Working paper

PRECAUTION, INFORMATION AND TIME-INCONSISTENCY: DO WE NEED A *PRECAUTIONARY PRINCIPLE*? [1]

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ABSTRACT. We study the optimal action plan chosen by a decision-maker who faces the possibility of an irreversible catastrophe. That event follows a non-homogenous Poisson process with a rate that depends on the stock of past actions. Passed a tipping point, the rate of arrival of such a disaster increases. We describe optimal trajectories under various informational scenarios with uncertainty on where the tipping point lies. When the mere fact of having passed the tipping point is immediately known, the optimal action plan is time-consistent. When having passed the tipping point remains unknown, a case of deeper uncertainty, a time-inconsistency problem arises. We characterize the unique Markov-perfect equilibrium among the decision-maker's different selves with feedback rules being conditioned only on the existing stock of past actions. We interpret the *Precautionary Principle* as an institutional restriction on actions which aims at solving this time-inconsistency problem. Unfortunately, such restriction is never optimal as it means refusing acting on useful information that will arrive in the future. We investigate and discuss the relevance of other potential solutions to restore commitment.

KEYWORDS. Precautionary Principle, Environmental Risk, Regulation, Learning.

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1. INTRODUCTION

ON THE PRECAUTIONARY PRINCIPLE. When dealing with the major environmental and health issues that pertain to our modern *risk society,*[1] policy decision-making is complicated by two features that make the standard tools of cost-benefit analysis of little value or even irrelevant. The first specificity is that consumption and/or production choices might entail a strong irreversibility component. The most salient example is given by global warming. Pollutants have been accumulating in the atmosphere from the inception of the industrial era, leading to a steady increase in temperature. All current or planned efforts against global warming consist in controlling the growth rate of temperature, not reducing it. Another example is given by GMO crops whose production may profoundly modify the surrounding biotope without any possibility of engineering back that biotope because of irreversible mutations.[2]

[1]We thank seminar participants to the *Environmental Economics Seminar* and to the *Frontier in Environmental Economics 2017 Workshop* at Paris School of Economics, ETH Zurich Workshop on Environmental Economics 2017, *World Congress of Environmental and Resource Economics 2018*, and Montpellier, CREST-ENSAE-Polytechnique, Columbia University and the *3rd IO Workshop* in Bergamo 2019 for useful comments on earlier versions of this paper. This work has benefitted of many fruitful discussions with Antoine Bommier, Ivar Ekeland and Eric Serre. The usual disclaimer applies.

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[1]Beck (1992).

[2]Other examples of irreversible choices include hydraulic fracturing to exploit shale gas (which implies irreversible pollution of underground water reserves), authorizing the use of Bisphenol A or glyphosate (which are both potential sources of cancers), relying exclusively on nuclear energy (with potential severe environmental destruction and potential severe health issues in case of an accident).

The second feature of those problems is that the costs and benefits of any decisions have to be assessed in a world of significant uncertainty. Although the consequences of acting might be detrimental to the environment, the extent to which it is so and the probability of harmful events, in other words the physical processes at play, remain to a large extent unknown to decision-makers at the time of acting.

The policy guidelines that have been adopted to rule decision-making and regulation in those contexts widely vary from one country to the other. To illustrate, while GMOs are authorized for human consumption in the U.S. without labeling, it is compulsory to label them in sixty four other countries throughout the world and they are actually forbidden in most of the European Union. Despite such variations in responses, a common concern has been to improve knowledge of the risky phenomenons at stake and thus to let scientific expertise play a significant role throughout decision-making process. When the issues of probabilities and consequences have to be assessed, learning about fundamentals of the physical process at stake becomes a key aspect of risk regulation.[3]

To further guide decision-making, a concept that has repeatedly been invoked is the so called *Precautionary Principle*. The original idea is due to the philosopher Hans Jonas' *Vorsorgeprinzip*, or *Principle of Foresight* - sometimes translated and referred to as the *Principle of Responsibility.* This *Principle* suggests that we should acknowledge the long-term irreversible consequences of present actions, and refrain from undertaking any such action if there is no proof that it would not negatively affect future generations' well-being. The *Precautionary Principle* was acknowledged by the United Nations in 1992, during the *Earth Summit* held in Rio, and expressed as: *"Where there are threats of serious and irreversible damage, lack of full scientific certainty shall not be used as a reason for postponing cost-effective measures to prevent environmental degradation."* The same idea was then developed and adopted by several other governments such as France, where a very similar principle was written in the 2004 *Charter on the Environment*,[4] that is now part of the French Constitution. Any risk regulation must comply with the legal framework that the *Principle* contributes to build, most often taking the form of a law that states a period during which a certain action cannot be undertaken, or only at a very limited level.

There has always been a lively debate on whether the *Precautionary Principle* provides a convenient guide for decision-making under deep uncertainty. Doubts exist on the fact that its adoption might actually do more harm, by hindering innovation and wealth creation, than good, by protecting human health or the environment.

The debate goes on by recognizing the contradictory views that pertains to the *Precautionary Principle.*[5] Giddens (2011) forcefully argues that preventing one risk may sometimes trigger another. A ban on GMOs may increase the risk of starvation and malnutrition. Following the 2011 disaster in Fukushima, powerful interest groups throughout Europe have been advocating for a complete ban on nuclear energy; but on the other hand, it would also mean relying on fossil energies even more at the cost of accelerating

[3]We shall leave aside the concerns about the reliability of information and how it can be manipulated or interpreted by groups of different backgrounds. For some related discussion of those considerations, we refer to Hood, Rothstein and Baldwin (2003, Chapter 2).

[4]Loi constitutionnelle n 2005-205 du 1 mars 2005 relative à la Charte de l'environnement.

[5]See Gardiner (2006) and O'Riordan (2013) for informed discussions.

global warming. Sunstein (2005) also points out that the *Precautionary Principle* is sometimes understood as meaning *not acting* because more of the act is also associated to a greater harm while *true* precaution might instead require taking large actions. Fighting global warming is here an example in order. Finally, commentators have also wondered about the exact definition of the *Precautionary Principle* which seems to be modified on a case-by-case basis.[6] One example of the fuzziness of the concept is given by the difficulty to agree on what is meant by *"full scientific certainty"*, or the absence of it. To illustrate, while the intensity of damages following a nuclear accident is unfortunately perfectly known, it is possible that at some point in the future, advances in science might make the probability of such catastrophe much smaller. But overturning precautionary stances, if written into the Law or the Constitution, will be extremely difficult[7].

That ethical considerations have entered the judicial arsenal and how such entry has been perceived by practitioners raises two important comments. The first one is that any ban on acting that the *Principle* implies is only justified and thus matters if the *laissez-faire* outcome would require, on the contrary, excessive actions. Any theory justifying the *Principle* in a *laissez-faire* economy is thus conceptually flawed. In other words, the *Precautionary Principle* can only be justified if it solves an agency problem between the constitutional level, which aims at preventing actions, and decision-makers who are in charge of implementing actions later on. Our analysis will unveil such conflict of interests and link it to the dynamics of actions in a context where information arrives over time. Indeed, it is a well-known tenet of dynamic decision-making under uncertainty that the preferred action that a decision-maker would choose *ex ante*, say *'not to act'*, under the veil of ignorance might no longer be optimal *ex post* when more information has been gathered, prospects on the riskiness of the project are better and acting becomes more attractive. Time-inconsistency may be a concern here and, in that respect, a *Precautionary Principle* may look as an attractive vehicle to restore some sort of commitment.

The second immediate comment is that, even if agency considerations do matter, it is not clear that a *Precautionary Principle* is the best way of solving this issue. Taking again the perspective of Agency Theory, the *Precautionary Principle* could be viewed as a rough and incomplete social contract[8] to solve a commitment problem. This raises the issue of whether one can find solutions to this commitment problem.

The objective of this paper is thus threefold. First, we show how time-consistency becomes a concern in a context of dynamic decision-making under deep uncertainty, learning and irreversibility. Yet, the underlying question remains of what is precisely meant by information learning and different scenarios are envisioned in the sequel. Those scenarios are meant to illustrate different degrees of scientific knowledge and discovery. Second, we demonstrate that a commitment to a fixed action for any period of time is suboptimal. In other words, the *Precautionary Principle* is useless. Lastly, we show that more complete contingent plans that would link actions to current beliefs, might be effective remedies to the commitment problem but we also cast doubts on the practical relevance of such a solution.

[6]See Immordino (2003) on this.
[7]Austria banned nuclear power in 1978, arguably before greenhouse gases emissions became a strong concern for citizens.
[8]Grossman and Hart (1986).

Model and Results. We consider the following dynamic environment. A decision-maker chooses at any point in time an action which yields a flow surplus. The past stock of actions affects the rate of arrival of an environmental disaster. Following such disaster, viewed as a major disruptive event, opportunities for surplus disappear and a flow damage is incurred from that date on. A disaster follows a non-homogenous Poisson process, and as such arrives with probability one in the long run. To capture the idea that past actions have an irreversible impact on the likelihood of such a disaster, the rate of arrival of such disaster depends on the stock of past actions. More precisely, when the stock reaches a tipping point the rate of arrival discontinuously jumps upwards from a fixed value to another fixed value.[9] To capture various kinds of information learning along the process, we investigate different scenarios for the degree of uncertainty surrounding decision-making.

Common knowledge of the tipping point. This is the simplest scenario. The decision-maker knows where the tipping point is. He can thus postpone the date at which this tipping point will be reached by reducing earlier actions. Indeed, those actions have now an opportunity cost since they contribute to approaching the tipping point, an *irreversibility effect.* Because future surpluses are discounted while earlier actions all contribute with the same intensity to the speed at which the tipping point is reached, optimal actions are reduced over time in a first phase. Once the tipping point has been passed, actions have no longer any impact on the rate of arrival and the decision-maker maximizes current benefits by jumping to a higher myopic action. In other words, the action profile is non-monotonic but distortions below the myopic optimum are driven by concerns for irreversibility.

Common knowledge of when the tipping point has been passed. Suppose now that the tipping point is not *a priori* known but instead known to be drawn from a common knowledge distribution.[10] In this first scenario of uncertainty, the mere fact of having passed the tipping point is immediately learned. A by-product of such information structure is that as long as the tipping point is known not to have been passed, the decision-maker also knows that the rate of arrival remains low. There is no learning along the process. The decision-maker's problem is now to find an optimal action plan that prevails as long as ignorance on the value of the tipping point remains. From a dynamic programming point of view, the state of the system is entirely determined by the stock of past actions. The decision-maker acts accordingly; taking into account the irreversibility of his earlier actions and the uncertainty on when regimes switch. The second consequence is that, upon revelation of having passed the tipping point, the decision-maker immediately switches to the myopic action forever just as in the common knowledge scenario.

The dynamic optimization problem has a recursive structure. The *Principle of Dynamic Programming* applies and the solution is thus time-consistent. We fully characterize the

[9]Tipping points are a cornerstone of many models in ecology and in climatology (Lenton et al., 2008). To illustrate, a recent report by the World Bank argues that *"As global warming approaches and exceeds 2-degrees Celsius, there is a risk of triggering nonlinear tipping elements. Examples include the disintegration of the West Antarctic ice sheet leading to more rapid sea-level rise.The melting of the Arctic permafrost ice also induces the release of carbon dioxide, methane and other greenhouse gases which would considerably accelerate global warming."* See http://whrc.org/project/arctic-permafrost.

[10]The case of an agnostic Laplace distribution is a particular example of some relevance in practice. Kriegler et al. (2009) offers a view of what experts might think of those distributions of tipping points. Roe and Baker (2007) argues that whether past actions have already triggered a change of regimes might remain unknown for a while.

optimal trajectory by means of a Hamilton-Bellman-Jacobi equation that is satisfied by the value function together with a feedback rule that determines how the current action (conditionally on not having yet passed the tipping point) varies with state of the system, i.e., the existing stock of past actions. We analyze the long-run behavior of this system, providing bounds on actions and value, and showing two main results. First, the value function always converges towards the myopic payoff. In the long run, the probability of having passed the tipping point converges to one and the myopic strategy applies afterwards. Second, the optimal action conditionally on not having passed the tipping point (an event with arbitrarily low probability in the long run) remains always bounded away from the myopic optimum. In other words, the *irreversibility effect* is still at play as long as the decision-maker knows that the tipping point has not been passed. Uncertainty on its location does not change the decision-maker's incentives to reduce actions before the tipping point is reached.

Deep uncertainty, irreversibility and learning. Suppose now that the decision-maker remains ignorant on whether the tipping point has already been passed. At any point in time t, the decision-maker evaluates the extra opportunity costs of having passed the tipping point around that date. This cost is expressed in terms of the possible loss of surplus associated to all future actions taken at dates $t+\tau$. This cost is thus forward-looking. Ideally, the decision-maker would like to decrease this cost by committing to lower future actions below the myopic optimum.

The presence of a forward-looking cost in the decision-maker's objective is a non-standard feature that makes our analysis differ from standard optimization problems. The optimization problem loses its recursive structure. It is where the time-inconsistency problem bites. To be more explicit, as time $t' \geq t$ comes, reducing actions at all future dates $t+\tau \geq t'$ is viewed as being less useful than in the past since the forward-looking loss that is considered in between t and t' no longer matters for the decision-maker at date t. A plan stipulating actions from a given date t on is not time-consistent; the decision-maker would like to further increase these actions later on. In other words, the decision-maker when choosing a higher action at date $t+\tau$ exerts a negative externality on his own selves acting at previous dates.

Accordingly and in the spirit of sequential optimality, we look for a time-consistent feedback rule and the associated *pseudo*-value function[11] by allowing the decision-maker to only commit for periods of lengths that are arbitrarily small. At any point in time, the decision-maker thus chooses an action that is optimal given the current stock and given that he expects his own selves to stick to the same feedback rule later on, when the stock will have evolved according to his own current choice. The pseudo-value function satisfies a functional equation that, although somewhat similar to a Hamilton-Bellman-Jacobi equation, is now non-local in nature. Indeed, this non-local aspect captures the fact that future actions are not committed to. It represents the externality that future selfs exert on the decision-maker's current payoff.

Characterizing the solution to such a functional equation is a difficult task that requires involved techniques. We indeed transform this functional equation into a pair of differential equations respectively for the pseudo-value function and the externality component of

[11] The qualifier *pseudo* captures the fact that this value function takes into account that future actions will be taken by the decision-maker with the same requirement of time-consistency.

the payoff. The properties of this system are analyzed by means of the Hartman-Grobman Theorem which helps us to show the uniqueness of the time-consistent feedback rule and the *pseudo*-value function. We are also able to provide an analysis of the asymptotic behavior of these variables and derive tight bounds. In particular, the equilibrium action (which by definition cannot be conditioned on whether the tipping has been passed or not) and the value function now converges towards their levels at the myopic optimum. Moreover, the optimal action is always positive and *not acting* is never optimal.

The irrelevance of the Precautionary Principle. This time-inconsistency problem is akin to a conflict of interests between the decision-maker's selves acting at different points in time. It provides a sound foundation for viewing the *Precautionary Principle* as a potentially attractive solution to such conflict. Committing to a fixed action before more information is learned (which in our context means that it becomes more likely that the tipping point has been passed after this commitment phase) indeed forces future selves to abide to the rule chosen earlier on. Yet, the cost of such commitment is that the action no longer depends on the current stock, i.e., on how much has been learned on the rate of arrival of a disaster. The trade-off is of course reminiscent of the *rules versus discretion* trade-off that arises (under different forms) in macroeconomics,[12] political science[13] and mechanism design[14] although, in our context, the conflict of interests is between the different selves of a given decision-maker, acting at different points in time in a context of time-inconsistency.

Yet, we demonstrate the sub-optimality of such strategy. Expanding the commitment period before switching to a time-consistent path is never optimal.The benefits of conditioning the action on the current stock always outweighs the cost of misaligned incentives between current and future selves. In other words, the *Precautionary Principle* is unfortunately a bad solution to a true problem.

Restoring commitment. This failure of the *Precautionary Principle* raises the question of knowing how to restore commitment under deep uncertainty. With deep uncertainty, the beliefs on what is the prevailing rate of arrival of a disaster now matters. Unfortunately, at any point in time, the current stock of past actions does not suffice to summarize past history. Feedback rules contingent on this sole state variable are thus of limited value. Instead, the whole past trajectory of the stock does affect beliefs. Different trajectories bring different amounts of information on whether the tipping point is likely to have been passed or not. Restoring commitment thus requires a *complete feedback rule* that would specify how the current action should depend on the current stock and (an index based on) current beliefs which stands for a sufficient statistics for this past history. Unfortunately, this approach is only feasible if the various selves can coordinate and redistribute utils among themselves.

ORGANIZATION OF THE PAPER. Section 2 reviews the literature. Section 3 presents the model. Section 4 presents two benchmarks: the case where the rate of disaster follows a homogenous Poisson process and the case of where the tipping point is known. Section

[12]See Kydland and Prescott (1977), Persson and Tabellini (1994) for a nice survey of applications, Stockey (2002) for a more recent overview and Halac and Yared (2014) for recent developments.

[13]Epstein and O'Halloran (1999), Huber and Shipan (2002).

[14]See the literature on delegation in organisations as developed in Melumad and Shibano (1991), Alonso and Matousheck (2008), Martimort and Semenov (2008) and Amador and Bagwell (2013).

5 introduces uncertainty on the tipping point but, also suppose that having passed the tipping point is immediately known. Section 6 deals with the case of deep uncertainty, stressing the time-inconsistency problem. Section 7 investigates the limited value of a *Precautionary Principle*. Section 8 shows how to restore commitment with more complete feed-back rules if some intertemporal coordination is possible. Section 9 briefly recaps our results and discusses possible extensions.

2. LITERATURE REVIEW

The paper contributes to several trends of the literature.

Irreversibility, Uncertainty and Information. Arrow and Fisher (1974), Henry (1974) and Freixas and Laffont (1984) were the first to show how a decision-maker should take more preventive stances when the consequences of irreversible choices are uncertain, the comparison being here with respect to the certainty case. Epstein (1980) has discussed general conditions under which this *Irreversibility Effect* prevails and proved that the value of waiting[15] increases when the decision-maker benefits of a more informative signal (in the sense of Blackwell) on the future realizations of uncertainty.

The main differences with our setting are twofold. First, irreversibility is hereafter encapsulated in the role played by the stock of past actions on triggering the tipping point and with it, a riskier state of the world. Second, in those earlier models, information is exogenous while in many contexts in environmental economics, earlier actions also determine whether information structures will indeed be finer. In contrast, information in our model is endogenous; the probability of having passed the tipping point depends on the stock of past actions. Reducing early actions improves flexibility but, in a model of deep uncertainty where the location of the tipping point remains unknown, those actions also make the non-occurence of a disaster less informative on whether the tipping point has been passed or not. Models with such endogenous information structures are scarce. Freixas and Laffont (1984) have studied a scenario in which more flexible actions increase the quality of future information, thus confirming the existence of the *Irreversibility Effect* while Miller and Lad (1984) have challenged this view in a model of conservation in which irreversible actions might also be more informative.[16] Salmi, Laiho and Murto (2019) study the trade-off faced by a decision-maker who must choose between acting now, which means taking a less informed decision but generating information for later, and acting later, more informed. Still, only the *speed* of learning is endogenous here. The quantity of action scales the variance of the belief process: a higher quantity means the belief converges faster to the true state.

The general framework proposed by the irreversibility literature has been applied to the economics of climate change with mixed success. Some authors have argued that this literature suggests that current abatements of greenhouse gaz emissions should be greater when more information will be available in the future (Chichilnisky and Heal, 1993; Beltratti, Chichilnisky and Heal, 1995; Kolstad, 1996; Gollier, Jullien and Treich, 2000; among others). Others like Ulph and Ulph (2012) have pointed out that the sufficient

[15] Later coined as the *quasi-option value* by Graham-Tomasi (1995). See also Jones and Ostroy (1984) and Haneman (1989).

[16] Charlier (1997), Ramani, Richard and Trommetter (1992) and Ramani and Richard (1993) have also provided such models of endogenous information structures specializing their analysis to the context of *GMOs* and their development.

conditions given by Epstein (1980) for the *Irreversibility Effect* to hold may fail even in simple models of global warming.

THE *Laissez-Faire* INTERPRETATION OF THE *Precautionary Principle*. Gollier, Jullien and Treich (2000) have built on the insights of the irreversibility literature to give some economic content to the *Precautionary Principle*. These authors interpret the *Precautionary Principle* as the incentives of a decision-maker to reduce his action below the level that would otherwise be optimal without uncertainty, when this action is taken before any information is learned. Much in the spirit of Kolstad (1996), Gollier, Jullien and Treich (2000) build a two-period model of pollution accumulation with exogenous information and draw conclusions on specific forms of utility functions that induce more of what they define as precaution. Asano (2010) has focused on the comparison of optimal environmental policies without and with ambiguity, showing that the decision-maker's lack of confidence forces him to hasten the adoption of a policy, rather than postpone it.

As we already pointed out, the decision-maker's behavior is there optimal and thus not constrained by the *Precautionary Principle* in any way.[17] In other words, there would be no reason for drafting such legal principle in this setting. The *Laissez-Faire* solution suffices. We depart from this approach by stressing the commitment value of the *Precautionary Principle* but also the sub-optimality of such commitment.

TIME-INCONSISTENCY. Our approach for characterizing a trajectory in a continuous time model with a time-inconsistency problem is similar to that developed in Ekeland and Lazrak (2010), Karp (2005, 2007) and Karp and Lee (2003), although details differ. These authors have analyzed macroeconomic growth models with time-inconsistent preferences in continuous time. The source of such time-inconsistency is the time-dependency of the discount factor.[18] Hereafter, the decision-maker has a constant discount factor so his preferences are instead *a priori* time-consistent. The time-inconsistency problem arises from the fact that feedback rules only depend on the current stock of past actions and cannot keep track of the evolution of beliefs. An action plan based on those simple feedback rules must thus be continuously re-optimized to capture, even imperfectly so, a rough dependence on how beliefs evolve.

Marcet and Marimon (2019) have presented a general theory of discrete-time optimization problems with forward-looking constraints, a feature that prevails in a number of macroeconomic and political economy contexts.[19] Our continuous time model is somewhat simpler since, in the scenario of deep uncertainty, payoffs themselves have a forward-looking component. Marcet and Marimon (2019) have shown how to recover a recursive structure to the optimization problems by adding multiplier of the forward-looking constraints as state variables which follow a specific evolution. In our context, a recursive structure can be found when beliefs are used as an extra state variable.

ON TIPPING POINTS. We are not the first ones to introduce tipping points in environmental economics. Sims and Finoff (2016) have analyzed how irreversibility in environmental damage and irreversibility in sunk cost investment do interact. Tsur and Zemel

[17]This feature is shared by other models in the field like Immordino (2000, 2005) and Gonzales (2008).

[18]An assumption that generalizes the discrete-time models of Strosz (1955), Laibson (1997), Harris and Laibson (2001) and O'Donoghue and Rabin (2003).

[19]Aiyagari et al. (2002), Acemoglu et al. (2011), Attanasio and Rios-Rull (2002) among others.

(1995) have investigated a problem of optimal resource extraction when extraction affects the probability that resource becomes obsolete passed a certain threshold. Under deep uncertainty (unknown threshold) the initial state affects the optimal path and the decision-maker might end up exploiting the resource less than under certainty, maybe up to the point of stopping exploitation; an extreme form of precaution. In our model, the probability of the catastrophe is never zero once the activity has been started[20] and foregoing it completely is never optimal. In a model of optimal control of atmospheric pollution, Tsur and Zemel (1996) have shown how uncertainty on a tipping point introduces a multiplicity of possible equilibrium values. Finally, and independently of us, Liski and Salanié (2018) have also studied a model with unknown tipping points and deep uncertainty, but with different concerns. Their analysis focuses on the commitment scenario. Instead, we stress an important time-inconsistency problem in a context of deep uncertainty.[21] Our approach, based on Bellman equations or a generalized version of those in the case of time-inconsistency, allows us to unveil the detailed structure of the solutions (with our without deep uncertainty/with or without commitment) and in particular to study their asymptotic behavior.

3. THE MODEL

TECHNOLOGY. A decision-maker (thereafter DM) has to decide *ex ante* (i.e., at date $t = 0^-$) whether he should undertake a project which puts the environment at risk. Time is continuous and $r > 0$ denotes the discount rate. This risky technology may induce a disaster. The project pays off till such disaster arises. Such a disaster follows a Poisson process with a (non-negative) rate $\theta(t)$. The probability that a disaster arises over an interval $[t, t + dt]$ is thus $\theta(t)e^{-\int_0^t \theta(\tau)d\tau}dt$ and the probability that there has been no disaster up to date t is $e^{-\int_0^t \theta(\tau)d\tau}$. Later, the rate $\theta(t)$ will be supposed to depend on the stock of past actions that have already been undertaken by DM before date t.

PREFERENCES. Let $\mathbf{x} = (x(\tau))_{\tau\geq 0}$ (resp. $\mathbf{x}_t = (x(\tau))_{\tau\geq t}$, $\mathbf{x}_t^{t'} = (x(\tau))_{t'\geq\tau\geq t}$) denote a plan of actions (resp. the continuation of such a plan from date t on, and the plan between dates t and t'). The action $x(t)$ belongs to an interval $\mathcal{X} = [0, \bar{x}]$ where $\bar{x}$ is supposed to be large enough to ensure interior solutions under all circumstances below. Action $x(t)$ yields a surplus (net of the action cost) at date t worth

$$\zeta x(t) - \frac{x^2(t)}{2}$$

where $\zeta > 0$. Had he been myopic, DM would maximize his current payoff by choosing $x^m(t) = \zeta$ at any date $t \geq 0$. This myopic action is an important benchmark to assess the impact and origins of the precautionary motives that pertain to the different scenarios we investigate below.

For future reference, we define the cumulated stock of past actions up to date t as:

$$X(t) = \int_0^t x(\tau)d\tau.$$

[20] In fact, the probability of the catastrophe in the long run is one

[21] Other differences are related to objective functions and that negative actions are feasible. Instead, the optimal trajectory in our setting is always strictly monotonic; meaning that optimal actions remain bounded away from zero.

