Learning Before Trading: On the Inefficiency of Ignoring Free Information*

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Abstract

This paper analyzes a bilateral trade model where the buyer's valuation for the object is uncertain and she can privately purchase any signal about her valuation. The seller makes a take-it-or-leave-it offer to the buyer. The cost of a signal is smooth and increasing in informativeness. We characterize the set of equilibria when learning is free and show that they are strongly Pareto ranked. Our main result is that, when learning is costly but the cost of information goes to zero, equilibria converge to the worst free-learning equilibrium.

1 Introduction

Recent developments in information technology have given consumers access to new information sources that allow them to learn about products prior to trading. For example, online resources enable buyers to learn about a mechanic's reputation, a contractor's ability, or an over-the-counter (OTC) asset's value. This information acquisition often takes

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place before the buyers learn the terms of trade. Indeed, in order to get a price quote, customers may need to bring their cars to the mechanic, have a contractor over, or waste their first contact with an OTC dealer.[1] Since the buyer's willingness-to-pay depends on her information about the product, the seller's price depends on what he expects the buyer to learn. The flip side is that the seller's pricing strategy determines what information is worth learning for the buyer. For example, there may be no point in knowing more about the value of an asset if the buyer is already sure that it is below its price. So, the buyer's information acquisition depends on the seller's expected prices. The goal of this paper is to study this mutual dependency between the buyer's learning strategy and the seller's pricing policy.

We consider a stylized model where the seller has a single object for sale and full bargaining power. Initially, the buyer does not know anything about the value of the object except its prior distribution. We model the buyer's learning as *flexible* information acquisition, that is, she can purchase *any* signal about her valuation privately. Then the seller, without observing the buyer's learning strategy and her signal realization, sets a price. Signals are costly and we assume that this cost is a *smooth and strongly increasing* function of the signal's informativeness. Below, we explain these assumptions in detail. Our aim is to characterize the set of equilibria of this game. We are particularly interested in the limit where the buyer's cost vanishes. This appears to be particularly relevant in a world where information is cheap and accessible to consumers. To this end, we parameterize the cost by a multiplicative constant and consider the limit when this parameter converges to zero.

We now describe the buyer's action space and the cost of information. The demand of the buyer, which is the probability of trade occurring at a given price, is fully determined by the distribution of her posterior value estimate. In turn, the seller's profit from any given price is pinned down by the buyer's demand. As a consequence, trade outcomes are fully determined by the distribution of the buyer's posterior estimate. The prior distribution is a mean-preserving spread of any such distribution since each signal contains less information than the valuation itself. Since the buyer can choose any signal, we identify her action space with the set of these distributions and define the cost of information acquisition on this set. To require this function to be smooth we appeal to a generalized notion of differentiability because the domain is a set of CDFs. We postulate that the

[1] A stylized feature of OTC markets is that prices quoted on a second call can be dramatically higher than the first one, see Bessembinder and Maxwell (2008), or Zhu (2012).

cost function is Fréchet-differentiable.[2]

Let us now turn to our main assumption on the cost of information. A signal is more informative than another if its induced distribution over posterior value estimates is a mean-preserving spread of that of the other. So, a cost function is said to be monotonic in the signal's informativeness if it respects this mean-preserving relationship. As will be argued, a cost function is monotonic whenever its Fréchet-derivative is convex.[3] Our main assumption is somewhat stronger than monotonicity: in addition to requiring the Fréchet-derivative to be convex, we assume that this derivative at a given CDF is strictly convex on the CDF's support. The additional property imposed by this assumption to monotonicity resembles stipulating that the derivative of a strictly increasing function is strictly positive everywhere.

Monotonicity of the learning cost implies that the seller randomizes in every equilibrium in which the buyer learns. To see this, suppose that there is an equilibrium in which the seller sets a fixed price and the buyer receives an informative signal about her valuation. Then this signal must be binary, indicating whether or not the buyer should trade. The reason is that any other signal can be made less informative, and hence cheaper, while still leading to the same trading decisions. The seller's best-response to such a binary signal is to charge the expected valuation of the buyer conditional on one of the two signal realizations. To get a contradiction, notice that the buyer is strictly better off by not learning irrespective of which of these prices is set. If the price is the lower signal realization, then the buyer always trades so learning yields no benefit. If the price is the higher signal realization, then the buyer's surplus from trade is zero, so she could again profitably deviate by saving the cost of learning and not trading.

Our aforementioned strong monotonicity assumption also has important implications on the buyer's equilibrium learning strategy. We show that the support of the buyer's equilibrium signal is an interval and the buyer's demand generated by this signal makes the seller indifferent between setting any price on its support. This indifference condition implies that the buyer's equilibrium CDF is a *truncated Pareto distribution* and hence, her equilibrium demand is unit-elastic.

As mentioned above, our main objective is to characterize equilibrium outcomes as the buyer's cost vanishes. To this end, we first consider the case where learning is free. We show that this case admits multiple equilibria, all of which can be Pareto ranked. In

[2]The Fréchet-derivative is a function itself.

[3]See Machina (1982) for a similar result.

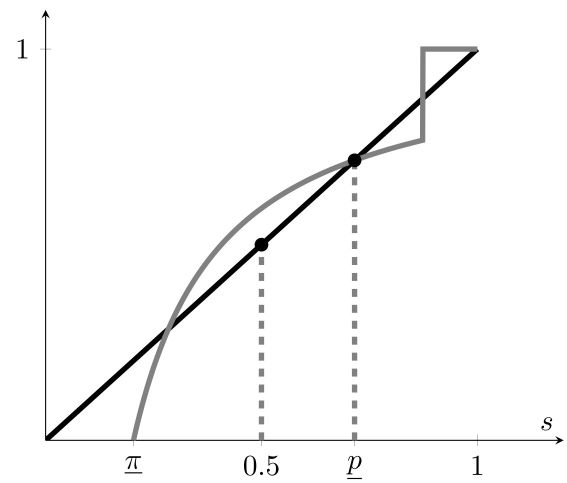

Figure 1: An Illustration of the best and worst equilibria in the uniform case.

the Pareto-best equilibrium, which maximizes both players' payoffs across all free-learning equilibria, the buyer learns her valuation perfectly. The Pareto-worst equilibrium turns out to be the unique equilibrium in which the buyer's posterior estimate is distributed according to a truncated Pareto distribution.

The best and worst free-learning equilibria are illustrated on Figure 1 for the case of the uniform prior on $[0, 1]$. In the Pareto-best equilibrium, the buyer learns her valuation perfectly so the distribution of her value estimate is also uniform, and so is represented by the 45-degree line on Figure 1. In this case, the seller's equilibrium price is .5, his profit is .25, and the buyer's payoff is .125. The buyer's CDF in the Pareto-worst equilibrium is depicted as a gray curve on Figure 1. In this worst equilibrium, the seller's profit, $\underline{\pi}$, is approximately .2, the price is $\underline{p}\,(\approx .715)$ and the buyer's payoff is only slightly above .04. So, the buyer's payoff is less than one third of her payoff in the perfect-learning equilibrium.

At first, it may appear counter-intuitive that there are equilibria in which the buyer does not learn perfectly although information is free. In the above described Pareto-worst equilibrium, the seller's price, $\underline{p}$, is defined by the highest intersection of the Pareto curve and the prior CDF. At this point, the mean-preserving spread constraint binds, that is, the integral of the Pareto curve and the prior CDF on $\left[0, \underline{p}\right]$ coincide. We call such a point *separating*. The important property of a separating point is that the buyer never confuses a value below such a point with a value above it. That is, a value below $\underline{p}$ never generates the same signal as a value above $\underline{p}$. This implies that the buyer would not gain anything by learning more because this Pareto signal already reveals if her valuation is above or below $\underline{p}$, which is the only information she needs to know in order to trade ex-post efficiently.

Our main result is that as the buyer's learning cost vanishes, equilibria converge to a Pareto-worst free-learning equilibrium. For intuition, recall that when learning is costly, the CDF of the buyer's posterior estimate is a truncated Pareto. The limit of truncated Pareto distributions is also a truncated Pareto, so the same must hold for the costless limit, which is a free-learning equilibrium. Hence, as costs shrink, we obtain a free learning equilibrium in which the buyer's demand is unit elastic. All that remains is to recall the fact mentioned above, namely that the unique such equilibrium yields the Pareto-worst free learning outcome.

The main takeaway from our paper is that possessing information might be significantly better than having cheap access to it. When information is costly, buyers must have incentives to acquire it. In equilibrium, prices fail to provide these incentives, so buyers choose to ignore large amounts of information even when costs are minuscule. In turn, this ignorance triggers prices that are too high compared to those in a full-information environment, leading to considerable welfare losses. Mitigating these losses may justify certain market features such as the existence of professional intermediaries. By making sure that traders are informed, intermediaries can substantially increase social surplus. More broadly, our results highlight the importance of regulating the provision of product information even when data is cheap, as cheap data does not necessarily approximate full information. Special care should be taken when designing the informational channels through which market participants learn. For example, mandatory information sessions appear to be more desirable than supplying brochures. The reason is that being able to know something is not the same as actually knowing it.

Our paper serves as a cautionary tale on interpreting recent papers characterizing consumer and producer surplus pairs which can arise as an equilibrium outcome for *some* information structure (e.g. Bergemann et al. (2015), Roesler and Szentes (2017)). Of particular relevance is Roesler and Szentes (2017), who consider a setting similar to ours in which the buyer's signal is observed by the seller before he sets a price. Their key result identifies the signal-equilibrium pair that maximize the buyer's payoff. It turns out that the buyer-optimal signal is the same Pareto signal as in our worst free-learning equilibrium. At first glance, this might seem surprising given that the worst free-information equilibrium minimizes the buyer's payoff. However, since the seller sets a price only after observing the buyer's signal in Roesler and Szentes's (2017) model, he can set any profit-maximizing price and, in the buyer-optimal equilibrium, he chooses the lowest such price. In contrast, in our model, the seller's price must also justify the buyer's signal choice, forcing him to choose a separating point. Thus, our analysis suggests that the same information structure can lead to two drastically different outcomes. Which outcome is selected depends on the mechanism through which trade occurs.

Our paper also adds to the recent literature on the relationship between free-learning equilibria and the vanishing-cost limits of equilibria. For example, Yang (2015) studies a 2-by-2 coordination game where players can learn about their stochastic benefits from coordination. When the learning cost is proportional to entropy reduction, infinitely many equilibria can be attained in the limit. Morris and Yang (2016) considers a related regime-switching game and show that there is a unique vanishing-cost limit if the learning cost admits a "continuous choice" property, that is, if only signals whose distribution varies continuously with the state are optimal.[4] This literature primarily focuses on static flexible learning models where all players have access to the same information. It turns out that, in these models, free information always yields a perfect-learning outcome. Therefore, the vanishing-cost limit can be considered as an equilibrium-selection device from a symmetric information game. In contrast, learning is asymmetric in our model since the seller cannot acquire information about the buyer's valuation. Consequently, as we explained above, perfect-learning corresponds to an asymmetric information game with a substantially smaller equilibrium set than our free-learning game. And indeed, our vanishing cost limit selects a free-learning outcome that is simply not an equilibrium under full information.

Costly consumer learning is extensively studied in the literature on rational inattention initiated by Sims (1998, 2003, 2006). In these models, information cost is proportional to the resulting expected reduction in entropy. For example, Matějka (2015) studies a dynamic pricing model with a consumer who is rationally inattentive to prices. He finds that rational inattention leads to rigid pricing because such pricing structures are easier to assess for the consumer. Ravid (2018) studies a dynamic, repeated-offer bargaining game in which the buyer is rationally inattentive and can learn about both her valuation

[4]Denti (2018) shows that allowing players to learn about others' information yields a unique vanishing cost limit in Yang's (2015) model, while Hoshino (2018) argues that the limits selected by Denti's (2018) model depend on the fine details of the cost function.

and the seller's offers. He finds that the buyer benefits from her inattention, and that such benefits remain large even when offers are frequent and costs vanish. In contrast to this literature, we treat the cost of information in an abstract way and do not assume such a particular form. Still, one can show that our results go through even when the buyer's information costs are given by expected entropy reduction.

Several papers examine buyers' incentives to acquire costly information about their valuations before participating in auctions. The buyers' learning strategies depend on the selling mechanism announced by the seller. Persico (2000) shows that if the buyers' signals are affiliated then they acquire more information in a first-price auction than in a second-price one. Compte and Jehiel (2007) show that dynamic auctions tend to generate higher revenue than simultaneous ones. Shi (2012) also analyses models where it is costly for the buyers to learn about their valuations and identifies the revenue-maximizing auction in private-value environments. In all of these setups, the seller is able to commit to a selling mechanism before the buyers decide how much information to acquire. In contrast, we consider environments where the monopolist cannot commit and best-responds to the buyer's signal structure.[5]

Condorelli and Szentes (2018) also consider a bilateral trade model. In contrast to our setup, the distribution of the buyer's valuation is not given exogenously. Instead, the buyer chooses her value-distribution supported on a compact interval and perfectly observes its realization. The seller observes the buyer's distribution but not her valuation and sets a price. The authors show that, as in our model, the equilibrium distribution generates a unit-elastic demand.

2 The Model

A seller, S, has an object to sell to a single buyer, B. B's valuation, $\mathbf{v}$, takes values in $[0,1]$ according to the CDF F_0 whose expected value is $\bar{v} = \int v \, \mathrm{d}F_0(v) > 0$. We assume that F_0 is **regular**, meaning that it has a strictly positive density, f_0, on $[0,1]$ and that $v - (1 - F_0(v))/f_0(v)$ is strictly increasing in v. B does not observe $\mathbf{v}$ but can choose to observe *any* signal, $\mathbf{s}$, at a cost that depends on the signal's informativeness. Below, we describe the set of signals available to the buyer and the associated cost in detail.

