cdot;\mu_1))$, where the constant C is chosen so that $s = m(\check{p}) + Cu(\check{p};\mu_1)$.

Fix $\varepsilon > 0$. As v converges uniformly to V_1^* as $\check{p} \to p_1^*$, we can choose $\check{p}$ such that $v \geq V_1^* - \varepsilon$. It suffices now to show that there is a $\bar{\Delta} > 0$ such that $T_*^\Delta v \geq v$ for $\Delta < \bar{\Delta}$. In fact, the monotonicity of T_*^Δ then implies $W_*^\Delta \geq v$ and hence $V_1^* - \varepsilon \leq v \leq W_*^\Delta \leq W_1^\Delta \leq V_1^*$ for all $\Delta < \bar{\Delta}$.

For $p \leq p_1^*$, we have $T_*^\Delta v(p) = (1-\delta)s + \delta v(p) \geq v(p)$ for all Δ, because $v \leq s$ in this range. For $p > p_1^*$,

$$\begin{aligned} T_*^\Delta v(p) &= (1-\delta)m(p) + \delta\mathcal{E}_1^\Delta v(p) \\ &= (1-\delta)m(p) + \delta\mathcal{E}_1^\Delta \left[m + Cu + \mathbb{1}_{[0,p^\natural]}(s - m - Cu)\right](p) \\ &= v(p) + \delta\mathcal{E}_1^\Delta \left[\mathbb{1}_{[0,p^\natural]}(s - m - Cu)\right](p), \end{aligned}$$

where the last equation uses that $\mathcal{E}_1^\Delta m(p) = m(p)$ and $\delta\mathcal{E}_1^\Delta u(p) = u(p)$. In particular, $T_*^\Delta v(1) = v(1)$.

The function $s - m - Cu$ is negative on the interval $(0, \check{p})$ and positive on $(\check{p}, p^\sharp)$, for some $p^\sharp > p_1^*$. The expectation of $s - m(p_\Delta) - Cu(p_\Delta)$ conditional on $p_0 = p$ and $p_\Delta \leq p^\natural$ is continuous in $(p, \Delta) \in [p_1^*, 1) \times (0, \infty)$ and converges to $s - m(p^\natural) - Cu(p^\natural) > 0$ as $p \to 1$ or $\Delta \to 0$ because the conditional distribution of p_Δ becomes a Dirac measure at $p^\natural$ in either limit. This implies existence of $\bar{\Delta} > 0$ such that this conditional expectation is positive for all $(p, \Delta) \in [p_1^*, 1) \times (0, \bar{\Delta})$. For these (p, Δ), we thus have

$$\mathcal{E}_1^\Delta \left[\mathbb{1}_{[0,p^\natural]}(s - m - Cu)\right](p) \geq \mathcal{E}_1^\Delta \left[\mathbb{1}_{[p^\flat,p^\natural]}(s - m - Cu)\right](p) \geq 0,$$

where $p^\flat = \frac{\check{p}+p^\natural}{2}$. As a consequence, $T_*^\Delta v \geq v$ for all $(p, \Delta) \in (p_1^*, 1) \times (0, \bar{\Delta})$. ■

Next, we turn to the payoff function associated with the good state of the automaton defined in Section 6. By the same arguments as invoked immediately before Lemma A.2, $\overline{w}^{\Delta}$ is the unique fixed point in $\mathcal{W}$ of the operator $\overline{T}^{\Delta}$ defined by

$$\overline{T}^{\Delta} w(p) = \begin{cases} (1-\delta)m(p) + \delta \mathcal{E}_N^{\Delta} w(p) & \text{if} \quad p > \underline{p}, \\ (1-\delta)s + \delta w(p) & \text{if} \quad p \leq \underline{p}. \end{cases}$$

Lemma A.3 *Let $\underline{p} > p_N^*$. Then $\overline{w}^{\Delta} \geq V_{N,\underline{p}}$ for Δ sufficiently small.*

PROOF: Because of the monotonicity of the operator $\overline{T}^{\Delta}$, it suffices to show that $\overline{T}^{\Delta} V_{N,\underline{p}} \geq V_{N,\underline{p}}$ for sufficiently small Δ. Recall that for $p > \underline{p}$, $V_{N,\underline{p}}(p) = m(p) + Cu(p;\mu_N)$ where the constant $C > 0$ is chosen to ensure continuity at $\underline{p}$.

For $p \leq \underline{p}$, we use exactly the same argument as in the penultimate paragraph of the proof of Lemma A.2; for $p > \underline{p}$, the argument is the same as in the last paragraph of that proof. ■

The next two results concern the payoff function associated with the bad state of the automaton defined in Section 6. Fix a cutoff $\bar{p} \in (p^m, 1)$ and let $K(p) = N-1$ when $p > \bar{p}$, and $K(p) = 0$ otherwise. Given $\Delta > 0$, and any bounded function w on $[0,1]$, define a bounded function $\underline{T}^{\Delta} w$ by

$$\underline{T}^{\Delta} w(p) = \max\left\{(1-\delta)m(p) + \delta \mathcal{E}_{K(p)+1}^{\Delta} w(p),\ (1-\delta)s + \delta \mathcal{E}_{K(p)}^{\Delta} w(p)\right\}.$$

The operator $\underline{T}^{\Delta}$ again satisfies Blackwell's sufficient conditions for being a contraction mapping with modulus δ on $\mathcal{W}$. Its unique fixed point in this space is the payoff function $\underline{w}^{\Delta}$ (introduced in Section 6) from playing a best response against $N-1$ opponents who all play risky when $p > \bar{p}$, and safe otherwise.

Lemma A.4 *Let $\underline{p} \in (p_N^*, p_1^*)$. Then there exists $p^{\diamond} \in [p^m, 1)$ such that for all $\bar{p} \in (p^{\diamond}, 1)$, the inequality $\underline{w}^{\Delta} \leq V_{N,(\underline{p}+p_1^*)/2}$ holds for Δ sufficiently small.*

PROOF: Let $\tilde{p} = (\underline{p} + p_1^*)/2$. For $p > \tilde{p}$, we have $V_{N,\tilde{p}}(p) = m(p) + Cu(p;\mu_N)$ where the constant $C > 0$ is chosen to ensure continuity at $\tilde{p}$. To simplify notation, we write $\tilde{v}$ instead of $V_{N,\tilde{p}}$ and u instead of $u(\cdot;\mu_N)$.

For $x > 0$, we define

$$p_x^* = \frac{\mu_x(s - m_0)}{(\mu_x + 1)(m_1 - s) + \mu_x(s - m_0)},$$

where μ_x is the unique positive root of

$$f(\mu; x) = \frac{\rho}{2}\mu(\mu+1) + (\lambda_1 - \lambda_0)\mu + \lambda_0 \left(\frac{\lambda_0}{\lambda_1}\right)^{\mu} - \lambda_0 - \frac{r}{x};$$

existence and uniqueness of this root follow from continuity and monotonicity of $f(\cdot; x)$ to-

gether with the fact that $f(0;x) < 0$ while $f(\mu;x) \to \infty$ as $\mu \to \infty$.[23] This extends our previous definitions of μ_N and p_N^* to non-integer numbers. It is immediate to verify now that $\frac{d\mu_x}{dx} < 0$ and hence $\frac{dp_x^*}{dx} < 0$. Thus, there exists $\breve{x} \in (1, N)$ such that $p_{\breve{x}}^* \in (\tilde{p}, p_1^*)$.

Having chosen such an $\breve{x}$, we fix a belief $\breve{p} \in (\tilde{p}, p_{\breve{x}}^*)$ and, on the open unit interval, consider the function $\breve{v}$ that solves

$$\mathcal{L}^1 v - \frac{r}{\breve{x}}(v - m) = 0$$

subject to the conditions $\breve{v}(\breve{p}) = s$ and $\breve{v}'(\breve{p}) = 0$. This function has the form

$$\breve{v}(p) = m(p) + \breve{u}(p),$$

with

$$\breve{u}(p) = A(1-p)\left(\frac{1-p}{p}\right)^{\breve{\mu}} + Bp\left(\frac{p}{1-p}\right)^{\hat{\mu}} = Au(p;\breve{\mu}) + Bu(1-p;\hat{\mu}).$$

Here, $\breve{\mu} = \mu_{\breve{x}}$ and $\hat{\mu}$ is the unique positive root of

$$g(\mu;x) = \frac{\rho}{2}\mu(\mu+1) - (\lambda_1 - \lambda_0)\mu + \lambda_1\left(\frac{\lambda_1}{\lambda_0}\right)^{\mu} - \lambda_1 - \frac{r}{x};$$

existence and uniqueness of this root follow along the same lines as above.

The constants of integration A and B are pinned down by the conditions $\breve{v}(\breve{p}) = s$ and $\breve{v}'(\breve{p}) = 0$. One calculates that $B > 0$ if, and only if, $\breve{p} < p_{\breve{x}}^*$, which holds by construction, and that $A > 0$ if, and only if,

$$\breve{p} < \frac{(1+\hat{\mu})(s-m_0)}{\hat{\mu}(m_1 - s) + (1+\hat{\mu})(s-m_0)}.$$

The right-hand side of this inequality is decreasing in $\hat{\mu}$ and tends to p^m as $\hat{\mu} \to \infty$. Therefore, we can conclude that the inequality holds, and $A > 0$ as well. Moreover, $A + B > 0$ implies that $\breve{v}$ is strictly increasing and strictly convex on $(\breve{p}, 1)$; as $B > 0$, finally, $\breve{v}(p) \to \infty$ for $p \to 1$.

So there exists a belief $p^\natural \in (\breve{p}, 1)$ such that $\breve{v}(p^\natural) = \tilde{v}(p^\natural)$ and $\breve{v} > \tilde{v}$ on $(p^\natural, 1)$. We now show that $\breve{v} < \tilde{v}$ in $(\breve{p}, p^\natural)$. Indeed, if this is not the case, then $\breve{v} - \tilde{v}$ assumes a non-negative local maximum at some $p^\sharp \in (\breve{p}, p^\natural)$. This implies:
(i) $\breve{v}(p^\sharp) \geq \tilde{v}(p^\sharp)$, *i.e.*,

$$Au(p^\sharp;\breve{\mu}) + Bu(1-p^\sharp;\hat{\mu}) \geq Cu(p^\sharp;\mu_N); \tag{A.1}$$

(ii) $\breve{v}'(p^\sharp) = \tilde{v}'(p^\sharp)$, *i.e.*,

$$-(\breve{\mu}+p^\sharp)Au(p^\sharp;\breve{\mu}) + (\hat{\mu}+1-p^\sharp)Bu(1-p^\sharp;\hat{\mu}) = -(\mu_N + p^\sharp)Cu(p^\sharp;\mu_N); \tag{A.2}$$

[23] *Cf.* Lemma 6 in Cohen & Solan (2013).

and (iii) $\breve{v}''(p^\sharp) \leq \tilde{v}''(p^\sharp)$, *i.e.*,

$$\breve{\mu}(\breve{\mu}+1)Au(p^\sharp;\breve{\mu}) + \hat{\mu}(1+\hat{\mu})Bu(1-p^\sharp;\hat{\mu}) \leq \mu_N(\mu_N+1)Cu(p^\sharp;\mu_N). \tag{A.3}$$

Solving for $Bu(1-p^\sharp;\hat{\mu})$ in (A.2) and inserting the result into (A.1) and (A.3), we obtain, respectively,

$$\frac{Cu(p^\sharp;\mu_N)}{Au(p^\sharp;\breve{\mu})} \leq \frac{\breve{\mu}+\hat{\mu}+1}{\mu_N+\hat{\mu}+1},$$

and

$$\frac{Cu(p^\sharp;\mu_N)}{Au(p^\sharp;\breve{\mu})} \geq \frac{\breve{\mu}(\breve{\mu}+1)(\hat{\mu}+1-p^\sharp)+\hat{\mu}(\hat{\mu}+1)(\breve{\mu}+p^\sharp)}{\mu_N(\mu_N+1)(\hat{\mu}+1-p^\sharp)+\hat{\mu}(\hat{\mu}+1)(\mu_N+p^\sharp)}.$$

This implies that

$$\frac{\breve{\mu}+\hat{\mu}+1}{\mu_N+\hat{\mu}+1} \geq \frac{\breve{\mu}(\breve{\mu}+1)(\hat{\mu}+1-p^\sharp)+\hat{\mu}(\hat{\mu}+1)(\breve{\mu}+p^\sharp)}{\mu_N(\mu_N+1)(\hat{\mu}+1-p^\sharp)+\hat{\mu}(\hat{\mu}+1)(\mu_N+p^\sharp)},$$

which one shows to be equivalent to $\breve{\mu} \leq \mu_N$. But $\breve{x} < N$ and $\frac{d\mu_x}{dx} < 0$ imply $\breve{\mu} > \mu_N$. This is the desired contradiction.

Now let $p^\diamond = \max\{p^m, p^\natural\}$, fix $\bar{p} \in (p^\diamond, 1)$ and define

$$v(p) = \begin{cases} \tilde{v}(p) & \text{if} \quad p > p^\natural, \\ \breve{v}(p) & \text{if} \quad \breve{p} \leq p \leq p^\natural, \\ s & \text{if} \quad p < \breve{p}. \end{cases}$$

By construction, $s \leq v \leq \min\{\tilde{v}, \breve{v}\}$. This immediately implies that $(1-\delta)s + \delta v \leq v$. We now show that $\underline{T}^\Delta v \leq v$, and hence $\underline{w}^\Delta \leq v$, for Δ sufficiently small.

First, let $p \in (\bar{p}, 1]$. We have

$$\begin{aligned} (1-\delta)m(p) + \delta\mathcal{E}_N^\Delta v(p) &\leq (1-\delta)m(p) + \delta\mathcal{E}_N^\Delta\left[m + Cu + \mathbb{1}_{(0,\breve{p})}(s-m-Cu)\right](p) \\ &= m(p) + Cu(p) + \delta\mathcal{E}_N^\Delta\left[\mathbb{1}_{(0,\breve{p})}(s-m-Cu)\right](p) \\ &\leq m(p) + Cu(p) \\ &= v(p), \end{aligned}$$

for Δ small enough that $\mathcal{E}_N^\Delta\left[\mathbb{1}_{(0,\breve{p})}(s-m-Cu)\right](\bar{p}) \leq 0$; that this inequality holds for small Δ follows from the fact that $s < m + Cu$ on $(\tilde{p}, \breve{p})$. By the same token,

$$\begin{aligned} (1-\delta)s + \delta\mathcal{E}_{N-1}^\Delta v(p) &\leq (1-\delta)s + \delta\mathcal{E}_{N-1}^\Delta(m+Cu)(p) + \delta\mathcal{E}_{N-1}^\Delta\left[\mathbb{1}_{(0,\breve{p})}(s-m-Cu)\right](p) \\ &= (1-\delta)s + \delta m(p) + \delta^{\frac{1}{N}}Cu(p) + \delta\mathcal{E}_{N-1}^\Delta\left[\mathbb{1}_{(0,\breve{p})}(s-m-Cu)\right](p) \\ &\leq m(p) + Cu(p) \\ &= v(p), \end{aligned}$$

for Δ small enough that $\mathcal{E}_{N-1}^\Delta\left[\mathbb{1}_{(0,\breve{p})}(s-m-Cu)\right](\bar{p}) \leq 0$, as $Cu(p) > 0$ and $s < m(p)$ for

$p > p^m$.

