Multi-dimensional communication with limited commitment

Sofia Moroni[*]

Preliminary draft

Abstract

We consider a contractual relationship between an uninformed principal and an informed agent. The agent observes a multi-dimensional piece of private information which affects the mapping from actions to both players' payoffs. Conditional on the information, the two parties' preferences are not perfectly aligned. The agent sends a message to the principal, who then chooses multiple actions. We ask what outcomes the principal can achieve if she has limited commitment. In particular, we allow precisely one dimension in which the principal is able to commit to an action as a function of the agent's message. In all other dimensions, the principal must take an action that is optimal given what she has learned. We show that there is always a mechanism that induces full revelation of the information about the non-commitment actions. Furthermore, as the principal and the agent's preferences become more and more divergent in the dimension of commitment, the principal's payoff approaches that in which she has full information pertaining to all the dimensions in which she lacks commitment power. Our results imply that incompleteness of contracts may be irrelevant for payoffs and information revelation as long as there is some contractable outcome in which there is large conflict.

[*]University of Pittsburgh. email: moroni@pitt.edu.

1 Introduction

Oftentimes in contractual relationships, multiple decisions must be made which rely on information held by different parties. In this paper we focus on the case in which an agent has private information that is relevant to each decision, but has preferences that are misaligned with what his principal would prefer. For instance, a manager and worker must decide on the worker's time allocation and decisions on multiple tasks, but the worker has certain tasks that he enjoys. A medical doctor, employed by an HMO, has to choose tests and procedures for multiple patients, but is inclined to order more procedures than the HMO would prefer.

One would expect that contracting on a bundle of decisions based on all recommendations would be desirable to elicit the most information. Nonetheless, most contracts are written on a small subset of the payoff-relevant features of the interaction. For example, labor contracts generally specify salary or wages but not the precise assignments the workers will receive. In many settings contracting on some decisions is infeasible. The HMO cannot condition the treatment a doctor prescribes to a patient on the treatment prescribed to another patient. A judge must take the most fair-minded action given the information that has been brought forward by the prosecutor in each case. Pay or work assignments, however, can generally be contracted upon as a function of this kind of recommendations.

Formally, we consider a contractual relationship between an agent an a principal who lacks commitment power. The agent knows n pieces of information each of which is informative about one action. The principal does not know the information that the agent holds. Conditional on the information the agent and the principal's preferences are misaligned. The agent sends a message to the principal who then chooses the n actions.

The principal's commitment power is limited. There is only one aspect of the complex decision to which the principal is able to commit as a function of what she learns. In all other dimensions of the decision, the non-commitment dimensions, the principal takes the action that is best given her information.

Our main result is that as long as the principal's preference is monotone in the agent's and separable across dimensions,[1] there is an allocation in which the agent reveals all information relevant to the dimensions of non-commitment and, hence, the principal attains the first best on those dimensions. As the divergence between the agent and the principal increases in the dimension of commitment, under this allocation the payoff of the principal converges to what she would obtain if she could observe the information in those dimensions. In the optimal allocation the principal trades off implementing her preferred action in the dimension of commitment and eliciting information in the other dimensions. Our result implies that as the divergence in preferences grows large, this trade off comes at a negligible cost.

The principal is better off if she can increase her commitment power. Thus, our results are also applicable to problems of full commitment (delegation) or partial commitment on any (non-empty) subset of actions.

To gain intuition for our result, notice that the principal sets the action in the dimension of commitment close to her preference. When the action chosen by the principal is far from the action that is preferred by the agent, small movements in the commitment action can generate large gains and losses to the agent. Think, for example, in the case in which there are only two dimensions of actions and information. In the limit, as the preferences in the dimension of commitment diverge, the indifference curves of the agent are flat in the dimension of non commitment. The agent, then, only cares about the action in the dimension of commitment and is willing to disclose her information on every other aspect of the decision in a exchange for small concessions in that dimension.

Our results are in contrast with the common finding in the literature of cheap talk and delegation that the principal or the receiver is better off communicating with a player whose preferences are closer to hers.[2] In a multidimensional environment in which a principal is able to contract in at least one action, our results show that greater disagreement has a flip side. It confers leverage to the principal and allows

[1] We assume that the utility is a quadratic, additively separable in each dimension. Additionally, we assume that the agent's preferred action in each dimension only affects the principal's preferred action in that dimension.

[2] See for example Holmström (1977, 1984), Crawford and Sobel (1982), Levy and Razin (2007), Alonso and Matouschek (2008), Koessler and Martimort (2012).

her to induce disclosure of other payoff relevant states.

We prove our result in two steps. We first show that many constraints are redundant. It is sufficient, for instance, for an allocation to satisfy all the adjacent constraints to obtain general incentive compatibility.

In the second step, we construct an allocation that satisfies incentive compatibility in the simpler case in which the principal and the agent have the same preferences in the dimensions of non-commitment. We construct this allocation by solving for a subset of incentive compatibility constraints in which there are no "cycles".[3] As the preferences in the dimension of commitment diverge, the principal and the agent's preferences become closer in the dimensions of non-commitment in relative terms. They become close enough that, by an Implicit Function Theorem, solving the problem at the agent's preferences implies it has a solution at the principal's preferences. Here is where the no "cycles" condition comes to play. It is crucial to guarantee that the Jacobian of the problem is non-singular.

The organization of the paper is as follows. Section 2 introduces the model. Section 3 presents our main results and provides the proof of the main theorem. Section 4 applies our results to a manager-employee relationship and the regulation of a multi-product monopolist.

Related literature. Our model is inspired by the seminal cheap-talk environment in Crawford and Sobel (1982). We extend their work by assuming there are at least two dimensions of information and actions and the receiver is able to commit to one action as a function of the message. To obtain our result we assume that the bliss points of the principal are monotone in the agent's bliss points as in Crawford and Sobel (1982).

Our paper is related to multi-dimensional cheap-talk models. Levy and Razin (2007) find bounds on the amount of communication in such settings. They show that for large divergence in preferences equilibria feature finitely many actions in equilibrium with large probability. Chakraborty and Harbaugh (2010) show that in

[3] Consider the undirected graph in which the nodes are types (states known by the agent) and the edges are the incentive constraints that are set to equality. A cycle is a sequence of incentive constraints that form a cycle in this graph.

a cheap talk model in which the expert has state independent preferences there are cheap talk equilibria in which the agent influences the decision of the principal. Under some assumptions they show that there can be full revelation in the "dimensions of agreement". In contrast, we assume that both expert and receiver have state dependent preferences but the principal has some limited commitment power. Carroll and Egorov (2017) study a multi-dimensional communication model in which the principal can verify one dimension of the agent's private information and identify conditions under which perfect revelation of information is possible. In contrast, we do not assume an ability to verify, but show that commitment on one dimension is sufficient to ensure truthful revelation.

Our paper is also related to the literature on delegation, which originated from Holmström (1977, 1984), and includes works by Melumad and Shibano (1991), Alonso and Matouschek (2008), Kováč and Mylovanov (2009), Martimort and Semenov (2006), and Goltsman et al. (2009) in the case of unidimensional delegation, and Koessler and Martimort (2012) and Frankel (2015) in the case of multidimensional delegation. Under delegation the principal has full commitment in all dimensions. Since the principal is always better when she is able to commit on more aspects of the decision, our results imply that as the discordance in one dimensions grows large the principal can attain the first best in all other dimensions.

Koessler and Martimort (2012) study multi-dimensional delegation where the agent has the same bias in each direction. In their model, the principal is better off when contracting with an agent with smaller bias. Frankel (2015) studies optimal multiple delegation when the principal faces an uncertain prior about the agent's preferences. He generalizes the "cap" formulation of the optimal delegation set in the unidimensional setting to a multidimensional one. Antić and Iaryczower (2016) analyze a delegation game where the principal not only chooses the set of actions, but also the scale at which each action can be implemented. They show that the principal may want to limit the scale of implementation for an agent biased towards too high an action but provide too much scale for an agent biased towards too low an action, which creates inefficiency.

In unidimensional setups, Ottaviani (2000) and Krishna and Morgan (2008) allow the principal to make monetary transfers to the agent, conditional on the messages sent by the agent. We do not allow for monetary transfers. Ambrus and Egorov (2017) allow for money burning, defined as contractual terms costly to the agent, but not beneficial to the principal. They find that money burning may be used by the principal when there is limited liability on monetary transfers. The intuition for the value of money burning is related to our intuition for why large disagreement can be beneficial to our principal. The chief difference, is that in our setting there is uncertainty in the dimension of disagreement and the principal does care about actions in the dimension of commitment. When her preferences differ from that of the agent she can achieve outcomes close to the first best in the dimension of non-commitment.

Bester and Strausz (2001) study the general problem of mechanism design under imperfect commitment and show that a version of the revelation principle holds. Bester and Strausz (2007) characterize the optimal mechanism and show that it often involves stochastic communication devices.

In the context of strategic information transmission, stronger incentive for information acquisition has often been identified as the reason why a principal may prefer a biased agent–Dewatripont and Tirole (1999) and Che and Kartik (2009) provide two such examples.

2 Model

We consider the relationship between a principal (she) and an agent (he). The agent is informed about the realization of a state $\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in \Theta$ with $\Theta = \times_{i=1}^n \Theta_i \subseteq \mathbb{R}^n$, a finite set. State θ contains information about the desirability, to the agent and the principal, of an n-dimensional action $(x_1, \ldots x_n) \in \mathbb{R}^n$. After learning his type, an agent of type $\theta \in \Theta$ sends message $m(\theta)$, an element of message space M, to the principal and the principal chooses an allocation $x = (x_1, x_2 \ldots, x_n)$ that consists of an action for each dimension.

The principal has the ability to commit to the action in dimension 1, $x_1(m)$, as a function of the message m but cannot commit on the choice of actions $x_2, \ldots, x_n$. This means that $x_2, \ldots, x_n$ must be optimal given any information the principal may learn from the message m.

The principal's utility from action $x = (x_i)_{i=1}^n \in \mathbb{R}^n$ if the type of the agent is θ is given by

$$V(x, \theta) = \sum_{i=1}^{n} -\alpha_i \left(y_i^P(\theta) - x_i(m) \right)^2$$

where $\alpha_i > 0$ and $y_i^P(\theta_i) \in \mathbb{R}$. $y_i^P(\theta)$ is the principal's bliss point in dimension i when θ is the realized state.

The principal has a prior $p(\theta)$ on each state θ.

The utility of the agent of type θ from action $x = (x_i)_{i=1}^n \in \mathbb{R}^n$ is given by[4]

$$U(x, \theta) = \sum_{i=1}^{n} -(\theta_i - x_i)^2 \equiv -d(x, \theta)^2.$$

Therefore, whenever $y_i^P(\theta) \neq \theta_i$ the principal and the agent's preferences are misaligned in dimension i in state θ.

The timing of the game is as follows. First the principal commits to a function $x_1 : M \to \mathbb{R}$. Then the agent learns his type and sends a message to the principal. After hearing the message the principal chooses the non-committal actions $(x_2, \ldots, x_n)$. An agent's strategy is a function $\sigma^A : \Theta \to \Delta M$. A principal's strategy is a function $\sigma^P : M \to \mathbb{R}^{n-1}$ that delivers for each message $m \in M$ a vector $(x_2, \ldots, x_n)$ of non-committal actions.[5] For a fixed contract, principal and agent play a Perfect Bayesian Equilibrium of this game.

[4]The function d is the distance between vectors. For $y, x \in \mathbb{R}^n$ the distance between x and y is given by,

$$d(x, y) = \left(\sum_{i=1}^{n} (y_i - x_i)^2 \right)^{1/2}.$$

[5]Due to the strict concavity of her preferences, the principal has a unique best response. Thus, it is without loss to assume that the principal uses only pure strategies.

2.1 The principal's problem

Bester and Strausz (2001) show that a version of the revelation principle holds in settings with partial commitment, such as ours. Namely, it is without loss to assume that $M = \Theta$ and to assume that the agent declares his type with positive probability. This probability may be less than one, in which case the agent lies about his type with positive probability.

Using this revelation principle we assume that the agent declares a type $\theta \in \Theta$ (not necessarily truthfully) and the principal chooses an allocation $x(\theta) = (x_1(\theta), \ldots, x_n(\theta))$.