[5]Another strand of the literature analyzes the seller's incentives to reveal information about the buyers' valuations prior to participating in an auction, see for example, Ganuza (2004), Bergemann and Pesendorfer (2007) and Ganuza and Penalva (2010).

Then S, without observing B's information acquisition strategy and signal, makes a take-it-or-leave-it price offer, $p \in [0,1]$, which B accepts if and only if her expected valuation conditional on her signal weakly exceeds p.[6] Both players are risk-neutral expected payoff maximizers.

Signal structures and B's action space.— Note that both B's trading decision and her expected payoff from trading depend only on her posterior expectation, $\mathbb{E}[\mathbf{v}|\mathbf{s}]$. Assuming that acquiring more information[7] is more costly, it is without loss of generality to restrict attention to signal structures for which B's posterior expectation is the signal itself – i.e., $\mathbb{E}[\mathbf{v}|\mathbf{s}] = \mathbf{s}$. As a consequence, both B and S only care about the signal's marginal distribution. Thus, we identify each signal with the CDF of its marginal. We let $\mathcal{F}$ be the space of all CDFs over $[0,1]$, which we endow with the $\mathcal{L}_1$-norm, denoted by $\|\cdot\|$.[8] For any subset $A \subseteq [0,1]$, we take $\mathbf{1}_A$ to be the indicator function that is equal to 1 on A and to zero otherwise. So, $\mathbf{1}_{[x,1]} \in \mathcal{F}$ is the CDF corresponding to a unit atom at $x \in [0,1]$.

It turns out to be useful to compare the informativeness of different signals. We say that $\mathbf{s}$ is more informative than $\mathbf{s}'$ if $\mathbf{s} = \mathbf{s}' + \mathbf{t}$ for some random variable $\mathbf{t}$ satisfying $\mathbb{E}[\mathbf{t}|\mathbf{s}'] = 0$, that is, observing $\mathbf{s}$ is the same as observing both $\mathbf{s}'$ and $\mathbf{t}$. In other words, $\mathbf{s}$ is a mean preserving spread of $\mathbf{s}'$. Hence, if $F, F' \in \mathcal{F}$, we say that F is **more informative** than F' (denoted by $F \succeq F'$) if and only if F is a mean preserving spread of F' – i.e.,[9]

$$\int_0^x (F - F')\, \mathrm{d}s \geq 0 \text{ for all } x \text{ with equality for } x = 1. \tag{1}$$

The CDF F is said to be **strictly more informative** than F' (denoted by $F \succ F'$) if both $F \succeq F'$ and $F' \neq F$.[10]

We allow B to choose *any* signal to learn about $\mathbf{v}$ and we identify B's action space with the set of those CDF's which correspond to a signal about her valuation. Of course, observing the valuation perfectly is more informative than any signal. Thus, B can choose any CDF F which is less informative than the prior F_0 – i.e., any $F \in \mathcal{F}$ such that $F_0 \succeq F$. We denote this set by $\mathcal{A}$, and refer to CDFs in $\mathcal{A}$ as signals. Letting $I_F(x)$ denote $\int_0^x (F_0 - F)\, \mathrm{d}s$, (1) implies that $F \in \mathcal{A}$ if and only if $I_F(x) \geq 0$ for all x and $I_F(1) = 0$.

[6] Assuming that B trades if indifferent has no effect on our results but makes the analysis simpler.

[7] Using Blackwell's 1953 information ranking.

[8] That is, the norm that maps any Borel measurable $\phi : [0,1] \to \mathbb{R}$ to $\|\phi\| = \int_0^1 |\phi(x)| \mathrm{d}x$. Restricted to the set of CDFs over $[0,1]$, this norm metrizes weak* convergence, see for example Machina (1982).

[9] See Rothschild and Stiglitz (1970) for the statement and Leshno et al. (1997) for the corrected proof. Blackwell and Girshick (1979) proves the result for discrete distributions.

[10] Notice that $\succeq$ is reflexive and anti-symmetric, meaning that $F \succeq F'$ and $F' \succeq F$ if and only if $F = F'$.

The cost of information acquisition.— Information acquisition is costly. In general, different information structures generating the same distribution of posterior expectation might come at different costs. However, since B's expected payoff from trading depends only on the distribution of this posterior expectation, F, she would always use the least expensive signal that leads to F. In fact, B may even randomize to get F. Thus, we can evaluate the cost of F by the expected cost of the cheapest randomization that leads to it. This results in a *convex* cost function,

$$C : \mathcal{A} \to \mathbb{R}_+.$$

We require the function C to be sufficiently smooth. More precisely, we assume that C is **Fréchet differentiable**; that is, it is continuous and for each $F \in \mathcal{A}$, there is a Lipschitz function, $c_F : [0,1] \to \mathbb{R}$, such that for every $F' \in \mathcal{A}$,

$$C(F') - C(F) = \int c_F \, \mathrm{d}(F' - F) + o\left(\left\|F' - F\right\|\right), \tag{2}$$

where o is a function that equals to zero at zero and $\lim_{x \searrow 0} [o(x)/x] = 0$. We refer to c_F as C's **derivative at** F.[11]

It is natural to assume that acquiring more information is more costly. We say that C is increasing if $C(F) \geq C(F')$ whenever F is strictly more informative than F'. Next, we show that C is increasing in the informativeness of the signal if and only if its Fréchet-derivative is convex.

Claim 1 *Let C be convex and Fréchet-differentiable. Then C is increasing if and only if c_F is convex for each $F \in \mathcal{A}$.*

Proof. See appendix. ■

For the intuition behind the claim and for better understanding the concept of Fréchet-differentiable, let us restrict attention to signals whose support lies in a finite set, say $\{s_1, \ldots, s_N\}$. Then each $F \in \mathcal{F}$, can be represented by the vector $(\alpha_1, \ldots, \alpha_N)$ such for which $F = \sum_{n=1}^{N} \alpha_n \mathbf{1}_{[s_n,1]}$. In this case, the Fréchet-derivative at F at s_n, $c_F(s_n)$, is C's partial derivative with respect to the probability of s_n, that is, $\partial C / \partial \alpha_n(F) = c_F(s_n)$. Thus, the marginal cost of a small shift from F to F' is the sum of the marginal cost at each signal realization times the change in each realization's probability, that is, $\int c_F \, \mathrm{d}(F' - F)$. Of course, if $F' \succeq F$ then this quantity is positive if c_F is convex.

[11] Formally, $c_F(x) = \int_0^x \phi_F \, \mathrm{d}s$ for some $\phi_F \in \mathcal{L}_\infty[0,1]$, and so, c_F is unique Lebesgue-a.e.

Our main assumption requires c_F to be not only convex but also strictly convex on the support of F.

Assumption 1 *For each $F \in \mathcal{A}$, c_F is convex and strictly convex on* co(supp F).

Strategies and payoffs.— A mixed strategy for S is a random price, represented by a CDF over prices, $H \in \mathcal{F}$, while a strategy for B is a signal, $F \in \mathcal{A}$.[12] If B's signal is F, S's expected payoff from the random price H is given by

$$\Pi(H, F) = \int p(1 - F(p-)) \, \mathrm{d}H(p).$$

We let π_F denote S's maximal profit, $\max_{p\in[0,1]} \Pi(p, F)$, and $P(F)$ denote the set of profit maximizing prices, $\arg\max_{p\in[0,1]} \Pi(p, F)$.[13]

If S's random price is H, B's expected payoff from the signal distribution F is

$$U_\kappa(H, F) = \int \int_0^s (s - p) \, \mathrm{d}H(p) \, \mathrm{d}F(s) - \kappa C(F),$$

where $\kappa \in \mathbb{R}_+$ is a constant parameterizing B's cost of information.

Equilibrium Definition and Existence.— An equilibrium is a pair, (H, F), such that:

1. H maximizes $\Pi(H, F)$ over $H \in \mathcal{F}$.

2. F maximizes $U_\kappa(H, F)$ over $F \in \mathcal{A}$.

We call an equilibrium **non-trivial** if B learns, that is, $F \neq \mathbf{1}_{[\bar{v},1]}$. The following theorem shows that a non-trivial equilibrium exists under general conditions whenever costs are sufficiently small.

Theorem 1 *Suppose C satisfies Assumption 1. Then, an equilibrium exists for all $\kappa \geq 0$. Moreover, there exists $\bar{\kappa} > 0$ such that all equilibria are non-trivial whenever $\kappa < \bar{\kappa}$.*

Proof. See appendix. ■

Truncated Pareto Distributions.— As mentioned in the introduction, the set of truncated Pareto distributions plays an important role in our analysis. To formally define this set, for each $\pi \in (0, 1]$ and $t \in [\pi, 1]$ let,

$$G_{\pi,t}(s) = \mathbf{1}_{[\pi,t)} \left(1 - \frac{\pi}{s}\right) + \mathbf{1}_{[t,1]}. \tag{3}$$

We refer the set $\{G_{\pi,t}\}$ as the set of truncated Pareto distributions and an element of $\{G_{\pi,t}\} \cap \mathcal{A}$ as a **Pareto signal**.

[12] Since C is convex, S's objective is linear, and $\mathcal{A}$ is convex, we can assume B uses a pure strategy.

[13] We slightly abuse notation and let $\Pi(p, F)$ denote $\Pi(\mathbf{1}_{[p,1]}, F)$.

2.1 Examples for the Cost of Learning

This section provides three examples for cost functions and characterizes their Fréchet derivatives.

Example 1. (Constant Marginal Cost) Fix some strictly convex function $c : [0,1] \to \mathbb{R}_+$. Then one can define the function

$$C(F) = \int c \, \mathrm{d}F.$$

Then, C's Fréchet derivative equals to c for all F.

Example 2. (Increasing Marginal Cost) Fix some strictly convex $c : [0,1] \to \mathbb{R}_+$ and a strictly increasing, convex and differentiable $\psi : \mathbb{R}_+ \to \mathbb{R}_+$. Then the function

$$C(F) = \psi \left(\int c \, \mathrm{d}F \right)$$

satisfies our assumptions. Indeed, by the chain rule, the above cost function is Fréchet differentiable, with the derivative being given by

$$c_F(\cdot) = \psi' \left(\int c \, \mathrm{d}F \right) c(\cdot),$$

which is convex for all F, and strictly convex for any $F \neq \mathbf{1}_{[\bar{v},1]}$.

Example 3. (Quadratic Costs) Let $c : [0,1] \times [0,1] \to \mathbb{R}_+$ be some strictly convex, symmetric function – i.e., such that $c(s_1, s_2) = c(s_2, s_1)$ for all $s_1, s_2 \in [0,1]$. Then the cost function

$$C(F) = \frac{1}{2} \int \int c(s_1, s_2) \, \mathrm{d}F(s_1) \mathrm{d}F(s_2)^{14}$$

is Fréchet differentiable, with the derivative being given by

$$c_F(\cdot) = \int c(\cdot, s_2) \, \mathrm{d}F(s_2).$$

As c_F is strictly convex, this cost satisfies our assumption.[15]

3 Costless learning

In this section, we analyze the set of equilibria when learning is free – i.e., when $\kappa = 0$. We first provide geometric characterizations of the best responses of B and S, respectively.

[14]This is essentially the functional form for quadratic preferences as introduced by Machina (1982).

[15]To ensure convexity of C, c must be positive semidefinite – i.e., $\int \int c \, \mathrm{d}(F - F')\mathrm{d}(F - F') \geq 0$ for all $F, F' \in \mathcal{F}$.

We then use these characterizations to identify the set of payoff profiles that arise in equilibrium. We also show that the free-learning equilibrium set can be strongly Pareto ranked, with the best equilibrium being the one given by perfect learning, that is, $F = F_0$. The worst equilibrium outcome is attainable with a Pareto signal.

3.1 The Buyer's Best Responses

If S sets price p and B learns her valuation perfectly, she makes an ex-post efficient trading decision. To make such decisions, B's signal must reveal whether the true valuation is above or below p. In what follows, we characterize the set of such signal distributions.

To this end, note that if B chooses F and the price is p then her expected payoff from trade is

$$\int_p^1 (s-p)\,dF(s) = (1-p) - \int_p^1 F(s)\,ds, \tag{4}$$

where the equality follows from integrating by parts. Of course, when information is free, perfect learning is a best response to any pricing strategy of the seller. Hence, this payoff is maximized by F_0. Therefore, the previous equation implies that F is also a best response and achieves the same payoff as that of perfect learning if and only if $\int_p^1 (F_0 - F)\ \mathrm{d}s = 0$ or equivalently $I_F(p) = 0$.[16] Intuitively, $I_F(p) = 0$ means that p separates the signal realizations in the sense that either both B's true valuation and her signal generated by F are larger or both of them are smaller than p. In what follows, we refer to such a price as **F-separating** and we denote the collection of such prices by $S(F)$, that is,

$$S(F) = \{p \in [0,1] : I_F(p) = 0\}.$$

In summary, if S sets price p the CDF F is B's best response if and only if $p \in S(F)$. The next lemma shows that the argument of this paragraph can be extended to the case where S randomizes over prices. We show that by choosing F, B achieves the same payoff as with perfect learning if each possible price of S is F-separating.

Lemma 1 *The signal F is a best response against H if and only if* $\operatorname{supp} H \subseteq S(F)$.

[16] To see this equivalence, recall that $I_F(p) = \int_0^p (F_0 - F)\,ds$ and note that

$$\int_p^1 (F_0 - F)\ \mathrm{d}s = I_F(1) - I_F(p) = -I_F(p),$$

where the last equality follows from $I_F(1) = 0$.