Second, let $p \in (p^\natural, \bar{p}]$. Now, we have

$$\begin{aligned}(1-\delta)m(p) + \delta\mathcal{E}_1^\Delta v(p) &\leq m(p) + \delta^{1-\frac{1}{N}} Cu(p) + \delta\mathcal{E}_1^\Delta \left[\mathbb{1}_{(0,\breve{p})}(s - m - Cu)\right](p) \\ &\leq m(p) + Cu(p) \\ &= v(p),\end{aligned}$$

for Δ small enough that $\mathcal{E}_1^\Delta \left[\mathbb{1}_{(0,\breve{p})}(s - m - Cu)\right](p^\natural) \leq 0$.

Third, let $p \in [\breve{p}, p^\natural]$. In this case,

$$\begin{aligned}(1-\delta)m(p) + \delta\mathcal{E}_1^\Delta v(p) &\leq (1-\delta)m(p) + \delta\mathcal{E}_1^\Delta \breve{v}(p) \\ &= m(p) + \delta\mathcal{E}_1^\Delta \breve{u}(p) \\ &= m(p) + \breve{u}(p) + \mathbb{E}\left[\int_0^\Delta e^{-rt}\left\{\mathcal{L}^1\breve{u}(p_t) - r\breve{u}(p_t)\right\} dt \,\middle|\, p_0 = p\right] \\ &\leq m(p) + \breve{u}(p) + \mathbb{E}\left[\int_0^\Delta e^{-rt}\left\{\mathcal{L}^1\breve{u}(p_t) - \frac{r}{\breve{x}}\breve{u}(p_t)\right\} dt \,\middle|\, p_0 = p\right] \\ &= m(p) + \breve{u}(p) \\ &= v(p),\end{aligned}$$

where the second equality follows from Dynkin's formula, the second inequality holds because $\breve{u}(p_t) > 0$ and $\breve{x} > 1$, and the third equality is a consequence of the identity $\mathcal{L}^1\breve{u} - r\breve{u}/\breve{x} = 0$.

Finally, let $p \in [0, \breve{p})$. By monotonicity of m and v (and the previous step), we see that $(1-\delta)m(p) + \delta\mathcal{E}_1^\Delta v(p) \leq (1-\delta)m(\breve{p}) + \delta\mathcal{E}_1^\Delta v(\breve{p}) \leq v(\breve{p}) = s = v(p)$. ■

Lemma A.5 *There exist $\breve{p} \in (p^m, 1)$ and $p^\ddagger \in (p_N^*, p_1^*)$ such that $\underline{w}^\Delta(p) = s$ for all $\bar{p} \in (\breve{p}, 1)$, $p \leq p^\ddagger$ and $\Delta > 0$. For any $\varepsilon > 0$, moreover, there exists $\breve{p}_\varepsilon \in (\breve{p}, 1)$ such that $\underline{w}^\Delta \leq V_1^* + \varepsilon$ for all $\Delta > 0$.*

Proof: Consider any $\bar{p} \in (p^m, 1)$ and an initial belief $p < \bar{p}$. We obtain an upper bound on $\underline{w}^\Delta(p)$ by considering a modified problem in which (i) the player can choose a best response in continuous time and (ii) the game is stopped with continuation payoff m_1 as soon as the belief $\bar{p}$ is reached. This problem can be solved in the standard way, yielding an optimal cutoff $p^\ddagger$. By construction, $\underline{w}^\Delta = s$ on $[0, p^\ddagger]$. As we take $\bar{p}$ close to 1, $p^\ddagger$ approaches p_1^* from the left and thus gets to lie strictly in between p_N^* and p_1^*. This proves the first statement.

The second follows from the fact that the value function of the modified problem converges uniformly to V_1^* as $\bar{p} \to 1$. ■

In the case of pure Poisson learning ($\rho = 0$), we need a sharper characterization of the payoff function $\underline{w}^\Delta$ as Δ becomes small. To this end, we define $V_{1,\bar{p}}$ as the continuous-time counterpart to $\underline{w}^\Delta$. The methods employed in Keller and Rady (2010) can be used to establish that $V_{1,\bar{p}}$ has the following properties for $\rho = 0$. First, there is a cutoff $p^\dagger < p^m$ such that $V_{1,\bar{p}} = s$ on $[0, p^\dagger]$, and $V_{1,\bar{p}} > s$ everywhere else. Second, $V_{1,\bar{p}}$ is continuously differentiable

everywhere except at $\bar{p}$. Third, $V_{1,\bar{p}}$ solves the Bellman equation

$$v(p) = \max\left\{m(p) + [K(p)+1]b(p,v),\ s + K(p)b(p,v)\right\},$$

where

$$b(p,v) = \frac{\lambda(p)}{r}\,[v(j(p)) - v(p)] - \frac{\lambda_1 - \lambda_0}{r}\,p(1-p)\,v'(p),$$

and $v'(p)$ is taken to mean the left-hand derivative of v. Fourth, $b(p, V_{1,\bar{p}}) \geq 0$ for all p. Fifth, because of smooth pasting at $p^\dagger$, the term $m(p) + b(p, V_{1,\bar{p}}) - s$ is continuous in p except at $\bar{p}$; it has a single zero at $p^\dagger$, being positive to the right of it and negative to the left. Finally, we note that $V_{1,\bar{p}} = V_1^*$ and $p^\dagger = p_1^*$ for $\bar{p} = 1$.

Lemma A.6 *Let $\rho = 0$. Then $V_{1,\bar{p}} \to V_1^*$ uniformly as $\bar{p} \to 1$. The convergence is monotone in the sense that $\bar{p}' > \bar{p}$ implies $V_{1,\bar{p}'} < V_{1,\bar{p}}$ on $\{p\colon s < V_{1,\bar{p}}(p) < \lambda_1 h\}$.*

As the closed-form solutions for the functions in question make it straightforward to establish this result, we omit the proof.

A key ingredient in the analysis of the pure Poisson case is uniform convergence of $\underline{w}^\Delta$ to $V_{1,\bar{p}}$ as $\Delta \to 0$, which we establish by means of the following result.[24]

Lemma A.7 *Let $\{T^\Delta\}_{\Delta>0}$ be a family of contraction mappings on the Banach space $(\mathcal{W}; \|\cdot\|)$ with moduli $\{\beta^\Delta\}_{\Delta>0}$ and associated fixed points $\{w^\Delta\}_{\Delta>0}$. Suppose that there is a constant $\nu > 0$ such that $1 - \beta^\Delta = \nu\Delta + o(\Delta)$ as $\Delta \to 0$. Then, a sufficient condition for w^Δ to converge in $(\mathcal{W}; \|\cdot\|)$ to the limit v as $\Delta \to 0$ is that $\|T^\Delta v - v\| = o(\Delta)$.*

Proof: As

$$\|w^\Delta - v\| = \|T^\Delta w^\Delta - v\| \leq \|T^\Delta w^\Delta - T^\Delta v\| + \|T^\Delta v - v\| \leq \beta^\Delta \|w^\Delta - v\| + \|T^\Delta v - v\|,$$

the stated conditions on β^Δ and $\|T^\Delta v - v\|$ imply

$$\|w^\Delta - v\| \leq \frac{\|T^\Delta v - v\|}{1 - \beta^\Delta} = \frac{\Delta f(\Delta)}{\nu\Delta + \Delta g(\Delta)} = \frac{f(\Delta)}{\nu + g(\Delta)},$$

with $\lim_{\Delta\to 0} f(\Delta) = \lim_{\Delta\to 0} g(\Delta) = 0$. ■

In our application of this lemma, $\mathcal{W}$ is again the Banach space of bounded real-valued functions on the unit interval, equipped with the supremum norm. The operator in question is $\underline{T}^\Delta$ as defined above; the corresponding moduli are $\beta^\Delta = \delta = e^{-r\Delta}$, so that $\nu = r$.

Lemma A.8 *Let $\rho = 0$. Then $\underline{w}^\Delta \to V_{1,\bar{p}}$ uniformly as $\Delta \to 0$.*

[24] To the best of our knowledge, the earliest appearance of this result in the economics literature is in Biais et al. (2007). A related approach is taken in Sadzik and Stacchetti (2015).

PROOF: To simplify notation, we write v instead of $V_{1,\bar{p}}$. For $K \in \{0, 1, \ldots, N\}$, a straightforward Taylor expansion of $\mathcal{E}_K^\Delta v$ with respect to Δ yields

$$\lim_{\Delta \to 0} \frac{1}{\Delta} \left\| \delta \, \mathcal{E}_K^\Delta v - v - r[Kb(\cdot, v) - v]\Delta \right\| = 0. \tag{A.4}$$

For $p > \bar{p}$, we have $K(p) = N - 1$, and (A.4) implies

$$\begin{aligned} (1-\delta)m(p) + \delta \mathcal{E}_N^\Delta v(p) &= v(p) + r\left[m(p) + Nb(p, v) - v(p)\right]\Delta + o(\Delta), \\ (1-\delta)s + \delta \mathcal{E}_{N-1}^\Delta v(p) &= v(p) + r\left[s + (N-1)b(p, v) - v(p)\right]\Delta + o(\Delta). \end{aligned}$$

As $m(p) > s$ on $[\bar{p}, 1]$ and $b(p, v) \geq 0$, there exists $\xi > 0$ such that

$$m(p) + Nb(p, v) - [s + (N-1)b(p, v)] > \xi,$$

on $(\bar{p}, 1]$. Thus, $\underline{T}^\Delta v(p) = (1-\delta)m(p) + \delta \mathcal{E}_N^\Delta v(p)$ for Δ sufficiently small, and the fact that $v(p) = m(p) + Nb(p, v)$ now implies $\underline{T}^\Delta v(p) = v(p) + o(\Delta)$ on $(\bar{p}, 1]$.

On $[0, \bar{p}]$, we have $K(p) = 0$, and (A.4) implies

$$\left\| (1-\delta)m + \delta \mathcal{E}_1^\Delta v - v - r[m + b(\cdot, v) - v]\Delta \right\| = \Delta \psi_R(\Delta), \tag{A.5}$$

$$\left\| (1-\delta)s + \delta \mathcal{E}_0^\Delta v - v - r[s - v]\Delta \right\| = \Delta \psi_S(\Delta), \tag{A.6}$$

for some functions $\psi_R, \psi_S : (0, \infty) \to [0, \infty)$ that satisfy $\psi_R(\Delta) \to 0$ and $\psi_S(\Delta) \to 0$ as $\Delta \to 0$.

First, let $p \in (p^\dagger, \bar{p}]$. We note that $\underline{T}^\Delta v(p) \geq (1-\delta)m(p) + \delta \mathcal{E}_1^\Delta v(p) \geq v(p) - \Delta \psi_R(\Delta)$ in this range, where the first inequality follows from the definition of $\underline{T}^\Delta$, and the second inequality is implied by (A.5) and $v(p) = m(p) + b(p, v)$ for $p \in (p^\dagger, \bar{p}]$. If the maximum in the definition of $\underline{T}^\Delta v(p)$ is achieved by the risky action, the first in the previous chain of inequalities holds as an equality, and (A.5) immediately implies that $\underline{T}^\Delta v(p) = v(p) + o(\Delta)$. If the maximum in the definition of $\underline{T}^\Delta v(p)$ is achieved by the safe action, however, we have $\underline{T}^\Delta v(p) = (1-\delta)s + \delta \mathcal{E}_0^\Delta v(p) \leq v(p) + r[s - v(p)]\Delta + \Delta \psi_S(\Delta) \leq v(p) + \Delta \psi_S(\Delta)$, where the second inequality follows from $v > s$ on $(p^\dagger, \bar{p}]$. Thus $v(p) - \Delta \psi_R(\Delta) \leq \underline{T}^\Delta v(p) \leq v(p) + \Delta \psi_S(\Delta)$, and we can conclude that $\underline{T}^\Delta v(p) = v(p) + o(\Delta)$ in this case as well.

Now, let $p \leq p^\dagger$. We note that $\underline{T}^\Delta v(p) \geq (1-\delta)s + \delta \mathcal{E}_0^\Delta v(p) \geq v(p) - \Delta \psi_S(\Delta)$ in this range, where the first inequality follows from the definition of $\underline{T}^\Delta$, and the second inequality is implied by (A.6) and $v(p) = s$ for $p \leq p^\dagger$. If the maximum in the definition of $\underline{T}^\Delta v(p)$ is achieved by the safe action, the first in the previous chain of inequalities holds as an equality, and (A.6) immediately implies that $\underline{T}^\Delta v(p) = v(p) + o(\Delta)$. If the maximum in the definition of $\underline{T}^\Delta v(p)$ is achieved by the risky action, however, we have $\underline{T}^\Delta v(p) = (1-\delta)m(p) + \delta \mathcal{E}_1^\Delta v(p) \leq v(p) + r[m(p) + b(p, v) - v(p)]\Delta + \Delta \psi_R(\Delta) \leq v(p) + \Delta \psi_R(\Delta)$, where the second inequality follows from $v = s \geq m(p) + b(p, v)$ on $[0, p^\dagger]$. Thus $v(p) - \Delta \psi_S(\Delta) \leq \underline{T}^\Delta v(p) \leq v(p) + \Delta \psi_R(\Delta)$, and we can again conclude that $\underline{T}^\Delta v(p) = v(p) + o(\Delta)$ in this case as well. ■

Our last two auxiliary results pertain to the case of pure Poisson learning.

Lemma A.9 *Let $\rho = 0$. There is a belief $\hat{p} \in [p_N^*, p_1^*]$ such that*

$$\lambda(\underline{p}) \left[N V_{N,\underline{p}}(j(\underline{p})) - (N-1) V_1^*(j(\underline{p})) - s \right] - rc(\underline{p})$$

is negative if $0 < \underline{p} < \hat{p}$, zero if $\underline{p} = \hat{p}$, and positive if $\hat{p} < \underline{p} < 1$. Moreover, $\hat{p} = p_N^*$ if, and only if, $j(p_N^*) \leq p_1^*$, and $\hat{p} = p_1^*$ if, and only if, $\lambda_0 = 0$.

PROOF: We start by noting that given the functions V_1^* and V_N^*, the cutoffs p_1^* and p_N^* are uniquely determined by

$$\lambda(p_1^*)[V_1^*(j(p_1^*)) - s] = rc(p_1^*), \tag{A.7}$$

and

$$\lambda(p_N^*)[N V_N^*(j(p_N^*)) - Ns] = rc(p_N^*), \tag{A.8}$$

respectively.

