Partnership with Persistence

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Abstract

We study a continuous-time model of partnership, with persistence and imperfect state monitoring. Partners exert private efforts to shape the stock of fundamentals, which drives the profits of the partnership. The near-optimal strongly symmetric equilibria are non-Markovian and are characterized by a novel differential equation that describes maximal equilibrium incentives for any level of *relational capital*: the value of partnership net of the fundamentals. Imperfect monitoring of the fundamentals helps sustain incentives, due to *precautionary motive,* and discontinuously increases the partnership's value (*Fine Sand in the Wheels).* Good profit outcomes *rally* the partners to further increase effort when relational capital is low, but lead them to *coast* and decrease effort when relational capital is high. In our equilibria, even partnerships with high fundamentals may unravel as a consequence of a short spell of terrible signals (*Beatles' Break-up).*

1 Introduction

Teams and partnerships are among the main forms of organizing a joint economic activity. Characterized by a fixed rule for sharing the benefits, partnerships are common among both individuals and businesses in the form of joint ventures. They constitute one of the dominant forms of structuring a firm, along with corporations and limited liability

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companies. Yet each partnership is built on an incentive problem: a partner exerts private effort to contribute to a common good. The success of a joint venture requires everyone to pull his weight, but each partner is tempted to free-ride and blame lack of luck for poor results. The key hurdle for the success of the venture is to properly motivate the partners.

The incentive problem is particularly complicated in the case of ongoing, dynamic ventures. Consider the example of a tech start-up. On a daily basis, each partner devotes his time and effort to improving the "fundamentals" of the venture: upgrade the quality of the product; broaden the customer base; facilitate access to external capital; improve the internal organization; and more. Each of these fundamentals depends on the partners' entire past stream of efforts and only gradually changes over time. Moreover, none of the fundamentals needs to be directly observed by the partners, who see only how they are reflected in profits, customer reviews, or internal audits. In such an environment, with persistence and imperfect state monitoring, the scope for free-riding widens: a partner can shirk today, observe the profit or customer review outcomes, and try to catch up if those are flagging. At the same time, the range of potential motivating mechanisms broadens.

In this paper, we present a dynamic model of partnership whose two central features are persistent effect of effort and imperfect state monitoring. We first develop a new method that allows us to characterize near-optimal strongly symmetric equilibria of the game. They are characterized by a one-dimensional differential equation that describes the maximal incentives achievable in an equilibrium, for any level of relational capital that captures the "soft" capital—goodwill or mutual trust—in the partnership. This single endogenous state variable evolves differently than the persistent "hard" fundamentals of the venture, and so the equilibrium is tractable yet non-Markovian. This helps generate novel predictions about the dynamics of effort, fundamentals and profits and identifies new channels for motivating partners.

In our continuous-time model, at any point in time, partners privately choose costly effort and evenly split the profits of their venture. The only payoff-relevant state variable captures the fundamentals, which determine the expected profit flows and, in turn, change in response to the total effort. Neither efforts nor fundamentals are observable, and profits, which follow Brownian diffusion, are the partners' only publicly available information.

Our minimal monitoring structure does not allow the signals to separately identify each partner's effort (Fudenberg et al. [1994]). Consequently, we focus on the strongly symmetric equilibria (SSE), without asymmetric punishments of presumed deviators. Those modeling assumptions are restrictive. Investment in the monitoring technology and separately policing each of the partners would be an alternative way to address the moral hazard problem, in the spirit of Alchian and Demsetz [1972]. In contrast, we are interested in the incentives that can be sustained by the information that is readily available in any venture.

The first step in our solution is the choice of a novel state space and an objective function. In our model, the state space can be reduced to a single variable—the continuation value of the partners net of the value of the fundamentals, which we call *relational capital*. Generalizing the notion of continuation value from the repeated games literature, it is the sufficient statistic for "good" and "bad" histories, after which partners coordinate on their efforts. The characterization is based on a novel differential equation, whose solution parametrizes the maximal level of marginal benefits of effort—or, simply, *incentives* deliverable in a SSE—for any level of relational capital.

Maximizing incentives might seem like a strange choice. After all, the goal is to find not the highest incentives, but the highest utilities attainable in SSE. Moreover, since the efficient level of effort is interior, it is possible to overincentivize the partners. Our approach is based on the idea of tracing out the upper boundary of the set of incentive-relational capital pairs achievable in SSE, as a way of getting to the rightmost point of maximal relational capital. In the proof, we establish that for interior levels of relational capital, this boundary is self-generating and smooth[1] and, thus, satisfies the differential equation.

[1] More precisely, subsets of the boundary that are parametrized by open intervals of relational capital are self-generating.

We would like to stress three technical difficulties associated with our solution. First, just as in the repeated game analysis, and unlike in the Hamilton-Jacobi-Belman (HJB) characterization of the optimal Markov equilibria, our state variable and its law of motion are not exogenously given, but endogenous and optimized over. Second, our equation is not an HJB equation associated with a dynamic stochastic control problem, since the change in relational capital depends on efforts, which depend on the value function (incentives). Nevertheless, our verification results establish that an HJB-like characterization is also valid in a setting in which the value function doubles as a state variable. Finally, the differential equation is a solution to a relaxed problem, only under local incentive constraints. In a separate result, we provide conditions on the primitives—roughly, the cost of effort being convex enough—so that the constructed strategies are fully incentive-compatible.

The characterization gives us a convenient tool for analyzing the value of a partnership, the dynamics of effort and fundamentals, and the underlying incentive mechanisms. Our first result here is that when either the partners are patient and the fundamentals persistent, or the profits are not too noisy, there exist equilibria that are strictly better than the repeated static Nash. This might seem surprising in light of the results in Sannikov and Skrzypacz [2007], which establish that when the noise in profits disappears completely, and, thus, the stock of fundamentals is observable, it is impossible to incentivize partners. Moreover, we show that as long as there is positive noise in the monitoring of the state, less noise results in better equilibria *("Fine Sand in the Wheels")*. Thus, the triviality result of Sannikov and Skrzypacz [2007] holds at the parameter value, at which the value function is not continuous.

The dynamics of equilibrium efforts depend on the level of relational capital. When relational capital is low, partnership is close to unraveling and reverting to the inefficient repeated static Nash equilibrium. In this case, high profit realizations, which always boost mutual trust and increase relational capital, *rally* the partnership away from the brink and encourage higher effort. In contrast, when partnership is close to the bliss point and relational capital is high, partners *coast*: high profit realizations decrease their effort.

Finally, for the intermediate values of the relational capital, the partners may *over-work* and exert effort higher than the first-best.

The two different ways in which effort responds to the change in relational capital highlight two mechanisms for providing partners' incentives. When relational capital is low and effort increasing in it, one partner's high effort brings about high profit realizations in the near future, and thus motivates higher, more efficient effort from my partner. This is related to the *encouragement effect* in the literature on experimentation in teams,[2] whereby good signals make partners believe that the exogenous success of the project is more likely. Here, good signals make partners believe that the endogenous failure (unraveling of the partnership) is less likely.

When the relational capital is sufficiently high, its sensitivity to profit outcomes must vanish, and encouragement dies out. In the models without persistence, or with perfect state monitoring, this results in the triviality of the equilibria. In our case, incentives are provided by a *precautionary motive:* partners work because doing so will help push profits and relational capital up in the bleaker future, once the partnership drifts down below the bliss point. Agents substitute their effort at the bliss point—when they work little and the marginal cost of effort is low—for their effort later in the cut-throat phase—when the marginal cost of effort is high. The same precautionary motive justifies equilibrium over-working.

Lastly, the equilibrium dynamics are non-Markovian. Stage game payoffs and profitability are driven by the public beliefs about fundamentals, whereas efforts are driven by the endogenous relational capital. The main empirical implication is that the dynamics of profitability and effort are not colinear. Moreover, when profits are noisy, beliefs about the fundamentals are sluggish and change slowly as the effects of effort accrue. Relational capital, and so the total partnership value, responds to the profit outcomes and is more volatile. One consequence is that a spell of sharp negative shocks will drain mutual trust and unravel the partnership, with hardly any effect on its fundamentals. Even very profitable partnerships collapse *("Beatles' Break-up").*

[2] For instance, see Bolton and Harris [1999].

1.1 Related Literature

This paper belongs to the literature on free-riding in groups, in dynamic environments.[3] The repeated partnership game was first studied in Radner [1985] and Radner et al. [1986], who demonstrate inefficiency of equilibria, and Fudenberg et al. [1994], who pin down the identifiability conditions violated in the model. Symmetric equilibria in this setting feature a "bang-bang" property (Abreu et al. [1986]; Abreu et al. [1990]), with effort changing only once on the equilibrium path. Lack of identifiability also hampers incentive provision in our model. However, it features true, gradual equilibrium dynamics, due to persistence and imperfect state monitoring, as the signals about effort accrue slowly over time.

Abreu et al. [1991] and Sannikov and Skrzypacz [2007] show how increasing the frequency of interactions may have detrimental effect on incentives. In particular, the Brownian model[4] of partnership or collusion in Sannikov and Skrzypacz [2007], which is closely related to ours but has no persistence, or, alternatively, perfectly monitored state, features only trivial repeated static Nash equilibrium.[5] Faingold and Sannikov [2011] and Bohren [2016] establish related results in models with one long-lived player in a competitive market setting. We show that when actions have a persistent effect and there is any amount of noise, nontrivial equilibria exist, with incentives restored via a novel precautionary motive. Rahman [2014] shows how incentives may be restored in the presence of a mediator, using secret monitoring and infrequent coordination.

Our paper ties into the literature on experimentation in teams, either in the exponential bandit (Keller et al. [2005], Keller and Rady [2010], Klein and Rady [2011], and Bonatti and Hörner [2011]) or Brownian model (Bolton and Harris [1999], Georgiadis [2014], and Cetemen et al. [2017]).[6] In those models, productivity of effort depends on

[3]See Olson [1971], Alchian and Demsetz [1972], Holmstrom [1982]. See also Legros and Matthews [1993] and Winter [2004] for the seminal contributions in static settings.

[4]More precisely, Sannikov and Skrzypacz [2007] consider models with short period lengths, approximating the Brownian model.

[5]With a perfectly monitored state, the sufficient signal for efforts is the instantaneous *change* in the state, much like the instantaneous public signal in a model without persistence; see Section 6 in Sannikov and Skrzypacz [2007].

[6]See, also, Décamps and Mariotti [2004], Rosenberg et al. [2007], Murto and Välimäki [2011], and

the exogenous observable state or on public beliefs about the state, such as profitability of a risky project, and agents are investing in a production technology tomorrow. The literature focuses on the effect of payoff or information externalities on incentives in Markov equilibria. We want to investigate the optimal dynamics of relational incentives instead, and so do not restrict attention to Markov equilibria. Our equilibrium characterization is equally tractable, with incentives driven by the endogenous relational capital of the partnership. Working to rally the partnership is related to the encouragement effect identified by Bolton and Harris [1999], and coasting is reminiscent of the "work-shirk" dynamics in the reputation model of Board and Meyer-ter Vehn [2013].

Beyond partnerships, persistence plays an important role in agency problems, most importantly in dynamic moral hazard models with learning (see Holmstrom [1982] and Cisternas [2017] for models without, and Williams [2011], Prat and Jovanovic [2014], DeMarzo and Sannikov [2016], and He et al. [2017], for models with commitment, in a Brownian setting similar to ours). In particular, Jarque [2010], Sannikov [2014], and Prat [2015] analyze payments in the optimal commitment contracts, when effort has a persistent effect. Although the questions and the incentive mechanisms are different from ours, the literature has long recognized the difficulty of accounting for the marginal benefits of deviations, or marginal incentives, as well as verifying that no global deviations exist. Our method of characterizing near-optimal equilibrium is new and is based on maximizing incentives, rather than including them as an additional dynamic constraint. Moreover, we provide conditions on the primitives of the model (in our case, the convexity of costs), so that the solution of the relaxed problem is fully incentive-compatible (see Edmans et al. [2012] and Cisternas [2017] for related results).[7]

Finally, there is a large literature on strategic management, documenting the recent growth in partnering and external collaboration between coorporations, as well as the particularities of managing joint ventures given the risks of shirking associated with those enterprises. For instance, see Powell et al. [1996], Luo [2002], and Reuer and Arino [2007].

Hopenhayn and Squintani [2011] for related stopping games with incomplete information.

[7]Williams [2011], Sannikov [2014], and Prat [2015] provide analytical conditions not on the primitives but on the solution of the relaxed problem, under which the first order approach is valid.

In particular, Madhok [2006] argues that overemphasis on the outcome of joint ventures has led to neglect of the importance of trust for the quality of the relationships. In a similar vein, a large literature in social psychology focuses on free-riding in teams. For instance, see Gersick [1988], McGrath [1991], Smith [2001], or Levi [2015] for a survey of team theory and the dynamics of teamwork. McGrath's Time, Interaction, and Performance (TIP) theory emphasizes the fact that different teams might follow different paths to reach the same point. This resonates with our results, especially with the non-Markovian nature of our equilibrium, as displayed in Figure 4. In contrast, Hackman [1987] explores different criteria to evaluate team success analyzing both the group and the individual perspectives, with emphasis on (i) completing the task, (ii) maintaining social relations, and (iii) benefiting the individual. While the first is related to the partnership's fundamentals in our model, the other two connect to our concept of relational capital.

Summarizing, we contribute to the literature in the following way. Persistence and imperfect monitoring are important for applications, yet known to lead to intractable solutions. First, on the theoretical side, we provide a model of partnership that includes those two features, and we show that i) it can sustain nontrivial relational incentives; ii) it has a tractable solution, characterized by a one-dimensional differential equation; and iii) it features true equilibrium dynamics. The solution method is new, and goes beyond the application of the stochastic optimal control. It allows for verification of global incentive compatibility under conditions directly on model's primitives. Second, we uncover new relational incentive mechanisms (precautionary motive and encouragement effects when the partnership is struggling), and we provide empirical predictions on the relational, non-Markovian dynamics of partnerships (rallying, over-working, coasting, and Beatles' Break-up).

2 Model

Two partners play a stochastic game with imperfect monitoring in continuous time. At every moment in time, $t \in [0, \infty)$, each partner privately and independently chooses nonnegative effort, $a_t^i \in \mathbb{R}_+$, $i = 1, 2$. Formally, each[8] $\{a_t^i\}$ is a process measurable with respect to a filtration $\{\mathcal{F}_t\}$ of public information, which includes the sigma-algebra generated by the process of cumulative profits $\{Y_t\}$ and allows for public randomization. Time t total effort, $a_t^1 + a_t^2$, contributes to the *fundamentals* μ_t of the partnership. The stock of fundamentals depreciates at a constant rate $\alpha > 0$, and is unobservable to the partners. At any point in time, it determines the publicly observable flow of partnership profits dY_t,

$$\begin{aligned} d\mu_t &= (r+\alpha)\left(a_t^1 + a_t^2\right)dt - \alpha\mu_t dt + \sigma_\mu dB_t^\mu, \\ dY_t &= \mu_t dt + \sigma_Y dB_t^Y, \end{aligned} \tag{1}$$

where $\{B_t^\mu\}_{t\geq 0}$ and $\left\{B_t^Y\right\}_{t\geq 0}$ are two independent Brownian Motions.