Proof. If S uses H and B chooses F, the difference between B's payoff generated by F and that of F_0 can be written as

$$U_0(H,F) - U_0(H,F_0) = \int \left[\int_p^1 (s-p)\, dF(s) - \int_p^1 (s-p)\, dF_0(s)\right] dH(p)$$
$$= \int \left[\int_p^1 F_0(s)\, ds - \int_p^1 F(s)\, ds\right] dH(p) = -\int I_F(p)\, dH(p),$$

where the first equality follows from (4) and the third one from $\int_p^1 (F_0 - F)\ \mathrm{d}s = -I_F(p)$. Since $I_F(\cdot)$ is continuous, we conclude that F generates the same payoff as perfect learning if and only if $I_F(p) = 0$ for all $p \in \text{supp } H$, that is, $\text{supp } H \subseteq S(F)$. ■

Next, we show that the graphs of F and F_0 must intersect at any F-separating price. Intuitively, p is F-separating if the signal reveals whether the valuation is above or below p. The probability that B observes a signal realization below p and the probability that the value is below p must be equal, that is, the CDFs F and F_0 must cross at p.

Lemma 2 *If $F \in \mathcal{A}$ and $p \in S(F)$ then F is continuous at p and*

$$F(p) = F_0(p). \tag{5}$$

Proof. Suppose $p \in S(F)$. Then, by the definition of $S(F)$, $I_F(p) = 0$. Recall that $I_F(x) \geq 0$ for all $x \in [0,1]$, so

$$p \in \arg\min_{x\in[0,1]} I_F(x). \tag{6}$$

Since $I_F(x) = \int_0^x (F_0 - F)\ \mathrm{d}s$, it can be differentiated from both sides at p. Therefore, (6) implies that

$$\begin{aligned} 0 &\geq I'_{F-}(p) = F_0(p-) - F(p-), \\ 0 &\leq I'_{F+}(p) = F_0(p) - F(p). \end{aligned}$$

From these two inequalities it follows that $F_0(p-) \leq F(p-) \leq F(p) \leq F_0(p)$. Since F_0 is regular, it does not have an atom at p, so $F_0(p-) = F(p)$. Hence, all the inequalities are equalities in the previous chain and the statement of the lemma follows. ■

3.2 The Seller's Best Responses

We now characterize the set of profit maximizing prices. To this end, we first describe S's iso-profit curves on the price-cumulative probability space. Note that if the price is

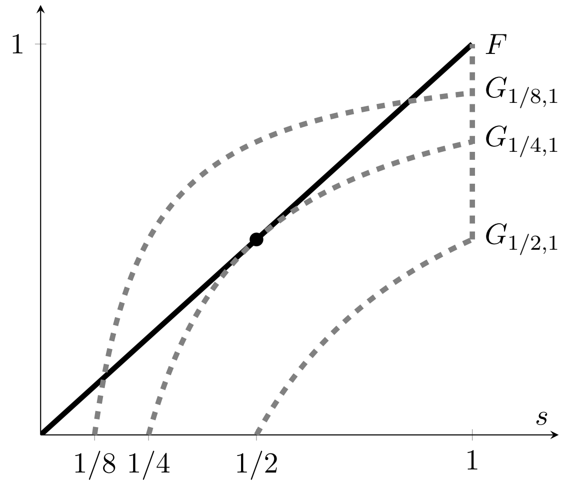

Figure 2: The seller's best response against the uniform distribution.

p and the probability that B's valuation is strictly less than p is x, then the S's profit is $p(1-x)$. Hence, the iso-profit curve in this space corresponding to a given profit, say $\pi\,(>0)$, is defined by

$$\{(p,x) : x \in [0,1]\,,\, p(1-x) = \pi\}\,.$$

Of course, if $p < \pi$ then the profit cannot exceed p and there is no $x \in [0,1]$ which generates π. Otherwise, for each $p \in [\pi, 1]$, the cumulative probability, x, which guarantees profit π is $1 - \pi/p$. Observe that $1 - \pi/p$ is the CDF corresponding to the Pareto distribution parameterized by π. Since $p \leq 1$, we conclude that the iso-profit curve of the seller corresponding to profit π is essentially identical to the truncated Pareto distribution, $G_{\pi,1}$.

These iso-profit curves can be used to analyze S's best-response against B's signal distribution as illustrated on Figure 2 for the case of a uniform F. Note that lower iso-profit curves correspond to larger profits. In addition, the set of feasible outcomes are $\{(p, F(p-)) : p \in [0,1]\}$. Therefore, S's profit is defined by the largest π such that the curve $G_{\pi,1}(s)$ is weakly below that of $F(s-)$. In Figure 2, three iso-profit curves are depicted as the gray dashed contours and the middle one, $G_{1/4,1}$, is the largest iso-profit curve below F so the profit of S is 1/4. Furthermore, the set of optimal prices, $P(F)$, are those values at which F is tangent to the largest iso-profit curve below it. In Figure 2, there is a only a single point of tangency at $p = 0.5$. The following lemma summarizes these observations.

Lemma 3 *Fix any $F \in \mathcal{A}$. Then,*

(i) for all $s \in [0,1]$, $F(s-) \geq G_{\pi_F,1}(s-)$; and

(ii) $P(F) = \{p \geq \pi_F : F(p-) = G_{\pi_F,1}(p-)\} \subseteq \text{supp} F$.

Part (i) states that B's CDF is first-order stochastically dominated by the Pareto distribution parameterized by S's profit, π_F. Part (ii) says the set of profit-maximizing prices are those signals at which B's CDF essentially coincides with this Pareto distribution.

Proof of Lemma 3. To prove part (i), note that S's profit from setting a certain price cannot exceed π_F, that is, for all $s \in [0,1]$, $s(1 - F(s-)) \leq \pi_F$. Rearranging this inequality yields

$$G_{\pi_F,1}(s-) = 1 - \frac{\pi_F}{s} \leq F(s-),$$

which proves part (i).

To see part (ii), note that $s \in P(F)$ if and only if the inequality in the previous displayed chain is an equality. Hence, $P(F) = \{p \geq \pi_F : F(p-) = G_{\pi_F}(p-)\}$. It remains to show that $P(F) \subseteq \text{supp } F$. Suppose, by contradiction, that there exists a p such that $p \in P(F) \setminus \text{supp } F$. Then there exist $p' > p$ such that $F(p'-) = F(p-)$. So,

$$\Pi(p, F) = p(1 - F(p-)) < p'(1 - F(p-)) = p'\left(1 - F(p'-)\right) = \Pi\left(p', F\right),$$

where the inequality follows from $p' > p$ and the second equality follows from $F(p'-) = F(p-)$. This inequality chain implies that S is strictly better off with setting price p' than price p, a contradiction to $p \in P(F)$. ■

3.3 Free-Learning Equilibrium Characterization

First, we show that S never randomizes in equilibrium. More specifically, we prove that if (H, F) is a free-learning equilibrium then H specifies an atom of size one at the largest price which would generate a profit of π_F if B learns perfectly. To state this result precisely, for each π, let X_π be the set of prices which yield profit π under F_0, that is,

$$X_\pi := \{p : \Pi(p, F_0) = \pi\}.$$

Observe that supp $H \subseteq P(F) \cap S(F)$ because any possible equilibrium price must be profit-maximizing as well as F-separating (see Lemma 1). We now explain that our characterizations of B's and S's best responses imply that the set of such prices, $P(F) \cap S(F)$, is contained in X_{π_F}. To see this, note that if $p \in P(F) \cap S(F)$ then

$$G_{\pi_F,1}(p-) = F(p-) = F(p) = F_0(p), \tag{7}$$

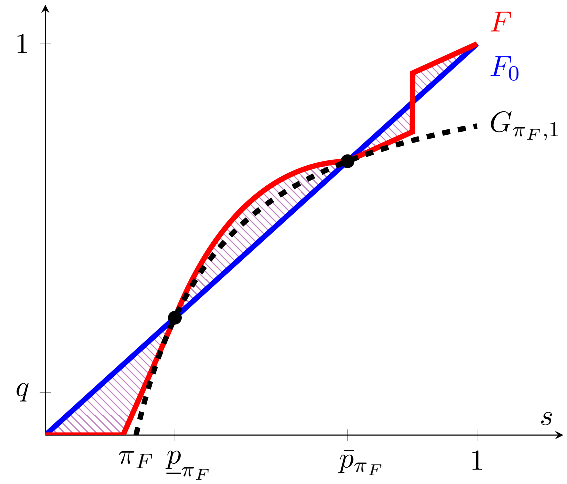

Figure 3: An illustration of Lemma 4. The blue line corresponds to the prior, F_0, the red curve is the the signal, F, and the dashed curve is the π_F iso-profit curve, G_{π_F}. While the signal is such that both prices in $X_{\pi_F} = \{\underline{p}_{\pi_F}, \bar{p}_{\pi_F}\}$ are profit maximizing, only $\bar{p}_F$ can be separating.

where the first equality holds because $p \in P(F)$ (see Lemma 3), and the last two equalities follow from $p \in S(F)$ and the continuity of F (see Lemma 2). Therefore,

$$P(F) \cap S(F) \subseteq \{p : G_{\pi_F,1}(p-) = F_0(p)\} = X_{\pi_F}. \tag{8}$$

Lemma 4 below states that, if S's profit is π in a free-learning equilibrium then his equilibrium price is the largest element of X_π. Before we state this result, note that since F_0 is regular, the function $\Pi(\cdot, F_0)$ is strictly concave, so there are at most two such prices for every π. Since $\Pi(\cdot, F_0)$ is continuous, it attains any value between 0 and π_{F_0}.[17] Therefore, for each $\pi \in [0, \pi_{F_0}]$, X_π is non-empty and contains at most two prices. Let $\bar{p}_\pi$ be the higher of those prices, that is, $\bar{p}_\pi = \max X_\pi$. The following lemma says that $\bar{p}_{\pi_F}$ is the unique price that can be both F-separating and profit maximizing.

Lemma 4 *Let (H, F) be a free-learning equilibrium. Then* supp $H = \{\bar{p}_{\pi_F}\}$.

Proof. See the Appendix. ■

For an explanation, recall that X_π has at most two elements. If X_{π_F} is a singleton then the statement of the lemma immediately follows from the observation that supp $H \subseteq$

[17]This follows from the Intermediate Value Theorem and the fact that charging zero generates zero profit.

$P(F) \cap S(F)$ and equation (8). Suppose now that X_{π_F} is binary, that is, $X_{\pi_F} = \{\underline{p}_{\pi_F}, \bar{p}_{\pi_F}\}$ and $\underline{p}_{\pi_F} < \bar{p}_{\pi_F}$. Figure 3 illustrates this case and depicts the prior, F_0, the signal, F, and the π_F-iso-profit curve, $G_{\pi_F,1}$. These three curves are drawn to intersect at $\underline{p}_{\pi_F}$ and $\bar{p}_{\pi_F}$. We now argue that

$$\int_{\underline{p}_{\pi_F}}^{\bar{p}_{\pi_F}} F_0(s)\, ds < \int_{\underline{p}_{\pi_F}}^{\bar{p}_{\pi_F}} G_{\pi_F,1}(s)\, ds \leq \int_{\underline{p}_{\pi_F}}^{\bar{p}_{\pi_F}} F(s)\, ds.$$

The first inequality follows from the observation that the strict concavity of $\Pi(\cdot, F_0)$ implies that $\Pi(\cdot, F_0)$ is strictly larger than π_F on $\left(\underline{p}_{\pi_F}, \bar{p}_{\pi_F}\right)$, so $F_0 < G_{\pi_F,1}$ on this interval. The second inequality follows from the fact that S's maximal profit is π_F if B's signal is F, so $F \geq G_{\pi_F,1}$. An immediate consequence of this inequality chain is that $I_F(\underline{p}_{\pi_F}) - I_F(\bar{p}_{\pi_F}) = \int_{\underline{p}_{\pi_F}}^{\bar{p}_{\pi_F}} F(s) - F_0(s)\, ds > 0$. Since $F \in \mathcal{A}$, $I_F(\bar{p}_{\pi_F}) \geq 0$ so it must be that $I_F(\underline{p}_{\pi_F}) > 0$, meaning that $\underline{p}_{\pi_F}$ is not F-separating.

We now turn to the main result of this section, which characterizes the set of payoff-profiles which can arise in equilibrium. Before stating this result, we introduce an additional piece of notation. Let $\underline{\pi}$ denote the smallest possible profit which can be generated by some learning strategy, that is, $\underline{\pi} = \inf_{F \in \mathcal{A}} \pi_F$.

Theorem 2 *A free-learning equilibrium (H, F) exists such that $\pi_F = \pi$ and $U_0(H, F) = u$ if and only if $\pi \in [\underline{\pi}, \pi_{F_0}]$ and $u = \int_{\overline{p}_\pi}^1 (v - \overline{p}_\pi)\, dF_0(v)$.*

Proof. See the Appendix. ■

The "only if" part of this theorem implies that in a free-learning equilibrium S can never attain a profit above his full-information profit. This is a straightforward consequence of Lemma 4. Recall that this lemma states that if B's signal is F then the equilibrium price is the largest price which generates profit π_F under perfect learning, $\overline{p}_{\pi_F}$. But if learning was perfect S can achieve π_{F_0} by setting the optimal price instead of $\overline{p}_{\pi_F}$, showing that $\pi_F \leq \pi_{F_0}$. The theorem also states that if B's signal is F then her equilibrium payoff the same as if she learns perfectly and S charges a price of $\bar{p}_{\pi_F}$. This follows from the facts that S sets price $\overline{p}_{\pi_F}$ in every equilibrium where his profit is π_F (see Lemma 4) and that perfect learning is always a best-response when information is free.