Consider the differentiable function f on $(0,1)$ given by

$$f(\underline{p}) = \lambda(\underline{p})[N V_{N,\underline{p}}(j(\underline{p})) - (N-1) V_1^*(j(\underline{p})) - s] - rc(\underline{p}).$$

For $\lambda_0 = 0$, we have $j(\underline{p}) = 1$ and $V_{N,\underline{p}}(j(\underline{p})) = V_1^*(j(\underline{p})) = m_1$ for all p, so $f(\underline{p}) = \lambda(\underline{p})[V_1^*(j(\underline{p})) - s] - rc(\underline{p})$, which is zero at $\underline{p} = p_1^*$ by (A.7), positive for $\underline{p} > p_1^*$, and negative for $\underline{p} < p_1^*$.

Assume $\lambda_0 > 0$. For $0 < \underline{p} < p \leq 1$, we have $V_{N,\underline{p}}(p) = m(p) + c(\underline{p}) u(p; \mu_N) / u(\underline{p}; \mu_N)$. Moreover, we have $V_1^*(p) = s$ when $p \leq p_1^*$, and $V_1^*(p) = m(p) + C u(p; \mu_1)$ with a constant $C > 0$ otherwise. Using the fact that

$$u(j(p); \mu) = \frac{\lambda_0}{\lambda(p)} \left(\frac{\lambda_0}{\lambda_1} \right)^{\mu} u(p; \mu),$$

we see that the term $\lambda(\underline{p}) N V_{N,\underline{p}}(j(\underline{p}))$ is actually linear in $\underline{p}$. When $j(\underline{p}) \leq p_1^*$, the term $-\lambda(\underline{p})(N-1) V_1^*(j(\underline{p}))$ is also linear in $\underline{p}$; when $j(\underline{p}) > p_1^*$, the nonlinear part of this term simplifies to $-(N-1) C \lambda_0^{\mu_1+1} u(\underline{p}; \mu_1) / \lambda_1^{\mu_1}$. This shows that f is concave, and strictly concave on the interval of all $\underline{p}$ for which $j(\underline{p}) > p_1^*$. As $\lim_{\underline{p} \to 1} f(\underline{p}) > 0$, this in turn implies that f has at most one root in the open unit interval; if so, f assumes negative values to the left of the root, and positive values to the right.

As $V_{N,p_1^*}(j(p_1^*)) > V_1^*(j(p_1^*))$, moreover, we have $f(p_1^*) > \lambda(p_1^*)[V_1^*(j(p_1^*)) - s] - rc(p_1^*) = 0$ by (A.7). Any root of f must thus lie in $[0, p_1^*)$. If $j(p_N^*) \leq p_1^*$, then $V_1^*(j(p_N^*)) = s$ and $f(p_N^*) = \lambda(p_N^*)[N V_N^*(j(p_N^*)) - Ns] - rc(p_N^*) = 0$ by (A.8). If $j(p_N^*) > p_1^*$, then $V_1^*(j(p_N^*)) > s$ and $f(p_N^*) < 0$, so f has a root in (p_N^*, p_1^*). ■

The following result is used in the proof of Proposition 2.

Lemma A.10 *Let $\rho = 0$. Then $\mu_1(\mu_1 + 1) > N \mu_N(\mu_N + 1)$.*

PROOF: We change variables to $\beta = \lambda_0/\lambda_1$ and $x = r/\lambda_1$, so that μ_N and μ_1 are implicitly defined as the positive solutions of the equations

$$\begin{aligned} \frac{x}{N} + \beta - (1-\beta)\mu_N &= \beta^{\mu_N+1}, \\ x + \beta - (1-\beta)\mu_1 &= \beta^{\mu_1+1}. \end{aligned}$$

Fixing $\beta \in [0,1)$ and considering μ_N and μ_1 as functions of $x \in (0,\infty)$, we obtain

$$\begin{aligned} \mu_N' &= \frac{N^{-1}}{1-\beta+\beta^{\mu_N+1}\ln\beta} = \frac{N^{-1}}{1-\beta+\left[\frac{x}{N}+\beta-(1-\beta)\mu_N\right]\ln\beta}, \\ \mu_1' &= \frac{1}{1-\beta+\beta^{\mu_1+1}\ln\beta} = \frac{1}{1-\beta+[x+\beta-(1-\beta)\mu_1]\ln\beta}. \end{aligned}$$

(All denominators are positive because $1-\beta+\beta^{\mu+1}\ln\beta \geq 1-\beta+\beta\ln\beta > 0$ for all $\mu \geq 0$.)

Let $d = \mu_1(\mu_1+1) - N\mu_N(\mu_N+1)$. As $\lim_{x\to 0}\mu_N = \lim_{x\to 0}\mu_1 = 0$, we see that $\lim_{x\to 0} d = 0$ as well. It is thus enough to show that $d' > 0$ at any $x > 0$. This is the case if, and only if, $(2\mu_1+1)\mu_1' > N(2\mu_N+1)\mu_N'$, that is,

$$(2\mu_1+1)\left\{1-\beta+\left[\tfrac{x}{N}+\beta-(1-\beta)\mu_N\right]\ln\beta\right\} > (2\mu_N+1)\left\{1-\beta+[x+\beta-(1-\beta)\mu_1]\ln\beta\right\}.$$

This inequality reduces to

$$(\mu_1-\mu_N)\left\{2(1-\beta)+\left[\tfrac{2x}{N}+1+\beta\right]\ln\beta\right\} > (2\mu_N+1)\left[x-\tfrac{x}{N}\right]\ln\beta.$$

It is straightforward to show that $\mu_1 > \mu_N + \frac{1}{1-\beta}\left[x-\frac{x}{N}\right]$. So $d' > 0$ if

$$2(1-\beta)+\left[\tfrac{2x}{N}+1+\beta\right]\ln\beta > (2\mu_N+1)(1-\beta)\ln\beta,$$

which simplifies to $1-\beta+\left[\frac{x}{N}+\beta-(1-\beta)\mu_N\right]\ln\beta > 0$ – an inequality that we have already established. ■

B Proofs

B.1 Main Results (Theorem 1 and Propositions 1–4)

PROOF OF THEOREM 1: For $\rho > 0$, this result is an immediate consequence of inequalities (2), the fact that $\liminf_{\Delta\to 0}\underline{W}^{\Delta}_{\text{PBE}} \geq V_1^*$ and $\overline{W}^{\Delta}_{\text{PBE}} \leq V_N^*$, and Proposition 6. For $\rho = 0$, the result follows from inequalities (2), the fact $\liminf_{\Delta\to 0}\underline{W}^{\Delta}_{\text{PBE}} \geq V_1^*$, and Propositions 7 and 10. ■

PROOF OF PROPOSITION 1: Keller and Rady (2010) establish that in the unique symmetric MPE of the continuous-time game, all experimentation stops at the belief $\tilde{p}_N$ implicitly defined by $rc(\tilde{p}_N) = \lambda(\tilde{p}_N)[\tilde{u}(j(\tilde{p}_N)) - s]$, where $\tilde{u}$ is the players' common equilibrium pay-

off function. The results of Keller and Rady (2010) further imply that $V_{N,\tilde{p}_N}(j(\tilde{p}_N)) > \tilde{u}(j(\tilde{p}_N)) > V_1^*(j(\tilde{p}_N))$, so that $NV_{N,\tilde{p}_N}(j(\tilde{p}_N)) - (N-1)V_1^*(j(\tilde{p}_N)) > \tilde{u}(j(\tilde{p}_N))$, and hence $\hat{p} < \tilde{p}_N$ by Lemma A.9. ■

Proof of Proposition 2: There is nothing to show for $\lambda_0 = 0$. Using the same change of variables as in the previous proof, we fix $\beta \in (0,1)$, therefore, and define

$$q = \beta \cdot \frac{1+\mu_N^{-1}}{1+\mu_1^{-1}},$$

so that $j(p_N^*) \leq p_1^*$ if, and only if, $q \geq 1$. As $\lim_{x\to\infty} \mu_N = \lim_{x\to\infty} \mu_1 = \infty$, we have $\lim_{x\to\infty} q = \beta < 1$. As $\lim_{x\to 0} \mu_N = \lim_{x\to 0} \mu_1 = 0$, moreover,

$$\lim_{x\to 0} q = \beta \lim_{x\to 0} \frac{\mu_1}{\mu_N} = \beta \lim_{x\to 0} \frac{\mu_1'}{\mu_N'} = \beta N$$

by l'Hôpital's rule. Finally, q' is easily seen to have the same sign as

$$-\mu_1(\mu_1+1)(1-\beta+\beta^{\mu_1+1}\ln\beta) + N\mu_N(\mu_N+1)(1-\beta+\beta^{\mu_N+1}\ln\beta).$$

As $\beta^{\mu_1+1}\ln\beta > \beta^{\mu_N+1}\ln\beta$, Lemma A.10 implies that q decreases strictly in x. This in turn implies that $q < 1$ at all $x \in (0,\infty)$ when $\beta N \leq 1$, which proves the first part of the corollary. Otherwise, there exists a unique $x^* \in (0,\infty)$ at which $q = 1$. The second part of the corollary thus holds with $(\lambda_1^*, \lambda_0^*) = (r/x^*, \beta r/x^*)$.

It is straightforward to see that x varies continuously with β and that $\lim_{\beta\to 1/N} x^* = 0$. So it remains to show that x^* remains bounded as $\beta \to 1$. Rewriting the defining equation for x^* as

$$1 + \frac{1}{(1-\beta)\mu_1(x^*(\beta),\beta)} = \frac{1}{(1-\beta)\mu_N(x^*(\beta),\beta)},$$

we see that $(1-\beta)\mu_N(x^*(\beta),\beta)$ must stay bounded as $\beta \to 1$. By the defining equation for μ_N, $x^*(\beta)$ must then also stay bounded. ■

Proof of Proposition 3: For the case that $\hat{p} = p_N^*$, this is shown in Keller and Rady (2010). Thus, in what follows we assume that $\hat{p} > p_N^*$.

Recall the defining equation for $\hat{p}$ from Lemma A.9,

$$\lambda(\hat{p})NV_{N,\hat{p}}(j(\hat{p})) - \lambda(\hat{p})s - rc(\hat{p}) = (N-1)\lambda(\hat{p})V_1^*(j(\hat{p})).$$

We make use of the closed-form expression for $V_{N,\hat{p}}$ to rewrite its left-hand side as

$$N\lambda(\hat{p})\lambda(j(\hat{p}))h + Nc(\hat{p})[\lambda_0 - \mu_N(\lambda_1 - \lambda_0)] - \lambda(\hat{p})s.$$

Similarly, by noting that $\hat{p} > p_N^*$ implies $j(\hat{p}) > j(p_N^*) > p_1^*$, we can make use of the closed-

form expression for V_1^* to rewrite the right-hand side as

$$(N-1)\lambda(\hat{p})\lambda(j(\hat{p}))h + (N-1)c(p_1^*)\frac{u(\hat{p};\mu_1)}{u(p_1^*;\mu_1)}[r+\lambda_0-\mu_1(\lambda_1-\lambda_0)].$$

Combining, we have

$$\frac{\lambda(\hat{p})\lambda(j(\hat{p}))h + Nc(\hat{p})[\lambda_0-\mu_N(\lambda_1-\lambda_0)] - \lambda(\hat{p})s}{(N-1)[r+\lambda_0-\mu_1(\lambda_1-\lambda_0)]c(p_1^*)} = \frac{u(\hat{p};\mu_1)}{u(p_1^*;\mu_1)}.$$

It is convenient to change variables to

$$\beta = \frac{\lambda_0}{\lambda_1} \quad \text{and} \quad y = \frac{\lambda_1}{\lambda_0}\,\frac{\lambda_1 h - s}{s-\lambda_0 h}\,\frac{\hat{p}}{1-\hat{p}}\,.$$

The implicit definitions of μ_1 and μ_N imply

$$N = \frac{\beta^{1+\mu_1} - \beta + \mu_1(1-\beta)}{\beta^{1+\mu_N} - \beta + \mu_N(1-\beta)}\,,$$

allowing us to rewrite the defining equation for $\hat{p}$ as the equation $F(y,\mu_N)=0$ with

$$\begin{aligned} F(y,\mu) &= 1 - y + [\beta(1+\mu)y - \mu]\,\frac{1-\beta}{\beta}\,\frac{\beta^{1+\mu_1} - \beta + \mu_1(1-\beta)}{(\mu_1-\mu)(1-\beta) + \beta^{1+\mu_1} - \beta^{1+\mu}} \\ &\quad - \frac{\mu_1^{\mu_1}}{(1+\mu_1)^{1+\mu_1}}\,y^{-\mu_1}. \end{aligned}$$

As y is a strictly increasing function of $\hat{p}$, we know from Lemma A.9 that $F(\cdot,\mu_N)$ admits a unique root, and that it is strictly increasing in a neighborhood of this root.

A straightforward computation shows that

$$\frac{\partial F(y,\mu_N)}{\partial \mu} = \frac{1-\beta}{\beta}\,\frac{\beta^{1+\mu_1} - \beta + \mu_1(1-\beta)}{((\mu_1-\mu_N)(1-\beta) + \beta^{1+\mu_1} - \beta^{1+\mu_N})^2}\;\zeta(y,\mu_N),$$

with

$$\zeta(y,\mu) = \beta(1-\beta)(1+\mu_1)y - (1-\beta)\mu_1 + (1-\beta y)(\beta^{1+\mu} - \beta^{1+\mu_1}) + \beta^{1+\mu}\,(\beta(1+\mu)y - \mu)\ln\beta.$$

As $p_N^* < \hat{p} < p_1^*$, we have

$$\frac{\mu_N}{1+\mu_N} < \beta y < \frac{\mu_1}{1+\mu_1}\,,$$

which implies

$$\zeta(y,\mu_1) = (\beta(1+\mu_1)y - \mu_1)\,(1 - \beta + \beta^{1+\mu_1}\ln\beta) < 0,$$

and

$$\frac{\partial \zeta(y,\mu)}{\partial \mu} = \beta^{1+\mu}[\beta(1+\mu)y - \mu](\ln\beta)^2 > 0,$$

for all $\mu \in [\mu_N, \mu_1]$. This establishes $\zeta(y,\mu_N) < 0$.