Exerting effort a entails a private flow cost $c(a)$. We assume a twice differentiable, increasing, and strictly convex cost of effort function $c(\cdot)$ with $c(0) = 0$. At each point in time, the partners split the profits evenly. Both are risk-neutral, and discount the future at a constant common rate $r > 0$. Thus, for fixed effort choices of both partners, $i's$ expected discounted continuation payoffs are given by

$$W_\tau^i = \mathbb{E}^{\{a_t^1, a_t^2\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\mu_t}{2} - c(a_t^i)\right)dt \,\middle|\, \mathcal{F}_t\right].$$

The dynamic game has one state variable, the fundamentals μ_t, which equals

$$\mu_t = e^{-\alpha t}\mu_0 + \int_0^t e^{-\alpha(t-s)}\left[\left(a_t^1 + a_t^2\right)dt + \sigma_\mu dB_s^\mu\right].$$

It is driven by the efforts of the partners and changes stochastically, subject to the *produc-*

[8]Unless otherwise specified, the processes are indexed by time $t \in [0, \infty)$.

tion noise $\sigma_\mu dB_t^\mu$. The only publicly observable signal informative of the fundamentals is the profit flow dY_t, which is subject to the *observation noise* $\sigma_Y dB_t^Y$. Regardless of the noise, persistence of fundamentals implies that actions have a persistent effect: total effort $a_s^1 + a_s^2$ at time s adds $(r+\alpha)e^{-\alpha(t-s)}\left(a_t^1 + a_t^2\right)$ to the fundamentals, and so profit flow dY_t at time $t > s$.[9]

The game has a novel structure of a *stochastic game with an imperfectly monitored state.* Generalizing the game to allow for persistence and imperfect state monitoring adds realism to the model of partnership, as follows. Without persistence, profits at any time would be determined solely by instantaneous actions. We allow for the profits of a partnership to be determined by a company's fundamentals, which is our umbrella term that captures such factors as the quality of the product, the breadth of the customer base, the relationship with the financing institutions, the efficiency of the partnership's internal management structure, and more. Each of these factors is persistent and typically changes only gradually in response to the partners effort and the external circumstances ("noise"). Heuristically, profits of a start-up are determined not by the lines of codes written at a given moment, but by the overall quality of its "app."Imperfect state monitoring is an analogue, and partly a consequence of imperfect action monitoring. Given the unobserved effort of the partner, the quality of the app (how few "bugs" there are) is also uncertain. The bugs in the code will be discovered only with some delay. The discovery may happen directly, as a consequence of an internal audit; it may be inferred from the customers' reviews; or, in our model, it may be inferred from the profit flows.[10]

The new structure of the game will have a dramatic effect on the provision of incentives. Persistence and imperfect observability of the state imply that the whole path of future profit realizations can be used as a signal of effort at time t. Intuitively, this prevents the optimality of the "bang-bang" symmetric equilibria seen in the literature without persistence (see Abreu et al. [1986] and Abreu et al. [1990]), and it will lead to a nontrivial dynamic of the partnership. On the other hand, when the state is per-

[9]The scaling factor $(r+\alpha)$ guarantees that the present value of unit effort is one.

[10]Adding an additional publicly observable signal, beside the profit flows, would hardly affect the results.

fectly monitored, $\sigma_Y = 0$, profit flow dY_t at time t is a sufficient statistic for time t efforts. Moreover, when there is no production noise, $\sigma_Y = \sigma_\mu = 0$, efforts are perfectly monitored, and the first-best is achievable in a continuous-time setting.

Free-Riding The defining feature of a partnership is a fixed rule for splitting the profits. This gives rise to an obvious free-riding problem. At any point in time, a partner incurs a private cost of effort, whereas the (expected discounted) benefits of effort will be split evenly between the two partners. It follows from (4) that the efficient effort level a^{EF} and the effort a_{NE} in the unique stationary equilibrium satisfy

$$c'\left(a_{EF}\right) = 1, \quad c'\left(a_{NE}\right) = \frac{1}{2}, \tag{2}$$

with the corresponding continuation values

$$W_\tau^X = \frac{\overline{\mu}_\tau}{2\left(r+\alpha\right)} + \frac{1}{r}\left(a_X - c(a_X)\right), \quad X \in \{EF, NE\}. \tag{3}$$

With some abuse of terminology, we call a pair of strategies that play a_{NE} after every history a *repeated static Nash equilibrium.* We also normalize $a_{NE} = 0$, by setting $c'(0) = \frac{1}{2}$, which implies that $W_\tau^{NE} = \frac{\overline{\mu}_\tau}{2(r+\alpha)}$, and, in the long term, $W_\tau^{NE} = 0$. A partnership *unravels* if, from that point on, partners exert no more effort—that is, play the repeated static Nash equilibrium.

Strongly Symmetric Equilibria Exploiting the exponential decay of the fundamentals, we may rewrite the continuation payoffs as

$$\begin{aligned} W_\tau^i &= \mathbb{E}^{\{a_t^1, a_t^2\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{u_\tau}{2}e^{-\alpha(t-\tau)} + \frac{1}{2}\int_\tau^t (\alpha+r)(a_s^1 + a_s^2)e^{-a(t-s)}ds - c(a_t^i)\right)dt \middle| \mathcal{F}_\tau\right] \\ &= \frac{\overline{\mu}_\tau}{2\left(r+\alpha\right)} + \mathbb{E}^{\{a_t, a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{a_t^1 + a_t^2}{2} - c(a_t^i)\right)dt \middle| \mathcal{F}_\tau\right], \end{aligned} \tag{4}$$

where

$$\overline{\mu}_\tau = \mathbb{E}^{\{a_t^1, a_t^2\}}\left[\mu_\tau | \mathcal{F}_\tau\right]$$

is the expected level of fundamentals at time τ, given the pair of strategies $\{a_t^1, a_t^2\}$ and the history of public signals $\{Y_t\}_{t\in[0,\tau)}$.

A pair of strategies $\{a_t^1, a_t^2\}$ is a *Perfect Public Equilibrium (PPE)* if for each partner i, at any time $\tau \geq 0$ and after any public history in $\mathcal{F}_\tau$

$$\mathbb{E}^{\{a_t^i, a_t^{-i}\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\mu_t}{2} - c(a_t^i)\right)dt \middle| \mathcal{F}_\tau\right] \geq \mathbb{E}^{\{\tilde{a}_t^i, a_t^{-i}\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\mu_t}{2} - c(\tilde{a}_t^i)\right)dt \middle| \mathcal{F}_\tau\right]. \tag{5}$$

They are a *local PPE* if they satisfy only the local version of the above incentive-compatibility constraint,

$$\frac{\partial}{\partial a_\tau^i}\mathbb{E}^{\{a_t^i, a_t^{-i}\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\mu_t}{2} - c(a_t^i)\right)dt \middle| \mathcal{F}_\tau\right] = 0. \quad i = 1, 2, \tau \geq 0 \tag{6}$$

Our partnership game is characterized by a parsimonious information structure. The only information about the partners' effort comes from the joint stream of profits, and the efforts enter the profits additively. Consequently, it is not possible to identify which of the partners did, and which one did not, contribute to the common good based solely on the public signals (Fudenberg et al. [1994]). This implies that, as in the classic analysis of repeated duopoly by Green and Porter [1984] or partnerships by Radner et al. [1986], it is not possible to provide incentives by continuation value "transfers" between the agents, shifting resources from likely deviators. Since, in addition, the costs of effort are convex, the efficient levels of effort are always symmetric.

Consequently, the sole benefit of asymmetric play, when partners choose different effort levels, is to increase total flow costs and, thus, "destroy value."While this might be beneficial for the partnership, in the paper, we restrict attention to symmetric equilibria. In Section 5, we show how our solution extends to the case in which partners are allowed to destroy value, in the direct form of observable unproductive effort.

Formally, we say that a pair of public strategies $\{a_t^1, a_t^2\}$ is *strongly symmetric* if, after every public history in $\mathcal{F}_t$,

$$a_\tau^1 \equiv a_\tau^2.$$

Finally, we have that a *(local) Strongly Symmetric Equilibrium (SSE)* is a (local) PPE in strongly symmetric strategies.

3 Solution

In this section, we characterize the optimal SSE and present the main technical results of the paper: characterization of near-optimal local SSE, as well as the supremum of continuation values achievable in local SSE, verification of global incentive compatibility, and existence. We postpone analyzing the properties of the partnership's value and its dynamics until Section 4.

Whenever observational noise σ_Y is strictly positive, partners do not observe the fundamentals. However, in an equilibrium, they share public beliefs about it. A simple application of the Kalman-Bucy filter yields that for a fixed pair of strategies $\{a_t^1, a_t^2\}$, the publicly expected fundamentals $\overline{\mu}_t$ follows

$$d\overline{\mu}_t = (r+\alpha)\left(a_t^1 + a_t^2\right)dt - \alpha\overline{\mu}_t dt + \gamma_t[dY_t - \overline{\mu}_t dt], \tag{7}$$

for an appropriate gain parameter γ_t. For simplicity, we will assume that at time zero, partners believe that μ_0 is Normally distributed with steady-state variance σ^2. This implies that both the posterior variance σ_t^2 of their estimate and the gain parameter γ_t remain constant throughout the game, $\sigma_t^2 = \sigma^2$ and $\gamma_t = \gamma$, and equal (see Liptser and Shiryaev [2013])

$$\gamma = \sqrt{\alpha^2 + \left(\frac{\sigma_\mu}{\sigma_Y}\right)^2} - \alpha,$$
$$\sigma^2 = \gamma \times \sigma_Y^2.$$

In what follows, we use the following endogenous state. For a fixed pair of strategies, define *relational capital* as the continuation value net of the expected value of the fundamentals,

$$w_\tau = W_\tau - \frac{\overline{\mu}_\tau}{2\,(r+\alpha)} = \mathbb{E}^{\{a_t,a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{a_t^1+a_t^2}{2} - c(a_t^i)\right)dt\right]. \tag{8}$$

The strongly symmetric equilibria that provide nontrivial incentives must have agents coordinate on relatively efficient effort after "good" histories, indicating high past effort, and on relatively inefficient effort after "bad" histories. Accounting for which histories are good and which histories are bad can be quite complicated. Since, due to persistence, rewards and punishments at any time help incentivize all past actions, the accounting should depend on the whole stream of past actions, as well as on the likelihood ratios of the profit paths.

In the paper, the relational capital will play precisely the role of such an accounting device. Our results establish that, even though relational capital is somewhat simple, it is an optimal accounting device, as the strategies in the most efficient equilibrium can be expressed solely as a function of it.

The use of relational capital as the state variable in the dynamic equilibrium will be formally justified in Proposition 2 below. The intuition relies on the fact that the inherited level of capital does not interact with the current effort to affect either the cost or the profitability of the partnership, as described in (1). The only channel through which fundamentals could help partners is informational, as a statistic of the public history of profit flows, since high beliefs $\overline{\mu}_t$ correlate with high past profit flows. However, this exogenous state variable is arbitrary. Instead, borrowing from the dynamic programming or the repeated games literature, it is optimal to build an equilibrium around the endogenous state variable that is the forward-looking future value of the partnership or the continuation value *net* of the believed effects of the inherited capital. This is how we defined relational capital.

Before we move on, let us establish the following benchmark result. With no observational noise, $\sigma_Y = 0$, partners observe the process of internal capital and, thus, play a stochastic game with perfect monitoring. While at first blush, no noise should benefit the partners, we have the following result (see, also, Sannikov and Skrzypacz [2007], Bohren [2016]).[11]

Proposition 1 *With no observational noise, $\sigma_Y = 0$, the essentially unique SSE is the repeated static Nash equilibrium.*

The intuition for the above result is similar to that for Abreu et al. [1991] and Sannikov and Skrzypacz [2007] results on discrete time models. On the one hand, with perfect monitoring of the fundamentals, the current innovation in the state, $d\mu_t$, is a sufficient statistic for the actions taken at time t. On the other hand, it follows from the Martingale Representation Theorem that the process of partners' relational capital is locally linear in $d\mu_t$ (see Karatzas [1991] and Sannikov [2008]). This implies that once relational capital is at the maximum, it may not respond to the signal $d\mu_t$, and so the partners' only incentive to exert effort comes from the direct effect on fundamentals, as in the inefficient repeated Nash Equilibrium.[12]

In what follows, we consider the model with observational noise, $\sigma_Y > 0$. The following proposition is the first step in the construction of the optimal equilibrium.

Proposition 2 *Consider a symmetric strategy profile $\{a_t, a_t\}$. It is a local SSE with the relational capital process $\{w_t\}$ if and only if there are L^2 processes $\{I_t\}$ and $\{J_t\}$ such that*

$$\begin{aligned} dw_t &= (rw_t - (a_t - c(a_t)))\, dt + I_t \times (dY_t - \overline{\mu}_t dt) + dM_t^w, \qquad (9) \\ dF_t &= (r + \alpha + \gamma)\, F_t - (r + \alpha)\, I_t dt + J_t \times (dY_t - \overline{\mu}_t dt) + dM_t^F, \\ a_t &= c'^{-1}(F_t + 1/2) =: a(F_t), \end{aligned}$$

where $\{M_t^w\}$ and $\{M_t^w\}$ are martingales orthogonal to $\{Y_t\}$, and the transversality conditions $\mathbb{E}\left[e^{-rt} w_t\right], \mathbb{E}\left[e^{-(r+\alpha)t} F_t\right] \to_{t\to\infty} 0$ hold.

[11] As is standard, by essential uniqueness we mean that any other SSE may not be different with nonzero probability.

[12] Sannikov and Skrzypacz [2007] establish the continuity result, showing that the impossibility holds in a discrete-time model with short period lengths, and Normal noise.

Formally, the proposition shows that constructing a local SSE is equivalent to constructing two L^2 processes $\{I_t\}$ and $\{J_t\}$, capturing how the relational capital and the stock of incentives respond to profit realizations, together with martingales $\{M_t^F\}$ and $\{M_t^w\}$ that capture the public randomization. Given strict convexity and continuity of the cost of effort function, they fully pin down local equilibrium efforts.

Process $\{I_t\}$ measures the sensitivity of relational capital to public profit signals. For example, in the repeated static Nash equilibrium, $\{I_t\}$ is the null process and the relational capital is constant. Intuitively, as partners' efforts translate into higher fundamentals, which, in turn, pushes profits up, positive sensitivities I_t will provide additional incentives for effort, beyond the direct marginal benefit of fundamentals in the repeated static Nash equilibrium. We call $\{I_t\}$ a *flow of incentives* process.

The proposition shows that in our model, the marginal benefit of exerting effort at time τ equals $1/2$ plus F_τ. We call $\{F_t\}$ a *(stock of) incentives* process. Notice that $\{F_t\}$ satisfies

$$F_\tau = (r+\alpha) \times \mathbb{E}^{\{a_t, a_t\}} \left[\int_\tau^\infty e^{-(r+\alpha+\gamma)(t-\tau)} I_t dt \middle| \{Y_t\}_{t \in [0,\tau)} \right], \tag{10}$$

and $\{J_t\}$ measures how sensitive the incentives are, or how much they change in response to the public profit signals.

To understand the accounting of incentives, consider the effect of increasing effort at time τ. This increases both the (expected) fundamentals $\overline{\mu}_t^{dev}$ and the wedge between the correct beliefs and the equilibrium beliefs about the fundamentals, $\overline{\mu}_t^{dev} - \overline{\mu}_t$ by a factor of $r+\alpha$. Given discounting and the mean reversion of the fundamentals, the total present value of the first effect is 1, half of which is captured by a partner.

The effect of the increased wedge results in an increased flow of the relational capital today by $(r+\alpha)I_\tau$. Moreover, given persistence, it follows from the Kalman formula (7) that the wedge reverts to the mean at rate $\alpha+\gamma$, and so in the next instant, $\tau+\Delta$ will scale down by $e^{-(\alpha+\gamma)\Delta}$. The term α is the mean reversion of the fundamentals. The second term γ follows from a *ratchet effect:* if the wedge is positive, and, thus, the equilibrium beliefs $\overline{\mu}_t$ are relatively low, the new profit realization will be surprisingly high, and so $\overline{\mu}_t$ will move up faster than the correct beliefs $\overline{\mu}_t^{dev}$, by a factor of γ. Given discounting, the overall effect of the increased wedge at time $\tau + \Delta$ is $(r+\alpha)e^{-(r+\alpha+\gamma)\Delta} I_{\tau+\Delta}$. Thus the integral in (6) captures the total marginal benefit of extra effort across time, net of the direct effect on the fundamentals.