If a disaster occurs at date t, DM incurs an irreversible flow of damages $-D$ from that date on. With a Poisson process, the long run probability that such disaster arises is one. The discounted welfare loss of such a sure-thing is $\frac{D}{r}$. To capture the detrimental and irreversible impact of a disaster, we also assume that, if such an event arises at date t, the flow surplus is no longer realized from that date on. A justification is that, the disaster is such a large event that the technology might no longer be used afterwards. We will often think of the benefit of not facing a disaster as the (not incurred) damage together with the surplus, namely:

$$u(x(t)) \equiv \zeta x(t) - \frac{x^2(t)}{2} + D$$

This reference will have some importance when writing the present value of DM's payoff.

Welfare. Using the Poisson specification of the arrival rate of a disaster, DM's expected discounted welfare at date 0 if he adopts a plan $\mathbf{x} = (x(t))_{t\geq 0}$ can be expressed as:

$$\mathcal{W}(\mathbf{x}) \equiv \int_0^{+\infty} e^{-\left(rt+\int_0^t \theta(\tau)d\tau\right)} u(x(t))dt - \frac{D}{r}. \tag{3.1}$$

DM enjoys the surplus plus the flow benefit of not incurring a disaster as long as there is no such disaster, i.e., with probability $e^{-\int_0^t \theta(\tau)d\tau}$. Since with a Poisson process a disaster occurs with probability one, everything happens as if $\frac{D}{r}$ was paid upfront and DM would also enjoy D, viewed as the current benefit of not having a disaster, at any point in time. Throughout the paper, we will specialize this expression to various possible kinds of Poisson processes and different informational environments.

4. BENCHMARKS

4.1. *No Irreversibility: Homogenous Poisson*

We start with the simplest case where DM has no control over the rate of arrival of a disaster, which is kept constant and equal to an exogenous parameter θ_0. Specializing our previous formula (3.1), expected welfare can thus be written as:

$$\mathcal{W}(\mathbf{x}) \equiv \int_0^{+\infty} e^{-\lambda_0 t} u(x(t))dt - \frac{D}{r} \tag{4.1}$$

where, for future reference, we denote $\lambda_0 = r + \theta_0$ the effective discount rate that applies once the possibility of a disaster is taken into account.

Since he cannot influence the rate of arrival of the disaster, DM only maximizes current surplus. The optimal action, *the myopic outcome,* is constant over time:

$$x^m(t) = \zeta \quad \forall t \geq 0.$$

The net present value of this project is positive, and the project so valuable, when

$$\mathcal{W}(\mathbf{x^m}) = \frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty - \frac{D}{r} \geq 0 \tag{4.2}$$

where, for future reference, we denote $\lambda_1 \mathcal{V}_\infty = u(\zeta) = D + \frac{\zeta^2}{2}$, $\lambda_1 = r + \theta_1$, and $\Delta = \theta_1 - \theta_0 > 0$.

4.2. *Known Tipping Point*

The simplest way to capture a nonlinear dynamics of the stock process is to assume that the Poisson process features a *tipping point*. When the cumulative stock of past actions $X(t) = \int_0^t x(\tau)d\tau$ passes a threshold X_0, the rate of arrival of a disaster thus increases from θ_0 to $\theta_1 > \theta_0$. Formally, we may define the time-dependent rate of arrival as:

$$\theta(t) = \theta_0 + \Delta \mathbb{1}_{\{X(t)>X_0\}}. \tag{4.3}$$

Although it remains quite close to a homogeneous Poisson process, and indeed it is so before and after the tipping point, this specification also implies a dependence on the past history of actions. In this respect, it is useful to define t_0 as the date at which the tipping point is reached for the first time, namely:

$$t_0 = \min\left\{t \geq 0 \text{ s.t. } X(t) = X_0\right\}.$$

Had DM chosen to always act myopically, the tipping point would be reached at $t^m = \frac{X_0}{\zeta}$.

With these notations at hands, we may rewrite DM's expected welfare as:

$$\mathcal{W}(\mathbf{x}) \equiv \int_0^{t_0} e^{-\lambda_0 t} u(x(t))dt + e^{-\lambda_0 t_0} \int_{t_0}^{+\infty} e^{-\lambda_1 (t-t_0)} u(x(t))dt - \frac{D}{r}. \tag{4.4}$$

The integrand in the first line stems for welfare before the tipping point. This term is identical to that found for an homogenous Poisson process as in (4.1), although now the upper bound of the interval is the date t_0 at which the tipping point is reached. The second line stands for welfare after the tipping point weighted by the probability of survival. The only difference is that the arrival rate has now jumped up.

Proposition 1 *Suppose that the non-homogenous Poisson process is defined by (4.3).*

- *The optimal action is decreasing over $t \in [0, t_0)$ with $x^*(t) < x^m$ for all $t \in [0, t_0)$:*

$$x^*(t) = \begin{cases} \zeta - e^{\lambda_0 (t-t_0)} \sqrt{2\Delta \mathcal{V}_\infty} & \text{for } t \in [0, t_0), \\ \zeta & \text{for } t \geq t_0 \end{cases} \tag{4.5}$$

where t_0 (with $t_0 > t^m$), the date at which the tipping point is reached, is the unique positive root for

$$\zeta t_0 - X_0 = \frac{\sqrt{2\Delta \mathcal{V}_\infty}}{\lambda_0}\left(1 - e^{-\lambda_0 t_0}\right). \tag{4.6}$$

- *The optimal stock $X^*(t)$ satisfies*

$$X^*(t) = \begin{cases} \zeta t - (\zeta t_0 - X_0)\frac{e^{\lambda_0 t} - 1}{e^{\lambda_0 t_0} - 1} & \text{for } t \in [0, t_0), \\ \zeta(t - t_0) + X_0 & \text{for } t \geq t_0\ . \end{cases} \tag{4.7}$$

Actions profile. The optimal action goes through two distinct phases. In the first *precautionary phase*, i.e., before reaching the tipping point, DM chooses an action which remains below the myopic optimum. The intuition is straightforward. Indeed, over this first phase, the stock remains below the tipping point, namely:

$$\int_0^{t_0} x^*(t)dt \leq X_0. \tag{4.8}$$

This condition puts a constraint on how to accumulate actions before reaching the tipping point. To relax this *irreversibility constraint,* DM chooses low actions early on. This feature of the optimal path comes from *an irreversibility effect.* The actions taken earlier on have a long-lasting impact since they may contribute to passing the tipping point. Reducing these actions keeps the probability that a disaster arises earlier at a low level.

The optimal action is decreasing over time before the tipping point is passed. All actions taken there have the same marginal contribution to the overall stock. Because of discounting and because the probability of not having yet faced a disaster decreases over time before reaching the tipping point, DM prefers to choose the highest actions earlier on and the lowest ones when approaching the tipping point.

Once the tipping point has been passed, DM knows that his actions will no longer impact the arrival rate of a disaster. We are back to the homogeneous case studied in Section 4.1. The optimal action is again set at its myopic level after the tipping point.

Henceforth, when the tipping point is known, the optimal action path is thus non-monotonic with actions decreasing over the first phase and then jumping up to the myopic outcome beyond the tipping point.

Tipping Point. Because actions are now of a lower magnitude than the myopically optimal one over the first phase, the tipping point t_0 is reached after date t^m. By pushing a bit further in the future the date at which the tipping point is reached by a small amount dt_0, DM incurs a welfare loss since, over the precautionary phase, the action is below the myopic optimum (DM is therefore getting less than the optimal surplus over a longer period of time). Taking into account discounting and the probability that no disaster has ever occurred before date t_0, this marginal loss can be expressed in terms of date 0 utils by discounting payoff at a rate $\lambda_0 = r + \theta_0$ as:

$$\underbrace{e^{-\lambda_0 t_0}\left[\zeta x - \frac{x^2}{2}\right]_{x^*(t_0^-)}^{x^*(t_0^+)=\zeta} dt_0}_{\text{Marginal loss from not choosing the myopic action over } [t_0, t_0+dt_0]}.$$

On the other hand, pushing a bit further that date t_0 by a small amount dt_0 maintains the rate of arrival of a disaster at its low level θ_0. By doing so, DM is less likely to losing not only the surplus $\frac{\zeta^2}{2}$ achieved with the myopic action that is optimal for $t \geq t_0$ but also the flow damage D in case a disaster occurs. Taking into account the discounted probability of a disaster from date t_0 on, the benefit (still expressed in terms of date 0 utils) of delaying the date at which the tipping point is reached by dt_0 can be written as:

$$\underbrace{\Delta\left(D + \frac{\zeta^2}{2}\right) e^{-\lambda_0 t_0}\left(\int_{t_0}^{+\infty} e^{-\lambda_1(t-t_0)}dt\right) dt_0}_{\text{Marginal benefit of delaying the tipping point by } dt_0} \equiv \Delta\mathcal{V}_\infty e^{-\lambda_0 t_0} dt_0.$$

At the optimum, DM balances the loss over the first phase and the above benefit. For future reference, we may thus define the net marginal benefit from pushing the tipping point further by dt_0 as the undiscounted difference between the two above quantities, namely

$$\left(\Delta\mathcal{V}_\infty - \frac{1}{2}\left(x^*(t_0^-) - \zeta\right)^2\right) dt_0.$$

The optimal time at which the tipping point is reached is thus given by (4.6). It is worth pointing out that, for the constellation of parameters under consideration, passing the tipping point is always optimal. In other words, D is not so large as to make the project not valuable upfront or along the course of actions.

Positive Net Present Value of the Project. DM chooses to run the risky technology when it yields a positive net present value. This condition writes as:

$$\mathcal{W}(\mathbf{x}^*) = \frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty - \frac{D}{r} + \frac{e^{-\lambda_0 t_0}}{\lambda_0}\left(e^{-\lambda_0 t_0} - 2\right)\Delta\mathcal{V}_\infty \geq 0. \tag{4.9}$$

That the rate of arrival of a disaster increases to θ_1 once the tipping point is passed, means that expected welfare is necessarily lower than with an homogenous Poisson process corresponding to a fixed arrival rate θ_0. The condition for running the technology is thus harder to satisfy as it can be seen from comparing (4.2) and (4.9).

In the sequel, we will ensure that the project has a positive NPV by assuming the slightly stronger condition

$$\mathcal{V}_\infty - \frac{D}{r} \geq 0. \tag{4.10}$$

5. Uncertainty on the Tipping Point

Consider the more realistic case where the tipping point is not known at the time of starting the project. We also suppose that, DM knows when the tipping point is passed, at which point the rate of arrival of a disaster changes from θ_0 to θ_1. To model uncertainty, we assume that the tipping point X is now a random variable drawn on the whole positive real line from a common knowledge (and atomless) distribution F. Let f be the corresponding (everywhere positive) density.

Dynamic Programming. Consider an action plan $\mathbf{x}_t = \{x(\tau)\}_{\tau \geq t}$ from date t onwards. If the stock at date t is X, this action plan induces a stock process $\tilde{X}(\tau; X, t)$. We will restrict to processes which are everywhere increasing and continuously differentiable. This stock evolves as:

$$\tilde{X}(\tau; X, t) = X + \int_t^\tau x(s)ds \tag{5.1}$$

As times passes, the stock goes through possible values of the tipping point. Formally, we may also describe this cumulative process by the time $\tilde{T}(\tilde{X}; X, t) \geq t$ at which the stock reaches a level $\tilde{X} \geq X$. Below, we will sometimes slightly abuse and simplify notations and write $\tilde{X}(\tau; X) \equiv \tilde{X}(\tau; X, 0)$.

Accordingly, we also define the value function $\tilde{\mathcal{V}}_u(X,t)$ as DM's optimal intertemporal payoff starting from date t onwards when the stock level at date t is X. We make two normalizations. First, the discounted damage $-\frac{D}{r}$ is again omitted but of course, it has to be counted to assess the *ex ante* value of the technology. Second, $\tilde{\mathcal{V}}_u(X,t)$ is in fact an expected payoff taking into account that, with probability $e^{-\theta_0 t}F(X)$, the tipping point has been passed and there has been no catastrophe up to date t. In that event, the optimal action from date t on is the myopic optimum and the rate of arrival of a disaster is θ_1. Taking into account discounting, we obtain the following expression for $\tilde{\mathcal{V}}_u(X,t)$:

$$\tilde{\mathcal{V}}_u(X,t) \equiv \sup_{\mathbf{x}_t,\tilde{X}(\cdot)\text{ s.t. }(5.1)} \int_0^X e^{-\lambda_0 t}\left(\int_t^{+\infty} e^{-\lambda_1(\tau-t)}\lambda_1\mathcal{V}_\infty d\tau\right)f(\tilde{X})d\tilde{X}+ \tag{5.2}$$

$$\int_X^{+\infty}\left(\int_t^{\tilde{T}(\tilde{X};X,t)} e^{-\lambda_0\tau}u(x(\tau))d\tau + e^{-\lambda_0\tilde{T}(\tilde{X};X,t)}\int_{\tilde{T}(\tilde{X};X,t)}^{+\infty} e^{-\lambda_1(\tau-\tilde{T}(\tilde{X};X,t))}\lambda_1\mathcal{V}_\infty d\tau\right)f(\tilde{X})d\tilde{X}$$

The first term stands for DM's expected payoff once DM has already learned before date t that the tipping point was less than X. This expression takes into account the fact that the optimal action from that point on is the myopic optimum and the rate of arrival of a disaster is θ_1. Therefore, when we look for the optimal path, it is only the optimal path conditional on not having passed the tipping point that we are looking for. The second term stands for the expected payoff when it is not yet known that the tipping point has been passed. It accounts for the regime shift at date $\tilde{T}(\tilde{X};X,t)$.

The difficulty here is that the maximand depends on both the action plan $\mathbf{x}_t$ and the inverse $\tilde{T}(\tilde{X};X,t)$ of the stock accumulation that this plan induces; a quite unusual feature. Next lemma provides a reduction of that problem that makes it look more familiar.

LEMMA 1 *The value function $\tilde{\mathcal{V}}_u(X,t)$ satisfies $\tilde{\mathcal{V}}_u(X,t) = e^{-\lambda_0 t}\mathcal{V}_u(X)$ for all (X,t) where $\mathcal{V}_u(X)$ is defined as:*

$$\mathcal{V}_u(X) \equiv \mathcal{V}_\infty + \sup_{\mathbf{x}_t,\tilde{X}(\cdot)\ s.t.\ (5.1)} \int_0^{+\infty} e^{-\lambda_0\tau}\left(1-F\left(\tilde{X}(\tau;X)\right)\right)\left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x(\tau)-\zeta)^2\right)d\tau \tag{5.3}$$ [22]

The important point to notice is that the maximization problem (5.3) has a recursive structure. As a consequence, the *Principle of Dynamic Programming* applies and an optimal action plan is necessarily time-consistent. In the scenario studied here, DM always knows the rate of arrival of a catastrophe. Indeed, the mere fact of knowing that the tipping point will be known when it is passed is enough to bring information on this rate of arrival either after or before the tipping point. There is nothing learned along the trajectory and the state of the system can be reduced to the current stock of past actions exactly as when the tipping point is known.

[22] From (5.3), it is straightforward to check that the curent value function $\tilde{\mathcal{V}}_u(X)$ is non-increasing in X and thus almost everywhere differentiable. In the sequel, we will look for a value function that is actually C^1. From there, we will deduce an Hamilton-Bellman-Jacobi equation that this C^1 value function satisfies. A *Verification Theorem* will provide sufficient conditions satisfied by the candidate solution. See the Appendix for details.

Of particular importance is the value $\mathcal{V}_u(0)$ since the intertemporal welfare for the optimal path $\mathbf{x}_u$ can be expressed as $\mathcal{V}_u(0) - \frac{D}{r}$. Rearranging terms, Lemma 1 shows that

$$\mathcal{V}_u(0) = \underbrace{\int_0^{+\infty} e^{-\lambda_0 t} u(x_u(t))dt}_{\text{Expected welfare with arrival rate } \theta_0}$$

$$- \underbrace{\int_0^{+\infty} e^{-\lambda_0 t}\left(\Delta\mathcal{V}_\infty - \frac{1}{2}\left(x_u(t) - \zeta\right)^2\right) F(X_u(t))dt}_{\text{Loss from having already passed the tipping point}}$$

The first term above is the familiar expression of expected welfare found for an homogeneous Poisson process with arrival rate θ_0. It would be maximized by adopting the myopic action at any point in time.

When choosing an action plan, i.e., how the stock of past actions $X_u(t)$ evolves over time, DM also implicitly chooses the dates at which all possible tipping points $X \leq X_u(t)$ are reached. To understand the second term in the above expression, first notice that $e^{-\lambda_0 t} F(X_u(t))$ is the (conveniently discounted) probability that the tipping point has been passed before date t and that no disaster has yet occurred. Second, we know from Section 4.2, that the expression $\Delta\mathcal{V}_\infty - \frac{1}{2}(x_u(t) - \zeta)^2$ stems for the opportunity cost of choosing earlier the date t at which a tipping point is reached.

Hamilton-Bellman-Jacobi (HBJ) Equation. Next proposition presents the HBJ equation satisfied by $\mathcal{V}_u(X)$, together with a characterization of the optimal feedback rule that determines which action $\sigma_u(X)$ is optimal at a given level of stock X.

Proposition 2 *If the function $\mathcal{V}_u(X)$ is C^1, it satisfies the following HBJ equation*

$$\mathcal{V}_u'(X) = (1 - F(X))\left(-\zeta + \left(\zeta^2 - 2\Delta\mathcal{V}_\infty + 2\lambda_0 \frac{\mathcal{V}_u(X) - \mathcal{V}_\infty}{1 - F(X)}\right)^{\frac{1}{2}}\right) \tag{5.4}$$

with the boundary condition

$$\lim_{X \to +\infty} \mathcal{V}_u(X) = \mathcal{V}_\infty. \tag{5.5}$$

The optimal feedback rule is

$$\sigma_u(X) = \zeta + \frac{\mathcal{V}_u'(X)}{1 - F(X)} \tag{5.6}$$

From (4.9), remember that $\mathcal{V}_\infty$ is in fact DM's payoff when the tipping point X_0 is known but arbitrarily large, so that the date t_0 at which this tipping point would be crossed also goes to infinity. From (5.2), it also follows that, as X grows large and it becomes very likely that the tipping point has been passed, the value function comes close to this limiting value $\mathcal{V}_\infty$.

Because $\mathcal{V}_u'(X) \leq 0$, we have $\sigma_u(X) \leq \zeta$ for all X. The optimal action is always below the myopic optimum. Even though there is uncertainty on where the tipping point lies, the optimal trajectory takes into account that passing the tipping point remains an irreversible act which leads to lower actions below the myopic outcome. Yet, Condition (5.8) below also shows that *not acting* is never optimal.[23]

PROPOSITION 3 *There exists a unique function C^1, the current value value function, $\mathcal{V}_u(X)$ satisfying the HBJ equation (5.4) and the boundary condition (5.5) with*

$$\mathcal{V}_\infty < \mathcal{V}_u(X) < \mathcal{V}_\infty \left(1 + \frac{\Delta}{\lambda_0}(1 - F(X))\right) \quad \forall X \geq 0, \tag{5.7}$$

$$\sqrt{\zeta^2 - 2\Delta\mathcal{V}_\infty} < \sigma_u(X) < \zeta \quad \forall X \geq 0.\text{[24]} \tag{5.8}$$

For what follows, it is important to remind the heuristic derivation of this HBJ equation. By the *Principle of Dynamic Programming,* the payoff $\mathcal{V}_u(X)$ is obtained by piecing together an optimal action path $\mathbf{x}_0^\varepsilon$ over an arbitrary interval $[0, \varepsilon]$ with a continuation path $\mathbf{x}_\varepsilon$ that yields the corresponding (non-discounted) continuation payoff $\mathcal{V}_u(\tilde{X}(t+\varepsilon; X, t))$. The HBJ equation is then obtained by making the commitment period ϵ arbitrarily small, taking Taylor expansions while assuming that the function $\mathcal{V}_u(X)$ is C^1.[25] Reciprocally, Proposition 3 shows that a C^1 solution to the HBJ equation satisfying the boundary condition (5.5) is the value function.

LONG-RUN BEHAVIOR. It is interesting to describe the long-run behavior of the solution. To this end, we first define the function $R(Y) = f(F^{-1}(1 - Y))$ for all $Y \in [0, 1]$ and assume that $\lim_{X\to+\infty} -\frac{f'(X)}{f(X)}$ exists and is positive. Let $R'(0) > 0$ denote this limit. We got the following approximations.

$$\sigma_u(X) \approx_{X\to+\infty} \sqrt{\left(\frac{\lambda_0}{R'(0)} + \zeta\right)^2 - 2\Delta\mathcal{V}_\infty} - \frac{\lambda_0}{R'(0)} < \zeta \tag{5.9}$$

(5.10)

$$\mathcal{V}_u(X) - \mathcal{V}_\infty \approx_{X\to+\infty} (1 - F(X)) \left(\frac{\lambda_0}{R'(0)^2} + \frac{\zeta}{R'(0)} - \sqrt{\left(\frac{\lambda_0}{R'(0)^2} + \frac{\zeta}{R'(0)}\right)^2 - \frac{2\Delta\mathcal{V}_\infty}{R'(0)^2}}\right)$$

To illustrate, suppose that X is drawn according to the logistic distribution with density $f(X) = \frac{ke^{-k(X-X_0)}}{\left(1+e^{-k(X-X_0)}\right)^2}$.[26] As k increases towards $+\infty$, this distribution puts more mass

[23] Observe also that having an increasing stock process, as requested to ensure that the smooth stock profile is invertible, requires $\sigma_u(X) > 0$, a condition that is implied by Condition (5.8).

[24] Observe that Condition (4.10) also writes as $\mathcal{V}_\infty > \frac{1}{r}\left(\lambda_1\mathcal{V}_\infty - \frac{\zeta^2}{2}\right)$ or $\frac{\zeta^2}{2} > \lambda_1\mathcal{V}_\infty - r\mathcal{V}_\infty = \theta_1\mathcal{V}_\infty$ which implies $\frac{\zeta^2}{2} > \Delta\mathcal{V}_\infty$ and $\sqrt{\zeta^2 - 2\Delta\mathcal{V}_\infty}$ exists.

[25] Of course, we do not know yet that the value function is of class C^1. We will prove below that the value function indeed solves the HBJ equation and is C^1. We postpone this step of the analysis to the Proof of Proposition 3.

[26] Admittedly, this density is defined over the whole real line but negative values have a very low probability when $k \to +\infty$.

around the threshold X_0 so as to come close to the complete information model. Yet, since $R'(0) = \lim_{X\to+\infty} -\frac{f'(X)}{f(X)} = k$, the optimal action converges again towards the lower bound $\sqrt{\zeta^2 - 2\Delta\mathcal{V}_\infty}$ and not towards the myopic optimum[27].

Evolution of the stock. Inserting the expression of the optimal stock $X_u(t) = \int_0^t \sigma_u(X_u(\tau))d\tau$ into (5.3), $X_u(t)$ satisfies:

(5.11)
$$\dot{X}_u(t) = \zeta - \frac{1}{1-F(X_u(t))}\int_0^{+\infty} e^{-\lambda_0\tau}\left(\Delta\mathcal{V}_\infty - \frac{1}{2}\left(\dot{X}_u(t+\tau)-\zeta\right)^2\right) f(X_u(t+\tau))d\tau.$$ [28]

The intuition behind (5.11) is straightforward. At any time t, suppose that the tipping point has not yet been reached; an event of probability $1 - F(X_u(t))$. Consider the possibility for DM to increase his action $x_u(t)$ in that event by a small amount dx over a small interval of length dt keeping all other actions $x_u(\tau)$ constant for $\tau \geq t + dt$. Counted in date 0 utils, the marginal benefit of doing so writes as:

$$\approx e^{-\lambda_0 t}\left(\zeta - \dot{X}_u(t)\right)(1 - F(X_u(t)))dxdt. \tag{5.12}$$

On the other hand, such a marginal increase in actions also shifts upward the whole path of future stocks $X_u(\tau)$ for $\tau \geq t + dt$ by an amount $dxdt$. It thus increases the probability that the tipping point might be passed. Up to terms of order more than two and still counted in date 0 utils, the corresponding marginal cost of such a change of the overall trajectory is thus:

$$\int_{t+dt}^{+\infty} e^{-\lambda_0\tau}\left(\frac{1}{2}\left(\dot{X}_u(\tau)-\zeta\right)^2 - \Delta\mathcal{V}_\infty\right)\left(F(X_u(\tau)+dxdt) - F(X_u(\tau))\right)d\tau$$

$$\approx \left(\int_t^{+\infty} e^{-\lambda_0\tau}\left(\frac{1}{2}\left(\dot{X}_u(\tau)-\zeta\right)^2 - \Delta\mathcal{V}_\infty\right) f(X_u(\tau))d\tau\right)dxdt. \tag{5.13}$$

Along the optimal path $X_u(t)$, the current marginal benefit (5.12) equals the future marginal cost (5.13) of slightly increasing current action. Simplifying yields (5.11).

Exponential distributions. For such functional forms, closed-form solutions are readily obtained. The optimal action is stationary, always positive and independent of the current stock while the stock evolves linearly over time.

Proposition 4 *Suppose that X is exponentially distributed over $\mathbb{R}_+$, i.e., $f(X) = ke^{-kX}$ and $F(X) = 1 - e^{-kX}$ for some $k > 0$. Notice that $R'(0) = k$. Closed forms for the current value function, the optimal feedback rule and the optimal stock are obtained respectively as*

$$\mathcal{V}_u(X) = \mathcal{V}_\infty + \left(\frac{\lambda_0}{k^2} + \frac{\zeta}{k} - \sqrt{\left(\frac{\lambda_0}{k^2}+\frac{\zeta}{k}\right)^2 - 2\frac{\Delta\mathcal{V}_\infty}{k^2}}\right)e^{-kX}, \tag{5.14}$$

[27] Conditional on not having passed the tipping point.