By the implicit function theorem, therefore, y is increasing in μ_N. Recalling from Keller and Rady (2010) that μ_N is decreasing in N, we have thus shown that y (and hence $\hat{p}$) are decreasing in N. ■

Proof of Proposition 4: Simple algebra yields

$$\frac{j(p_N^*)}{p_1^*} = \frac{\lambda_1}{\lambda_0}\,\frac{\mu_N}{\mu_1}\,\frac{(\mu_1+1)(\lambda_1 h - s) + \mu_1(s-\lambda_0 h)}{(\mu_N+1)(\lambda_1 h - s) + (\lambda_1/\lambda_0)\mu_N(s-\lambda_0 h)}\,.$$

From the implicit definitions of μ_1 and μ_N, we obtain $\lim_{r\to 0}\mu_1 = \lim_{r\to 0}\mu_N = 0$ (so that the third fraction in the previous expression converges to 1) and

$$\lim_{r\to 0}\frac{\partial \mu_1}{\partial r} = \left[\lambda_1 - \lambda_0 + \lambda_0 \ln\frac{\lambda_0}{\lambda_1}\right]^{-1} = N \lim_{r\to 0}\frac{\partial \mu_N}{\partial r}\,,$$

implying

$$\lim_{r\to 0}\frac{\mu_N}{\mu_1} = \frac{1}{N},$$

by l'Hôpital's rule.

Furthermore, we note that we may write equivalently

$$\frac{j(p_N^*)}{p_1^*} = \frac{\lambda_1}{\lambda_0}\,\frac{(1+\frac{1}{\mu_1})(\lambda_1 h - s) + (s-\lambda_0 h)}{(1+\frac{1}{\mu_N})(\lambda_1 h - s) + (\lambda_1/\lambda_0)(s-\lambda_0 h)}\,.$$

As $\lim_{r\to\infty}\mu_1 = \lim_{r\to\infty}\mu_N = \infty$, we can immediately conclude that this ratio converges to the stated limit for $r\to\infty$. ■

B.2 Learning with a Brownian Component (Propositions 5–6)

The proof of Proposition 5 rests on a sequence of lemmas that prove incentive compatibility of the proposed strategies on various subintervals of $[0,1]$. When no assumption on the signal-to-noise ratio ρ is stated, the respective result holds irrespectively of whether $\rho > 0$ or $\rho = 0$.

In view of Lemmas A.4 and A.5, we take $\underline{p}$ and $\bar{p}$ such that

$$p_N^* < \underline{p} < p^\ddagger < p_1^* < p^m < \max\{p^\diamond, \check{p}\} < \bar{p} < 1. \tag{B.9}$$

The first two lemmas deal with the safe action ($\kappa = 0$) on the interval $[0,\bar{p}]$.

Lemma B.1 *For all $p \le p^\ddagger$,*

$$(1-\delta)s + \delta\overline{w}^\Delta(p) \ge (1-\delta)m(p) + \delta\mathcal{E}_1^\Delta\underline{w}^\Delta(p).$$

Proof: As $\overline{w}^\Delta(p) \ge s = \underline{w}^\Delta(p)$ for $p \le p^\ddagger$, we have $(1-\delta)s + \delta\overline{w}^\Delta(p) \ge s$ whereas $s \ge (1-\delta)m(p) + \delta\mathcal{E}_1^\Delta\underline{w}^\Delta(p)$ by the functional equation for $\underline{w}^\Delta$. ■

Lemma B.2 *There exists $\Delta_{(p^\ddagger,\bar{p}]} > 0$ such that*

$$(1-\delta)s + \delta\overline{w}^\Delta(p) \geq (1-\delta)m(p) + \delta\mathcal{E}_1^\Delta\underline{w}^\Delta(p),$$

for all $p \in (p^\ddagger, \bar{p}]$ and $\Delta < \Delta_{(p^\ddagger,\bar{p}]}$.

PROOF: By Lemmas A.3 and A.4, there exist $\nu > 0$ and $\Delta_0 > 0$ such that $\overline{w}^\Delta(p) - \underline{w}^\Delta(p) \geq \nu$ for all $p \in [p^\ddagger, \bar{p}]$ and $\Delta < \Delta_0$. Further, there is a $\Delta_1 \in (0, \Delta_0]$ such that $|\mathcal{E}_1^\Delta\underline{w}^\Delta(p) - \underline{w}^\Delta(p)| \leq \frac{\nu}{2}$ for all $p \in [p^\ddagger, \bar{p}]$ and $\Delta < \Delta_1$. For these p and Δ, we thus have

$$(1-\delta)s + \delta\overline{w}^\Delta(p) - \left[(1-\delta)m(p) + \delta\mathcal{E}_1^\Delta\underline{w}^\Delta(p)\right] \geq (1-\delta)[s - m(p)] + \delta\,\frac{\nu}{2}.$$

Finally, there is a $\Delta_{(p^\ddagger,\bar{p}]} \in (0, \Delta_1]$ such that the right-hand side of this inequality is positive for all $p \in (p^\ddagger, \bar{p}]$ and $\Delta < \bar{\Delta}$. ■

We establish incentive compatibility of the risky action ($\kappa = 1$) to the immediate right of $\underline{p}$ by means of the following result.

Lemma B.3 *Let X be a Gaussian random variable with mean m and variance V.*

1. *For all $\eta > 0$,*
$$\mathbb{P}[X - m > \eta] < \frac{V}{\eta^2}.$$

2. *There exists $\overline{V} \in (0,1)$ such that for all $V < \overline{V}$,*
$$\mathbb{P}\left[V^{\frac{3}{4}} \leq X - m \leq V^{\frac{1}{4}}\right] \geq \frac{1}{2} - V^{\frac{1}{4}}.$$

PROOF: The first statement is a trivial consequence of Chebysheff's inequality. The proof of the second relies on the following inequality (13.48) of Johnson et al. (1994) for the standard normal cumulative distribution function:

$$\frac{1}{2}\left[1 + (1 - e^{-x^2/2})^{\frac{1}{2}}\right] \leq \Phi(x) \leq \frac{1}{2}\left[1 + (1 - e^{-x^2})^{\frac{1}{2}}\right].$$

Letting Φ^V denote the cdf of the Gaussian distribution with variance V (and mean 0), and using the above upper and lower bounds, we have

$$\frac{\frac{1}{2} + \Phi^V(V^{\frac{3}{4}}) - \Phi^V(V^{\frac{1}{4}})}{\sqrt[4]{V}} \leq \frac{1 - \sqrt{1 - e^{-\frac{1}{2\sqrt{V}}}} + \sqrt{1 - e^{-\sqrt{V}}}}{2\sqrt[4]{V}}.$$

Writing $x = \sqrt{V}$ and using the fact that $1 - \sqrt{1-y} \leq \sqrt{y}$ for $0 \leq y \leq 1$, moreover, we have

$$\frac{1 - \sqrt{1 - e^{-\frac{1}{2x}}} + \sqrt{1 - e^{-x}}}{2\sqrt{x}} \leq \frac{1}{2}\sqrt{\frac{e^{-\frac{1}{2x}}}{x}} + \frac{1}{2}\sqrt{\frac{1 - e^{-x}}{x}} \to \frac{1}{2},$$

as $x \to 0$. Thus,

$$\frac{\frac{1}{2} + \Phi^V(V^{\frac{3}{4}}) - \Phi^V(V^{\frac{1}{4}})}{\sqrt[4]{V}} \leq 1,$$

for sufficiently small V, which is the second statement of the lemma. ■

We apply this lemma to the log odds ratio ℓ associated with the current belief p. For later use, we note that $dp/d\ell = p\,(1-p)$.

Lemma B.4 *Let $\rho > 0$. There exist $\varepsilon \in (0, p^\ddagger - \underline{p})$ and $\Delta_{(\underline{p},\underline{p}+\varepsilon]} > 0$ such that*

$$(1-\delta)m(p) + \delta\mathcal{E}_N^\Delta \overline{w}^\Delta(p) \geq (1-\delta)s + \delta\mathcal{E}_{N-1}^\Delta \underline{w}^\Delta(p),$$

for all $p \in (\underline{p}, \underline{p}+\varepsilon]$ and $\Delta < \Delta_{(\underline{p},\underline{p}+\varepsilon]}$.

PROOF: Consider a belief $p_0 = p$ and the corresponding log odds ratio ℓ. Let K players use the risky arm on the time interval $[0, \Delta)$ and consider the resulting belief $p_\Delta^{(K)}$ and the associated log odds ratio $\ell_\Delta^{(K)}$.

Let $\mathbb{P}_\theta$ denote the probability measure associated with state $\theta \in \{0,1\}$. Expected continuation payoffs are computed by means of the measure $\mathbb{P}_p = p\mathbb{P}_1 + (1-p)\mathbb{P}_0$.

Let J_0^Δ denote the event that no lump-sum arrives by time Δ. The probability of J_0^Δ under the measure $\mathbb{P}_\theta$ is $e^{-\lambda_\theta \Delta}$. Note that

$$e^{-\lambda_\theta\Delta}\mathbb{P}_\theta[A \mid J_0^\Delta] \leq \mathbb{P}_\theta[A] \leq e^{-\lambda_\theta\Delta}\mathbb{P}_\theta[A \mid J_0^\Delta] + 1 - e^{-\lambda_\theta\Delta},$$

for any event A.

As we have seen in Appendix A.1, conditional on J_0^Δ, the random variable $\ell_\Delta^{(K)}$ is normally distributed with mean $\ell - K\left(\lambda_1 - \lambda_0 - \frac{\rho}{2}\right)\Delta$ and variance $K\rho\Delta$ under $\mathbb{P}_1$, and normally distributed with mean $\ell - K\left(\lambda_1 - \lambda_0 + \frac{\rho}{2}\right)\Delta$ and variance $K\rho\Delta$ under $\mathbb{P}_0$.

Now choose $\varepsilon > 0$ such that $\underline{p} + \varepsilon < p^\ddagger$. Write $\underline{\ell}$, ℓ_ε, $\ell^\ddagger$ and $\bar{\ell}$ for the log odds ratios associated with $\underline{p}$, $\underline{p}+\varepsilon$, $p^\ddagger$ and $\bar{p}$, respectively. Choose $\Delta_0 > 0$ such that

$$\nu_0 = \min_{(\Delta,\ell)\in[0,\Delta_0]\times[\underline{\ell},\ell_\varepsilon]} \left[\ell^\ddagger - \ell + (N-1)\left(\lambda_1 - \lambda_0 - \frac{\rho}{2}\right)\Delta\right]^2 > 0.$$

For all $p \in (\underline{p}, \underline{p}+\varepsilon]$ and $\Delta \in (0, \Delta_0)$, the first part of Lemma B.3 now implies

$$\begin{aligned}
\mathbb{P}_p\Big[p_\Delta^{(N-1)} > p^\ddagger\Big] &= \mathbb{P}_p\Big[\ell_\Delta^{(N-1)} > \ell^\ddagger\Big] \\
&\le p\left\{e^{-\lambda_1\Delta}\mathbb{P}_1\Big[\ell_\Delta^{(N-1)} > \ell^\ddagger \Big| J_0^\Delta\Big] + 1 - e^{-\lambda_1\Delta}\right\} \\
&\quad + (1-p)\left\{e^{-\lambda_0\Delta}\mathbb{P}_0\Big[\ell_\Delta^{(N-1)} > \ell^\ddagger \Big| J_0^\Delta\Big] + 1 - e^{-\lambda_0\Delta}\right\} \\
&\le p\left\{\frac{e^{-\lambda_1\Delta}(N-1)\rho\Delta}{\nu_0} + 1 - e^{-\lambda_1\Delta}\right\} \\
&\quad + (1-p)\left\{\frac{e^{-\lambda_0\Delta}(N-1)\rho\Delta}{\nu_0} + 1 - e^{-\lambda_0\Delta}\right\} \\
&\le \frac{(N-1)\rho\Delta}{\nu_0} + 1 - e^{-\lambda_1\Delta} \\
&\le \left\{\frac{(N-1)\rho}{\nu_0} + \lambda_1\right\}\Delta.
\end{aligned}$$

As $\underline{w}^\Delta \le s + (m_1 - s)\mathbb{1}_{(p^\ddagger,1]}$, moreover,

$$\mathcal{E}_{N-1}^\Delta \underline{w}^\Delta(p) \le s + (m_1 - s)\,\mathbb{P}_p\Big[p_\Delta^{(N-1)} > p^\ddagger\Big].$$

So there exists $C_0 > 0$ such that $\mathcal{E}_{N-1}^\Delta \underline{w}^\Delta(p) \le s + C_0\Delta$ for all $p \in (\underline{p}, \underline{p}+\varepsilon]$ and $\Delta \in (0, \Delta_0)$.

Next, define $\nu_1 = \min_{\underline{p}\le p\le\bar{p}} p\,(1-p)$ and note that for $\underline{p} \le p \le \bar{p}$ (and thus for $\underline{\ell} \le \ell \le \bar{\ell}$),

$$V_{N,\underline{p}}(p) \ge s + \max\left\{0, V'_{N,\underline{p}}(\underline{p}+)(p-\underline{p})\right\} \ge s + \max\left\{0, V'_{N,\underline{p}}(\underline{p}+)\nu_1(\ell-\underline{\ell})\right\}.$$

By the second part of Lemma B.3, there exists $\Delta_1 > 0$ such that $N\rho\Delta_1 < 1$ and

$$\mathbb{P}_1\Big[(N\rho\Delta)^{\frac{3}{4}} \le \ell_\Delta^{(N)} - \ell + N\left(\lambda_1 - \lambda_0 - \frac{\rho}{2}\right)\Delta \le (N\rho\Delta)^{\frac{1}{4}} \Big| J_0^\Delta\Big] \ge \frac{1}{2} - (N\rho\Delta)^{\frac{1}{4}},$$

for arbitrary ℓ and all $\Delta \in (0, \Delta_1)$. In particular,

$$\begin{aligned}
&\mathbb{P}_p\Big[(N\rho\Delta)^{\frac{3}{4}} \le \ell_\Delta^{(N)} - \ell + N\left(\lambda_1 - \lambda_0 - \frac{\rho}{2}\right)\Delta \le (N\rho\Delta)^{\frac{1}{4}}\Big] \\
&\quad \ge p\mathbb{P}_1\Big[(N\rho\Delta)^{\frac{3}{4}} \le \ell_\Delta^{(N)} - \ell + N\left(\lambda_1 - \lambda_0 - \frac{\rho}{2}\right)\Delta \le (N\rho\Delta)^{\frac{1}{4}}\Big] \\
&\quad \ge pe^{-\lambda_1\Delta}\mathbb{P}_1\Big[(N\rho\Delta)^{\frac{3}{4}} \le \ell_\Delta^{(N)} - \ell + N\left(\lambda_1 - \lambda_0 - \frac{\rho}{2}\right)\Delta \le (N\rho\Delta)^{\frac{1}{4}} \Big| J_0^\Delta\Big] \\
&\quad \ge pe^{-\lambda_1\Delta}\left(\frac{1}{2} - (N\rho\Delta)^{\frac{1}{4}}\right),
\end{aligned}$$

for these Δ. Taking Δ_1 smaller if necessary, we can also ensure that

$$\underline{\ell} < \ell - N\left(\lambda_1 - \lambda_0 - \frac{\rho}{2}\right)\Delta + (N\rho\Delta)^{\frac{3}{4}} < \ell - N\left(\lambda_1 - \lambda_0 - \frac{\rho}{2}\right)\Delta + (N\rho\Delta)^{\frac{1}{4}} < \bar{\ell},$$

for all $\ell \in (\underline{\ell}, \ell_\varepsilon]$ and all $\Delta \in (0, \Delta_1)$.