The "if" part of the theorem's proof is constructive. Specifically, we find an equilibrium for each $\pi \in [\underline{\pi}, \pi_{F_0})$ such that S's profit is π. Figure 4 illustrates our construction, which obtains an equilibrium by applying two modifications to the π-iso profit curve, $G_{\pi,1}$. The

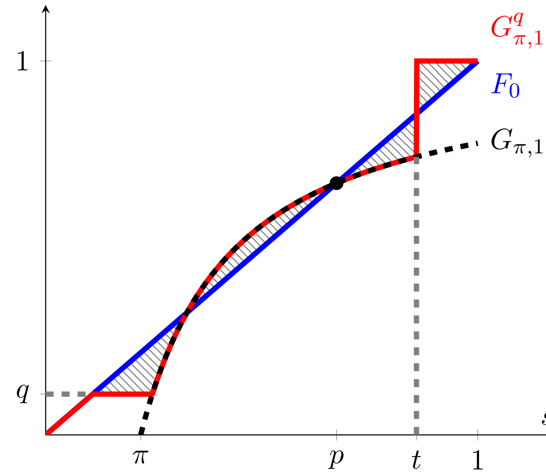

Figure 4: A constructed free learning equilibrium, $(\mathbf{1}_{[p,1]}, F)$.

first modification creates a CDF with separating and profit maximizing price p that gives S a profit of π. To get this CDF, we replace the realizations in the lowest q quantiles of $G_{\pi,1}$ with realizations from same quantiles of F_0. The resulting CDF is equal to F_0 at any x such that $F_0(x) \leq q$, to $G_{\pi,1}$ when $G_{\pi,1}(x) \geq q$, and to q otherwise. This CDF, however, fails to be a signal due to having too large of a mean. To make the CDF into a signal, we reduce the mean using the second modification: truncating the distribution at the top at some value t. The result is a signal corresponding to the red curve, $G^q_{\pi,t}$, in Figure 4. Noting that the truncation point t is larger than p means that p still yields S a profit of π, and remains separating and profit maximizing. Thus, having S offer p and B use $G^q_{\pi,t}$ gives a free learning equilibrium.

Using Theorem 2, we can deduce that free-learning equilibria are strongly Pareto ranked – i.e, B prefers one free-learning equilibrium to another if and only if S does as well.

Corollary 1 *All free-learning equilibria are strongly Pareto ranked. – i.e., for any two free learning equilibria,* (H, F) *and* (H', F'),

$$\Pi(H, F) \geq \Pi(H', F') \textit{ if and only if } U_0(H, F) \geq U_0(H', F').$$

Proof. We prove the corollary by showing that $\bar{p}_\pi$ is strictly decreasing in π over the interval $[\underline{b}, \pi_{F_0}]$. To see why this is sufficient, recall that B's free learning equilibrium payoff is equal to $\int_{\bar{p}_\pi}^{1}(s - \bar{p}_\pi)\, \mathrm{d}F_0$, where π is S's profit. Hence, B's utility decreases in S's price. If S's price decreases with her profit, then we get that higher profits correspond to lower prices and therefore higher B utility. We now show that $\bar{p}_\pi$ decreases over the range of feasible free learning equilibrium profits. Thus, take any $\pi < \pi'$ in $[\underline{b}, \pi_{F_0}]$. We prove that $\bar{p}_{\pi'} < \bar{p}_\pi$ by showing that X_π contains a price strictly larger than $\bar{p}_{\pi'}$. To find such a price, we make two observations. First, since $\pi < \pi'$, we have that

$$F_0(\bar{p}_{\pi'}) = G_{\pi',1}(\bar{p}_{\pi'}-) = 1 - \frac{\bar{p}_{\pi'}}{\pi'} < 1 - \frac{\bar{p}_{\pi'}}{\pi} = G_{\pi,1}(\bar{p}_{\pi'}-).$$

Second, since F_0 is regular, $G_{\pi,1}(1-) = 1 - \frac{1}{\pi} < 1 = F_0(1-)$. Combining the two observations, we have that $G_{\pi,1}(\bar{p}_{\pi'}) - F_0(\bar{p}_{\pi'}) > 0 > G_{\pi,1}(1-\epsilon) - F_0(1-\epsilon)$ for any small positive ϵ. As the difference $G_{\pi,1} - F_0$ is continuous on $[0, 1)$, we can apply the intermediate value theorem to find some $p \in (\bar{p}_{\pi'}, 1)$ for which $G_{\pi,1}(p) - F_0(p) = 0$. Therefore, $p \in X_\pi$, meaning that $\bar{p}_\pi \geq p$. We have thus concluded that $\bar{p}_\pi \geq p > \bar{p}_{\pi'}$, meaning that the higher profit level corresponds to a lower price, thereby proving the corollary. ■

We have thus shown that when learning is free, our model admits a continuum of equilibria, all of which can be Pareto ranked. In the next section we discuss the shape of equilibria when learning is costly and show that, as costs vanish, the equilibrium must converge to a Pareto worst free-learning equilibrium.

4 Costly Learning

This section accomplishes two goals. First, we provide an equilibrium characterization in our model of costly learning. In particular, B's equilibrium signal is shown to belong to the family of Pareto signals. Second, we prove the main result of this paper: as the cost of learning vanishes, equilibria converge to the worst free-learning equilibrium.

4.1 Equilibrium Characterization

The next result provides a partial characterization of the equilibrium when B's learning cost satisfy Assumption 1.

Proposition 1 *Suppose Assumption 1 holds and that* (H, F) *is an equilibrium. Then,*

(i) supp $H =$ supp $F =$ co(supp F)*, and*

(ii) F *is a Pareto signal.*

Proof. See the Appendix. ■

Part (i) of this proposition states that the supports of B's signal and the randomization of S coincide. Furthermore, this support is an interval. From these two observations, it is straightforward to conclude part (ii). The reason is that S must be indifferent on supp H, so each price in supp H must generate the same profit. Therefore, part (i) implies that B's equilibrium signal, F, must coincide with an iso-profit curve over its support. Since the iso-profit curve is a Pareto distribution truncated at one, F must be a Pareto signal.

Next, we explain how to establish part (i). The key step is to show that S charges every price between any two possible signal realizations, that is,

$$\text{co}(\text{supp } F) \subseteq \text{supp } H. \tag{9}$$

If this inclusion does not hold then there exists an interval (x, y), $x, y \in$ co(supp F), $x < y$, such that supp $H \cap (x, y) = \{\emptyset\}$. In fact, we show that if (x, y) is maximal among such intervals then $x, y \in$ supp F. So, in order to prove (9), it is enough to show that supp $H \cap (x, y) \neq \{\emptyset\}$ if $x, y \in$ supp F. Suppose first that F places atoms at both x and y. Then B can profitably deviate by bunching together all the signals x and y, that is, instead of observing these signals, she only learns that the signal is in $\{x, y\}$. By Assumption 1, this bunching strictly reduces B's learning cost. Moreover, since S never sets a price in (x, y), such a bunching leaves B's trade surplus unchanged. To see this note that, conditional on the original signal being x, the buyer trades if and only if the price is weakly less than x, irrespective of whether the signals are bunched together or not. The only difference in trading decisions is that if the original signal is y, B trades if the price is y but rejects this price after the bunching. Since the buyer breaks even, this does not change her payoff. We conclude that when F has atoms at both x and y, it cannot be a best response against H if supp $H \cap (x, y) = \{\emptyset\}$. If either x or y have zero mass according to F, one can construct a profitable deviation in a similar fashion by pooling together small neighborhoods of x and y. Finally, notice that

$$\text{co}(\text{supp } F) \subseteq \text{supp } H \subseteq \text{supp } F \subseteq \text{co}(\text{supp } F),$$

where the first inclusion is just (9), the second follows from the observation that S never sets a price which is not a possible signal realization (see part (ii) of Lemma 3). This chain of inclusion implies part (i) of the theorem.

4.2 Vanishing Learning Cost

We are now ready to state and prove the main result of the paper: as the cost of learning vanishes, equilibria converge to a free-learning equilibrium that is worst for both players.

Theorem 3 *For $\kappa > 0$, let (H_κ, F_κ) be any equilibrium of the κ-game. Then, a $\bar{t} > \bar{v}$ exists such that*

$$\lim_{\kappa \to 0} (H_\kappa, F_\kappa) = (\mathbf{1}_{[\bar{p}_{\underline{\pi}}, 1]}, G_{\underline{\pi}, \bar{t}}).$$

Recall that $\underline{\pi}$ denotes the smallest possible profit in a free-learning equilibrium and $\bar{p}_{\underline{\pi}}$ is the largest price which generates profit $\underline{\pi}$ if B learns perfectly. So, this theorem says that in the limit as learning becomes free, B uses a Pareto signal which generates the lowest profit across all signals. In turn, S charges the higher of the two prices yielding $\underline{\pi}$ when B collects full information.

The proof of this theorem is based on connecting our analysis of costly learning with our observations regarding free-learning equilibria. When costs are positive, B uses a Pareto signal (see Proposition 1). As the set of Pareto signals is closed, she must also be using a Pareto signal in the limit, say $G_{\pi,t}$. In turn, when learning is free, S must set a $G_{\pi,t}$-separating price in the support of $G_{\pi,t}$ (see Lemma 1 and part (ii) of Lemma 3). The key step in the proof is to show that a Pareto signal that has a non-empty set of separating prices in its support is associated with the lowest profit– i.e., $\pi = \underline{\pi}$. We state this result in the next lemma and provide an explanation afterwards. Since the mean generated by $G_{\underline{\pi},t}$ is strictly increasing in t and F_0 is a mean-preserving spread of B's signal, there is a unique $\bar{t}$ such that $G_{\underline{\pi},\bar{t}}$ is a Pareto signal. Finally, we note that if S's profit is $\underline{\pi}$, he must charge $\bar{p}_{\underline{\pi}}$ by Lemma 4.

Lemma 5 *For a Pareto signal, $G_{\pi,t} \in \mathcal{A}$,* $\text{supp } G_{\pi,t} \cap S(G_{\pi,t}) \neq \emptyset$ *only if $\pi = \underline{\pi}$.*

Proof. See the Appendix. ■

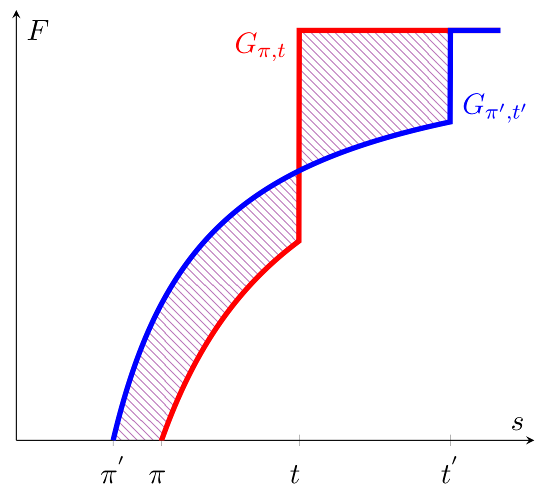

Figure 5: The area between two Pareto signals, $G_{\pi,t}$ and $G_{\pi',t'}$, where $\pi < \pi' < t < t'$.

We now explain that Pareto signals yielding a profit above $\underline{\pi}$ have no separating prices in their support. To do so, we first observe that for each Pareto signal, $G_{\pi,t} \in \mathcal{A}$, with $\underline{\pi} < \pi$, there is another one, $G_{\pi',t'} \in \mathcal{A}$, which is a mean-preserving spread of $G_{\pi,t}$ such that $[\pi, t] \subset (\pi', t')$. We now argue that the information constraint of $G_{\pi',t'}$ is point-wise slacker than that of $G_{\pi,t}$, that is, $I_{G_{\pi,t}} > I_{G_{\pi',t'}}$ on supp $G_{\pi,t}$. In other words, we want to prove that for all $x \in$ supp $G_{\pi,t}$

$$0 < I_{G_{\pi,t}}(x) - I_{G_{\pi',t'}}(x) = \int_0^x \left[G_{\pi',t'}(s) - G_{\pi,t}(s)\right] ds. \tag{10}$$

The right-hand side is just the area between the CDFs $G_{\pi,t}$ and $G_{\pi',t'}$ on $[0, x]$. Figure 5 illustrates these CDFs and the area between them. Note that this area is zero if $x \in [0, \pi']$, strictly increases on $x \in [\pi', t]$ and strictly decreases on $[t, t']$. Moreover, since $G_{\pi',t'}$ is a mean-preserving spread of $G_{\pi,t}$, the area is zero for all $x \geq t'$. Therefore, the area must be strictly positive for all $x \in [\pi, t] =$ supp $G_{\pi,t}$, so we obtain (10). Since $I_{G_{\pi',t'}}(x) \geq 0$ (because $G_{\pi',t'} \in \mathcal{A}$), we conclude that for all $x \in$ supp $G_{\pi,t}$, $I_{G_{\pi,t}}(x) > 0$, implying that $G_{\pi,t}$ has no separating prices in its support.

Armed with Lemma 5, we are now ready to prove Theorem 3.

Proof of Theorem 3. As a preliminary step, note that there is at most one Pareto signal associated with any feasible profit level. Said differently, $G_{\pi,t}$ and $G_{\pi,t'}$ are both in $\mathcal{A}$ only if $t = t'$. This can be seen by noting that the mean of a truncated Pareto $G_{\pi,t}$ is strictly increasing in the truncation point t, and that all signals must have the same mean. It follows that there can be only one $\bar{t}$ such that $G_{\underline{\pi},\bar{t}} \in \mathcal{A}$.

Now, let $\{\kappa_n\}_{n\geq 0}$ be a strictly positive sequence that converges to zero, and take $\{(H_n, F_n)\}_{n\geq 0}$ to be a corresponding sequence of equilibria. As $\mathcal{F}$ and $\mathcal{A}$ are both compact, $\{(H_n, F_n)\}_{n\geq 0}$ can be seen as a union of convergent subsequences. Without loss, let one of these subsequences be the sequence itself, and let $(H_\infty, F_\infty) \in \mathcal{F} \times \mathcal{A}$ be its limit. To prove the theorem, it is sufficient to show that $(H_\infty, F_\infty) = (\mathbf{1}_{[\bar{p}_{\underline{\pi}},1]}, G_{\underline{\pi},\bar{t}})$.

To this end, we begin by noting that, since B's objective is a continuous function of (κ, H, F), B's best response correspondence is upper hemicontinuous in (κ, H). Therefore, $F_\infty \in \arg\max_{F\in\mathcal{A}} U_0(H_\infty, F)$, meaning that supp $H_\infty \subseteq S(F_\infty)$ by Lemma 1.