For a given strategy σ^A, of the agent, define $\mathbf{S}_{\sigma^A}(\theta) = \{\theta' \in \Theta | \sigma^A(\theta)(\theta') > 0\}$ and define $p_{\sigma^A}^{\theta|\theta'}$ to be the probability of state θ, induced by the strategy σ^A, when the agent sends message θ'. By Bayes' rule,

$$p_{\sigma^A}^{\theta|\theta'} = \frac{p(\theta)\sigma^A(\theta)(\theta')}{\sum_{\tilde{\theta}\in\Theta} p(\tilde{\theta})\sigma^A(\tilde{\theta})(\theta')}.$$

The principal's problem is to find a contract $x_1(\theta)$, and strategies σ^A and $\sigma^P = (x_2(\theta), \ldots, x_n(\theta))$ so as to maximize

$$-\sum_{\theta\in\Theta, \theta'\in\mathbf{S}_{\sigma^A}(\theta)} p(\theta)\sigma^A(\theta)(\theta') \left[\sum_{i=1}^{n} \alpha_i \left(y_i^P(\theta) - x_i(\theta')\right)^2\right]$$

subject to the IC constraints of the agent

$$-d\left(y^A(\theta), x(\theta')\right) \geq -d\left(y^A(\theta), x(\tilde{\theta})\right), \qquad (\theta,\tilde{\theta})$$

for each $\theta, \tilde{\theta} \in \Theta$ and $\theta' \in \mathbf{S}_{\sigma^A}(\theta)$, and the equilibrium condition for the principal

$$(x_i(\theta))_{i=2}^n \in \text{argmax}_{(\hat{x}_i(\theta))_{i=2}^n} - \sum_{\theta\in\Theta} p_{\sigma^A}^{\theta|\theta'} \sum_{i=2}^{n} \alpha_i \left(y_i^P(\theta) - \hat{x}_i(\theta')\right)^2,$$

for each $\theta' \in \Theta$.

The IC constraint requires that the agent be willing to declare the types in the support of his strategy. The principal's equilibrium condition requires that the action chosen by the principal is optimal given her posterior about the agent's type after hearing his message.

3 Main Results

Definition 1. We say that the agent and the principal's preferences are *weakly aligned* if for every $\theta, \theta' \in \Theta$ and $i \in \{1, \dots, n\}$

$$\theta_i > \theta_i' \text{ if and only if } y_i^P(\theta) > y_i^P(\theta').$$

We say that an allocation $x = \{x(\theta)\}_{\theta \in \Theta}$ is *weakly aligned* with the agent's preferences in dimension i if

$$\theta_i > \theta_i' \text{ if and only if } x_i(\theta) > x_i(\theta').$$

Weak alignment of preferences implies that the preferences are *separable across dimensions.* In fact, as $\theta_i = \theta_i'$ if and only if $y_i^P(\theta) = y_i^P(\theta')$, we can write $y_i^P(\theta) = y_i^P(\theta_i)$. That is, $y_i^P(\theta)$ only depends on θ_i and the bliss points of the agent and the principal live on a grid.

Definition 2. The *bias in dimension 1* is defined as $b_1(\Theta) = \min_{\theta \in \Theta} |y_1^P(\theta) - \theta_1|$.

The bias measures the minimum distance between the principal and the agent's preferences in the commitment dimension.

We are now ready to state our main result.

Theorem 1. *Fix Θ_i and y_i^P for $i \in \{2, \dots, n\}$. If the principal and the agent's preferences are weakly aligned, there is $b > 0$ such that if $b_1(\Theta) > b$, there is an incentive compatible allocation $x = (x(\theta))_{\theta \in \Theta}$ with $x_i(\theta) = y_i^P(\theta)$ for $i \in \{2, \dots, n\}$. Moreover, as $b_1(\Theta) \to \infty$, $\max_{\theta \in \Theta} \left| \mathbb{E}\left(y_1^P(\theta)\right) - x_1(\theta) \right| \to 0$.*

Theorem 1 says that for large enough divergence between the principal and the agent's preferences there is an allocation that satisfies IC, such that the principal receives her preferred actions in the dimensions with no commitment, and such the allocation in the dimension of commitment approaches the principal's preferred action in that dimension. The principal trades off implementing her preferred action in dimension 1, the one in which she can commit, with inducing information disclosure in all the other dimensions. Theorem 1 says that she can do this at nearly no cost as the divergence between preferences grows large. In the limit her payoff approaches what she would obtain if she knew all the information in the dimensions of no commitment.

To gain intuition for our main result consider Figure 1 below. In the picture, there is no uncertainty in the dimension of commitment. The green lines depict indifference curves of agent type (θ_1, θ_2^1), the red lines the curves of type (θ_1, θ_2^2) and the blue lines the indifference curves of type (θ_1, θ_2^3). The red dots represent allocations for the different types. Because the principal does not have commitment power in dimension 2, when she learns the dimension 2 type is θ_2^i, the allocation that is implemented must be on the horizontal line that passes through $y_2^P(\theta_2^i)$. When the principal's preference in the dimension of commitment is $y_1^P(\theta_1)$, i.e when it is closest to the agent's preference, there is no allocation that satisfies $|y_1^P(\theta) - x_1(\theta)| \leq \varepsilon/2$ for every $\theta \in \Theta$ in which the agent reveals his type in every state. In fact, if $x_2(\theta_1, \theta_2^2) = y_2(\theta_2^2)$ in any allocation in which $\left|y_1^P(\theta_1^2) - x_1\left(\theta_1, \theta_1^2\right)\right| \leq \varepsilon/2$, where ε is the length of the orange intervals at $\left(y_1^P(\theta_1), y_2^P(\cdot)\right)$, the agent of type (θ_1, θ_2^1) has an incentive to deviate. In the figure, this observation follows from the fact that the green indifference curve does not intersect the allocations within the orange interval at $\left(y_1^P(\theta_1), y_2^P(\theta_2^1)\right)$. Note that the red point on the orange interval at $\left(y_1^P(\theta_1), y_2^P(\theta_2^2)\right)$ is the worst possible allocation within the bounds that can be awarded to type (θ_1, θ_2^2).

When the principal's preference in dimension 1 is given by ${y'_1}^P(\theta_1)$, the agent's indifference curves are flatter at the allocations that are close to ${y'_1}^P(\theta_1)$. The principal, thus, needs to forgo less dimension-1 utility to separate two dimension-2 types. The figure shows that there is an allocation that is incentive compatible and satisfies $|{y'_1}^P(\theta) - x_1(\theta)| \leq \varepsilon/2$ for every θ. At the preference that is the furthest, ${y''_1}^P(\theta_1)$, the principal can implement an allocation satisfying $\left|{y''_1}^P(\theta) - x_1(\theta)\right| \leq \varepsilon/4$.

In the example illustrated in Figure 1, due to the dimensionality of the problem, we could find an incentive compatible allocation by finding allocations that satisfied the

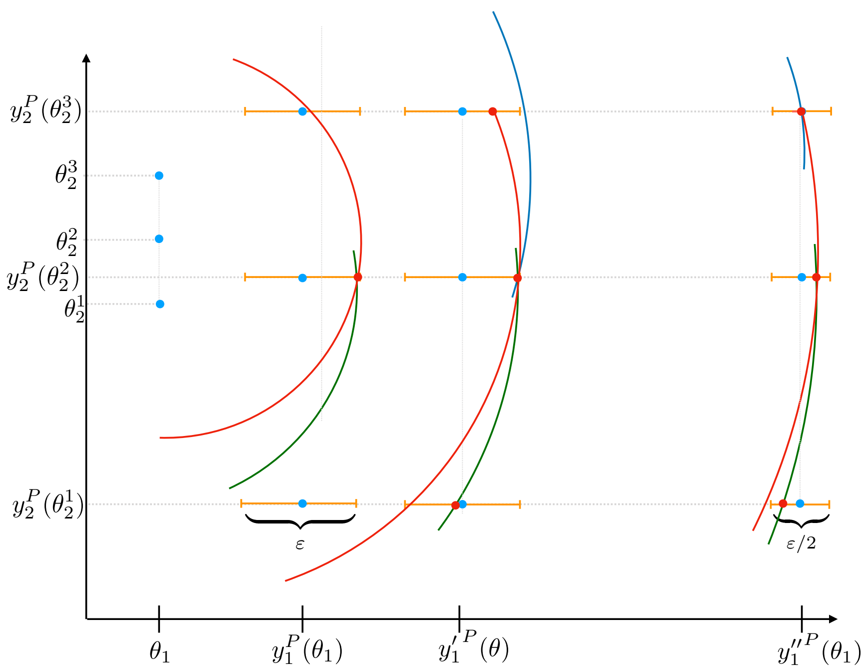

Figure 1: Constructing incentive compatible allocations for different preference divergences.

adjacent (or local) incentive constraints.[6] In more dimensions (3 or more) finding an incentive compatible allocation is more involved as many more incentive constraints must be satisfied for every additional dimension that is added into the problem.

To prove our result we first show that many constraints are redundant. It is sufficient, for instance, for an allocation that is aligned with the agent's preferences in dimensions 2 through n to satisfy all the adjacent constraints to obtain general incentive compatibility. Due to the weak alignment of the agent and the principal's preferences this is true for any allocation in which the principal receives her preferred allocation in these dimensions.[7]

In the second step, we construct an allocation that satisfies incentive compatibility in the simpler case in which the principal and the agent have the same preferences in the dimensions of non-commitment. We construct this allocation by solving for a subset of incentive compatibility constraints in which there are no "cycles".[8] As the preferences in the dimension of commitment diverge, the principal and the agent's preferences become closer in the dimensions of non-commitment, in relative terms. They become close enough that, by an Implicit Function Theorem, if the the problem has a solution when the principal and the agent's preferences coincide in the dimensions of non-commitment then it has a solution at the principal's original preferences. The size of the vicinity in which a solution can be found by the IFT provides an upper bound on the distance between $x_1(\theta)$ and the principal's preferred action. This vicinity shrinks as divergence in dimension 1 increases implying that $x_1(\theta)$ converges to the principal's preferred action. The no "cycles" condition is crucial as it implies that the Jacobian of the implicit function is non-singular.

[6]See for example Carroll (2012).

[7]If and allocation $x(\theta)$ is aligned with the agent's preferences in dimensions 2 through n then $x_2^n(\theta) = (x_2(\theta), \ldots, x_n(\theta))$ satisfies the sum of cycles condition in Rochet (1987) and condition a) in Theorem 3.1 in Jehiel and Moldovanu (2001). Thus, if the agent had quasi-linear preferences (i.e. the action in dimension 1 is a transfer) there is a transfer function that implements $x_2^n(\theta)$. It is not surprising, then, that $x_2^n(\theta)$ can be implemented via choices in dimension 1 when Θ_1 consists of one point. Our approach allows us to show that it can be implemented when Θ_1 is not a singleton and that the principal's payoff is close of what she obtains when she knows all information in dimensions 2 through n when $b_1(\Theta)$ is large.

[8]Consider the undirected graph in which the nodes are types (states known by the agent) and the edges are the incentive constraints that are set to equality. A cycle is a sequence of incentive constraints that form a cycle in this graph.

4 Examples

4.1 Red Tape

Consider the contractual relationship between a doctor (she) that is employed by an HMO. For illustration, assume that the doctor sees two patients and spends time t doing tedious paperwork. Each patient can have two possible diagnosis. Only the doctor observes their diagnosis. Let $\Theta_2 = \{b_2, 1 + b_2\}$ and $\Theta_3 = \{b_3, 1 + b_3\}$ be the treatments, in terms of expenditure to the HMO, that the doctor would ideally prescribe to the patients as a function of their diagnosis. The HMO would like to implement $Y_2 = \{0, 1\}$ and $Y_3 = \{0, 1\}$ instead. Note that in one dimensional cheap talk there is no communication when $b_i > 1/2$.