The second step in our solution relies on the following novel parametrization. Our problem does not yield a natural choice of objective function to be maximized, such as one player's continuation value (in the Principal-Agent problems, or asymmetric equilibria in dynamic games). In order to construct local SSE. we (i) use the relational capital as a state variable, and (ii) parametrize maximal incentives as a function of it.

To highlight the role of the new parametrization, we split the problem of characterizing the optimal local SSE into two steps. First, the following proposition suggests a strategy of constructing any, not necessarily optimal, equilibria.

Proposition 3 *Consider a bounded measurable* $I : [\underline{w}, \overline{w}] \to R_+$ *and a* C^2 *function* $F : [\underline{w}, \overline{w}] \to R$ *that satisfy the differential equation*

$$(r+\alpha+\gamma)F(w) = (r+\alpha)I(w) + F'(w) \times (rw - (a(F(w)) - c(a(F(w))))) + \frac{F''(w)}{2}\sigma_Y^2 I^2(w), \tag{11}$$

such that each boundary point $w^\partial \in \{\underline{w}, \overline{w}\}$ *is either achievable by a local SSE or satisfies*

$$(r+\alpha+\gamma)F(w^\partial) = F'(w^\partial) \times \left(rw^\partial - \left(a(F(w^\partial)) - c(a(F(w^\partial)))\right)\right), \tag{12}$$
$$sgn\left(\frac{\overline{w}+\underline{w}}{2} - w^\partial\right) = sgn\left(rw^\partial - \left(a(F(w^\partial)) - c(a(F(w^\partial)))\right)\right).$$

Then, for each $w_0 \in [\underline{w}, \overline{w}]$, *there is a local SSE* $\{a_t\}$ *achieving* w_0. *For* $t < \tau$, *the efforts satisfy* $a_t = a(F(w_t))$, *and the relational capital process* $\{w_t\}_{t\in[0,\tau)}$ *is as in Proposition 2, with* $M_t^w = 0$, *where* τ *is the stopping time of reaching a boundary point that does not satisfy (12).*

The proposition reduces the problem of finding local SSE to solving the Ordinary Differential Equation (11). Function I parametrizes the flow of incentives, $I_t = I(w)$ for w in the interval $[\underline{w}, \overline{w}]$. The differential equation then characterizes the incentive function $F_t = F(w)$, defined by (10), when the relational capital follows the process (9). It follows from the Ito formula that the process $\{F(w_t)\}_{t\geq 0}$ satisfies the conditions of Proposition 2, for the sensitivities $J_t = F'(w_t) \times I(w_t)$.

When the boundary point is a relational capital known to be achievable by a local SSE, upon reaching this point, the game simply follows this local SSE. The alternative boundary conditions (12) are more complicated. The first clause requires that, at each boundary point, the flow of incentives $I(\overline{w})$ is zero. Since the flow of incentives is the sensitivity of relational capital with respect to the flow profits, the condition is necessary so that the relational capital does not escape out of the set with positive probability (see Proposition 1). Consequently, at the boundaries, the (stock of) incentives $F\left(w^{\partial}\right)$ is made up only of the discounted stock of incentives in the next instant, due to persistence. Similarly, the second clause requires that at the boundary points, the relational capital does not drift outside of the set.

Proposition 3 is a useful tool for constructing some nontrivial, not necessarily optimal, equilibria (see Proposition 5). The following result characterizes the highest relational capital achievable in a local equilibrium and is one of the main results of this paper.

Theorem 1 *Let w^* be the supremum of the relational capital achievable in a local SSE. Then, there exists a C^2 strictly concave function F on $[0, w^*)$ that satisfies*

$$(r+\alpha+\gamma)F(w) = \max_I \left\{ (r+\alpha)I + F'(w)\left(rw - (a(F(w)) - c(a(F(w))))\right) + \frac{F''(w)\sigma_Y^2}{2}I^2 \right\} \quad (13)$$
$$= F'(w)\left(rw - (a(F(w)) - c(a(F(w))))\right) - \frac{(r+\alpha)^2}{2\sigma_Y^2 F''(w)},$$

as well as the boundary conditions

$$F(0) = 0, \quad (14)$$
$$\lim_{w\uparrow w^*}(r+\alpha+\gamma)F(w) = \lim_{w\uparrow w^*}\left\{F'(w)\times\left(rw - (a(F(w)) - c(a(F(w))))\right)\right\},$$
$$rw^* - (a(F(w^*)) - c(a(F(w^*)))) = 0.$$

The novel differential equation (13) characterizes the solution to the problem of maximizing the stock of incentives F, over all possible incentive flows I, given the "promise keeping" law of motion of the relational capital (9). The first term in the maximization problem is proportional to the flow of incentives; the second term is the change of the stock due to the change in the relational capital; and the last term is the change in incentives due to the second-order variation. The maximization problem is easily solved, with the incentive flow inversely proportional to the curvature of F,

$$I^*(w) = -\frac{r+\alpha}{\sigma_Y^2 F''(w)}. \tag{15}$$

The boundary conditions are the same as in Proposition 3, and capture the fact that the flow of incentives dies out at the extremes.

Despite similarities, we stress that the equation is *not* the Hamilton-Belmann-Jacobi (HJB) equation for the solution of a stochastic control problem. The reason is that, in our problem, the law of motion of the state variable (relational capital) depends, via actions chosen, on the value function F (see (9)). This is not allowed in a stochastic control problem, and so we may not rely on the existing verification theorems. Nevertheless, our proof establishes that an HJB-like characterization (13) of the solution is also available for a problem in which the value function doubles as a state variable.

The result provides a procedure for finding the supremum of relational capital w^* achievable in a local SSE. It is characterized by a solution of the differential equation (13) passing through the starting point $(w_{NE}, 0)$ that reaches furthest to the right. The idea behind the result is to trace out the upper boundary of the set of the relational capital and stock of incentives pairs achievable across all the local SSE. The right end of this boundary is, by definition, the highest relational capital w^* achievable in a local SSE. The proof of the theorem establishes, among other things, that the boundary is smooth and must satisfy equation (13).

There are two difficulties with the solution. First, equation (13) is not uniformly elliptic since $I(w)$ can be arbitrarily small. Indeed, the equations in (14) imply that close to w^*, $I^*(w)$ vanishes and F'' converges to negative infinity. This causes computational problems. Second, the supremum w^* is not achievable, and so the optimal local SSE does not exist. This follows from the last two equations in (14), which imply that the point

$(w^*, F(w^*))$ would have to be self-generating, and so the curve F cannot be extended to w^*.[13] Intuitively, at the right boundary, the volatility of the relational capital must be zero, and its drift must be negative, in order to satisfy self-generation (see Proposition 3). If the drift were strictly negative, one could extend the curve F further right, with the stock of incentives provided with zero flow, just by "waiting it out" and the relational capital drifting down. Consequently, the curve F is not self-generating, and we cannot invoke Proposition 3 to construct the optimal equilibrium.

The following approximation solves both problems. For an arbitrary $\varepsilon > 0$, consider the local SSE with policies $I(w)$ constrained to be either zero or above ε (see Proposition 2). The proposition below shows a uniformly elliptic differential equation that characterizes a near-optimal value of this problem. The solution to the equation satisfies the constraints of Proposition 3, and so defines a local SSE that achieves this value.

In the rest of this section, we assume that the cost of effort is quadratic:[14]

$$\text{(Quadratic Cost)} \qquad c(a) = \frac{1}{2}a + \frac{C}{2}a^2.$$

Proposition 4 *For $\varepsilon > 0$, let w^*_ε be the upper bound on relational capital achievable in a local SSE with policies $I(w)$ constrained to be either zero or above ε. Then, there exists a C^2 strictly concave function F_ε on $[0, \overline{w}_\varepsilon]$ that satisfies*

$$\begin{aligned}(r+\alpha+\gamma)F_\varepsilon(w) = {} & F'_\varepsilon(w)\left(rw - (a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))\right) \\ & - \frac{(r+\alpha)^2}{2\sigma_Y^2 F''_\varepsilon(w)}\mathbf{1}_{F''_\varepsilon(w) \geq -\frac{r+\alpha}{\sigma_Y^2 \varepsilon}} + \varepsilon\left(r+\alpha+\varepsilon\frac{F''_\varepsilon(w)\sigma_Y^2}{2}\right)\mathbf{1}_{F''_\varepsilon(w) < -\frac{r+\alpha}{\sigma_Y^2 \varepsilon}},\end{aligned} \tag{16}$$

together with the boundary conditions (14) at 0 and $\overline{w}_\varepsilon$, as well as

$$F''_\varepsilon(w) \geq -\frac{2}{\varepsilon}, \quad w \in [0, \overline{w}_\varepsilon],$$
$$w^*_\varepsilon - \overline{w}_\varepsilon = O(\varepsilon^{2/3}).$$

[13] Formally, $\lim_{w\uparrow w^*} I^*(w) = 0$ and $\lim_{w\uparrow w^*} dw_t = 0$.

[14] Quadratic costs greatly simplify deriving the bounds in Propositions 4, 5, and in Theorem 2, but we are confident that the result can be extended to more general cost functions, with appropriate bounds on third derivatives. The linear term in the cost of effort is solely to normalize repeated Nash Equilibrium effort to zero; see Section 2.

So far we have characterized only local equilibria. The following result shows conditions on the primitives, under which local equilibria satisfy full incentive compatibility constraints.[15] For a C^2 strictly concave function $F : [\underline{w}, \overline{w}] \to R$, we say that $I : [\underline{w}, \overline{w}] \to R_+$ is a *D-optimal* policy function, for $D > 0$, if

$$\frac{|I(w) - I^*(w)|}{I^*(w)} \leq D, \tag{17}$$

where $I^*(w)$ is the locally optimal flow of incentives, defined in (15).

Theorem 2 *Consider a local SSE $\{a_t\}$ as in Proposition 3, such that the flow of incentives function I is D-optimal, for some $D \geq 0$. Then $\{a_t\}$ is a SSE when the second derivative C of cost function is sufficiently high.*

Specifically, there is $\underline{C} > 0$ such that global incentive compatibility holds as long as

$$C \geq \underline{C} \times \max\left\{\frac{1}{\sqrt{r+\alpha+\gamma}}, \frac{1}{\sqrt{\sigma_Y}}\right\}. \tag{18}$$

Condition (17) requires that the flow of incentives in the local SSE is proportional to the flow of incentives that is locally optimal if the continuation stock of incentives is given by F. For example, the condition is satisfied by the approximately optimal equilibria characterized in Proposition 4, with $D = 1$.[16]

The problem consists in showing that, after any history, the effort choice is concave. Given strict convexity of the effort cost function, this boils down to establishing bounds on how convex the expected benefit of effort is. Crucially, in a dynamic environment with persistence, like ours, a deviation affects the strength of incentives that the agent faces in the future. This "knock-on" effect makes accounting for the benefits of deviations much more involved than in a static setting or without persistence.

Following up on this intuition, in order to bound how convex the benefit of effort is, it is sufficient to establish a uniform bound on how sensitive the incentives $F(w)$ are with respect to public signals. The first part of the proof is related to the results

[15] The proof in the Appendix yields an explicit formula for a lower bound $C(r, \alpha, \sigma_Y^2, \sigma_\mu^2)$ on C in the statement of the Theorem.

[16] For those equilibria, we have $I(w) = I^*(w)$ when $I^*(w) \geq \varepsilon$, and $I(w) = \varepsilon$, when $I^*(w) \in [\varepsilon/2, \varepsilon]$.

in the literature and shows that there are no global deviations from a local SSE if this sensitivity of incentives $F(w)$ is uniformly bounded (see Williams [2011], Sannikov [2014], and Cisternas [2017]).

In the second part of the proof, we bound this endogenous sensitivity of incentives by a function of the primitives of the model. This part of the proof relies heavily on the analytical tractability of our solution. The sensitivity equals $F'(w) \times I(w)$ (see discussion under Proposition 3 and Theorem 1). Thus, for example, when $F'(w)$ is bounded, the result follows simply from a uniform boundedness of the stock of incentives, which implies that the flow $I(w)$ is bounded too (see (13)).

We conclude this section by establishing the existence of an SSE that is better than the repeated static Nash. The proposition below shows that nontrivial local SSE exist in either of the two cases: when the partners are patient, fundamentals persistent, and there is little production noise (and so γ is small); or when public monitoring is not too noisy (σ_Y small). Full incentive compatibility follows from Theorem 2.

Proposition 5 *The supremum w^* of relational capitals achievable in local SSE is strictly positive, as long as, for appropriate $\overline{C} > 0$,*

$$C \leq \overline{C} \times \frac{1}{(r + \alpha + \gamma)\sigma_Y^2}.$$

4 Value of Partnership and Equilibrium Dynamics

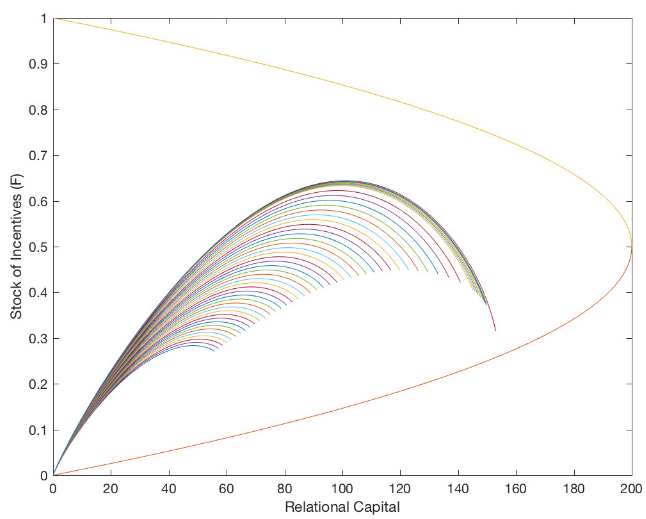
The results in the previous section provide a convenient tool with which to explore the comparative statics of the value of the partnership. They also provide a complete characterization of the dynamics of the fundamentals, or the profitability of the partnership, together with the level of effort that the partnership sustains. In this section, we explore some of their properties. Given nonexistence of the optimal local SSE (see discussion below Theorem 1), in this section we refer to the near-optimal equilibria characterized in Proposition 4.

4.1 Dynamics of Effort

Figure 1 illustrates a near-optimal SSE. The horizontal parabola is the locus of the relational capital-stock of incentive pairs $(w, \widetilde{F})$ that can be achieved by symmetric play in a stage game, absent any incentive constraints. This is an analogue of feasible stage game payoffs. Specifically, define the lower and upper arms of the parabola, $\underline{F}$ and $\overline{F}$, such that for any $w \leq w_{EF}$,

$$rw = a(\underline{F}(w)) - c\left(a(\underline{F}(w))\right) = a(\overline{F}(w)) - c\left(a(\overline{F}(w))\right), \; \underline{F}(w) < \overline{F}(w). \tag{19}$$

Relative to the first best, in which the efficient stock of incentives F_{EF} equals 1/2 (see (2)), $\underline{F}$ traces out the pairs at which the partners have too little incentives, whereas at each $\overline{F}$, they have too much incentives.