We now show that supp $H_\infty \subseteq P(F_\infty)$. On the one hand, supp$(\cdot)$ is lower hemicontinuous, and so $p_\infty \in$ supp H_∞ only if a sequence $p_n \in$ supp H_n exists that attains p_∞ as its limit. On the other hand, $P(\cdot)$ is upper hemicontinuous,[18] and so the limit of any convergent sequence $p_n \in$ supp $H_n \subseteq P(F_n)$ is in $P(F_\infty)$. Therefore, $p_\infty \in$ supp H_∞, only if $p_\infty \in P(F_\infty)$ – i.e., supp $H_\infty \subseteq P(F_\infty)$.

The above establishes that the limit (H_∞, F_∞) is a free learning equilibrium. As such, F_∞ must have a separating price in its support. Moreover, F_∞ must be a Pareto signal, since it is the limit of Pareto signals (Proposition 1). Hence, F_∞ is a Pareto signal, G_{π_∞,t_∞} that has separating price in its support. However, Lemma 5 says G_{π_∞,t_∞} has a separating price in its support only if $\pi_\infty = \underline{\pi}$ – i.e., $F_\infty = G_{\underline{\pi},\bar{t}}$. As $\max \Pi(\cdot, G_{\underline{\pi},\bar{t}}) = \underline{\pi}$, we therefore have that $(H_\infty, F_\infty) = (H_\infty, G_{\underline{\pi},\bar{t}})$ is a free learning equilibrium in which S's profit is $\underline{\pi}$. That $H_\infty = \mathbf{1}_{[\bar{p}_{\underline{\pi}},1]}$ then follows from Lemma 4. ■

5 Discussion

To conclude, we discuss some of our assumptions and how they can be relaxed.

Production costs.— We assumed that S's production cost is zero. We now discuss how our results generalize to the case where S has to incur a positive production cost upon trade. Thus, suppose that S's payoff when trading is $p - c$, where $c \in (0, 1)$. For $c \in (0, \bar{v})$ our analysis goes through with the c-**shifted truncated Pareto** signal

$$\hat{G}^c_{\pi,t}(s) = \mathbf{1}_{[\pi+c,t)}\left(1 - \frac{\pi}{s-c}\right) + \mathbf{1}_{[t,1]} \qquad t \geq \pi + c,\ \pi \geq 0.$$

replacing the truncated Pareto, $G_{\pi,t}$. Other than this replacement, all results hold as stated.

[18] We provide a proof of this fact in Lemma 12 in Appendix G.

For $c \geq \bar{v}$, our analysis implies that trade breaks down: in the costless limit B collects no information, and there is no trade. To see why, note that even when $c > 0$, Proposition 1's part (i) continues to hold for any non-trivial costly learning equilibrium. In other words, in any costly learning equilibrium in which B learns, the support of S's price and of B's signal must equal the same interval. As such, if B learns, her signal must be a c-shifted truncated Pareto one. But when $c \geq \bar{v}$, no informative signal can have a c-shifted truncated Pareto distribution.[19] Hence, B acquires no information when learning is costly, and so the same must hold in the costless limit. However, if $p < 1$, B is strictly better off with learning. Thus, the vanishing-cost limit is autarky with no learning.

Robustness and purification: random production costs.— Our main result appears to rely on the observation that, if information is free, B learns whether her valuation is above or below the equilibrium price but chooses to ignore large amounts of information. If there were many possible equilibrium prices, B may need to learn more and compare her valuation with any of these prices. So, one may wonder whether our results extend to environments where the price is stochastic. Another concern is that, when learning is costly, S randomizes in equilibrium and it is not obvious that S's strategy can be purified without affecting our main conclusion. To address these issues, we describe what happens if S has a random production cost with full support in $[0, 1]$ which is independent of B's valuation. S privately observes the cost realization, c, before setting a price. Then his utility from trade at price p is $p - c$, where c is the production-cost realization. In this case, free-learning equilibria are still strictly Pareto ranked and are indexed by the price S charges when $c = 0$. This price is offered for all values of c for which S would set a lower price under perfect learning and B's signal distribution above this price agrees with the CDF of her prior. For higher values of c, S sets the same price as he would under perfect learning. It turns out that both players strictly prefer equilibria in which the price is lower conditional on $c = 0$. As this price must be separating in equilibrium, its maximum across all B signals is $\bar{p}_{\underline{\pi}}$, whereas its minimum is attained when B learns perfectly. As such, perfect learning is still a Pareto-best equilibrium. In the Pareto-worst equilibrium, the CDF of B's signal coincides with the truncated Pareto, $G_{\underline{\pi},\bar{t}}$, for all values below $\bar{p}_{\underline{\pi}}$. One can show that this is the only free-learning equilibrim in which B uses this CDF, and that the same CDF is attained at the vanishing cost limit. Hence, even when the

[19]To see this, suppose that $F = \hat{G}^c_{\pi,t}$ for some signal $F \in \mathcal{A}$. Then supp $F =$ supp $\hat{G}^c_{\pi,t} \subseteq [c, 1] \subseteq [\bar{v}, 1]$. Therefore, $\int s \, \mathrm{d}F \geq \bar{v}$, with equality only if supp $F = \{\bar{v}\}$, that is, if F is uninformative.

production costs is stochastic, our main result is valid and the costless limit still selects the Pareto worst free-learning equilibrium.

Random prices as general mechanisms.— We argue that it is without the loss of generality to assume that S sets a price instead of a more general mechanism. Consider a more general model, where S and B simultaneously choose a mechanism and a signal, respectively. Then B observes her signal's realization and decides whether to participate in S's mechanism. A mechanism constitutes a set of messages for B and each message is associated with a transfer and a probability of trade. Note that B's interim expected payoff from any of the messages is fully determined by her posterior value estimate. Hence, by the Revelation Principle, it is without loss to restrict attention to individually rational and incentive compatible mechanisms in which B truthfully reports her posterior value estimate. Then standard arguments imply that any mechanism is equivalent to setting a random price, see, for example, Börgers (2015) Proposition 2.5.

Non-regular prior.— Most of our results generalize to the case where B's prior value distribution is not regular. When learning is free, equilibrium requires S's price to be separating and the full-information outcome remains profit-maximizing regardless of the prior. Similarly, the regularity of the prior plays no role in showing that B uses a Pareto signal when learning is costly and the same holds in the costless limit. Since the costless limit is a free-learning equilibrium, the limit Pareto signal still has a separating price in its support, so this signal is still profit-minimizing. Therefore, even without regularity, the costless limit still minimizes S's profits across all signal structures and generates the lowest profit across all free-learning equilibria.

However, a non-regular prior does impact the conclusion that the costless limit minimizes B's payoff for two reasons. First, a non-regular prior can result in Pareto incomparable free-learning equilibria, and so the profit minimizing equilibrium may not minimize B's payoff. Second, when the prior is non-regular, the profit-minimizing Pareto signal may have more than one separating price in its support, so there may be many free-learning equilibria in which B uses the profit minimizing Pareto signal. In fact, one can show that, under Assumption 1, each such equilibrium is a limit of some equilibrium sequence with vanishing costs. As a consequence, without regularity, B may obtain different outcomes in the vanishing-cost limit depending on the fine details of the prior and the converging equilibrium sequence.

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Appendix

A Proof of Claim 1

We begin by proving the following useful lemma, which shows for every F, $w, z \in \operatorname{int}(\operatorname{co}(\operatorname{supp} F))$, and $\alpha \in (0,1)$, two distributions, F', F'' exist such that $F \succeq F' \succ F''$ and

$$F' - F'' = \gamma\left(\alpha \mathbf{1}_{[w,1]} + (1-\alpha)\mathbf{1}_{[z,1]} - \mathbf{1}_{[\alpha w + (1-\alpha)z, 1]}\right)$$

for some $\gamma > 0$.

Lemma 6 *Fix some $F \in \mathcal{F} \setminus \{\mathbf{1}_{[x,1]} : x \in [0,1]\}$, let $[x, x'] = \operatorname{co}(\operatorname{supp}\ F)$, and take $\bar{w} = \int s \, \mathrm{d}F$. Take any $w, y, z \in (x, x')$, and $\alpha \in (0,1)$ such that $y = \alpha w + (1-\alpha)z$. For $\lambda, \beta \in [0,1)$ define $x_\lambda = \frac{\bar{w} - \lambda y}{1-\lambda}$, and*

$$F_{\lambda,\beta} := (1-\lambda)\mathbf{1}_{[x_\lambda,1]} + \lambda(1-\beta)\mathbf{1}_{[y,1]} + \lambda\beta\left[\alpha\mathbf{1}_{[w,1]} + (1-\alpha)\mathbf{1}_{[z,1]}\right].$$

Then there exists $\beta, \lambda \in (0,1)$ such that $F \succeq F_{\lambda,\beta} \succ F_{\lambda,0}$.

Proof. Suppose without loss that $z > w$. Note that $F_{\lambda,0} \succeq \mathbf{1}_{[\bar{w},1]}$ for all $\lambda > 0$ since $\lambda y + (1-\lambda)x_\lambda = \bar{w}$. We now show that $F_{\lambda,\beta} \succeq F_{\lambda,0}$ for every $\beta \geq 0$. For this purpose, notice that

$$F_{\lambda,\beta} - F_{\lambda,0} = \lambda\beta[\alpha\mathbf{1}_{[w,1]} + (1-\alpha)\mathbf{1}_{[z,1]}] - \lambda\beta\mathbf{1}_{[y,1]}.$$

Therefore, for all $\bar{s} \in [0,1]$,

$$\int_0^{\bar{s}} (F_{\lambda,\beta} - F_{\lambda,0}) \, \mathrm{d}s = \lambda\beta \int_0^{\bar{s}} (\alpha\mathbf{1}_{[w,1]} + (1-\alpha)\mathbf{1}_{[z,1]} - \mathbf{1}_{[y,1]}) \, \mathrm{d}s \geq 0,$$

in view of $\left(\alpha\mathbf{1}_{[w,1]} + (1-\alpha)\mathbf{1}_{[z,1]}\right) \succeq \mathbf{1}_{[y,1]}$. Because $\bar{s}$ was arbitrary, we have $F_{\lambda,\beta} \succeq F_{\lambda,0}$.

Let us introduce some helpful definitions, which rely on x_λ being continuous in λ and $x_0 = \bar{w}$. Fixing some $\epsilon > 0$ for which $(\bar{w} - \epsilon, \bar{w} + \epsilon) \subseteq (x, x')$, choose a $\bar{\lambda}$ to be such that $\{x_\lambda\}_{\lambda \in [0,\bar{\lambda}]} \subseteq (\bar{w} - \epsilon, \bar{w} + \epsilon) \subseteq (x, x')$. Let $x^* = \max\left(\{z\} \cup \{x_\lambda\}_{\lambda \in [0,\bar{\lambda}]}\right)$ and $x_* = \min\left(\{w\} \cup \{x_\lambda\}_{\lambda \in [0,\bar{\lambda}]}\right)$, and define the function

$$\begin{aligned} \varphi : [x_*, x^*] \times [0, \bar{\lambda}]^2 &\to \mathbb{R} \\ (\bar{s}, \lambda, \beta) &\mapsto \int_0^{\bar{s}} (F - F_{\lambda,\beta}) \, \mathrm{d}s. \end{aligned}$$

Taking $(\cdot)_+ := \max\{\cdot, 0\}$, we can write

$$\begin{aligned}\varphi(\bar{s}, \lambda, \beta) = \int_0^{\bar{s}} F \, \mathrm{d}s &- (1-\lambda)(\bar{s} - x_\lambda)_+ - \lambda(1-\beta)(\bar{s} - y)_+ \\ &- \lambda\beta\alpha(\bar{s} - w)_+ - \lambda\beta(1-\alpha)(\bar{s} - z)_+,\end{aligned}$$

and so φ is continuous in the product topology. Therefore,

$$\begin{aligned}\varphi^* : [0, \bar{\lambda}]^2 &\to \mathbb{R} \\ (\lambda, \beta) &\mapsto \min_{s \in [x_*, x^*]} \varphi(s, \lambda, \beta)\end{aligned}$$

is also continuous by Berge's Theorem.