[28] It immediately follows from Proposition 3 that there exists a unique solution to the above integralo-differential equation with the initial condition $X_u(0) = 0$ which is C^1.

$$\sigma_u(X) = \sqrt{\left(\frac{\lambda_0}{k} + \zeta\right)^2 - 2\Delta\mathcal{V}_\infty} - \frac{\lambda_0}{k} > 0 \quad \forall X \geq 0, \tag{5.15}$$

$$X_u(t) = \left(\sqrt{\left(\frac{\lambda_0}{k} + \zeta\right)^2 - 2\Delta\mathcal{V}_\infty} - \frac{\lambda_0}{k}\right) t \quad \forall t \geq 0. \tag{5.16}$$

These expressions provide important insights on how uncertainty shapes optimal trajectories. By varying the parameter k, we may go from the pure uninformative Laplacian distribution over the positive real line ($k \to 0$) to the Dirac distribution putting mass one at zero ($k \to +\infty$); meaning the tipping point is passed almost immediately. Moving towards the Laplacian world ($k \to 0$) can admittedly be viewed as a metaphor for a context where DM is agnostic on where tipping points lies. The feedback $\sigma_u(X)$ then converges towards the myopic optimum $x^m = \zeta$. Indeed, with such large ignorance on where the tipping point lies, the probability that the tipping point has been passed remains always the same at any point in time, namely almost zero. In other words, actions that have already been taken have no impact on the probability of having passed the tipping point and DM is as well off always opting for the myopic action.

When the distribution comes closer to a Dirac distribution at zero ($k \to +\infty$), the feedback rule $\sigma_u(X)$ converges towards the lower bound $\sqrt{\zeta^2 - 2\Delta\mathcal{V}_\infty}$. Intuitively, DM refrains from taking large actions because he expects that, otherwise, the evolving stock will quickly cross almost all values of the tipping point so as the likelihood of a disaster increases.

6. DEEP UNCERTAINTY

Suppose now that DM does not even know whether the tipping point has been passed or not; a scenario thereafter coined as being one of deep uncertainty. The key difference with the less extreme scenario investigated in Section 5 is that DM can no longer switch to the myopic optimum once the tipping point has been passed since he ignores this event. Yet, DM must account for that possibility when choosing his action plan.

DYNAMIC PROGRAMMING. We first define the value function $\tilde{\mathcal{V}}^c(X,t)$ (again gross of the term $\frac{D}{r}$) as DM's optimal intertemporal payoff starting from date t onwards when the stock level at date t is X and evolves thereafter as $\tilde{X}(\tau; X, t)$. By definition, we have:

$$\tilde{\mathcal{V}}^c(X,t) \equiv \sup_{\mathbf{x}_t, \tilde{X}(\cdot) \text{ s.t. } (5.1)} \int_0^X e^{-\lambda_0 t} \left(\int_t^{+\infty} e^{-\lambda_1(\tau - t)} u(x(\tau)) d\tau\right) f(\tilde{X}) d\tilde{X} \tag{6.1}$$

$$+ \int_X^{+\infty} \left(\int_t^{\tilde{T}(\tilde{X};X,t)} e^{-\lambda_0 \tau} u(x(\tau)) d\tau\right.$$

$$\left. + e^{-\lambda_0 \tilde{T}(\tilde{X};X,t)} \int_{\tilde{T}(\tilde{X};X,t)}^{+\infty} e^{-\lambda_1(\tau - \tilde{T}(\tilde{X};X,t))} u(x(\tau)) d\tau\right) f(\tilde{X}) d\tilde{X}.$$

The first bracketed term stands for DM's expected payoff if the tipping point was less than X and was already passed so that the rate of arrival of a disaster is θ_1. This expression takes into account the fact that DM can no longer condition his action on this non-observable event. The second bracketed term stands for the expected payoff when the tipping point has not yet been passed. It accounts for the regime shift that will take place at date $\tilde{T}(\tilde{X}; X, t)$, still assuming that this event remains non-observable.

LEMMA 2 *The value function $\tilde{\mathcal{V}}^c(X, t)$ satisfies*

(6.2)

$$\tilde{\mathcal{V}}^c(X,t) = \sup_{\mathbf{x}_t, \tilde{X}(\cdot) \; s.t. \; (5.1)} \int_t^{+\infty} e^{-\lambda_0 \tau} \left(u(x(\tau)) - \Delta F(\tilde{X}(\tau; X, t)) \int_\tau^{+\infty} e^{-\lambda_1(s-\tau)} u(x(s)) ds \right) d\tau \; \forall (X, t)$$

The integrand not only depends on the stock of past actions X, the current action taken at date $\tau \geq t$ but also on the forward-looking stock of future actions $\mathbf{x}_t^\infty$ from that date τ on. Indeed, the term

$$\Delta F(\tilde{X}(\tau; X, t)) e^{-\lambda_0 \tau} \int_\tau^{+\infty} e^{-\lambda_1(s-\tau)} u(x(s)) ds$$

stands for the current value of an increase in the cost of having passed the tipping point at date $\tau \geq t$ (which arises with probability $F(\tilde{X}(\tau; X, t))$ along the path $\tilde{X}(\tau; X, t)$ if no accident has occurred up to date τ (with probability $e^{-\lambda_0 \tau}$). This expression highlights the intertemporal externality that the choices of DM's future selves at dates $\tau \geq t$ exert on his payoff at date t. To illustrate, the externality is most extreme term if all those selves choose the myopic action. *A contrario*, had those selves taken into account their impact on date t's payoff, they would certainly reduce their own actions.

From a dynamic programming viewpoint, the maximization problem now loses its recursive structure and the time-consistency property that ensures that an optimal plan from date t on starting with a stock level X, say $\mathbf{x}_t^\infty(X)$, would remain optimal when the stock reaches a level $\tilde{X}(\tau; X, t)$ at a future date $\tau = t + \varepsilon$. Indeed, when date $t + \varepsilon$ comes, DM now views all future actions $\mathbf{x}_{t+\varepsilon}^\infty(X)$ as being less costly since their contribution to the loss in welfare has diminished by an amount

$$\Delta \int_t^{t+\varepsilon} F(\tilde{X}(\tau; X, t)) e^{-\lambda_0 \tau} \left(\int_\tau^{+\infty} e^{-\lambda_1(s-\tau)} u(x(s)) ds \right) d\tau$$

There are several possible ways of solving this time-inconsistency problem. The first one is to unveil the extra constraints that would be satisfied by a time-consistent action plan. This route is followed for the rest of this section. The second solution, investigated in Section 7 below, is to look for institutional responses to this commitment problem. It is where a *Precautionary Principle* that would regulate actions could *a priori* be attractive. The last avenue is to expand the state that describes the evolution of the system and observe that this state should not be reduced to the sole stock X but instead should includ DM's beliefs on the rate of arrival of a catastrophe since those beliefs may have evolved along the past history. We will come back on this issue in Section 8 below.

Time-Consistent Value Function and Feedback Rule. To find a time-consistent plan of actions, we follow an approach which is similar in spirit although different in details to that developed in Ekeland and Lazrak (2010), Karp (2005, 2007) and Karp and Lee (2003). These authors have analyzed macroeconomic models with time-consistency problems. Roughly speaking their approach consists in importing an equilibrium notion, familiar in discrete-time model, to a continuous time setting. To figure out how it can be done, consider a discrete version of our model where DM would thus commit to an action over each period $[t, t+\varepsilon]$, $[t+\varepsilon, t+2\varepsilon]$, ...$[t+n\varepsilon, t+(n+1)\varepsilon]$ (with $n \in \mathbb{N}$). It is then natural to focus on stationary Markov-perfect subgame equilibria for such a discrete game. In such an equilibrium, DM follows a feedback rule $\tilde{\sigma}^*_\varepsilon(X)$ that defines his current action in terms of the existing stock. This strategy yields a (current value) ε-value function $\tilde{\mathcal{V}}^*_\varepsilon(X,t) = \mathcal{V}^*_\varepsilon(X)e^{-\lambda_0 t}$. Of course, the equilibrium requirement imposes that this feedback rule is a best-response to DM's anticipations of his own future actions, which should themselves obey to the same time-consistent feedback rule although, of course, the stock at those future dates has evolved according to past actions.

The second step would consist in looking at the continuous time case by making the length of the commitment period ε arbitrarily small. When the pair $(\tilde{\mathcal{V}}^*_\varepsilon(X,t), \sigma^*_\varepsilon(X,t))$ converges as ε goes to zero towards a limit $(\mathcal{V}^*(X), \sigma^*(X))$, this limit satisfies a number of important properties that echo those found for $(\mathcal{V}_u(X), \sigma_u(X))$ in Section 5. To get an heuristic derivation of those properties, observe first that, when DM adopts a stationary feedback rule $\sigma^*(X)$, the stock evolves according to

$$\tilde{X}(t;X) = X + \int_0^t \sigma^*(\tilde{X}(\tau;X))d\tau \tag{6.3}$$

We denote the (current value) *pseudo-value function* $\mathcal{V}^*(X)$ associated to this stationary trajectory as:

(6.4)

$$\mathcal{V}^*(X) = \int_0^{+\infty} e^{-\lambda_0 \tau}\left(u(\sigma^*(\tilde{X}(\tau;X))) - \Delta F(\tilde{X}(\tau;X))\int_\tau^{+\infty} e^{-\lambda_1(s-\tau)}u(\sigma^*(\tilde{X}(s;X)))ds\right)d\tau \ \forall X$$

From there, we define accordingly $\tilde{\mathcal{V}}^*(X,t) = \mathcal{V}^*(X)e^{-\lambda_0 t}$ for all (X,t). The qualifier *pseudo* comes from the fact that $\mathcal{V}^*(X)$ does not necessarily reach the maximum feasible payoff starting from a stock X. In other words, $\tilde{\mathcal{V}}^*(X,t)$ may differ from $\tilde{\mathcal{V}}^c(X,t)$ (the optimal commitment value) precisely because of the intertemporal externality stressed above.

To evaluate the equilibrium conditions, we need to assess the benefits that DM may have when deviating from the putative feedback rule. To this end, consider a possible deviation that would consist in committing to an action x for a period of length ε, reaching a stock level $X + x\varepsilon$, before jumping back to the above feedback rule. With such deviation, the whole trajectory is modified and becomes

$$\tilde{X}_{(x,\varepsilon)}(\tau;X) = \begin{cases} X + x\tau & \text{if } \tau \in [0,\varepsilon], \\ X + x\varepsilon + \int_0^\tau \sigma^*(\tilde{X}_{(x,\varepsilon)}(s;X))ds & \text{if } \tau \geq \varepsilon. \end{cases} \tag{6.5}$$

Next definition explains which requirement are satisfied by a time-consistent action plan and its associated pseudo-value function.

DEFINITION 1 *A stationary feedback rule $\sigma^*(X)$ and the associated pseudo-value function $\mathcal{V}^*(X)$ are time-consistent if $\mathcal{V}^*(X)$ cannot be improved upon by deviations of the form (6.5) for ε arbitrarily small.*

A FUNCTIONAL EQUATION SATISFIED BY $\mathcal{V}^*(X)$. Writing the equilibrium condition suggested by Defintion 1 gives us some properties satisfied by a candidate pseudo-value function $\mathcal{V}^*(X)$.

PROPOSITION 5 *If the time-consistent pseudo-value function $\mathcal{V}^*(X)$ is C^1, it satisfies the following functional equation:*

(6.6)
$$\mathcal{V}^{*'}(X) = -\zeta + \left(\zeta^2 + 2\lambda_0\left(\mathcal{V}^*(X) - \mathcal{V}_\infty\right) - 2\Delta\mathcal{V}_\infty(1 - F(X)) - \Delta F(X)\int_0^{+\infty} e^{-\lambda_1\tau}\mathcal{V}^{*'2}(\tilde{X}(\tau; X))d\tau\right)^{\frac{1}{2}}$$

with the boundary condition

$$\lim_{X\to+\infty} \mathcal{V}^*(X) = \mathcal{V}_\infty. \tag{6.7}$$

The time-consistent feedback rule is

$$\sigma^*(X) = \zeta + \mathcal{V}^{*'}(X). \tag{6.8}$$

The functional equation (6.6) looks like a HBJ differential equation at first glance, although it is strikingly different. Indeed, it is *non-local* and forward-looking. It depends not only on the current stock but also on future values of the stock along the equilibrium trajectory. Once contemplating a deviation over an interval of an arbitrarily small length, DM takes as given the fact that, in the future, he will stick to the time-consistent feedback rule. The whole profile of these future actions which, from optimality of the feedback rule, depends on future values of the marginal current-value function, is thus taken as given to assess the cost and benefit of any putative deviation.

Next result provides our key existence result.

PROPOSITION 6 *There exists a unique function C^1, $\mathcal{V}^*(X)$, satisfying the functional equation (6.6) and the boundary condition (6.7).*

Characterizing the solution to the functional equation (6.6) together with the boundary condition (6.7) requires involved techniques. We indeed transform this functional equation into a pair of differential equations respectively for the pseudo-value function and the externality component of the payoff, namely

$$\varphi(X) = \frac{1}{2}\int_0^{+\infty} e^{-\lambda_1\tau}\mathcal{V}^{*'2}(\tilde{X}(\tau; X))d\tau.$$

The properties of this system are then analyzed by means of the Hartman-Grobman Theorem which helps us to show the uniqueness of a unique stable manifold. From there, the uniqueness of the time-consistent feedback rule and the *pseudo*-value function follows.

In passing, this analysis provides interesting properties of the asymptotic behavior of these variables and allows us to derive rather tight bounds. In particular, the optimal action (which by definition cannot be conditioned on whether the tipping has been passed or not) now converges towards the myopic optimum while the value function still converges towards the corresponding myopic payoff.

PROPOSITION 7 *$\mathcal{V}^*(X)$ and $\sigma^*(X)$ admit the following bounds:*

$$\mathcal{V}_\infty < \mathcal{V}^*(X) < \mathcal{V}_\infty\left(1+\frac{\Delta}{\lambda_0}(1-F(X))\right) \quad \forall X \geq 0, \quad \forall X \geq 0, \tag{6.9}$$

$$\sqrt{\zeta^2 - 2\Delta\mathcal{V}_\infty} < \sigma^*(X) < \zeta \quad \forall X \geq 0. \tag{6.10}$$

$\mathcal{V}^*(X)$ and $\sigma^*(X)$ admit the following approximations when X is large:

$$\mathcal{V}^*(X) - \mathcal{V}_\infty \approx_{+\infty} \frac{\Delta\mathcal{V}_\infty(1-F(X))}{\zeta R'(0)+\lambda_0}, \tag{6.11}$$

$$\sigma^*(X) \approx_{+\infty} \zeta - \frac{\Delta\mathcal{V}_\infty f(X)}{\zeta R'(0)+\lambda_0}. \tag{6.12}$$

The bounds for the pseudo-value function and the feed-back rule are the same as in the scenario of Section 5. The dynamics are quite similar. To illustrate, the upper bound on $\mathcal{V}^*(X)$ is readily obtained by following a sub-optimal strategy consisting in adopting the myopic action under all circumstances.

In the long run, the stock is likely to have gone through most possible values of the tipping point. The choice of the action then has almost no longer any influence on the rate of arrival of a disaster which is almost surely θ_1. The optimal action thus converges towards the myopically optimal decision as shown in (6.12). At the same time, the value function converges towards its value $\mathcal{V}_\infty$ under a myopic scenario.

EXPONENTIAL DISTRIBUTIONS. Unfortunately, closed-form solutions can no longer be obtained when there is deep uncertainty. Yet, some simple comparisons with the more informative scenario of Section 5 can be readily obtained.

PROPOSITION 8 *Suppose that X is exponentially distributed over $\mathbb{R}_+$, i.e., $f(X) = ke^{-kX}$ and $F(X) = 1-e^{-kX}$ for some $k>0$. For X large enough, the following comparisons hold:*

$$\mathcal{V}_u(X) > \mathcal{V}^*(X) > \mathcal{V}_\infty, \tag{6.13}$$

$$\zeta > \sigma^*(X) > F(X)\zeta + (1-F(X))\sigma_u(X). \tag{6.14}$$

The first condition captures the fact that being ignorant on whether the tipping point has been passed or not reduces DM's value function. In a sense, the difference $\mathcal{V}_u(X) -$

$\mathcal{V}^*(X)$ stems for the value of learning whether the tipping point is passed or not. The value of such information is thus positive. The comparison of the feedback rules given in (6.14) is more interesting. The right-hand side is the average action taken in the scenario where the tipping point is learned. With probability $F(X)$ the tipping point has been passed and DM has switched to the myopic action. With probability $1 - F(X)$ instead, DM knows that the tipping point has not been passed and still chooses $\sigma_u(X)$. This average action is lower than the action taken under full ignorance. To grasp the intuition behind such comparative statics, remember that, under deep uncertainty, DM finds less reasons to commit to distort future actions downwards as time passes. As no disaster has ever arisen, the perceived cost of choosing future actions away from the myopic optimum diminishes. This force pushes actions upwards.

7. THE IRRELEVANCE OF THE *PRECAUTIONARY PRINCIPLE*

The *Precautionary Principle* can be viewed as imposing a legal restriction on the set of actions available to DM. As long as not much information on the arrival rate of the disaster has been revealed, i.e., as long as the accumulated stock remains low and few possible values of the tipping point might have already been passed, actions are constrained. Later, a *"laissez-faire"* solution will prevail and DM will freely choose actions with no restriction beyond the equilibrium conditions embodied in a time-consistent plan.

The simplest way of modeling this issue is to suppose that DM is forced to choose a fixed action x_0 over an interval of (not necessarily infinitesimal) length $\varepsilon > 0$. The stock then reaches a level $X_0 = x_0\varepsilon$ at the end of this phase. Afterwards, DM can follow the time-consistent feedback rule $\sigma^*(X)$ for $X \geq X_0$ and gets the payoff $\mathcal{V}^*(X_0)$ from that date on. The benefit of such policy is to be able to commit to a given action over $[0, \varepsilon]$. The cost is that such a commitment is independent of where the stock lies during that interval while, even though it is imperfectly so, the time-consistent solution keeps track of such information.

Of course, DM should optimize over the fixed action x_0 and the length of the commitment phase. A first and intuitive result is that, at the optimum, the following smooth-pasting condition should hold:

$$\sigma^*(X_0) = x_0 = \frac{X_0}{\varepsilon}. \tag{7.1}$$

This condition just says that, at the end of the commitment period, DM should move continuously from his committed action to the time-consistent feedback rule that will be followed from that date on. If that equality were not to hold, it would have been optimal to extend or contract the length of the commitment period.

This condition also implies that DM's intertemporal payoff under such restriction, say $\Omega(X_0)$, only depends on the final stock at the end of the commitment period. Of course, we have also $\Omega(0) \equiv \mathcal{V}^*(0)$. Next proposition shows how that payoff actually varies with X_0 in that neighborhood.

Proposition 9 *For all X_0 small enough,*

$$\Omega(X_0) \leq \mathcal{V}^*(0)$$

with equality only at $X_0 = 0$.

From this, it follows that slightly expanding the commitment period beyond zero has no *positive value*. Restricting actions over an interval with positive measure is sub-optimal. There are two consequences of this result. The first one is mostly technical. It provides a justification for our previous analysis of the time-consistent scenario that was developed under the assumption that the commitment period was made arbitrarily small.

The second consequence lies at the core of our analysis. Proposition 9 also shows that the *Precautionary Principle* has no rationale. Committing over a period of positive (albeit small) length is never optimal. The benefits of letting future selves act upon having learned a bit more of information on the rate of arrival of the disaster always exceeds the cost of not having them internalizing the impact of their actions on earlier selves.

8. HOW TO RESTORE COMMITMENT? ADDING STATE VARIABLES

Our previous analysis has shown that the decisions that DM's future selves will exert an externality on his current self's payoff. If his future selves adopt the time-consistent feedback rule $\sigma^*(X)$, DM is as well off doing the same today. In this section, we investigate whether there is a way to make future selves internalize the impact of their future actions on DM's current self.

8.1. *The Accounting Approach: Beliefs as a State Variable*

UPDATED BELIEFS. We first compute DM's updated beliefs that a disaster occurs over an interval $[t, t+dt]$ if, starting from an initial stock $X \geq 0$ at date 0, the action plan $\mathbf{x}^t = (x(\tau))_{\tau \leq t}$ has been followed up to date t and no disaster has yet occurred. The corresponding stock $\tilde{X}(t; X)$ is defined as

$$\tilde{X}(t; X) = X + \int_0^t x(\tau) d\tau.\text{[29]} \tag{8.1}$$

Let $\tilde{T}(\tilde{X}; X)$ be the corresponding inverse function. The updated density function that a disaster occurs over an interval $[t, t+dt]$, say $g(t|\mathbf{x}^t, X)$, writes as

$$g(t|\mathbf{x}^t, X) = (1 - F(\tilde{X}(t; X))\theta_0 e^{-\theta_0 t} + \int_0^{\tilde{X}(t;X)} \theta_1 e^{-(\theta_0 \tilde{T}(\tilde{X};X) + \theta_1(t - \tilde{T}(\tilde{X};X)))} f(\tilde{X}) d\tilde{X}.$$

This expression takes into account that, for any date $t \geq 0$, all tipping points $\tilde{X}$ such that $\tilde{X} \leq \tilde{X}(t; X)$ have already been passed and the arrival rate of a disaster has thus increased from θ_0 to θ_1. If instead $\tilde{X} > \tilde{X}(t; X) \geq X$, the arrival rate remains θ_0. Let also $1 - G(t|\mathbf{x}^t, X) = 1 - \int_0^t g(\tau|\mathbf{x}^t, X) d\tau$ denote DM's beliefs that a disaster has not yet occurred up to date t if a path $\tilde{X}(t; X)$ has so far been followed.

LEMMA 3 *DM's beliefs that a disaster does not occur up to date t when an increasing path $\tilde{X}(t; X)$ has been followed up to that date, is given by:*

$$1 - G(t|\mathbf{x}^t, X) = e^{-\theta_0 t} \left(1 - \Delta e^{-\Delta t} \int_0^t F(\tilde{X}(\tau; X)) e^{\Delta \tau} d\tau \right). \tag{8.2}$$

[29] Again, we slightly abuse notations and adopt our previous convention, namely $\tilde{X}(\tau; X) \equiv \tilde{X}(\tau; X, 0)$.

This formula shows how beliefs evolve along the trajectory. At the beginning, $\tilde{X}(t;X)$ is close to X and the likelihood of having passed the tipping point close to $F(X)$. DM still believes that the expected rate of arrival of a disaster is close to θ_0. As $\tilde{X}(t;X)$ increases, it becomes more likely that the tipping point has been passed and beliefs evolve towards thinking that this expected rate is now close to θ_1. Of course, the shape of the distribution function F matters to evaluate such Bayesian updating. As F puts more mass around 0, it becomes more likely that the tipping point has been passed early on and DM is more encline to think that the rate of arrival will quickly shift to θ_1. Instead, if F puts more mass for higher values of X, DM believes that this rate remains θ_0 for a longer period. The shape of F and the choice of actions along the trajectory jointly determine how much has been learned by DM through the process.

A consequence of (8.2) and the fact that $F(X) \leq F(\tilde{X}(t;X)) \leq 1$ for all $t \geq 0$ is that

$$e^{-\theta_1 t} \leq 1 - G(t|\mathbf{x}^t, X) \leq (1-F(X))e^{-\theta_0 t} + F(X)e^{-\theta_1 t} \quad \forall (X,t) \tag{8.3}$$

The right-hand side is the expected probability that no disaster has happened at date t if DM takes a naive view and considers that only the initial stock matters to assess this probability. This inequality thus means that, as the stock evolves, a fully Bayesian DM becomes less optimistic than with such naive stance.

Complete Value Function. By definition, the value function $\tilde{\mathcal{V}}^c(X,t)$ is the maximum intertemporal payoff achievable from any date t on when the stock at that date is X. This payoff is necessarily obtained by committing to an action plan for the whole future.[30] Our previous analysis has shown that such commitment is not time-consistent. We now propose an alternative formulation of this value function which makes it possible to restore time-consistency.

Lemma 4 *The value function* $\tilde{\mathcal{V}}^c(X,t)$ *satisfies* $\tilde{\mathcal{V}}^c(X,t) = \mathcal{V}^c(X)e^{-\lambda_0 t}$ *for all* (X,t) *where*

$$\mathcal{V}^c(X) = \sup_{\mathbf{x},\tilde{X}(\cdot) \ s.t.\ (8.1)} \int_0^{+\infty} e^{-rt}\left(1 - G(t|\mathbf{x}^t, X)\right) u(x(t))d\tau \quad \forall X. \tag{8.4}$$

What is remarkable on this formula is that the commitment value function only depends on the stock X through the impact that the whole trajectory $\tilde{X}(t;X)$ has on $G(t|\mathbf{x}^t,X)$ (see (8.3)). To understand how this result is obtained, remember Definition (6.1). Technically, $\tilde{\mathcal{V}}^c(X,t)$ is actually a double integral, taken first over all possible values of the tipping point and second over time. Adding up the probabilities of all potential scenarios that ensure there has not been a disaster up to date t amounts in fact to re-organizing this double integral; first along time and second along possible values of the tipping point that have already been passed up to that time. Counting paths this way shows that this overall probability is precisely $1-G(t|\mathbf{x}^t,X)$. Yet, expressing DM's intertemporal payoff as (8.4) implicitly assumes that all DM's selves are able to re-organize their payoffs across time; which amounts to assuming that DM can indeed commit. Indeed, (8.4) shows that the optimization problem has a simple recursive structure and thus, the *Principle of Dynamic Programming* now applies.

[30] For instance, $\tilde{\mathcal{V}}^c(0,0) - \frac{D}{r}$ is the ex ante payoff that DM could achieve by committing from date 0 on.

The expression of $\mathcal{V}^c(X)$ as (8.4) also suggests that the state of the system is best described by adding to the value of the current stock X another state variable that would reflect DM's updated beliefs. Indeed, those beliefs summarize how the whole past trajectory matters for current decision. In other words, two trajectories that reach the same value for the current stock at date t and keep those beliefs the same should be optimally continued the same way. Instead, two trajectories that have reached the same stock of past actions at that date but have induced different beliefs might be pursued along two different paths.

The logic behind the necessity of keeping track on how beliefs evolve can be understood by coming back on the one-sided externality that DM's future selves exert on the current one. Roughly, beliefs provide a measure of this externality. When DM follows a feedback rule that depends on his updated beliefs, he implicitly takes care of the impact of his own action on the welfare loss that have been perceived by his previous selves. In other words, internalizing the intertemporal externality is achieved with feedback rules that condition actions not only on stock but also on beliefs.