By Lemma A.3, there exists $\Delta_2 \in (0, \Delta_1)$ such that $\overline{w}^\Delta \ge V_{N,\underline{p}}$ for $\Delta \in (0, \Delta_2)$. For such

Δ and $p \in (\underline{p}, \underline{p}+\varepsilon]$, we now have

$$\begin{aligned}\mathcal{E}_N^\Delta \overline{w}^\Delta(p) &\geq s + pe^{-\lambda_1\Delta}\left(\frac{1}{2} - (N\rho\Delta)^{\frac{1}{4}}\right) V'_{N,\underline{p}}(\underline{p}+)\,\nu_1\left[\ell - N\left(\lambda_1 - \lambda_0 - \frac{\rho}{2}\right)\Delta + (N\rho\Delta)^{\frac{3}{4}} - \underline{\ell}\right] \\ &\geq s + \underline{p}(1-\lambda_1\Delta)\left(\frac{1}{2} - (N\rho\Delta)^{\frac{1}{4}}\right) V'_{N,\underline{p}}(\underline{p}+)\,\nu_1\left[-N\left(\lambda_1 - \lambda_0 - \frac{\rho}{2}\right)\Delta + (N\rho\Delta)^{\frac{3}{4}}\right].\end{aligned}$$

This implies the existence of $\Delta_3 \in (0, \Delta_2)$ and $C_1 > 0$ such that

$$\mathcal{E}_N^\Delta \overline{w}^\Delta(p) \geq s + C_1\Delta^{\frac{3}{4}},$$

for all $p \in (\underline{p}, \underline{p}+\varepsilon]$ and $\Delta \in (0, \Delta_3)$.

For $p \in (\underline{p}, \underline{p}+\varepsilon]$ and $\Delta \in (0, \min\{\Delta_0, \Delta_3\})$, finally,

$$\begin{aligned}&(1-\delta)m(p) + \delta\mathcal{E}_N^\Delta \overline{w}^\Delta(p) - \left[(1-\delta)s + \delta\mathcal{E}_{N-1}^\Delta \underline{w}^\Delta(p)\right] \\ &\quad\geq\ (1-\delta)[m(\underline{p}) - s] + \delta\left\{C_1\Delta^{\frac{3}{4}} - C_0\Delta\right\} \\ &\quad=\ C_1\Delta^{\frac{3}{4}} - \left\{r[s - m(\underline{p})] + C_0\right\}\Delta + o(\Delta).\end{aligned}$$

As the term in $\Delta^{\frac{3}{4}}$ dominates as Δ becomes small, there exists $\Delta_{(\underline{p},\underline{p}+\varepsilon]} \in (0, \min\{\Delta_0, \Delta_3\})$ such that this expression is positive for all $p \in (\underline{p}, \underline{p}+\varepsilon]$ and $\Delta < \Delta_{(\underline{p},\underline{p}+\varepsilon]}$. ■

Lemma B.5 *For all $\varepsilon \in (0, p^\ddagger - \underline{p})$, there exists $\Delta_{(\underline{p}+\varepsilon,\bar{p}]} > 0$ such that*

$$(1-\delta)m(p) + \delta\mathcal{E}_N^\Delta \overline{w}^\Delta(p) \geq (1-\delta)s + \delta\mathcal{E}_{N-1}^\Delta \underline{w}^\Delta(p),$$

for all $p \in (\underline{p}+\varepsilon, \bar{p}]$ and $\Delta < \Delta_{(\underline{p}+\varepsilon,\bar{p}]}$.

PROOF: First, by Lemma A.3, there exists $\Delta_0 > 0$ such that $\overline{w}^\Delta \geq V_{N,\underline{p}}$ on the unit interval. Second, by Lemma A.4, there exist $\nu > 0$, $\eta > 0$ and $\Delta_1 \in (0, \Delta_0)$ such that $V_{N,\underline{p}}(p) - \underline{w}^\Delta(p) \geq \nu$ for all $p \in [\underline{p} + \frac{\varepsilon}{2}, \bar{p} + \eta]$ and $\Delta < \Delta_1$. For these p and Δ, and by convexity of $V_{N,\underline{p}}$, we then have

$$\begin{aligned}\mathcal{E}_N^\Delta \overline{w}^\Delta(p) - \mathcal{E}_{N-1}^\Delta \underline{w}^\Delta(p) &\geq \mathcal{E}_N^\Delta V_{N,\underline{p}}(p) - \mathcal{E}_{N-1}^\Delta \underline{w}^\Delta(p) \\ &\geq \mathcal{E}_{N-1}^\Delta V_{N,\underline{p}}(p) - \mathcal{E}_{N-1}^\Delta \underline{w}^\Delta(p) \\ &\geq \chi^\Delta(p)\nu + [1 - \chi^\Delta(p)](s - m_1),\end{aligned}$$

where $\chi^\Delta(p)$ denotes the probability that the belief $p_{t+\Delta}$ lies in $[\underline{p} + \frac{\varepsilon}{2}, \bar{p} + \eta]$ given that $p_t = p$ and $N-1$ players use the risky arm for a length of time Δ. Next, there exists $\Delta_2 \in (0, \Delta_1)$ such that

$$\chi^\Delta(p) \geq \frac{\frac{\nu}{2} + m_1 - s}{\nu + m_1 - s},$$

for all $p \in (\underline{p}+\varepsilon, \bar{p}]$ and $\Delta < \Delta_2$. For these p and Δ, we thus have

$$(1-\delta)m(p) + \delta\mathcal{E}_N^\Delta \overline{w}^\Delta(p) - \left[(1-\delta)s + \delta\mathcal{E}_{N-1}^\Delta \underline{w}^\Delta(p)\right] \geq (1-\delta)[m(p) - s] + \delta\,\frac{\nu}{2}.$$

Finally, there is a $\Delta_{(\underline{p}+\varepsilon,\bar{p}]} \in (0, \Delta_2)$ such that the right-hand side of this inequality is positive for all $p \in (\underline{p}+\varepsilon, \bar{p}]$ and $\Delta < \Delta_{(\underline{p}+\varepsilon,\bar{p}]}$. ■

Lemma B.6 *There exists $\Delta_{(\bar{p},1]} > 0$ such that*

$$(1-\delta)m(p) + \delta\mathcal{E}_N^\Delta \overline{w}^\Delta(p) \geq (1-\delta)s + \delta\mathcal{E}_{N-1}^\Delta \underline{w}^\Delta(p),$$

for all $p > \bar{p}$ and $\Delta < \Delta_{(\bar{p},1]}$.

Proof: By Lemmas A.3 and A.4, there exists $\Delta_{(\bar{p},1]} > 0$ such that $\overline{w}^\Delta \geq \underline{w}^\Delta$ for all $\Delta < \Delta_{(\bar{p},1]}$. For such Δ and all $p > \bar{p}$, we thus have

$$(1-\delta)m(p) + \delta\mathcal{E}_N^\Delta \overline{w}^\Delta(p) = \overline{w}^\Delta(p) \geq \underline{w}^\Delta(p) \geq (1-\delta)s + \delta\mathcal{E}_{N-1}^\Delta \underline{w}^\Delta(p),$$

with the last inequality following from the functional equation for $\underline{w}^\Delta$. ■

Proof of Proposition 5: Given $\underline{p}$ and $\bar{p}$ as in (B.9), choose $\varepsilon > 0$ and $\Delta_{(\underline{p},\underline{p}+\varepsilon]}$ as in Lemma B.4, and $\Delta_{(p^\ddagger,\bar{p}]}$, $\Delta_{(\underline{p}+\varepsilon,\bar{p}]}$ and $\Delta_{(\bar{p},1]}$ as in Lemmas B.2, B.5 and B.6. The two-state automaton is an SSE for all

$$\Delta < \min\left\{\Delta_{(p^\ddagger,\bar{p}]}, \Delta_{(\underline{p},\underline{p}+\varepsilon]}, \Delta_{(\underline{p}+\varepsilon,\bar{p}]}, \Delta_{(\bar{p},1]}\right\}.$$

So the statement of the proposition holds with $p^\flat = p^\ddagger$ and $p^\sharp = \max\{\check{p}, p^\diamond\}$. ■

Proof of Proposition 6: Let $\varepsilon > 0$ be given. First, the explicit representation for $V_{N,\underline{p}}$ in Section 5 and Lemma A.5 allow us to choose $\underline{p} \in (p_N^*, p^\flat)$ and $\bar{p} \in (p^\sharp, 1)$ such that $V_{N,\underline{p}} > V_N^* - \varepsilon$ and $\underline{w}^\Delta < V_1^* + \varepsilon$ for all $\Delta > 0$. Second, Lemmas A.2 and A.3 and Proposition 5 imply the existence of a $\Delta^\dagger > 0$ such that for all $\Delta \in (0, \Delta^\dagger)$: $W_1^\Delta > V_1^* - \varepsilon$, $\overline{w}^\Delta \geq V_{N,\underline{p}}$, and $\overline{w}^\Delta$ and $\underline{w}^\Delta$ are SSE payoff functions of the game with period length Δ. Third, $\overline{W}_{\text{PBE}}^\Delta \leq V_N^*$ for all $\Delta > 0$ because any discrete-time strategy profile is feasible for a planner who maximizes the players' average payoff in continuous time.

For $\Delta \in (0, \Delta^\dagger)$, we thus have

$$V_N^* - \varepsilon < V_{N,\underline{p}} \leq \overline{w}^\Delta \leq \overline{W}_{\text{SSE}}^\Delta \leq \overline{W}_{\text{PBE}}^\Delta \leq V_N^*,$$

and

$$V_1^* - \varepsilon < W_1^\Delta \leq \underline{W}_{\text{PBE}}^\Delta \leq \underline{W}_{\text{SSE}}^\Delta \leq \underline{w}^\Delta < V_1^* + \varepsilon,$$

so that $\|\overline{W}_{\text{PBE}}^\Delta - V_N^*\|$, $\|\overline{W}_{\text{SSE}}^\Delta - V_N^*\|$, $\|\underline{W}_{\text{PBE}}^\Delta - V_1^*\|$ and $\|\underline{W}_{\text{SSE}}^\Delta - V_1^*\|$ are all smaller than ε, which was to be shown. ■

B.3 Pure Poisson Learning (Propositions 7–10)

Proof of Proposition 7: For any given $\Delta > 0$, let $\tilde{p}^\Delta$ be the infimum of the set of beliefs at which there is some PBE that gives a payoff $w_n(p) > s$ to at least one player. Let

$\tilde{p} = \liminf_{\Delta \to 0} \tilde{p}^{\Delta}$.

For any fixed $\varepsilon > 0$ and $\Delta > 0$, consider the problem of maximizing the players' average payoff subject to no use of the risky arm at beliefs $p \leq \tilde{p} - \varepsilon$. Denote the corresponding value function by $\widetilde{W}^{\Delta,\varepsilon}$. By the definition of $\tilde{p}$, there exists a $\tilde{\Delta}_\varepsilon > 0$ such that for $\Delta \in (0, \tilde{\Delta}_\varepsilon)$, the function $\widetilde{W}^{\Delta,\varepsilon}$ provides an upper bound on the players' average payoff in any PBE, and so $\overline{W}^{\Delta}_{\text{PBE}} \leq \widetilde{W}^{\Delta,\varepsilon}$. The value function of the continuous-time version of this maximization problem is V_{N,p_ε} with $p_\varepsilon = \max\{\tilde{p} - \varepsilon, p^*_N\}$. As the discrete-time solution is also feasible in continuous time, we have $\widetilde{W}^{\Delta,\varepsilon} \leq V_{N,p_\varepsilon}$, and hence $\overline{W}^{\Delta}_{\text{PBE}} \leq V_{N,p_\varepsilon}$ for $\Delta < \tilde{\Delta}_\varepsilon$.

Consider a sequence of such Δ's converging to 0 such that the corresponding beliefs $\tilde{p}^{\Delta}$ converge to $\tilde{p}$. For each Δ in this sequence, select a belief $p^{\Delta} > \tilde{p}^{\Delta}$ with the following two properties: (i) starting from p^{Δ}, a single failed experiment takes us below $\tilde{p}^{\Delta}$; (ii) given the initial belief p^{Δ}, there exists a PBE for reaction lag Δ in which at least one player plays risky with positive probability in the first round. Select such an equilibrium for each Δ in the sequence and let L^{Δ} be the number of players in this equilibrium who, at the initial belief p^{Δ}, play risky with positive probability. Let L be an accumulation point of the sequence of L^{Δ}'s. After selecting a subsequence of Δ's, we can assume without loss of generality that player $n = 1, \ldots, L$ plays risky with probability $\pi^{\Delta}_n > 0$ at p^{Δ}, while player $n = L+1, \ldots, N$ plays safe; we can further assume that $(\pi^{\Delta}_n)_{n=1}^{L}$ converges to a limit $(\pi_n)_{n=1}^{L}$ in $[0,1]^L$.