Now, under Lemma 4, we set one particular set of constraints to be binding, ensuring that all the incentive compatibility conditions are satisfied. In particular, we make $(0, 0) \rightarrow (0, 1)$, $(0, 0) \rightarrow (1, 0)$, and $(1, 0) \rightarrow (1, 1)$ binding.

$$\begin{aligned}
-\left(t_{00}+b_1\right)^2-\left(-b_2\right)^2-\left(-b_3\right)^2 &= -\left(t_{01}+b_1\right)^2-\left(-b_2\right)^2-\left(1-b_3\right)^2, \\
-\left(t_{00}+b_1\right)^2-\left(-b_2\right)^2-\left(-b_3\right)^2 &= -\left(t_{10}+b_1\right)^2-\left(1-b_2\right)^2-\left(-b_3\right)^2, \\
-\left(t_{10}+b_1\right)^2-\left(-b_2\right)^2-\left(-b_3\right)^2 &= -\left(t_{11}+b_1\right)^2-\left(-b_2\right)^2-\left(1-b_3\right)^2.
\end{aligned}$$

We set t_{01} to 0, divide every variable by b_1, and rearrange terms, and obtain

$$\begin{aligned}
(\widetilde{t}_{00})^2+2\widetilde{t}_{00}-1/\left(b_1\right)^2+2\widetilde{b}_3/b_1 &= 0, \\
(\widetilde{t}_{10})^2+2\widetilde{t}_{10}-2\widetilde{b}_2/b_1+2\widetilde{b}_3/b_1 &= 0, \\
(\widetilde{t}_{11})^2+2\widetilde{t}_{11}+1/\left(b_1\right)^2-2\widetilde{b}_2/b_1 &= 0,
\end{aligned}$$

where $\widetilde{t}_{ij} = t_{ij}/b_1$ and $\widetilde{b}_k = b_k/b_1$ $(k = 2, 3)$. If b_1 is sufficiently large such that it satisfies

$$\begin{aligned}
{b_1}^2 &> 2b_3 - 1, \\
{b_1}^2 &> 2\left(b_3 - b_2\right),
\end{aligned}$$

we obtain the following solutions:

$$
\begin{aligned}
\widetilde{t}_{00} &= -1+\sqrt{{b_1}^2+1-2b_3}/b_1,\\
\widetilde{t}_{01} &= 0,\\
\widetilde{t}_{10} &= -1+\sqrt{{b_1}^2+2b_2-2b_3}/b_1,\\
\widetilde{t}_{11} &= -1+\sqrt{{b_1}^2-1+2b_3}/b_1.
\end{aligned}
$$

Note that they all converge to 0 as b_1 approaches infinity.

To simplify our demonstration, consider the special case $b_2 = b_3 = b$. Substituting the solutions into the objective function, the organization head's objective function becomes

$$
-b_1\left(-b_1+\frac{1}{2}\sqrt{{b_1}^2+(1-2b)}+\frac{1}{2}\sqrt{{b_1}^2-(1-2b)}\right).
$$

It can be shown that as b_1 becomes larger, the above payoff becomes larger, and it converges to the maximum value, 0, as b_1 goes to infinity.

4.2 Regulation of a multi-product monopolist

Consider the example below from Baron and Myerson (1982), as described in by Alonso and Matouschek (2008). Consider a monopolist, who is active in one big market and several small markets, which are distinct from one another. In market i, the monopolist faces an inverse demand curve $y_i = A_i - Bq_i$, where $A_i > 2$. To keep consistency with Alonso and Matouschek (2008)'s and our own notation, we use y to represent price, as it is the action to be chosen by the monopolist. The monopolist holds private information about his constant marginal cost in each market i, θ_i, which can be either 0 or 1. Note that the monopolist's profits in market i, as a function of price, can be written

$$\begin{aligned}\Pi_i(y) &= y\left(\frac{A_i - y}{B}\right) - \theta_i \frac{A_i - y}{B}, \\ &= -\frac{1}{B}\left[\left(y - \frac{A_i + \theta_i}{2}\right)^2 - \left(\frac{A_i - \theta_i}{2}\right)^2\right].\end{aligned}$$

On the other hand, the regulator's objective function in market i is the social surplus

$$\begin{aligned}S_i(y) &= \frac{1}{2}(A_i + y)\left(\frac{A_i - y}{B}\right) - \theta_i \frac{A_i - y}{B}, \\ &= -\frac{1}{2B}\left[(y - \theta_i)^2 - (A_i - \theta_i)^2\right].\end{aligned}$$

Thus, the monopolist maximizes

$$\Pi(\mathbf{y}) = -\frac{1}{B}\sum_{i=1}^{n}\left[\left(y_i - \frac{A_i + \theta_i}{2}\right)^2 - \left(\frac{A_i - \theta_i}{2}\right)^2\right],$$

while the regulator maximizes

$$S(\mathbf{y}) = -\frac{1}{2B}\sum_{i=1}^{n}\left[(y_i - \theta_i)^2 - (A_i - \theta_i)^2\right].$$

Note that both objective functions can be made simpler by removing the factor $1/B$ and the terms that do not involve y, as those are constants that are irrelevant to incentives.

Let us consider a special case of three markets. There is no uncertainty about the monopolist's cost in the first market. Assume in addition that $A_2 = A_3 = A$. Without loss of generality, $\theta_1 = 0$. Let y_{01} be the decision taken by the principal in market 1 when the monopolist claims that his marginal cost is 0 in market 2 and 1 in market 3. The incentive constraints of the monopolist can be written

$$-\left(y_{ab} - \frac{A_1}{2}\right)^2 - (a - \frac{a+A}{2})^2 - (b - \frac{b+A}{2})^2 \geq -\left(y_{\tilde{a}\tilde{b}} - \frac{A_1}{2}\right)^2 - (\tilde{a} - \frac{a+A}{2})^2 - (\tilde{b} - \frac{b+A}{2})^2,$$

for $a, b, \tilde{a}, \tilde{b} \in \{0, 1\}$. Under Lemma 4, and given our particular payoff functions, we set one particular set of constraints to be binding (namely, the upward adjacent constraints), ensuring that all the incentive compatibility conditions are satisfied. That is, we make $(0,0) \to (0,1)$, $(0,0) \to (1,0)$, and $(1,0) \to (1,1)$ binding. Setting $y_{1,1} = 0$ and solving these equations, as long as A_1 is large enough, we may obtain valid solutions:

$$y_{01} = y_{10} = \frac{A_1}{2} - \sqrt{\frac{{A_1}^2}{4} + \left(1 - \frac{A}{2}\right)^2 - \left(\frac{A}{2}\right)^2},$$
$$y_{00} = \frac{A_1}{2} - \sqrt{\frac{{A_1}^2}{4} + 2\left[\left(1 - \frac{A}{2}\right)^2 - \left(\frac{A}{2}\right)^2\right]}.$$

Note that as $A_1 \to +\infty$, $y_{ab} \to 0$ for all $a, b \in \{0,1\}$, which means that the allocation approaches the regulator's first best outcome.

In a large enough market, A_1, the regulator is not be able to induce disclosure of the firm's private information about the price, due to the divergence of the ideal points. However, by committing to setting the prices in the large market as a function of the prices set on other markets, the regulator can bring the price in these latter markets to the marginal cost in every eventuality.

4.3 Proof of Theorem 1

We first introduce some notation. It is useful to denote the states in the support by an index that represents where they "stand" relative to the other states. Suppose the set Θ_i has n_i elements, then we can write it as $\Theta_i = \{\theta_i^j\}_{j=1}^{n_i}$ with $\theta_i^j \geq \theta_i^l$ whenever $j \geq l$. $\mathbf{I} \equiv \times_{i=1}^{n}\{1, \ldots n_i\}$ is a set of *indices*. For $\mathbf{k} = (\mathbf{k}_1, \ldots, \mathbf{k}_n) \in \mathbf{I}$ let $\theta^{\mathbf{k}}$ denote type $(\theta_1^{\mathbf{k}_1}, \ldots, \theta_n^{\mathbf{k}_n})$. Thus, we can write $\Theta = \left\{\theta^{\mathbf{k}}\right\}_{\mathbf{k}\in\mathbf{I}}$. We also write $y_i^{P,\mathbf{k}}$ for $y_i^P(\theta^{\mathbf{k}})$, $x^{\mathbf{k}}$ for $x(\theta^{\mathbf{k}})$, and we refer to IC constraint $(\theta^{\mathbf{k}}, \theta^{\mathbf{j}})$ as IC constraint $(\mathbf{k}, \mathbf{j})$.

The set of incentives constraints is given by $\mathbf{C} = \{(\mathbf{k}, \mathbf{j}) | \mathbf{k}, \mathbf{j} \in \mathbf{I}\}$. We define the *slack of constraint* $(\mathbf{k}, \mathbf{j})$ under an allocation x as

$$S_{\mathbf{k},\mathbf{j}}(x) = d\left(\theta^{\mathbf{k}}, x^{\mathbf{j}}\right)^2 - d\left(\theta^{\mathbf{k}}, x^{\mathbf{k}}\right)^2.$$

Clearly, the IC constraints are satisfied if and only if $S_{\mathbf{k},\mathbf{j}}(x) \geq 0$ for every $\mathbf{k}$ and $\mathbf{j}$.

In what follows we assume that z is an allocation that is aligned with the agent's preference across dimensions 2 through n. Note that an allocation such that $x_i = \theta_i$ or $x_i = y_i^P(\theta_i)$ for every $\theta \in \Theta$ and $i \in \{2, \ldots, n\}$ is aligned with the agent's preference in those dimensions.

Definition 3. We say that two index points $\mathbf{j}, \mathbf{k} \in \mathbf{I}$, with $\mathbf{k}_1 = \mathbf{j}_1$, are *adjacent* if $\mathbf{k} \neq \mathbf{j}$ and $\sum_{i \in \{2,\ldots,n\}} |\mathbf{j}_i - \mathbf{k}_i| = 1$. If $\mathbf{k}$ and $\mathbf{j}$ are adjacent we call $(\mathbf{j}, \mathbf{k})$ and $(\mathbf{k}, \mathbf{j})$ *adjacent constraints.*

That is to say, two index points are adjacent if they only differ in only one dimension $i \in \{2, \ldots, n\}$, and their types are adjacent in the dimension in which they differ.

The set of adjacent constraints is denoted $\mathbf{C}^{ad}$.

Lemma 1. *Let* $(\mathbf{k}, \mathbf{j}) \in \mathbf{C}^{ad}$. *If* $S_{\mathbf{k},\mathbf{j}}(x) = \eta$, *with* $\eta \in \left(0, \left(\theta_i^{\mathbf{k}} - \theta_i^{\mathbf{j}}\right)\left(x_i^{\mathbf{k}} - x_i^{\mathbf{j}}\right)\right)$, *then* $S_{\mathbf{j},\mathbf{k}}(x) \geq 0$.

Proof of Lemma 1. Let m be the dimension in which $\mathbf{k}_i \neq \mathbf{j}_i$, where $i \in \{2, \ldots, n\}$. Note that

$$\frac{\partial \left[d\left(\left(y_{-i}^A, y_i\right), x^{\mathbf{k}}\right)^2 - d\left(\left(y_{-i}^A, y_i\right), x^{\mathbf{j}}\right)^2 \right]}{\partial y_i} = 2\left(x_i^{\mathbf{j}} - x_i^{\mathbf{k}}\right). \tag{1}$$

Since $\theta^{\mathbf{k}}$ and $\theta^{\mathbf{j}}$ differ only in dimension i, by integrating this derivative from $\theta^{\mathbf{k}}$ to $\theta^{\mathbf{j}}$ we obtain

$$\begin{aligned} S_{\mathbf{j},\mathbf{k}}(x) &= d\left(\theta^{\mathbf{j}}, x^{\mathbf{k}}\right)^2 - d\left(\theta^{\mathbf{j}}, x^{\mathbf{j}}\right)^2 \\ &= -S_{\mathbf{k},\mathbf{j}}(x) + \int_{\theta_i^{\mathbf{k}}}^{\theta_i^{\mathbf{j}}} 2\left(x_i^{\mathbf{j}} - x_i^{\mathbf{k}}\right) dy_i. \end{aligned}$$

Because z is aligned with the agent's preferences in dimension i

$$\left(\theta_i^{\mathbf{k}} - \theta_i^{\mathbf{j}}\right)\left(x_i^{\mathbf{k}} - x_i^{\mathbf{j}}\right) > 0. \tag{2}$$

Therefore, we conclude

$$S_{\mathbf{j},\mathbf{k}}(x) \quad \geq \quad \eta. \quad \square$$

□

The set of constraints between types that differ in only one dimension $i \in \{2, \ldots, n\}$ (but not necessarily adjacent in the dimension in which they differ) is defined as,

$$\mathbf{C}^{ad+} = \{(\mathbf{k}, \mathbf{j}) | \exists r \in \{2, \ldots, n\}, \mathbf{k}_r \neq \mathbf{j}_r, \mathbf{k}_s = \mathbf{j}_s \text{ for } s \in \{1, \ldots, n\} \backslash \{r\}\}.$$

Lemma 2. *If all constraints in* $\mathbf{C}^{ad}$ *then all constraints in* $\mathbf{C}^{ad+}$ *hold.*

Proof of Lemma 2. We show the following claim which establishes the desired result: For fixed $r \in \{2, \ldots, n\}$, if $\mathbf{k}, \mathbf{j}, \mathbf{m} \in \mathbf{I}$ are such that $\mathbf{k}_r > \mathbf{j}_r > \mathbf{m}_r$ and $\mathbf{k}_s = \mathbf{j}_s = \mathbf{m}_s$ for $s \in \{2, \ldots n\}/\{r\}$, then, the constraints $(\mathbf{k}, \mathbf{m})$ and $(\mathbf{m}, \mathbf{k})$ are satisfied if the constraints $(\mathbf{k}, \mathbf{j})$, $(\mathbf{j}, \mathbf{m})$, $(\mathbf{m}, \mathbf{j})$, and $(\mathbf{j}, \mathbf{k})$ are satisfied.