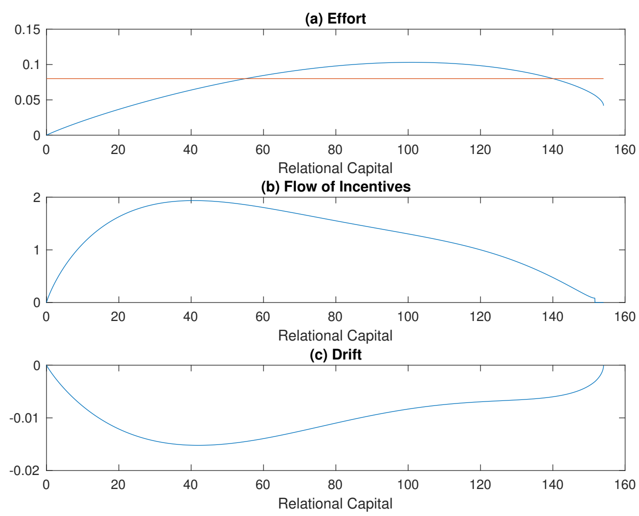


This figure displays the stock of incentives as a function of the relational capital of the partnership. We fix a parametrization of the model,[17] and find many local SSE as described in Proposition 3. The parametrization used satisfies the second-order condition in Theorem 2.

Figure 1: Stock of Incentives in Near-optimal SSE

Function F describes the partners' incentives in a near-optimal SSE and fully characterizes the equilibrium dynamics. Given the quadratic cost of effort, the level of incentives F translates linearly into the level of efforts taken by the partners. The effort, together with the level of relational capital w determine the drift of the relational capital (see Proposition 2), whereas its sensitivity with respect to profit flows, or flow of incentives $I(w)$, is proportional to the inverse of F''. The nontrivial equilibrium dynamics contrasts with the results in the repeated game literature, in which, with "continuous signals," one of the optimal equilibria has the "bang-bang" structure and actions changing only once on the equilibrium path (Abreu et al. [1986] and Abreu et al. [1990]).

The graph of F starts on the left at the stationary repeated Nash equilibrium point $(0,0)$, as displayed in Figure 1. Proposition 2 implies that at any internal point, where the graph lies "within" the parabola, the drift of the relational capital is negative.[18] Concavity of F implies that if it ever drops below $\underline{F}$, relational capital drifts up at all high values, violating transversality conditions. Instead, on the right, the graph approaches a stationary point $(w^*, \underline{F}(w^*))$, and close to this limit, the negative drift dies out to zero (see Figure 2).



This figure displays in panel (a) the effort of a partner in the near-optimal SSE, as a function of the relational capital. The horizontal line represents the efficient level of effort. Panels (b) and (c) display the flow of incentives, and the drift as a function of the relational capital of the partnership, in the near-optimal SSE.

Figure 2: Effort, Flow of Incentives, and Drift in Near-optimal SSE

[18] While it is possible for the graph of F to reach above $\overline{F}$, where the drift is positive, we show in the proof of Theorem 2 that this does not happen when the cost of effort is sufficiently convex.

Effort is increasing in the stock of incentives. Thus, concavity of the stock of incentives, F, has the following implications for the dynamics of effort as a function of the relational capital of the partnership.

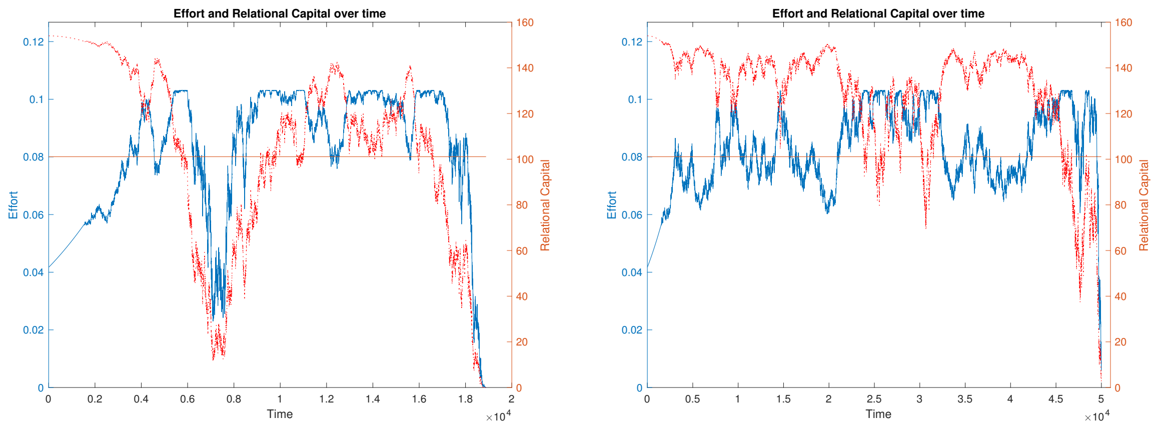
Corollary 1 (Rallying, Over-Working and Coasting) *In the near-optimal local SSE from Proposition 4, effort $a(w)$ is increasing to the left, and decreasing to the right of $w^{\#}$, for $w^{\#} < w^*_\varepsilon$. For some parameter values, there is a neighborhood of $w^{\#}$ in which $a(w) > a_{EF}$.*

Note that in the local SSE, relational capital always increases after high profit flows, since $I(w)$ is positive (see Figure 2). Thus, the corollary says that when relational capital is low, the partners' effort increases after good outcomes. Roughly speaking, when relational capital is low, the possibility of the partnership unraveling looms large. In this case, effort will likely not matter for the partnership beyond the direct effect on the fundamentals. As a consequence, a good profit outcome *rallies* the partnership further away from the brink and encourages higher effort.

On the other hand, when relational capital is sufficiently high, the partners' effort decreases after good outcomes.[19] In other words, partners *coast* on their past good performance. The reason is that, as the relational capital approaches the highest achievable value, it can no longer respond to profit realizations, and so the flow of incentives $I(w)$ dies out. At these high values, the provision of incentives today come almost entirely from discounting back those incentives from the future. Thus, the closer to the bliss point, the longer partners must wait for the relational capital to drift down to the range where the flow of incentives picks up.

Figure 3 displays two sample paths for the evolution of effort and relational capital over time. In both sample paths, the relationship starts with the same relational capital, w^*. In the beginning, players coast, and as I is very small, the relational capital drifts down, undisturbed by shocks. When relational capital is above the horizontal line, good profit outcomes that increase relational capital lead players to exert less effort. In these

[19]In the optimal SSE of the (discrete-time) version of the model without persistence, good signals never lead to lower effort.



(a) A short-lived partnership

(b) A long-lived partnership

This figure displays two sample paths of effort (on the left axis) and relational capital (on the right axis) over time. Panel (a) displays an eventful sample path, where the relationship reached the brink of dissolution and was rallied back by partners' efforts. Panel (b) displays a long-lived relationship, in which the partners exerted effort for a long time. The horizontal line represents the efficient effort (on the left axis), and the relational capital where effort is maximized (on the right axis).

Figure 3: Effort and Relational Capital over Time

coasting phases, changes in effort and relational capital are negatively correlated. For instance, the sample path in Panel (b) shows, almost everywhere, that when relational capital goes up, effort goes down. Also note that on both sample paths, players exert effort higher than the stationary efficient quite frequently. When relational capital is below the horizontal line, changes in effort and relational capital are positively correlated. For instance, in Panel (a) at around time 0.7, the relational capital reached a very low point, with a very low effort as well, and the partnership was on the brink of dissolution. Good profit realizations raised the relational capital and partners rallied their effort, giving the partnership more time.

The two different ways in which effort responds to the change in relational capital highlight two different mechanisms for providing incentives to partners. When relational capital is low and effort increasing in it, a player's high effort brings about high profit realizations in the near future and, thus, motivates higher, more efficient effort from her partner. This is related to the *encouragement effect* defined in Bolton and Harris [1999].

The difference is that in the literature on experimentation in teams, the effect comes from good signals making partners believe that the exogenous success of the project is more likely, whereas in our case, it makes partners believe that the endogenous failure (the dissolution of the partnership) is less likely.

When the relational capital is sufficiently high, players know that it will drift down over time and eventually reach the range where effort is high, and so the marginal cost of effort is also high. Thus, for sufficiently high relational capital, agents exert *precautionary effort:* even close to the bliss point, partners work because it will be useful in the bleaker future. More precisely, incentives are provided directly by this precautionary motive. It pays off to work close to the bliss point—as the marginal cost of effort is low—in order to save on effort later, in a cut-throat phase, when the marginal cost of effort is high. Precaution would motivate effort in any environment, as long as an agent can substitute effort across time to smooth its marginal cost. In our partnership model, the intertemporal substitution of effort relies on the imperfect state monitoring and the resulting fact that incentive benefits of effort accrue in the future (compared with Proposition 1).

Another implication of the precautionary motive is that in the near-optimal local SSE, partners may exert inefficiently high effort. This is illustrated in Figure 1, where the stock of incentives F reaches above the efficient level of $1/2$, and in both sample paths of Figure 3. The paradox is, of course, that the incentive problem is to find ways to provide nontrivial incentives, rather than to curb excessive incentives. In a model without persistence (and discrete time), such *over-working* may not happen.[20] In our case, high incentives for average levels of relational capital lead to inefficiently high effort, but this is outweighed by the benefit of sustaining nontrivial incentives at higher levels of relational capital, due to precautionary motives.

We believe that there are two ways to interpret the empirical implications of those results, depending on how one thinks about relational capital. In the model, relational capital is an endogenous variable that is not directly observable; however, it has a clear

[20] While we are not aware of a reference for this result, the intuition is clear: as long as the set of equilibrium payoff vectors is convex, scaling down the incentives is feasible in the Bellman problem.

interpretation, facilitating the search for observable instruments. Those range from informal expressions of partners' optimism or "bad blood" to the goodwill of the joint venture, which is recorded in financial statements. Alternatively, one can calculate relational capital directly from the observable profit flows. Under this interpretation, for example, coasting means that after a string of good outcomes, the partners' effort starts decreasing. Although their effort is not observable, the joint effort can be identified from the changes in profitability of the partnership.

4.2 Fine Sand in the Wheels

We focus on the comparative statics of the value of partnership along the two novel dimensions of our model: persistence of the fundamentals α, and observational noise σ_Y.

Increasing the mean reversion parameter to infinity, our model approximates the one from the repeated game literature, with no persistence. In this case, we have the following continuity result.

Proposition 6 *The upper bound w^* on relational capital achievable in a local SSE converges to 0 as the persistence of the internal capital vanishes, $\alpha \to \infty$.*

In the limiting model without persistence, the impossibility is immediate from the fact that, at the optimal level of continuation value, any amount of nontrivial incentives equivalent to strictly positive flow $I_t > 0$ would result in the value escaping to the right. Sannikov and Skrzypacz [2007] show that this result is also true with short period lengths. In our case, the proof requires an additional step. Just as above, the flow of incentives must be vanishing close to the bliss point w^*, but this still leaves the possibility of incentives delivered by the precautionary motive, as discounted incentives from the future. Given little persistence, however, this would require unbounded incentives delivered for average levels of relational capital, which is not possible (the stock of incentives is an integral of discounted volatilities of the relational capital process, belonging to a bounded interval).

We agree with Sannikov and Skrzypacz [2007] that in the continuous-time setting, a model with persistence, in which the effect of actions does not vanish instantaneously, is more realistic.[21] For this model, either with short periods (Sannikov and Skrzypacz [2007]) or directly in continuous time (Proposition 1), perfect monitoring of fundamentals ($\sigma_Y = 0$) prevents any nontrivial equilibria. However, we have the following:

Proposition 7 *The upper bound w^* on relational capital achievable in a local SSE increases discontinuously at $\sigma_Y = 0$, and is decreasing in the variance of the monitoring technology σ_Y, for $\sigma_Y > 0$.*

The result is stated only for the solutions of the relaxed problem, assuming only local incentive-compatibility constraints. Given Theorem 2, however, this implies that for any fixed cost function, for the range of observational noises σ_Y for which the local equilibrium is fully incentive-compatible, the same comparative statics holds for the supremum of relational capital achievable in an SSE.[22]

It is intuitive that the equilibrium becomes more efficient as monitoring technology improves. With better monitoring, for the fixed sensitivity of relational capital with respect to the signals—and so for fixed level of incentives—the volatility of relational capital goes down. This allows the partnership to operate longer before it reaches the absorbing inefficient state 0. Eliminating the monitoring noise, however, eliminates the possibility of providing any incentives at the most efficient state w^*, and so the equilibrium unravels. The result highlights that the impossibility of collusion in Sannikov and Skrzypacz [2007] and in Proposition 1 is non-robust with respect to adding the monitoring noise. Formally, the impossibility holds for the parameter value at which the equilibrium correspondence is not upper-hemicontinuous.[23]

[21] See Sannikov and Skrzypacz [2007], Section 6.

[22] The tractable analytical characterization of the parameters, for which the First Order Approach is valid, seems unlikely. Consequently, the comparative statics of the SSE payoffs for a fixed cost function and the *entire* range of observational noise seems analytically intractable.

[23] Lack of upper-hemicontinuity of the (local) equilibrium correspondence is surprising in itself. We stress that a continuous-time model is a limiting model. We suspect that for a discrete time model with short period lengths, the graph of $w^*(\sigma_Y)$ is continuous at zero, with a "hump" at small values - the "hump" being pushed against the vertical axis as period length shrinks. This does not, we believe, contradict the statement that the result on the triviality of equilibria for the model with persistence is an anomaly.

4.3 Beatles' Break-up

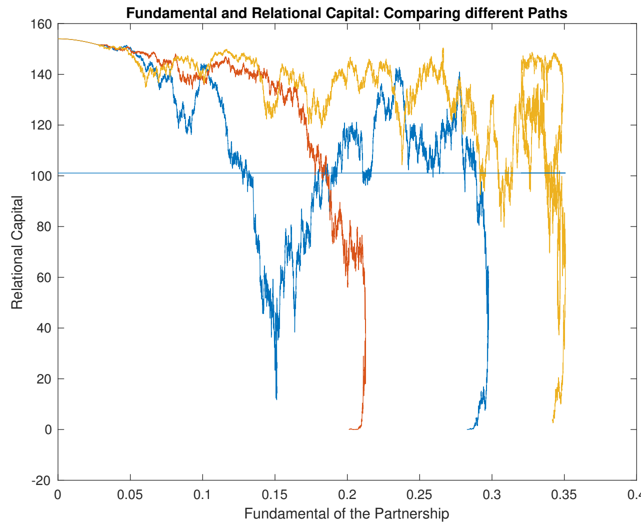
The only state that determines the flow ("stage game") payoffs is the mean fundamentals, $\overline{\mu}_t$. It seems natural to restrict attention to the Markov equilibria, in which this is also the only variable that determines the equilibrium actions and serves as an accounting device for the good and bad histories (with high fundamentals indicative of past high effort). As our results demonstrate, however, the near-optimal equilibria are not Markovian. While stage game payoffs are driven by the exogenous mean fundamentals, it is optimal to have actions driven by the endogenous relational capital.

The dynamics of the two fundamentals and relational capital are different, and this has several empirical implications. The sensitivity of fundamentals with respect to the profit flows is constant, $\frac{d\overline{\mu}_t}{dY_t} = \gamma = \sqrt{\alpha^2 + \left(\frac{\sigma_\mu}{\sigma_Y}\right)^2} - \alpha$. The sensitivity of relational capital with respect to profits is variable and equals $I(w)$. Thus, unlike in Markov equilibria the relationship between the mean profitability and relational capital, and so effort is not deterministic: two equally profitable partnerships can differ on the level of relational capital, and so effort, as Figure 4 shows.

For example, sensitivity $I(w)$ vanishes close to the boundaries. When one partnership that is on the brink of unraveling experiences positive shocks, and another one that is close to the bliss point w^* experiences negative shocks, both can end up with similar levels of fundamentals, even though relational capital hardly budges.

