Bargaining and Competition in Thin Markets

Francesc Dilmé[*]

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Abstract

This paper studies markets where buyers and sellers gradually arrive over time, bargain in bilateral encounters and leave the market when they trade. We obtain that, differently from big markets with many traders, these markets feature trade delay and price dispersion even when buyers and sellers are homogeneous and bargaining frictions are small. Transaction prices are mostly determined by the endogenous evolution of the number of traders in the market, and not much by the particular bargaining protocol used in each meeting. We show that the market price drifts towards the price in a balanced market and, under some conditions, increments on the interest rate generate mean-preserving spreads of its ergodic distribution.

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[*]University of Bonn. `fdilme@uni-bonn.de`. Previous versions of the paper were titled "Dynamic Asset Trade a la Bertrand". I thank Syed Nageeb Ali, Stephan Lauermann, Francesco Nava, Peter Norman, Balázs Szentes and Caroline Thomas for their useful comments, as well as the audiences in the EEA 2016 conference in Geneva and the Workshop Decentralized Markets with Informational Asymmetries 2016 (Turin), and the seminar workshops in London School of Economics, UC Berkeley, UNC Chapel Hill, and University of Oxford.

1 Introduction

This paper studies dynamic thin markets. These are markets where buyers and sellers gradually and endogenously arrive over time, bargain in bilateral encounters and leave the market when they trade. Examples include housing/rental markets in given locations, job markets for specific occupations, or some over-the-counter (OTC) financial markets. In them, the number of trading opportunities at a given moment in time is limited. Still, each trader has the possibility of waiting for the arrival of new traders; it is costly but serves her as an outside option. Hence, in a thin market, the bargaining power of each trader not only depends on the bargaining protocol or the the current trade opportunities, but also on the endogenous expectation about the future ones.

Our goal is to characterize the trade outcome of a thin market—i.e., the timing and price of transactions—and analyze how it differs from those previously obtained for big markets. To achieve it, we first develop a general tractable model of a thin market. We then analyze how the bargaining protocol and the endogenous arrival process determine the endogenous market evolution, the future equilibrium prices and the trade delay. We obtain two main departures from the usual findings in the "big markets" literature (reviewed below). First, thin markets feature trade delay even when buyers and sellers are homogeneous and do not have private information. Second, they also feature a significant price dispersion even when the bargaining frictions are small. We characterize the resulting price dynamics, which are shown to mostly depend on the evolution of the composition of the market and not much on the specific features of the bargaining protocol.

We introduce a thin-market version of Gale (1987) model with an endogenously-evolving market composition. At any given moment in time, it consists of a finite number of sellers who own one unit of a homogeneous indivisible good, and a finite number of homogeneous buyers with a unit demand. Once in the market, each trader keeps meeting traders from the other side of the market. In the base model, within each meeting, one of the traders is randomly chosen to make a take-it-or-leave-it offer. Either the other trader accepts the offer, and both leave the market, or rejects it, and both continue. The arrival process of buyers and sellers, the matching rate and the probability of making offers are allowed to depend on the market composition, that is, the numbers of buyers and sellers in the market. We later show that our results apply to more general arrival process or bargaining protocols, or to modeling the outcome of a meeting as a general Nash-bargaining outcome. We focus on Markov perfect equilibria using the market composition as state variable, where all sellers and all buyers play the same strategy.

Our first result claims that trade delay may occur in equilibrium. In other words, even though traders are homogeneous and they have no private information, equilibrium offers may be rejected. To illustrate how trade delay arises, consider a market containing, at a given moment in time, one buyer and two sellers. Assume that if a transaction occurs before a trader arrives, it makes the market visible to buyers, so their arrival rate increases afterwards. If, conversely, a trader arrives before a transaction takes place, it is likely to be a seller. In this case, the buyer in the market does not accept a high price, since he can wait for the seller's competition to increase. Also, each seller obtains high continuation payoff when the other seller trades. A war of attrition between the sellers arises as a result, where both of them delay trade hoping that the other seller will trade first. More generally, we obtain that trade delay will tend to arise when traders on the long side benefit from other traders' transactions, while traders on the short side benefit from the arrival of new traders. In this case, the joint continuation value of a buyer and a seller from not agreeing may be bigger than their joint surplus from trade, and trade delay may occur. We show that, nonetheless, there is never a "market breakdown"; as long as there are buyers and sellers in the market, there is a strictly positive probability that they will trade. Furthermore, at times when the number of buyers and sellers in the market is the same, equilibrium offers are accepted for sure.

The second result establishes that the price dispersion is sizable even when bargaining frictions are small. In the limit when traders in the market meet frequently, there is one transaction price for each market composition, but the dispersion across different market compositions remains. In this limit, trade delay may stay significantly large. We show that when the numbers of buyers and sellers in the market differ, traders on the long side of the market Bertrand compete, and the transaction price is close to their endogenous continuation value from not trading. When, instead, there are the same number of buyers and sellers, the market clears fast and the transaction price is determined by the stochastic Rubinstein bargaining game played by a buyer and a seller when they are alone in the market. In this game, the players' reservation value is endogenously determined by the stochastic arrival of new traders. We use these results to show that transaction prices can be approximated by changing the probability measure that determines the evolution of the market, akin of the use of the so called *risk-neutral measure* to study some financial markets. Under such a measure, the market evolves *as if* a trader on the long side of the market deviated to not trading. The transaction price for a given composition of the market is shown to be proportional to the discounted future time the market exhibits excess demand under the risk-neutral measure, adjusted by the bargaining power of the seller when only a buyer and a seller are present in the market.

We provide conditions for the equilibrium trade delay to shrink to zero as the meeting frequency increases. They take the form of bounds on the effect that current transactions have on future arrivals. The fact that, under these conditions, the equilibrium outcome—that is, transaction probabilities and prices—is unique eases obtaining comparative statics results. The continuation play is determined by the net supply, that is, the difference between the number of sellers and the number of buyers in the market. We obtain that, even though the market composition may not drift toward being balanced, the market price always moves, in expectation, towards the price of a balanced market. On average, it increases when there is excess supply, and it decreases when there is excess demand. Also, increasing the interest (or discount) rate increases the price dispersion: for a given size of the excess supply in the market, a higher interest rate increases the discounted time it takes for the market to clear, and therefore depresses the price. When, instead, the discount rate becomes low, the distribution of transaction prices degenerates towards a "competitive price", which is proportional to the ergodic probability of the market exhibiting excess demand. In this instance, waiting to trade with future arrivals is cheap, so the effective market accessible to each trader increases and the equilibrium outcome approaches that of a big market. Nevertheless, differently from what happens in a large market, the absence of both bargaining and delay frictions does not necessarily imply that the surplus from trade is fully captured by one side of the market; the endogeneity of the arrival process prevents the market to become permanently unbalanced.

Our results are robust to some extensions of our model. We first show that they hold for arrival processes following a general multi-dimensional Markov chain. We allow some components to be exogenously-evolving, such as the economic cycle of the economy or legislation changes, and some others to evolve endogenously, such as idiosyncratic demand or supply shocks. We also consider the effect of changing the bargaining protocol to a general Nash bargaining. In this case, when bargaining frictions are small, the change only affects the price through the relative bargaining power of sellers and buyers when the market is balanced. Hence, the bargaining protocol does not affect prices significantly if, for example, the market is rarely balanced.

The organization of the paper is as follows. After this introduction, we review the literature related to our paper. Section 2 introduces our model, and Section 3 provides the equilibrium analysis. In Section 4, we obtain conditions that guarantee that trade delay shrinks as the bargaining frictions disappear and we provide some comparative statics results. Finally, Section 5 discusses general arrival processes and bargaining protocols, and Section 6 concludes. The Appendix provides the proofs of the results.

1.1 Literature review

Our paper contributes to the literature on thin markets with stochastic arrival of traders. The paper closest to ours, Taylor (1995), analyzes a centralized market where buyers and sellers arrive over time. In every period, traders on the short side of the market make offers, while each side makes an offer with probability $\frac{1}{2}$ when the market is balanced. Coles and Muthoo (1998) consider a similar model where buyers and sellers arrive in pairs, and they allow for heterogeneity in both buyers and goods. Similarly, Said (2011) studies dynamic market in which buyers compete in a sequence of private-value second-price auctions for differentiated goods. These papers analyze price dynamics under different price mechanisms in centralized markets with constant arrival rates of traders. Our focus is, instead, on analyzing decentralized bargaining with an endogenous arrival process. We characterize how the arrival process and bargaining asymmetries affect price dynamics and trade delay. This allows us to compare our results with some of the literature on big markets (see below).