We show the claim for $(\mathbf{k}, \mathbf{m})$ (the argument for $(\mathbf{m}, \mathbf{k})$ is similar). Note that

$$d(\theta^{\mathbf{k}}, x^{\mathbf{m}})^2 - d(\theta^{\mathbf{k}}, x^{\mathbf{k}})^2 \quad = \quad d(\theta^{\mathbf{k}}, x^{\mathbf{m}})^2 - d(\theta^{\mathbf{k}}, x^{\mathbf{j}})^2 + d(\theta^{\mathbf{k}}, x^{\mathbf{j}})^2 - d(\theta^{\mathbf{k}}, x^{\mathbf{k}})^2. \quad (3)$$

In addition, by incentive compatibility

$$\begin{aligned} d(\theta^{\mathbf{k}}, x^{\mathbf{j}})^2 - d(\theta^{\mathbf{k}}, x^{\mathbf{k}})^2 &\geq 0, \\ d(\theta^{\mathbf{j}}, x^{\mathbf{m}})^2 - d(\theta^{\mathbf{j}}, x^{\mathbf{j}})^2 &\geq 0. \end{aligned}$$

Observe that

$$\begin{aligned} & d(\theta^{\mathbf{k}}, x^{\mathbf{m}})^2 - d(\theta^{\mathbf{k}}, x^{\mathbf{j}})^2 \\ = \quad & d(\theta^{\mathbf{j}}, x^{\mathbf{m}})^2 - d(\theta^{\mathbf{j}}, x^{\mathbf{j}})^2 + \int_{\theta_r^{\mathbf{j}}}^{\theta_r^{\mathbf{k}}} 2\left(x_r^{\mathbf{j}} - x_r^{\mathbf{m}}\right) dy_r \\ \geq \quad & d(\theta^{\mathbf{j}}, x^{\mathbf{m}})^2 - d(\theta^{\mathbf{j}}, x^{\mathbf{j}})^2, \end{aligned}$$

where the equality follows from (1) and the inequality from, $\theta_r^{\mathbf{k}} > \theta_r^{\mathbf{j}} > \theta_r^{\mathbf{m}}$ and (2), as x is aligned with the agent's preference in dimension r. From (3), we conclude that the IC constraint $(\mathbf{k}, \mathbf{m})$ is satisfied. $\square$

Thus, Lemma 2 implies that if x is weakly aligned with the agent's preferences in dimensions 2 through n and all the adjacent constraints hold then any constraint $(\mathbf{k}, \mathbf{j})$ in which $\mathbf{k}$ and $\mathbf{j}$ differ in only one dimension also holds.

Definition 4. We say that a constraint $(\mathbf{k}, \mathbf{j})$ is *diagonal* if $(\mathbf{k}, \mathbf{j}) \in \mathbf{C} \backslash \mathbf{C}^{ad+}$.

Definition 5. We say an allocation is *independent of dimension 1* if $\mathbf{k}_i = \mathbf{j}_i$ for $i \in \{2, \ldots, n\}$ implies $x^{\mathbf{k}} = x^{\mathbf{j}}$.

Lemma 3. *Suppose x is an allocation that is independent of dimension 1. If all adjacent constraints hold under allocation x, then all diagonal constraints hold as well.*

Proof. Let $(\mathbf{k}, \mathbf{m})$ be a diagonal constraint with $\mathbf{k}_1 = \mathbf{m}_1$. For $i \in \{1, \ldots, n\}$ define $\mathbf{j}^i_s$ for $s \in \{1, \ldots, n\}$ as $\mathbf{j}^i_s = \mathbf{k}_s$ for $s \in \{i+1, \ldots, n\}$ and $\mathbf{j}^i_s = \mathbf{m}^i_s$ for $s \in \{1, \ldots, i\}$. Thus, $\mathbf{j}^1 = \mathbf{k}$, $\mathbf{j}^n = \mathbf{m}$. Also, $(\mathbf{j}^i, \mathbf{j}^{i+1}) \in \mathbf{C}^{ad+}$ since indices $\mathbf{j}^i$ and $\mathbf{j}^{i+1}$ differ at most in dimension $i+1$. By hypothesis,

$$
\begin{aligned}
0 \quad \leq \quad & S_{\mathbf{j}^i, \mathbf{j}^{i+1}}(x) = d\left(\theta^{\mathbf{j}^i}, x^{\mathbf{j}^{i+1}}\right)^2 - d\left(\theta^{\mathbf{j}^i}, x^{\mathbf{j}^i}\right)^2 \\
= \quad & d\left(\theta^{\mathbf{k}}, x^{\mathbf{j}^{i+1}}\right)^2 - d\left(\theta^{\mathbf{k}}, x^{\mathbf{j}^i}\right)^2 + \sum_{r=2}^{i} \int_{\theta_r^{\mathbf{j}^{r-1}}}^{\theta_r^{\mathbf{j}^r}} \frac{\partial \left[d\left((y_r, \theta^{\mathbf{j}^r}_{-r}), x^{\mathbf{j}^{i+1}}\right)^2 - d\left((y_r, \theta^{\mathbf{j}^r}_{-r}), x^{\mathbf{j}^i}\right)^2 \right]}{\partial y_r} dy_r, \\
= \quad & d\left(\theta^{\mathbf{k}}, x^{\mathbf{j}^{i+1}}\right)^2 - d\left(\theta^{\mathbf{k}}, x^{\mathbf{j}^i}\right)^2 + \sum_{r=2}^{i} 2\left(\theta_r^{\mathbf{j}^r} - \theta_r^{\mathbf{j}^{r-1}}\right)\left(x_r^{\mathbf{j}^i} - x_r^{\mathbf{j}^{i+1}}\right) \\
= \quad & d\left(\theta^{\mathbf{k}}, x^{\mathbf{j}^{i+1}}\right)^2 - d\left(\theta^{\mathbf{k}}, x^{\mathbf{j}^i}\right)^2,
\end{aligned}
$$

where the last inequality follows from $\mathbf{j}^i_r = \mathbf{j}^{i+1}_r$ for $r \leq i$ and x weakly aligned with the agent's preferences, which implies $x_r^{\mathbf{j}^i} = x_r^{\mathbf{j}^{i+1}}$ for $r \in \{2, \ldots, i\}$.[9]

[9]To understand the first equality note that for any differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ $f(x_1, \ldots, x_n) = f(x^0_1, \ldots, x^0_n) + \sum_{i=1}^n \int_{x_{i0}}^{x_i} f_{x_i}(x^0_1, \ldots, x^0_{i-1}, \tilde{x}_i, x_{i+1}, \ldots, x_n)\, d\tilde{x}_i$

Now,

$$S_{\mathbf{k},\mathbf{m}}(x) = d\left(\theta^{\mathbf{k}}, x^{\mathbf{j}^n}\right)^2 - d\left(\theta^{\mathbf{k}}, x^{\mathbf{j}^1}\right)^2 = \sum_{i=1}^{n-1}\left(d\left(\theta^{\mathbf{k}}, x^{\mathbf{j}^{i+1}}\right)^2 - d\left(\theta^{\mathbf{k}}, x^{\mathbf{j}^i}\right)^2\right) \tag{4}$$
$$= \sum_{i=1}^{n-1} S_{\mathbf{j}^i,\mathbf{j}^{i+1}}(x) \geq 0.$$

This las inequality shows that constraint $(\mathbf{k}, \mathbf{m})$ is satisfied.

Finally, if $\mathbf{k}_1 \neq \mathbf{j}_1$, IC constraint $(\mathbf{k}, (\mathbf{k}_1, \mathbf{j}_{-1}))$ implies constraint $(\mathbf{k}, \mathbf{j})$ as the allocations for types $(\mathbf{k}_1, \mathbf{j}_{-1})$ and $\mathbf{j}$ are identical under x by independence of dimension 1. □

Define $\mathbf{C}^{ad}_{+1} = \{(\mathbf{k}, \mathbf{j}) \in \mathbf{C}^{ad} | \mathbf{j}_2 = \mathbf{k}_2 + 1\}$ and $\mathbf{C}^{ad}_{=1} = \{(\mathbf{k}, \mathbf{j}) \in \mathbf{C}^{ad} | \mathbf{j}_2 = \mathbf{k}_2 = 1\}$.

Lemma 4. *Suppose that the allocation x is independent of dimension 1 and that the following conditions hold*

1. *There is $\eta \in \left(0, \frac{1}{n_2} \min_{(\mathbf{k},\mathbf{j}) \in \mathbf{C}^{ad}_{+}} \left(\theta^{\mathbf{k}}_2 - \theta^{\mathbf{j}}_2\right)\left(x^{\mathbf{k}}_2 - x^{\mathbf{j}}_2\right)\right)$ such that $S_{\mathbf{k},\mathbf{j}}(x) \in [0, \eta]$ for $(\mathbf{k}, \mathbf{j}) \in \mathbf{C}^{ad}_{+1}$;*
2. *$S_{\mathbf{k},\mathbf{j}}(x) \geq n_2\eta$ for $(\mathbf{k}, \mathbf{j}) \in \mathbf{C}^{ad}_{=1}$.*

then x is incentive compatible.

Proof. We will show inductively that all the adjacent constraints hold, which, by lemmas 2 and 3, implies that the allocation is incentive compatible.

Define $\mathbf{C}^{ad}_{=m} = \{(\mathbf{k}, \mathbf{j}) \in \mathbf{C}^{ad} | \mathbf{j}_2 = \mathbf{k}_2 = m\}$. We need to show that $\mathbf{C}^{ad}_{=m}$ holds for every $m \in \{2, \ldots, n_2\}$.

Let $(\mathbf{k}, \mathbf{j}) \in \mathbf{C}^{ad}$. Note that the assumptions imply that $(\mathbf{k}, \mathbf{j})$ holds if $\mathbf{k}_2 = 1$. Suppose $S_{\mathbf{k},\mathbf{j}}(x) \geq (n_2 - (m-1))\eta$ for $(\mathbf{k}, \mathbf{j}) \in \mathbf{C}^{ad}_{=\tilde{m}}$ for $\tilde{m} \leq m$. We will show that $S_{\mathbf{k},\mathbf{j}}(x) \geq (n_2 - m)\eta$ for $(\mathbf{k}, \mathbf{j}) \in \mathbf{C}^{ad}_{=m+1}$.

Let $(\mathbf{k}, \mathbf{j}) \in \mathbf{C}^{ad}$ be such that $\mathbf{k}_2 = m + 1$. If $\mathbf{j}_2 = m$, then $(\mathbf{k}, \mathbf{j})$ holds by Lemma 1 as $S_{\mathbf{j},\mathbf{k}}(x) = \eta \leq \left(\theta^{\mathbf{k}}_2 - \theta^{\mathbf{j}}_2\right)\left(x^{\mathbf{k}}_2 - x^{\mathbf{j}}_2\right)$. If $\mathbf{j}_2 \neq m$, because $(\mathbf{k}, \mathbf{j})$ is an adjacent constraint, we must have $\mathbf{k}_2 = \mathbf{j}_2 = m + 1$. Let $\bar{\mathbf{k}}, \bar{\mathbf{j}} \in \mathbf{I}$ be such that $\bar{\mathbf{k}}_2 = \bar{\mathbf{j}}_2 = m$ and $\bar{\mathbf{k}}_s = \mathbf{k}_s, \bar{\mathbf{j}}_s = \mathbf{j}_s$ for $s \in \{3, \ldots, n\}$. Then by equation (4) in the proof of Lemma

3 we have $S_{\bar{\mathbf{k}},\mathbf{j}}(x) = S_{\bar{\mathbf{k}},\bar{\mathbf{j}}}(x) + S_{\bar{\mathbf{j}},\mathbf{j}}(x) = S_{\bar{\mathbf{k}},\mathbf{k}}(x) + S_{\mathbf{k},\mathbf{j}}(x)$. We Since $(\bar{\mathbf{j}}, \mathbf{j}) \in \mathbf{C}^{ad}_{+1}$ and $S_{\bar{\mathbf{k}},\bar{\mathbf{j}}}(x) \geq (n_2 - m + 1)\eta$ (by the induction hypothesis) we obtain $S_{\bar{\mathbf{k}},\mathbf{j}}(x) \geq (n_2 - m + 1)\eta$. Therefore, $S_{\bar{\mathbf{k}},\mathbf{k}}(x) \leq \eta$ implies $S_{\mathbf{k},\mathbf{j}}(x) \geq (n_2 - m)\eta$. □

Proof of Theorem 1.