In contrast, when the production noise σ_μ is low or absent, fundamentals barely responds to profit outcomes and, thus, is much more sluggish than relational capital. In this case, a short string of sharp, low profit realizations will unravel the partnership, with hardly any effect on profitability. In other words, even a very profitable partnership may unravel, when its goodwill is tested by a series of adverse outcomes, even if they have a negligible effect on the partnership's profitability.

Figure 4 displays the differences in the dynamics of the two capitals. It shows three different sample paths, highlighting that the profitability of a partnership and its relational capital are not colinear. This result speaks to the literature on persistent productivity



This figure displays three different sample paths of the relational capital of a partnership, as a function of the fundamentals of the relationship. The three different paths highlight that the relationship between fundamentals and relational capital is not a one-to-one relationship. Furthermore, the three partnerships unravel, when relational capital drops to zero, at different levels of fundamentals. The drops are sharp, with small effect on fundamentals. The horizontal line marks the relational capital where effort is maximized.

Figure 4: Relational Capital and Fundamentals of a Partnership

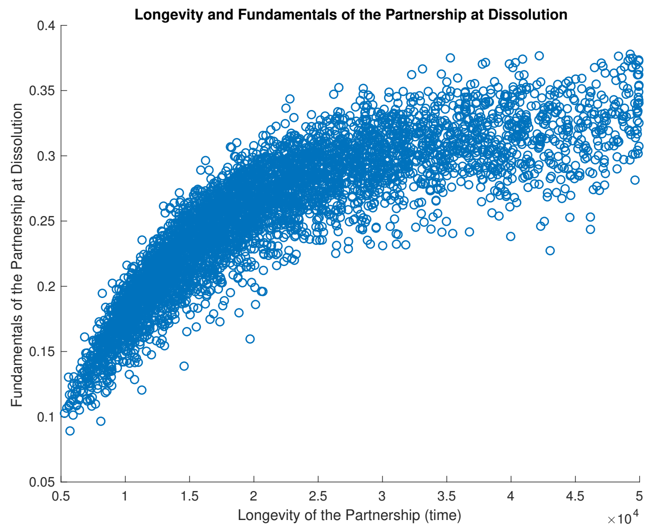
differences.[24] Partnerships not only have different levels of productivity, but those differences also persist over time. Furthermore, even at dissolution, different partnerships have different levels of productivity. This intuition is formalized in the next proposition.

Proposition 8 *In the near-optimal local SSE from Proposition 4, at any time $t > 0$, the distribution of mean fundamentals $\overline{\mu}_t$ and relational capital w_t has full support. Starting at any level of mean fundamentals $\overline{\mu}_t$, a partnership may unravel in an arbitrarily short period of time, with an arbitrarily small change in $\overline{\mu}_t$, when production noise σ_μ is sufficiently small.*

Finally, Figure 5 displays the relationship between the longevity of the partnership and its fundamentals at the moment of unraveling. Longer partnerships have, in general, better fundamentals when they unravel, as can be seen by the positive relationship

[24] See Cusolito and Maloney [2018], Bloom et al. [2013], and, in particular, the review in Syverson [2011].

displayed in Figure 5. This suggests that the partnerships are relatively unstable. They last if relational capital stays at intermediate levels, when partners work hard and keep fundamentals at high levels. Partnerships lingering on with low relational capital and little effort are rare.



This figure displays the relationship between the longevity of the partnership and the partnership's fundamentals at dissolution. It displays the time to dissolution-fundamentals at dissolution pairs of five thousand simulated paths of the near-optimum SSE.

Figure 5: Longevity and Fundamentals of a Partnership

5 Concluding Remarks

In this paper, we present a dynamic model of partnership whose two central features are persistent effect of effort and imperfect state monitoring. We develop a method that allows us to characterize near-optimal strongly symmetric equilibria of the game with a simple differential equation. Its solution describes the supremum of incentives achievable in an SSE for a given level of relational capital, and fully characterizes equilibrium dynamics in near-optimal equilibria. Relational capital, which captures the goodwill, or mutual trust in the partnership, is the only state variable, which evolves differently than the persistent "hard" fundamentals of the venture. The non-Markovian dynamics generates novel predictions about the dynamics of effort, fundamentals and profits in a partnership. We also identify new channels for motivating partners in the setting with persistence and imperfect state monitoring.

Both the model of partnership and the solution method can be generalized. Below, we briefly discuss three extensions.

Observable effort. We may allow agents to also exert observable effort. It may be productive and drive profits up, or it may be unproductive, with the only effect of "burning value". We conjecture that the only difference in the resulting differential equation (13) is the extra term in the drift of the relational capital: when $F' < 0$, partners exert the efficient level of observable effort $\overline{o} > 0$, which drives relational capital down. When relational capital is low and $F' > 0$, partners exert the most unproductive effort $\underline{o} < 0$. This "conspicuous toiling" is best viewed as an investment in relational capital, which moves up quickly in response.[25]

Selling partnership. An alternative interpretation of the unraveling partnership is that partners sell it. If the venture's market value is the value of its fundamentals, partners part with it when relational capital and, thus, their value added dries up. Realistically, the market value of a venture may well exceed its fundamental value. In this case, if partners cannot commit not to sell the partnership, the scope for incentives diminishes. For example, when the market offers a fixed markup above the fundamental value, the relational capital may not decrease below this level. Formally, the left boundary condition in the equation (13) changes. When the market's markup is a fixed fraction of the fundamental value, the boundary condition includes both the fundamental value and relational capital. This requires adding the fundamental value as the second state variable in equation (13).[26]

Direct incentives. In our model, the effect of effort on fundamentals does not depend on the level of fundamentals. Allowing for such interdependence would affect the incentives, much like in dynamic decision problems or in Markov equilibria. For example,

[25] Similar investment in the value of a partnership has been documented in the equilibrum setting by Fujiwara-Greve and Okuno-Fujiwara [2009] and verified in the lab setting by Lee [2018].

[26] When there is no production noise, and so fundamentals change deterministically, the new differential equation does not have additional second-order derivatives.

when marginal productivity of effort is decreasing in fundamentals, incentives are dampened, since higher effort today makes effort less productive tomorrow. In the opposite case, one exerts extra effort to invest in a better production technology tomorrow. Formally, as with the selling of the partnership, equation (13) needs to include fundamentals as the additional state variable. How the introduction of such direct incentives affects the equilibrium behavior is a subject for future research.

6 Appendix: Proofs

6.1 Proofs of Propositions 1, 2 and 3.

Proof of Proposition 1. Fix a SSPPE and let $\overline{w}$ be the maximal relational capital it achieves. As in the proof of Proposition 2 below, it follows from the Martingale Representation Theorem (Karatzas [1991], Sannikov [2008]) that the relational capital W_t follows a diffusion process:

$$dw_t = (rw_t - (a_t - c(a_t)))\,dt + I_t \times (dY_t - \mu_t dt) + M_t, \tag{20}$$

where I_t is a nonnegative adaptive process, a_t is the effort taken by each agent and M_t is a martingale orthogonal to Y_t. Given the process for μ_t and that the continuation value $W_t = w_t + \frac{\mu_t}{2(r+\alpha)}$, it follows that the marginal benefit of effort at time t equals $I_t + \frac{1}{2}$ (see Sannikov [2008] or Bohren [2016]). Since $\overline{w}$ is maximal, in the event when $w_t = \overline{w}$ we must have $I_t = 0$, and so $a_t = a_{NE}$. But then it follows from (20) that $\overline{w} \leq w_{NE}$.

Proof of Proposition 2. Fix a strategy profile $\{a_t, a_t\}$ and let $\{w_t\}$ be the process of relational capital defined by (8). Since the process $dw_t - (rw_t - (a_t - c(a_t)))\,dt$ is a martingale, it follows from the Martingale Representation Theorem that the relational capital process can be represented as in (9) for some L^2 processes $\{I_t\}$ and a martingale $\{M_t\}$ orthogonal to $\{Y_t\}$.

For any $\varepsilon \geq 0$ let us define the process $\{V_t\}_{t \geq 0}$

$$\begin{aligned}
V_\tau(\widetilde{w}_\tau, \varepsilon_\tau) &= \mathbb{E}^{\{a_t, a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\,(a_t - c(a_t))\,dt \,\middle|\, \{Y_t\}_{t\in[0,\tau)}\right],\\
d\widetilde{w}_t &= (r\widetilde{w}_t - (a_t - c(a_t)))\,dt + I_t \times (dY_t - \overline{\mu}_t dt + \varepsilon_t dt)\,,\\
d\varepsilon_t &= -(\alpha+\gamma)\varepsilon_t dt,\\
\widetilde{w}_0 &= w_0, \quad \varepsilon_0 = \varepsilon.
\end{aligned}$$

V_τ is the process of relational capital, when the agents stick to the original strategies, but the initial stock of fundamentals is greater by ε than the one believed by the agents,

$\overline{\mu}_t$. In particular, it follows from the Kalman formula (7) that this wedge is mean reverting with parameter $\alpha+\gamma$. Let $\{\lambda_t^w\}_{t\geq 0}$ and $\{\lambda_t^\varepsilon\}_{t\geq 0}$ be the processes of the adjoint (shadow) variables with respect to V_t, evaluated at $\varepsilon=0$. They follow the following stochastic differential equations for $flow_t = a_t - c(a_t)$,

$$\begin{bmatrix} d\lambda_t^w \\ d\lambda_t^\varepsilon \end{bmatrix} = -\begin{bmatrix} r - flow_{t,w} & 0 \\ I_t & -(\alpha+\gamma) \end{bmatrix}\begin{bmatrix} \lambda_t^w \\ \lambda_t^\varepsilon \end{bmatrix} dt - \begin{bmatrix} e^{-rt} flow_{t,w} \\ 0 \end{bmatrix} dt + \begin{bmatrix} \phi_t^w \\ \phi_t^\varepsilon \end{bmatrix} (dY_t - \overline{\mu}_t dt) + \begin{bmatrix} dN_t^w \\ dN_t^\varepsilon \end{bmatrix},$$

for appropriate L^2 processes ϕ_t^w and ϕ_t^ε and martingales $\{N_t^w\}$ and $\{N_t^\varepsilon\}$ orthogonal to $\{Y_t\}$. As w_t is the relational capital, we can verify that indeed $\lambda_t^w = e^{-rt}$ solves the first differential equation, with $\phi_t^w = N_t^w = 0$. The second equation implies that for $F_t = (r+\alpha)e^{rt}\lambda_t^\varepsilon$

$$\begin{aligned} dF_t &= (r+\alpha)\left[re^{rt}\lambda_t^\varepsilon dt - e^{rt}\left(I_t\lambda_t^w - (\alpha+\gamma)\lambda_t^\varepsilon\right)dt + e^{rt}\phi_t^\varepsilon(dY_t - \overline{\mu}_t dt) + e^{rt} dN_t^\varepsilon\right] \\ &= (r+\alpha+\gamma)\,F_t - (r+\alpha)\,I_t dt + (r+\alpha)\left[e^{rt}\phi_t^\varepsilon(dY_t - \overline{\mu}_t dt) + e^{rt}dN_t^\varepsilon\right]. \end{aligned}$$

This agrees with the second differential equation in (9), with $J_t = (\alpha+r)e^{rt}\phi_t^\varepsilon$ and $dM_t^F = (\alpha+r)e^{rt}dN_t^\varepsilon$.

This establishes that $\{w_t\}_{t\geq 0}$ and $\{F_t\}_{t\geq 0}$ are processes of relational capital and adjoint variable of fundamentals with respect to relational capital (at present value and scaled by a constant $\alpha+r$), if and only if they can be represented as in (9). Finally, since effort increases the fundamentals by a factor of $\alpha+r$, and from the decomposition of the net continuation value into the internal and relational capital as in (8), the effort process is a local SSPPE exactly when $c'(a_t) = F_t + 1/2$.

Proof of Proposition 3. It follows from the Ito formula that before τ the process $F_t = F(w_t)$, where $\{w_t\}_{t\in[0,\tau)}$ is the continuation value process defined in (9) with $w_0 \in [\underline{w}, \overline{w}]$, $I_t = I(w_t)$ and $J_t = F'(w_t)\times I(w_t)$ satisfies the differential equation in (9), where τ is the stopping time of reaching a boundary point that does not satisfy (12). Since $w_t \in [w_{NE}, \overline{w}]$ and $F_t \in \{F(w) | w \in [w_{NE}, \overline{w}]\}$ the transversality conditions are satisfied.

We may extend the processes $\{w_t\}, \{I_t\}$ and $\{F_t\}$ beyond τ by letting them follow the paths of a SSPPE that achieves value w_τ. Then the processes defined until time ∞ satisfy the conditions of Proposition 2.

6.2 Proof of Theorem 1.

Let us define $\mathcal{E}$ to be the set of the pairs of relational capital and stocks of incentives, $(w, F(w))$, achievable in local SSPPEE, and let $E : \mathbb{R} \to \mathbb{R}$ parametrize the upper boundary of this set, as a function of w.

The proof of the Theorem follows in several steps. Lemma 1 below establishes convexity and the bounds on the set $\mathcal{E}$. Lemma 3 establishes the novel "local" escape argument, which is the key step of the proof. It is a version of the escape arguments used in the stochastic control verification theorems, adapted to our setting, in which the law of motion of a state variable w_t depends also on the value of the problem, $F(w_t)$.

Using those results, together with Proposition 3, Proposition 9 shows that E is differentiable at any interior w, whereas Propositions 10 and 11 show that at relational capital w in the right neighborhood of w_{NE}, E locally satisfies the differential equation (13) from the Theorem. Proposition 12 then shows that the solution of the equation can be extended to the right, all the way till w^*, the highest relational capital achievable in a local SSPPE.

Lemma 1 *The set $\mathcal{E}$ is convex. Moreover $\mathcal{E}$ is included in the upper envelope of $\underline{F}$:*

$$(w, F) \in \mathcal{E} \text{ implies } w \leq w_{EF} \text{ and } F \geq \underline{F}(w).$$

Proof. Convexity is immediate from the possibility of public randomization. The inequality $w \leq w_{EF}$ follows from the definitions of the relational capital in (8) and w_{EF} in (3). Finally, when $F < \underline{F}(w)$ then the process of relational capital with $w_t = w$ has a strictly positive drift (see (9)). Together with strict convexity of $\underline{F}$ and concavity of E this establishes that with strictly positive probability the stock of incentives process $\{F(w_t)\}$ remains bounded away below F, and the process of relational capital escapes above w_{EF}, establishing the contradiction. ■

Lemma 2 *If* $(1, E')$ *is a tangent vector at* $(w_0, E(w_0))$ *then in must be that*

$$(r + \alpha)E(w_0) \geq E' \times (rw_0 - (a(E(w_0)) - c(a(E(w_0))))) . \tag{21}$$

Proof. Suppose that (21) is violated, and $rw_0 - (a(E(w_0)) - c(a(E(w_0)))) < 0$ (when the inequality is reversed the proof is analogous). Let $\overline{E}' > E'$ be such that inequality (21) is violated, with $\overline{E}'$ in place of E'. Consider a function F defined over $[w_0, w']$, where w' is in the right neighborhood of w_0, such that

$$F(w) = E(w_0) + \overline{E}' \times (w - w_0) .$$

The function satisfies the conditions of Proposition, together with $I \equiv 0$, and so there are local SSPPE that achieve it. However, the function lies strictly above the boundary E, yielding the contradiction. ■

For $\varepsilon > 0$ consider a differential equation related to the one in Theorem 1, but with an extra "slack" of ε,

$$(r + \alpha + \gamma)F_\varepsilon(w) = F'_\varepsilon(w)\,(rw - (a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))) - \frac{(r+\alpha)^2}{2\sigma_Y^2 F''_\varepsilon(w)} + \varepsilon, \tag{22}$$

where function a is defined, as before, by $a(F) := c'^{-1}(F + 1/2)$. Intuitively, for given initial conditions $(w, F_\varepsilon, F'_\varepsilon)$ the solution F_ε of (22), is more concave (or, when $F''_\varepsilon < 0$, has more curvature) than the solution of (13).