Our paper is also related to the extensive literature on bargaining and matching in large markets, reviewed in Osborne and Rubinstein (1990) and Gale (2000).[1] Models in this literature typically contain a continuum of traders and feature non-stochastic population dynamics, many times assumed to be in a stationary state. We focus, instead, on how the endogenous change of the number of traders on each side of the market affects and is affected by the trade outcome, and how both are also determined by the bargaining protocol. In Section 4.1 we consider the limit where traders become patient, which can be interpreted as the market growing by replication, and we compare the results on convergence to the competitive outcome of this literature.

Finally, there has been some recent interest on thin markets in a network of traders. For example, Condorelli, Galeotti, and Renou (2016), Talamàs (2016) and Elliott and Nava (forthcoming) look at bargaining in networks without arrival and with replacement, and allow for differences in the valuation of the good by sellers and buyers. Our analysis, instead, focuses on understanding how the dynamics of the population determines the price process and bargaining outcomes in an endogenously-growing complete network.

[1] Important contributions in this literature are Rubinstein and Wolinsky (1985), Gale (1987), Burdett and Coles (1997), Shimer and Smith (2000), Atakan (2006), Satterthwaite and Shneyerov (2007), Manea (2011) and Lauermann (2012).

2 The model

In this section we introduce a model similar to Rubinstein and Wolinsky (1985) and Gale (1987). The main distinguishing feature of our model is that the market is assumed to be "small", that is, the number of traders in the market at each moment in time is a non-negative integer number—instead of a mass—which endogenously and stochastically changes over time.

State of the market. Time is continuous with an infinite horizon, $t \in \mathbb{R}_+$. There is an infinite number of potential buyers and sellers. At a given moment in time t, there are $B_t \in \{0, \dots, \bar{B}\}$ buyers and $S_t \in \{0, \dots, \bar{S}\}$ sellers in the market, for some large $\bar{B}, \bar{S} > 0$. The state (of the market) at time t is defined to be (B_t, S_t).[2]

Arrival process. Buyers arrive into the market at a Poisson rate $\gamma_b \equiv \gamma_b(B_t, S_t) \in \mathbb{R}_+$, and sellers arrive into the market at a Poisson rate $\gamma_s \equiv \gamma_s(B_t, S_t) \in \mathbb{R}_+$. The total rate at which the state exogenously changes is denoted $\gamma \equiv \gamma_b + \gamma_s$. Section 5.1 considers a more general arrival process. Note that $\gamma_b(\bar{B}, \cdot) \equiv \gamma_s(\cdot, \bar{S}) \equiv 0$.

Bargaining. In our base model we focus, for the sake of clarity, on a simplistic (yet canonical) bargaining protocol. As it is pointed in Section 5.2, our results can be straightforwardly generalized to allowing for general Nash bargaining.

If, at time t, there are buyers and sellers in the market (i.e., $B_t, S_t > 0$), meetings occur at a Poisson arrival rate $\lambda(B_t, S_t) > 0$. When a meeting occurs, nature selects one of the buyers and one of the sellers in the market uniformly randomly, and also chooses the trader who makes a price offer. The probability that the seller is chosen is $\xi(B_t, S_t) \in (0, 1)$.[3] The trading counterparty decides then whether to accept the offer or not. If the offer is accepted, transaction happens and the traders leave the market, while if it is rejected they continue in the market.

Payoffs. Both buyers and sellers discount the future at rate $r > 0$. If a buyer and a seller trade at time t at price p they obtain, respectively, $e^{-rt}(1-p)$ and $e^{-rt}p$. If they never trade they both obtain 0. Both buyers and sellers are risk-neutral and expected-utility maximizers. Even though the formal expressions for the payoffs (and the conditions for the optimality of a strategy profile) are obtained using standard recursive analysis, their length makes it convenient to leave them to Appendix A.1.

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[2]The assumption that the number of traders in the market is bounded is technical and simplifies the intuitions and the proofs. Standard arguments—that is, taking sequences of models where $\bar{B}$ and $\bar{S}$ tend to $+\infty$—permit showing that our results apply when $\bar{B} = \bar{S} = \infty$, requiring arrival rates to be bounded.

[3]The assumption that $\xi(B_t, S_t) \notin \{0, 1\}$ avoids the Diamond's paradox (see Remark 3.2).

Strategies. To simplify the model setting, we focus directly on Markov strategies, the state variable being the state of the market. Thus, the strategy of a trader (buyer or seller) maps each state (B,S) with $B,S > 0$ into a price offer distribution in $\Delta(\mathbb{R}_+)$, and to a probability of acceptance for each offer received, interpreted to be his/her strategy in the bargaining stage when he/she is matched and the market state is (B,S).[4]

Equilibrium concept. We focus on Markov perfect equilibria in symmetric strategies, where all traders on each side of the market use the same strategy (see Appendix A.1 for the formal definition). From now on, we refer to symmetric Markov perfect equilibria as just "equilibria".

Remark 2.1. Our specification includes the possibility that the arrival rates of traders depend on their endogenous (equilibrium) continuation values from entering the market. This could be the case if, for example, buyers and sellers became active at some respective (state-independent) rates $\bar{\gamma}_b$ and $\bar{\gamma}_s$ instead of directly entering the market. Once a θ-trader would become activate, he/she would draw a cost c from some distribution F_θ. If, for example, the trader was a seller and the state was (B,S), she would enter the market if the net payoff of doing so, $V_s(B,S+1)-c$, was above a fixed outside option (choosing to sell in another market or keeping the good for herself). This would imply that $\gamma_s(B,S) = \bar{\gamma}_s F_s(V_s(B,S+1))$. Given that our results hold for general arrival rates (further generalized in Section 5.1), any equilibrium outcome of such a model would correspond to an equilibrium outcome of ours.

3 Equilibrium analysis

3.1 Equilibrium payoff functions and preliminary results

We begin this section by presenting the equations that the continuation values of each type of trader satisfy in an equilibrium, and stating the existence of an equilibrium. We will then use these expressions to obtain some preliminary results, as well as to provide some intuition on why they hold.

[4]We implicitly assume traders observe the state of the market. Markov perfect equilibria (see the definition below) remain equilibria independently of the information structure as long as the current state of the market is known to the traders in the market.

Equilibrium continuation values and existence of equilibria

Fix an equilibrium. To write the expressions for both types of traders we will sometimes use N_b and N_s to denote, respectively, B and S. We will some times refer to buyers and sellers as, respectively, b-traders and s-traders. Also, for a fixed trader's type $\theta \in \{b, s\}$, we use $\bar{\theta}$ to denote the complementary type, so $\{\theta, \bar{\theta}\} = \{b, s\}$. The continuation value of a θ-trader, for $\theta \in \{b, s\}$, at some state (B, S) is given by

$$V_\theta = \overbrace{\frac{\frac{1}{N_\theta}\lambda}{\lambda+\gamma+r}\, V_\theta^{\mathrm{m}}}^{\text{match}} + \overbrace{\frac{\frac{N_\theta-1}{N_\theta}\lambda}{\lambda+\gamma+r}\, V_\theta^{\mathrm{o}}}^{\text{others match}} + \overbrace{\frac{\gamma}{\lambda+\gamma+r}\, V_\theta^{\mathrm{a}}}^{\text{arrival}}\,, \tag{3.1}$$

where we omitted the dependence of all V_θ's, λ, and γ on the state of the market.[5] As we see, the payoff is divided into the following three pieces:

1. **Match:** Consider, for example, a seller who is matched with a buyer. If she is chosen to make the offer, she can make an unacceptable price offer (above 1, for example), which provides her with a continuation value equal to V_s. The seller can alternatively make an offer intended to be accepted by the buyer. Since the continuation value of a buyer from rejecting the offer is V_b, he accepts for sure price offers strictly lower than $1 - V_b$, and rejects offers strictly above $1-V_b$. Using the standard argument for take-it-or-leave-it offers, equilibrium offers by the seller which are accepted with positive probability are equal to $1-V_b$. If the buyer, instead, is chosen to make the offer, the seller receives payoff is equal to V_s: in equilibrium, if the offer is acceptable, the buyer makes her indifferent between accepting it or not. Hence, we have

$$V_s^{\mathrm{m}} = \xi \max\{V_s, 1-V_b\} + (1-\xi)\, V_s\,. \tag{3.2}$$

The analogous equation for the buyers is given by

$$V_b^{\mathrm{m}} = \xi\, V_b + (1-\xi) \max\{V_b, 1-V_s\}\,. \tag{3.3}$$

2. **Others match:** The continuation value of a θ-trader if other traders match depends on the acceptance probability of equilibrium offers. It can be written as

$$V_\theta^{\mathrm{o}}(B,S) = \alpha\, V_\theta(B-1, S-1) + (1-\alpha)\, V_\theta(B,S)\,, \tag{3.4}$$

where $\alpha \equiv \alpha(B,S)$ is the equilibrium probability that there is trade in a meeting between a buyer and a seller in state (B,S). It is important to notice that, if the net surplus from

[5] To keep the expressions simple we will often not write the dependence of some variables on the state of the market. When we do this the state of the market (B, S) will be clear.

trade is positive, $1-V_s-V_b>0$, the equilibrium offer is accepted for sure in any meeting in state (B,S) (so $\alpha=1$), while if it is negative, $1-V_s-V_b<0$, the equilibrium offer is rejected for sure (so $\alpha=0$).