For player $n = 1, \ldots, L$ to play optimally at p^{Δ}, it must be the case that

$$\begin{aligned}
&(1-\delta)\left[\pi^{\Delta}_n \lambda(p^{\Delta})h + (1-\pi^{\Delta}_n)s\right] + \delta\left\{\Pr^{\Delta}(\emptyset) w^{\Delta}_{n,\emptyset} + \sum_{K=1}^{L} \sum_{|I|=K} \Pr^{\Delta}(I) \sum_{J=0}^{\infty} \Lambda^{\Delta}_{J,K}(p^{\Delta}) w^{\Delta}_{n,I,J}\right\} \\
&\geq \quad (1-\delta)s + \delta\left\{\Pr^{\Delta}_{-n}(\emptyset) w^{\Delta}_{n,\emptyset} + \sum_{K=1}^{L-1} \sum_{|I|=K,\, n\notin I} \Pr^{\Delta}_{-n}(I) \sum_{J=0}^{\infty} \Lambda^{\Delta}_{J,K}(p^{\Delta}) w^{\Delta}_{n,I,J}\right\},
\end{aligned}$$

where we write $\Pr^{\Delta}(I)$ for the probability that the set of players experimenting is $I \subseteq \{1, \ldots, L\}$, $\Pr^{\Delta}_{-n}(I)$ for the probability that among the $L-1$ players in $\{1, \cdots, L\} \setminus \{n\}$ the set of players experimenting is I, and $w^{\Delta}_{n,I,J}$ for the conditional expectation of player n's continuation payoff given that exactly the players in I were experimenting and had J successes ($w^{\Delta}_{n,\emptyset}$ is player n's continuation payoff if no one was experimenting). As $\Pr^{\Delta}(\emptyset) = (1-\pi^{\Delta}_n)\Pr^{\Delta}_{-n}(\emptyset) \leq \Pr^{\Delta}_{-n}(\emptyset)$, the inequality continues to hold when we replace $w^{\Delta}_{n,\emptyset}$ by its lower bound s. After subtracting $(1-\delta)s$ from both sides, we then have

$$\begin{aligned}
&(1-\delta)\pi^{\Delta}_n\left[\lambda(p^{\Delta})h - s\right] + \delta\left\{\Pr^{\Delta}(\emptyset)s + \sum_{K=1}^{L} \sum_{|I|=K} \Pr^{\Delta}(I) \sum_{J=0}^{\infty} \Lambda^{\Delta}_{J,K}(p^{\Delta}) w^{\Delta}_{n,I,J}\right\} \\
&\geq \quad \delta\left\{\Pr^{\Delta}_{-n}(\emptyset)s + \sum_{K=1}^{L-1} \sum_{|I|=K,\, n\notin I} \Pr^{\Delta}_{-n}(I) \sum_{J=0}^{\infty} \Lambda^{\Delta}_{J,K}(p^{\Delta}) w^{\Delta}_{n,I,J}\right\}.
\end{aligned}$$

Summing up these inequalities over $n = 1, \ldots, L$ and writing $\bar{\pi}^\Delta = \frac{1}{L}\sum_{n=1}^{L} \pi_n^\Delta$ yields

$$\begin{aligned}
&(1-\delta)L\bar{\pi}^\Delta \left[\lambda(p^\Delta)h - s\right] + \delta \left\{ \Pr^\Delta(\emptyset) Ls + \sum_{K=1}^{L} \sum_{|I|=K} \Pr^\Delta(I) \sum_{J=0}^{\infty} \Lambda_{J,K}^\Delta(p^\Delta) \sum_{n=1}^{L} w_{n,I,J}^\Delta \right\} \\
&\quad \geq \quad \delta \left\{ \sum_{n=1}^{L} \Pr_{-n}^\Delta(\emptyset) s + \sum_{n=1}^{L} \sum_{K=1}^{L-1} \sum_{|I|=K,\, n\notin I} \Pr_{-n}^\Delta(I) \sum_{J=0}^{\infty} \Lambda_{J,K}^\Delta(p^\Delta) w_{n,I,J}^\Delta \right\}.
\end{aligned}$$

By construction, $w_{n,I,0}^\Delta = s$ whenever $I \neq \emptyset$. For $|I| = K > 0$ and $J > 0$, moreover, we have $w_{n,I,J}^\Delta \geq W_1^\Delta(B_{J,K}^\Delta(p^\Delta))$ for *all* players $n = 1, \ldots, N$, and hence

$$\begin{aligned}
\sum_{n=1}^{L} w_{n,I,J}^\Delta &\leq N\overline{W}_{\mathrm{PBE}}^\Delta(B_{J,K}^\Delta(p^\Delta)) - (N-L)W_1^\Delta(B_{J,K}^\Delta(p^\Delta)) \\
&\leq NV_{N,p_\varepsilon}(B_{J,K}^\Delta(p^\Delta)) - (N-L)W_1^\Delta(B_{J,K}^\Delta(p^\Delta)).
\end{aligned}$$

So, for the preceding inequality to hold, it is necessary that

$$\begin{aligned}
&(1-\delta)L\bar{\pi}^\Delta \left[\lambda(p^\Delta)h - s\right] + \delta \left\{ \Pr^\Delta(\emptyset) Ls + \sum_{K=1}^{L} \sum_{|I|=K} \Pr^\Delta(I) \Lambda_{0,K}^\Delta(p^\Delta) Ls \right. \\
&\qquad \left. + \sum_{K=1}^{L} \sum_{|I|=K} \Pr^\Delta(I) \sum_{J=1}^{\infty} \Lambda_{J,K}^\Delta(p^\Delta) \left[NV_{N,p_\varepsilon}(B_{J,K}^\Delta(p^\Delta)) - (N-L)W_1^\Delta(B_{J,K}^\Delta(p^\Delta)) \right] \right\} \\
&\quad \geq \quad \delta \left\{ \sum_{n=1}^{L} \Pr_{-n}^\Delta(\emptyset) s + \sum_{n=1}^{L} \sum_{K=1}^{L-1} \sum_{|I|=K,\, n\notin I} \Pr_{-n}^\Delta(I) \Lambda_{0,K}^\Delta(p^\Delta) s \right. \\
&\qquad \left. + \sum_{n=1}^{L} \sum_{K=1}^{L-1} \sum_{|I|=K,\, n\notin I} \Pr_{-n}^\Delta(I) \sum_{J=1}^{\infty} \Lambda_{J,K}^\Delta(p^\Delta) W_1^\Delta(B_{J,K}^\Delta(p^\Delta)) \right\}.
\end{aligned}$$

As

$$\Pr^\Delta(\emptyset) + \sum_{K=1}^{L} \sum_{|I|=K} \Pr^\Delta(I) = 1 \quad \text{and} \quad \sum_{K=1}^{L} \sum_{|I|=K} \Pr^\Delta(I) K = L\bar{\pi}^\Delta,$$

we have the first-order expansions

$$\begin{aligned}
&\Pr^\Delta(\emptyset) + \sum_{K=1}^{L} \sum_{|I|=K} \Pr^\Delta(I) \Lambda_{0,K}^\Delta(p^\Delta) \\
&\quad = \quad \Pr^\Delta(\emptyset) + \sum_{K=1}^{L} \sum_{|I|=K} \Pr^\Delta(I) \left(1 - K\lambda(p^\Delta)\Delta\right) + o(\Delta) \\
&\quad = \quad 1 - L\bar{\pi}^\Delta \lambda(p^\Delta)\Delta + o(\Delta),
\end{aligned}$$

and

$$\sum_{K=1}^{L}\sum_{|I|=K}\Pr^{\Delta}(I)\Lambda^{\Delta}_{1,K}(p^{\Delta})=\sum_{K=1}^{L}\sum_{|I|=K}\Pr^{\Delta}(I)K\lambda(p^{\Delta})\Delta+o(\Delta)=L\bar{\pi}^{\Delta}\lambda(p^{\Delta})\Delta+o(\Delta),$$

so, by uniform convergence $W_1^{\Delta}\to V_1^*$ (Lemma A.2), the left-hand side of the last inequality expands as

$$Ls+L\left\{r\bar{\pi}\left[\lambda(\tilde{p})h-s\right]-rs+\bar{\pi}\lambda(\tilde{p})\left[NV_{N,p_\varepsilon}(j(\tilde{p}))-(N-L)V_1^*(j(\tilde{p}))-Ls\right]\right\}\Delta+o(\Delta),$$

with $\bar{\pi}=\lim_{\Delta\to 0}\bar{\pi}^{\Delta}$. In the same way, the identities

$$\Pr^{\Delta}_{-n}(\emptyset)+\sum_{K=1}^{L-1}\sum_{|I|=K,\,n\notin I}\Pr^{\Delta}_{-n}(I)=1\quad\text{and}\quad\sum_{K=1}^{L-1}\sum_{|I|=K,\,n\notin I}\Pr^{\Delta}_{-n}(I)K=L\bar{\pi}^{\Delta}-\pi_n^{\Delta}$$

imply

$$\sum_{n=1}^{L}\Pr^{\Delta}_{-n}(\emptyset)+\sum_{n=1}^{L}\sum_{K=1}^{L-1}\sum_{|I|=K,\,n\notin I}\Pr^{\Delta}_{-n}(I)\Lambda^{\Delta}_{0,K}(p^{\Delta})=L-L(L-1)\bar{\pi}^{\Delta}\lambda(p^{\Delta})\Delta+o(\Delta),$$

and

$$\sum_{n=1}^{L}\sum_{K=1}^{L-1}\sum_{|I|=K,\,n\notin I}\Pr^{\Delta}_{-n}(I)\Lambda^{\Delta}_{1,K}(p^{\Delta})=L(L-1)\bar{\pi}^{\Delta}\lambda(p^{\Delta})\Delta+o(\Delta),$$

and so the right-hand side of the inequality expands as

$$Ls+L\left\{-rs+(L-1)\bar{\pi}\lambda(\tilde{p})\left[V_1^*(j(\tilde{p}))-s\right]\right\}\Delta+o(\Delta).$$

Comparing terms of order Δ, dividing by L and letting $\varepsilon\to 0$, we obtain

$$\bar{\pi}\left\{\lambda(\tilde{p})\left[NV_{N,\check{p}}(j(\tilde{p}))-(N-1)V_1^*(j(\tilde{p}))-s\right]-rc(\tilde{p})\right\}\geq 0.$$

By Lemma A.9, this means $\tilde{p}\geq\hat{p}$ whenever $\bar{\pi}>0$.

For the case that $\bar{\pi}=0$, we write the optimality condition for player $n\in\{1,\dots,L\}$ as

$$\begin{aligned}&(1-\delta)\lambda(p^{\Delta})h+\delta\left\{\sum_{K=0}^{L-1}\sum_{|I|=K,\,n\notin I}\Pr^{\Delta}_{-n}(I)\sum_{J=0}^{\infty}\Lambda^{\Delta}_{J,K+1}(p^{\Delta})w^{\Delta}_{n,I\dot{\cup}\{n\},J}\right\}\\ \geq\;&(1-\delta)s+\delta\left\{\Pr^{\Delta}_{-n}(\emptyset)w^{\Delta}_{n,\emptyset}+\sum_{K=1}^{L-1}\sum_{|I|=K,\,n\notin I}\Pr^{\Delta}_{-n}(I)\sum_{J=0}^{\infty}\Lambda^{\Delta}_{J,K}(p^{\Delta})w^{\Delta}_{n,I,J}\right\}.\end{aligned}$$

As above, $w^{\Delta}_{n,\emptyset}\geq s$, and $w^{\Delta}_{n,I,0}=s$ whenever $I\neq\emptyset$. For $|I|=K>0$ and $J>0$, moreover, we have $w^{\Delta}_{n,I,J}\geq W_1^{\Delta}(B^{\Delta}_{J,K}(p^{\Delta}))$, $w^{\Delta}_{n,I\dot{\cup}\{n\},J}\geq W_1^{\Delta}(B^{\Delta}_{J,K+1}(p^{\Delta}))$ and $w^{\Delta}_{n,I\dot{\cup}\{n\},J}\leq$

$NV_{N,p_\varepsilon}(B^\Delta_{J,K+1}(p^\Delta)) - (N-1)W^\Delta_1(B^\Delta_{J,K+1}(p^\Delta))$. So, for the optimality condition to hold, it is necessary that

$$\begin{aligned}
&(1-\delta)\lambda(p^\Delta)h + \delta\left\{\sum_{K=0}^{L-1}\sum_{|I|=K,\, n\notin I}\Pr^\Delta_{-n}(I)\Lambda^\Delta_{0,K+1}(p^\Delta)s\right.\\
&\left.+\sum_{K=0}^{L-1}\sum_{|I|=K,\, n\notin I}\Pr^\Delta_{-n}(I)\sum_{J=1}^{\infty}\Lambda^\Delta_{J,K+1}(p^\Delta)\left[NV_{N,p_\varepsilon}(B^\Delta_{J,K+1}(p^\Delta)) - (N-1)W^\Delta_1(B^\Delta_{J,K+1}(p^\Delta))\right]\right\}\\
&\geq\quad (1-\delta)s + \delta\left\{\Pr^\Delta_{-n}(\emptyset)s + \sum_{K=1}^{L-1}\sum_{|I|=K,\, n\notin I}\Pr^\Delta_{-n}(I)\Lambda^\Delta_{0,K}(p^\Delta)s\right.\\
&\qquad\left.+\sum_{K=1}^{L-1}\sum_{|I|=K,\, n\notin I}\Pr^\Delta_{-n}(I)\sum_{J=1}^{\infty}\Lambda^\Delta_{J,K}(p^\Delta)W^\Delta_1(B^\Delta_{J,K}(p^\Delta))\right\}.
\end{aligned}$$

Now,

$$\sum_{K=1}^{L-1}\sum_{|I|=K,\, n\notin I}\Pr^\Delta_{-n}(I)K = L\bar\pi^\Delta - \pi^\Delta_n \to 0,$$

as Δ vanishes. Therefore, the left-hand side of the above inequality expands as

$$s + \left\{r\left[\lambda(\tilde p)h - s\right] + \lambda(\tilde p)\left[NV_{N,p_\varepsilon}(j(\tilde p)) - (N-1)V^*_1(j(\tilde p)) - s\right]\right\}\Delta + o(\Delta),$$

and the right-hand side as $s + o(\Delta)$. Comparing terms of order Δ, letting $\varepsilon \to 0$ and using Lemma A.9 once more, we again obtain $\tilde p \geq \hat p$.

The statement about the range of experimentation now follows immediately from the fact that for $\Delta < \tilde\Delta_\varepsilon$, we have $\overline{W}^\Delta_{\rm PBE} \leq V_{N,p_\varepsilon}$, and hence $\overline{W}^\Delta_{\rm PBE} = V_{N,p_\varepsilon} = s$ on $[0, \tilde p - \varepsilon] \supseteq [0, \hat p - \varepsilon]$.

The statement about the supremum of equilibrium payoffs follows from the inequality $\overline{W}^\Delta_{\rm PBE} \leq V_{N,p_\varepsilon}$ for $\Delta < \tilde\Delta_\varepsilon$, convergence $V_{N,p_\varepsilon} \to V_{N,\tilde p}$ as $\varepsilon \to 0$, and the inequality $V_{N,\tilde p} \leq V_{N,\hat p}$. ■

We now turn to the proof of Proposition 8. The only difference to the case with a Brownian component is the proof of incentive compatibility to the immediate right of $\underline{p}$.

In view of Lemmas A.9, A.4 and A.5, we consider $\underline{p}$ and $\bar p$ such that

$$\hat p < \underline{p} < p^\ddagger < p^*_1 < p^m < \max\{p^\diamond, \check p\} < \bar p < 1. \tag{B.10}$$

Lemma B.7 *Let $\rho = 0$ and $\lambda_0 > 0$. There exists $p^\sharp \in (\max\{p^\diamond, \check p\}, 1)$ such that for all $\bar p \in (p^\sharp, 1)$, there exist $\varepsilon \in (0, p^\ddagger - \underline{p})$ and $\Delta_{(\underline{p}, \underline{p}+\varepsilon]} > 0$ such that*

$$(1-\delta)m(p) + \delta\mathcal{E}^\Delta_N\overline{w}^\Delta(p) \geq (1-\delta)s + \delta\mathcal{E}^\Delta_{N-1}\underline{w}^\Delta(p),$$

for all $p \in (\underline{p}, \underline{p}+\varepsilon]$ and $\Delta < \Delta_{(\underline{p}, \underline{p}+\varepsilon]}$.