We now show $\varphi(\bar{s}, 0, 0) > 0$ for all $\bar{s} \in [x_*, x^*]$. To do so, notice $x_0 = \bar{w}$, and therefore $F_{0,0} = \mathbf{1}_{[x_0,1]} = \mathbf{1}_{[\bar{w},1]}$. Because $\bar{w} > x_* > x$ (by choice of F), we also have $F(s) > 0 = \mathbf{1}_{[\bar{w},1]}(s)$ for all $s \in [x, \bar{w})$. As such, if $\bar{s} \in [x_*, \bar{w}]$ then $\int_0^{\bar{s}} (F - \mathbf{1}_{[\bar{w},1]})(s) \, \mathrm{d}s = \int_x^{\bar{s}} F(s) \, \mathrm{d}s > 0$. Similarly, for all $s \in [\bar{w}, x')$, $F(s) < 1 = \mathbf{1}_{[\bar{w},1]}(s)$. As such, if $\bar{s} \in [\bar{w}, x^*]$, $\int_{\bar{s}}^1 (1 - F(s)) \, \mathrm{d}s > 0 = \int_{\bar{s}}^1 (1 - \mathbf{1}_{[\bar{w},1]}(s)) \, \mathrm{d}s$, and so $\int_{\bar{s}}^1 (F - \mathbf{1}_{[\bar{w},1]})(s) \, \mathrm{d}s < 0$. Since $\int_0^1 (F - \mathbf{1}_{[\bar{w},1]})(s) \, \mathrm{d}s = 0$, we obtain $\int_0^{\bar{s}} (F - \mathbf{1}_{[\bar{w},1]})(s) \, \mathrm{d}s > 0$ for all $\bar{s} \in [\bar{w}, x^*]$ as well

We are now in a position to complete the proof; that is, show $F \succeq F_{\lambda,\beta}$ for all small $\lambda, \beta > 0$. By the previous paragraph, $\varphi(\bar{s}, 0, 0) > 0$ for all $\bar{s} \in [x_*, x^*]$. As such, $\varphi^*(0,0) = \min_{s \in [x_*, x^*]} \varphi(s, 0, 0) > 0$, and so by continuity of φ^*, one must then have $\varphi^*(\lambda, \beta) > 0$ for all $\lambda, \beta > 0$ small enough. Fixing any such λ and β, we now show that $\int_0^{\bar{s}} (F - F_{\lambda,\beta}) \, \mathrm{d}s \geq 0$ for all $\bar{s}$ by considering three cases. First, if $\bar{s} \in [x_*, x^*]$,

$$\int_0^{\bar{s}} (F - F_{\lambda,\beta}) \, \mathrm{d}s \geq \varphi^*(\lambda, \beta) > 0.$$

Second, if $\bar{s} \in [x, x_*)$, $F(x) \geq 0 = F_{\lambda,\beta}(x)$, and so $\int_0^{\bar{s}} (F - F_{\lambda,\beta}) \, \mathrm{d}s = \int_0^{\bar{s}} F \, \mathrm{d}s \geq 0$. Third, if $\bar{s} \in (x^*, 1]$,

$$\begin{aligned}\int_0^{\bar{s}} (F - F_{\lambda,\beta}) \, \mathrm{d}s &= \int_0^{x^*} (F - F_{\lambda,\beta}) \, \mathrm{d}s + \int_{x^*}^{\bar{s}} (F - 1) \, \mathrm{d}s \\ &\geq \int_0^{x^*} (F - F_{\lambda,\beta}) \, \mathrm{d}s + \int_{x^*}^{1} (F - 1) \, \mathrm{d}s = \int_0^1 (F - F_{\lambda,\beta}) \, \mathrm{d}s = 0,\end{aligned}$$

in view of $\operatorname{supp} F_{\lambda,\beta} \subseteq [x_*, x^*]$ and $F_{\lambda,\beta} \succeq \mathbf{1}_{[\bar{w},1]}$. We have therefore shown that for all sufficiently small λ and β, $\int_0^{\bar{s}} (F - F_{\lambda,\beta}) \, \mathrm{d}s \geq 0$ for all $\bar{s} \in [0,1]$, with equality holding at $\bar{s} = 1$ (because $F_{\lambda,\beta} \succeq \mathbf{1}_{[\bar{w},1]}$). Therefore, $F \succeq F_{\lambda,\beta}$, thereby completing the proof. ■

We are now ready to prove Claim 1.

Proof of Claim 1. Suppose first that c_F is convex for all F. Fix some $F' \succeq F$. Since C is convex, we have that

$$C(F') - C(F) \geq \int c_F \; \mathrm{d}(F' - F) \geq 0,$$

where the last inequality follows from c_F being convex. Hence, C is monotone.

Suppose now that C is monotone. Fix any $w, y, z \in \mathrm{co}(\mathrm{supp}\; F_0)$ such that $y = \alpha w + (1-\alpha)z$ for some $\alpha \in (0,1)$. Because c_F is only unique Lebesgue almost everywhere (see footnote 11), we may as well assume $w, y, z \in \mathrm{int}\,(\mathrm{co}(\mathrm{supp}\; F_0))$. Our task is to show that $c_F(y) \leq \alpha c_F(w) + (1-\alpha)c_F(z)$.

By Lemma 6, an F' and F'' exist such that $F_0 \succeq F' \succ F''$ and

$$F' - F'' = \left(\alpha \mathbf{1}_{[w,1]} + (1-\alpha)\mathbf{1}_{[z,1]} - \mathbf{1}_{[\alpha w + (1-\alpha)z,1]}\right),$$

for some $\gamma > 0$. Because $\succeq$ respects convex combinations,

$$F + \epsilon(F' - F) \succeq F + \epsilon(F'' - F)$$

must hold for all $\epsilon \in [0,1]$. Appealing to monotonicity, convexity and Fréchet differentiability of C then yields that, for all $\epsilon \in (0,1)$,

$$0 \leq C(F+\epsilon(F'-F)) - C(F+\epsilon(F''-F)) = \left[C(F + \epsilon(F' - F)) - C(F)\right] - \left[C(F + \epsilon(F'' - F)) - C(F)\right].$$

Dividing by $\epsilon > 0$, taking $\epsilon \searrow 0$ and substituting for F' and F'' then yields

$$\begin{aligned} 0 &\leq \frac{1}{\epsilon}\left[C(F + \epsilon(F' - F)) - C(F)\right] - \frac{1}{\epsilon}\left[C(F + \epsilon(F'' - F)) - C(F)\right] \\ &\to \int c_F \; \mathrm{d}(F' - F) - \int c_F \; \mathrm{d}(F'' - F) = \alpha c_F(w) + (1-\alpha)c_F(z) - c_F(y), \end{aligned}$$

thereby concluding the proof. ■

B Proof of Theorem 1

We prove a slightly more general result, showing that the theorem holds for any cost function $C : \mathcal{A} \to [0, \infty]$ that is convex, monotone ($C(F) \geq C(F')$ whenever $F \succeq F'$), lower semicontinuous, and satisfies $\mathrm{cl}\left(C^{-1}[0,\infty)\right) = \mathcal{A}$. Because of the last property $C(F) < \infty$ for some $F \in \mathcal{A}$. Therefore, C being monotone and $F' \succeq \mathbf{1}_{[\bar{v},1]}$ for all $F' \in \mathcal{A}$ imply $C(\mathbf{1}_{[\bar{v},1]}) = \min C(\mathcal{A}) < \infty$. Thus, we may as well normalize $C(\mathbf{1}_{[\bar{v},1]}) = 0$.

We begin by proving that an equilibrium exists. Let $U_\kappa^M(H,F) := \max\{U_\kappa(H,F), M\}$ for some $M < 0$. Note that U_κ^M is upper semicontinuous and quasiconcave since it is a composition of a continuous and increasing function on a concave and upper semicontinuous function. Consider the game in which the S's action set equals $\mathcal{F}$, B's action set is $\mathcal{A}$, and the player's payoffs are given by Π and U_κ^M. We prove that the modified game has an equilibrium, after which we show that this equilibrium corresponds to an equilibrium of the original game. We then conclude the proof by showing that the equilibrium must be non-trivial whenever κ is low enough.

We prove that the modified game has an equilibrium using Corollary 3.3 of Reny (1999). Both U_κ^M and Π are quasiconcave, lie in $[M,1]$, and are upper semicontinuous over $\mathcal{F} \times \mathcal{A}$, which is a compact subset of a topological vector space. Thus, this is a compact, quasiconcave, and reciprocally upper-semicontinuous game. To show existence, it is therefore sufficient to show the game is payoff secure.

Fix any (H,F) and $\epsilon > 0$. Since U_κ is continuous in H, F secures a payoff of $U_\kappa(H,F) - \epsilon$ from $U_\kappa(H,F)$ (otherwise one can find a sequence $H_n \to H$ such that $U_\kappa(H_n,F)$ does not converge to $U_\kappa(H,F)$), and so B can secure $\max\{U_\kappa(H,F)-\epsilon, M\} \geq U_\kappa^M(H,F) - \epsilon$ from $U_\kappa^M(H,F)$ using F. To show that S can secure $\Pi(H,F)$, take H_ϵ to be the distribution of $\max\{p-\epsilon, 0\}$, where p is drawn according to H. Note that for any sequence $\{F_n\}_{n\geq 0}$ that converges to F,

$$\begin{aligned}
\liminf_{n\to\infty} \Pi(H_\epsilon, F_n) &\geq \liminf_{n\to\infty} \int (p-\epsilon)[1 - F_n((p-\epsilon)-)]\mathrm{d}H(p) \\
&\geq \liminf_{n\to\infty} \int p[1 - F_n((p-\epsilon)-)]\mathrm{d}H(p) - \epsilon \\
&\geq \int \liminf_{n\to\infty} p[1 - F_n((p-\epsilon)-)]\mathrm{d}H(p) - \epsilon \\
&\geq \int p[1 - F((p-\epsilon)-)]\mathrm{d}H(p) - \epsilon \\
&\geq \int p[1 - F(p)]\mathrm{d}H(p) - \epsilon = \Pi(H,F) - \epsilon,
\end{aligned}$$

where the first inequality follows from $\max\{p-\epsilon, 0\} \geq p - \epsilon$, the second inequality from $\{s > p-\epsilon\} \subseteq \{s \geq p - \epsilon\}$ and probabilities being less than 1, the third inequality from Fatou's lemma, and the fourth inequality from Portmonteau theorem. Thus, the modified game is payoff secure and therefore has an equilibrium.

We now show that any equilibrium of the modified game must be an equilibrium of the original game. To do so, let (H,F) be an equilibrium of the modified game. Clearly S is best responding, as his objective is the same in both games. To see that B best responds, notice first that $C(\mathbf{1}_{[\bar{v},1]}) = 0$ implies that $U_\kappa^M(H, \mathbf{1}_{[\bar{v},1]}) = U_\kappa(H, \mathbf{1}_{[\bar{v},1]}) \geq 0$, and so $U_\kappa^M(H,F) = U_\kappa(H,F)$ due to F being optimal in the modified game. Combined with $U_\kappa^M \geq U_\kappa$, F being optimal in the modified game also implies that $U_\kappa(H,F) = U_\kappa^M(H,F) \geq U_\kappa^M(H,F') \geq U_\kappa(H,F')$ for all $F' \in \mathcal{A}$. In other words, F maximizes $U_\kappa(H,\cdot)$ over $\mathcal{A}$. Thus, (H,F) is an equilibrium of the original game.

All that remains is to show that there is no trivial equilibrium for sufficiently low κ. Note that in any trivial equilibrium, $F = \mathbf{1}_{[\bar{v},1]}$, meaning that $H = \mathbf{1}_{[\bar{v},1]}$ by S optimality. As such, B's utility is zero in any trivial equilibrium. To prove the result it is therefore sufficient to show that there is a $\bar{\kappa} > 0$ and F such that $\kappa < \bar{\kappa}$ implies $U_\kappa(\mathbf{1}_{[\bar{v},1]}, F) > 0$. By non-degeneracy of F_0, there exists $\epsilon > 0$ such that $\int_{\bar{v}}^1 (v - \bar{v})\ \mathrm{d}F_0(v) > 2\epsilon$. Since $F_0 \in \mathrm{cl}\ (C^{-1}[0,\infty))$, there exists a sequence $\{F_n\}_{n\geq 0}$ such that $F_n \to F_0$ and $C(F_n) < \infty$ for all n. Because $F_n \to F_0$, an n exists such that $\int_{\bar{v}}^1 (v - \bar{v})\ \mathrm{d}F_n(v) > \epsilon$. But then one must have $U_\kappa(\mathbf{1}_{[\bar{v},1]}, F_n) > 0$ for all $\kappa < \bar{\kappa} = \epsilon / C(F_n) > 0$, as required.

C Proof of Lemma 4: Free-learning equilibrium prices

If (H,F) is an equilibrium then F is best-response to H and hence, by Lemma 1, supp $H \subseteq S(F)$. Furthermore, since H is a best-response to F, each price in the support of H must be profit-maximizing, that is, supp $H \subseteq P(F)$. Therefore, it is enough to prove that $P(F) \cap S(F) = \{\bar{p}_{\pi_F}\}$. We have already shown that $P(F) \cap S(F) \subseteq X_{\pi_F}$, see equation (8). Thus, it remains to show that if $p \in X_{\pi_F}$ but $p < \bar{p}_{\pi_F}$ then $p \notin S(p)$.

To this end, note that for all $s \in (p, \bar{p}_{\pi_F})$ it must be that

$$G_{\pi_F,1}(s) > F_0(s). \tag{11}$$

The reason is that since F_0 is regular, the profit function $\Pi(\cdot, F_0)$ is strictly concave and hence, any price between p and $\bar{p}_{\pi_F}$ generates a profit strictly above π_F $(= \Pi(p,F) = \Pi(p,F_0))$. This means that F_0 is strictly below the π_F-iso-profit curve at these prices, that is, (11) holds. Now, observe that

$$I_F(p) = I_F(\bar{p}_{\pi_F}) - \int_p^{\bar{p}_{\pi_F}} (F_0 - F)\, ds \geq I_F(\bar{p}_{\pi_F}) - \int_p^{\bar{p}_{\pi_F}} (F_0 - G_{\pi_F,1})\, ds > I_F(\bar{p}_{\pi_F}) \geq 0,$$

where the first inequality follows from part (i) of Lemma 3, the strict inequality follows from (11), and the last inequality is implied by $F \in \mathcal{A}$. Thus, we have shown that $I_F(p) > 0$ and hence, $p \notin S(p)$.

D Proof of Theorem 2: Free-learning equilibrium payoffs

We begin by noting that, if (H, F) is a free-learning equilibrium and F_0 is regular, then B's expected utility is $\int_{\bar{p}_{\pi_F}}^{1} (v - \bar{p}_{\pi_F})\, \mathrm{d}F_0(v)$. This is a simple consequence of two facts. First, Lemma 4 implies that H puts a unit mass on $\bar{p}_{\pi_F}$; that is, $H = \mathbf{1}_{[\bar{p}_{\pi_F},1]}$. Second, full information is always optimal for B when learning is costless, meaning that her expected utility in equilibrium must be the same as her expected utility when collecting full information – i.e., $U_0(\mathbf{1}_{[\bar{p}_{\pi_F},1]}, F) = U_0(\mathbf{1}_{[\bar{p}_{\pi_F},1]}, F_0) = \int_{\bar{p}_{\pi_F}}^{1} (v - \bar{p}_{\pi_F})\, \mathrm{d}F_0(v)$.