3. **Arrival:** An arriving trader is a buyer with probability $\frac{\gamma_b}{\gamma_s+\gamma_b}$, and is a seller with probability $\frac{\gamma_s}{\gamma_s+\gamma_b}$. This implies that the continuation value of a θ-trader conditional on the arrival of a trader in the market can be written as

$$V_\theta^{\mathrm{a}}(B,S)=\tfrac{\gamma_b}{\gamma_b+\gamma_s}V_\theta(B+1,S)+\tfrac{\gamma_s}{\gamma_b+\gamma_s}V_\theta(B,S+1) \tag{3.5}$$

for both $\theta\in\{b,s\}$.

We begin stating the existence of equilibria. Its proof follows relatively standard fixed-point arguments.

Proposition 3.1. *An equilibrium exists. The continuation values in an equilibrium are uniquely determined by the probability of agreement α, and satisfy equations* (3.1)-(3.5).

Preliminary results

We continue our analysis with some preliminary results which set some important features of equilibrium behavior. The first establishes that there is no equilibrium and state where equilibrium offers are rejected for sure. Hence, even though we will see that equilibrium offers may be rejected with a strictly positive probability, there is never a "market breakdown". That is, in equilibrium, there are no periods of time where trade happens with zero probability even though there are both buyers and sellers in the market.

Result 3.1. *In any equilibrium, there is a strictly positive probability of trade in every meeting, that is, $\alpha(B,S)>0$ whenever $B,S>0$.*

The proof of the lemma proceeds by contradiction, that is, by assuming that there is an equilibrium and a state (B,S) where equilibrium offers are rejected for sure. This implies that the joint continuation value of a buyer and a seller at state (B,S) is weakly higher than the trade surplus:

$$V(B,S)\equiv V_b(B,S)+V_s(B,S)\geq 1\ .$$

Therefore, there exists a state (B',S') (maybe equal to (B,S)) satisfying that $V(B',S')$ is maximal across all states and such that $\alpha(B',S')=0$. Nevertheless, in this case we have a contradiction:

$$V(B',S')=\tfrac{\gamma(B',S')}{\gamma(B',S')+r}\,V^{\mathrm{a}}(B',S')\leq\tfrac{\gamma(B',S')}{\gamma(B',S')+r}V(B',S')<V(B',S')\ .$$

The next result establishes that when there is a meeting and market is balanced (so $B = S$), there is trade with probability one.

Result 3.2. *In any equilibrium, if the market is balanced then there is trade for sure.*

When the market is balanced, a buyer and a seller "agree" on the relative likelihood of the three events that potentially change the state (matching, others matching and arrival). Using that (by Result 3.1) their joint surplus is never higher than 1, we have

$$V = \frac{\frac{1}{S}\lambda}{\lambda+\gamma+r} \underbrace{V^{\mathrm{m}}}_{\leq 1} + \frac{\frac{S-1}{S}\lambda}{\lambda+\gamma+r} \underbrace{V^{\mathrm{o}}}_{\leq 1} + \frac{\gamma}{\lambda+\gamma+r} \underbrace{V^{\mathrm{a}}}_{\leq 1} \leq \frac{\lambda+\gamma}{\lambda+\gamma+r} < 1 .$$

As we see, their joint surplus from not agreeing is strictly lower than 1 since they discount the time where next event occurs.

Our last result in this section establishes that, if equilibrium offers are rejected with a positive probability at some state (B, S), then a trader on the long side of the market benefits from other traders' transactions.

Result 3.3. *Assume* (B,S) *is such that* $\alpha(B,S) < 1$. *Then, if the* θ*-traders are on the long side of the market,* $V^{\mathrm{o}}_{\theta}(B,S) > V_{\theta}(B,S)$ *and* $V^{\mathrm{o}}_{\bar{\theta}}(B,S) < V_{\bar{\theta}}(B,S)$.

To shed some light on Result 3.3 consider the case where sellers are on the long side of the market, that is, $S > B$. As equation (3.1) shows, the rate at which there is a match involving other traders is, from a seller's perspective, $\frac{S-1}{S}\lambda$. This rate is lower from a buyer's perspective, which equals $\frac{B-1}{B}\lambda$. Thus, the weight of the event where other traders match is higher in determining the sellers' continuation value than in determining the buyers' (see equation (3.1)). If state (B, S) is such that there is a positive probability that the equilibrium offer is rejected (so $\alpha < 1$), it is necessarily the case that $V(B, S) = 1$. Also, we know from Result 3.1 that the join continuation value of a buyer and a seller is weakly lower than 1 at any state. Hence, we can write

$$1 = V = \frac{\lambda}{\lambda+\gamma+r} \Big(\overbrace{\tfrac{1}{S} V_s + \tfrac{S-1}{S} V^{\mathrm{o}}_s + \tfrac{1}{B} V_b + \tfrac{B-1}{B} V^{\mathrm{o}}_b}^{(*)} \Big) + \frac{\gamma}{\lambda+\gamma+r} V^{\mathrm{a}} .$$

Since V^{m}, V^{o} and V^{a} are weakly lower than 1, the previous equation holds only if $V^{\mathrm{o}}_s > V_s$ and $V^{\mathrm{o}}_b < V_b$. In this case, the higher weight that a seller assigns to a meeting involving other traders occurs makes the term $(*)$ in the previous expression strictly bigger than 1 (which is necessary for V to be equal to 1). In fact, it can be written as

$$1 < (*) = \underbrace{\tfrac{S-B}{BS}(V^{\mathrm{o}}_s - V_s)}_{>0} + \underbrace{\tfrac{1}{B} V + \tfrac{B-1}{B} V^{\mathrm{o}}}_{\leq 1} = \underbrace{\tfrac{S-B}{BS}(V_b - V^{\mathrm{o}}_b)}_{>0} + \underbrace{\tfrac{1}{S} V + \tfrac{S-1}{S} V^{\mathrm{o}}}_{\leq 1} .$$

3.2 An example with trade delay

In our setting, all sellers and buyers are homogeneous and do not have private information. Thus, given our focus on symmetric equilibria it is, a priori, unclear whether there exist equilibria where some equilibrium offers are rejected with a positive probability. In this section we illustrate how equilibria with trade delay may arise. In order to keep the example simple, we focus on a given state of the market, and we exogenously fix the continuation payoffs when such a state changes without explicitly modeling the continuation play. Considering this "reduced version" of our model simplifies the expressions and arguments, and it is easy to verify that there exist full specifications of our model with the same equilibrium features.