By dividing by $\bar{y}_1^P = \mathbb{E}\left(y_1^{P,\mathbf{k}}\right)$, the IC constraints can be written as

$$-d\left(\tilde{y}_i^{A,\mathbf{k}}, z^{\mathbf{k}_\sigma}\right) \geq -d\left(\tilde{y}_i^{A,\mathbf{k}}, z^{\mathbf{j}}\right) \qquad (\mathbf{k}, \mathbf{j})$$

for each $\mathbf{k}, \mathbf{j} \in \mathbf{I}$ and $\mathbf{k}_\sigma \in \mathbf{I}_{\sigma^A}(\mathbf{k}) = \{\mathbf{j} \in \mathbf{I} | \sigma^A(\theta^{\mathbf{k}})(\theta^{\mathbf{j}}) > 0\}$ and $\tilde{y}_i^{A,\mathbf{k}} = \frac{\theta_i^{\mathbf{k}}}{\bar{y}_1^P}$, where z is a a *normalized allocation* (that is to say $x = \bar{y}_1^P \cdot z$ is an allocation of the original problem). The set of normalized allocations is called Z. We also define the principal's normalized bliss points as $\tilde{y}^{P,\mathbf{k}} = \frac{y^{P,\mathbf{k}}}{\bar{y}_1^P}$.

Without loss of generality we may assume that $\min_{\theta_1 \in \Theta_1} \theta_1 = 0$. Under this normalization $b_1(\Theta) \to \infty$ implies $\bar{y}_1^P \to \infty$ as well.

Define $\mathbf{C}^{ad,1} = \mathbf{C}^{ad} \cap \{(\mathbf{k}, \mathbf{j}) | \mathbf{k}_1 = \mathbf{j}_1 = 1\}$, $\mathbf{C}^{ad,1}_{+1} = \mathbf{C}^{ad}_{+1} \cap \{(\mathbf{k}, \mathbf{j}) | \mathbf{k}_1 = \mathbf{j}_1 = 1\}$, $\mathbf{C}^{ad,1}_{=1} = \mathbf{C}^{ad}_{=1} \cap \{(\mathbf{k}, \mathbf{j}) | \mathbf{k}_1 = \mathbf{j}_1 = 1\}$, and $\mathbf{I}^{\mathbf{1}} \equiv \{\mathbf{r} \in \mathbf{I} | \mathbf{r}_1 = 1\}$. These sets of constraints are the subsets of $\mathbf{C}^{ad,1}$, $\mathbf{C}^{ad,1}_{+1}$ and $\mathbf{C}^{ad,1}_{=1}$, respectively, involving types with $\theta_1 = \theta_1^1$ (the smallest type in dimension 1).

The following Lemma shows that we can find an allocation that is incentive compatible for players of type θ_1^1 (by showing that it satisfies the conditions in Lemma 4), that provides the agent his preferred allocation in dimensions 2 through n and is such that allocation in dimension 1 can be arbitrarily close to the the principal's preferred allocation $\bar{y}_1^P$ as $b_1(\Theta)$ converges to ∞.

Lemma 5. *Fix Θ_i and y_i^P for $i \in \{2, \ldots, n\}$. For every $\nu \in (0, 1)$ there is $\bar{b} > 0$ and constants η with $\eta \in \left(0, \frac{1}{n_2} \min_{(\mathbf{k},\mathbf{j}) \in \mathbf{C}^{ad}_{+}} \left(\theta_2^{\mathbf{k}} - \theta_2^{\mathbf{j}}\right)\left(x_2^{\mathbf{k}} - x_2^{\mathbf{j}}\right)\right)$ and $\alpha \in \left(0, \frac{1}{n_2} \min_{(\mathbf{k},\mathbf{j}) \in \mathbf{C}^{ad}_{+}} \left(\theta_2^{\mathbf{k}} - \theta_2^{\mathbf{j}}\right)\left(x_2^{\mathbf{k}} - x_2^{\mathbf{j}}\right) - \eta\right)$, such that for each $b_1(\Theta) \geq \bar{b}$ there is a normalized allocation $z = \left(z_i^{\mathbf{k}}\right)_{i \in \{2,\ldots,n\}, \mathbf{k} \in \mathbf{I}}$ such that*

1. *$z_i^{\mathbf{k}} = \tilde{y}_i^{A,\mathbf{k}}$ for $i \in \{2, \ldots, n\}$ and $\mathbf{k} \in \mathbf{I}^{\mathbf{1}}$,*

2. $S_{\mathbf{k},\mathbf{j}}(z) = \eta / \left(\bar{y}_1^P\right)^2$ *for* $(\mathbf{k},\mathbf{j}) \in \mathbf{C}^{ad,1}_{+1}$,

3. $S_{\mathbf{k},\mathbf{j}}(z) \geq n_2 \eta / \left(\bar{y}_1^P\right)^2 + \alpha / \left(\bar{y}_1^P\right)^2$ *for* $(\mathbf{k},\mathbf{j}) \in \mathbf{C}^{ad,1}_{=1}$, *and*

4. $\left|z_1^{\mathbf{k}} - 1\right| < \frac{\nu}{\left|\bar{y}_1^P\right|}$ *for* $\mathbf{k} \in \mathbf{I^1}$.

Proof of Lemma 5.

We define $z_i^{\mathbf{k}} = \tilde{y}_i^{A,\mathbf{k}}$, for $i \in \{2,\ldots,n\}$ and set $z_1^{\mathbf{k}} = 1$ for every $\mathbf{k} \in \mathbf{I^1} \cap \{\mathbf{j} | \mathbf{j}_2 = 1\}$.

Let $(\mathbf{k},\mathbf{j}) \in \mathbf{C}^{ad,1}_{+1}$ with $\mathbf{k}_2 = 1$. We define $z_1^{\mathbf{j}}$ so that $S_{\mathbf{k},\mathbf{j}}(z) = \tilde{\eta} = \eta / \bar{y}_1^P$ for $(\mathbf{k},\mathbf{j}) \in \mathbf{C}^{ad,1}_{+1}$. Then, $z_1^{\mathbf{j}}$ that must satisfy the following,

$$-1 = -\left(z_1^{\mathbf{k}}\right)^2 = -\left(z_1^{\mathbf{j}}\right)^2 - \sum_{i=2}^{n} \left(\tilde{y}_i^{A,\mathbf{k}} - z_i^{\mathbf{j}}\right)^2 + \tilde{\eta} = -\left(z_1^{\mathbf{j}}\right)^2 - \left(\tilde{y}_2^{A,\mathbf{k}} - \tilde{y}_2^{A,\mathbf{j}}\right)^2 + \tilde{\eta}$$

Inductively, for $(\mathbf{k},\mathbf{j}) \in \mathbf{C}^{ad}_{+1} \cap \{(\mathbf{k},\mathbf{j}) | \mathbf{k}_1 = \mathbf{j}_1 = 1\}$ with $\mathbf{k}_2 = m-1$ and $\mathbf{j}_2 = m$ we can write

$$\left(z_1^{\mathbf{j}}\right)^2 = \left(z_1^{\mathbf{k}}\right)^2 - \left(\tilde{y}_2^{A,\mathbf{k}} - \tilde{y}_2^{A,\mathbf{j}}\right)^2 + \tilde{\eta},$$

which yields for every $\mathbf{k} \in \mathbf{I}$,

$$\left(z_1^{\mathbf{k}}\right)^2 = 1 + m\tilde{\eta} - \sum_{j=1}^{m} \left(\tilde{y}_2^{A,(j,\mathbf{k}_{-2})} - \tilde{y}_2^{A,(j+1,\mathbf{k}_{-2})}\right)^2,$$

where $(j, \mathbf{k}_{-2})$ denotes the index in which the second coordinate is j and all other coordinates are that of index $\mathbf{k}$. For small enough $\tilde{\eta}$ these equations have a solution.

By construction $S_{\mathbf{k},\mathbf{j}}(z) = \tilde{\eta}$ for $(\mathbf{k},\mathbf{j}) \in \mathbf{C}^{ad,1}_{+1}$. For $(\mathbf{k},\mathbf{j}) \in \mathbf{C}^{ad,1}_{=1}$, since $\mathbf{k}_2 = \mathbf{j}_2 = 1$, we obtain $z_{-1}^{\mathbf{k}} = \tilde{y}_{-1}^{A,\mathbf{k}}$ and $z_{-1}^{\mathbf{j}} = \tilde{y}_{-1}^{A,\mathbf{j}}$ and $z_1^{\mathbf{k}} = z_1^{\mathbf{j}} = 1$, and, therefore, for η and α small enough $\min\{S_{\mathbf{k},\mathbf{j}}(z) | (\mathbf{k},\mathbf{j}) \in \mathbf{C}^{ad,1}_{=1}\} > n_2\tilde{\eta} + \alpha / \bar{y}_1^P$.

Note that $\tilde{y}_2^{A,\mathbf{k}}$ is of the order of $\left(\bar{y}_1^P\right)^{-1}$, for each $\mathbf{k} \in \mathbf{I}$ and $\tilde{\eta}$ is of the order of $\left(\bar{y}_1^P\right)^{-2}$. Thus, for $\nu \in (0,1)$ and $\left|\bar{y}_1^P\right| \geq 1$, there exists $\bar{b}$, such that for all $b_1(\Theta) > \bar{b}$,

$$\sum_{j=1}^{m}\left(\tilde{y}_2^{A,(j,\mathbf{k}_{-2})}-\tilde{y}_2^{A,(j+1,\mathbf{k}_{-2})}\right)^2-m\tilde{\eta}<\frac{\nu}{|\bar{y}_1^P|}<1-\left(1-\frac{\nu}{|\bar{y}_1^P|}\right)^2.$$

Thus, the desired result follows. □

Lemma 6. *Fix Θ_i and y_i^P for $i\in\{2,\ldots,n\}$. For every $\nu\in(0,1)$ there is $\bar{b}>0$ and constants η with $\eta\in\left(0,\frac{1}{n_2}\min_{(\mathbf{k},\mathbf{j})\in\mathbf{C}_+^{ad}}\left(\theta_2^{\mathbf{k}}-\theta_2^{\mathbf{j}}\right)\left(x_2^{\mathbf{k}}-x_2^{\mathbf{j}}\right)\right)$ and $\alpha\in\left(0,\frac{1}{n_2}\min_{(\mathbf{k},\mathbf{j})\in\mathbf{C}_+^{ad}}\left(\theta_2^{\mathbf{k}}-\theta_2^{\mathbf{j}}\right)\left(x_2^{\mathbf{k}}-x_2^{\mathbf{j}}\right)-\eta\right)$, such that for each $b_1(\Theta)\geq\bar{b}$ there is a normalized allocation $z=\left(z_i^{\mathbf{k}}\right)_{i\in\{2,\ldots,n\},\mathbf{k}\in\mathbf{I}}$ such that*

1. $z_i^{\mathbf{k}}=\tilde{y}_i^{P,\mathbf{k}}$ for $i\in\{2,\ldots,n\}$ and $\mathbf{k}\in\mathbf{I^1}$,
2. $S_{\mathbf{k},\mathbf{j}}(z)=\eta/\left(\bar{y}_1^P\right)^2$ for $(\mathbf{k},\mathbf{j})\in\mathbf{C}_{+1}^{ad,1}$,
3. $S_{\mathbf{k},\mathbf{j}}(z)\geq n_2\cdot\eta/\left(\bar{y}_1^P\right)^2+\alpha/\left(\bar{y}_1^P\right)^2$ for $(\mathbf{k},\mathbf{j})\in\mathbf{C}_{=1}^{ad,1}$, and
4. $\left|z_1^{\mathbf{k}}-1\right|<\frac{\nu}{|\bar{y}_1^P|}$ for $\mathbf{k}\in\mathbf{I^1}$.