To capture this possibility formally, we now introduce a new state variable, $\tilde{Z}^c$, which can be thought of as measuring the extent to which updated beliefs about the probability of a disaster arising evolves over time. At time τ, when the stock is X, the ratio between updated and prior beliefs at the beginning of the trajectory provides such index:

$$\frac{1 - G(\tau|\mathbf{x}^\tau, X)}{e^{-\theta_0 \tau}}$$

To be able to track this quantity in all possible configurations of the system, we consider the following law of motion and initial condition:

$$\frac{\partial \tilde{Z}^c}{\partial \tau}(\tau; X, Z) = \Delta(1 - F(\tilde{X}(\tau; X, Z)) - \tilde{Z}^c(\tau; X, Z)) \text{ with } \tilde{Z}(0; X, Z) = Z. \tag{8.5}$$

Notice that the stock trajectory $\tilde{X}(\tau; X, Z)$ is now also contingent on the initial condition Z for the new state variable. Integrating, we immediately get

$$\tilde{Z}^c(\tau; X, Z) = (Z-1)e^{-\Delta\tau} + 1 - \Delta e^{-\Delta\tau} \int_0^\tau F(\tilde{X}^c(s; X, Z))e^{\Delta s} ds. \tag{8.6}$$

In particular, taking $Z = 1$ yields

$$\tilde{Z}^c(\tau; X, 1) = (1 - G(\tau|\mathbf{x}^\tau, X))e^{\theta_0 \tau}. \tag{8.7}$$

Let us now define the *complete value function* $\mathcal{W}^c(X, Z)$ for any $X \geq 0$ and any $Z \in [0, 1]$ as

$$\mathcal{W}^c(X, Z) = \sup_{\mathbf{x}, \tilde{X}(\cdot), \tilde{Z}(\cdot) \text{ s.t. } (8.1) \text{ and } (8.6)} \int_0^{+\infty} e^{-\lambda_0 \tau} \tilde{Z}^c(\tau; X, Z) u(x(\tau)) d\tau. \tag{8.8}$$

Together with the *complete feedback rule* $\sigma^c(X, Z)$, this value function defines the full trajectory of the system both in terms of the overall stock $\tilde{X}^c(t; X, Z)$ but also of the belief index $\tilde{Z}^c(\tau; X, Z)$. In particular, the commitment payoff $\mathcal{V}^c(0) = \mathcal{W}^c(0, 1)$ can be achieved by adopting the feedback rule $\sigma^c(\tilde{X}^c(t; 0, 1), \tilde{Z}(t; 0, 1))$ starting from the initial conditions of the system $X = 0$ and $Z = 1$. Of course, the simple identity $\mathcal{V}^c(X) \equiv \mathcal{W}^c(X, 1)$ holds.

PROPOSITION 10 *If the complete value function $\mathcal{W}^c(X,Z)$ is C^1, $\mathcal{V}^c(X)$ satisfies the following differential equation:*

$$\mathcal{V}^{c'}(X) = -\zeta + \left(\zeta^2 + 2\lambda_0\left(\mathcal{V}^c(X) - \mathcal{V}_\infty\right) - 2\Delta\mathcal{V}_\infty\left(1 - \frac{\lambda_1}{\lambda_0}F(X)\right) - \Delta F(X)\int_0^{+\infty} e^{-\lambda_0\tau}\left(\frac{1}{\tilde{Z}^c(\tau;X,1)}\frac{\partial\mathcal{W}^c}{\partial X}(\tilde{X}^c(\tau;X,1),\tilde{Z}^c(\tau;X,1))\right)^2 d\tau\right)^{\frac{1}{2}} \tag{8.9}$$

with the boundary condition

$$\lim_{X\to+\infty}\mathcal{V}^c(X) = \mathcal{V}_\infty. \tag{8.10}$$

The complete feedback rule is

$$\sigma^c(X,Z) = \zeta + \frac{1}{Z}\frac{\partial\mathcal{W}^c}{\partial X}(X,Z). \tag{8.11}$$

The boundary condition (8.10) is quite intuitive. Whether DM can commit to an action plan or not, all possible values of the tipping point have been passed in the long run and the myopic action becomes optimal. This yields a payoff worth $\mathcal{V}_\infty$, exactly as in the previous scenarios we envisioned earlier on.

The complete value function $\mathcal{W}^c(X,Z)$ satisfies a (bi-dimensional) HBJ equation, whose details are given in the Appendix. The differential equation (8.9) is the trace of this HBJ equation on the manifold $Z = 1$. It is strikingly similar to (6.6) found in the no-commitment case. The difference lies in the last correcting terms where

$$\frac{1}{\tilde{Z}^c(\tau;X,1)}\frac{\partial\mathcal{W}^c}{\partial X}(\tilde{X}^c(\tau;X,1),\tilde{Z}^c(\tau;X,1)) = \sigma^c(\tilde{X}^c(\tau;X,1),\tilde{Z}^c(\tau;X,1)) - \zeta$$

now replaces

$$\mathcal{V}^{*'}(\tilde{X}(\tau;X)) = \sigma^*(\tilde{X}(\tau;X)) - \zeta.$$

If the feedback rule $\sigma^c(X,Z)$ was independent of Z, these two terms would be identical and, as the result of also having the same boundary conditions, $\mathcal{V}^c(X)$ would be equal to $\mathcal{V}^*(X)$. In other words, the incomplete feedback rule $\sigma^*(X)$ would be enough to achieve the full commitment solution. Unfortunately, this cannot be true. The complete feedback rule necessarily keeps track of how beliefs evolve.

PROPOSITION 11 *$\sigma^c(X,Z)$ cannot be a function of X only.*

This analysis shows that payoffs can be re-organized so as to restore a recursive structure and ensure that the *Principle of Dynamic Programming* applies. Yet, this re-organization is somewhat artificial, purely based on counting all paths that avoid a disaster. Implicitly, it amounts to assuming that the different selves meet at once and trade utils before any action plan is chosen. In other words, this accounting approach *de facto* posits that the commitment problem is solved. In practice, payoffs are hard to trade intertemporally, which cast doubts on the validity of this approach.

PROPOSITION 12 *$\mathcal{V}^c(X)$ and $\sigma^c(X,1)$ admit the following bounds:*

$$\mathcal{V}_\infty < \mathcal{V}^c(X) < \mathcal{V}_\infty\left(1+\frac{\Delta}{\lambda_0}(1-F(X))\right) \quad \forall X\geq 0 \tag{8.12}$$

$$\sqrt{\zeta^2-2\Delta\mathcal{V}_\infty} < \sigma^c(X,1) < \zeta \quad \forall X\geq 0. \tag{8.13}$$

We observe that these bounds are exactly the same as those obtained above for $\mathcal{V}^*(X)$ and $\sigma^*(X)$. It suggests that, at least for large values of X, the welfare loss from being unable to commit might not be such a curse, especially in comparison of the requirements in terms of computational cost that come with the commitment solution.

8.2. *The Game Theoretic Approach: The Failure of Using Promises as a State Variable*

Following the seminal work of Abreu, Pearce and Stacchetti (1990), the literature on repeated games has shown how, in the presence of forward-looking incentive constraints that characterize equilibrium play, the whole set of subgame-perfect equilibrium payoffs of a repeated game can be described by using the promise of future utilities as state variables. This approach restores a recursive structure for the game. There are two differences with our more specific context. First, ours is not a repeated game but instead a dynamic game whose state is also defined in terms of the stock of past actions. Second, promised utilities left to future selves are here of little interest. Instead, the expression of DM's payoff suggests that the overall externality that future DM's selves may inflict on the current one is a more attractive state variable. Simple but tedious computations show that the value function $\mathcal{V}^c(X)$ can indeed be expressed as

$$\mathcal{V}^c(X) = \sup_{\mathbf{x},\tilde{X}(\cdot)\text{ s.t. }(8.1)} \mathcal{V}_\infty + \int_0^{+\infty} e^{-\lambda_0\tau}\Bigg(\Delta\mathcal{V}_\infty(1-F(\tilde{X}(\tau;X))-\frac{1}{2}(\zeta-x(\tau))^2 \\ +\Delta F(\tilde{X}(\tau;X))\int_\tau^{+\infty} e^{-\lambda_1(s-\tau)}\frac{1}{2}\left(\zeta-x(s)\right)^2 ds\Bigg)d\tau. \tag{8.14}$$

The term $\frac{1}{2}\int_\tau^{+\infty} e^{-\lambda_1(s-\tau)}\left(\zeta-x(s)\right)^2 d\tau$ captures the whole impact of future selves' actions on DM's current payoff. Of course, the worst scenario for the current self is that his followers adopt the myopic action ζ, a strategy that nullifies this externality component.

Consider a new state variable $\tilde{\varphi}(\cdot)$, capturing the intertemporal externality that might be exerted by future selves and evolving as

$$\frac{\partial\tilde{\varphi}}{\partial\tau}(\tau;X,\varphi) = \lambda_1\tilde{\varphi}(\tau;X,\varphi) - \frac{1}{2}\left(\zeta-x(\tau,X,\varphi)\right)^2 \text{ with } \tilde{\varphi}(0;X,\varphi)=\varphi.\text{[31]} \tag{8.15}$$

Integrating, we immediately get

$$\tilde{\varphi}(\tau;X,\varphi) = e^{\lambda_1\tau}\left(\varphi - \int_0^\tau e^{-\lambda_1 s}\frac{1}{2}\left(\zeta-x(s,X,\varphi)\right)^2 ds\right). \tag{8.16}$$

[31] Notice that the stock trajectory $\tilde{X}(\tau;X,\varphi)$ and the action $x(\tau,X,\varphi)$ should now also be made contingent on the initial conditions (X,φ) imposed on the two states variables $\tilde{X}(\cdot)$ and $\tilde{\varphi}(\cdot)$.

This formulation suggests to define a *promise-keeping value function* $\mathcal{W}^\varphi(X,\varphi)$ for any $X \geq 0$ and any φ as

(8.17)
$$\mathcal{W}^\varphi(X,\varphi) = \sup_{\mathbf{x},\tilde{X}(\cdot),\tilde{\varphi}(\cdot) \text{ s.t. } (8.1) \text{ and } (8.16)} \mathcal{V}_\infty + \int_0^{+\infty} e^{-\lambda_0\tau}\Bigg(\Delta\mathcal{V}_\infty(1-F(\tilde{X}(\tau;X,\varphi)) -\frac{1}{2}(\zeta - x(\tau))^2 + \Delta F(\tilde{X}(\tau;X,\varphi))\tilde{\varphi}(\tau;X,\varphi)\Bigg)d\tau.$$

The structure of the maximization problem (8.17) is now recursive. The *Principle of Dynamic Programming* would thus apply and the solution would be time-consistent with the proviso that such a solution *should exist*. Intuitively, had he been *promised* a given level of externality φ by his future selves and the current stock of past actions is X, DM would optimally choose an action $\sigma^\varphi(X,\varphi)$. This decision would be part of a consistent plan since, by construction of the above value function, in the continuation, future selves would themselves find optimal to stick to the same rule when they inherit of the new stock and promise.

Unfortunately, the dynamics of the new state variable $\tilde{\varphi}(\cdot)$ (8.16) does not ensure that this *promise-keeping value function* is well-defined. Indeed, the integral in (8.17) converges only when $\varphi = \varphi_0(X) = \int_{T^c(X)}^{+\infty} e^{-\lambda_1(s-T^c(X))}\frac{1}{2}\left(\zeta - x^c(s)\right)^2 ds$ where $x^c(t) = \sigma^c(\tilde{X}^c(\tau;X,1),\tilde{Z}^c(\tau;X,1))$ is the full commitment action plan found in Section 8.1 and $T^c(X)$ is defined as $X = \int_0^{T^c(X)} x^c(t)dt$. For $\varphi \neq \varphi_0(X)$, the integral is improper and $\mathcal{W}^\varphi(X,\varphi)$ cannot be defined. Intuitively, suppose that the state of the system is $(X,\varphi_0(X))$ at some date $t = T^c(X)$. The full commitment solution would be to choose an action $x^c(t)$ at this stage. Yet, by choosing instead an action closer to ζ, $\tilde{\varphi}(\tau,X,\varphi_0(X))$ would be slightly higher than $\tilde{\varphi}(\tau,X^c(\tau),\varphi_0(X^c(\tau)))$ in a right-neighborhood of t. In other words, DM could increase arbitrarily his continuation payoff. An unpalatable conclusion showing that promises of externality cannot be used as state variable.

9. CONCLUSION

This paper has discussed the relevance of the *Precautionary Principle,* a controversial legal framework that aims at preventing the reexamination of decisions for certain periods of time. We first argued that such a ban on actions only makes sense when there exists a conflict of interests between decision-makers acting at different points in time. We proposed a simple dynamic setting where a single decision-maker control actions whose cumulative stock over time increases the risk of an environmental catastrophe. Deep uncertainty on the location of the tipping point of such a physical process generates a time-inconsistency problem. By generalizing Bellman techniques in a context where dynamic programming fails, we have characterized the unique non-commitment equilibrium path of actions when the decision-maker can only commit for arbitrarily small lengths of time. We have also provided various comparative statics, especially comparing equilibrium actions with and without deep uncertainty. We have shown that there is no value in extending the commitment period to a fixed action for a non-infinitesimal amount of time; the optimal commitment period is infinitesimal, therefore contradicting the idea contained in the *Precautionary Principle* that commitment helps.

This negative conclusion should of course be taken with a grain of caution. Indeed, it might be reversed in alternative settings, especially when political considerations are at play. To illustrate, consider the possibility that rotating decision-makers with different preferences are democratically elected for periods of finite length. If a first decision-maker knows he is about to step down from power and be replaced with another decision-maker who cares less about the cost of a catastrophe (or has less power to decide, for example if the future involves free trade agreement with countries that care less about the catastrophe) he might enact laws that stipulate that actions may not increase beyond her own optimal control rule[32]. Now the *Precautionary Principle* is akin to a political constraint on future decision-makers. Although attractive, such political considerations would also suggest that a decision-maker who instead does not care much about the catastrophe would force more prudent followers to adopt a minimal level of actions. In fact, we do not observe such a *reverse Precautionary Principle*, which in our view casts doubt on the validity of such political economy foundations for the *Precautionary Principle*.

In the context of our model, the fact that expanding the commitment period to a fixed action beyond an infinitesimal length is of no value also suggests that there might not be so much value to improve commitment by other means. In particular, bringing in a second state variable on top of the stock of past actions (i.e., the beliefs about the state of the system) to condition feedback rules, although it certainly restores commitment, might in fact be of limited value. Indeed, the bounds expressed in Propositions 7 and 12 are identical, meaning that, in the long term, there is not so much difference between the value functions and the feed-back rules with and without commitment.The difference certainly matters more at the very start of the process.

Finally, our analysis also relates to the behavioral economics literature. With an incomplete feed-back rule that only depends on the stock of past actions and does not keep track of the evolution of beliefs, the decision-maker faces a time-consistency problem. Yet, this decision-maker remains rational in the sense that first, he fully understands that his future selves will adopt a similar feed-back rule at equilibrium and second, he takes into account this feature when choosing his own current action. In other words, the time-inconsistency problem comes from some sort of bounded rationality constraint that limits feedback rules but this constraint does not invalidate optimizing and forward-looking behavior.

The conflict of interests between the different selves of the decision-maker that is inherited from such constrained feed-back rules also raises the question of the relevant definition of welfare. Following the steps of the literature on hyperbolic discounting,[33] our approach has been to define welfare as date 0-self's expected utility along the equilibrium path of actions.[34] Our analysis of the *Precautionary Principle*, i.e., a constraint on actions for a-non infinitesimal period, indeed relied on such criterion.

[32]Historical precedents include the Second Amendment to the U.S. Constitution (1791) which guarantees individual citizens' right to bear arms and has prevented reforms despite frequent political interventions; more recently Austria has written in its constitution its refusal of nuclear energy (1999). It will be interesting to follow potential trade disputes between the EU, where some rules are (loosely) based on the Precautionary Principle, and Canada, who refuses precautionary arguments, following the implementation of the CETA.

[33]Strosz (1955), Laibson (1997), Harris and Laibson (2001) and O'Donoghue and Rabin (2003).

[34]Bernheim and Rangel's (2005) reminds us that any welfare analysis is actually based on revealed

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preferences, i.e., the idea that individuals prefer what they choose over what they do not choose, whereas behavioral economics posit that the "true" underlying preferences may not suffice to explain observed choices. Behavioral considerations complicate the task of defining and implementing welfare-improving actions.

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APPENDIX: MAIN PROOFS

PROOF OF PROPOSITION 1: DM's problem is to maximize over all possible continuous differentiable profiles $X(t)$, non-negative action profiles $x(t)$ and switching time t_0, the expression of expected welfare $\mathcal{W}(\mathbf{x})$ given in (4.4) subject to the following constraints:

$$\dot{X}(t) = x(t) \quad \forall t \geq 0, \tag{A.1}$$

$$X(0) = 0, \tag{A.2}$$

$$X(t_0) = X_0, \tag{A.3}$$

$$\text{No condition on } X(\infty), \tag{A.4}$$

$$t_0 \text{ free.} \tag{A.5}$$

We denote by (X^*, x^*, t_0) a solution to this problem. To characterize this optimum, we shall decompose the optimization into two pieces: a first initial phase over $[0, t_0]$ before the tipping point has been reached and a second phase over $[t_0, +\infty)$ after the tipping point. Over each phase, we proceed to a standard optimization with two boundary conditions (A.2) and (A.3) over the first phase and, for the second phase, a terminal condition (A.3) and an initial condition inherited from the first phase. Then, we characterize the optimal date t_0 at which the tipping point is reached.

FIRST PHASE ON $[0, t_0]$. On $[0, t_0]$, we write the Hamiltonian of the maximization problem as:

$$\mathcal{H}_1(X, x, \lambda, t, t_0) = \mathcal{L}_1(X, x, t, t_0) + p_1 x$$

where p_1 is the costate variable for (A.1) on $[0, t_0]$ and the Lagrangean writes as

$$\mathcal{L}_1(X, x, t, t_0) = e^{-\lambda_0 t} u(x)).$$

Pontryagin Principle gives us the following necessary conditions for optimality.[35]

Costate variable $p_1(t)$. $p_1(t)$ is continuously differentiable on $[0, t_0]$ with

$$-\dot{p}_1(t) = \frac{\partial \mathcal{H}_1}{\partial X}(X^*(t), x^*(t), p_1(t), t, t_0) \Leftrightarrow \dot{p}_1(t) = 0 \quad \forall t \in [0, t_0]\,. \tag{A.6}$$

[35]Seierstad and Sydsaeter (1987).

From this, there exists μ such that

(A.7) $\quad p_1(t) = -p \quad \forall t \in [0, t_0]\,.$

Transversality conditions. The boundary conditions (A.2) and (A.3) imply that there are no conditions on $p_1(t)$ at either 0 or t_0.

Control variable $x^*(t)$.

$$\frac{\partial \mathcal{H}_1}{\partial x}(X^*(t), x^*(t), p_1(t), t, t_0) = \begin{cases} 0 & x^*(t) > 0, \\ \leq 0 & x^*(t) = 0 \end{cases}$$

or

(A.8) $\quad x^*(t) = \sup\left\{\zeta - pe^{\lambda_0 t}, 0\right\} \quad \forall t \in [0, t_0]\,.$

Observe that $\zeta - pe^{\lambda_0 t}$ is decreasing in t when $\lambda > 0$ (a claim to be proved below). Of course, it cannot be optimal to have $x^*(t) = 0$ for some non-empty interval $[t_1, t_0]$ since DM would find it optimal to start the second phase earlier at some $t_1 < t_0$ rather than at t_0 because of discounting. Henceforth, we have:

(A.9) $\quad x^*(t) = \zeta - pe^{\lambda_0 t} \quad \forall t \in [0, t_0]\,.$

Second phase on $[t_0, +\infty)$. On this interval, we now write the Hamiltonian of the maximization problem as:

(A.10) $\quad \mathcal{H}_2(X, x, \lambda_2, t, t_0) = \mathcal{L}_2(X, x, t, t_0) + p_2 x$

where p_2 is now the costate variable for (A.1) on $[t_0, +\infty)$ and where the Lagrangean writes as

(A.11) $\quad \mathcal{L}_2(X, x, t, t_0) = e^{-\lambda_1 t} e^{\Delta t_0} u(x).$

Again, Pontryagin Principle gives us the following necessary conditions for optimality.

Costate variable $p_2(t)$. $p_2(t)$ is continuously differentiable on $[t_0, +\infty)$ with

(A.12) $\quad -\dot{p}_2(t) = \dfrac{\partial \mathcal{H}_2}{\partial X}(X^*(t), x^*(t), p_2(t), t, t_0) \Leftrightarrow \dot{p}_2(t) = 0 \quad \forall t \geq t_0.$

From this, we deduce that there exists p_2 such that:

(A.13) $\quad p_2(t) = -p_2 \quad \forall t \geq 0.$

Transversality conditions. The boundary condition (A.3) implies that there are no conditions on $p_2(t_0)$. Instead, a necessary condition for (A.4) is

(A.14) $\quad \lim\limits_{t \to +\infty} p_2(t) X^*(t) = 0.$

Because $X^*(t)$ is non-decreasing by assumption, it follows from (A.13) that necessarily, we should have:

(A.15) $\quad p_2 = 0.$

Control variable $x^*(t)$.

$$\frac{\partial \mathcal{H}_2}{\partial x}(X^*(t), x^*(t), p_2(t), t, t_0) = 0$$

or

$$x^*(t) = \zeta \quad \forall t \in [t_0, +\infty). \tag{A.16}$$

Tipping point. Optimizing with respect to t_0 and taking into account that (A.3) holds yields the following necessary condition:

$$\mathcal{L}_1(X^*(t_0), x^*(t_0^-), t_0, t_0) - \mathcal{L}_2(X^*(t_0), x^*(t_0^+), t_0, t_0) + \int_{t_0}^{+\infty} \frac{\partial \mathcal{L}_2}{\partial t_0}(X^*(t), x^*(t), t, t_0)dt = 0 \tag{A.17}$$

where $x^*(t_0^+)$ and $x^*(t_0^-)$ denote respectively the right-hand side and the left-hand side limits of $x^*(t)$ at t_0.

Taking into account (A.16), we rewrite (A.17) as:

$$e^{-\lambda_0 t_0}\left[\zeta x - \frac{x^2}{2}\right]_{x^*(t_0^-)}^{x^*(t_0^+)} = \Delta\left(\frac{\zeta^2}{2} + D\right)\int_{t_0}^{+\infty} e^{-\lambda_1 t} e^{\Delta t_0} dt$$

or

$$\left[\zeta x - \frac{x^2}{2}\right]_{x^*(t_0^-)}^{x^*(t_0^+)} = \Delta \mathcal{V}_\infty. \tag{A.18}$$

From (A.9) and (A.16), we compute:

$$\left[\zeta x - \frac{x^2}{2}\right]_{x^*(t_0^-)}^{x^*(t_0^+)} = \frac{1}{2}\left(x^*(t_0^-) - \zeta\right)^2 = \frac{p^2}{2} e^{2\lambda_0 t_0}.$$

Inserting into (A.18) yields:

$$p = e^{-\lambda_0 t_0}\sqrt{2\Delta \mathcal{V}_\infty}. \tag{A.19}$$

On the other hand, t_0 also satisfies the condition:

$$X_0 = \int_0^{t_0} x^*(t)dt.$$

Using the expression of $X^*(t)$ obtained by integrating (A.9) between 0 and t_0, we obtain:

$$X_0 = \zeta t_0 - \frac{p}{\lambda_0}\left(e^{\lambda_0 t_0} - 1\right).$$

Inserting into (A.19) yields (4.6).

Unicity. Consider

$$\delta(t) \equiv \frac{\sqrt{2\Delta \mathcal{V}_\infty}}{\lambda_0}\left(1 - e^{-\lambda_0 t}\right) - \zeta t + X_0.$$

We have

$$\delta'(t) = \sqrt{2\Delta \mathcal{V}_\infty} e^{-\lambda_0 t} - \zeta \text{ and } \delta''(t) = -\lambda_0 \sqrt{2\Delta \mathcal{V}_\infty} e^{-\lambda_0 t} < 0.$$

Hence, δ is strictly concave and thus cross zero at most twice. Since, $\delta(0) = X_0 > 0$ and $\lim_{t_0\to+\infty}\delta(t) = -\infty$, there is a unique positive root $t_0 \in \left(\frac{X_0}{\zeta} + \infty\right)$ for (4.6).

MONOTONICITY. Observe that $x^*(t)$ is decreasing with t for $t \in (0, t_0)$, constant thereafter.

STOCK. The expression of $X^*(t)$ in (4.7) is obtained by integrating (A.9) between 0 and t when $t \leq t_0$. The case $t \geq t_0$ is similar.