Consider the following reduced version of our model. Initially, there is one buyer and two sellers in the market. We assume that $\gamma_s(1,2) > \gamma_b(1,2) = 0$, and denote $\gamma \equiv \gamma_s(1,2)$ and $\lambda \equiv \lambda(1,2)$. If a transaction occurs before the arrival of a seller, the market becomes visible to other buyers. The remaining seller obtains a high continuation payoff, which for simplicity is assumed to be equal to 1.[6] If, instead, a seller arrives, the strong competition between sellers gives the buyer a high continuation payoff, which is again assumed to be 1 (see footnote 6), and the sellers obtain 0.[7]

We first compute the continuation values of the buyer and sellers under the assumption that, in each meeting, the price offer is equal to the continuation value of the trader receiving the offer, and such an offer is accepted for sure (i.e., equations (3.1)-(3.5) hold with $\alpha = 1$). They solve the following system of equations:

$$V_b(1,2) = \tfrac{\lambda}{\lambda+\gamma+r}\left(\xi\, V_b(1,2)+(1-\xi)\,(1-V_s(1,2))\right) + \tfrac{\gamma}{\lambda+\gamma+r}\,1\,,$$
$$V_s(1,2) = \tfrac{\lambda/2}{\lambda+\gamma+r}\left(\xi\,(1-V_b(1,2)) + (1-\xi)\,V_s(1,2)\right) + \tfrac{\lambda/2}{\lambda+\gamma+r}\,1\,.$$

Solving the previous system of equations, and using simple algebra, it is easy to show that

$$V_b(1,2) + V_s(1,2) = 1 + \tfrac{\gamma\,(\lambda-2\,r)-2\,r^2}{(\gamma+\lambda+r)\,(2\,\gamma+(1-\xi)\,\lambda+2\,r)}\,.$$

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[6] All values can be perturbed while keeping the same features of the example. A continuation payoff for the seller arbitrarily close to 1 when the state is $(0,1)$ can be supported assuming that $\gamma_b(0,1) \gg \gamma_s(0,1)$, that $\gamma_b(1,1) \gg \gamma_s(1,1)$ and that the arrival sellers is very low afterwards. Analogously, a high continuation value for the buyer in state $(1,3)$ can be supported if, for example, no more buyers arrive afterwards.

[7] In Section 5.1, using a more general state of the market and arrival process, we argue that trade delay arises in a wider set of situations where traders on the short side of the market benefit from some events (arrival of traders, changes in the economic cycle, legislation reforms, etc), while traders on the short side of the market benefit from transactions of other traders (as they can make the market more visible).

If λ is big or r is low (so the right hand side of the previous equation is strictly bigger than 1), an equilibrium where there is trade in every meeting does not exist. Thus, any equilibrium of this reduced version of our model involves randomization in the acceptance of offers. Using α to denote the probability of agreement in a meeting in state $(1,2)$, in any equilibrium of the (reduced) game, we have

$$\alpha = \min\left\{1, \frac{2\,r\,(\gamma+r)}{\gamma\,\lambda}\right\}.$$

Notice that the rate at which an agreement occurs in state $(1,2)$ (which equals $\alpha\,\lambda$) converges to $\frac{2\,r\,(\gamma+r)}{\gamma}$ as λ becomes big, that is, a significant trade delay remains even in the limit where bargaining frictions disappear.

Our example shows that, in some specifications, traders on one side of the market benefit from other traders' transactions, while traders on the other side of the market benefit from the arrival of new traders. In the example, sellers obtain a high continuation payoff if a transaction occurs, and the buyer gets a high payoff if a trader arrives. The buyer is unwilling to accept a price above $\frac{\gamma}{\gamma+r}$, given that he has the option of waiting for the arrival of another seller and then obtain a high payoff. As a result, immediate agreement is not possible: otherwise, each seller would have the incentive to let the other seller trade at a low price, and obtain a high continuation payoff afterwards. The equilibrium behavior of the sellers in the market resembles then a war of attrition: each of them trades at the rate that makes the other seller indifferent between trading at price $\frac{\gamma}{\gamma+r}$ or not. Such delay lowers the value of making unacceptable offers from each seller's perspective, since doing so comes at the risk of another seller arriving. As time passes, either one of the sellers trades (and the remaining seller obtains a high payoff), or another seller arrives (and all sellers obtain a low continuation payoff).

Remark 3.1. Inefficient delay can also be found in other bargaining models with complete information. For example, Cai (2000) analyzes a model of one-to-many bargaining between farmers and a railroad company, where the gains from trade are realized only if all farmers agree. Similar to us, farmers want other farmers to trade, to gain monopsony power. Also, in models of bargaining in networks such as Elliott and Nava (forthcoming), delay may happen because traders are heterogeneous. Our example illustrates that trade delay may appear even when bargaining is decentralized and traders are homogeneous, the reason being that some traders may benefit from other traders' trades, while others benefit from arrivals.

3.3 Small bargaining frictions

We continue with our analysis by focussing on the case where the bargaining frictions are small, that is, where traders in the market meet frequently. This may be a plausible assumption in some thin markets such as localized housing markets or job markets for specific occupations, where the rate at which traders (can) meet once they are in the market is much higher than the arrival rate into the market. As in the large markets literature, studying the case where frictions are small will allow us to provide a sharper characterization of the equilibrium outcome.

In order to analyze the case where bargaining frictions are small, we now separate each state's meeting rate $\lambda(B,S)$ into two parts. The first is a state-independent common factor $k>0$, which will taken to be big. The second is a function $\ell(B,S)$, measuring the relative frequency with which traders meet in each state. Thus, from now on, we use $\lambda(B,S)$ and $k\,\ell(B,S)$ interchangeably.

Given that we will compare equilibria for different values of k, the following notation is convenient to simplify the presentation of the results. In the expressions below, the notation "$\simeq$" indicates that terms on each of the sides are equal in any equilibrium except for terms that go to 0 as k increases (sometimes, for the sake of clarity, we add "as $k\to\infty$").[8] Our first result establishes that when bargaining frictions are small, the joint continuation value of a buyer and a seller is close to the joint surplus they obtain from trade.

Result 3.4. *As $k\to\infty$, $V_b(B,S)+V_s(B,S)\simeq 1$ for all states (B,S) with $B,S>0$.*

To get an intuition for Result 3.4 note that, for a fixed equilibrium, there are three kinds of states. The first kind contains all states with $B,S\geq 1$ where equilibrium offers are rejected with a positive probability, in which case $V=1$ and the result holds. The second kind contains all states where either $B=1$ or $S=1$ (or both), and there is trade for sure in every meeting. In this case, if for example there is one buyer, $S\geq B=1$, his continuation value can be approximated as follows:

$$V_b \simeq \xi\, V_b + (1-\xi)\,(1-V_s) \;\;\Rightarrow\;\; V_b \simeq 1-V_s\,.$$

Intuitively, given that meetings happen very frequently, the buyer can almost costlessly wait until he makes the offer and obtain $1-V_s\geq V_b$. Finally, there are states where $B,S>1$ and there is immediate trade. Simple algebra shows that, in this case, when the meeting frequency

[8] For example, the statement of Result 3.4 should be read as "For all $\varepsilon>0$ there is a $\bar{k}>0$ such that if $k>\bar{k}$ then, for any equilibrium and state (B,S), $|V_b(B,S)+V_s(B,S)-1|<\varepsilon$.".

is high, we can write the net surplus from trade as

$$1 - V \simeq \tfrac{(B-1)\,(S-1)}{B\,S - \xi\,S - (1-\xi)B}\left(1 - V(B-1, S-1)\right). \tag{3.6}$$

As we see, if there is immediate trade at state (B, S) and the net surplus from trade is small after a transaction occurs, the net surplus is necessarily small in (B, S). Define then m as the lowest value $m \geq 1$ such that $(B-m, S-m)$ belongs to one of the first two kinds of states. Since by our previous arguments the net surplus from trade is small in state $(B-m, S-m)$ (so $1 - V(B-m, S-m) \simeq 0$), we can iteratively use equation (3.6) to obtain that $1 - V(B, S) \simeq 0$.

An immediate and important consequence of Result 3.4 is that, when bargaining frictions are low, a seller is approximately indifferent between trading or not in all states (B, S) with $S > B$. This is obviously true if $\alpha < 1$ (the first kind of states defined before). When, instead, $\alpha = 1$ and $S > B = 1$, the payoff of a seller is

$$V_s(1, S) \simeq \tfrac{1}{S}\, V_s(1, S) + \tfrac{S-1}{S}\, V_s(0, S-1)\,.$$

Thus, from the previous equation, $V_s(1, S) \simeq V_s(0, S-1)$, and therefore not trading is close-to-optimal for a seller. As we argued before, when bargaining frictions are low, the third kind of states (in this case, states where $\alpha = 1$ and $S > B \geq 1$) change very fast to states of one of the first two kinds, so the result holds.

Another implication of Result 3.4 is that the price dispersion of the transactions that occur in a given state is low when the bargaining frictions are small. Indeed, the equilibrium price in state (S, B) is either $V_s(S, B)$ (if the buyer makes the offer) or $1 - V_b(S, B)$ (if the seller makes the offer). Since $V(S, B) \simeq 1$, we have $V_s(S, B) \simeq 1 - V_b(S, B)$. The following section shows that the price dispersion across states remains.