PROOF: By Lemma A.3, there exists $\Delta_0 > 0$ such that $\overline{w}^\Delta \geq V_{N,\underline{p}}$ for $\Delta \in (0, \Delta_0)$.

By Lemma A.9,

$$\lambda(p)[NV_{N,p}(j(p)) - (N-1)V_1^*(j(p)) - s] - rc(p) > 0$$

on $[\underline{p}, 1]$. As $V_{N,p}(j(p)) \leq V_{N,\underline{p}}(j(p))$ for $p \geq \underline{p}$, this implies

$$\lambda(p)[NV_{N,\underline{p}}(j(p)) - (N-1)V_1^*(j(p)) - s] - rc(p) > 0$$

on $[\underline{p}, 1]$. By Lemma A.6, there exists a belief $p^\sharp > \max\{p^\diamond, \check{p}\}$ such that for all $\bar{p} > p^\sharp$,

$$\lambda(p)[NV_{N,\underline{p}}(j(p)) - (N-1)V_{1,\bar{p}}(j(p)) - s] - rc(p) > 0$$

on $[\underline{p}, 1]$. Fix a $\bar{p} \in (p^\sharp, 1)$, define

$$\nu = \min_{p \in [\underline{p}, 1]} \left\{ \lambda(p)[NV_{N,\underline{p}}(j(p)) - (N-1)V_{1,\bar{p}}(j(p)) - s] - rc(p) \right\} > 0,$$

and choose $\varepsilon > 0$ such that $\underline{p} + \varepsilon < p^\ddagger$ and

$$(N\lambda(\underline{p} + \varepsilon) + r) \left[V_{N,\underline{p}}(\underline{p} + \varepsilon) - s \right] < \nu/3.$$

In the remainder of the proof, we write p_J^K for the posterior belief starting from p when K players use the risky arm and J lump-sums arrive within the length of time Δ.

For $p \in (\underline{p}, \underline{p} + \varepsilon]$ and $\Delta \in (0, \Delta_0)$,

$$\begin{aligned}
&(1-\delta)m(p) + \delta \mathcal{E}_N^\Delta \overline{w}^\Delta(p) \\
&\quad \geq \quad (1-\delta)m(p) + \delta \mathcal{E}_N^\Delta V_{N,\underline{p}}(p) \\
&\quad = \quad r\Delta\, m(p) + (1 - r\Delta) \left\{ N\lambda(p)\Delta\, V_{N,\underline{p}}(p_1^N) + (1 - N\lambda(p)\Delta)\, V_{N,\underline{p}}(p_0^N) \right\} + O(\Delta^2) \\
&\quad = \quad V_{N,\underline{p}}(p_0^N) + \left\{ rm(p) + N\lambda(p) V_{N,\underline{p}}(p_1^N) - (N\lambda(p) + r) V_{N,\underline{p}}(p_0^N) \right\} \Delta + O(\Delta^2),
\end{aligned}$$

while

$$\begin{aligned}
&(1-\delta)s + \delta \mathcal{E}_{N-1}^\Delta \underline{w}^\Delta(p) \\
&\quad = \quad r\Delta\, s + (1 - r\Delta) \left\{ (N-1)\lambda(p)\Delta\, \underline{w}^\Delta(p_1^{N-1}) + [1 - (N-1)\lambda(p)\Delta]\, \underline{w}^\Delta(p_0^{N-1}) \right\} + O(\Delta^2) \\
&\quad = \quad \underline{w}^\Delta(p_0^{N-1}) + \left\{ rs + (N-1)\lambda(p)\underline{w}^\Delta(p_1^{N-1}) - [(N-1)\lambda(p) + r]\underline{w}^\Delta(p_0^{N-1}) \right\} \Delta + O(\Delta^2).
\end{aligned}$$

As $V_{N,\underline{p}}(p_0^N) \geq s = \underline{w}^\Delta(p_0^{N-1})$, the difference $(1-\delta)m(p) + \delta \mathcal{E}_N^\Delta \overline{w}^\Delta(p) - \left[(1-\delta)s + \delta \mathcal{E}_{N-1}^\Delta \underline{w}^\Delta(p)\right]$ is no smaller than Δ times

$$\lambda(p) \left[NV_{N,\underline{p}}(p_1^N) - (N-1)\underline{w}^\Delta(p_1^{N-1}) - s \right] - rc(p) - (N\lambda(p) + r) \left[V_{N,\underline{p}}(p_0^N) - s \right],$$

plus terms of order Δ^2 and higher.

Let $\xi = \frac{\nu}{6(N-1)\lambda_1}$. By Lemma A.8 as well as Lipschitz continuity of $V_{N,\underline{p}}$ and $V_{1,\bar{p}}$, there exists $\Delta_1 \in (0, \Delta_0)$ such that $\|\underline{w}^\Delta - V_{1,\bar{p}}\|$, $\max_{\underline{p} \le p \le p^\ddagger} |V_{N,\underline{p}}(p_1^N) - V_{N,\underline{p}}(j(p))|$ and $\max_{\underline{p} \le p \le p^\ddagger} |V_{1,\bar{p}}(p_1^{N-1}) - V_{1,\bar{p}}(j(p))|$ are all smaller than ξ when $\Delta < \Delta_1$. For such Δ and $p \in (\underline{p}, p^\ddagger]$, we thus have $V_{N,\underline{p}}(p_1^N) > V_{N,\underline{p}}(j(p)) - \xi$ and $\underline{w}^\Delta(p_1^{N-1}) < V_{1,\bar{p}}(j(p)) + 2\xi$, so that the expression displayed above is larger than $\nu - 2(N-1)\lambda(p)\xi - \nu/3 > \nu/3$. This implies existence of a $\Delta_{(\underline{p}, \underline{p}+\varepsilon]} \in (0, \Delta_1)$ as in the statement of the lemma. ■

Proof of Proposition 8: Given $\underline{p}$ as in (B.10), take $p^\sharp$ as in Lemma B.7 and fix $\bar{p} > p^\sharp$. Choose $\varepsilon > 0$ and $\Delta_{(\underline{p}, \underline{p}+\varepsilon]}$ as in Lemma B.7, and $\Delta_{(p^\ddagger, \bar{p}]}$, $\Delta_{(\underline{p}+\varepsilon, \bar{p}]}$ and $\Delta_{(\bar{p}, 1]}$ as in Lemmas B.2, B.5 and B.6. The two-state automaton is an SSE for all

$$\Delta < \min\left\{\Delta_{(p^\ddagger, \bar{p}]}, \Delta_{(\underline{p}, \underline{p}+\varepsilon]}, \Delta_{(\underline{p}+\varepsilon, \bar{p}]}, \Delta_{(\bar{p}, 1]}\right\}.$$

So the statement of the proposition holds with $p^\flat = p^\ddagger$ and the chosen $p^\sharp$. ■

For the proof of Proposition 9, we modify notation slightly, writing Λ for the probability that, conditional on $\theta = 1$, a player has at least one success on his own risky arm in any given round, and g for the corresponding expected payoff per unit of time.[25]

Consider an SSE played at a given prior p, with associated payoff W. If $K \ge 1$ players unsuccessfully choose the risky arm, the belief jumps down to a posterior denoted p_K. Note that an SSE allows the continuation play to depend on the identity of these players. Taking the expectation over all possible combinations of K players who experiment, however, we can associate with each posterior p_K, $K \ge 1$, an expected continuation payoff W_K. If $K = 0$, so that no player experiments, the belief does not evolve, but there is no reason that the continuation strategies (and so the payoff) should remain the same. We denote the corresponding payoff by W_0. In addition, we write $\pi \in [0, 1]$ for the probability with which each player experiments at p, and q_K for the probability that at least one player has a success, given p, when K of them experiment. The players' common payoff must then satisfy the following optimality equation:

$$\begin{aligned} W \;=\; \max\Bigg\{ & (1-\delta)p_0 g + \delta \sum_{K=0}^{N-1} \binom{N-1}{K} \pi^K (1-\pi)^{N-1-K} [q_{K+1} g + (1-q_{K+1}) W_{K+1})] \,, \\ & (1-\delta)s + \delta \sum_{K=1}^{N-1} \binom{N-1}{K} \pi^K (1-\pi)^{N-1-K} (q_K g + (1-q_K) W_K) + \delta (1-\pi)^{N-1} W_0) \Bigg\}. \end{aligned}$$

The first term corresponds to the payoff from playing risky, the second from playing safe.

As it turns out, it is more convenient to work with odds ratios

$$\omega = \frac{p}{1-p} \quad \text{and} \quad \omega_K = \frac{p_K}{1-p_K},$$

[25] *I.e.*, $\Lambda = 1 - e^{-\lambda_1 \Delta}$ and $g = m_1$.

which we refer to as "belief" as well. Note that

$$p_K = \frac{p\,(1-\omega)^K}{p\,(1-\omega)^K + 1 - p}$$

implies that $\omega_K = (1-\Lambda)^K\omega$. Note also that

$$1 - q_K = p\,(1-\Lambda)^K + 1 - p = (1-p)(1+\omega_K), \quad q_K = p - (1-p)\omega_K = (1-p)(\omega - \omega_K).$$

We define

$$m = \frac{s}{g-s}, \quad \upsilon = \frac{W-s}{(1-p)(g-s)}, \quad \upsilon_K = \frac{W_K - s}{(1-p_K)(g-s)}.$$

Note that $\upsilon \geq 0$ in any equilibrium, as s is a lower bound on the value. Simple computations now give

$$\begin{aligned} \upsilon \;\; = \;\; \max\Bigg\{ & \omega - (1-\delta)m + \delta \sum_{K=0}^{N-1} \binom{N-1}{K} \pi^K (1-\pi)^{N-1-K} (\upsilon_{K+1} - \omega_{K+1}) \,, \\ & \delta\omega + \delta \sum_{K=0}^{N-1} \binom{N-1}{K} \pi^K (1-\pi)^{N-1-K} (\upsilon_K - \omega_K) \Bigg\}. \end{aligned}$$

It is also useful to introduce $w = \upsilon - \omega$ and $w_K = \upsilon_K - \omega_K$. We then obtain

$$\begin{aligned} w \;\; = \;\; \max\Bigg\{ & -(1-\delta)m + \delta \sum_{K=0}^{N-1} \binom{N-1}{K} \pi^K (1-\pi)^{N-1-K} w_{K+1} \,, \\ & -(1-\delta)\omega + \delta \sum_{K=0}^{N-1} \binom{N-1}{K} \pi^K (1-\pi)^{N-1-K} w_K \Bigg\}. \end{aligned} \tag{B.11}$$

We define

$$\omega^* = \frac{m}{1 + \frac{\delta}{1-\delta}\Lambda}\,.$$

This is the odds ratio corresponding to the single-agent cutoff p_1^Δ, *i.e.*, $\omega^* = p_1^\Delta/(1-p_1^\Delta)$. Note that $p_1^\Delta > p_1^*$ for $\Delta > 0$.

As stated in Section 6.2, no PBE involves experimentation below p_1^Δ or, in terms of odds ratios, ω^*. For all beliefs $\omega < \omega^*$, therefore, any equilibrium has $w = -\omega$, or $\upsilon = 0$, for each player.

Proof of Proposition 9: Following terminology from repeated games, we say that we can *enforce* action $\pi \in \{0,1\}$ at belief ω if we can construct an SSE for the prior belief ω in which players prefer to choose π in the first round rather than deviate unilaterally.

Our first step is to derive sufficient conditions for enforcement of $\pi \in \{0,1\}$. The conditions to enforce these actions are intertwined, and must be derived simultaneously.

Enforcing $\pi = 0$ at ω. To enforce $\pi = 0$ at ω, it suffices that one round of using the safe arm followed by the best equilibrium payoff at ω exceeds the payoff from one round of using

the risky arm followed by the resulting continuation payoff at belief ω_1 (as only the deviating player will have experimented). See below for the precise condition.

Enforcing $\pi = 1$ at ω. If a player deviates to $\pi = 0$, we jump to w_{N-1} rather than w_N in case all experiments fail. Assume that at ω_{N-1} we can enforce $\pi = 0$. As explained above, this implies that at ω_{N-1}, a player's continuation payoff can be pushed down to what he would get by unilaterally deviating to experimentation, which is at most $-(1-\delta)m + \delta w_N$ where w_N is the highest possible continuation payoff at belief ω_N. To enforce $\pi = 1$ at ω, it then suffices that

$$w = -(1-\delta)m + \delta w_N \geq -(1-\delta)\omega + \delta(-(1-\delta)m + \delta w_N),$$

with the same continuation payoff w_N on the left-hand side of the inequality. The inequality simplifies to

$$\delta w_N \geq (1-\delta)m - \omega;$$

by the formula for w, this is equivalent to $w \geq -\omega$, *i.e.*, $\upsilon \geq 0$. Given that

$$\upsilon = \omega - (1-\delta)m + \delta(\upsilon_N - \omega_N) = (1 - \delta(1-\Lambda)^N)\omega - (1-\delta)m + \delta\upsilon_N,$$

to show that $\upsilon \geq 0$, it thus suffices that

$$\omega \geq \frac{m}{1 + \frac{\delta}{1-\delta}(1-(1-\Lambda)^N)} = \tilde{\omega},$$

and that $\upsilon_N \geq 0$, which is necessarily the case if υ_N is an equilibrium payoff. Note that $(1-\Lambda)^N\tilde{\omega} \leq \omega^*$, so that $\omega_N \geq \omega^*$ implies $\omega \geq \tilde{\omega}$. In summary, to enforce $\pi = 1$ at ω, it suffices that $\omega_N \geq \omega^*$ and $\pi = 0$ be enforceable at ω_{N-1}.

Enforcing $\pi = 0$ at ω (continued). Suppose we can enforce it at $\omega_1, \omega_2, \ldots, \omega_{N-1}$, and that $\omega_N \geq \omega^*$. Note that $\pi = 1$ is then enforceable at ω from our previous argument, given our hypothesis that $\pi = 0$ is enforceable at ω_{N-1}. It then suffices that

$$-(1-\delta)\omega + \delta(-(1-\delta)m + \delta w_N) \geq -(1-\delta^N)m + \delta^N w_N,$$

where again it suffices that this holds for the highest value of w_N. To understand this expression, consider a player who deviates by experimenting. Then the following period the belief is down one step, and if $\pi = 0$ is enforceable at ω_1, it means that his continuation payoff there can be chosen to be no larger than what he can secure at that point by deviating and experimenting again, etc. The right-hand side is then obtained as the payoff from N consecutive unilateral deviations to experimentation (in fact, we have picked an upper bound, as the continuation payoff after this string of deviations need not be the maximum w_N). The left-hand side is the payoff from playing safe one period before setting $\pi = 1$ and getting the maximum payoff w_N, a continuation strategy that is sequentially rational given that $\pi = 1$ is enforceable at ω by our hypothesis that $\pi = 0$ is enforceable at ω_{N-1}.