Lemma 6 shows that there is an allocation that is incentive compatible for players of type θ_1^1 (by showing that it satisfies the conditions in Lemma 4), that implements *the principal's* preferred allocation in dimensions 2-n and is such that allocation in dimension 1 can be arbitrarily close to the the principal's preferred allocation $\bar{y}_1^P$ as $b_1(\Theta)$ converges to ∞.

The argument uses the Implicit Function Theorem, "around" an allocation that provides the agent's preferences in dimensions 2-n, and satisfies conditions 2) and 3) (which we can find from Lemma 5). The IFT implies we can find an incentive compatible allocation that also satisfies conditions 2) and 3) that provides any allocation in dimensions 2-n that is within a vicinity of the agent's preferences. This vicinity expands as $b_1(\Theta)\to\infty$ and as it expands, it eventually covers the principal's preferences in dimensions 2-n.

To understand the argument in more detail, consider Figure 2 (left), which represents a setting with $n=3$ and $\Theta_1=\{\theta_1\}$. The dots represent allocations in dimension

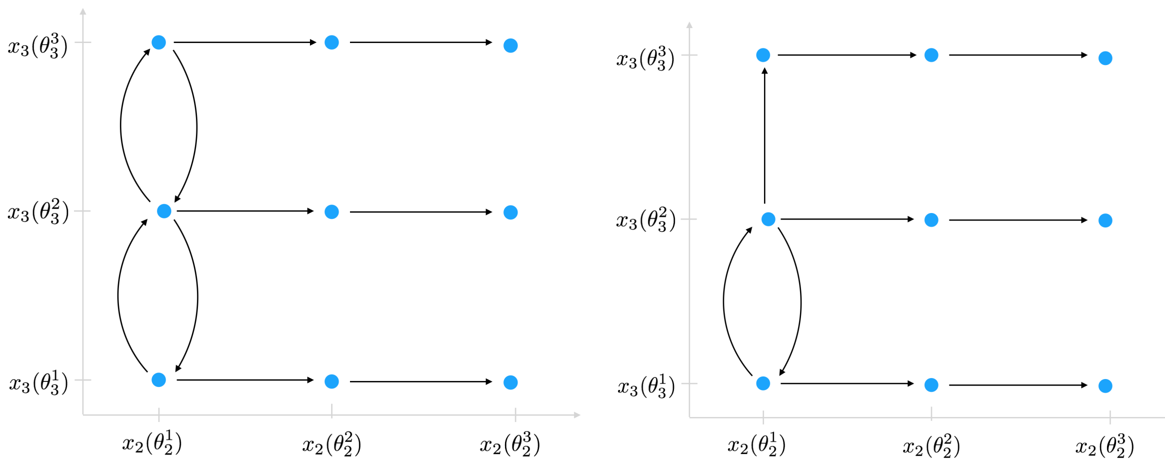

Figure 2: Illustration of argument. Step 1 (left): set constraints in $\mathbf{C}^{ad}_{=1}$ to be slack (curved lines) and those in $\mathbf{C}^{ad}_{+1}$ (straight lines) to be satisfied with equality. Step 2 (right): set constraints that bind "first" to equality.

2 and 3 (dimension 1 allocations are not in the picture). The curved arrows represent constraints in $\mathbf{C}^{ad}_{=1}$ and the straight arrows arrows constraints in $\mathbf{C}^{ad}_{+1}$. The arrows point towards the type to which the agent may deviate. From Lemma 4 if an allocation satisfies the constraints corresponding to the straight arrows with slack less than or equal than a small enough η and those that correspond to the curved arrows with slack greater or equal than $n_2\eta$ then the allocation is IC. An allocation z^1 from Lemma 5 satisfies these conditions.

In order to apply the IFT we start by defining a function g^1 that takes a normalized allocation and some auxiliary valuables and delivers a function of the slack of constraints in $\mathbf{C}^{ad}_{=1} \cup \mathbf{C}^{ad}_{+1}$. When this function is zero the slack in constraints in $\mathbf{C}^{ad}_{+1}$ is exactly η while the slack in constraints in $\mathbf{C}^{ad}_{+1}$ may greater or less than $n_2\eta$ (the auxiliary variables represent the value of these slacks). Using the IFT, for large enough $b_1(\Theta)$, starting from z^1 there is a differentiable function $z(t), t \in [0,1]$ such that $z_{2:n}(0) = \left(\tilde{y}^A\right)_{2:n}$, $z_{2:n}(1) = \left(\tilde{y}^P\right)_{2:n}$ and such that for every $t \in [0,1]$ all the constraints in $\mathbf{C}^{ad}_{+1}$ are set to η.[10]. If at $z(1)$ the constraints have slack greater or equal than $n_2\eta$ we have proven our theorem. If not, since $z(t)$ is continuous and at $z(0)$ the constraints in $\mathbf{C}^{ad}_{=1}$ have slack at least $n_2\eta$ there must be a time t^2 at which

[10] If x is an allocation, $x_{2:n}$ denotes $\left\{(x_2^{\mathbf{k}}, \ldots, x_n^{\mathbf{k}})\right\}_{\mathbf{k}\in\mathbf{I}}$.

in allocation $z(t^2)$ the slack of a subset of constraints in $\mathbf{C}^{ad}_{=1}$ first becomes equal to $\eta \cdot n_2$. We can then define a new function g^2, that is just like g^1, except that at a zero the slack of constraints in $\mathbf{C}^{ad}_{=1}$ that bind at t^2 is exactly $n_2\eta$. This step is illustrated in figure 2 (right). Some curved arrows in $\mathbf{C}^{ad}_{=1}$ are now straight, representing the constraints that are now required to be satisfied with equality. This step may deliver a solution, if not we apply the IFT again and find yet another allocation $z(t^3)$. Since $\mathbf{C}^{ad}_{+1}$ is finite, and we set at least one constraint to equality in each round, the recursion must deliver the desired allocation in finitely many steps.

Proof of Lemma 6.

Let $\tilde{y}_i^I$ for $I \in \{A, P\}$ denote the vector $(\tilde{y}_i^{i,\mathbf{k}})_{\mathbf{k}\in\mathbf{I}}$ and define $z_i^{A,P}(t) = \tilde{y}_i^A(1-t) + \tilde{y}_i^P t$, where $\tilde{y}_i^A$ and $\tilde{y}_i^P$ are the vectors of normalized dimension-i bliss points indexed by $\mathbf{k} \in \mathbf{I}$. $z_i^{A,P}(t)$ is a vector indexed by the set of types $\mathbf{I}$. Clearly $z_i^{A,P}(0) = \tilde{y}_i^A$ and $z_i^{A,P}(1) = \tilde{y}_i^P$ for $i \in \{2, \ldots, n\}$. Note that an allocation z such that $z_i = z_i(t)$ for $i \in \{2, \ldots, n\}$ is aligned with the agent's preferences in dimensions 2 through n.

Let $b_1(\Theta)$ be at least large enough that $\min_{\mathbf{j}\in\mathbf{I}} |\tilde{y}_1^{\mathbf{A},\mathbf{j}} - 1| > 2/3$ and $\left|\bar{y}_1^P\right| \geq 1$.

Fix constants $\eta, \alpha, \nu > 0$ with $\eta \in \left(0, \frac{1}{n_2} \min_{(\mathbf{k},\mathbf{j})\in\mathbf{C}^{ad}_+} \left(\theta_2^{\mathbf{k}} - \theta_2^{\mathbf{j}}\right)\left(x_2^{\mathbf{k}} - x_2^{\mathbf{j}}\right)\right)$ and $\alpha \in \left(0, \frac{1}{n_2} \min_{(\mathbf{k},\mathbf{j})\in\mathbf{C}^{ad}_+} \left(\theta_2^{\mathbf{k}} - \theta_2^{\mathbf{j}}\right)\left(x_2^{\mathbf{k}} - x_2^{\mathbf{j}}\right) - \eta\right)$ and let $\tilde{\eta} \equiv \eta / \left(\bar{y}_1^P\right)^2$ and $\tilde{\alpha} \equiv \eta / \left(\bar{y}_1^P\right)^2$

We now provide a recursive argument that shows that the desired allocation exists for large enough $b_1(\Theta)$.

k'th Inductive Step

Let $k \leq |\mathbf{C}^{ad}_{=1}|$, assume that there are sets $\mathbf{C}^k_{sl}, \mathbf{C}^k_{=1,eq}, \mathbf{C}^k_{+1,eq}, \mathbf{C}^k_{eq} \subseteq \mathbf{C}^{ad,1}$, $\mathbf{I}^k_{fix} \subseteq \{\mathbf{r} \in \mathbf{I} | \mathbf{r}_1 = \mathbf{r}_2 = 1\}$ and there are an allocation $z^k \in Z$ and $t^k \in (0, 1)$ such that the following four conditions hold:

1. $\mathbf{I^1} = \mathbf{I}^k_{fix} \cup \{\mathbf{k} \in \mathbf{I} | (\mathbf{k}, \mathbf{j}) \in \mathbf{C}^k_{=1,eq}, \mathbf{j} \in \mathbf{I}\} \cup \{\mathbf{j} \in \mathbf{I} | (\mathbf{k}, \mathbf{j}) \in \mathbf{C}^k_{+1,eq}, \mathbf{k} \in \mathbf{I}\}$
2. The slack constraints are such that
   (a) $S_{\mathbf{k},\mathbf{j}}(z^k) \geq n_2\tilde{\eta} + \tilde{\alpha}$ if $(\mathbf{k}, \mathbf{j}) \in \mathbf{C}^k_{sl}$,
   (b) $S_{\mathbf{k},\mathbf{j}}(z^k) = n_2\tilde{\eta} + \tilde{\alpha}$ if $(\mathbf{k}, \mathbf{j}) \in \mathbf{C}^k_{=1,eq}$
   (c) $S_{\mathbf{k},\mathbf{j}}(z^k) = \tilde{\eta}$ if $(\mathbf{k}, \mathbf{j}) \in \mathbf{C}^k_{+1,eq}$ and,

(d) $z_1^{k,\mathbf{j}} = 1$ if $\mathbf{j} \in \mathbf{I}^k_{fix}$.

3. $z_i^{k,\mathbf{j}} = z_i^{A,P,\mathbf{j}}(t^k)$, for $\mathbf{j} \in \mathbf{I}$ and $i \in \{2,\dots,n\}$ where $z_i^{A,P,\mathbf{j}}(t^k)$ is the $\mathbf{j}$'th element of $z_i^{A,P}(t)$, and

4. $\max_{\mathbf{j}\in\mathbf{I}} \left| z_1^{k,\mathbf{j}} - 1 \right| < \frac{k}{3|\mathbf{C}^{ad}_{=1}|} \cdot \frac{\nu}{\left|\bar{y}_1^P\right|}$,

For the first step of the induction, we set $\mathbf{C}^1_{sl} = \mathbf{C}^{ad,1}_{=1}$, $\mathbf{C}^1_{=1,eq} = \emptyset$ $\mathbf{C}^1_{eq} = \mathbf{C}^{ad,1}_{+1}$, $\mathbf{I}^1_{fix} = \{\mathbf{r} \in \mathbf{I} | \mathbf{r}_1 = \mathbf{r}_2 = 1\}$ and $t^1 = 0$.[11]

From Lemma 5 there is z^1 that satisfies conditions 1 through 3 for large enough $b_1(\Theta)$.