Change of measure

As we argue above, Result 3.4 establishes that, when bargaining frictions are small, traders on the long side of the market are close-to-indifferent on trading or letting other traders trade as long as there are traders on both sides of the market. We use such indifference to provide a characterization of the equilibrium price by changing the probability measure that determines the evolution of the state of the market. This approach is in the same spirit of the use of risk-neutral measures in the study of financial markets. The main difference, apart from the thinness of our market, is the fact that the side of the market with more traders changes over time.

Fix an equilibrium. Consider a measure for which the state of the market (B_t, S_t) evolves according to a Markov chain as follows. At Poisson rates $\gamma_b(B_t, S_t)$ and $\gamma_s(B_t, S_t)$ the state changes to (B_t+1, S_t) and (B_t, S_t+1), respectively. Additionally, at rate $\tilde{\delta}(B_t, S_t)$, the state changes to (B_t-1, S_t-1), where

$$\tilde{\delta}(B_t, S_t) \equiv \begin{cases} \frac{B_t-1}{B_t}\,\alpha(B_t, S_t)\,\lambda(B_t, S_t) & \text{if } B_t \geq S_t, \\ \frac{S_t-1}{S_t}\,\alpha(B_t, S_t)\,\lambda(B_t, S_t) & \text{if } B_t < S_t. \end{cases}$$

With some abuse of language, we call this measure the *risk-neutral measure* (of the fixed equilibrium). Notice that the evolution of (B_t, S_t) under the risk-neutral measure corresponds to the evolution of the state of the market "when, at each time, one trader on the long side of the market deviates to not trading". Note also that the dynamics of the state of the market under the risk-neutral measure can be entirely determined from—and therefore uniquely obtained by an external observer who only observes—the equilibrium dynamics of the state of the market.

Proposition 3.2. *For any t and (B_0, S_0) we have, as $k \to \infty$,*

$$V_s(B_0, S_0) \simeq \tilde{\mathbb{E}}\Big[\int_0^{\infty} e^{-r\,t}\,(\mathbb{I}_{B_t > S_t} + \xi(1,1)\,\mathbb{I}_{B_t = S_t})\,r\,\mathrm{d}t\Big]\,, \tag{3.7}$$

where $\tilde{\mathbb{E}}$ is the expectation using the risk-neutral measure.

Proposition 3.2 gives an approximation of the transaction price at each state (B, S) (which is approximately equal to $V_s(B, S)$) in terms of the equilibrium dynamics of the state, and the probability that a seller makes an offer when there is only one buyer and one seller in the market. As we see, it is a discounted average (under the risk-neutral measure) of the future time the market exhibits excess supply, adjusted by the times it is balanced.

To obtain some intuition for Proposition 3.2 consider a state (B, S) where the market is imbalanced. If there are more sellers than buyers, $B < S$, sellers are approximately indifferent on trading or not, and this implies

$$V_s \simeq \frac{\frac{S-1}{S}\,\alpha\,\lambda}{\frac{S-1}{S}\,\alpha\,\lambda + \gamma + r}\,V_s^{\mathrm{m}} + \frac{\gamma}{\frac{S-1}{S}\,\alpha\,\lambda + \gamma + r}\,V_s^{\mathrm{a}}\,. \tag{3.8}$$

A similar equation can be obtained when there are more buyers than sellers in the market (replacing s by b and S by B). Using Result 3.4 we can write, when $B > S$,

$$\overbrace{1 - V_b}^{\simeq V_s} \simeq \frac{r}{\frac{B-1}{B}\,\alpha\,\lambda + \gamma + r} + \frac{\frac{B-1}{B}\,\alpha\,\lambda}{\frac{B-1}{B}\,\alpha\,\lambda + \gamma + r}\,\overbrace{(1 - V_b^{\mathrm{m}})}^{\simeq V_s^{\mathrm{m}}} + \frac{\gamma}{\frac{B-1}{B}\,\alpha\,\lambda + \gamma + r}\,\overbrace{(1 - V_b^{\mathrm{a}})}^{\simeq V_s^{\mathrm{a}}}\,. \tag{3.9}$$

Hence, when the market is imbalanced, the outcome of the market resembles the outcome typically obtained in models of Bertrand competition. Indeed, in a match, the payoff of a trader on the long side of the market if he/she trades is very close to his/her continuation value from not trading until the state of the market changes. Importantly, in a dynamic market, the continuation value is endogenous, and driven by the expectation about the future trade opportunities.[9]

When the market is balanced, Result 3.2 establishes that there is trade in every meeting. Consequently, when $S > 1$, we have

$$V_s(S,S) \simeq \tfrac{1}{S}\, V_s(S,S) + \tfrac{S-1}{S}\, V_s(S-1,S-1)\ ,$$

so $V_s(S,S) \simeq V_s(S-1,S-1)$. Each seller is close-to-indifferent on trading or letting other traders trade until she is alone in the market with a single buyer. When there are only one buyer and one seller in the market, the reservation value of the seller (i.e., her value from not trading) is $\frac{\gamma(1,1)}{\gamma(1,1)+r}\, V_s^{\mathrm{a}}(1,1)$. Similarly, the reservation value of the buyer is $\frac{\gamma(1,1)}{\gamma(1,1)+r}\, V_b^{\mathrm{a}}(1,1)$. As the bargaining frictions become small, the transaction price is determined by the limit outcome of a two-player bargaining game *á la* Rubinstein (1982) with randomly arriving outside options (given by the potential arrival of other traders). The "size of the pie" over which they bargain is not 1, but the trade surplus net of the sum of the outside options, which is

$$1 - \tfrac{\gamma(1,1)}{\gamma(1,1)+r}\, (V_b^{\mathrm{a}}(1,1) + V_s^{\mathrm{a}}(1,1)) \simeq \tfrac{r}{\gamma(1,1)+r}\ .$$

As in the standard Rubinstein bargaining game, the seller obtains, on top of her reservation value, a fraction of the size of the pie equal to the probability with which she makes offers, $\xi(1,1)$. Hence, the Rubinstein payoff of the seller, which is approximately equal to the transaction price, is given by

$$V_s \simeq \tfrac{r}{\gamma(1,1)+r}\, \xi(1,1) + \tfrac{\gamma(1,1)}{\gamma(1,1)+r}\, V_s^{\mathrm{a}}(1,1)\ . \tag{3.10}$$

Equations (3.8)-(3.10) show that V_s approximately follows, under the risk neutral measure, the same equations as the continuation payoff of a fictitious agent who receives a flow payoff of 1 when there is excess supply (i.e., $B_t > S_t$), a flow payoff of 0 when there is excess demand (i.e., $B_t < S_t$) and a flow payoff of $\xi(1,1)$ when the market is balanced (i.e., $B_t = S_t$). The right hand side of equation (3.7) gives an expression for such a continuation value.

An implication of Proposition 3.2 is that only the evolution of the sign of the net amount of sellers (or buyers) in the market, which we call *balancedness of the market*, is relevant for determining the market price. This is because the intensity of the competition between traders

[9]This result can be interpreted to micro-found, using a decentralized approach, the assumption in Taylor (1995) that, at any given time where the market is imbalanced, the transaction price is equal to the one corresponding to a static market with Bertrand-competition in the long side.

on the long side of the market is irrelevant for determining the price when the market is unbalanced: the price equals their reservation value independently of their number. Thus, the price is not directly affected by the expected amount of future transactions, but by the expected evolution of the balancedness of the market. The only dependence of the price on the details of the bargaining protocol comes from the relative bargaining powers when there are only one buyer and one seller in the market.

Remark 3.2 (No Diamond's paradox). Corollary 3.2 shows that, in the limit where bargaining frictions disappear, the payoff of each trader in each state is strictly positive as long as there is a positive probability that his or her side of the market becomes the short side of the market in the future. This may be surprising since, in bargaining models with one-sided offers (which in our model would correspond to $\xi \equiv 0$ or $\xi \equiv 1$), the side of the market making the offers obtains all surplus from trade, independently of the degree of balancedness of the market, usually known as the Diamond's paradox (see Diamond, 1971). In our model, the order of limits matters: our claim implicitly takes the takes the limit of small bargaining frictions first, and the limit of one-sided offers afterwards. This result would not hold if we first assumed that $\xi(\cdot,\cdot)$ is constant and equal to either 0 or 1, and then we took the limit where the bargaining frictions disappear: in this case, the type of traders making all offers would obtain all gains from trade.