CONDITION FOR A POSITIVE NPV. We decompose the expression of expected welfare into two pieces:

$$\mathcal{W}([t_0,+\infty)) = e^{\Delta t_0}\int_{t_0}^{+\infty} e^{-\lambda_1 t}\left(\frac{\zeta^2}{2} + D\right)dt - \frac{D}{r} = \mathcal{V}_\infty e^{-\lambda_0 t_0} - \frac{D}{r}$$

and

$$\mathcal{W}([0,t_0]) = \int_0^{t_0} e^{-\lambda_0 t}u(x^*(t))dt = \int_0^{t_0} e^{-\lambda_0 t}\left(\frac{\zeta^2}{2} + D - \frac{1}{2}(x^*(t)-\zeta)^2\right)dt.$$

Taking into account (A.9) and (A.19), we compute:

$$\mathcal{W}([0,t_0]) = \frac{\lambda_1\mathcal{V}_\infty}{\lambda_0}\left(1 - e^{-\lambda_0 t_0}\right) - \frac{\Delta\mathcal{V}_\infty}{\lambda_0}e^{-\lambda_0 t_0}\left(1 - e^{-\lambda_0 t_0}\right).$$

Computing $\mathcal{W}(\mathbf{x}_t^*) = \mathcal{W}([0,t_0]) + \mathcal{W}([t_0,+\infty)))$ yields the non-negativity NPV condition (4.9). *Q.E.D.*

PROOF OF LEMMA 1: Observe that

$$\int_0^X e^{\Delta t}\left(\int_t^{+\infty} e^{-\lambda_1\tau}\left(\frac{\zeta^2}{2} + D\right)d\tau\right)f(\tilde{X})d\tilde{X} = F(X)\mathcal{V}_\infty e^{-\lambda_0 t}.$$

Integrating by parts the second integral in the maximand on the right-hand side of (5.2), we thus obtain:

$$\text{(A.20)}\quad \int_X^{+\infty}\left(\int_t^{\tilde{T}(\tilde{X};X,t)} e^{-\lambda_0\tau}u(x(\tau))d\tau + e^{\Delta\tilde{T}(\tilde{X};X,t)}\int_{\tilde{T}(\tilde{X};X,t)}^{+\infty} e^{-\lambda_1\tau}\lambda_1\mathcal{V}_\infty d\tau\right)f(\tilde{X})d\tilde{X}$$

$$= (1-F(X))\int_t^{+\infty} e^{-\lambda_0\tau}u(x(\tau))d\tau$$

$$-\int_X^{+\infty}(F(\tilde{X}) - F(X))\frac{\partial\tilde{T}}{\partial t}(\tilde{X};X,t)e^{-\lambda_0\tilde{T}(\tilde{X};X,t)}\left(u(x(\tilde{T}(\tilde{X};X,t))) - \lambda_0\mathcal{V}_\infty\right)d\tilde{X}.$$

Changing variables and taking now time as the relevant variable (i.e., setting $\tilde{X} = \tilde{X}(\tau; X, t)$ with $d\tilde{X} = \dot{\tilde{X}}(\tau; X, t)d\tau$), we rewrite

$$\text{(A.21)}\quad \int_X^{+\infty}(F(\tilde{X}) - F(X))\frac{\partial\tilde{T}}{\partial t}(\tilde{X};X,t)e^{-\lambda_0\tilde{T}(\tilde{X};X,t)}\left(u(x(\tilde{T}(\tilde{X};X,t))) - \lambda_0\mathcal{V}_\infty\right)d\tilde{X}$$

$$= \int_t^{+\infty}(F(\tilde{X}(\tau;X,t)) - F(X))e^{-\lambda_0\tau}\left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x(\tau)-\zeta)^2\right)d\tau.$$

Gathering (A.20) and (A.21), the maximand on the right-hand side of (5.2) is expressed as

(A.22)

$$\tilde{\mathcal{V}}_u(X,t) = \sup_{\mathbf{x}_t,\tilde{X}(\cdot) \text{ s.t. } (5.1)} \mathcal{V}_\infty e^{-\lambda_0 t} + \int_t^{+\infty} e^{-\lambda_0 \tau}(1-F(\tilde{X}(\tau;X,t)))\left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x(\tau)-\zeta)^2\right) d\tau \quad \forall (X,t).$$

It immediately follows from (A.22) that we can look for a solution of the form

$$\tilde{\mathcal{V}}_u(X,t) \equiv e^{-\lambda_0 t}\mathcal{V}_u(X) \quad \forall (X,t).$$

Indeed, if the trajectory $\tilde{X}(\tau;X,t)$ with the associated actions $\tilde{x}(\tau;X,t) = \frac{\partial \tilde{X}}{\partial \tau}(\tau;X,t)$ were to maximize the right-hand side of (A.22), the trajectory $\tilde{X}(\tau-t;X,0)$ with the associated actions $\tilde{x}(\tau-t;X,0) = \frac{\partial \tilde{X}}{\partial \tau}(\tau-t;X,0)$ would achieve the maximand for $\tilde{\mathcal{V}}_u(X,0)$. From (A.22), we can thus define the current value function as (5.3).

Q.E.D.

Proof of Proposition 2: Let us denote an action plan over the interval $[t,t+\varepsilon]$ as $\mathbf{x}_t^{t+\varepsilon} = \{x(\tau)\}_{t+\varepsilon\geq\tau\geq t}$ for any arbitrary $\varepsilon \geq 0$. Using (A.22), it is straightforward to check that $\tilde{\mathcal{V}}_u(X,t)$ solves the following recursive condition

$$\text{(A.23)} \quad \tilde{\mathcal{V}}_u(X,t) = \sup_{\mathbf{x}_t^{t+\varepsilon},\tilde{X}(\cdot) \text{ s.t. } (5.1)} \mathcal{V}_\infty e^{-\lambda_0 t}\left(1-e^{-\lambda_0\varepsilon}\right)$$
$$+\int_t^{t+\varepsilon} e^{-\lambda_0\tau}(1-F(\tilde{X}(\tau;X,t)))\left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x(\tau)-\zeta)^2\right)d\tau + \tilde{\mathcal{V}}(\tilde{X}(t+\varepsilon;X,t),t+\varepsilon).$$

Existence of such value function (not necessarily C^1) easily follows from Ekeland and Turnbull (1983, Corollary 2, p.92).[36] As far as smoothness is concerned, we provide the following Lemma.

Lemma A.1 *If the function $\tilde{\mathcal{V}}_u(X,t)$ is C^1, it satisfies the following HBJ equation:*

(A.24)

$$-\frac{\partial\tilde{\mathcal{V}}_u}{\partial t}(X,t) = \sup_{x\in\mathcal{X}}\left\{x\frac{\partial\tilde{\mathcal{V}}_u}{\partial X}(X,t) + e^{-\lambda_0 t}\left(\lambda_0\mathcal{V}_\infty + (1-F(X))\left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x-\zeta)^2\right)\right)\right\}.$$

Proof of Lemma A.1: Consider ε small enough. When $\tilde{\mathcal{V}}_u(X,t)$ is C^1, we can take a first-order Taylor expansion in ε of the maximand in (A.23) to write it as:

$$\tilde{\mathcal{V}}_u(X,t)+$$
$$\varepsilon\left(\frac{\partial\tilde{\mathcal{V}}_u}{\partial t}(X,t) + x\frac{\partial\tilde{\mathcal{V}}_u}{\partial X}(X,t) + e^{-\lambda_0 t}\left(\lambda_0\mathcal{V}_\infty - (1-F(X))\left(\frac{1}{2}(x-\zeta)^2 - \Delta\mathcal{V}_\infty\right)\right)\right) + o(\varepsilon)$$

where $\lim_{\varepsilon\to 0} o(\varepsilon)/\varepsilon = 0$.

Inserting into (A.23) gives us the necessary condition (A.24). *Q.E.D.*

Expressed in terms of the current value value function $\mathcal{V}_u(X)$, the HBJ equation (A.24) now writes as

$$\text{(A.25)} \quad \lambda_0(\mathcal{V}_u(X) - \mathcal{V}_\infty) = \sup_{x\in\mathcal{X}}\left\{x\mathcal{V}_u'(X) + (1-F(X))\left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x-\zeta)^2\right)\right\}$$

and simplifying as

$$\text{(A.26)} \quad \lambda_0(\mathcal{V}_u(X) - \mathcal{V}_\infty) = \zeta\mathcal{V}_u'(X) + \frac{\mathcal{V}_u'^2(X)}{2(1-F(X))} + \Delta\mathcal{V}_\infty(1-F(X)).$$

Feedback rule. Maximizing the right-hand side of (A.25) yields (5.6). *Q.E.D.*

[36] Similar existence arguments can be used throughout the paper and won't be repeated.

The boundary condition (5.5) is immediately obtained from (5.3) and the fact that $1-F(X+\int_0^t x(\tau)d\tau)$ converges towards zero for any positive action profile when $X \to +\infty$.

Proof of Proposition 3: We rewrite (A.26) using (5.5) as:

$$\lambda_0(\mathcal{V}_u(X)-\mathcal{V}_\infty)(1-F(X)) = \frac{\mathcal{V}_u'^2(X)}{2} + \zeta(1-F(X))\mathcal{V}_u'(X) + \Delta\mathcal{V}_\infty(1-F(X))^2. \tag{A.27}$$

Taking the highest root of this second-degree equation in $\mathcal{V}_u'(X)$ (so as to ensure that the feedback rule defined below is positive leading to a stock profile which is increasing over time), we rewrite this ordinary differential equation as (5.4). Let denote the locus of points where $V_u'(X)=0$ as

$$\hat{\mathcal{V}}(X) = \mathcal{V}_\infty + (1-F(X))\frac{\Delta\mathcal{V}_\infty}{\lambda_0}.$$

Observe that, $\hat{\mathcal{V}}(0) = \frac{\lambda_1\mathcal{V}_\infty}{\lambda_0}$, $\hat{\mathcal{V}}(X)$ is decreasing and converges to $\mathcal{V}_\infty$ as X goes to $+\infty$.

Existence. Consider the domain $\mathcal{D} = \{(V,X)|\hat{\mathcal{V}}(X) \geq V \geq \mathcal{V}_\infty \text{ for some } X \geq 0\}$. The boundaries of $\mathcal{D}$ is made of the vertical segment $V \in [\mathcal{V}_\infty, \hat{\mathcal{V}}(0)]$, the horizontal axis $\{V = \mathcal{V}_\infty\}$ and the curve $\{V = \hat{\mathcal{V}}(X), X \geq 0\}$. Let the flow defined by (5.4) $\gamma : (\mathcal{V}_0, X) \to \mathcal{V}(X|\mathcal{V}_0)$ where $\mathcal{V}(X|\mathcal{V}_0)$ is the solution to (5.4) for some fixed initial value $\mathcal{V}_0$. This flow is of course continuous. By construction, any solution $\mathcal{V}(X|\mathcal{V}_0)$ that crosses $\hat{\mathcal{V}}(X)$ at some $X_0 \geq 0$ is such that $\mathcal{V}(X|\mathcal{V}_0)$ is decreasing for $X < X_0$ and increasing for $X > X_0$ and thus can only cross $\hat{\mathcal{V}}(X)$ once. Hence, such solution cannot satisfy the boundary condition (5.5). Take any solution $\mathcal{V}(X|\mathcal{V}_0)$ that crosses the horizontal axis $\{V = \mathcal{V}_\infty\}$ for some $X_2 \geq 0$. At such point, (5.4) indicates that $\mathcal{V}'(X_2|\mathcal{V}_0) < 0$. Such solution cannot converge towards $\mathcal{V}_\infty$ either since, otherwise, there would exist a point $X_3 > X_2$ such that $\mathcal{V}'(X_3|\mathcal{V}_0) = 0$ and $\mathcal{V}(X_3|\mathcal{V}_0) < 0$. At such point, we should also have $\mathcal{V}(X_3|\mathcal{V}_0) = \hat{\mathcal{V}}(X_3)$ which yields a contradiction with $\mathcal{V}(X_3|\mathcal{V}_0) < 0 < \hat{\mathcal{V}}(X_3)$.

From these observations, and from the continuity of the flow γ, we deduce that the reciprocal image of the horizontal line $\{V = \mathcal{V}_\infty\}$ is of the form $[\mathcal{V}_\infty, V_{02})$. Similarly, the reciprocal image of $\{V = \hat{\mathcal{V}}(X), X \geq 0\}$ is of the form $(V_{01}, \hat{\mathcal{V}}(0)]$ with necessarily $V_{02} \leq V_{01}$. Of course, $[\mathcal{V}_\infty, V_{02})$ and $(V_{01}, \hat{\mathcal{V}}(0)]$ cannot overlap because it would violate the local uniqueness of the solution $\mathcal{V}(X|\mathcal{V}_0)$ to (5.4) around $X = 0$ (Cauchy-Lipschitz Unicity Theorem). Thus $[V_{02}, V_{01}]$ is non-empty and necessarily a solution with $\mathcal{V}_0 \in [V_{02}, V_{01}]$ is such that:

$$\lim_{X \to +\infty} \mathcal{V}(X|\mathcal{V}_0) = \mathcal{V}_\infty.$$

This proves existence of a solution $\mathcal{V}_u(X)$ to (5.4) that satisfies the boundary condition (5.5).

Uniqueness. To prove uniqueness of the solution to (5.4) with the boundary condition (5.5), consider two putative distinct solutions to (5.4), say $\mathcal{V}_1$ and $\mathcal{V}_2$ satisfying this boundary condition with $V_1(0) \in [V_{02}, V_{01}]$ and $V_2(0) \in [V_{02}, V_{01}]$. Denote $\Delta\mathcal{V} = \mathcal{V}_1 - \mathcal{V}_2$ and suppose w.l.o.g that $\Delta\mathcal{V}(0) > 0$. Observe that necessarily $\Delta\mathcal{V}(X) > 0$ for all $X \geq 0$ (otherwise there would be a contradiction with Cauchy-Lipschitz Unicity Theorem at a putative date X_4 where $\mathcal{V}_1(X_4) = \mathcal{V}_2(X_4)$ would be supposed). We may compute:

$$\Delta\mathcal{V}'(X) = \frac{2\lambda_0\Delta\mathcal{V}(X)}{\sqrt{\zeta^2 - 2\Delta\mathcal{V}_\infty + 2\lambda_0\frac{\mathcal{V}_1(X)-\mathcal{V}_\infty}{1-F(X)}} + \sqrt{\zeta^2 - 2\Delta\mathcal{V}_\infty + 2\lambda_0\frac{\mathcal{V}_2(X)-\mathcal{V}_\infty}{1-F(X)}}}.$$

Integrating, we get:

$$\Delta\mathcal{V}(X) = \Delta\mathcal{V}(0)e^{\int_0^X \frac{2\lambda_0 d\tilde{X}}{\sqrt{\zeta^2-2\Delta\mathcal{V}_\infty+2\lambda_0\frac{\mathcal{V}_1(\tilde{X})-\mathcal{V}_\infty}{1-F(\tilde{X})}}+\sqrt{\zeta^2-2\Delta\mathcal{V}_\infty+2\lambda_0\frac{\mathcal{V}_2(\tilde{X})-\mathcal{V}_\infty}{1-F(\tilde{X})}}}}.$$

Observe that both $\mathcal{V}_1$ and $\mathcal{V}_2$ satisfy (5.7) when $\mathcal{V}_1(0) \in [V_{02}, V_{01}]$ and $\mathcal{V}_2(0) \in [V_{02}, V_{01}]$. It implies that

$$\int_0^X \frac{2\lambda_0 d\tilde{X}}{\sqrt{\zeta^2 - 2\Delta\mathcal{V}_\infty + 2\lambda_0 \frac{\mathcal{V}_1(\tilde{X}) - \mathcal{V}_\infty}{1-F(\tilde{X})}} + \sqrt{\zeta^2 - 2\Delta\mathcal{V}_\infty + 2\lambda_0 \frac{\mathcal{V}_2(\tilde{X}) - \mathcal{V}_\infty}{1-F(\tilde{X})}}} \geq \frac{\lambda_0}{\zeta} X.$$

Hence, $\Delta\mathcal{V}(0) > 0$ also implies

$$\lim_{X \to +\infty} \Delta\mathcal{V}(X) = +\infty.$$

A contradiction with our assumption that $\mathcal{V}_1$ and $\mathcal{V}_2$ both satisfy the boundary condition (5.5). It follows that there exists a unique solution to (5.4) satisfying (5.5).

Comparative statics. The above analysis shows that any solution $\mathcal{V}_u(X)$ also satisfies

$$\mathcal{V}_\infty < \mathcal{V}_u(X) < \hat{\mathcal{V}}(X).$$

Rearranging terms on the right-hand side gives (5.7). Inserting now (5.7) into (5.4), we obtain

$$(1 - F(X))\left(-\zeta + \sqrt{\zeta^2 - 2\Delta\mathcal{V}_\infty}\right) < \mathcal{V}_u'(X) < 0.$$

Inserting into (5.6) yields (5.8).

To look at the long-term behavior, we first change variables and define

$$Y = 1 - F(X) \in [0,1] \text{ and } \mathcal{V}_u(X) - \mathcal{V}_\infty = \mathcal{U}(Y) \text{ and } R(Y) = f(F^{-1}(1-Y))$$

From this, we get

$$\mathcal{V}_u'(X) = -\mathcal{U}'(Y)R(Y).$$

Inserting into (5.4) yields

$$\mathcal{U}'(Y) = \frac{Y}{R(Y)}\left(\zeta - \sqrt{\zeta^2 - 2\Delta\mathcal{V}_\infty + 2\lambda_0 \frac{\mathcal{U}(Y)}{Y}}\right). \tag{A.28}$$

From (5.5), we deduce that

$$\mathcal{U}(0) = 0. \tag{A.29}$$

Observe that $R(0) = 0$ and $R'(0) = \lim_{Y\to 0} \frac{R(Y)}{Y}$. Taking the limit of (A.28) to $(Y = 0, \mathcal{U}(0) = 0)$, we find that $\mathcal{U}'(0)$ must solve

$$\mathcal{U}'(0) = \frac{1}{R'(0)}\left(\zeta - \sqrt{\zeta^2 - 2\Delta\mathcal{V}_\infty + 2\lambda_0 \mathcal{U}'(0)}\right).$$

After manipulations, we find that $\mathcal{U}'(0)$ must solve (A.36) for $k = R'(0)$. From this, (5.9) immediately follows. Further, notice that

$$\frac{V_u(X) - V_\infty}{1 - F(X)} = \frac{\mathcal{U}(Y)}{Y} \text{ so } \lim_{X\to\infty} \frac{V_u(X) - V_\infty}{1 - F(X)} = \mathcal{U}'(0)$$

Which yields (5.10).

A Verification Theorem. Proposition A.1 shows that the conditions given Proposition 3 to characterize a value function by means of an HBJ equation together with a boundary conditions are in fact sufficient. We follow Ekeland and Turnbull (1983, Theorem 1, p. 6) and derive a *Verification Theorem.*

PROPOSITION A.1 *Assume first that there exists a C^1 function $\mathcal{V}_0(X)$ which satisfies:*

$$\lambda_0(\mathcal{V}_0(X)-\mathcal{V}_\infty) \geq x\mathcal{V}_0'(X)+(1-F(X))\left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x-\zeta)^2\right) \quad \forall(x,X); \tag{A.30}$$

and, second, that there exists an action profile $\mathbf{x}_0$ and $X_0(t)=\int_0^t x_0(\tau)d\tau$ such that

$$\lambda_0(\mathcal{V}_0(X_0(t))-\mathcal{V}_\infty) = x_0(t)\mathcal{V}_0'(X_0(t))+(1-F(X_0(t)))\left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x_0(t)-\zeta)^2\right) \quad \forall t\geq 0. \tag{A.31}$$

Then $\mathbf{x}_0$ is an optimal action profile with its associated path $X_0(t)$.

PROOF OF PROPOSITION A.1: First observe that $\mathcal{V}_u(X)$ as characterized in Proposition 3 is C^1. It is our candidate for the function $\mathcal{V}_0(X)$ in the statement of Proposition A.1. By definition (A.27), we have

$$\lambda_0(\mathcal{V}_u(X)-\mathcal{V}_\infty) = \sigma_u(X)\mathcal{V}_u'(X)+(1-F(X))\left(\Delta\mathcal{V}_\infty - \frac{1}{2}(\sigma_u(X)-\zeta)^2\right)$$

and thus

$$\lambda_0(\mathcal{V}_u(X)-\mathcal{V}_\infty) \geq x\mathcal{V}_u'(X)+(1-F(X))\left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x-\zeta)^2\right) \quad \forall(x,X) \tag{A.32}$$

where the inequality comes from the concavity in x of the right-hand side.

To get (A.31), we use again (A.27) but now applied to the path $(x_0(t),X_0(t))\equiv(x_u(t),X_u(t))$ where $X_u(t)$ is such that $\dot{X}_u(t)=x_u(t)=\sigma_u(X_u(t))$ with $X_u(0)=0$.

Define now a value function $\tilde{\mathcal{V}}_u(X,t)=e^{-\lambda_0 t}\mathcal{V}_u(X)$. By (A.32), we get

$$0 \geq \frac{\partial\tilde{\mathcal{V}}_u}{\partial t}(X,t)+x\frac{\partial\tilde{\mathcal{V}}_u}{\partial X}(X,t)+e^{-\lambda_0 t}\left(\lambda_0\mathcal{V}_\infty+(1-F(X))\left(\Delta\mathcal{V}_\infty-\frac{1}{2}(x-\zeta)^2\right)\right) \quad \forall(x,X). \tag{A.33}$$

Using $x_u(t)=\sigma_u(X_u(t))$ and (A.31), we also get

$$0 = \frac{\partial\tilde{\mathcal{V}}_u}{\partial t}(X_u(t),t)+x_u(t)\frac{\partial\tilde{\mathcal{V}}_u}{\partial X}(X_u(t),t)+e^{-\lambda_0 t}\left(\lambda_0\mathcal{V}_\infty+(1-F(X_u(t)))\left(\Delta\mathcal{V}_\infty-\frac{1}{2}(x_u(t)-\zeta)^2\right)\right) \quad \forall t\geq 0. \tag{A.34}$$

Take now an arbitrary action plan $\mathbf{x}$ with the associated path $X(t)=\int_0^t x(\tau)d\tau$. Let us fix an arbitrary $T>0$. Integrating (A.33) along the path $(x(t),X(t))$, we compute

$$0 \geq \int_0^T \left(\frac{\partial\tilde{\mathcal{V}}_u}{\partial t}(X(t),t)+x(t)\frac{\partial\tilde{\mathcal{V}}_u}{\partial X}(X(t),t)+e^{-\lambda_0 t}\left(\lambda_0\mathcal{V}_\infty+(1-F(X(t)))\left(\Delta\mathcal{V}_\infty-\frac{1}{2}(x(t)-\zeta)^2\right)\right)\right)dt$$

or

$$0 \geq \int_0^T \left(\frac{d\tilde{\mathcal{V}}_u}{dt}(X(t),t)+e^{-\lambda_0 t}\left(\lambda_0\mathcal{V}_\infty+(1-F(X(t)))\left(\Delta\mathcal{V}_\infty-\frac{1}{2}(x(t)-\zeta)^2\right)\right)\right)dt \quad \forall T\geq 0$$

By definition of the total derivative of $\tilde{\mathcal{V}}_u$ with respect to time, we thus get

$$\tilde{\mathcal{V}}_u(0,0) \geq \tilde{\mathcal{V}}_u(X(T),T)+\mathcal{V}_\infty\left(1-e^{-\lambda_0 T}\right)+\int_0^T(1-F(X(t)))\left(\Delta\mathcal{V}_\infty-\frac{1}{2}(x(t)-\zeta)^2\right)dt \quad \forall T\geq 0.$$

Because $\tilde{\mathcal{V}}_u(X,t) = e^{-\lambda_0 t}\mathcal{V}_u(X) \geq 0$ for all (X,t), we obtain:

$$\tilde{\mathcal{V}}_u(0,0) \geq e^{-\lambda_0 T}\left(\mathcal{V}_u(X) - \mathcal{V}_\infty\right) + \mathcal{V}_\infty + \int_0^T e^{-\lambda_0 t}(1-F(X(t)))\left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x(t)-\zeta)^2\right)dt \quad \forall T \geq 0$$

and thus

$$\tilde{\mathcal{V}}_u(0,0) \geq \mathcal{V}_\infty + \int_0^T e^{-\lambda_0 t}(1-F(X(t)))\left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x(t)-\zeta)^2\right)dt \quad \forall T \geq 0$$

since $\mathcal{V}_u(X) \geq \mathcal{V}_\infty$ for all $X \geq 0$ from our above findings.

Because $|e^{-\lambda_0 t}(1-F(X(t)))\left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x(t)-\zeta)^2\right)| \leq Me^{-\lambda_0 t}$ for some M when $x \in \mathcal{X} = [0,\zeta]$ the above integral converges for any feasible path $(x(t), X(t))$ as T goes to $+\infty$. Hence, we can write

$$\tilde{\mathcal{V}}_u(0,0) \geq \mathcal{V}_\infty + \int_0^{+\infty} e^{-\lambda_0 t}(1-F(X(t)))\left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x(t)-\zeta)^2\right)dt.$$

Moreover, integrating (A.34), the inequality above is indeed an equality for $(x_u(t), X_u(t))$:

$$\tilde{\mathcal{V}}_u(0,0) = \mathcal{V}_\infty + \int_0^{+\infty} e^{-\lambda_0 t}(1-F(X(t)))\left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x(t)-\zeta)^2\right)dt.$$

which shows that $(x_u(t), X_u(t))$ is indeed an optimal path. *Q.E.D.*

Q.E.D.