Lemma 3 *Let* F_ε *be a concave solution of the differential equation (22) on an interval* $[\underline{w}, \overline{w}]$*, with* $|F'_\varepsilon| \leq M$*. Then there is* $\delta(\varepsilon, M) > 0$ *such the following conditions may not be satisfied:*

i) $F_\varepsilon(\underline{w}) = E(\underline{w})$ *and* $F_\varepsilon(\overline{w}) = E(\overline{w})$*,*

ii) $0 < E(w) - F_\varepsilon(w) \leq \delta(\varepsilon, M)$*. for* $w \in (\underline{w}, \overline{w})$

Roughly speaking, a standard escape argument would have F_ε solve the original differential equation (13) and no upper bound on the distance from $E(w)$ to $F_\varepsilon(w)$ in condition ii). The idea is that the only way to generate a value strictly above the solution of the HJB equation, which captures the maximum given the state w_t and shadow values F' and F'', is for the value to drift higher. This (together with no escape via the endpoints, guaranteed by i)) establishes that the value must be able to grow without bound, with positive probability, establishing the contradiction. In our case, however, the law of motion of w_t also depends on the value F. This implies that, given the state w_t and shadow values F' and F'', the differential equation captures only a *local* maximum, for a given value of F. The idea of the modified result is to show that with F in the vicinity of the HJB equation, the law of motion of w_t does not change much. Thus, in the vicinity of F the value must be close to the solution of the original differential equation (13), and so below the solution of the differential equation (22) with an appropriate slack. This implies that the value of the problem that starts above but sufficiently close to F_ε must drift up, out of the neighborhood (violating the second inequality in ii)).

Proof. Fix $(w_0, E(w_0))$ with $w_0 \in (\underline{w}, \overline{w})$ together with a local SSPPE achieving it, and let $\{w_t\}$ and $\{F_t\}$ be the processes of relational capital and stocks of incentives it gives rise to. Define $D(w_t, F_t)$ as $F_t's$ distance from the solution F_ε of the differential equation (22),

$$D(w_t, F_t) = F_t - F_\varepsilon(w_t).$$

Using Ito's lemma together with the Proposition 2, the drift of the process $D(w_t, F_t)$ equals, for appropriate process $\{I_t\}$,

$$\begin{aligned}\frac{\mathbb{E}\left[dD(w_t, F_t)\right]}{dt} &= (r+\alpha+\gamma)\,F_t - (r+\alpha)I_t - F_\varepsilon'(w_t)\times(rw_t - (a(F_t) - c(a(F_t)))) \quad (23)\\ &- \frac{F_\varepsilon''(w)\left[\sigma_Y^2 I_t^2 + d\left\langle M_t^w\right\rangle\right]}{2}\\ &\geq (r+\alpha+\gamma)\,F_t - (r+\alpha)I_t - F_\varepsilon'(w_t)\times(rw_t - (a(F_\varepsilon(w_t)) - c(a(F_\varepsilon(w_t)))))\\ &- \frac{F_\varepsilon''(w)\left[\sigma_Y^2 I_t^2 + d\left\langle M_t^w\right\rangle\right]}{2} - \frac{\varepsilon}{2}\\ &\geq (r+\alpha+\gamma)\,(F_t - F_\varepsilon(w_t)) + \varepsilon - \frac{\varepsilon}{2} > (r+\alpha+\gamma)\times D(w_t, F_t),\end{aligned}$$

The first inequality holds because $|F'_\varepsilon(w_t)| \leq M$, functions a and c are Lipschitz continuous and $|F_t - F_\varepsilon(w_t)| \leq \delta(\varepsilon, M)$, which is assumed to be sufficiently small. The second inequality follows because F_ε satisfies

$$(r+\alpha+\gamma)F_\varepsilon(w) = \max_I \left\{ (r+\alpha)I + F'(w)\left(rw - (a(F(w)) - c(a(F(w))))\right) + \frac{F''(w)\sigma_Y^2}{2} I^2 \right\} + \varepsilon.$$

Since $D(w_0, F_0) > 0$, inequality (23) shows that with positive probability $D(w_t, F_t)$ remains strictly positive and in finite time exceeds $\delta(\varepsilon, M)$. The contradiction establishes the proof. ■

Proposition 9 *The upper boundary E of the set of relational capital and stocks of incentives achievable in a local SSPPE is differentiable everywhere.*

Proof. Suppose to the contrary that $(w_0, E(w_0))$ is a kink. If follows from Lemma there exists an interior tangent vector $(1, E')$ at $(w_0, E(w_0))$ such that

$$(r+\alpha)E(w_0) > E' \times (rw_0 - (a(E(w_0)) - c(a(E(w_0))))) .$$

In this case the differential equation (13), written as $F''(w) = \mathcal{F}(w, F, F')$, has the right hand side Lipschitz continuous in the neighborhood of the point $(w_0, E(w_0), E')$, with $F'' < 0$. Continuous dependence on the initial parameters implies that there exists $\varepsilon > 0$ such that F^*_ε solving (22) with the same initial conditions is strictly above curve E in a neighborhood of w_0 (excluding point w_0). Invoking the continuous dependence once again, this time shifting the initial condition $(w_0, E(w_0), E')$ down to $(w_0, E(w_0) - \delta, E')$, for $0 < \delta << \varepsilon$, we construct a function F_ε that satisfies the conditions of Lemma 3, yielding the contradiction. ■

Let us distinguish points on the boundary E at which the solution to the differential equation (13) would require F'' infinite. Specifically, we will say that $(w, E, E') \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ is *nondegenerate* if

$$(r+\alpha)E \neq \frac{r+\alpha}{2} + E' \times \left(rw - \left(\frac{a(E)}{\alpha + r} - c(a(E)) \right) \right) . \qquad (24)$$

Proposition 10 *Suppose* $(w_0, E(w_0), E'(w_0))$ *is nondegenerate. Then the solution* F *to the differential equation (13) with this initial condition is weakly above curve* E *in the neighborhood of* w_0.

Proof. Suppose to the contrary that $F < E$ in, say, the right neighborhood of w_0 (the case of the left neighborhood is analogous). Given that $(w_0, E(w_0), E'(w_0))$ is nondegenerate, and so the solution F is continuous in the initial conditions in the neighborhood of $(w_0, E(w_0), E'(w_0))$, there exists $\overline{E}' > E'(w_0)$ such that the solution $\overline{F}$ of (13) with initial conditions $(w_0, E(w_0), \overline{E}')$ "comes back to " E, meaning $\overline{F}(\overline{w}) = E(\overline{w})$ for some $\overline{w} > w_0$. But then the function $\overline{F}$ defined on $[w_0, \overline{w}]$ satisfies the conditions of Proposition 3, and so its graph is achievable by local SSPPE. However, since $\overline{E}' > E'(w_0)$ it follows that $\overline{F}$ is strictly above E in the right neighborhood of w_0, yielding the contradiction. ■

For the next Proposition we will need the following technical Lemma.

Lemma 4 *Let* $F, E : [\underline{w}, \overline{w}) \to \mathbb{R}$ *be two concave functions such that*

i) $E \leq F$,

ii) $E(\underline{w}) = F(\underline{w})$ *and* $E'_+(\underline{w}) = F'_+(\underline{w})$,

iii) $F''_+(\underline{w})$ *exists*

Then either $E''_+(\underline{w})$ *exists and equals* $F''_+(\underline{w})$ *or there is* G *with* $G(\underline{w}) = E(\underline{w})$, $G'_+(\underline{w}) = E'_+(\underline{w})$ *and* $G''_+(\underline{w}) < F''_+(\underline{w})$ *such that* $E \leq G$ *in a right neighborhood of* $\underline{w}$.

Proof. Suppose that $E''_+(\underline{w})$ does not exist or is not equal to $F''_+(\underline{w})$. From i), this means that there is a $\varepsilon > 0$ and a decreasing sequence $\{w_n\} \to \underline{w}$ such that

$$E(w_n) \leq F(\underline{w}) + F'_+(\underline{w}) \times (w_n - \underline{w}) + \left(F''_+(0) - \varepsilon\right) \times (w_n - \underline{w})^2 .$$

However, concavity of E implies that the above inequality holds not only for the sequence $\{w_n\}$ but in a right neighborhood of $\underline{w}$. This implies the result, with $G(w) = F(w) - \varepsilon(w - \underline{w})^2$ in a neighborhood of $\underline{w}$. ■

Proposition 11 *Suppose* $(w_0, E(w_0), E'(w_0))$ *is nondegenerate. Then* $E''(w_0)$ *exists and* E *satisfies the differential equation (13) at* w_0.

Proof. Let F satisfy (13) with initial conditions $(w_0, E(w_0), E'(w_0))$ and suppose that $E''_+(w_0) \neq F'_+(w_0)$ (the case of left second derivative is analogous). Propositions 9 and 10 establish that the conditions of Lemma 4 are satisfied with w_0, and so in the right neighborhood of w E is bounded above by $F(w) - \bar{\varepsilon}(w - w_0)^2$, for appropriate $\bar{\varepsilon} > 0$. Continuous dependence on the initial parameters implies that there exists $\varepsilon > 0$ such that F^*_ε solving (22) with the same initial conditions $(w_0, E(w_0), E'(w_0))$ as F is strictly above curve E in a right neighborhood of w_0 (excluding point w_0). Invoking the continuous dependence once again, this time turning the initial condition $(w_0, E(w_0), E'(w_0))$ right to $(w_0, E(w_0), E'(w_0) - \delta)$, for $0 < \delta << \varepsilon$, we construct a function F_ε that satisfies the conditions of Lemma 3, yielding the contradiction. ■

Together with Proposition 11, the following proposition establishes the proof of Theorem 1.

Proposition 12 *For every* $w_0 \in (w_{NE}, w^*)$ *the point* $(w_0, E(w_0), E'(w_0))$ *is nondegenerate.*

Proof. From Lemma 1, E lies above $\underline{F}$, defined in (19), and $w^* \leq w_{EF}$. Since $\underline{F}'(w_{NE}) > 0$, it follows that in the right neighborhood of w_{NE}, when the drift of relational capital is weakly negative and $E'(w_0) \geq 0$, the points $(w_0, E(w_0), E'(w_0))$ are nondegenerate, and so the boundary E solves the differential equation (13). Let us show that the results extends to all $w_0 < w^*$.

Note that if $E(w_0)$ equals either $\underline{F}(w_0)$ or $\overline{F}(w_0)$ then it may not be degenerate. We are thus left with two cases.

Case 1). A degenerate point $(w_0, E(w_0), E'(w_0))$ satisfies $E(w_0) \in \left(\underline{F}(w_0), \overline{F}(w_0)\right)$.

In this case $rw_0 - (a(E(w_0)) - c(a(E(w_0)))) < 0$ and so $E'(w_0) < 0$. We argue that the rate of change of E' in the right neighborhood of w_0 is infinite, which will contradict Lemma 2.

Otherwise, a quadratic function G^* such that $G^*(w_0) = E(w_0)$, $G^{*\prime}(w_0) = E'(w_0)$ and $G^{*\prime\prime} < 0$ "cuts inside" to the right of E. But then, by increasing slightly $G^{*\prime}(w_0)$, we may construct a quadratic function G over an inteval $[w_0, \overline{w}]$ such that $G(w_0) = E(w_0)$, $G'(w_0) > E'(w_0)$, $G(\overline{w}) = E(\overline{w})$ and

$$(r + \alpha + \gamma)\, G\,(w) < G'(w)\,(rw - (a(G(w)) - c(a(G(w))))) - \frac{(r+\alpha)^2}{2\sigma_Y^2 G''(w)}.\ \ w \in [w_0, \overline{w}]$$

There exists then a function $I : [w_0, \overline{w}] \to \mathbb{R}$, with $I(w) > -\frac{(r+\alpha)^2}{\sigma_Y^2 G''(w)}$, such that

$$(r + \alpha + \gamma)\, G\,(w) = I(w) + G'(w)\,(rw - (a(G(w)) - c(a(G(w))))) + \frac{G''(w)\sigma_Y^2}{2} I^2.\ \ w \in [w_0, \overline{w}]$$

Applying Proposition 2, each point $(w, G(w))$, for $w \in [w_0, \overline{w}]$, can be achieved by a local SSPPE. Since $G'(w_0) > E'(w_0)$, this yields the desired contradiction.

Case 2) A degenerate point $(w_0, E(w_0), E'(w_0))$ satisfies $E(w_0) > \overline{F}(w_0)$.

$$\begin{aligned}(r+\alpha+\gamma)F(w) &= \max_I \left\{(r+\alpha)I + F'(w)\,(rw - (a(F(w)) - c(a(F(w))))) + \frac{F''(w)\sigma_Y^2}{2} I^2\right\} \\ &= F'(w)\,(rw - (a(F(w)) - c(a(F(w))))) - \frac{(r+\alpha)^2}{2\sigma_Y^2 F''(w)},\end{aligned}$$

The differential equation (13) can be rewritten as

$$\frac{1}{F''(w)} = \frac{2\sigma_Y^2}{(r+\alpha)^2} \times \left[F'(w)\,(rw - (a(F(w)) - c(a(F(w))))) - (\alpha + r + \gamma)F\right].$$

We have

$$\frac{d}{dw}\frac{1}{F''(w)} = \frac{2\sigma_Y^2}{(r+\alpha)^2} \times \left[\begin{array}{c} F''(w) \times (rw - (a(F(w)) - c(a(F(w))))) + \\ F'(w)\,(r - (1 - c'(a(F(w)))))\, F'(w) a'(F(w)) - (r+\alpha+\gamma)F' \end{array}\right].$$

Since in the left neighborhood of w^* we have $F'(w) \in (0, F'(w^x))$, where w^x is such that $F(w^x) = \overline{F}(w^x)$, and also $(rw - (a(F(w)) - c(a(F(w))))) \geq d$ and $(r - (1 - c'(a(F(w)))))\, a'(F(w)$

D for appropriate $d, D > 0$, then we get

$$\frac{d}{dw}\frac{1}{F''(w)} \leq \frac{2\sigma_Y^2}{(r+\alpha)^2} \times \left[F''(w) \times d + D \times \overline{F}(w^x)^2\right].$$

This implies that $\frac{1}{F''(w)}$ must remain bounded away below zero to the left of w^*, yielding the contradiction. ■

Lemma 5 *As w approximates the upper bound w^* then stock of incentives in any SSPPE approximates the boundary of the feasible set,*

$$\lim_{w \uparrow w^*} E(w) = \underline{F}(w^*), \tag{25}$$

and the second equation (14) in Theorem 1 is satisfied.

Proof. Regarding (25), from Lemma 1 there are only three other possibilities:

Case 1). $\lim_{w\uparrow w^*} E(w) \in \left(\underline{F}(w_0), \overline{F}(w_0)\right)$. In this case, using Proposition 2 it would be possible to extend the solution to the right, with $I(w) = 0$ for $w > w^*$, contradiction.

Case 2) $\lim_{w\uparrow w^*} E(w) = \overline{F}(w_0)$. Whether E approaches $\overline{F}$ from above or below, the differential equation (13) would be violated in the left neighborhood of w_*.

Case 3) $\lim_{w\uparrow w^*} E(w) > \overline{F}(w_0)$. In this case any local SSPPE achieving points close to $(w^*, \lim_{w\uparrow w^*} E(w))$ has a relational capital with a strictly positive drift, bounded away from zero. This would lead to the escape of w to the right of w^*, with positive probability.