Changes in continuation values

The risk-neutral measure is typically defined as such that "the current value of a financial asset is equal to its expected payoffs in the future discounted at the risk-free rate". If $B_0 < S_0$ it is easy to see that, indeed, the transaction price in state (B_0, S_0) (which is approximately equal to $V_s(B_0, S_0)$) is approximately equal to the discounted price at which, in equilibrium, a seller at time 0 expects to sell the good (if she follows an optimal strategy). As we argued before, a close-to-optimal strategy for a seller at time 0 when the bargaining frictions are small consists on not trading until the market is balanced. It is then easy to see (see the proof of Corollary 3.1) that

$$V_s(B_0, S_0) \simeq \tilde{\mathbb{E}}[e^{r\,\tau_0}\, V_s(1,1)] \text{ whenever } B_0 < S_0 \,, \tag{3.11}$$

where τ_0 is the (stochastic) time it takes for the market to balance. When, instead $B_0 > S_0$, equation (3.11) holds for the continuation value of the buyers instead of the sellers' one. Hence, the risk-neutral measure makes "the current continuation value of a trader on the long side of the market equal to his/her expected surplus from trade in the future, discounted at the risk-free rate". This indifference allows us to establish the following result:

Corollary 3.1. *For all t and θ, if θ-traders are on the long side of the market,*

$$\lim_{\Delta \searrow 0} \frac{\mathbb{E}_t[V_{\theta,t+\Delta} - V_{\theta,t}]}{\Delta} \gtrsim \lim_{\Delta \searrow 0} \frac{\tilde{\mathbb{E}}_t[V_{\theta,t+\Delta} - V_{\theta,t}]}{\Delta} \simeq r\, V_{\theta,t}\,, \tag{3.12}$$

where $V_{\theta,t} \equiv V_\theta(B_t, S_t)$ and "$\gtrsim$" means higher except for terms that vanish as $k \to \infty$.

As we argued before, the equality in equation (3.12) derives from the definition of the risk-neutral measure. The inequality derives from Result 3.3. To see this, assume that at time t there is excess supply, $B_t < S_t$. Assume also that there is trade delay at state (B_t, S_t). Thus, the rate at which transactions happen if all sellers follow the equilibrium strategy is higher than the one of the sellers deviates and decides not to trade ($\alpha\,\lambda$ vs $\frac{S_t-1}{S_t}\,\alpha\,\lambda$). Given that sellers are close to approximately indifferent on trading, and by Result 3.3 they benefit from other sellers' transactions, the equilibrium expected increase on the continuation payoff of the sellers is higher than under the risk-neutral measure.

Equation (3.11), as well as the converse inequality when buyers are on the long side of the market, is helpful to set approximate bounds on the transaction prices when bargaining frictions are small. For example, setting an approximate upper bound for the discounted transaction price in a state (B_0, S_0) with $S_0 > B_0 \geq 1$ only requires knowing the equilibrium dynamics of the state of the market and the price when the market is balanced, which is approximately $V_s(1,1)$:[10]

$$V_s(B_0, S_0) \lesssim \mathbb{E}[e^{r\,\tau_0}\, V_s(1,1)]\,.$$

4 No delay

We devote this section to studying equilibrium behavior when trade delay disappears as the bargaining frictions vanish. When this happens, the limit outcome of the model is uniquely defined, and this eases obtaining comparative statics results.

We focus on a simple setting where the arrival rates of traders only depend on the net supply in the market, that is, the difference between sellers and buyers. This assumption simplifies the analysis, and clarifies the results. We now assume that $\bar{B} = \bar{S} = \infty$ to avoid constraining the set of arrival processes too much. In Section 4.2 we extend the results to a more general arrival process.

[10]Result 3.2 establishes that when the market is balanced (so $B = S$) there is trade for sure in every meeting. This implies that, when k is large, $V_\theta(S,S) \simeq \frac{1}{S}\,V_\theta(S,S) + \frac{S-1}{S}\,V_\theta(S-1,S-1)$, so $V_\theta(S,S) \simeq V_\theta(S-1,S-1)$. Given that transactions happen fast, it is the case that $V_\theta(S,S) \simeq V_\theta(1,1)$.

Condition 1. We assume, with some abuse of notation, that $\gamma_\theta(B,S) = \gamma_\theta(S-B)$ for all states (B,S) and types $\theta \in \{b,s\}$.

Condition 1 requires that only the net supply is relevant to determine the arrival rates of the different types of traders. Under this condition, the dynamics of the net supply are autonomous. The reason is that transactions between traders do not alter the net supply, and the arrival rates of traders are only a function of it. This implies, in particular, that the net supply (but not the state of the market) evolves equally under the equilibrium and risk-neutral measures.

Proposition 4.1. *Under Condition 1 there exists some $\bar{k}$ such that if $k > \bar{k}$ then there is no equilibrium with trade delay. There exists an increasing function $p : \mathbb{Z} \to [0,1]$ such that $V_s(B,S) \simeq p(S-B)$ for all states (S,B).*

Proposition 4.1 establishes that Condition 1 is sufficient for trade delay to disappear when bargaining frictions are low. To see this note that, by Result 3.3, trade delay occurs in a given state (B,S) with $S > B$ only if sellers gain from other traders' transactions, that is, $V_s(B–1,S–1)-V_s > 0$. The proof of Proposition 4.1 shows that this difference is bounded away from 0 when k is large. Furthermore, since the evolution of $S_t - B_t$ is autonomous under Condition 1, the right hand of expression (3.7) is only a function of the initial net supply $S_0 - B_0$. Therefore, when bargaining frictions are small, transaction prices are only a function of the net supply in the market, so $V_s(B-1,S-1) \simeq V_s$. This prevents delay to be part of equilibrium behavior.

We will refer to the function $p(\cdot)$ in the statement of Proposition 4.1 as the *market price*, and we will interpret $p(N)$ as the transaction price in a state (B,S) when the net supply is $N \equiv S-B$. Since the sellers' continuation value is close to p, we can characterize the market price obtaining expressions analogous to equations (3.8)-(3.10). If $N > 0$, that is, if there are more sellers than buyers in the market, sellers are indifferent on trading now (and obtaining a payoff equal to $p(N)$), or refusing to trade and waiting for the state to change (and trade then). Thus, the market price at the net state $N > 0$ satisfies the following equation

$$p(N) = \tfrac{\gamma_b(N)}{\gamma(N)+r}\, p(N-1) + \tfrac{\gamma_s(N)}{\gamma(N)+r}\, p(N+1)\ . \tag{4.1}$$

If $N < 0$, the situation is reversed: buyers are indifferent on trading now (obtaining a payoff equal to $1-p(N)$) or letting other buyers trade and waiting for the state to change. Rearranging some terms in the expression analogous to equation (4.1) for the buyers, we obtain that the price at net state N satisfies

$$p(N) = \tfrac{r}{\gamma(N)+r} + \tfrac{\gamma_b(N)}{\gamma(N)+r}\, p(N-1) + \tfrac{\gamma_s(N)}{\gamma(N)+r}\, p(N+1)\ . \tag{4.2}$$

Finally, when the market is balanced, the price is determined by computing the outcome of the frictionless-bargaining limit of a two-player bargaining game *á la* Rubinstein (1982) with randomly arrival outside options (given by the potential arrival of other traders). The resulting price is equal to

$$p(0) = \tfrac{r}{\gamma(0)+r}\,\xi(1,1) + \tfrac{\gamma_b(0)}{\gamma(0)+r}\,p(-1) + \tfrac{\gamma_s(0)}{\gamma(0)+r}\,p(1)\ . \tag{4.3}$$

One can use equations (4.1)-(4.3) to write the market price as the right hand side of equation (3.7), that is, as the future expected discounted time the market exhibits excess demand and, additionally, the future expected discounted time the market is balanced multiplied by $\xi(1,1)$. Similarly, using τ_0 is the (stochastic) time it takes for the market to balance as in equation (3.11), we can write

$$p(N) = \begin{cases} 1 - \mathbb{E}[e^{-r\tau_0}|N_0 = N]\,(1 - p(0)) & \text{if } N \le 0, \\ \mathbb{E}[e^{-r\tau_0}|N_0 = N]\,p(0) & \text{if } N > 0. \end{cases} \tag{4.4}$$

In words, traders on the long side of the market are indifferent on waiting to trade until the market is balanced. Consequently, when the market is imbalanced, the market price changes, in expectation, toward the price of a balanced market (recall also Corollary 3.1). Remarkably, this is the case independently of whether N_t tends in expectation to 0 or not, that is, whether the market tends towards being balanced.