Proof of Proposition 4: The HBJ equation (A.26) now writes as:

$$\lambda_0(\mathcal{V}_u(X) - \mathcal{V}_\infty)e^{-kX} = \frac{1}{2}\mathcal{V}_u'^2(X) + \zeta e^{-kX}\mathcal{V}_u'(X) + \Delta\mathcal{V}_\infty e^{-2kX}. \tag{A.35}$$

This expression suggests looking for a solution of the form

$$\mathcal{V}_u(X) = \mathcal{V}_\infty + \alpha e^{-kX}$$

for some $\alpha > 0$. Inserting into (A.35), it is immediate to check that such α is a root to the following second-order equation:

$$\frac{\alpha^2}{2} - \left(\frac{\lambda_0}{k^2} + \frac{\zeta}{k}\right)\alpha + \frac{\Delta\mathcal{V}_\infty}{k^2} = 0.$$

To ensure that the stock $X(t)$ is an increasing function, we select the lowest non-negative root, namely

$$\alpha = \frac{\lambda_0}{k^2} + \frac{\zeta}{k} - \sqrt{\left(\frac{\lambda_0}{k^2} + \frac{\zeta}{k}\right)^2 - 2\frac{\Delta\mathcal{V}_\infty}{k^2}}. \tag{A.36}$$

From there, (5.14), (5.15) and (5.16) immediately follow. *Q.E.D.*

Proof of Lemma 2: Starting from the definition (6.1) and integrating by parts, we obtain

$$\tilde{\mathcal{V}}^c(X,t) = \left[e^{\Delta t}\left(\int_t^{+\infty} e^{-\lambda_1\tau}u(x(\tau))d\tau\right)\left(F(\tilde{X})-F(X)\right)\right]_0^X$$

$$+\left[\left(\int_t^{\tilde{T}(\tilde{X};X,t)} e^{-\lambda_0\tau}u(x(\tau))d\tau\right)\left(F(\tilde{X})-F(X)\right)\right]_X^{+\infty}$$

$$+\left[\left(e^{\Delta\tilde{T}(\tilde{X};X,t)}\int_{\tilde{T}(\tilde{X};X,t)}^{+\infty} e^{-\lambda_1\tau}u(x(\tau))d\tau\right)\left(F(\tilde{X})-F(X)\right)\right]_X^{+\infty}$$

$$-\int_X^{+\infty}\frac{\partial\tilde{T}}{\partial t}(\tilde{X};X,t)e^{-\lambda_0\tilde{T}(\tilde{X};X,t)}u(x(\tilde{T}(\tilde{X};X,t)))\left(F(\tilde{X})-F(X)\right)d\tilde{X}$$

$$+\int_X^{+\infty}\frac{\partial\tilde{T}}{\partial t}(\tilde{X};X,t)e^{-\lambda_0\tilde{T}(\tilde{X};X,t)}u(x(\tilde{T}(\tilde{X};X,t)))\left(F(\tilde{X})-F(X)\right)d\tilde{X}$$

$$-\int_X^{+\infty}\frac{\partial\tilde{T}}{\partial t}(\tilde{X};X,t)\Delta e^{\Delta\tilde{T}(\tilde{X};X,t)}\left(\int_{\tilde{T}(\tilde{X};X,t)}^{+\infty} e^{-\lambda_1\tau}u(x(\tau))d\tau\right)\left(F(\tilde{X})-F(X)\right)d\tilde{X}.$$

Simplifying and changing variables by using $\tilde{X}(\tau;X,t)=\tilde{X}$ and $\tilde{T}(\tilde{X};X,t)=\tau$, we obtain:

$$\tilde{\mathcal{V}}^c(X,t) = F(X)e^{\Delta t}\int_t^{+\infty} e^{-\lambda_1\tau}u(x(\tau))d\tau + (1-F(X))\int_t^{+\infty} e^{-\lambda_0\tau}u(x(\tau))d\tau$$

$$-\Delta\int_t^{+\infty}(F(\tilde{X}(\tau;X,t))-F(X))e^{\Delta\tau}\left(\int_\tau^{+\infty} e^{-\lambda_1 s}u(x(s))ds\right)d\tau.$$

Integrating by part the terms in $F(X)$ in the last line, this simplifies to

$$\tilde{\mathcal{V}}^c(X,t) = F(X)e^{\Delta t}\int_t^{+\infty} e^{-\lambda_1\tau}u(x(\tau))d\tau + (1-F(X))\int_t^{+\infty} e^{-\lambda_0\tau}u(x(\tau))d\tau$$

$$-\Delta\int_t^{+\infty}F(\tilde{X}(\tau;X,t))e^{\Delta\tau}\left(\int_\tau^{+\infty} e^{-\lambda_1 s}u(x(s))ds\right)d\tau$$

$$+F(X)\left[e^{\Delta\tau}\left(\int_\tau^{+\infty} e^{-\lambda_1 s}u(x(s))d\tau\right)\right]_t^{+\infty} + F(X)\int_t^{+\infty} e^{-\lambda_0 t}u(x(\tau))d\tau.$$

After simplifications, the first and fourth terms cancel out, and, from the second, the third and the fifth terms, we finally obtain (6.2). *Q.E.D.*

Proof of Proposition 5: Observe that stationarity implies

$$\tilde{X}_{(x,\varepsilon)}(\tau;X) = \tilde{X}(\tau-\varepsilon;X+x\varepsilon) \text{ if } \tau\geq\varepsilon. \tag{A.37}$$

DM's payoff (expressed in current value) changes accordingly from (6.4) as

$$\mathcal{V}^*(X+x\varepsilon)e^{-\lambda_0\varepsilon}+\int_0^\varepsilon e^{-\lambda_0\tau}u(x)d\tau \tag{A.38}$$

$$-\Delta\int_0^\varepsilon F(X+x\tau)e^{\Delta\tau}\left(\int_\tau^\varepsilon e^{-\lambda_1 s}u(x)ds+\int_\varepsilon^{+\infty} e^{-\lambda_1 s}u(\sigma^*(\tilde{X}(s-\varepsilon;X+x\varepsilon)))ds\right)d\tau.$$

If the function $\mathcal{V}^*(X)$ is C^1, we can write a first-order Taylor expansion in ε of this quantity and get

$$\mathcal{V}^*(X)+\varepsilon\left(x\mathcal{V}^{*'}(X)-\lambda_0\mathcal{V}^*(X)+u(x)-\Delta F(X)\int_0^{+\infty}e^{-\lambda_1\tau}u(\sigma^*(\tilde{X}(\tau;X))))d\tau\right)+o(\varepsilon) \tag{A.39}$$

where $\lim_{\varepsilon\to 0} o(\varepsilon)/\varepsilon = 0$.

Consistency implies that this payoff should be no more than $\mathcal{V}^*(X)$ itself. A necessary condition is thus that the maximum over x of the bracketed term should be non-positive and achieved at $\sigma^*(X)$. Therefore, the following conditions should hold:

$$\lambda_0\mathcal{V}^*(X)=\sup_{x\in\mathcal{X}} x\mathcal{V}^{*'}(X)+u(x)-\Delta F(X)\int_0^{+\infty}e^{-\lambda_1\tau}u(\sigma^*(\tilde{X}(\tau;X))))d\tau \tag{A.40}$$

and

$$\sigma^*(X)\in\arg\sup_{x\in\mathcal{X}} x\mathcal{V}^{*'}(X)+u(x). \tag{A.41}$$

From (A.41), a first-order condition immediately yields (6.8). From there, we find the corresponding definition of the trajectory (6.3). Now, manipulating the right-hand side of (A.40), we find

$$\int_0^{+\infty}e^{-\lambda_1\tau}u(\mathcal{V}^{*'}(\tilde{X}(\tau;X))+\zeta)d\tau=\mathcal{V}_\infty-\frac{1}{2}\int_0^{+\infty}e^{-\lambda_1\tau}\mathcal{V}^{*'2}(\tilde{X}(\tau;X))d\tau.$$

Putting things together and inserting into (A.40), we obtain

$$\lambda_0\left(\mathcal{V}^*(X)-\mathcal{V}_\infty\right)=\sup_{x\in\mathcal{X}} x\left(\mathcal{V}^{*'}(X)+\zeta\right)-\frac{x^2}{2}-\frac{\zeta^2}{2}+\Delta\mathcal{V}_\infty(1-F(X))+\frac{\Delta}{2}F(X)\int_0^{+\infty}e^{-\lambda_1\tau}\mathcal{V}^{*'2}(\tilde{X}(\tau;X))d\tau \tag{A.42}$$

which is finally written as

$$\lambda_0\left(\mathcal{V}^*(X)-\mathcal{V}_\infty\right)=\zeta\mathcal{V}^{*'}(X)+\frac{1}{2}\mathcal{V}^{*'2}(X)+\Delta\mathcal{V}_\infty(1-F(X))+\frac{\Delta}{2}F(X)\int_0^{+\infty}e^{-\lambda_1\tau}\mathcal{V}^{*'2}(\tilde{X}(\tau;X))d\tau. \tag{A.43}$$

Taking the relevant positive root (remember that, from (6.8), the feedback rule $\sigma^*(X)=\mathcal{V}^{*'}(X)+\zeta$ must always remain non-negative), we now rewrite (A.43) as

$$\mathcal{V}^{*'}(X)=-\zeta+\sqrt{\zeta^2+2\lambda_0\left(\mathcal{V}^*(X)-\mathcal{V}_\infty\right)-2\Delta\mathcal{V}_\infty(1-F(X))-2\Delta F(X)\varphi(X)} \tag{A.44}$$

where define $\varphi(X)$ as

$$\varphi(X)=\frac{1}{2}\int_0^{+\infty}e^{-\lambda_1\tau}\mathcal{V}^{*'2}(\tilde{X}(\tau;X))d\tau. \tag{A.45}$$

We will demonstrate below (see Condition (A.83) below) that the quantity in the square root remains positive for all $X\geq 0$. Finally, (6.6) follows from (A.45) and (A.44).

To get the limiting behavior (6.7), we prove the following Lemma.

LEMMA A.2 *$\mathcal{V}^*(X)$ is non-increasing and satisfies*

$$\lim_{X\to+\infty}\mathcal{V}^*(X)=\mathcal{V}_\infty. \tag{A.46}$$

PROOF OF LEMMA A.2: From (A.42) and taking $x = \zeta$ on the right-hand side, we get

$$\lambda_0 \left(\mathcal{V}^*(X) - \mathcal{V}_\infty\right) \geq \zeta \mathcal{V}^{*'}(X) + \Delta \mathcal{V}_\infty (1 - F(X)) + \frac{\Delta}{2} F(X) \int_0^{+\infty} e^{-\lambda_1 \tau} \mathcal{V}^{*'2}(\tilde{X}(\tau; X)) d\tau$$

and thus

$$\mathcal{V}^{*'}(X) \leq -\frac{\Delta \mathcal{V}_\infty}{\zeta}(1 - F(X)) + \frac{\lambda_0}{\zeta}\left(\mathcal{V}^*(X) - \mathcal{V}_\infty\right). \tag{A.47}$$

Now, we rewrite (6.4) after an integration by parts as

$$\mathcal{V}^*(X) = \int_0^{+\infty} e^{-\lambda_0 \tau} \left(1 - \Delta e^{-\Delta \tau} \int_0^\tau F(\tilde{X}(s; X)) e^{\Delta s} ds\right) u(\sigma^*(\tilde{X}(\tau; X))) d\tau \quad \forall X.$$

Because $\tilde{X}(s; X) \geq X$ and $u(\sigma^*(\tilde{X}(\tau; X))) \leq \lambda_1 \mathcal{V}_\infty$, we then obtain

$$\mathcal{V}^*(X) \leq \lambda_1 \mathcal{V}_\infty \int_0^{+\infty} e^{-\lambda_0 \tau} \left(1 - e^{-\Delta \tau} F(X) \left(e^{\Delta \tau} - 1\right)\right) d\tau = \mathcal{V}_\infty \left(F(X) + \frac{\lambda_1}{\lambda_0}(1 - F(X))\right) \quad \forall X \tag{A.48}$$

and thus

$$\mathcal{V}^*(X) - \mathcal{V}_\infty \leq \frac{\Delta \mathcal{V}_\infty}{\lambda_0}(1 - F(X)) \quad \forall X.$$

This yields the upper bound in (6.9).

Inserting into (A.47) and simplifying, yields

$$\mathcal{V}^{*'}(X) \leq 0 \quad \forall X.$$

Because $\mathcal{V}^*(X) \geq 0$, $\mathcal{V}^*(X)$ converges when $X \to +\infty$. Let l be the corresponding limit. From (A.48), it follows that

$$l \leq \mathcal{V}_\infty. \tag{A.49}$$

Applying Gronwall's Lemma to (A.47) yields

$$(\mathcal{V}^*(X) - \mathcal{V}_\infty) e^{-\frac{\lambda_0}{\zeta} X} \geq \frac{\Delta \mathcal{V}_\infty}{\zeta} \int_X^{+\infty} (1 - F(\tilde{X})) e^{-\frac{\lambda_0}{\zeta} \tilde{X}} d\tilde{X}.$$

Thus

$$\mathcal{V}^*(X) \geq \mathcal{V}_\infty + \frac{\Delta \mathcal{V}_\infty}{\zeta} e^{\frac{\lambda_0}{\zeta} X} \int_X^{+\infty} (1 - F(\tilde{X})) e^{-\frac{\lambda_0}{\zeta} \tilde{X}} d\tilde{X}.$$

Taking limits, we get

$$l \geq \mathcal{V}_\infty. \tag{A.50}$$

Taking together (A.49) and (A.50) yields (A.46). *Q.E.D.*

Q.E.D.

Proofs of Propositions 6 and 7: Differentiating (A.45) with respect to X yields

$$\varphi'(X) = \int_0^{+\infty} e^{-\lambda_1 \tau} \mathcal{V}^{*'}(\tilde{X}(\tau;X))\mathcal{V}^{*''}(\tilde{X}(\tau;X))\frac{\partial \tilde{X}}{\partial X}(\tau;X)d\tau. \tag{A.51}$$

The following is a standard result on flows that is here recast in the case of a stationary flow.

Lemma A.3

$$\frac{\partial \tilde{X}}{\partial X}(\tau;X) = \frac{\sigma^*(\tilde{X}(\tau;X))}{\sigma^*(X)}. \tag{A.52}$$

Proof of Lemma A.3: Starting with the definition of $\tilde{X}(\tau;X)$ we get:

$$\frac{\partial \tilde{X}}{\partial \tau}(\tau;X) = \sigma^*(\tilde{X}(\tau;X))$$

and

$$\frac{\partial \tilde{X}}{\partial \tau}(\tau;X+dX) = \sigma^*(\tilde{X}(\tau;X+dX)).$$

Taking dX small and using a first-order Taylor approximation, we get:

$$\sigma^*(\tilde{X}(\tau;X+dX)) = \sigma^*(\tilde{X}(\tau;X)) + \sigma^{*'}(\tilde{X}(\tau;X))\frac{\partial \tilde{X}}{\partial X}(\tau;X)dX + o(dX)$$

where $\lim_{dX\to 0} o(dX)/X = 0$. Therefore, we get:

$$\frac{\partial \tilde{X}}{\partial \tau}(\tau;X+dX) - \frac{\partial \tilde{X}}{\partial \tau}(\tau;X) = \sigma^{*'}(\tilde{X}(\tau;X))\frac{\partial \tilde{X}}{\partial X}(\tau;X)dX + o(dX).$$

Using a first-order Taylor approximation of the left-hand side and simplifying, we get:

$$\frac{\partial}{\partial \tau}\left(\frac{\partial \tilde{X}}{\partial X}(\tau;X)\right) = \sigma^{*'}(\tilde{X}(\tau;X))\frac{\partial \tilde{X}}{\partial X}(\tau;X).$$

Thus,

$$\frac{\partial \log}{\partial \tau}\left(\frac{\partial \tilde{X}}{\partial X}(\tau;X)\right) = \sigma^{*'}(\tilde{X}(\tau;X)).$$

Integrating and taking into account that $\tilde{X}(0;X) = X$ yields

$$\frac{\partial \tilde{X}}{\partial X}(\tau;X) = e^{\int_0^\tau \sigma^{*'}(\tilde{X}(s;X))ds}. \tag{A.53}$$

Using the stationarity of the feedback rule and differentiating with respect to t yields

$$\sigma^{*'}(\tilde{X}(\tau;X)) = \frac{\frac{\partial^2 \tilde{X}}{\partial \tau^2}(\tau;X)}{\frac{\partial \tilde{X}}{\partial \tau}(\tau;X)}. \tag{A.54}$$

Inserting into (A.53) and integrating yields

$$\frac{\partial \tilde{X}}{\partial X}(\tau;X) = exp\left(ln\left(\frac{\frac{\partial \tilde{X}}{\partial \tau}(\tau;X)}{\frac{\partial \tilde{X}}{\partial \tau}(0;X)}\right)\right)$$

and thus

$$\frac{\partial \tilde{X}}{\partial X}(\tau;X) = \frac{\sigma^*(\tilde{X}(\tau;X))}{\sigma^*(\tilde{X}(0;X))}.$$

Noticing that $\tilde{X}(0;X) = X$ yields (A.52). *Q.E.D.*

From (A.52), it follows that

$$\frac{\partial \tilde{X}}{\partial X}(\tau; X) = \frac{\frac{\partial \tilde{X}}{\partial \tau}(\tau; X)}{\sigma^*(X)}.$$

Inserting now into (A.51) yields

$$\sigma^*(X)\varphi'(X) = \int_0^{+\infty} e^{-\lambda_1 \tau} \mathcal{V}^{*'}(\tilde{X}(\tau; X))\mathcal{V}^{*''}(\tilde{X}(\tau; X))\frac{\partial \tilde{X}}{\partial \tau}(\tau; X)d\tau. \tag{A.55}$$

Integrating by parts we obtain

$$\int_0^{+\infty} e^{-\lambda_1 \tau} \mathcal{V}^{*'}(\tilde{X}(\tau; X))\mathcal{V}^{*''}(\tilde{X}(\tau; X))\frac{\partial \tilde{X}}{\partial \tau}(\tau; X)d\tau = \left[\frac{1}{2}e^{-\lambda_1 \tau}\mathcal{V}^{*'2}(\tilde{X}(\tau; X))\right]_0^{+\infty} + \lambda_1\varphi(X).$$

Inserting into (A.55) and taking into account (6.8), we finally obtain

$$\varphi'(X) = \frac{\lambda_1\varphi(X) - \frac{1}{2}\mathcal{V}^{*'2}(X)}{\zeta + \mathcal{V}^{*'}(X)} \tag{A.56}$$

with the boundary condition

$$\lim_{X\to+\infty} \varphi(X) = 0. \tag{A.57}$$

This condition follows from the fact that $\lim_{X\to+\infty} \mathcal{V}^{*'}(X) = 0$ and the definition (A.45).

We now consider the system of first-order differential equations (A.56)-(A.44) together with the boundary conditions (6.7)-(A.57). Consider the new variables

$$Y = 1 - F(X) \in [0, 1]\,, \mathcal{V}^*(X) - \mathcal{V}_\infty = \mathcal{U}^*(Y) \text{ and } \varphi(X) = \psi(Y).\ R(Y) = f(F^{-1}(1 - Y)).$$

We rewrite (A.56)-(A.44) respectively as

$$\psi'(Y) = \frac{-\lambda_1\psi(Y) + \frac{1}{2}\mathcal{U}^{*'2}(Y)R^2(Y)}{R(Y)\left(\zeta - \mathcal{U}^{*'}(Y)R(Y)\right)} \tag{A.58}$$

$$\mathcal{U}^{*'}(Y) = \frac{1}{R(Y)}\left(\zeta - \sqrt{\zeta^2 + 2\lambda_0\mathcal{U}^*(Y) - 2\Delta V_\infty Y - 2\Delta(1 - Y)\psi(Y)}\right); \tag{A.59}$$

while the boundary conditions (6.7)-(A.57) become

$$\psi(0) = \mathcal{U}^*(0) = 0. \tag{A.60}$$

Local behavior. We transform this system as an autonomous system by introducing a new time scale z and express $\mathcal{U}^*$, φ and Y as functions of z (denoted with a *tilda*) so that:

$$\tilde{\mathcal{U}}^{*'}(z) = -\zeta + \sqrt{\zeta^2 + 2\lambda_0\tilde{\mathcal{U}}^*(z) - 2\Delta\mathcal{V}_\infty\tilde{Y}(z) - 2\Delta(1 - \tilde{Y}(z))\tilde{\psi}(z)} \tag{A.61}$$

$$\tilde{\psi}'(z) = \frac{\lambda_1\tilde{\psi}(z) - \frac{1}{2}\tilde{\mathcal{U}}^{*'2}(z)R^2(\tilde{Y}(z))}{\zeta - \tilde{\mathcal{U}}^{*'}(z)R(\tilde{Y}(z))} \tag{A.62}$$

$$\tilde{Y}'(z) = -R(\tilde{Y}(z)) \tag{A.63}$$

with the boundary conditions

$$\lim_{z\to+\infty}\tilde{\psi}(z) = \lim_{z\to+\infty}\tilde{\mathcal{U}}^*(z) = \lim_{z\to+\infty}\tilde{Y}(z) = 0. \tag{A.64}$$

We now linearize this system around its equilibrium $(0,0,0)$ to get:

$$\tilde{\mathcal{U}}^{*\prime}(z) = \frac{\lambda_0}{\zeta}\tilde{\mathcal{U}}^*(z) - \frac{\Delta}{\zeta}\tilde{\psi}(z) - \frac{\Delta V_\infty}{\zeta}\tilde{Y}(z) \tag{A.65}$$

$$\tilde{\psi}'(z) = \frac{\lambda_1}{\zeta}\tilde{\psi}(z) \tag{A.66}$$

$$\tilde{Y}'(z) = -R'(0)\tilde{Y}(z). \tag{A.67}$$

The properties of the linear system above are thus determined by those of the following matrix

$$A = \begin{pmatrix} \frac{\lambda_0}{\zeta} & -\frac{\Delta}{\zeta} & -\frac{\Delta V_\infty}{\zeta} \\ 0 & \frac{\lambda_1}{\zeta} & 0 \\ 0 & 0 & -R'(0) \end{pmatrix}$$

A has two positive eigenvalues and one negative one. The system is hyperbolic and its equilibrium $(0,0,0)$ is thus a saddle. The plane $(\tilde{\mathcal{U}}^*, \tilde{\psi})$ is unstable while the axis Y is stable.

From the Hartman-Grobman Theorem,[37] the nonlinear system (A.61)-(A.62)-(A.63) and the linear system (A.65)-(A.66)-(A.67) are topologically equivalent. More formally, let ϕ_z be the flow for the nonlinear system (A.61)-(A.62)-(A.63). Because A has non-zero eigenvalues, there exists a homeomorphism H on an open neighborhood U of $(0,0,0)$, such that for each $(\tilde{\mathcal{U}}_0^*, \tilde{\psi}_0, Y_0) \in U$ there is an open interval $I_0 \subset \mathbb{R}$ containing zero such that for all $(\tilde{\mathcal{U}}_0^*, \tilde{\psi}_0, Y_0) \in U$ and $z_0 \in I_0$, $H(\phi_z(\tilde{\mathcal{U}}_0^*, \tilde{\psi}_0, Y_0)) = e^{Az}H(\tilde{\mathcal{U}}_0^*, \tilde{\psi}_0, Y_0)$.

From this homeomorphism, it follows that the stable manifold for the nonlinear system (A.61)-(A.62)-(A.63) is also one-dimensional. This means that there is a one-to relationship between $\tilde{\mathcal{U}}^*$ and Y on that manifold. Henceforth, the solution $\mathcal{U}^*(Y)$ is also unique and thus the value function $\mathcal{V}^*(X)$ is thus also unique. This ends the proof of uniqueness of the equilibrium.

To give an approximation of the solution. Observe that the linear system (A.65)-(A.66)-(A.67) can be solved recursively by noticing first that

$$\tilde{Y}(z) = \tilde{Y}_0 e^{-R'(0)z} \tag{A.68}$$

for some arbitrary $\tilde{Y}_0$ since all such solutions satisfy (A.64).

We have also

$$\tilde{\psi}(z) = \tilde{\psi}_0 e^{\frac{\lambda_1}{\zeta}z} \tag{A.69}$$

but the only solution consistent with (A.64) has $\tilde{\psi}_0 = 0$. Finally inserting those findings into (A.65), we get

$$\tilde{\mathcal{U}}^{*\prime}(z) = \frac{\lambda_0}{\zeta}\tilde{\mathcal{U}}^*(z) - \frac{\Delta V_\infty \tilde{Y}_0}{\zeta}e^{-R'(0)z} \tag{A.70}$$

[37] Perko (1991, Section 2.8).

Integrating yields

$$\tilde{\mathcal{U}}^*(z) = e^{\frac{\lambda_0}{\zeta}z}\left(\tilde{\mathcal{U}}_0^* - \frac{\Delta\mathcal{V}_\infty\tilde{Y}_0}{\zeta}\int_0^z e^{-\left(R'(0)+\frac{\lambda_0}{\zeta}\right)\tilde{z}}d\tilde{z}\right). \tag{A.71}$$

The only solution consistent with (A.64) has thus

$$\tilde{\mathcal{U}}_0^* = \frac{\Delta\mathcal{V}_\infty\tilde{Y}_0}{\zeta R'(0)+\lambda_0} \tag{A.72}$$

which gives us a first-order approximation for the stable manifold. Expressed in terms of our original variable, we find that the one-dimensional stable manifold can be approximated as (6.11) when $X \to +\infty$.

Global Behavior and Comparative Statics. The local analysis above ensures the existence of a unique solution $(\mathcal{U}^*(Y), \psi(Y))$ to (A.58)-(A.59) with the boundary conditions (A.60). This system has a singularity at 0. Yet, simple applications of L'Hôspital's rule give us the values of the following derivatives:

$$\psi'(0) = 0 \tag{A.73}$$

and

$$\mathcal{U}^{*'}(0) = \frac{\Delta\mathcal{V}_\infty}{\zeta R'(0)+\lambda_0}. \tag{A.74}$$

Define now $\hat{\mathcal{U}}(Y)$ as the solution to

$$\hat{\mathcal{U}}(Y) = \frac{\Delta\mathcal{V}_\infty}{\lambda_0}Y + \frac{\Delta}{\lambda_0}(1-Y)\psi(Y). \tag{A.75}$$

Observe that $\hat{\mathcal{U}}(1) = \frac{\Delta\mathcal{V}_\infty}{\lambda_0}$ and $\hat{\mathcal{U}}(0) = 0$. Moreover, we have

$$0 < \mathcal{U}^{*'}(0) < \frac{\Delta\mathcal{V}_\infty}{\lambda_0} = \hat{\mathcal{U}}'(0).$$

Hence, locally in a right-neighborhood of 0, we have:

$$0 \le \mathcal{U}^*(Y) \le \hat{\mathcal{U}}(Y) \tag{A.76}$$

with these inequalities being strict for $Y > 0$ in that right-neighborhood.

To study some of the global properties of $\mathcal{U}^*(Y)$, we take $\psi^*(Y)$ as exogenous for a while and consider all the solutions to (A.59), viewed as a backward differential equation for $Y \le 1$, taking possibles values $\mathcal{U}_1$ in $\left[0, \hat{\mathcal{U}}(1)\right]$. Consider the domain $\mathcal{D} = \{(U,Y) | \quad \hat{\mathcal{U}}(Y) \ge U \ge 0 \text{ for } Y \in [0,1]\}$. The boundaries of $\mathcal{D}$ is made of the vertical segment $U \in [0, \hat{\mathcal{U}}(1)]$ at $Y = 1$, the horizontal axis $\{U = 0\}$ and the curve $\{U = \hat{\mathcal{U}}(Y), Y \in [0,1]\}$. Let the (backward) flow defined by (A.59) $\pi : (\mathcal{U}_1, Y) \to \mathcal{U}(Y|\mathcal{U}_1)$ where $\mathcal{U}(Y|\mathcal{U}_1)$ is the backward solution to (A.59) for some fixed initial value $\mathcal{U}_1$ at $Y = 1$. This flow is of course continuous.

Take any solution $\mathcal{U}(Y|\mathcal{U}_1)$ that crosses $\hat{\mathcal{U}}(Y)$ at a point $Y_1 \in (0,1)$ such that $\mathcal{U}^{*'}(Y_0|\mathcal{U}_1) = 0$. For $Y \le Y_1$, this solution is decreasing and cannot converge towards 0 when $Y \to 0^+$. Take now any solution $\mathcal{U}(Y|\mathcal{U}_1)$ that crosses the horizontal line at a point $Y_2 \in (0,1)$. We have $\mathcal{U}'(Y_2|\mathcal{U}_1) > 0$. Such solution cannot converge towards 0 either since, otherwise, there would exist a point $Y_3 < Y_2$ such that $\mathcal{U}'(Y_3|\mathcal{U}_1) = 0$ and $\mathcal{U}(Y_3|\mathcal{U}_1) < 0$. At such point, we should also have $\mathcal{U}'(Y_3|\mathcal{U}_1) = \hat{\mathcal{U}}(Y_3)$ which yields a contradiction with $\mathcal{U}(Y_3|\mathcal{U}_1) < 0 < \hat{\mathcal{U}}(Y_3)$.