If the equation (14) was violated, then $\lim_{w\uparrow w^*} E''(w) > -\infty$ and so $\lim_{w\uparrow w^*} E'(w) > -\infty$. This however would imply that $(w^*, E(w^*))$ is achievable in a SSPPE (by extending it as in Case 1), contradicting $I(w^*) = 0$ and $E(w^*) > 0$. ■

6.3 Proof of Proposition 4

An analogue to Theorem 1 establishes existence of a concave C^2 function $\widetilde{F}_\varepsilon$ on $[0, w_\varepsilon^*)$ that satisfies

$$(r+\alpha+\gamma)\widetilde{F}_\varepsilon(w) = \max_{I=0 \text{ or } I\geq\varepsilon}\left\{(r+\alpha)I + \widetilde{F}_\varepsilon'(w)\left(rw - \left(a(\widetilde{F}_\varepsilon(w)) - c(a(\widetilde{F}_\varepsilon(w)))\right)\right) + \frac{\widetilde{F}_\varepsilon''(w)\sigma_Y^2}{2}I^2\right\} \tag{26}$$

$$= \widetilde{F}_\varepsilon'(w)\left(rw - \left(a(\widetilde{F}_\varepsilon(w)) - c(a(\widetilde{F}_\varepsilon(w)))\right)\right)$$
$$- \frac{(r+\alpha)^2}{2\sigma_Y^2\widetilde{F}_\varepsilon''(w)}\mathbf{1}_{\widetilde{F}_\varepsilon''(w)\geq -\frac{r+\alpha}{\sigma_Y^2\varepsilon}} + \varepsilon\left(r+\alpha+\varepsilon\frac{\widetilde{F}_\varepsilon''(w)\sigma_Y^2}{2}\right)\mathbf{1}_{\widetilde{F}_\varepsilon''(w)\in[-2\frac{r+\alpha}{\sigma_Y^2\varepsilon}, -\frac{r+\alpha}{\sigma_Y^2\varepsilon}]},$$

together with

$$rw_\varepsilon^* - \left(a(\widetilde{F}_\varepsilon(w_\varepsilon^*)) - c(a(\widetilde{F}_\varepsilon(w_\varepsilon^*)))\right) = 0. \tag{27}$$

The only difference with (16) is that for $\widetilde{F}_\varepsilon'' < -\frac{2}{\varepsilon}$ the last term in equation disappears (with $I^*(w)$ set to zero). It is easy to establish that $F_\varepsilon''(0) = -\frac{2}{\varepsilon}$, since with any other value, the equation (26) would be violated around zero. In what follows we establish that if $\widetilde{F}_\varepsilon''(w^0) = -\frac{2}{\varepsilon}$, for some $w^0 \in [0, w_\varepsilon^*)$, then $\widetilde{F}_\varepsilon'(w^0) << 0$ and $rw^0 - \left(a(\widetilde{F}_\varepsilon(w^0)) - c(a(\widetilde{F}_\varepsilon(w^0)))\right) \approx 0$. Given concavity of $\widetilde{F}_\varepsilon$, $w^0 - w_\varepsilon^*$ is small, and so this will establish the proof for F_ε that equals $\widetilde{F}_\varepsilon$ restricted to $[0, \overline{w}_\varepsilon]$, where $\overline{w}_\varepsilon$ is the first point at which $\widetilde{F}_\varepsilon''$ equals $-\frac{2}{\varepsilon}$.

Consider w^0 such that $\widetilde{F}_\varepsilon'' \leq -\frac{2}{\varepsilon}$ in a neighborhood of w^0. Differentiating (26), we get

$$\widetilde{F}_\varepsilon''(w) = \frac{\widetilde{F}_\varepsilon'(w)\left(\alpha+\gamma+\left(a(\widetilde{F}_\varepsilon(w)) - c(a(\widetilde{F}_\varepsilon(w)))\right)'\right)}{rw - a(\widetilde{F}_\varepsilon(w)) - c(a(\widetilde{F}_\varepsilon(w)))}, \tag{28}$$

where

$$\left(a(\widetilde{F}_\varepsilon(w)) - c(a(\widetilde{F}_\varepsilon(w)))\right)' = \left(1 - c'(a(\widetilde{F}_\varepsilon(w)))\right)a'(\widetilde{F}_\varepsilon(w))\widetilde{F}_\varepsilon'(w)$$
$$= \frac{1}{C}\left(1/2 - \widetilde{F}_\varepsilon(w)\right)\widetilde{F}_\varepsilon'(w).$$

First, suppose that $\widetilde{F}_\varepsilon'(w^0) < 0$. Since $\widetilde{F}_\varepsilon$ satisfies (28), with $\widetilde{F}_\varepsilon''(w), \widetilde{F}_\varepsilon'(w)$ and denominator negative, it follows that $\widetilde{F}_\varepsilon(w) \leq 1/2$. Since it also satisfies (26), with second derivative terms absent, it follows that

$$\widetilde{F}'_\varepsilon(w^0)\left(rw^0 - \left(a(\widetilde{F}_\varepsilon(w^0)) - c(a(\widetilde{F}_\varepsilon(w^0)))\right)\right) \le \frac{r+\alpha+\gamma}{2},$$
$$\frac{\widetilde{F}'^2_\varepsilon(w^0)}{rw^0 - a(\widetilde{F}_\varepsilon(w^0)) - c(a(\widetilde{F}_\varepsilon(w^0)))} \ge -\frac{4C}{\varepsilon}.$$

The two inequalities imply that as ε converges to zero, then $rw^0 - a(\widetilde{F}_\varepsilon(w^0)) - c(a(\widetilde{F}_\varepsilon(w^0)))$ converges to zero and $\widetilde{F}'_\varepsilon(w^0)$ to negative infinity. Specifically, fix $\varepsilon > 0$ and let $D > 0$ be such that $\frac{d}{dx}\left(rw - a(\widetilde{F}_\varepsilon(w)) - c(a(\widetilde{F}_\varepsilon(w)))\right) \ge D|\widetilde{F}'_\varepsilon(w)|$, for $w \ge w^0$. The two inequalities imply that

$$\left(\frac{\widetilde{F}'_\varepsilon(w^0)}{rw^0 - a(\widetilde{F}_\varepsilon(w^0)) - c(a(\widetilde{F}_\varepsilon(w^0)))}\right)^3 \ge \left(\frac{4C}{\varepsilon}\right)^2 \frac{2}{r+\alpha+\gamma},$$

and, using (27),

$$\begin{aligned} w^*_\varepsilon - w^0 &\le \frac{1}{D}\frac{rw^0 - a(\widetilde{F}_\varepsilon(w^0)) - c(a(\widetilde{F}_\varepsilon(w^0)))}{\widetilde{F}'_\varepsilon(w^0)} \\ &\le \frac{1}{D}\left(\frac{r+\alpha+\gamma}{2}\right)^{1/3}\left(\frac{\varepsilon}{4C}\right)^{2/3}. \end{aligned}$$

Let us now show that the case $\widetilde{F}'_\varepsilon(w^0) \ge 0$ is not possible. In the neighborhood in which $\widetilde{F}''_\varepsilon \le -\frac{2}{\varepsilon}$, we have, by differentiating (28),

$$\widetilde{F}'''_\varepsilon(w) = \left(\frac{\widetilde{F}'_\varepsilon(w)}{rw - a(\widetilde{F}_\varepsilon(w)) - c(a(\widetilde{F}_\varepsilon(w)))}\right)'\left(\alpha+\gamma+\left(a(\widetilde{F}_\varepsilon(w)) - c(a(\widetilde{F}_\varepsilon(w)))\right)'\right) \tag{29}$$
$$\begin{aligned} &+ \frac{\widetilde{F}'_\varepsilon(w)}{rw - a(\widetilde{F}_\varepsilon(w)) - c(a(\widetilde{F}_\varepsilon(w)))}\left(a(\widetilde{F}_\varepsilon(w)) - c(a(\widetilde{F}_\varepsilon(w)))\right)'' \\ &> \frac{\widetilde{F}'_\varepsilon(w)}{rw - a(\widetilde{F}_\varepsilon(w)) - c(a(\widetilde{F}_\varepsilon(w)))}\left(a(\widetilde{F}_\varepsilon(w)) - c(a(\widetilde{F}_\varepsilon(w)))\right)'' \\ &=_{sgn} \left(a(\widetilde{F}_\varepsilon(w)) - c(a(\widetilde{F}_\varepsilon(w)))\right)'' \end{aligned}$$

where the inequality follows from the fact that we have

$$rw - a(\widetilde{F}_\varepsilon(w)) - c(a(\widetilde{F}_\varepsilon(w))), \left(rw - a(\widetilde{F}_\varepsilon(w)) - c(a(\widetilde{F}_\varepsilon(w)))\right)' > 0,$$

implied by (26), $\widetilde{F}''_\varepsilon(w) < 0$ and $\alpha + \gamma + \left(a(\widetilde{F}_\varepsilon(w)) - c(a(\widetilde{F}_\varepsilon(w)))\right)' < 0$. Moreover,

$$\begin{aligned}\left(a(\widetilde{F}_\varepsilon(w)) - c(a(\widetilde{F}_\varepsilon(w)))\right)'' &= \left(\frac{1}{C}\left(1/2 - \widetilde{F}_\varepsilon(w)\right)\widetilde{F}'_\varepsilon(w)\right)' \\ &=_{sgn} \left(\left(1/2 - \widetilde{F}_\varepsilon(w)\right)\widetilde{F}''_\varepsilon(w) - \left(\widetilde{F}'_\varepsilon(w)\right)^2\right).\end{aligned} \tag{30}$$

As in the previous step, it follows from (28) that $\widetilde{F}'_\varepsilon(w)$ is large and $rw - a(\widetilde{F}_\varepsilon(w)) - c(a(\widetilde{F}_\varepsilon(w)))$ is small, for ε small. From concavity of $\widetilde{F}_\varepsilon$, it follows that w is small, and so $\widetilde{F}_\varepsilon(w) \approx 1$, or $1/2 - \widetilde{F}_\varepsilon(w) \approx -1/2$. On the other hand, it also follows from (28) that

$$\widetilde{F}''_\varepsilon(w) \approx -\frac{1}{2C}\frac{\left(\widetilde{F}'_\varepsilon(w)\right)^2}{rw - a(\widetilde{F}_\varepsilon(w)) - c(a(\widetilde{F}_\varepsilon(w)))}.$$

This, together with (29) and (30) establishes that $\widetilde{F}''_\varepsilon(w^0) \leq -\frac{2}{\varepsilon}$ implies $\widetilde{F}'''_\varepsilon(w^0) > 0$, and so case $\widetilde{F}'_\varepsilon(w^0) \geq 0$ is not possible. This concludes the proof.

6.4 Proof of Theorem 2

Recall from Proposition 2 and the discussion below Proposition 3 that the stock of incentives process $\{F_t\} = \{F(w_t)\}$ follows the Ito diffusion

$$dF(w_t) = (r + \alpha + \gamma)\,F(w_t) - (r + \alpha)I(w_t)dt + J(w_t) \times (dY_t - \overline{\mu}_t dt)\,,$$

where

$$J(w) = F'(w) \times I(w).$$

Fix time τ, equilibrium level of relational capital w_τ and a wedge $\widetilde{\mu}_\tau - \overline{\mu}_\tau$ between the correct beliefs (after deviation) and the equilibrium beliefs.

Step 1. First, we show that as long as

$$J(w_\tau) \leq \frac{C\left(r+2\left(\alpha+\gamma\right)\right)}{8\left(r+\alpha\right)} =: C_0, \tag{31}$$

then for an appropriate $X > 0$ the relational capital (expected discounted payoffs net of the effect of inherited fundamentals) to the deviating agent from any deviation strategy are bounded from above by

$$\widetilde{w}_\tau(\widetilde{\mu}_\tau - \overline{\mu}_\tau, w_\tau) = w_\tau + \frac{F(w_\tau)}{r+\alpha}(\widetilde{\mu}_\tau - \overline{\mu}_\tau) + X(\widetilde{\mu}_\tau - \overline{\mu}_\tau)^2, \tag{32}$$

where w_τ is the equilibrum relational capital, for appropriate $X > 0$.

Fix a deviation strategy $\{\widetilde{a}_t\}$ and consider the process

$$v_\tau = \int_0^\tau e^{-rs}\left(\frac{\widetilde{a}_t + a_t}{2} - c(\widetilde{a}_t)\right)dt + e_\tau^{-r\tau}\widetilde{w}(\widetilde{\mu}_\tau - \overline{\mu}_\tau, w_\tau),$$

where, from (7), the wedge process $\{\widetilde{\mu}_t - \overline{\mu}_t\}$ follows

$$d\left(\widetilde{\mu}_t - \overline{\mu}_t\right) = \left(r+\alpha\right)(\widetilde{a}_t - a_t)dt - \left(\alpha+\gamma\right)\left(\widetilde{\mu}_t - \overline{\mu}_t\right)dt.$$

In order to establish the bound (32) it is enough to show that the process $\{v_t\}$ has negative drift. We have$\sim$

$$\begin{aligned}
e^{-rt}dv_t = &\left(\frac{\widetilde{a}_t + a_t}{2} - c(\widetilde{a}_t)\right)dt - r\left(w_t + \frac{F(w_t)}{r+\alpha}(\widetilde{\mu}_t - \overline{\mu}_t) + X(\widetilde{\mu}_t - \overline{\mu}_t)^2\right) \\
&+ (rW_t - (a_t + c(a_t)))dt + I(w_t) \times (dY_t - \overline{\mu}_t dt) \\
&+ \frac{\widetilde{\mu}_t - \overline{\mu}_t}{r+\alpha}\left((r+\alpha+\gamma)\,F_t(w_t) - (r+\alpha)I(w_t)dt + J(w_t) \times (dY_t - \overline{\mu}_t dt)\right) \\
&+ \left(\frac{F(w_t)}{r+\alpha} + 2X(\widetilde{\mu}_t - \overline{\mu}_t)\right)\left((r+\alpha)\,(\widetilde{a}_t - a_t)dt - (\alpha+\gamma)(\widetilde{\mu}_t - \overline{\mu}_t)dt\right).
\end{aligned}$$

Given that the drift of $dY_t - \overline{\mu}_t dt$ is $(\widetilde{\mu}_t - \overline{\mu}_t)dt$,the drift of the $e^{-rt}dv_t$ process equals

$$\begin{aligned}
&\frac{\widetilde{a}_t - a_t}{2} + c(a_t) - c(\widetilde{a}_t) + F(w_t)(\widetilde{a}_t - a_t) \\
&+ (\widetilde{\mu}_t - \overline{\mu}_t)^2 \left(\frac{J(w_t)}{r+\alpha} - X\left(r + 2\left(\alpha+\gamma\right)\right)\right) + (\widetilde{\mu}_t - \overline{\mu}_t)(\widetilde{a}_t - a_t) 2X\left(r+\alpha\right) \\
&= -\frac{C}{2}\left(a_t - \widetilde{a}_t\right)^2 + (\widetilde{\mu}_t - \overline{\mu}_t)^2 \left(\frac{J(w_t)}{r+\alpha} - X\left(r + 2\left(\alpha+\gamma\right)\right)\right) \\
&+ (\widetilde{\mu}_t - \overline{\mu}_t)(\widetilde{a}_t - a_t) 2X\left(r+\alpha\right),
\end{aligned}$$

where we used that $c(a) = \frac{1}{2}a + \frac{C}{2}a^2$ and $F_t(w_t) = Ca$. Note that when the matrix

$$\begin{bmatrix} -\frac{C}{2} & X\left(r+\alpha\right) \\ X\left(r+\alpha\right) & \frac{J_t(w_t)}{r+\alpha} - X\left(r+2\left(\alpha+\gamma\right)\right) \end{bmatrix}$$

has a positive determinant, then the trace is negative, and the matrix is negative semidefinite, guaranteing negative drift.