4.1 Comparative statics

One of the salient questions in the literature on decentralized bargaining in large markets is whether lowering the frictions in the market leads to a competitive outcome. This exercise allows analyzing whether and how frictions may be magnified or mitigated by the equilibrium behavior of the traders in the market, and therefore shed light on how robust the predictions of models with markets without frictions are.[11] This section asks a similar question for a thin

[11]For example, Gale (1987) characterizes the trade outcome in the large-market version of our model in the limit where the discount rate converges to 0, and obtains that it converges to that of a competitive market. In this limit, the price is either 0 (if there are more buyers than sellers) or equal to 1 (if there are more buyers than sellers). Other papers have identified some reasons for the failure of convergence. It may be caused, for example, by asymmetric information between traders (Satterthwaite and Shneyerov, 2007; Lauermann and Wolinsky, 2016), the heterogeneity in each side of the market (Lauermann, 2012), or lack of knowledge about the state of the market (Lauermann, Merzyn, and Virág, 2017). See also Lauermann (2013) for an analysis of other causes of delay.

market, analyzing the role of the friction that remains in the market when the meeting frequency is high: the delay cost that a trader incurs when he or she is on the long side of the market.

In this section we assume that N_t has an ergodic distribution. More concretely, we assume that there is some distribution F on $\mathbb{Z}$ such that, for each net supply value N, $\lim_{t\to\infty} \Pr(N_t = N|N_0) = F(N)$ independently of N_0. Note that, when Condition 1 holds, the ergodic distribution of N_t in the limit where bargaining frictions are small is independent of the discount rate, but the ergodic distribution of market prices does.

Corollary 4.1. *An increase in r generates a mean-preserving spread of the ergodic distribution of market prices.*

Corollary 4.1 illustrates the effect that an increase in the discount rate (or the interest rate) has on the trade outcome: it increases the ergodic dispersion of the market price distribution, while keeping its mean the same. The fact that the average long run price is independent of the discount rate follows from equation (3.7). Indeed, from time-0 perspective, the expected price at time t gets increasingly close to the ergodic probability that the market exhibits excess demand (i.e., $N_t < 0$) plus the probability that the market is balanced (i.e., $N_t = 0$) multiplied by $\xi(1,1)$. The increase in the price dispersion follows from Corollary 3.1. For example, when $N_t > 0$, $p(N_t)$ drifts towards the price in a balanced market at a rate $r\,p(N_t)$. This is necessary to make sellers willing to delay trade instead of trading now when they are on the long side of the market. For a fixed $N > 0$, an increase in the discount rate r lowers the discount factor of the time it takes the market to become balanced. Thus, in the limit where $r \to \infty$, we have $p(N) \to 0$ for all $N > 0$, and $p(N) \to 1$ for all $N < 0$.

Patient traders

In a thin market there is no natural analogous of a "competitive outcome", since the number of traders on each side of the market is, at each given moment in time, finite. Still, when traders are patient, it becomes less costly for them wait to trade (and compete) with future traders, enlarging the effective market that each trader faces. As we will see, this implies that, when traders are patient, the outcome of a thin market shares many features with that of a competitive market: the price is (approximately) constant, and depends only the (expected) of balancedness of the market. Nevertheless, the endogenous arrival process in a thin market implies that, in general, no side of the market obtains the full surplus from trade.

As traders become more patient, the current state of the market becomes progressively less relevant to determine the price, since each trader in the market can wait for the state of the market to change without incurring a big cost. In particular, the delay cost from not trading and waiting until the net state reaches some given state in the support of the ergodic distribution of N tends to 0 as $r \to 0$. As a result, as the discount rate r shrinks, it may seem that, for each given state, the payoff from not trading becomes increasingly attractive to each of the traders in the market. Nevertheless, this is not possible when bargaining frictions are small: the sum of the continuation values of a seller and a buyer in the market is always close to 1, independently of their discount rate. Even though waiting is increasingly cheap, it also becomes increasingly invaluable, since also the price variation across states becomes increasingly small. The following result characterizes how the waiting options of buyers and sellers affect the market's outcome when the discount rate decreases.

Corollary 4.2. *As $r \searrow 0$, the ergodic distribution of transaction prices converges to a distribution degenerated at price $p^* \equiv \lim_{t\to\infty} \mathbb{E}[\mathbb{I}_{N_t<0} + \xi(1,1)\, \mathbb{I}_{N_t=0}]$.*

Corollary 4.2 establishes that, as traders become more patient, the distribution of transaction prices converges to a distribution degenerated at some "competitive" price p^*. This is intuitive: since waiting for the state to change (instead of trading now) is increasingly cheap as $r \to 0$, the market price in all states of the market converges to a single price. Such a price can be obtained using equation (3.7), from which it is clear that when r is small the market price is close to the ergodic probability of the market having an excess demand, adjusted by the probability with which the market is balanced. It is then immediate to see that, as $r \to 0$, the distribution of transaction prices also becomes degenerated to the competitive price p^*.

To provide further intuition on the previous results, let B_t^Σ and S_t^Σ denote, respectively, the number of buyers and sellers into the market from 0 to t, including the ones "arrived" (or present) at time 0. Then, trivially, $N_t = S_t - B_t = S_t^\Sigma - B_t^\Sigma$, so equation (3.7) is valid replacing B_t and S_t by B_t^Σ and $\bar{S}_t^\Sigma$, respectively. As traders become more patient, the price (at time 0, for example) approximates the (ergodic) probability that more sellers than buyers arrive in the future. Hence, as r decreases, the effective market that a trader (at time 0) faces grows intertemporally. The endogenity of the arrival process implies that, in general (and differently from the big market case), the price is not degenerated at 0 (when there is excess supply) or 1 (when there is excess demand). Instead, in a thin market, the competitive price is a convex combination of the two extremes, each of them weighted according to the probability that the market features excess supply and demand.

Remark 4.1. In our model, making traders more patient (through decreasing r by a factor $1/M < 1$) can be reinterpreted as enlarging the market by replication, that is, increasing the arrival rates (of buyers and sellers) by a factor $M > 1$. Indeed, increasing the arrival rates accelerates the pace of our model: the distribution of (B_t, S_t) under arrival rates (γ_b, γ_s) is the same as the distribution of $(B_{t/M}, S_{t/M})$ under arrival rates $(M\gamma_b, M\gamma_s)$. Using equation (3.7), it is not difficult to see that the value of $p(S_0 - B_0)$ is the same under (r, γ_b, γ_s) and under $(r/M, M\gamma_b, M\gamma_s)$. So, replicating the market M times is equivalent (in terms of the ergodic distribution of prices) to making it M times faster or, equivalently, to multiply its traders' discount factor by a factor $1/M$.

Our interpretation of market replication may correspond, in practice, to the unification of similar markets into bigger ones. This may be result of, for example, from the introduction of websites providing information on local rental or housing prices in close locations, or used durable goods. The introduction of such webpages may make it easier for buyers to compare prices across markets, which may *de facto* transform them into a single market. Our result implies that even though the unification of markets may not change the ergodic distribution of the market composition much, it may make prices fluctuate faster (in the sense that changes in the market price happen more frequently), and may make their ergodic distribution gets more concentrated around a given value.

4.2 A more general condition for no delay

Proposition 4.1 shows that, when the net supply evolves following an autonomous process (Condition 1), there is no trade delay when the bargaining frictions are small enough. We now show that the same result holds under a more general condition, and we illustrate the robustness of the results presented above.

Condition 2. For each state (B, S) with $B, S > 0$, $\frac{\gamma_\theta(B-1,S-1)}{\gamma(B-1,S-1)+r} \leq \frac{\gamma_\theta}{\gamma+r} + \frac{r}{\gamma+r}\frac{1}{2}$.

Condition 2 relaxes Condition 1. It requires that single transactions do not delay much the expected time until th arrival of each type of traders. This limits the possibility that, as in the example in Section 3.2, traders on the long side of the market benefit significantly from transactions of other traders. Consequently, as the next result establishes, it prevents trade delay to occur in equilibrium.

Proposition 4.2. *Proposition 4.1 holds under Condition 2.*

As k increases, delay disappears and the time it takes the short side of the market to clear is increasingly small. Consequently, in the limit dynamics of the state of the market, one of the sides of the market is always empty. The evolution of the net supply can then be approximated by an autonomous process satisfying a condition similar to Condition 1, now with $\gamma_\theta(N) =$

$\gamma_\theta(0,N)$ if $N \geq 0$ and $\gamma_\theta(N) = \gamma_\theta(-N,0)$ if $N < 0$, for each $\theta \in \{b,s\}$. Similarly, the net supply under the risk neutral measure also evolves approximately autonomously when k increases.