From these observations, and from the continuity of the flow π, we deduce that the reciprocal image of the segment $\{(0,Y)| \quad Y \in (0,1]\}$ is a set of the form $[0, U_{02})$. Similarly, the reciprocal image of the set $\{U = \hat{\mathcal{U}}(Y), Y \in (0,1]\}$ is also of the form $(U_{01}, \hat{\mathcal{U}}(1)]$ with necessarily $U_{02} \leq U_{01}$. Of course, $[0, U_{02})$ and $(U_{01}, \hat{\mathcal{U}}(1)]$ cannot overlap because it would violate the local uniqueness of the solution $\mathcal{U}(X|\mathcal{U}_1)$ to (A.59) around $Y = 1$ (Cauchy-Lipschitz Unicity Theorem). Thus $[U_{02}, U_{01}]$ is non-empty and necessarily a solution with an initial condition $\mathcal{U}_1 \in [U_{02}, U_{01}]$ is such that:

$$\lim_{Y \to 0^+} \mathcal{U}(Y|\mathcal{U}_1) = 0.$$

This confirms the existence of a solution $\mathcal{U}^*(Y)$ to the backward differential equation (A.59), corresponding to an initial value $\mathcal{U}_1 \in [U_{02}, U_{01}]$ that satisfies the boundary condition (5.5). From the local uniqueness result, we also know that $\mathcal{U}^*(Y)$ is unique.

From the analysis above, it also follows that this solution satisfies

$$\text{(A.77)} \quad 0 \leq \mathcal{U}^*(Y) \leq \hat{\mathcal{U}}(Y) \quad \forall Y \in [0,1]$$

with both inequalities being equalities at $Y = 0$. Expressed in terms of our initial variables, this inequality becomes

$$\text{(A.78)} \quad \mathcal{V}_\infty \leq \mathcal{V}^*(X) \leq \hat{\mathcal{V}}^*(X) \quad \forall X \geq 0$$

where

$$\text{(A.79)} \quad \hat{\mathcal{V}}^*(X) = \mathcal{V}_\infty + \frac{\Delta \mathcal{V}_\infty}{\lambda_0}(1 - F(X)) + \frac{\Delta}{\lambda_0} F(X)\varphi(X).$$

From the left-hand side inequality above, it also follows that

(A.80)

$$\zeta^2 + 2\lambda_0(\mathcal{V}^*(X) - \mathcal{V}_\infty) - 2\Delta\mathcal{V}_\infty(1 - F(X)) - 2\Delta F(X)\varphi(X) \geq \zeta^2 - 2\Delta\mathcal{V}_\infty + 2\Delta F(X)(\mathcal{V}_\infty - \varphi(X)).$$

Now, observe that $\zeta^2 - 2\Delta\mathcal{V}_\infty > 0$ (see Footnote 3). From (A.77) and (A.44), we also deduce

$$\text{(A.81)} \quad 0 \leq -\mathcal{V}^{*'}(X) \leq \zeta.$$

Notice that the left-hand side inequality of (A.81) implies the upper bound for the feedback rule (6.10).

Moreover, the right-hand side inequality of (A.81) also implies that

$$\text{(A.82)} \quad \varphi(X) \leq \frac{\zeta^2}{2\lambda_1} < \mathcal{V}_\infty \quad \forall X.$$

Gathering those findings and inserting into (A.80) yields the non-negativity condition:

$$\text{(A.83)} \quad \zeta^2 + 2\lambda_0\mathcal{V}^*(X) - 2\Delta\mathcal{V}_\infty(1 - F(X)) - 2\Delta F(X)\varphi(X) > 0 \quad \forall X \geq 0.$$

Hence, the solution to (A.43) never crosses the manifold where the square root is zero.

Moreover, inserting again the right-hand-side inequality of (A.82) into (A.80) immediately gives us

$$\begin{aligned}
&(\sigma^*(X)-\zeta)^2 = \mathcal{V}^{*'2}(X)\\
&= \zeta^2 + 2\lambda_0(\mathcal{V}^*(X)-\mathcal{V}_\infty) - 2\Delta\mathcal{V}_\infty(1-F(X)) - 2\Delta F(X)\varphi(X)\\
&\geq \zeta^2 - 2\Delta\mathcal{V}_\infty + 2\Delta F(X)(\mathcal{V}_\infty - \varphi(X))\\
&\geq \zeta^2 - 2\Delta\mathcal{V}_\infty.
\end{aligned}$$

From which, we get the lower bound for the feedback rule (6.10).

SUFFICIENT CONDITION. Coming back on the first-order Taylor expansion (A.39), observe that the term in ε writes as

$$\text{(A.84)}\quad \varepsilon\left(x\mathcal{V}^{*'}(X) - \lambda_0\mathcal{V}^*(X) + u(x) - \Delta F(X)\left(\int_0^{+\infty} e^{-\lambda_1\tau}u(\sigma^*(\tilde{X}(\tau;X))))ds\right)d\tau\right)$$

where $(\mathcal{V}^*(X), \sigma^*(X))$ solves (A.43)-(6.7)-(6.8). This expression is strictly concave in x. Thus the maximum is indeed achieved for $\sigma^*(X)$ as characterized by (6.8).

Q.E.D.

PROOF OF PROPOSITION 8: Remember that $R'(0) = k$ for an exponential distribution. The proof then hinges on the following comparison that holds for all $A \leq \frac{1}{2}$:

$$\sqrt{1-2A} \leq 1 - A.$$

Using $A = \frac{\Delta\mathcal{V}_\infty}{\left(\zeta+\frac{\lambda_0}{k}\right)^2}$ and noticing that $\Delta\mathcal{V}_\infty < \frac{\zeta^2}{2}$ is a sufficient condition for $A \leq \frac{1}{2}$, we obtain

$$\zeta + \frac{\lambda_0}{k} - \sqrt{\left(\zeta + \frac{\lambda_0}{k}\right)^2 - 2\Delta\mathcal{V}_\infty} \geq \frac{\frac{\Delta\mathcal{V}_\infty}{k}}{\zeta + \frac{\lambda_0}{k}}$$

Inserting into (5.14) and comparing with (6.11) yields the left-hand side inequality of (6.13). The right-hand side follows from our earlier findings that $\mathcal{V}^*(X) < 0$ is non-decreasing with limit $\mathcal{V}_\infty$. Inserting into (5.15) and comparing with (6.12) also yields

$$(1-F(X))(\zeta - \sigma_u(X)) = (1-F(X))\left(\zeta + \frac{\lambda_0}{k} - \sqrt{\left(\zeta + \frac{\lambda_0}{k}\right)^2 - 2\Delta\mathcal{V}_\infty}\right) > \zeta - \sigma^*(X)$$

which finally writes as the right-hand side inequality of (6.14). The left-hand side follows from $\mathcal{V}^{*'}(X) < 0$ and the definition of the feedback rule $\sigma^*(X)$. *Q.E.D.*

PROOF OF PROPOSITION 9: We first write DM's intertemporal payoff $\omega(x_0, \varepsilon)$ in terms of (x_0, ε) and $X_0 = x_0\varepsilon$ as

$$\begin{aligned}
\omega(x_0,\varepsilon) = \int_0^{\varepsilon}\Bigg(&e^{-\lambda_0 t}\left(\lambda_1\mathcal{V}_\infty - \frac{1}{2}(\zeta - x_0)^2\right) - \Delta F(x_0 t)e^{\Delta t}\Bigg(\int_t^{\varepsilon} e^{-\lambda_1\tau}\left(\lambda_1\mathcal{V}_\infty - \frac{1}{2}(\zeta - x_0)^2\right)d\tau\\
&+ \int_{\varepsilon}^{+\infty} e^{-\lambda_1\tau}\left(\lambda_1\mathcal{V}_\infty - \frac{1}{2}\left(\zeta - \sigma^*(\tilde{X}(\tau-\varepsilon;X_0))\right)^2\right)d\tau\Bigg)\Bigg)dt + e^{-\lambda_0\varepsilon}\mathcal{V}^*(X_0)
\end{aligned}$$

where

$$e^{-\lambda_0\varepsilon}\mathcal{V}^*(X_0)=\int_\varepsilon^{+\infty}\Bigg(e^{-\lambda_0 t}\Bigg(\lambda_1\mathcal{V}_\infty-\frac{1}{2}\left(\zeta-\sigma^*(\tilde{X}(t-\varepsilon;X_0))\right)^2\Bigg)$$

$$-\Delta F(\tilde{X}(t-\varepsilon;X_0))e^{\Delta t}\int_t^{+\infty}e^{-\lambda_1\tau}\left(\lambda_1\mathcal{V}_\infty-\frac{1}{2}(\zeta-\sigma^*(\tilde{X}(\tau-\varepsilon;X_0)))^2\right)d\tau\Bigg)dt.$$

PARTIAL DERIVATIVES. We first differentiate with respect to ε and find

$$\frac{\partial\omega}{\partial\varepsilon}(x_0,\varepsilon)=e^{-\lambda_0\varepsilon}\left(\lambda_1\mathcal{V}_\infty-\frac{1}{2}(\zeta-x_0)^2\right)-\Delta F(X_0)e^{\Delta\varepsilon}\int_\varepsilon^{+\infty}e^{-\lambda_1\tau}\left(\lambda_1\mathcal{V}_\infty-\frac{1}{2}\left(\zeta-\sigma^*(\tilde{X}(\tau-\varepsilon;X_0))\right)^2\right)d\tau$$

$$+e^{-\lambda_1\varepsilon}\left[\frac{1}{2}(\zeta-x)^2\right]_{\sigma^*(\tilde{X}(\varepsilon;X_0))}^{x_0}\int_0^\varepsilon\Delta F(x_0t)e^{\Delta t}dt$$

$$-\int_0^\varepsilon\Delta F(x_0t)e^{\Delta t}\left(\int_\varepsilon^{+\infty}e^{-\lambda_1\tau}x_0\sigma^{*'}(\tilde{X}(\tau-\varepsilon;X_0))\frac{\partial\tilde{X}}{\partial X}(\tau-\varepsilon;X_0)\left(\zeta-\sigma^*(\tilde{X}(\tau-\varepsilon;X_0))\right)d\tau\right)dt$$

$$+\int_0^\varepsilon\Delta F(x_0t)e^{\Delta t}\left(\int_\varepsilon^{+\infty}e^{-\lambda_1\tau}\sigma^{*'}(\tilde{X}(\tau-\varepsilon;X_0))\frac{\partial\tilde{X}}{\partial\tau}(\tau-\varepsilon;X_0)\left(\zeta-\sigma^*(\tilde{X}(\tau-\varepsilon;X_0))\right)d\tau\right)dt$$

$$-\lambda_0e^{-\lambda_0\varepsilon}\mathcal{V}^*(X_0)+x_0e^{-\lambda_0\varepsilon}\mathcal{V}^{*'}(X_0).$$

Simplifying, we obtain:

$$\frac{\partial\omega}{\partial\varepsilon}(x_0,\varepsilon)=e^{-\lambda_0\varepsilon}\Bigg(-\lambda_0\mathcal{V}^*(X_0)+\lambda_1\mathcal{V}_\infty-\frac{1}{2}(\zeta-x_0)^2-\Delta\mathcal{V}_\infty F(X_0)\tag{A.85}$$

$$+\frac{\Delta}{2}F(X_0)e^{\lambda_1\varepsilon}\int_\varepsilon^{+\infty}e^{-\lambda_1\tau}\left(\zeta-\sigma^*(\tilde{X}(\tau;X_0))\right)^2d\tau+x_0\mathcal{V}^{*'}(X_0)\Bigg)$$

$$+e^{-\lambda_1\varepsilon}\left[\frac{1}{2}(\zeta-x)^2\right]_{\sigma^*(\tilde{X}(\varepsilon;X_0))}^{x_0}\int_0^\varepsilon\Delta F(x_0t)e^{\Delta t}dt$$

$$-e^{-\lambda_1\varepsilon}\left(\int_0^\varepsilon\Delta F(x_0t)e^{\Delta t}dt\right)$$

$$\times\left(\int_0^{+\infty}e^{-\lambda_1\tau}\sigma^{*'}(\tilde{X}(\tau;X_0))\left(x_0\frac{\partial\tilde{X}}{\partial X}(\tau;X_0)-\frac{\partial\tilde{X}}{\partial\tau}(\tau;X_0)\right)\left(\zeta-\sigma^*(\tilde{X}(\tau;X_0))\right)d\tau\right).$$

From (A.42), we know that

$$\lambda_0\mathcal{V}^*(X_0)=\lambda_1\mathcal{V}_\infty+\sigma^*(X_0)\mathcal{V}^{*'}(X_0)-\frac{1}{2}(\zeta-\sigma^*(X_0))^2\tag{A.86}$$

$$-\Delta\mathcal{V}_\infty F(X_0)+\frac{\Delta}{2}F(X_0)\int_0^{+\infty}e^{-\lambda_1\tau}(\zeta-\sigma^*(\tilde{X}(\tau-\varepsilon;X_0)))^2d\tau.$$

By stationarity, we also know that

$$e^{\lambda_1\varepsilon}\int_\varepsilon^{+\infty}e^{-\lambda_1\tau}\left(\zeta-\sigma^*(\tilde{X}(\tau-\varepsilon;X_0))\right)^2d\tau=\int_0^{+\infty}e^{-\lambda_1\tau}\left(\zeta-\sigma^*(\tilde{X}(\tau;X_0))\right)^2d\tau.\tag{A.87}$$

Finally, from Lemma A.3, we know that

$$\text{(A.88)}\quad \frac{\partial \tilde{X}}{\partial X}(\tau;X_0)=\frac{\sigma^*(\tilde{X}(\tau;X_0))}{\sigma^*(X_0)}=\frac{\frac{\partial \tilde{X}}{\partial \tau}(\tau;X_0)}{\sigma^*(X_0)}.$$

Inserting (A.86), (A.87) and (A.88) into (A.85) yields

(A.89)

$$\frac{\partial \omega}{\partial \varepsilon}(x_0,\varepsilon)=e^{-\lambda_0\varepsilon}\Bigg(-\frac{1}{2}\left[(\zeta-x)^2\right]_{\sigma^*(X_0)}^{x_0}\left(1-\Delta e^{-\Delta\varepsilon}\int_0^\varepsilon F(x_0t)e^{\Delta t}dt\right)+(x_0-\sigma^*(X_0))\mathcal{V}^{*\prime}(X_0)-$$

$$\left(\frac{x_0}{\sigma^*(X_0)}-1\right)\left(\Delta e^{-\Delta\varepsilon}\int_0^\varepsilon F(x_0t)e^{\Delta t}dt\right)\int_\varepsilon^{+\infty}e^{-\lambda_1\tau}\sigma^{*\prime}(\tilde{X}(\tau;X_0))\sigma^*(\tilde{X}(\tau;X_0))\left(\zeta-\sigma^*(\tilde{X}(\tau;X_0))\right)d\tau\Bigg).$$

We now differentiate with respect to x_0 and find

$$\begin{aligned}\text{(A.90)}\quad \frac{\partial \omega}{\partial x_0}(x_0,\varepsilon)&=(\zeta-x_0)\int_0^\varepsilon e^{-\lambda_0 t}dt-\int_0^\varepsilon \Delta t f(x_0t)e^{\Delta t}\Bigg(\int_t^\varepsilon e^{-\lambda_1\tau}\left(\lambda_1\mathcal{V}_\infty-\frac{1}{2}(\zeta-x_0)^2\right)d\tau\\
&+\int_\varepsilon^{+\infty}e^{-\lambda_1\tau}\left(\lambda_1\mathcal{V}_\infty-\frac{1}{2}\left(\zeta-\sigma^*(\tilde{X}(\tau-\varepsilon;X_0))\right)^2\right)d\tau\Bigg)dt\\
&-\int_0^\varepsilon \Delta F(x_0t)e^{\Delta t}\Bigg((\zeta-x_0)\int_t^\varepsilon e^{-\lambda_1\tau}d\tau\\
&+\varepsilon\int_\varepsilon^{+\infty}e^{-\lambda_1\tau}\sigma^{*\prime}(\tilde{X}(\tau-\varepsilon;X_0))\frac{\partial \tilde{X}}{\partial X}(\tau-\varepsilon;X_0)\left(\zeta-\sigma^*(\tilde{X}(\tau-\varepsilon;X_0))\right)d\tau\Bigg)dt\\
&+\varepsilon e^{-\lambda_0\varepsilon}\mathcal{V}^{*\prime}(X_0).\end{aligned}$$

Now, we notice that

(A.91)

$$\int_t^\varepsilon e^{-\lambda_1\tau}\left(\lambda_1\mathcal{V}_\infty-\frac{1}{2}(\zeta-x_0)^2\right)d\tau+\int_\varepsilon^{+\infty}e^{-\lambda_1\tau}\left(\lambda_1\mathcal{V}_\infty-\frac{1}{2}\left(\zeta-\sigma^*(\tilde{X}(\tau-\varepsilon;X_0))\right)^2\right)d\tau$$

$$=\left(\lambda_1\mathcal{V}_\infty-\frac{1}{2}(\zeta-x_0)^2\right)\frac{e^{-\lambda_1 t}-e^{-\lambda_1\varepsilon}}{\lambda_1}+e^{-\lambda_1\varepsilon}\int_0^{+\infty}e^{-\lambda_1\tau}\left(\lambda_1\mathcal{V}_\infty-\frac{1}{2}\left(\zeta-\sigma^*(\tilde{X}(\tau;X_0))\right)^2\right)d\tau$$

and

(A.92)

$$(\zeta-x_0)\int_t^\varepsilon e^{-\lambda_1\tau}d\tau+\varepsilon\int_\varepsilon^{+\infty}e^{-\lambda_1\tau}\sigma^{*\prime}(\tilde{X}(\tau-\varepsilon;X_0))\frac{\partial \tilde{X}}{\partial X}(\tau-\varepsilon;X_0)\left(\zeta-\sigma^*(\tilde{X}(\tau-\varepsilon;X_0))\right)d\tau$$

$$=(\zeta-x_0)\frac{e^{-\lambda_1 t}-e^{-\lambda_1\varepsilon}}{\lambda_1}+\varepsilon e^{-\lambda_1\varepsilon}\int_0^{+\infty}e^{-\lambda_1\tau}\sigma^{*\prime}(\tilde{X}(\tau;X_0))\frac{\partial \tilde{X}}{\partial X}(\tau;X_0)\left(\zeta-\sigma^*(\tilde{X}(\tau;X_0))\right)d\tau.$$

Insert (A.91) and (A.92) into (A.90) and taking into account (6.8) and (7.1) then yields

$$\text{(A.93)}\quad \frac{\partial \omega}{\partial x_0}\left(\sigma^*(X_0),\frac{X_0}{\sigma^*(X_0)}\right)=(\zeta-\sigma^*(X_0))\left(\frac{1-e^{-\frac{\lambda_0X_0}{\sigma^*(X_0)}}}{\lambda_0}-e^{-\frac{\lambda_0X_0}{\sigma^*(X_0)}}\frac{X_0}{\sigma^*(X_0)}\right)$$

$$
-\int_0^{\frac{X_0}{\sigma^*(X_0)}} \Delta t f\left(\sigma^*(X_0)t\right) e^{\Delta t}\Bigg(\left(\lambda_1 \mathcal{V}_\infty - \frac{1}{2}(\zeta-\sigma^*(X_0))^2\right) \frac{e^{-\lambda_1 t} - e^{-\frac{\lambda_1 X_0}{\sigma^*(X_0)}}}{\lambda_1}
$$

$$
+e^{-\frac{\lambda_1 X_0}{\sigma^*(X_0)}} \int_0^{+\infty} e^{-\lambda_1 \tau}\left(\lambda_1 \mathcal{V}_\infty - \frac{1}{2}\left(\zeta - \sigma^*(\tilde{X}(\tau;X_0))\right)^2\right) d\tau \Bigg) dt
$$

$$
-\int_0^{\frac{X_0}{\sigma^*(X_0)}} \Delta F\left(\sigma^*(X_0)t\right) e^{\Delta t}\Bigg((\zeta-\sigma^*(X_0))\frac{e^{-\lambda_1 t} - e^{-\frac{\lambda_1 X_0}{\sigma^*(X_0)}}}{\lambda_1}
$$

$$
+\frac{X_0}{\sigma^*(X_0)} e^{-\frac{\lambda_1 X_0}{\sigma^*(X_0)}} \int_0^{+\infty} e^{-\lambda_1 \tau} \sigma^{*'}(\tilde{X}(\tau;X_0)) \frac{\partial \tilde{X}}{\partial X}(\tau;X_0)\left(\zeta - \sigma^*(\tilde{X}(\tau;X_0))\right) d\tau \Bigg) dt.
$$

Optimality conditions. Clearly, the first-order optimality condition $\frac{\partial \omega}{\partial \varepsilon}(x_0,\varepsilon) = 0$ writes as the smooth-pasting condition (7.1) (assuming quasi-concavity of the objective in ε for a fixed value of x_0). This leads us to alternatively express DM's intertemporal payoff in terms of X_0 only as

$$
\Omega(X_0) = \omega\left(\sigma^*(X_0), \frac{X_0}{\sigma^*(X_0)}\right).
$$

We thus compute

$$
\Omega'(X_0) = \frac{\partial \omega}{\partial x_0}\left(\sigma^*(X_0), \frac{X_0}{\sigma^*(X_0)}\right)\sigma^{*'}(X_0) + \frac{\partial \omega}{\partial \varepsilon}\left(\sigma^*(X_0), \frac{X_0}{\sigma^*(X_0)}\right)\left(\frac{\sigma^*(X_0) - X_0\sigma^{*'}(X_0)}{\sigma^{*2}(X_0)}\right).
$$

Taking into account the smooth-pasting condition (7.1), the second term is 0 and thus

$$
\Omega'(X_0) = \frac{\partial \omega}{\partial x_0}\left(\sigma^*(X_0), \frac{X_0}{\sigma^*(X_0)}\right)\sigma^{*'}(X_0).
$$

To evaluate the local behavior of $\Omega'(X_0)$ around $X_0 = 0$, we now rely on a second-order Taylor expansion, namely:

$$
\Omega'(X_0) = \Omega'(0) + \Omega''(0)X_0 + \Omega'''(0)\frac{X_0^2}{2} + o(X_0^2) \tag{A.94}
$$

where $\lim o(X_0^2)/X_0^2 = 0$. To this end, we thus compute

$$
\Omega'(0) = \frac{\partial \omega}{\partial x_0}\left(\sigma^*(X_0), \frac{X_0}{\sigma^*(X_0)}\right)\Bigg|_{X_0=0} \sigma^{*'}(0),
$$

$$
\Omega''(0) = \frac{d}{dX_0}\left(\frac{\partial \omega}{\partial x_0}\left(\sigma^*(X_0), \frac{X_0}{\sigma^*(X_0)}\right)\right)\Bigg|_{X_0=0} \sigma^{*'}(0) + \frac{\partial \omega}{\partial x_0}\left(\sigma^*(X_0), \frac{X_0}{\sigma^*(X_0)}\right)\Bigg|_{X_0=0} \sigma^{*''}(0)
$$

$$
\Omega'''(0) = \frac{d^2}{dX_0^2}\left(\frac{\partial \omega}{\partial x_0}\left(\sigma^*(X_0), \frac{X_0}{\sigma^*(X_0)}\right)\right)\Bigg|_{X_0=0} \sigma^{*'}(0)
$$

$$
+2\,\frac{d}{dX_0}\left(\frac{\partial \omega}{\partial x_0}\left(\sigma^*(X_0), \frac{X_0}{\sigma^*(X_0)}\right)\right)\Bigg|_{X_0=0} \sigma^{*''}(0) + \frac{\partial \omega}{\partial x_0}\left(\sigma^*(X_0), \frac{X_0}{\sigma^*(X_0)}\right)\Bigg|_{X_0=0} \sigma^{*'''}(0).
$$

Taking into account that $\sigma^*(0) > \sqrt{\zeta^2 - 2\Delta \mathcal{V}_\infty} > 0$ (which comes from (6.10)), it follows that

$$
\lim_{X_0 \to 0} \frac{X_0}{\sigma^*(X_0)} = 0. \tag{A.95}
$$

Looking at (A.93), we can conclude that

$$\frac{\partial \omega}{\partial x_0}\left(\sigma^*(X_0), \frac{X_0}{\sigma^*(X_0)}\right)|_{X_0=0} = 0$$

and thus

$$\Omega'(0) = 0. \tag{A.96}$$

Differentiating (A.93) with respect to X_0 and using again (A.95), tedious computations show that

$$\frac{d}{dX_0}\left(\frac{\partial \omega}{\partial x_0}\left(\sigma^*(X_0), \frac{X_0}{\sigma^*(X_0)}\right)\right)|_{X_0=0} = 0$$

and thus

$$\Omega''(0) = 0. \tag{A.97}$$

Finally, we also compute

$$\frac{d^2}{dX_0^2}\left(\frac{\partial \omega}{\partial x_0}\left(\sigma^*(X_0), \frac{X_0}{\sigma^*(X_0)}\right)\right)_{X_0=0} = \frac{1}{\sigma^{*2}(0)}\left(\lambda_0(\zeta - \sigma^*(0)) - \Delta f(0)(\mathcal{V}_\infty - \varphi(0))\right). \tag{A.98}$$

Differentiating (A.43) with respect to X yields

$$\lambda_0 \mathcal{V}'^*(X) = (\zeta + \mathcal{V}^{*'}(X))\mathcal{V}^{*''}(X) - \Delta \mathcal{V}_\infty f(X) + \Delta(F(X)\varphi'(X) + f(X)\varphi(X)). \tag{A.99}$$

Taking into account (6.8) but also $\mathcal{V}^{*''}(X) = \sigma^{*'}(X)$, (A.99) taken for $X = 0$ becomes

$$\sigma^{*'}(0)\sigma^*(0) = \lambda_0(\sigma^*(0) - \zeta) + \Delta f(0)(\mathcal{V}_\infty - \varphi(0)).$$

Inserting into (A.98) yields

$$\frac{d^2}{dX_0^2}\left(\frac{\partial \omega}{\partial x_0}\left(\sigma^*(X_0), \frac{X_0}{\sigma^*(X_0)}\right)\right)_{X_0=0} = -\frac{\sigma^{*'}(0)}{\sigma^*(0)}.$$

Inserting the latter condition into (A.94), while taking into account (A.96) and (A.97), yields

$$\Omega'(X_0) = -\frac{\sigma^{*'2}(0)}{\sigma^*(0)}\frac{X_0^2}{2} + o(X_0^2) \tag{A.100}$$

From this, it follows that $\Omega'(X_0) \leq 0$ in a right-neighborhood of zero. Hence, expanding the commitment period is, locally at least, suboptimal.