Given the above, it remains to be shown that there exists a free-learning equilibrium (H, F) such that $\pi = \pi_F$ if and only if $\pi \in [\underline{\pi}, \pi_{F_0}]$. To do so, we first establish that $\underline{\pi} \leq \Pi(H, F) \leq \pi_{F_0}$ whenever (H, F) is a free learning equilibrium. Because $\underline{\pi} \leq \Pi(H, F)$ by definition of $\underline{\pi}$, it remains to show that $\Pi(H, F) \leq \pi_{F_0}$. To do so, notice that, since supp $H \subseteq S(F)$, we have by Lemma 2 that $F(p-) \geq F_0(p-)$ for every $p \in$ supp H. Since H maximizes S's profit, S's profit must be the same from all prices in supp H. We therefore have that for any $p \in$ supp H,

$$\Pi(H, F) = \Pi(p, F) = p(1 - F(p-)) \leq p(1 - F_0(p-)) = \Pi(p, F_0) \leq \pi_{F_0},$$

as required.

We now show that for every $\pi \in [\underline{\pi}, \pi_{F_0}]$ a free-learning equilibrium, (H, F), exists such that $\Pi(H, F) = \pi$. Note that the vanishing cost limit of Theorem 3 is a free learning equilibrium which gives S a profit of $\underline{\pi}$, while having B collect full information and S best respond is an equilibrium yielding S a profit of π_{F_0}. It thus remains to construct a free learning equilibrium for any profit $\pi \in (\underline{\pi}, \pi_{F_0})$. Fix such a π, and define for $q \in [0, 1]$ and $t \in [\pi, 1]$ the following CDF,

$$\begin{aligned} G^q_{\pi,t} &: [0,1] \to [0,1] \\ & x \mapsto \max\{G_{\pi,t}(x), \min\{q, F_0(x)\}\}. \end{aligned}$$

Let $[\underline{x}, \bar{x}] = \mathrm{co}(\text{supp } F_0)$. Below we prove the following lemma:

Lemma 7 *There exists q^* such that $I_{G^{q^*}_{\pi,1}} \geq 0$, with equality holding for some $\hat{x} \in [\pi, \bar{x}]$ such that $G^{q^*}_{\pi,1}(\hat{x}) = G_{\pi,1}(\hat{x}) \geq q^*$.*

Before providing the lemma's proof, let us show how to use the lemma to obtain an equilibrium. Take q^* and $\hat{x}$ to be as in the lemma. We explain how to find a $t \geq \hat{x}$ such that $G^{q^*}_{\pi,t}$ is a signal. Let $y = \max\{x \in [\underline{x}, \bar{x}] : I_{G^{q^*}_{\pi,1}}(x) = 0\}$. Since $I_{G^{q^*}_{\pi,1}}(\hat{x}) = 0$

and $\hat{x} \in [\pi, \bar{x}] \subseteq [\underline{x}, \bar{x}]$, $y \geq \hat{x}$. As such, $x \in [y, 1]$ implies $G_{\pi,1}(x) \geq q^*$, and therefore $G^{q^*}_{\pi,1}(x) = G_{\pi,1}(x)$. Thus,

$$I_{G^{q^*}_{\pi,y}}(1) = \int_y^{\bar{x}} (F_0(s) - 1) \, \mathrm{d}s \leq 0 \leq I_{G^{q^*}_{\pi,1}}(1).$$

As $x \mapsto I_{G^{q^*}_{\pi,x}}(1)$ is continuous, we have that a $t \in [y, 1]$ exists such that $I_{G^{q^*}_{\pi,t}}(1) = 0$. It remains to verify that $G^{q^*}_{\pi,t}$ is a signal. For $x \leq t$, $G^{q^*}_{\pi,t}(x-) = G^{q^*}_{\pi,1}(x-)$, and so $I_{G^{q^*}_{\pi,t}}(x) = I_{G^{q^*}_{\pi,1}}(x) \geq 0$. For $x > t$,

$$I_{G^{q^*}_{\pi,t}}(x) = I_{G^{q^*}_{\pi,t}}(t) + \int_t^x (F_0 - 1) \geq I_{G^{q^*}_{\pi,t}}(t) + \int_t^1 (F_0 - 1) = I_{G^{q^*}_{\pi,t}}(1) = 0.$$

Thus $G^{q^*}_{\pi,t}$ is a signal. We now argue that $(\mathbf{1}_{[\hat{x},1]}, G^{q^*}_{\pi,t})$ is a free learning equilibrium yielding S a profit of π. To do so, notice first that $G^{q^*}_{\pi,t}(x-) \geq G_{\pi,1}(x-)$ for all x, with equality holding for $x = \hat{x} \geq \pi$. Therefore, $\hat{x} \in P(G^{q^*}_{\pi,t})$, and

$$\pi_{G^{q^*}_{\pi,t}} = \Pi(\hat{x}, G^{q^*}_{\pi,t}) = \Pi(\hat{x}, G_{\pi,t}) = \pi.$$

Moreover, $I_{G^{q^*}_{\pi,t}}(\hat{x}) = I_{G^{q^*}_{\pi,1}}(\hat{x}) = 0$ by choice of $\hat{x}$ and in view of $t \geq y \geq \hat{x}$. Hence, $\hat{x} \in S(I_{G^{q^*}_{\pi,t}}(\hat{x}))$, and so $G^{q^*}_{\pi,t}$ is optimal for B given $\mathbf{1}_{[\hat{x},1]}$.

Hence, all that remains to prove Lemma 7, which we do below.

D.1 Proof of Lemma 7

We first show that mean preserving spreads increase the convex hull of a CDF's support.

Lemma 8 *Suppose $F \succeq G$. Then* $\mathrm{co}\,(\mathrm{supp}\ F) \supseteq \mathrm{co}\,(\mathrm{supp}\ G)$.

Proof. Let $[x, y] = \mathrm{co}\,(\mathrm{supp}\ F)$ and $[w, z] = \mathrm{co}\,(\mathrm{supp}\ G)$, and suppose that $w < x$ for a contradiction (the proof for $z > y$ is analogous). Take $\epsilon > 0$ to be such that $w + \epsilon < x$. Because w must be in G's support, $G(w + \epsilon) > 0$. In contrast, $F(w + \epsilon) = 0$ as $w + \epsilon$ is below F's support. Since these observations are true for every $\epsilon \in (0, x - w)$, we have $\int_0^x F \ \mathrm{d}s = 0 < \int_0^x G \ \mathrm{d}s$, contradicting that $F \succeq G$. ■

Because the support of every signal is contained in $[\underline{x}, \bar{x}] = \mathrm{co}(\mathrm{supp}\ F_0)$ (by Lemma 8), and there is a truncated Pareto signal associated with $\underline{\pi}$ (which follows from Theorem 3), $\pi > \underline{\pi} \geq \underline{x}$. We begin with the following lemma.

Lemma 9 $I_{G_{\pi,1}}(x) \geq 0$ *for all x, with a strict inequality whenever $x > \underline{x}$.*

Proof. note that $\pi > \underline{\pi}$ implies $G_{\pi,1}(s) \leq G_{\underline{\pi},1}(s)$ for all s, with a strict inequality for $s > \underline{\pi} \geq \underline{x}$. As such, for every $x > \underline{x}$,

$$I_{G_{\pi,1}}(x) = \int_0^x (F_0 - G_{\pi,1})\ \mathrm{d}s \geq \int_0^x (F_0 - G_{\underline{\pi},1})\ \mathrm{d}s \geq \int_0^x (F_0 - G_{\underline{\pi}})\ \mathrm{d}s = I_{G_{\underline{\pi}}}(x) \geq 0,$$

where the first inequality is strict whenever $x \geq \underline{\pi}$. Since $I_{G_{\pi,1}}(\cdot)$ is continuous, we also have that $I_{G_{\pi,1}}(\underline{x}) \geq 0$. ∎

Let

$$A = \{x \in [\pi, \bar{x}] : G_{\pi,1}(x) \geq F_0(x-)\}.$$

Note that A is closed in view of upper semicontinuity of $G_{\pi,1}(\cdot)$ and lower semicontinuity of $x \mapsto F_0(x-)$. We now show that A is non-empty. In particular, we show that $A \subseteq P(F_0)$, which is non-empty due to upper-semicontinuity of $\Pi(\cdot, F_0)$. By Lemma 3 and $\pi < \pi_{F_0}$, $P(F_0) \subseteq [\pi_{F_0}, \bar{x}] \subseteq [\pi, \bar{x}]$. Moreover, for any $x \in P(F_0)$, $\pi < \pi_{F_0}$ implies

$$F_0(x-) = G_{\pi_{F_0},1}(x-) < G_{\pi,1}(x-) \leq G_{\pi,1}(x).$$

That $P(F_0) \subseteq A$ follows.

In view of the above, $x^* := \min A$ is well-defined. We now prove that a q^* exists such that the minimal value of $I_{G^{q^*}_{\pi,1}}$ over A is zero.

Lemma 10 *There exists* $q^* \leq F_0(x^*-)$ *such that* $\min I_{G^{q^*}_{\pi,1}}(A) = 0$.

Proof. The proof is based on the intermediate value theorem. Observe that $\min I_{G^0_{\pi,1}}(A) = \min I_{G_{\pi,1}}(A) \geq 0$. Moreover, as $G_{\pi,1}(s) < F_0(s-)$ for all $s < x^*$, we have that:

$$\begin{aligned} I_{G^{F_0(x^*-)}_{\pi,1}}(x^*) &= \int_0^{x^*} (F_0 - \max\{G_{\pi,1}(s), \min\{F_0(x^*-), F_0(s)\}\})\ \mathrm{d}s \\ &= \int_0^{x^*} (F_0 - \max\{G_{\pi,1}(s), F_0(s)\})\ \mathrm{d}s = 0. \end{aligned}$$

Since $x^* \in A$, $\min\ I_{G^{F_0(x^*-)}_{\pi,1}}(A) \leq I_{G^{F_0(x^*-)}_{\pi,1}}(x^*) = 0$. Thus, we have shown that $\min\ I_{G^{F_0(x^*-)}_{\pi,1}}(A) \leq 0 = \min I_{G^0_{\pi,1}}(A)$. Now, observe that the mapping

$$(q, x) \mapsto I_{G^q_{\pi,t}}(x) = \int_0^x (F_0 - G^q_{\pi,t})\ \mathrm{d}s,$$

is continuous, being the difference of two continuous functions of (q, x). As such, $q \mapsto \min\ I_{G^q_{\pi,1}}(A)$ is continuous in view of the maximum theorem. The lemma follows from the intermediate value theorem. ∎

The next lemma assures us that $G^q_{\pi,1}$ is not a signal only if it has too high of a mean.

Lemma 11 *For all* $x \in [0,1]$*:* $I_{G^{q^*}_{\pi,1}}(x) \geq 0$.

Proof. Divide $[0,1]$ into three subintervals, $[0,\pi)$, $[\pi, x^*]$, and $(x^*, 1]$, showing that the desired inequality holds for each at a time. We first show that $\inf I_{G^{q^*}_{\pi,1}}([0,\pi)) \geq 0$. To see this, recall that $\pi \geq \underline{x}$, meaning that $x < \pi$ only if $G_{\pi,1}(x) = 0$. As such, whenever $x < \pi$,

$$G^{q^*}_{\pi,1}(x) = \max\{0, \min\{q^*, F_0(x)\}\} = \min\{q^*, F_0(x)\} \leq F_0(x).$$

Thus, $I_{G^{q^*}_{\pi,1}}(x) \geq \int_0^x (F_0 - F_0)\, \mathrm{d}s = 0$ for all $x \in [0,\pi)$. We now show that $\min\ I_{G^{q^*}_{\pi,1}}([\pi, x^*]) \geq 0$. For this, let $x \in [\pi, x^*]$, and recall that $G_{\pi,1}(s-) < F_0(s-) \leq F_0(s)$ must hold for all $s < x$ by choice of x^*. As a consequence,

$$\begin{aligned} I_{G^{q^*}_{\pi,1}}(x) &= \int_0^x F_0(s) - \max\{G_{\pi,1}(s), \min\{q^*, F_0(s)\}\}\ \mathrm{d}s \\ &\geq \int_0^x F_0(s) - \max\{G_{\pi,1}(s), F_0(s)\}\ \mathrm{d}s \\ &= \int_0^x F_0(s) - F_0(s)\ \mathrm{d}s = 0. \end{aligned}$$

We thus have that $\min I_{G^{q^*}_{\pi,1}}([0,x^*]) \geq 0$. To complete the proof that $\min I_{G^{q^*}_{\pi,1}}([0,1]) \geq 0$, suppose for a contradiction that there exists $x \in (x^*, 1]$ such that $I_{G^{q^*}_{\pi,1}}(x) < 0$. Take

$$x_0 \in \arg\min_{x \in [0,1]} I_{G^{q^*}_{\pi,1}}(x) = \arg\min_{x \in (x^*,1]} I_{G^{q^*}_{\pi,1}}(x).$$

As $I_{G^{q^*}_{\pi,1}}(x)$ is right differentiable we have that

$$0 \leq \partial_- I_{G^{q^*}_{\pi,1}} = F_0(x_0-) - G_{\pi,1}(x_0-),$$

in view of $q^* \leq F_0(x^*) \leq G_{\pi,1}(x^*)$. Therefore, $F_0(x_0) \geq F(x_0)$ – i.e., $x_0 \in A$, in contradiction to $\min\ I_{G^{q^*}_{\pi,1}}(A) = 0$. Thus, $I_{G^{q^*}_{\pi,1}}(x) \geq 0$ for all x. ■

To conclude the proof of Lemma 7, notice that $x \in A$ only if $G_{\pi,1}(x) \geq F_0(x-) \geq F_0(x^*-) \geq q^*$. Taking $x_1 \in \arg\min_{x \in A} I_{G^{q^*}_{\pi,1}}(x)$, we therefore have

$$G^{q^*}_{\pi,1}(x_1) = \max\{G_{\pi,1}(x_1), \min\{q^*, F_0(x_1)\}\} = \max\{G_{\pi,1}(x_1), q^*\} = G_{\pi,1}(x_1),$$

Thus, x_1 is in $A \subseteq [\pi, \bar{x}]$, has $I_{G^{q^*}_{\pi,1}}(x) = 0$, and satisfies $G^{q^*}_{\pi,1}(x) = G_{\pi,1}(x) \geq q^*$; that is, our proof is complete.