Since

$$\begin{aligned}
&\max_X \left\{ -\frac{C}{2} \times \left(\frac{J_t(w_t)}{r+\alpha} - X\left(r+2\left(\alpha+\gamma\right)\right)\right) - X^2\left(r+\alpha\right)^2 \right\} \\
&= \frac{C}{2\left(r+\alpha\right)} \left(\frac{C\left(r+2\left(\alpha+\gamma\right)\right)}{8\left(r+\alpha\right)} - J_t(w_t)\right),
\end{aligned}$$

it follows that, indeed, when $J_t(w_t)$ is bounded as in (31), then the bound (32) holds for X that maximizes the above expression. Consequently, using the bound with $\widetilde{\mu}_t = \overline{\mu}_t$, it follows that the local SSPPE strategies are globally incentive compatible.

Step 2. In this step we show that when C is sufficiently large, then for any w_t the sensitivity $J(w_t)$ of the stock of incentives is bounded as in (31).

For the proof we will need the following Lemma.

Lemma 6 *If C is sufficiently high then the drift of relational capital is negative,*

$$rw - (a(F(w)) - c(a(F(w)))) \leq 0,$$

for $w \in [\underline{w}, \overline{w}]$.

Proof. Negative drift of relational capital is equivalent to $F(w) \leq \overline{F}(w)$, for $w \in [\underline{w}, \overline{w}]$, where $\overline{F}$ is the stock of incentives achievable in a static play, in which agents are overincentivized relative to the first best (see (19)).

Otherwise, for all $w \in [\underline{w}, \overline{w}]$ with $F(w) \geq \overline{F}(w)$ we have

$$-\frac{(r+\alpha)^2}{2\sigma_Y^2 F''(w)} > (r+\alpha+\gamma)F(w) > \frac{r+\alpha+\gamma}{2},$$

which provides a lower bound on $-F''(w)$. Therefore, since $\overline{w} \leq w_{EF} = \frac{1}{8rC}$, it follows that as long as

$$C \geq \frac{r+\alpha}{4r\sigma_Y\sqrt{r+\alpha+\gamma}} =: C_1, \tag{33}$$

then

$$F'(w) \geq -2rC = \overline{F}'(0) \geq \overline{F}'(w),$$

and consequently $F(w) \geq \overline{F}(w)$, for all $w \in [\underline{w}, \overline{w}]$ with $F(w) \geq \overline{F}(w)$ and $F'(w) \leq 0$. But then F violates the boundary condition (12), establishing contradiction. ■

In the rest of the proof we assume that (33) is satisfied.

Function F is strictly concave, and its derivatives are potentially infinitely steep at the extremes. Consider $A > 0$ (to be determined later) and let's establish a bound on $J^*(w_t)$,

$$J^*(w) = F'(w) \times I^*(w) = -\frac{(r+\alpha)F'(w)}{\sigma_Y^2 F''(w)},$$

in two cases, depending on whether $F'(w) \in (0, A)$, or $F'(w) \geq A$. (Note that $F'(w) \leq 0$ implies $J(w_t) \leq 0$, and the bound (31) holds).

Case 1. $F'(w) \in (0, A)$.

We have

$$J^*(w) = F'(w) \times I^*(w) = \frac{2F'(w)}{r+\alpha}\left(-\frac{(r+\alpha)^2}{2\sigma_Y^2 F''(w)}\right) \tag{34}$$
$$= \frac{2F'(w)}{r+\alpha}\left[(r+\alpha+\gamma)F(w) - F'(w)\left(rw - (a(F(w)) - c(a(F(w))))\right)\right]$$
$$\leq \frac{2A}{r+\alpha}\left[r+\alpha+\gamma+\frac{A}{8rC}\right] =: J_1,$$

where the last inequality uses Lemma 6, $\overline{F}(w) \leq \overline{F}(0) = 1$ and $rw - (a - c(a)) \geq -\frac{1}{8rC}$.

Case 2. $F'(w) \geq A$.

Let us first bound $F(w)$ from above in this range. For any $w' \geq w$ such that $F'(w') \geq 0$ we have

$$-\frac{(r+\alpha)^2}{2\sigma_Y^2 F''(w')} \geq (r+\alpha)I(w) + \frac{F''(w)}{2}\sigma_Y^2 I^2(w) > (r+\alpha+\gamma)F(w),$$

and so

$$F'(w') \geq A - (w'-w) \times \frac{(r+\alpha)^2}{2\sigma_Y^2(r+\alpha+\gamma)F(w)},$$
$$F(w') \geq F(w) + A \times (w'-w) - \frac{(w'-w)^2}{2} \times \frac{(r+\alpha)^2}{2\sigma_Y^2(r+\alpha+\gamma)F(w)}.$$

Since for w'' such that $F'(w'') = 0$ we must have $F(w'') \leq \overline{F}(w'') \leq \overline{F}(0) = 1$ (Lemma 6), indeed it follows that

$$F(w) \leq \frac{(r+\alpha)^2}{2\sigma_Y^2(r+\alpha+\gamma)A^2}. \tag{35}$$

Now, we have

$$J^{*\prime}(w) = \left(-\frac{(r+\alpha)F'(w)}{\sigma_Y^2 F''(w)}\right)' = -\frac{r+\alpha}{\sigma_Y^2} + \frac{2}{r+\alpha}F'(w) \times \left(\frac{(r+\alpha)^2}{-2\sigma_Y^2 F''(w)}\right)',$$

where, from (13),

$$
\begin{aligned}
&\left(\frac{(r+\alpha)^2}{-2\sigma_Y^2 F''(w)}\right)' && (36)\\
&= (\alpha+\gamma)F'(w) - F''(w)\left(rw - \left(a(F(w)) - c(a(F(w)))\right)\right) && (37)\\
&+ F'^2(w)\left(1 - c'(a(F(w)))\right)a'(F(w))\\
&\geq (\alpha+\gamma)F'(w) + \frac{F''(w)}{8C} + \frac{F'^2(w)}{C}\left(\frac{1}{2} - F(w)\right)\\
&\geq F'(w) \times \left(\alpha+\gamma+\frac{1}{C}\left(\frac{F'(w)}{2}\left(1 - \frac{(r+\alpha)^2}{\sigma_Y^2(r+\alpha+\gamma)A^2}\right) - \frac{r+\alpha}{8\sigma_Y^2}\frac{1}{J(w)}\right)\right)
\end{aligned}
$$

The first inequality follows from $a(F) = \frac{F}{C}$, $c'(a) = 1/2 + Ca$ and $rw - (a - c(a)) \geq -\frac{1}{8rC}$, whereas the last one follows from (35).

Let A_1 be such that the right hand side of (35) is 1/2, and let J_2 be the value of $J^*(w)$ so that the above derivative is zero, in the case when $F'(w) = A_1$. It follows that if for any w' with $F'(w') \geq A_1$ $J^*(w')$ is larger than $J_3 = J_2 + \frac{r+\alpha}{\sigma_Y^2} \times \frac{1}{8rC}$, then $J^{*\prime}(w) \geq -\frac{r+\alpha}{\sigma_Y^2}$ to the right of w', and so $J^*(w'') \geq J^*(w') - \frac{r+\alpha}{\sigma_Y^2} \times \frac{1}{8rC}$, at w'' with $F'(w'') = A_1$. This would violate case 1 as long as $J^*(w') \geq J_1 + \frac{r+\alpha}{\sigma_Y^2} \times \frac{1}{8rC} =: J_4$. Thus, we get the bound

$$
J(w) = F'(w)I(w) \leq (1+D)J^*(w) \leq (1+D)\max\{J_3, J_4\}, \tag{38}
$$

where D is as in (17).

Summarizing the proof, for $A_1 = \frac{r+\alpha}{\sigma_Y\sqrt{r+\alpha+\gamma}}$ and $J_2 = \frac{\sqrt{r+\alpha+\gamma}}{2\sigma_Y}$, the three bounds: $C \geq C_1$ (see (33)), $(1+D)J_3 \leq C_0$, and $(1+D)J_4 \leq C_0$ (see (31), and (38); note that $J_1 \leq J_4$) boil down to

$$
\begin{aligned}
C &\geq \frac{r+\alpha}{4r\sigma_Y\sqrt{r+\alpha+\gamma}}, && (39)\\
(1+D)\left(\frac{\sqrt{r+\alpha+\gamma}}{2\sigma_Y} + \frac{r+\alpha}{8rC\sigma_Y^2}\right) &\leq \frac{C\left(r+2\left(\alpha+\gamma\right)\right)}{8\left(r+\alpha\right)},\\
(1+D)\left(\frac{2\sqrt{r+\alpha+\gamma}}{\sigma_Y} + \frac{r+\alpha}{8rC\sigma_Y^2}\left[1+\frac{2}{r+\alpha+\gamma}\right]\right) &\leq \frac{C\left(r+2\left(\alpha+\gamma\right)\right)}{8\left(r+\alpha\right)}.
\end{aligned}
$$

It follows that for sufficiently high C the three inequalities are satisfied. Specifically, the bound (18) holds when $r+\alpha+\gamma$ converges to zero. In the case when σ_Y converges to zero, the bound (18) follows from

$$\gamma = \sqrt{\alpha^2 + \left(\frac{\sigma_\mu}{\sigma_Y}\right)^2} - \alpha = O\left(\frac{1}{\sigma_Y}\right).$$

6.5 Proofs of Propositions 5, 6 and 7.

Proof of Proposition 5. For simplicity, suppose that the cost of effort is quadratic, $c(a) = \frac{a}{2} + \frac{C}{2}a^2$. This implies the flow payoffs

$$a(F) - c(a(F)) = \frac{F(w)}{2C}\left(1 - F(w)\right).$$

The proof strategy is to construct a a C^2 function $F : [0, \overline{w}] \to R$ that satisfies the differential inequality

$$(r+\alpha+\gamma)F(w) \leq F'(w) \times \left(rw - \left(a(F(w)) - c(a(F(w)))\right)\right) - \frac{(r+\alpha)^2}{2\sigma_Y^2 F''(w)}, \tag{40}$$

together with the boundary conditions (12). Given such an F, it is always possible to find an $I(w)$ for which the equation (11) in Proposition 3 holds at every $w \in (0, \overline{w})$.

Given the quadratic cost of effort, the function $\underline{F}(w)$ providing the lower bound on the feasible (w, F) pairs satisfies (see (19))

$$rw = \frac{\underline{F}(w)}{2C}\left(1 - \underline{F}(w)\right),$$

and in particular the static Nash pair $(0,0)$ lies on the curve, with $\underline{F}'(0) = 2Cr$.

We will construct a curve F over $[0, \varepsilon/r]$, with $\varepsilon > 0$ small, constant second derivative and with the right boundary condition

$$F(\overline{w}) = 2\underline{F}(\overline{w}) \approx 4C\varepsilon,$$

where the approximation follows from $\underline{F}(\overline{w}) \approx 2C\varepsilon$. This implies

$$F'(\overline{w}) = \frac{(r+\alpha+\gamma)F(\overline{w})}{r\overline{w} - \frac{F(\overline{w})}{2C}(1-F(\overline{w}))} = \frac{(r+\alpha+\gamma)(4C\varepsilon + o(\varepsilon))}{\varepsilon - (2\varepsilon(1-4C\varepsilon) + o(\varepsilon))} \approx -4C(r+\alpha+\gamma).$$

The constant second derivative D is pinned down by

$$\begin{aligned} 4C\varepsilon &= \int_0^{\overline{w}} F'(x)dx = \int_0^{\overline{w}} [F'(\overline{w}) - D(\overline{w}-x)]\,dx \\ &= F'(\overline{w}) \times \frac{\varepsilon}{r} - \frac{D}{2}\left(\frac{\varepsilon}{r}\right)^2, \\ D &\approx -\left(4C\varepsilon + 4C(r+\alpha+\gamma) \times \frac{\varepsilon}{r}\right)\left(2\frac{r^2}{\varepsilon^2}\right) = -8C(2r+\alpha+\gamma)\frac{r}{\varepsilon}. \end{aligned}$$

It follows that:

$$\begin{aligned} F(w) &= O(C(r+\alpha+\gamma)\frac{\varepsilon}{r}), \\ (r+\alpha+\gamma)F(w) - F'(w)\left(rw - \frac{F(w)}{2C}(1-F(w))\right) &= O(C(r+\alpha+\gamma)^2\frac{\varepsilon}{r}), \\ -\frac{(r+\alpha)^2}{2\sigma_Y^2 F''(w)} &= O\left(\sigma_Y^{-2}(r+\alpha+\gamma)\frac{\varepsilon}{r}\right), \end{aligned} \tag{41}$$

and so inequality (40) is satisfied, when $\sigma_Y^2 C(r+\alpha+\gamma)$ is small.

Proof of Proposition 6. Suppose to the contrary that the sequence $\{w_\alpha^*\}$ is bounded from below away from zero. The proof establishes the desired contradiction in the following two steps.

First, we show that for sufficiently high α the function F_α characerizing the maximal incentives, as in Theorem 1, crosses the upper boundary $\overline{F}$ at which the drift of the relational capital is zero (see (19)), and

$$\lim_{\alpha\to\infty} |F_\alpha'(w_\alpha)| = \infty, \tag{42}$$

where w_α is the crossing point, $F_\alpha(w_a) = \overline{F}(w_\alpha)$.

Indeed, if the $\{w_\alpha^*\}$ is bounded from below away from zero, then the is $d > 0$ such that for any α and w with $F_\alpha'(w) < 0$ we have

$$F_\alpha(w) \geq d.$$

Note also that for every α and w at which the drift of the relational capital is negative, the drift is uniformly bounded by

$$\begin{aligned}|rw - (a(F_\alpha(w)) - c(a(F_\alpha(w))))\,| &\leq a(F_\alpha(w)) - c(a(F_\alpha(w))) \\ &\leq a_{EF} - c_{EF}.\end{aligned}$$

Since each F_α satisfies the differential equation (13), the two inequalities above imply that there is a sequence of negative numbers $\{\underline{F}_\alpha'\}$, $\underline{F}_\alpha' \to -\infty$, such that for every α, if $F_\alpha'(w) \in [\underline{F}_\alpha', 0]$, then $F_\alpha''(w) \geq -1$. This in turn implies (42).

Second, we lead (42) to a contradiction. Equation (13) implies that when $F_\alpha(w) \geq \overline{F}(w)$ (and so the relational capital has positive drift) and $F_\alpha'(w) < 0$, then $F_\alpha''(w)$ is bounded from below by a constant, for all α.[27] Together with (42) and the fact that $w_\alpha \leq w_{EF}$, for all α, this implies that for sufficiently high α $F_\alpha'(w) < 0$, as long as $F_\alpha(w) \geq \overline{F}(w)$. It follows that for sufficiently high α $F_\alpha(0) > F_\alpha(w_\alpha) > 0$, establishing contradiction.

Proof of Proposition 7

In order to establish monotonicity, note that decreasing σ_Y changes the equation (13) only by increasing the last term. This means that the function F solving the differential equation (13) and boundary conditions (14), given σ_Y, will satisfy the same boundary conditions and the left inequality version of the equation (13) for any $\sigma_Y' < \sigma_Y$. As in the proof of Proposition 5, with appropriately increased the policy function $I(w)$, the pair (F, I) satisfies the conditions of Proposition 3 for σ_Y', and so gives rise to a local SSE. This establishes the proof.

[27] Indeed, not only are F_α'' bounded from below, but converge to zero, as $\alpha \to \infty$, given that the left hand side of the equation (13) blows up.

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