Let $\mathbf{C}^k_{eq} = \mathbf{C}^k_{=1,eq} \cup \mathbf{C}^k_{+1,eq}$, $\mathbf{I}^k_{=1,eq} = \{\mathbf{k} \in \mathbf{I} | (\mathbf{k},\mathbf{j}) \in \mathbf{C}^k_{=1,eq}, \mathbf{j} \in \mathbf{I}\}$, $\mathbf{I}^k_{+1,eq} = \{\mathbf{j} \in \mathbf{I} | (\mathbf{k},\mathbf{j}) \in \mathbf{C}^k_{+1,eq}, \mathbf{k} \in \mathbf{I}\}$, and $\mathbf{I}^k_{1,eq} = \{\mathbf{k} \in \mathbf{I} | (\mathbf{k},\mathbf{j}) \in \mathbf{C}^k_{eq}, \mathbf{j} \in \mathbf{I}\}$. Because $\mathbf{I}^k_{fix}, \mathbf{I}^k_{=1,eq}, \mathbf{I}^k_{+1,eq}$ and $\mathbf{C}^k_{sl}$ are finite, we can enumerate them and write $\mathbf{I}^k_{fix} = \{\mathbf{r}^m\}_{m=1}^{|\mathbf{I}^k_{fix}|}$, $\mathbf{I}^k_{=1,eq} = \{\mathbf{r}^m\}_{m=|\mathbf{I}^k_{fix}|+1}^{|\mathbf{I}^k_{fix}|+|\mathbf{I}^k_{=1,eq}|}$, $\mathbf{I}^k_{+1,eq} = \{\mathbf{r}^m\}_{m=|\mathbf{I}^k_{fix}|+|\mathbf{I}^k_{=1,eq}|+1}^{|\mathbf{I}^k_{fix}|+|\mathbf{I}^k_{=1,eq}|+|\mathbf{I}^k_{+1,eq}|}$, and $\mathbf{C}^k_{sl} = \{(\mathbf{k}^m,\mathbf{j}^m)\}_{m=1}^{|\mathbf{C}^k_{sl}|}$. From condition 1, $\mathbf{I^1} = \{\mathbf{r}^m\}_{m=1}^{|\mathbf{I^1}|}$ and we can write $\mathbf{C}^k_{eq} = \{(\bar{\mathbf{r}}^m, \bar{\bar{\mathbf{r}}}^m)\}_{m=|\mathbf{I}^k_{fix}|+1}^{|\mathbf{I^1}|}$ where $\bar{\mathbf{r}}^m = \mathbf{r}^m$ if $(\bar{\mathbf{r}}^m, \bar{\bar{\mathbf{r}}}^m) \in \mathbf{C}^k_{=1,eq}$ and $\bar{\bar{\mathbf{r}}}^m = \mathbf{r}^m$ if $(\bar{\mathbf{r}}^m, \bar{\bar{\mathbf{r}}}^m) \in \mathbf{C}^k_{+1,eq}$.

Let $a = \left(\left(z_1^{\mathbf{r}^m}\right)_{m=1}^{|\mathbf{I^1}|}, \left(\varepsilon_{(\mathbf{k}^m,\mathbf{j}^m)}\right)_{m=1}^{|\mathbf{C}^k_{sl}|}\right) \in \mathbb{R}^{|\mathbf{I^1}|+|\mathbf{C}^k_{sl}|}$. We define the function

$$g^k : [t^k, 1] \times \mathbb{R}^{|\mathbf{I^1}|+|\mathbf{C}^k_{sl}|} \to \mathbb{R}^{|\mathbf{I^1}|+|\mathbf{C}^k_{sl}|}, \text{ with } g^k = \left(\left(g^{1,k}_{\mathbf{r}^m}\right)_{m=1}^{|\mathbf{I}^k_{fix}|}, \left(g^{2,k}_{(\bar{\mathbf{r}}^m,\bar{\bar{\mathbf{r}}}^m)}\right)_{m=|\mathbf{I}^k_{fix}|+1}^{|\mathbf{I^1}|}, \left(g^{2,k}_{(\mathbf{k}^m,\mathbf{j}^m)}\right)_{m=1}^{|\mathbf{C}^k_{sl}|}\right)$$

where g^1 and g^2 are defined as

$$g^{1,k}_{\mathbf{r}}(t,a) = z_1^{\mathbf{r}} - 1 \text{for } \mathbf{r} \in \mathbf{I}^k_{fix},$$

and

$$g^{2,k}_{(\mathbf{k},\mathbf{j})}(t,a) = \begin{cases} S_{\mathbf{k},\mathbf{j}}\left(\left(z_1^{\mathbf{h}}, \left(z_i^{A,P,\mathbf{h}}(t)\right)_{i\in\{2,\dots,n\}}\right)_{\mathbf{h}\in\mathbf{I}}\right) - \tilde{\eta} & \text{if } (\mathbf{k},\mathbf{j}) \in \mathbf{C}^k_{+1,eq} \\ S_{\mathbf{k},\mathbf{j}}\left(\left(z_1^{\mathbf{h}}, \left(z_i^{A,P,\mathbf{h}}(t)\right)_{i\in\{2,\dots,n\}}\right)_{\mathbf{h}\in\mathbf{I}}\right) - n_2\tilde{\eta} - \tilde{\alpha} & \text{if } (\mathbf{k},\mathbf{j}) \in \mathbf{C}^k_{=1,eq} \\ S_{\mathbf{k},\mathbf{j}}\left(\left(z_1^{\mathbf{h}}, \left(z_i^{A,P,\mathbf{h}}(t)\right)_{i\in\{2,\dots,n\}}\right)_{\mathbf{h}\in\mathbf{I}}\right) - \varepsilon_{(\mathbf{k},\mathbf{j})} & \text{if } (\mathbf{k},\mathbf{j}) \in \mathbf{C}^k_{sl} \end{cases}$$

[11] To see that $\mathbf{I^1} = \mathbf{I}^1_{fix} \cup \{\mathbf{k} \in \mathbf{I} | (\mathbf{k},\mathbf{j}) \in \mathbf{C}^1_{=1,eq}, \mathbf{j} \in \mathbf{I}\} \cup \{\mathbf{j} \in \mathbf{I} | (\mathbf{k},\mathbf{j}) \in \mathbf{C}^1_{+1,eq}, \mathbf{k} \in \mathbf{I}\}$ note that there is one element in $\mathbf{C}^{ad}_{+1}$ for each $\mathbf{k} \in (\mathbf{I}^1_{fix})^c \cap \mathbf{I^1}$ and $\mathbf{C}^1_{=1,eq} = \emptyset$.

Since $|\mathbf{I^1}| + |\mathbf{C}^k_{sl}| = |\mathbf{I}^k_{fix}| + |\mathbf{C}^k_{eq}| + |\mathbf{C}^k_{sl}| \equiv N^k$, for fixed t, $g^k(t, \cdot)$ is a function from $\mathbb{R}^{N^k}$ to $\mathbb{R}^{N^k}$.

Let $\varepsilon^k_{\mathbf{k},\mathbf{j}} = S_{\mathbf{k},\mathbf{j}}\left(\left(z_1^{k,\mathbf{h}}, \left(z_i^{A,P,\mathbf{h}}(t^k)\right)_{i\in\{2,\dots,n\}}\right)_{\mathbf{h}\in\mathbf{I}}\right)$ for $(\mathbf{k},\mathbf{j}) \in \mathbf{C}^k_{sl}$ and let $a^k = \left(\left(z_1^{k,\mathbf{r}^m}\right)_{m=1}^{|\mathbf{I^1}|}, \left(\varepsilon^k_{(\mathbf{k}^m,\mathbf{j}^m)}\right)_{m=1}^{|\mathbf{C}^k_{sl}|}\right)$. Define $V_0^k = \{(t,a) \in [t^k, 1] \times \mathbb{R}^{|\mathbf{I^1}|+|\mathbf{C}^k_{=1,sl}|} : |a - a^k| < 1/3\}$.

By the induction hypothesis, we have $g^k(t^k, a^k) = 0$.

$D_a g^k$ denotes the Jacobian matrix of g^k with respect to a. For $p \in (\mathbf{C}^k_{eq} \cup \mathbf{C}^k_{sl}) \cup \mathbf{I}^k_{fix}$ and $q \in \mathbf{I^1} \cup \mathbf{C}^k_{sl}$.

$$\left(D_a g^k\right)_{p,q} = \begin{cases} 2(\tilde{y}_1^{\mathbf{A},\mathbf{k}} - z_1^{\mathbf{k}}) & \text{if } p = (\mathbf{k},\mathbf{j}) \in (\mathbf{C}^k_{eq} \cup \mathbf{C}^k_{sl}), \text{ and } q = \mathbf{k}, \\ -2(\tilde{y}_1^{\mathbf{A},\mathbf{j}} - z_1^{\mathbf{k}}) & \text{if } p = (\mathbf{j},\mathbf{k}) \in (\mathbf{C}^k_{eq} \cup \mathbf{C}^k_{sl}), \text{ and } q = \mathbf{k}, \\ -1 & \text{if } p = (\mathbf{k},\mathbf{j}) \in \mathbf{C}^k_{sl}, \text{ and } q = (\mathbf{k},\mathbf{j}), \\ 1 & \text{if } p = \mathbf{r} \in \mathbf{I}^k_{fix}, \text{ and } q = \mathbf{r}, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

$D_a g^k$ is non-singular if $(\tilde{y}_1^{\mathbf{A},\mathbf{k}} - z_1^{\mathbf{k}}), (\tilde{y}_1^{\mathbf{A},\mathbf{j}} - z_1^{\mathbf{k}}) \neq 0$ for every $\mathbf{k},\mathbf{j} \in \mathbf{I^1}$ (which holds for every $a \in V_0^k$ since $|z_1^{\mathbf{k}} - 1| < 1/3$ and $\min_{\mathbf{j}\in\mathbf{I}} |\tilde{y}_1^{\mathbf{A},\mathbf{j}} - 1| > 2/3$). In fact, in such case, the rows in $D_a g^k$ are linearly independent. To see this, note that the rows corresponding to $p \in \mathbf{I}^k_{fix}$ form an identity sub-matrix so they are trivially linearly independent. The rows corresponding to $p \in \mathbf{C}^k_{sl}$ are also linearly independent because the derivatives with respect to the $\varepsilon_{\mathbf{k},\mathbf{j}}$ also form an identity sub-matrix. Finally, the rows corresponding $p \in \mathbf{C}^k_{eq}$ are linearly independent because the sets of constraint do not have cycles.[12]

We now find a bound on $\left\|\left(D_a g^k\right)^{-1} D_t g^k\right\|_\infty$ in V_0^k. By the Implicit Function Theorem 4.2.1 in Krantz and Parks (2012), this bound allows us to determine the size of the vicinity of z^k in which g^k has a solution.

[12] A cycle would be a chain $(\mathbf{k},\mathbf{k}^1), (\mathbf{k}^1,\mathbf{k^2}), \dots, (\mathbf{k}^j,\mathbf{k}) \in \left(\mathbf{C}^k_{eq}\right)$. Only if there is such a cycle can a weighted sum of rows in the sub-matrix $\left(D_a g^k\right)_{p\in\left(\mathbf{C}^k_{eq}\right)}$. be zero.

By a known result, we can write

$$\left(D_a g^k\right)^{-1} = \frac{1}{\det(D_a g^k)} \mathrm{adj}(D_a g^k) \tag{6}$$

where $(\mathrm{adj}(A))_{i,j} = (-1)^{i+j} \det A_{-ji}$ and A_{-ji} the matrix obtained from deleting row j and column i from matrix A. From (5), $D_a g^k$ is a lower triangular matrix with

$$\left|\det\left(D_a g^k\right)\right| \geq \left(2 \min_{(\mathbf{k},\mathbf{j})\in\mathbf{C}^k_{eq}\cup\mathbf{C}^k_{sl}} \min\{|\tilde{y}_1^{A,\mathbf{k}} - z_1^{\mathbf{j}}|, |\tilde{y}_1^{A,\mathbf{k}} - z_1^{\mathbf{k}}|\}|\right)^{|\mathbf{C}^k_{eq}|} \geq (2/3)^{|\mathbf{C}^k_{eq}|}.$$

Since $\min_{\mathbf{j}\in\mathbf{I}} |\tilde{y}_1^{\mathbf{A},\mathbf{j}} - 1| > 1/3$ and $\max_{\mathbf{j}\in\mathbf{I}} \left|z_1^{k,\mathbf{j}} - 1\right| < \frac{k}{3|\mathbf{C}^{ad}_{=1}|} < 1/3$, $\hat{m}_1$ is bounded away from zero.