E Proof of Proposition 1: Costly learning equilibria

We show supp H = supp F = co(supp F), meaning that supp F is a convex set overwhich S is indifferent; that is, F is a truncated Pareto. Because supp $H \subseteq$ supp $F \subseteq$ co(supp F) by Lemma 3, our task is to show co(supp F) $\subseteq$ supp H.

Letting $[w, z] :=$ co(supp F), we wish to show that $[w, z] \subseteq$ supp H. Suppose otherwise for a contradiction – i.e., $[w, z] \cap$ supp $H \neq [w, z]$. Note that supp $H \cap [w, z]$ is a closed set, meaning that $[w, z] \setminus$ supp H is open (in $[w, z]$), and so must contain a non-empty open subinterval of $[w, z]$. Let (x, y) be a maximal such subinterval with respect to set containment; that is, (x, y) is such that $(x', y') \cap$ supp $H \neq \emptyset$ for all $(x', y') \supsetneq (x, y)$.[20] Because supp H is closed, if $x \neq w$ then $x \in$ supp H: otherwise, $(x - \epsilon, x + \epsilon) \subseteq [w, z] \setminus$ supp H for all small $\epsilon > 0$, meaning that $(x, y) \subsetneq (x - \epsilon, y) \subseteq [w, z] \setminus$ supp H, a contradiction to maximality of (x, y). An analogous argument gives $y \neq z$ only if $y \in$ supp H. Because supp $H \subseteq$ supp F (Lemma 3) and $\{w, z\} \subseteq$ supp F, we thus have that $x, y \in$ supp F.

We now construct a family of deviations indexed by $\epsilon > 0$, F^*_ϵ, and obtain a contradiction by showing that these deviations must be strictly profitable for B when $\epsilon > 0$ is sufficiently small.

Fix a small $\epsilon > 0$, and note that the following are all well-defined due to $x, y \in$ supp F:

$$\begin{aligned}
F_{1,\epsilon} &= F(\cdot|s \in [x - \epsilon, x + \epsilon]),\\
F_{2,\epsilon} &= F(\cdot|s \in [y - \epsilon, y + \epsilon]),\\
\beta_{1,\epsilon} &= F(x + \epsilon) - F((x - \epsilon)-) > 0,\\
\beta_{2,\epsilon} &= F(y + \epsilon) - F((y - \epsilon)-) > 0.
\end{aligned}$$

Moreover, take

$$\begin{aligned}
\beta_{0,\epsilon} &= 1 - \beta_{1,\epsilon} - \beta_{2,\epsilon},\\
F_{0,\epsilon} &= \begin{cases} F(\cdot|s \notin [x - \epsilon, y + \epsilon]) & \text{if } \beta_{0,\epsilon} > 0,\\ \text{arbitrary } F' \in \mathcal{A} & \text{otherwise.} \end{cases}
\end{aligned}$$

Clearly, $F = \sum_{i=0}^{2} \beta_{i,\epsilon} F_{i,\epsilon}$. Moreover, since $x, y \in$ supp F, both $\beta_{1,\epsilon}$ and $\beta_{2,\epsilon}$ are strictly

[20] One can find the subinterval (x, y) by fixing some $(x', y') \subset [w, z] \setminus$ supp H, and taking the union of all $(x'', y'') \subseteq [w, z] \setminus$ supp H that contain (x', y').

positive for all $\epsilon > 0$. Define

$$s_\epsilon = \int s \, \mathrm{d}\left(\alpha F_{1,\epsilon} + (1-\alpha)F_{2,\epsilon}\right),$$
$$\eta_\epsilon = \min\{\beta_{1,\epsilon}, \beta_{2,\epsilon}\} > 0,$$
$$F_\epsilon^* = \beta_{0,\epsilon}F_{0,\epsilon} + \eta_\epsilon\delta_{s_\epsilon} + (\beta_{1,\epsilon} - \alpha\eta_\epsilon)F_{1,\epsilon} + (\beta_{2,\epsilon} - (1-\alpha)\eta_\epsilon)F_{2,\epsilon}.$$

In words, F_ϵ^* takes $\alpha\eta_\epsilon$ mass from the ϵ-ball around x and $(1-\alpha)\eta_\epsilon$ mass from the ϵ-ball around y and pools them to create an $\eta_\epsilon > 0$ mass on s_ϵ. Since $\alpha F_{1,\epsilon} + (1-\alpha)F_{2,\epsilon} \succ \delta_{s_\epsilon}$, F_ϵ^* is less informative than F which, in turn, is less informative that F_0. By transitivity of the information ordering, F_0 is more informative than F_ϵ^*; that is, $F_\epsilon^* \in \mathcal{A}$.

Let $T_H(s) = \int_0^s (s-p) \, \mathrm{d}H(p)$ denote B's expected trade surplus conditional on signal realization s. Below we prove

$$\lim_{\epsilon \searrow 0} \int \frac{T_H}{\eta_\epsilon} \, \mathrm{d}(F - F_\epsilon^*) = 0, \tag{12}$$
$$\lim_{\epsilon \searrow 0} \left(\frac{C(F_\epsilon^*) - C(F)}{\eta_\epsilon}\right) < 0, \tag{13}$$

and so obtain the following contradiction to F maximizing $U_\kappa(H, F)$,

$$0 \leq \lim_{\epsilon \searrow 0} \frac{U_\kappa(H,F) - U_\kappa(H, F_\epsilon^*)}{\eta_\epsilon} = \lim_{\epsilon \searrow 0} \left[\int \frac{T_H}{\eta_\epsilon} \, \mathrm{d}(F - F_\epsilon^*) + \kappa \frac{C(F_\epsilon^*) - C(F)}{\eta_\epsilon}\right] < 0, \tag{14}$$

hence completing the proof.

We now explain why (12) and (13) both hold. Because $(x, y) \cap \operatorname{supp} H = \emptyset$, B's trading surplus from receiving a signal $s \in [x, y]$ is given by

$$T_H(s) = \int_0^s (s-p) \, \mathrm{d}H(p) = \int_0^x (s-p) \, \mathrm{d}H(p) = H(x)s - \int_0^x p \, \mathrm{d}H(p). \tag{15}$$

As such, T_H, is affine over $[x, y]$, and so (12) obtains as follows:

$$\begin{aligned}\int \frac{T_H}{\eta_\epsilon} \, \mathrm{d}(F - F_\epsilon^*) &= \alpha \int T_H \;\; \mathrm{d}F_{1,\epsilon} + (1-\alpha)\int T_H \, \mathrm{d}F_{2,\epsilon} - T_H(s_\epsilon) \\ &\to \alpha T_H(x) + (1-\alpha)T_H(y) - T_H(\alpha x + (1-\alpha)y) = 0,\end{aligned}$$

where convergence follows from continuity of $T_H(\cdot)$, $s_\epsilon \to \alpha x + (1-\alpha)y$, $F_{1,\epsilon} \to \mathbf{1}_{[x,1]}$, and $F_{2,\epsilon} \to \mathbf{1}_{[y,1]}$. We now use the latter three convergences to obtain (13). To do so, notice that these convergences imply that

$$\frac{\|F_\epsilon^* - F\|}{\eta_\epsilon} = \left\|\mathbf{1}_{[s_\epsilon,1]} - (\alpha F_{1,\epsilon} + (1-\alpha)F_{2,\epsilon})\right\| \to \left\|\mathbf{1}_{[\alpha x + (1-\alpha)y,1]} - \left(\alpha\mathbf{1}_{[x,1]} + (1-\alpha)\mathbf{1}_{[y,1]}\right)\right\| =: M.$$

As such, Fréchet differentiability of C, and strict convexity of c_F over co(supp F) $\subseteq [x, y]$ yield

$$\begin{aligned}\frac{1}{\eta_\epsilon}\left[C(F_\epsilon^*) - C(F)\right] &= \frac{1}{\eta_\epsilon}\left[\int c_F \text{ d}(F_\epsilon^* - F) + o\left(\|F_\epsilon^* - F\|\right)\right] \\ &= \int c_F \text{ d}\left[\mathbf{1}_{[s_\epsilon,1]} - (\alpha F_{1,\epsilon} + (1-\alpha)F_{2,\epsilon})\right] + \frac{\|F_\epsilon^* - F\|}{\eta_\epsilon}\left[\frac{o\left(\|F_\epsilon^* - F\|\right)}{\|F_\epsilon^* - F\|}\right] \\ &\to c_F(\alpha x + (1-\alpha)y) - (\alpha c_F(x) + (1-\alpha)c_F(y)) + M0 < 0.\end{aligned}$$

Thus, we have (12) and (13), which together yield the contradiction (14). In other words, the proof is complete.

F Proof of Lemma 5: Separating prices and Pareto signals

Suppose $G_{\pi,t} \in \mathcal{A}$. We begin by showing that supp $G_{\pi,t} \cap S(G_{\pi,t})$ is empty whenever $\pi > \underline{\pi}$; that is, In other words, we need to show that $I_{G_{\pi,t}}(x) > 0$ for all $x \in$ supp $G_{\pi,t}$. We do so by comparing the information constraint of $G_{\pi,t}$ to the constraint of a different Pareto signal, $G_{\pi',t'}$, where $\pi' \in [\underline{\pi}, \pi)$ and $t' > t$. Existence of a Pareto signal associated with such a π' follows from Roesler and Szentes (2017). That the truncation point associated with π' is strictly larger than t follows from $G_{\pi',t'}$ having the same mean as $G_{\pi,t}$ and the mean of a truncated Pareto distribution $G_{\pi'',t''}$ being strictly increasing (π'', t'').

Given that $G_{\pi',t'}$ exists, showing that

$$I_{G_{\pi,t}}(x) - I_{G_{\pi',t'}}(x) = \int_0^x (G_{\pi',t'} - G_{\pi,t}) \text{ d}s > 0 \text{ for all } x \in (\pi', t'), \tag{16}$$

is sufficient for proving that supp $G_{\pi,t} \cap S(G_{\pi,t})$ is empty whenever $\pi > \underline{\pi}$. To see that (16) must hold for $x \in (\pi', t)$, note that

$$(G_{\pi',t'} - G_{\pi,t})(s) = \begin{cases} 0 & \text{if } s \le \pi' \\ 1 - \pi'/s & \text{if } s \in (\pi', \pi] \\ (\pi - \pi')/s & \text{if } s \in (\pi, t). \end{cases}$$

As such, $(G_{\pi',t'} - G_{\pi,t})(s) \ge 0$ for all $s \le \pi'$, and, more importantly, $(G_{\pi',t'} - G_{\pi,t})(s) > 0$ for all $s \in (\pi', t)$. It follows that (16) holds for all $x \in (\pi', t]$. To see that (16) holds for

$x \in (t, t')$, notice that for any such x,

$$\begin{aligned}\int_0^x (G_{\pi',t'} - G_{\pi,t})\ \mathrm{d}s &= \int_0^1 (G_{\pi',t'} - G_{\pi,t})\ \mathrm{d}s - \int_x^1 (G_{\pi',t'} - G_{\pi,t})\ \mathrm{d}s \\ &= \int_x^1 (G_{\pi,t} - G_{\pi',t'})\ \mathrm{d}s \\ &= \int_x^{t'} (G_{\pi,t} - G_{\pi',t'})\ \mathrm{d}s \\ &= \int_x^{t'} \frac{\pi'}{s}\ \mathrm{d}s > 0,\end{aligned}$$

where the second equality follows from $I_{G_{\pi,t}}(1) = I_{G_{\pi',t'}}(1) = 0$, and the third equality following from $G_{\pi,t}(s) = G_{\pi',t'}(s) = 1$ for all $s \geq t' > t$.

G Upper hemicontinuity of S's best response

In this section we prove the following lemma about S's best response correspondence and maximal value.

Lemma 12 *S's maximal profit, $F \mapsto \pi_F$, is continuous, and $P(\cdot)$ is upper hemicontinuous.*

Proof. Let $\{F_n\}_{n\geq 0}$ be some sequence attaining F_∞ as its limit. We show that $\lim_{n\to\infty} \pi_{F_n} = \pi_{F_\infty}$. Since Π is upper-semicontinuous, $F \mapsto \pi_F$ is also upper-semicontinuous.[21] As such, it is sufficient to show that $\liminf_{n\to\infty} \pi_{F_n} \geq \pi_\infty$. To do so, take any $p \in P(F_\infty)$. Then for all $\epsilon > 0$,

$$\pi_{F_n} \geq \Pi(p - \epsilon, F_n) \geq (p - \epsilon)(1 - F_n(p - \epsilon)).$$

Thus,

$$\liminf_n \pi_{F_n} \geq \liminf_n (p - \epsilon)(1 - F_n(p - \epsilon)) \geq (p - \epsilon)(1 - F_\infty(p - \epsilon)) \geq p(1 - F_\infty(p-)) - \epsilon.$$

where the second inequality follows from the Portmanteau theorem. As ϵ above is arbitrary, the result follows.

To see that $P(\cdot)$ is upper hemicontinuous, take any convergent sequence $p_n \in P(F_n)$ attaining p_∞ as its limit. Since Π is upper semicontinuous and $F \mapsto \pi_F$ is continuous,

$$\pi_{F_\infty} = \lim \pi_{F_n} = \limsup \Pi(p_n, F_n) \leq \Pi(p_\infty, F_\infty) \leq \pi_{F_\infty}.$$

Thus, $\Pi(p_\infty, F_\infty) = \pi_{F_\infty}$; that is, $p_\infty \in P(F_\infty)$. ■

[21] See Aliprantis and Border (2006), Lemma 17.30, for example.