Plugging in the definition of v_N, this inequality simplifies to

$$(\delta^2 - \delta^N)v_N \geq (\delta^2 - \delta^N)(\omega_N - m) + (1-\delta)(\omega - m),$$

which is always satisfied for beliefs $\omega \leq m$, *i.e.*, below the myopic cutoff ω^m (which coincides with the normalized payoff m).

To summarize, if $\pi = 0$ can be enforced at the $N-1$ consecutive beliefs $\omega_1, \ldots, \omega_{N-1}$, with $\omega_N \geq \omega^*$ and $\omega \leq \omega^m$, then both $\pi = 0$ and $\pi = 1$ can be enforced at ω. By induction, this implies that if we can find an interval of beliefs $[\omega_N, \omega)$ with $\omega_N \geq \omega^*$ for which $\pi = 0$ can be enforced, then $\pi = 0, 1$ can be enforced at all beliefs $\omega' \in (\omega, \omega^m)$.

Our second step is to establish that such an interval of beliefs exists. This second step involves itself three steps. First, we derive some "simple" equilibrium, which is a symmetric Markov equilibrium. Second, we show that we can enforce $\pi = 1$ on sufficiently (finitely) many consecutive values of beliefs building on this equilibrium; third, we show that this can be used to enforce $\pi = 0$ as well.

It will be useful to distinguish beliefs according to whether they belong to the interval $[\omega^*, (1+\lambda_1\Delta)\omega^*), [(1+\lambda_1\Delta)\omega^*, (1+2\lambda_1\Delta)\omega^*), \ldots$ For $\tau \in I\!N$, let $I_{\tau+1} = [(1+\tau\lambda_1\Delta)\omega^*, (1+(\tau+1)\lambda_1\Delta)\omega^*)$. For fixed Δ, every $\omega \geq \omega^*$ can be uniquely mapped into a pair $(x, \tau) \in [0,1) \times I\!N$ such that $\omega = (1+\lambda_1(x+\tau)\Delta)\omega^*$, and we alternatively denote beliefs by such a pair. Note also that, for small enough $\Delta > 0$, one unsuccessful experiment takes a belief that belongs to the interval $I_{\tau+1}$ to (within $O(\Delta^2)$ of) the interval I_τ. (Recall that $\Lambda = \lambda_1\Delta + O(\Delta^2)$.)

Let us start with deriving a symmetric Markov equilibrium. Hence, because it is Markovian, $v_0 = v$ in our notation, that is, the continuation payoff when nobody experiments is equal to the payoff itself.

Rewriting the equations, using the risky arm gives the payoff[26]

$$v = \omega - (1-\delta)m - \delta(1-\Lambda)(1-\pi\Lambda)^{N-1}\omega + \delta\sum_{K=0}^{N-1}\binom{N-1}{K}\pi^K(1-\pi)^{N-1-K}v_{K+1},$$

while using the safe arm yields

$$v = \delta(1-(1-\pi\Lambda)^{N-1})\omega + \delta(1-\pi)^{N-1}v + \delta\sum_{K=1}^{N-1}\binom{N-1}{K}\pi^K(1-\pi)^{N-1-K}v_K.$$

In the Markov equilibrium we derive, players are indifferent between both actions, and so their payoffs are the same. Given any belief ω or corresponding pair (τ, x), we conjecture an equilibrium in which $\pi = a(\tau,x)\Delta^2 + O(\Delta^3)$, $v = b(\tau,x)\Delta^2 + O(\Delta^3)$, for some functions a, b of the pair (τ, x) only. Using the fact that $\Lambda = \lambda_1\Delta + O(\Delta^2), 1-\delta = r\Delta + O(\Delta^2)$, we replace

[26] To pull out the terms involving the belief ω from the sum appearing in the definition of v, use the fact that $\sum_{K=0}^{N-1}\binom{N-1}{K}\pi^K(1-\pi)^{N-1-K}(1-\Lambda)^K = (1-\pi\Lambda)^N/(1-\pi\Lambda)$.

this in the two payoff expressions, and take Taylor expansions to get, respectively,

$$0 = \left(rb(\tau, x) + \frac{\lambda_1 m}{\lambda_1 + r}(N-1)a(\tau, x) \right) \Delta^3 + O(\Delta^4),$$

and

$$0 = \left[b(\tau, x) - rm\lambda_1(\tau + x) \right] \Delta^2 + O(\Delta^3).$$

We then solve for $a(\tau, x)$, $b(\tau, x)$, to get

$$\pi_- = \frac{r(\lambda_1 + r)(x + \tau)}{N-1} \Delta^2 + O(\Delta^3),$$

with corresponding value

$$\upsilon_- = \lambda_1 mr(x + \tau)\Delta^2 + O(\Delta^3).$$

This being an induction on K, it must be verified that the expansion indeed holds at the lowest interval, I_1, and this verification is immediate.[27]

We now turn to the second step and argue that we can find $N-1$ consecutive beliefs at which $\pi = 1$ can be enforced. We then verify that incentives can be provided to do so, assuming that υ_- are the continuation values used by the players whether a player deviates or not from $\pi = 1$. Assume that $N-1$ players choose $\pi = 1$. Consider the remaining one. His incentive constraint to choose $\pi = 1$ is

$$-(1-\delta)m + \delta\upsilon_N - \delta(1-\Lambda)^N\omega \geq -(1-\delta)\omega - \delta(1-\Lambda)^{N-1}\omega + \delta\upsilon_{N-1}, \tag{B.12}$$

where $\upsilon_N, \upsilon_{N-1}$ are given by υ_- at ω_N, ω_{N-1}. The interpretation of both sides is as before, the payoff from abiding with the candidate equilibrium action vs. the payoff from deviating. Fixing ω and the corresponding pair (τ, x), and assuming that $\tau \geq N-1$,[28] we insert our formula for υ_-, as well as $\Lambda = \lambda_1\Delta + O(\Delta), 1-\delta = r\Delta + O(\Delta)$. This gives

$$\tau \geq (N-1)\left(2 + \frac{\lambda_1}{\lambda_1 + r}\right) - x.$$

Hence, given any integer $N' \in I\!N$, $N' > 3(N-1)$, there exists $\bar{\Delta} > 0$ such that for every $\Delta \in (0, \bar{\Delta})$, $\pi = 1$ is an equilibrium action at all beliefs $\omega = \omega^*(1 + \tau\Delta)$, for $\tau = 3(N-1), \ldots, N'$ (we pick the factor 3 because $\lambda_1/(\lambda_1 + r) < 1$).

Fix $N-1$ consecutive beliefs such that they all belong to intervals I_τ with $\tau \geq 3(N-1)$ (say, $\tau \leq 4N$), and fix Δ for which the previous result holds, *i.e.*, $\pi = 1$ can be enforced at all these beliefs. We now turn to the third step, showing how $\pi = 0$ can be enforced as well

[27] Note that this solution is actually continuous at the interval endpoints. It is not the only solution to these equations; as mentioned in the text, there are intervals of beliefs for which multiple symmetric Markov equilibria exist in discrete time. It is easy to construct such equilibria in which $\pi = 1$ and the initial belief is in (a subinterval of) I_1.

[28] Considering $\tau < N-1$ would lead to $\upsilon_N = 0$, so that the explicit formula for υ_- would not apply at ω_N. Computations are then easier, and the result would hold as well.

for these beliefs.

Suppose that players choose $\pi = 0$. As a continuation payoff, we can use the payoff from playing $\pi = 1$ in the following round, as we have seen that this action can be enforced at such a belief. This gives

$$\delta\omega + \delta(-(1-\delta)m - \delta(1-\Lambda)^N l + \delta v_-(\omega_N)).$$

(Note that the discounted continuation payoff is the left-hand side of (B.12).) By deviating from $\pi = 0$, a player gets at most

$$\omega + (-(1-\delta)m - \delta(1-\Lambda)\omega + \delta v_-(\omega_1)).$$

Again inserting our formula for v_-, this reduces to

$$\frac{mr(N-1)\lambda_1}{\lambda_1 + r}\Delta \geq 0.$$

Hence we can also enforce $\pi = 0$ at all these beliefs. We can thus apply our induction argument: there exists $\bar{\Delta} > 0$ such that, for all $\Delta \in (0, \bar{\Delta})$, both $\pi = 0, 1$ can be enforced at all beliefs $\omega \in (\omega^*(1 + 4N\Delta), \omega^m)$.

Note that we have not established that, for such a belief ω, $\pi = 1$ is enforced with a continuation in which $\pi = 1$ is being played in the next round (at belief $\omega_N > \omega^*(1+4N\Delta)$). However, if $\pi = 1$ can be enforced at belief ω, it can be enforced when the continuation payoff at ω_N is highest possible; in turn, this means that, as $\pi = 1$ can be enforced at ω_N, this continuation payoff is at least as large as the payoff from playing $\pi = 1$ at ω_N as well. By induction, this implies that the highest equilibrium payoff at ω is at least as large as the one obtained by playing $\pi = 1$ at all intermediate beliefs in $(\omega^*(1 + 4N\Delta), \omega)$ (followed by, say, the worst equilibrium payoff once beliefs below this range are reached).

Similarly, we have not argued that, at belief ω, $\pi = 0$ is enforced by a continuation equilibrium in which, if a player deviates and experiments unilaterally, his continuation payoff at ω_1 is what he gets if he keeps on experimenting alone. However, because $\pi = 0$ can be enforced at ω_1, the lowest equilibrium payoff that can be used after a unilateral deviation at ω must be at least as low as what the player can get at ω_1 from deviating unilaterally to risky again. By induction, this implies that the lowest equilibrium payoff at belief ω is at least as low as the one obtained if a player experiments alone for all beliefs in the range $(\omega^*(1 + 4N\Delta), \omega)$ (followed by, say, the highest equilibrium payoff once beliefs below this interval are reached).

Note that, as $\Delta \to 0$, these bounds converge (uniformly in Δ) to the cooperative solution (restricted to no experimentation at and below $\omega = \omega^*$) and the single-agent payoff, respectively, which was to be shown. (This is immediate given that these values correspond to precisely the cooperative payoff (with N or 1 player) for a cutoff that is within a distance of order Δ of the cutoff ω^*, with a continuation payoff at that cutoff which is itself within Δ

times a constant of the safe payoff.)

This also immediately implies (as for the case $\lambda_0 > 0$) that for fixed $\omega > \omega^m$, both $\pi = 0, 1$ can be enforced at all beliefs in $[\omega^m, \omega]$ for all $\Delta < \bar{\Delta}$, for some $\bar{\Delta} > 0$: the gain from a deviation is of order Δ, yet the difference in continuation payoffs (selecting as a continuation payoff a value close to the maximum if no player unilaterally defects, and close to the minimum if one does) is bounded away from 0, even as $\Delta \to 0$.[29] Hence, all conclusions extend: fix $\omega \in (\omega^*, \infty)$; for every $\varepsilon > 0$, there exists $\bar{\Delta} > 0$ such that for all $\Delta < \bar{\Delta}$, the best SSE payoff starting at belief ω is at least as much as the payoff from all players choosing $\pi = 1$ at all beliefs in $(\omega^* + \varepsilon, \omega)$ (using s as a lower bound on the continuation once the belief $\omega^* + \varepsilon$ is reached); and the worst SSE payoff starting at belief ω is no more than the payoff from a player whose opponents choose $\pi = 1$ if, and only if, $\omega \in (\omega^*, \omega^* + \varepsilon)$, and 0 otherwise.

The first part of the proposition follows immediately, picking arbitrary $\underline{p} \in (p_1^*, p^m)$ and $\bar{p} \in (p^m, 1)$. The second part follows from the fact that (i) $p_1^* < p_1^\Delta$, as noted, and (ii) for any $p \in [p_1^\Delta, \underline{p}]$, player i's payoff in any equilibrium is weakly lower than his best-reply payoff against $\kappa(p) = 1$ for all $p \in [p_1^*, \underline{p}]$, as easily follows from (B.11), the optimality equation for w.[30] ■

Proof of Proposition 10: For $\lambda_0 > 0$, the proof is the same as that of Proposition 6, except for the fact that it deals with $V_{N,\underline{p}}$ rather than V_N^* and relies on Proposition 8 rather than Proposition 5.

For $\lambda_0 = 0$, the proof of Proposition 9 establishes that there exists a natural number M such that, given $\underline{p}$ as stated, we can take $\bar{\Delta}$ to be $(\underline{p} - p_1^*)/M$. Equivalently, $p_1^* + M\bar{\Delta} = \underline{p}$. Hence, Proposition 9 can be restated as saying that, for some $\bar{\Delta} > 0$, and all $\Delta \in (0, \bar{\Delta})$, there exists $p_\Delta \in (p_1^*, p_1^* + M\Delta)$ such that the two conclusions of the proposition hold with $\underline{p} = p_\Delta$. Fixing the prior, let $\overline{w}^\Delta, \underline{w}^\Delta$ denote the payoffs in the first and second SSE from the proposition, respectively.[31] Given that $\underline{p} \to p_1^*$ and $\overline{w}^\Delta(p) \to s, \underline{w}^\Delta(p) \to s$ for all $p \in (p_1^*, p_\Delta)$ as $\Delta \to 0$, it follows that we can pick $\Delta^\dagger \in (0, \bar{\Delta})$ such that for all $\Delta \in (0, \Delta^\dagger)$, $\overline{W}^\Delta_{\text{PBE}} \leq V_{N,\hat{p}} + \varepsilon$, $\overline{w}^\Delta \geq V_{N,\underline{p}} - \varepsilon$, $\|W_1^\Delta - V_1^*\| < \varepsilon$ and $\|\underline{w}^\Delta - V_{1,\bar{p}}\| < \frac{\varepsilon}{2}$. The obvious inequalities follow as in the proof of Proposition 6 with the subtraction of an additional ε from the left-hand side of the first one; and the conclusion follows as before, using 2ε as an upper bound. ■

[29] This follows by contradiction. Suppose that for some $\Delta \in (0, \bar{\Delta})$, there is $\hat{\omega} \in [\omega^m, \omega]$ for which either $\pi = 0$ or 1 cannot be enforced. Consider the infimum over such beliefs. Continuation payoffs can then be picked as desired, which is a contradiction as it shows that at this presumed infimum belief $\pi = 0, 1$ can in fact be enforced.

[30] Consider the possibly random sequence of beliefs visited in an equilibrium. At each belief, a flow loss of either $-(1-\delta)m$ or $-(1-\delta)\omega$ is incurred. Note that the first loss is independent of the number of other players' experimenting, while the second is necessarily lower when at each round all other players experiment.

[31] Hence, to be precise, these payoffs are only defined on those beliefs that can be reached given the prior and the equilibrium strategies.

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