For each row j and column i, $\left(D_a g^k\right)_{-ij}$ is a lower triangular matrix and, by an analogous computation,

$$\left(D_a g^k\right)_{-ij} \leq \left(2 \max_{(\mathbf{k},\mathbf{j})\in\mathbf{C}^k_{eq}\cup\mathbf{C}^k_{sl}} \max\{|\tilde{y}_1^{A,\mathbf{k}} - z_1^{\mathbf{j}}|, |\tilde{y}_1^{A,\mathbf{k}} - z_1^{\mathbf{k}}|\}|\right)^{|\mathbf{C}^k_{eq}|+|\mathbf{C}^k_{sl}|} \leq (8/3)^{|\mathbf{C}^k_{eq}|+|\mathbf{C}^k_{sl}|}$$

Since $\left|z_{0,1}^{\mathbf{k}} - 1\right| < \nu$, from equation (6) and the previous calculations for large enough $b_1(\Theta)$,

$$\max_{(t,a)\in V_0^k} \left\|\left(D_a g^k(t,a)\right)^{-1}\right\|_\infty \leq \frac{(8/3)^{|\mathbf{C}^k_{eq}|+|\mathbf{C}^k_{sl}|}}{(2/3)^{|\mathbf{C}^k_{eq}|}} \equiv \hat{M}$$

Now, for $(\mathbf{k},\mathbf{j}) \in (\mathbf{C}^k_{eq} \cup \mathbf{C}^k_{sl})$,

$$\begin{aligned}\left|D_t g^{2,k}_{\mathbf{k},\mathbf{j}}\right| &= \left|-2\sum_{i=2}^n \left(\tilde{y}_i^{A,\mathbf{k}} - z_i^{\mathbf{k}}(t)\right)\left(\tilde{y}_i^{P,\mathbf{k}} - \tilde{y}_i^{A,\mathbf{k}}\right) + 2\sum_{i=2}^n \left(\tilde{y}_i^{A,\mathbf{k}} - z_i^{\mathbf{j}}(t)\right)\left(\tilde{y}_i^{P,\mathbf{j}} - \tilde{y}_i^{A,\mathbf{j}}\right)\right| \\ &= \left|-2\sum_{i=2}^n t\left(\tilde{y}_i^{A,\mathbf{k}} - \tilde{y}_i^{P,\mathbf{k}}\right)\left(\tilde{y}_i^{P,\mathbf{k}} - \tilde{y}_i^{A,\mathbf{k}}\right) + 2\sum_{i=2}^n \left(\tilde{y}_i^{A,\mathbf{k}} - \tilde{y}_i^{A,\mathbf{j}} + t\left(\tilde{y}_i^{A,\mathbf{j}} - \tilde{y}_i^{P,\mathbf{j}}\right)\right)\left(\tilde{y}_i^{P,\mathbf{j}} - \tilde{y}_i^{A,\mathbf{j}}\right)\right| \\ &\leq \max_{i\in\{2,\ldots,n\},\mathbf{k},\mathbf{j}\in\mathbf{I}} \left(4|\tilde{y}_i^{P,\mathbf{k}} - \tilde{y}_i^{A,\mathbf{k}}|^2 + \left|\tilde{y}_i^{P,\mathbf{k}} - \tilde{y}_i^{A,\mathbf{k}}\right|\left|\tilde{y}_i^{A,\mathbf{k}} - \tilde{y}_i^{A,\mathbf{j}}\right|\right),\end{aligned}$$

and $D_t g^{1,k}_{\mathbf{r}} = 0$ for $\mathbf{r} \in \mathbf{I}^k_{fix}$. The right hand side of the previous inequality converges to zero as $b_1(\Theta) \to \infty$.

We obtain

$$\left|\left(D_a g^k(t,a)\right)^{-1} D_t g^k\right| \leq \hat{M}\cdot\sqrt{N^k}\cdot \max_{i\in\{2,\ldots,n\},\mathbf{k},\mathbf{j}\in\mathbf{I}}\left(4|\tilde{y}_i^{P,\mathbf{k}}-\tilde{y}_i^{A,\mathbf{k}}|^2+\left|\tilde{y}_i^{P,\mathbf{k}}-\tilde{y}_i^{A,\mathbf{k}}\right|\left|\tilde{y}_i^{A,\mathbf{k}}-\tilde{y}_i^{A,\mathbf{j}}\right|\right)$$
$$\equiv R(\tilde{y}_{-1}^{P,\mathbf{k}},\tilde{y}_{-1}^{A,\mathbf{k}}).$$

Since $R(\tilde{y}_{-1}^{P,\mathbf{k}},\tilde{y}_{-1}^{A,\mathbf{k}})$ is of order $(\bar{y}_1^P)^{-2}$, for $b_1(\Theta)$ large enough that

$$R(\tilde{y}_{-1}^{P,\mathbf{k}},\tilde{y}_{-1}^{A,\mathbf{k}}) \leq \frac{1}{3|\mathbf{C}_{=1}^{ad}|}\cdot\frac{\nu}{|\bar{y}_1^P|}. \tag{7}$$

By Theorem 4.2.1 in Krantz and Parks (2012) there is $\bar{a}^k(t) = \left(\left(\bar{z}_1^{k,\mathbf{k}}(t)\right)_{\mathbf{k}\in\mathbf{I}},\left(\bar{\varepsilon}_{\mathbf{k},\mathbf{j}}^k(t)\right)_{(\mathbf{k},\mathbf{j})\in\mathbf{C}_{slack}^1}\right)$ with

$$|\bar{a}^k(t)-a^k| \leq R(\tilde{y}_{-1}^{P,\mathbf{k}},\tilde{y}_{-1}^{A,\mathbf{k}}) \tag{8}$$

such that

$$g(1,\bar{a}^k(1))=0.$$

If $\bar{\varepsilon}_{\mathbf{k},\mathbf{j}}^k(1)\geq n_2\tilde{\eta}+\tilde{\alpha}$ for $(\mathbf{k},\mathbf{j})\in\mathbf{C}_{sl}^k$ then $\left(\bar{z}_1^{k,\mathbf{k}}(1)\right)_{\mathbf{k}\in\mathbf{I}}$ is incentive compatible and the allocations in dimensions 2 through n coincide with the principal's bliss points. If not, for each $(\mathbf{k},\mathbf{j})\in\mathbf{C}_{sl}^k$ define

$$\bar{t}(\mathbf{k},\mathbf{j}) = \min\left\{t\in[t^k,1]|S_{\mathbf{k},\mathbf{j}}\left(\left(\bar{z}_1^{k,\mathbf{r}}(t),\left(z_i^{A,P,\mathbf{r}}(t)\right)_{i=2}^n\right)_{\mathbf{r}\in\mathbf{I}}\right)=0\right\}$$

and set $t^{k+1}=\min\{\bar{t}(\mathbf{r},\mathbf{s})|(\mathbf{r},\mathbf{s})\in\mathbf{C}_{sl}^k\}$.

Let $\tilde{C}^k = \left\{(\mathbf{k},\mathbf{j})\in\mathbf{C}_{sl}^k|\bar{t}(\mathbf{k},\mathbf{j})=t^{k+1}\right\}$, $\tilde{\tilde{C}}^k = \left\{(\mathbf{k},\mathbf{j})\in\tilde{C}^k|(\mathbf{j},\mathbf{k})\in\tilde{C}^k,\mathbf{k}\leq\mathbf{j}\right\}$, $\mathbf{C}_{add}^k = \tilde{C}^k\setminus\tilde{\tilde{C}}^k$, and define $\mathbf{C}_{sl}^{k+1} = \mathbf{C}_{sl}^k\setminus\{(\mathbf{k},\mathbf{j})|(\mathbf{k},\mathbf{j})\in\mathbf{C}_{add}^k \text{ or } (\mathbf{j},\mathbf{k})\in\mathbf{C}_{add}^k\}$, $\mathbf{C}_{eq}^{k+1}=\mathbf{C}_{eq}^k\cup\mathbf{C}_{add}^k$ and $\mathbf{I}_{fix}^{k+1}=\mathbf{I}_{fix}^k\setminus\{\mathbf{k}|(\mathbf{k},\mathbf{j})\in\mathbf{C}_{add}^k\}$. Clearly $\mathbf{I^1}=\mathbf{I}_{fix}^{k+1}\cup\{\mathbf{k}\in\mathbf{I}|(\mathbf{k},\mathbf{j})\in\mathbf{C}_{=1,eq}^{k+1},\mathbf{j}\in\mathbf{I}\}\cup\{\mathbf{j}\in\mathbf{I}|(\mathbf{k},\mathbf{j})\in\mathbf{C}_{+1,eq}^{k+1},\mathbf{k}\in\mathbf{I}\}$, also $N^{k+1}=|\mathbf{I^1}|+|\mathbf{C}_{eq}^{k+1}|\leq N^k$. Let $z^{k+1}=\left(\bar{z}_1^{k,\mathbf{r}}(t^k),\left(z_i^{A,P,\mathbf{r}}(t^k)\right)_{i\in\{2,\ldots,n\}}\right)_{\mathbf{r}\in\mathbf{I}}$. By construction conditions 1), 2) and 3) hold for z^{k+1}. Condition 1) holds for large enough $b_1(\Theta)$ as

$$\max_{\mathbf{j}\in\mathbf{I}}\left|z_1^{k+1,\mathbf{j}}-1\right| < \max_{\mathbf{j}\in\mathbf{I}}\left|z_1^{k+1,\mathbf{j}}-z_1^{k,\mathbf{j}}\right|+\max_{\mathbf{j}\in\mathbf{I}}\left|z_1^{k,\mathbf{j}}-1\right| < \frac{k+1}{3|\mathbf{C}_{=1}^{ad}|}\cdot\frac{\nu}{|\bar{y}_1^P|},$$

where the inequality follows from (8) and (7).

Since $\mathbf{C}_{sl}^{k}=\emptyset$ for $k\geq|\mathbf{C}_{=1}^{ad}|$ the iterative argument ends in at most $|\mathbf{C}_{=1}^{ad}|$ steps and delivers an allocation $\bar{z}$ that satisfies the requirements of the Lemma. □

Let z be a normalized allocation satisfying the conditions of Lemma 6. Define the allocation $\{x^{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{I}}$ as

$$x^{\mathbf{k}}=\begin{cases}\bar{y}_1^P z_1^{\mathbf{k}} & \text{if } \mathbf{k}\in\mathbf{I^1}\\ \bar{y}_1^P z_1^{(1,\mathbf{k}_{-1})} & \text{if } \mathbf{k}\notin\mathbf{I^1}\end{cases}.$$

We now show that x is incentive compatible for every $\theta^{\mathbf{k}}$, $\mathbf{k}\in\mathbf{I}$ for large enough $b_1(\Theta)$. Let $\mathbf{k},\mathbf{k'}\in\mathbf{I}$, and let $\tilde{\mathbf{k}}=(1,\mathbf{k}_{-1})$ and $\tilde{\mathbf{k}}'=(1,\mathbf{k}_{-1})$.

$$\begin{aligned}S_{\mathbf{k},\mathbf{k'}}(x)-S_{\tilde{\mathbf{k}},\tilde{\mathbf{k}}'}(x)&=\left(x_1^{\tilde{\mathbf{k}}'}-\theta_1^{\mathbf{k}_1}\right)^2-\left(x_1^{\tilde{\mathbf{k}}}-\theta_1^{\mathbf{k}_1}\right)^2-\left(x_1^{\tilde{\mathbf{k}}}-\theta_1^{1}\right)^2+\left(x_1^{\tilde{\mathbf{k}}'}-\theta_1^{1}\right)^2\\&=2\left(x_1^{\tilde{\mathbf{k}}'}-x_1^{\tilde{\mathbf{k}}}\right)\left(\theta_1^{\mathbf{k}_1}+\theta_1^{1}\right).\end{aligned}$$

Condition 4) in Lemma 6 implies that $\left|x_1^{\tilde{\mathbf{k}}'}-x_1^{\tilde{\mathbf{k}}}\right|$ converges to 0 as $b_1(\Theta)\to\infty$. Therefore, for large enough $b_1(\Theta)$,

$$\begin{aligned}&S_{\mathbf{k},\mathbf{k'}}(x)\in\left[0,\eta+\frac{\alpha}{2n_2}\right] \text{ if } (\mathbf{k},\mathbf{k}')\in\mathbf{C}_{+1}^{ad}\\&S_{\mathbf{k},\mathbf{k'}}(x)\geq n_2\left(\eta+\frac{\alpha}{2n_2}\right) \text{ if } (\mathbf{k},\mathbf{k}')\in\mathbf{C}_{=1}^{ad}.\end{aligned}$$

From Lemma 4 x is incentive compatible, which concludes the proof of Theorem 1. □

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