Q.E.D.

Proof of Lemma 3: Integrating by parts, we obtain:

$$\int_0^{\tilde{X}(t;X)} \theta_1 e^{-(\theta_0 \tilde{T}(\tilde{X};X) + \theta_1(t - \tilde{T}(\tilde{X};X))} f(\tilde{X}) d\tilde{X} =$$

$$F(\tilde{X}(t;X))\theta_1 e^{-\theta_0 t} - \Delta \int_0^{\tilde{X}(t;X)} F(\tilde{X})\frac{\partial \tilde{T}}{\partial \tilde{X}}(\tilde{X};X)\theta_1 e^{-(\theta_0 \tilde{T}(\tilde{X};X) + \theta_1(t - \tilde{T}(\tilde{X};X)))} d\tilde{X}.$$

Now changing variables and setting $\tilde{X} = \tilde{X}(\tau; X)$ (with $d\tilde{X} = \frac{\partial X}{\partial \tau}(\tau; X)d\tau$) in the integral, we obtain:

$$F(\tilde{X}(t;X))\theta_1 e^{-\theta_0 t} - \Delta\theta \int_0^{\tilde{X}(t;X)} F(\tilde{X})\frac{\partial \tilde{T}}{\partial \tilde{X}}(\tilde{X};X)\theta_1 e^{-(\theta_0 \tilde{T}(\tilde{X};X)+\theta_1(t-\tilde{T}(\tilde{X};X)))}d\tilde{X}$$

$$= \theta_1 e^{-\theta_0 t}\left(F(\tilde{X}(t;X)) - \Delta\theta e^{-\Delta\theta t}\int_0^t F(\tilde{X}(\tau;X))e^{-\Delta\tau}d\tau\right).$$

From this, it follows that:

$$g(t|\mathbf{x}^t, X) = (1-F(\tilde{X}(t;X))\theta_0 e^{-\theta_0 t} + \theta_1 e^{-\theta_0 t}\left(F(\tilde{X}(t;X)) - \Delta e^{-\Delta t}\int_0^t F(\tilde{X}(\tau;X))e^{\Delta\tau}d\tau\right).$$

Integrating by parts, we finally obtain:

$$g(t|\mathbf{x}^t, X) = (1-F(\tilde{X}(t;X)))\theta_0 e^{-\theta_0 t} + \theta_1 e^{-\theta_1 t}\int_0^t f(\tilde{X}(\tau;X))\frac{\partial \tilde{X}}{\partial \tau}(\tau;X)e^{\Delta\tau}d\tau.$$

Integrating, we obtain:

$$G(t|\mathbf{x}^t, X) = \int_0^t \left((1-F(\tilde{X}(\tau;X))\theta_0 e^{-\theta_0 \tau} + \theta_1 e^{-\theta_1 \tau}\int_0^\tau f(\tilde{X}(s;X))\frac{\partial \tilde{X}}{\partial s}(s;X)e^{\Delta s}ds\right)d\tau$$

Integrating by parts and simplifying yields:

$$1 - G(t|\mathbf{x}^t, X) = (1-F(\tilde{X}(t;X))e^{-\theta_0 t} + e^{-\theta_1 t}\int_0^t f(\tilde{X}(\tau;X))\frac{\partial X}{\partial \tau}(\tau;X)e^{\Delta\tau}d\tau.$$

Integrating by parts the last term, we now obtain:

$$1 - G(t|\mathbf{x}^t, X) = (1-F(\tilde{X}(t;X))e^{-\theta_0 t} + e^{-\theta_1 t}\left(F(\tilde{X}(t;X))e^{\Delta t} - \Delta\int_0^t F(\tilde{X}(\tau;X))e^{\Delta\tau}d\tau\right)$$

which finally yields (8.2). *Q.E.D.*

PROOF OF LEMMA 4: Integrating by parts the maximand on the right-hand side of (6.2), we rewrite the value function $\tilde{\mathcal{V}}^c(X,t)$ as

(A.101)
$$\tilde{\mathcal{V}}^c(X,t) = \sup_{\mathbf{x}_t, \tilde{X}(\cdot) \text{ s.t. } (5.1)} \int_t^{+\infty} e^{-\lambda_0 \tau}\left(1 - \Delta e^{-\Delta\tau}\int_t^\tau F(\tilde{X}(s;X,t))e^{\Delta s}ds\right)u(x(\tau))d\tau \quad \forall (X,t).$$

Changing variables in the integral and letting $\tau' = \tau - t$, we get

$$\int_t^{+\infty} e^{-\lambda_0 \tau}\left(1 - \Delta e^{-\Delta\tau}\int_t^\tau F(\tilde{X}(s;X,t))e^{\Delta s}ds\right)u(x(\tau))d\tau$$

$$= e^{-\lambda_0 t}\int_0^{+\infty} e^{-\lambda_0 \tau'}\left(1 - \Delta e^{-\Delta(\tau'+t)}\int_t^{\tau'+t} F(\tilde{X}(s;X,0))e^{\Delta s}ds\right)u(x(\tau'+t))d\tau'$$

$$= e^{-\lambda_0 t}\int_0^{+\infty} e^{-\lambda_0 \tau'}\left(1 - \Delta e^{-\Delta\tau'}\int_0^{\tau'} F(\tilde{\tilde{X}}(s';X,0))e^{\Delta s'}ds'\right)u(x(\tau'))d\tau'$$

where we have used the change of variables $s' = s - t$ and where $\tilde{x}(\tau') = x(\tau)$ and $\tilde{\tilde{X}}(s'; X, 0) = \tilde{X}(s' + t; X, t)$. Henceforth, if the actions plan $\tilde{x}(\tau')$ and the corresponding stock $\tilde{\tilde{X}}(s'; X, 0) = X + \int_0^{s'} \tilde{x}(\tau')d\tau'$ reach the maximum of $\mathcal{V}^c(X)$ which is defined as

$$\mathcal{V}^c(X) = \sup_{\mathbf{x}, \tilde{X}(\cdot)} \int_0^{+\infty} e^{-\lambda_0 t} \left(1 - \Delta e^{-\Delta t} \int_0^t F(\tilde{X}(\tau; X)) e^{\Delta \tau} d\tau \right) u(x(t)) d\tau \quad \forall X.$$

Then $x(\tau)$ and the corresponding stock $\tilde{X}(s; X, t) = X + \int_t^s x(\tau)d\tau$ reach the maximum of the right-hand side of (A.101). Taking into account (8.2), we obtain (8.4) which ends the proof of the lemma. *Q.E.D.*

Proof of Proposition 10: Notice first that (8.3) implies the following inequality:

(A.102) $e^{-\Delta t} \leq \tilde{Z}(t; X, 1) \leq F(X) e^{-\Delta t} + 1 - F(X).$

From which it follows the state variable Z takes value in the domain $[0, 1]$.

We now proceed with proving the proposition below and will conclude with a Verification Theorem (Proposition A.3):

Proposition A.2 *If the complete value function $\mathcal{W}^c(X, Z)$ is C^1, it satisfies the following HBJ equation:*

(A.103)

$$\lambda_0 \mathcal{W}^c(X, Z) = Z \lambda_1 \mathcal{V}_\infty + \zeta \frac{\partial \mathcal{W}^c}{\partial X}(X, Z) + \frac{1}{2Z} \left(\frac{\partial \mathcal{W}^c}{\partial X}(X, Z) \right)^2 + \Delta(1 - F(X) - Z) \frac{\partial \mathcal{W}^c}{\partial Z}(X, Z)$$

with the boundary conditions

(A.104) $Z \mathcal{V}_\infty \leq \lim_{X \to +\infty} \mathcal{W}^c(X, Z) \leq \mathcal{V}_\infty.$

The complete feedback rule is given by (8.11)

Proof of Proposition A.2: Using previous notations, it is straightforward to check that $\mathcal{W}^c(X, Z)$ solves the following recursive condition

(A.105) $$\mathcal{W}^c(X, Z) = \sup_{\mathbf{x}, \tilde{X}(\cdot), \tilde{Z}(\cdot) \text{ s.t. (8.1) and (8.5)}} \int_0^\varepsilon e^{-\lambda_0 t} \tilde{Z}(t; X, Z) u(x(t)) dt$$

$$+ e^{-\lambda_0 \varepsilon} \mathcal{W}^c(\tilde{X}(\varepsilon; X, Z), \tilde{Z}(\varepsilon; X, Z)).$$

Consider now ε small enough and denote by x a fixed action over the interval $[0, \varepsilon]$. From (8.1) and (8.5), we get

$$\tilde{X}(\varepsilon; X, Z) = X + \varepsilon x + o(\varepsilon)$$

and

$$\tilde{Z}(\varepsilon; X, Z) = Z + \varepsilon \Delta(1 - F(X) - Z) + o(\varepsilon)$$

where $\lim_{\varepsilon \to 0} o(\varepsilon)/\varepsilon = 0$.

When $\mathcal{W}^c(X,Z)$ is C^1, we can take a first-order Taylor expansion in ε of the maximand in (A.105) to write it as:

$$\mathcal{W}^c(X,Z)+$$
$$e^{-\lambda_0\epsilon}\varepsilon\left(Zu(x)+x\frac{\partial\mathcal{W}^c}{\partial X}(X,Z)+\Delta(1-F(X)-Z)\frac{\partial\mathcal{W}^c}{\partial Z}(X,Z)-\lambda_0\mathcal{W}^c(X,Z)\right)+o(\varepsilon).$$

Inserting into (A.105) yields the following HBJ equation:

$$\lambda_0\mathcal{W}^c(X,Z)=\sup_{x\in\mathcal{X}}\left\{Zu(x)+x\frac{\partial\mathcal{W}^c}{\partial X}(X,Z)+\Delta(1-F(X)-Z)\frac{\partial\mathcal{W}^c}{\partial Z}(X,Z)\right\}. \quad \text{(A.106)}$$

Simplifying using the feedback rule finally yields (A.103).

Boundary condition. Now, observe that (8.5) and $F(X)\leq F(\tilde{X}(t;X))\leq 1$ imply

$$0\leq\frac{\partial}{\partial t}\left(\tilde{Z}(t;X,Z)e^{\Delta t}\right)\leq\Delta(1-F(X))e^{\Delta t}.$$

Integrating between 0 and t yields

$$0\leq Ze^{-\Delta t}\leq\tilde{Z}(t;X,Z)\leq Ze^{-\Delta t}+(1-F(X))\left(1-e^{-\Delta t}\right).$$

From this and the fact that $0\leq Z\leq 1$, it follows that

$$0\leq Ze^{-\Delta t}\leq\tilde{Z}(t;X,Z)\leq F(X)e^{-\Delta t}+1-F(X)\leq 1. \quad \text{(A.107)}$$

Henceforth, the whole trajectory $\tilde{Z}(t;X,Z)$ always remains in the stable domain $[0,1]$.

From the third inequality in (A.107), taking maximum of the value function (A.105), it also follows that

$$\mathcal{W}^c(X,Z)\leq F(X)\mathcal{V}_\infty+(1-F(X))\frac{\lambda_1\mathcal{V}_\infty}{\lambda_0}. \quad \text{(A.108)}$$

Which can be rewritten to explain the right-hand side inequality of (8.12).

Passing to the limit yields the right-hand side inequality of (A.104)

$$\lim_{X\to+\infty}\mathcal{W}^c(X,Z)\leq\mathcal{V}_\infty.$$

From the second inequality in (A.107), and again taking maximum, we get

$$\mathcal{W}^c(X,Z)\geq Z\mathcal{V}_\infty$$

Passing to the limit yields the left-hand side inequality of (A.104) which holds for any Z. For $Z=1$, we obtain

$$\lim_{X\to\infty}\mathcal{V}^*(X)=\mathcal{V}_\infty$$

and therefore the left-hand side inequality of (8.12).

Feedback rule. Maximizing the right-hand side of (A.106) yields (8.11).

Q.E.D.

A Verification Theorem. Proposition A.3 below shows that the conditions given Proposition 10 to characterize the commitment value function by means of an HBJ equation together with boundary conditions are in fact sufficient. We again follow Ekeland and Turnbull (1983, Theorem 1, p. 6) to derive a *Verification Theorem.*

Proposition A.3 *Assume first that there exists a C^1 function $\mathcal{W}_0(X,Z)$ which satisfies:*

$$\lambda_0 \mathcal{W}_0(X,Z) \geq \tilde{Z}(t;X,Z)u(x) + x\frac{\partial \mathcal{W}_0}{\partial X}(X,Z) + \Delta(1-F(X)-\tilde{Z}(t;X,Z))\frac{\partial \mathcal{W}_0}{\partial Z}(X,Z) \quad \forall (x,X,Z); \tag{A.109}$$

and, second, that there exists an action profile $\mathbf{x}_0$ and a path $X_0(t) = \int_0^t x_0(\tau)d\tau$, $Z_0(t) = 1 - \Delta e^{-\Delta t}\int_0^t F(X_0(\tau))e^{\Delta\tau}d\tau$ such that

$$\begin{aligned} \lambda_0 \mathcal{W}_0(X_0(t), Z_0(t)) &= Z_0(t)u(x_0(t)) \\ &+ x_0(t)\frac{\partial \mathcal{W}_0}{\partial X}(X_0(t),Z_0(t)) + \Delta(1-F(X_0(t))-Z_0(t))\frac{\partial \mathcal{W}_0}{\partial Z}(X_0(t),Z_0(t)) \quad \forall t \geq 0. \end{aligned} \tag{A.110}$$

Then $\mathbf{x}_0$ is an optimal action profile with its associated path $(X_0(t), Z_0(t))$.

Proof of Proposition A.3: Suppose thus that $\mathcal{W}^c(X,Z)$ as characterized in Proposition A.2 is C^1. It is our candidate for the function $\mathcal{W}_0(X,Z)$ in the statement of Proposition A.3. By definition (A.106), we have

$$\lambda_0 \mathcal{W}^c(X,Z) = Zu(\sigma^c(X,Z)) + \sigma^c(X,Z)\frac{\partial \mathcal{W}^c}{\partial X}(X,Z) + \Delta(1-F(X)-Z)\frac{\partial \mathcal{W}^c}{\partial Z}(X,Z), \quad \forall (X,Z)$$

and thus

$$\lambda_0 \mathcal{W}^c(X,Z) \geq Zu(x) + x\frac{\partial \mathcal{W}^c}{\partial X}(X,Z) + \Delta(1-F(X)-Z)\frac{\partial \mathcal{W}^c}{\partial Z}(X,Z), \quad \forall (x,X,Z) \tag{A.111}$$

where the inequality comes from the concavity in x of the right-hand side.

To get (A.110), we use again (A.106) but now applied to the path $(x^c(t), X^c(t), Z^c(t))$ where $X^c(t)$ is such that $\dot{X}^c(t) = x^c(t) = \sigma^c(X^c(t), Z^c(t))$ with $X^c(0) = 0$ and $Z^c(t) = 1 - \Delta e^{-\Delta t}\int_0^t F(X^c(\tau))e^{\Delta\tau}d\tau$.

Define now a value function $\tilde{\mathcal{W}}^c(X,Z,t) = e^{-\lambda_0 t}\mathcal{W}^c(X,Z)$. By (A.111), we get

$$0 \geq \frac{\partial \tilde{\mathcal{W}}^c}{\partial t}(X,Z,t) + x\frac{\partial \tilde{\mathcal{W}}^c}{\partial X}(X,Z,t) + \Delta(1-F(X)-Z)\frac{\partial \tilde{\mathcal{W}}^c}{\partial Z}(X,Z) + e^{-\lambda_0 t}Zu(x) \quad \forall (x,X,Z). \tag{A.112}$$

Using $x^c(t) = \sigma^c(X^c(t), Z^c(t))$, $Z^c(t) = 1 - \Delta e^{-\Delta t}\int_0^t F(X^c(\tau))e^{\Delta\tau}d\tau$ and (A.31), we also get

$$\begin{aligned} 0 &= \frac{\partial \tilde{\mathcal{W}}^c}{\partial t}(X^c(t), Z^c(t), t) + x^c(t)\frac{\partial \tilde{\mathcal{W}}^c}{\partial X}(X^c(t), Z^c(t), t) \\ &+ \Delta(1-F(X^c(t))-Z^c(t))\frac{\partial \tilde{\mathcal{W}}^c}{\partial Z}(X^c(t), Z^c(t), t) + e^{-\lambda_0 t}Z^c(t)u(x^c(t)) \quad \forall t \geq 0. \end{aligned} \tag{A.113}$$

Take now an arbitrary action plan $\mathbf{x}$ with the associated path $X(t) = \int_0^t x(\tau)d\tau$ and $Z(t) = 1 - \Delta e^{-\Delta t}\int_0^t F(X(\tau))e^{\Delta\tau}d\tau$. Let us fix an arbitrary $T > 0$. Integrating (A.112) along the path $(x(t), X(t), Z(t))$, we compute

$$0 \geq \int_0^T \left(\frac{\partial \tilde{\mathcal{W}}^c}{\partial t}(X(t), Z(t), t) + x(t)\frac{\partial \tilde{\mathcal{W}}^c}{\partial X}(X(t), Z(t), t) \right.$$

$$+\Delta(1-F(X(t))-Z(t))\frac{\partial\tilde{\mathcal{W}}^c}{\partial Z}(X(t),Z(t),t)+e^{-\lambda_0 t}Z(t)u(x(t))\Bigg)dt$$

or

$$0\geq\int_0^T\left(\frac{d\tilde{\mathcal{W}}^c}{dt}(X(t),Z(t),t)+e^{-\lambda_0 t}Z(t)u(x(t))\right)dt\quad\forall T\geq 0.$$

By definition of the total derivative of $\tilde{\mathcal{W}}^c(X(t),Z(t),t)$ with respect to time, we thus get

$$\tilde{\mathcal{W}}^c(0,0,0)\geq\tilde{\mathcal{W}}^c(X(T),Z(T),T)+\int_0^T e^{-\lambda_0 t}Z(t)u(x(t))dt\quad\forall T\geq 0.$$

Because $\tilde{\mathcal{W}}^c(X,Z,t)=e^{-\lambda_0 t}\mathcal{W}^c(X,Z,t)\geq 0$ for all (X,Z,t), we obtain:

$$\mathcal{W}^c(0,0)\geq e^{-\lambda_0 T}\mathcal{W}^c(X(T),Z(T))+\int_0^T e^{-\lambda_0 t}Z(t)u(x(t))dt\quad\forall T\geq 0.$$

Because of the boundary conditions (A.104), $e^{-\lambda_0 T}\mathcal{W}^c(X(T),Z(T))$ converges towards zero as $T\to+\infty$ for any feasible path. Moreover, for any such feasible path $\int_0^{+\infty}e^{-\lambda_0 t}Z(t)u(x(t))dt$ exists. Henceforth, we get:

$$\tilde{\mathcal{V}}_u(0,0)\geq\sup_{\mathbf{x}}\int_0^{+\infty}e^{-\lambda_0 t}Z(t)u(x(t))dt$$

which shows that $(x^c(t),X^c(t),Z^c(t))$ is indeed an optimal path. *Q.E.D.*

Boundary condition. An immediate corollary of (A.104) is thus (8.10).

Differential equation (8.9) for $\mathcal{V}^c(X)$. A simple application of the Envelope Theorem on (8.8) demonstrates that

$$\frac{\partial\mathcal{W}^c}{\partial Z}(X,Z)=\int_0^{+\infty}e^{-\lambda_0 t}u(\sigma^c(\tilde{X}^c(t;X,Z),\tilde{Z}(t;X,Z)))dt$$

or using (8.11)

$$\frac{\partial\mathcal{W}^c}{\partial Z}(X,Z)=\frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty-\frac{1}{2}\int_0^{+\infty}e^{-\lambda_0 t}\left(\frac{1}{\tilde{Z}^c(t;X,Z)}\frac{\partial\mathcal{W}^c}{\partial X}(\tilde{X}^c(t;X,Z),\tilde{Z}^c(t;X,Z))\right)^2dt.$$

Inserting into (A.103), taking $Z=1$ and using $\mathcal{V}^c(X)\equiv\mathcal{W}^c(X,1)$ yields

$$\lambda_0(\mathcal{V}^c(X)-\mathcal{V}_\infty)=\Delta\mathcal{V}_\infty\left(1-F(X)\frac{\lambda_1}{\lambda_0}\right)+\zeta\mathcal{V}^{c'}(X)+\frac{1}{2}\mathcal{V}^{c'2}(X)$$

$$+\frac{\Delta F(X)}{2}\int_0^{+\infty}e^{-\lambda_0 t}\left(\frac{1}{\tilde{Z}^c(t;X,1)}\frac{\partial\mathcal{W}^c}{\partial X}(\tilde{X}^c(t;X,1),\tilde{Z}^c(t;X,1))\right)^2dt$$

Solving this second-order equation for $\mathcal{V}^{c'}(X)$ and keeping (as in our previous analysis) the positive root that corresponds to a positive action $\sigma^c(X,1)$ yields (8.9).
Q.E.D.

Proof of Proposition 11: Suppose, the contrary, then differentiating (8.11) with respect to Z yields

$$\frac{\partial \sigma^c}{\partial Z}(X,Z) = \frac{\partial}{\partial Z}\left(\frac{1}{Z}\frac{\partial \mathcal{W}^c}{\partial X}(X,Z)\right) = 0 \quad \forall (X,Z)$$

This means that the complete value function $\mathcal{W}^c(X,Z)$ would be affine in Z and thus of the form

$$\mathcal{W}^c(X,Z) = Z\mathcal{W}_1(X) + \mathcal{W}_2(X).$$

Inserting into the HBJ equation (A.103) satisfied by $\mathcal{W}^c(X,Z)$ for all (X,Z) immediately gives

$$\mathcal{W}_1(X) = \mathcal{W}_2(X) = 0 \quad \forall X,$$

which is a contradiction with the boundary condition (A.104). *Q.E.D.*

Proof of Proposition 12: From (8.8), we have:

$$\mathcal{V}^c(X) = \sup_{\mathbf{x},\tilde{X}(\cdot),\tilde{Z}(\cdot) \text{ s.t. } (8.1) \text{ and } (8.6)} \int_0^{+\infty} e^{-\lambda_0 \tau}\tilde{Z}^c(\tau;X,1)u(x(\tau))d\tau.$$

From (8.6) and the fact that $\tilde{X}^c(s;X,1) \geq X$, we also have:

$$\tilde{Z}^c(\tau;X,1) \leq 1 - \Delta e^{-\Delta\tau}\int_0^{\tau} F(X)e^{\Delta s}ds.$$

Therefore, the following inequality holds:

$$\mathcal{V}^c(X) \leq \lambda_1 \int_0^{+\infty} e^{-\lambda_0\tau}\left(1 - \Delta e^{-\Delta\tau}\int_0^{\tau} F(X)e^{\Delta s}ds\right)\mathcal{V}_\infty d\tau.$$

Developing and integrating the right-hand side term yields the right-hand side inequality of (8.12). The left-hand side inequality follows from observing that

$$\tilde{Z}^c(\tau;X,1) \geq e^{-\Delta\tau}.$$

Thus, we get

$$\mathcal{V}^c(X) \geq \int_0^{+\infty} e^{-\lambda_1\tau}u(\zeta)d\tau = \mathcal{V}_\infty \tag{A.114}$$

From the Envelope Theorem applied to (8.8), we get

$$\frac{\partial \mathcal{W}^c}{\partial X}(X,Z) = -\Delta\int_0^{+\infty} e^{-\lambda_0\tau}\left(e^{-\Delta\tau}\int_0^{\tau} f(\tilde{X}^c(s;X,Z))e^{\Delta s}ds\right)u(\sigma^c(\tilde{X}^c(\tau;X,Z),\tilde{Z}^c(\tau;X,Z)))d\tau < 0$$

where $\tilde{X}^c(s;X,Z)$ is the stock trajectory associated to the optimal complete feedback rule $\sigma^c(\tilde{X}^c(\tau;X,Z),\tilde{Z}^c(\tau;X,Z))$. Hence, (8.11) implies

$$\sigma^c(X,Z) < \zeta \tag{A.115}$$

and thus the right-hand side inequality of (8.13) follows.

We now use this condition to show that

$$\int_0^{+\infty} e^{-\lambda_0\tau}\left(\frac{1}{\tilde{Z}^c(t;X,Z)}\frac{\partial \mathcal{W}^c}{\partial X}(\tilde{X}^c(t;X,Z),\tilde{Z}^c(t;X,Z))\right)^2 d\tau$$

$$= \int_0^{+\infty} e^{-\lambda_0 \tau} \left(\sigma^c(\tilde{X}^c(t; X, Z), \tilde{Z}^c(t; X, Z)) - \zeta \right)^2 d\tau \leq \frac{\zeta^2}{\lambda_0}.$$

Using that inequality and (A.114), we now obtain

$$(\sigma^c(X, 1) - \zeta)^2 = \mathcal{V}^{c\prime 2}(X)$$

$$= \zeta^2 + 2\lambda_0(\mathcal{V}^c(X) - \mathcal{V}_\infty) - 2\Delta\mathcal{V}_\infty \left(1 - \frac{\lambda_1}{\lambda_0} F(X) \right)$$

$$-\Delta F(X) \int_0^{+\infty} e^{-\lambda_0 \tau} \left(\frac{1}{\tilde{Z}^c(t; X, Z)} \frac{\partial \mathcal{W}^c}{\partial X} (\tilde{X}^c(t; X, Z), \tilde{Z}^c(t; X, Z)) \right)^2 d\tau$$

$$\geq \zeta^2 - 2\Delta\mathcal{V}_\infty.$$

Taking roots ends the proof of the left-hand side inequality of (8.13). *Q.E.D.*