It is important to note that the equilibrium and risk-neutral dynamics of the net supply do not necessarily coincide in the limit where bargaining fictions vanish. When the market is imbalanced, their law of motion coincides so, for example, equation (4.4) still holds. Nevertheless, when the market is balanced, the arrival rate of θ-traders is $\gamma_\theta(0,0)$ in equilibrium, while it is $\gamma_\theta(1,1)$ under the risk-neutral measure. Consequently, Corollary 4.1 is true for the ergodic distribution under the risk-neutral measure. Still, if the arrival rates in states $(0,0)$ and $(1,1)$ are close, or if the likelihood that the market is balanced is low, the result gives a good approximation of the effects of changes in the interest rate, and the competitive price is close to that obtained in Corollary 4.2.

Remark 4.2. Note that Conditions 1 and 2 hold trivially in the big markets studied Rubinstein and Wolinsky (1985) and Gale (1987), which exhibit not trade delay. Indeed, the equilibrium arrival rate of traders—which, in their models, is a discrete-time flow—is independent of whether a given trader trades or not. Then, delaying trade does not change the continuation value of the traders and while postpones the realization of the gains from trade. This argument cannot be applied, in general, to a thin market: as each transaction affects the aggregate state of the market, traders may have the incentive to let other traders trade, and trade when his/her bargaining power is higher.

Remark 4.3. Condition 2 ensures that trade delay disappears when bargaining frictions are small. It is not difficult to find conditions that, on the contrary, ensure that *all* equilibria exhibit trade delay. An example is the following. Let $p(\cdot)$ be the solution of equations (4.1)-(4.3) with $\gamma_\theta(N) = \gamma_\theta(0,N)$ if $N \geq 0$ and $\gamma_\theta(N) = \gamma_\theta(-N,0)$ if $N < 0$, for each $\theta \in \{b,s\}$. Then, if there is some $N < 0$ such that

$$p(N) < \tfrac{\gamma_b(1-N,1)}{\gamma(1-N,1)+r}\, p(N-1) + \tfrac{\gamma_s(1-N,1)}{\gamma(1-N,1)+r}\, p(N+1)\ .$$

there is no equilibrium without trade delay if k is high enough. (A similar condition can be found for $N > 0$.) This is the case because, if the previous condition holds and there was an equilibrium without trad delay, a seller would not be willing to trade in state $(1-N,1)$. By doing so, she would obtain the right hand side of the expression instead of her payoff from trading, $p(N)$.

5 Generalizations and extensions

5.1 General market process

In our base model we assume that the arrival rates of traders depend only on the current number of buyers and sellers in the market. In practice, the arrival of traders in markets may depend on other factors, like the state of the economy (economic booms or downturns), changes in similar markets, idiosyncratic demand/supply shocks in the market, changes on the legislation affecting the bargaining powers of the different types of traders, etc. In this section, we argue that our results are robust to enriching the arrival process to depend on a multi-dimensional state.

We now assume that the state of the market at time t is (B_t, S_t, ω_t), where ω_t is the value of a stochastic process taking values in some set $\Omega \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$. We call ω_t the *market's cycle* at time t. Hence, with some abuse of notation, we use $\gamma_b \equiv \gamma_b(B_t, S_t, \omega_t)$ and $\gamma_s \equiv \gamma_s(B_t, S_t, \omega_t)$ to denote, respectively, the arrival rates of buyers and sellers of buyers into the market at time t, and $\gamma = \gamma_b + \gamma_s$ as before. The probability that the seller makes an offer in a meeting when the state of the market is (B_t, S_t, ω_t) is denoted $\xi \equiv \xi(B_t, S_t, \omega_t)$.

We assume that the market's cycle ω_t changes when there is a transaction, when there is an arrival, or exogenously at a Poisson rate $\eta \equiv \eta(B_t, S_t, \omega_t)$, where $\eta : \mathbb{Z}_+ \times \mathbb{Z}_+ \times \Omega \to \mathbb{R}_+$. In each of these evens, the new state is determined by a random variable $\tilde{\omega}$ which depends only on (B_t, S_t, ω_t) and the type of event. Some components of the market's cycle can be assumed to evolve exogenously (like, for example, the state of the economy or regulation changes) and some may depend on the endogenous variables of the market (such as the number of traders in the market, its visibility, or regional economic conditions if the market is geographically located).

Equations (3.1)-(3.5) can be adapted to the general market process. Now, transactions and arrivals have the potential to change (some components) of the market's cycle. Also, at each moment in time, there is the possibility of an exogenous change in the market's cycle. In an equilibrium, now the continuation value of a θ-trader at market state (B, S, ω) satisfies

$$V_\theta = \frac{\frac{1}{N_\theta}\lambda}{\lambda+\gamma+r+\eta} V_\theta^{\mathrm{m}} + \frac{\frac{N_\theta-1}{N_\theta}\lambda}{\lambda+\gamma+r+\eta} V_\theta^{\mathrm{o}} + \frac{\gamma}{\lambda+\gamma+r+\eta} V_\theta^{\mathrm{a}} + \frac{\eta}{\lambda+\gamma+r+\eta} V_\theta^{\mathrm{c}} , \tag{5.1}$$

where we omitted the dependence of the V_θ's, λ and γ on the state of the market (B, S, ω), where V_θ^{m} is defined as in (3.2) and (3.3), and where

$$\begin{aligned}
&V_\theta^{\text{o}} \equiv \alpha\, \mathbb{E}_{\tilde{\omega}}\left[V_\theta(B-1,S-1,\tilde{\omega})\middle|\text{trade}\right] + (1-\alpha)\, V_\theta\ ,\\
&V_\theta^{\text{a}} \equiv \tfrac{\gamma_b}{\gamma_b+\gamma_s}\, \mathbb{E}_{\tilde{\omega}}\left[V_\theta(B+1,S,\tilde{\omega})\middle|\text{buyer arrives}\right] + \tfrac{\gamma_s}{\gamma_b+\gamma_s}\, \mathbb{E}_{\tilde{\omega}}[V_\theta(B,S+1,\tilde{\omega})\middle|\text{seller arrives}]\ ,\\
&V_\theta^{\text{c}} \equiv \mathbb{E}_{\tilde{\omega}}\left[V_\theta(B,S,\tilde{\omega})\middle|\text{exogenous change}\right]\ ,
\end{aligned}$$

are the expected continuation value if the market's cycle changes (all expectations are conditional on the state (B,S,ω)).

The same logic as in our base model can be used to show that Results 3.1-3.4 and Corollary 3.1 still hold. Indeed, the additional last two terms on the right hand side of equation (5.1) play a similar role: they contain, in each state, the effect of exogenous changes of the state of the market. One can then see that the arguments for Results 3.1-3.4 and Corollary 3.1 are independent on the particular form of this term. Proposition 3.2, instead, has to be adapted as follows. Now, when the market is balanced, the flow payoff of the fictitious agent described in the paragraph following equation (3.10) is equal to $\xi(1,1,\omega_t)$, that is, potentially depends on the market cycle.

The generalized process for the state of the market expands the range of settings where equilibria feature trade delay. Indeed, as we see in the example in Section 3.2, the crucial feature for trade delay to occur is that, for a fixed state (B,S), traders on the short side of the market benefit from the arrival of traders, while traders on the long side of the market benefit from the endogenously determined transactions. Hence, trade delay may occur when traders on the short side of the market expect to benefit from exogenous changes in the market evolution, such as changes on the economic cycle or on legislation regarding their relative bargaining power. Conversely, endogenous changes in the market, driven for example by transactions, may change the arrival rates of new traders, as they may make the market more visible.

5.2 Nash bargaining

In this section we argue that our results can be straightforwardly generalized to allowing the outcome of each meeting to be the outcome of a general Nash bargaining problem.

In our base model, the bargaining protocol within each meeting consists on a take-it-or-leave-it offer by a randomly-chosen trader. In a more general bargaining protocol, such as Nash bargaining, a meeting results in some (potentially stochastic) transfers, and a probability of agreement. Then, we can write the payoffs for traders when they meet as

$$V_s^{\mathrm{m}} = \alpha\,\mathbb{E}[p|\text{agree}] + (1-\alpha)\,(V_s + \mathbb{E}[p|\text{disagree}])\,,$$
$$V_b^{\mathrm{m}} = \alpha\,\mathbb{E}[1-p|\text{agree}] + (1-\alpha)\,(V_b - \mathbb{E}[p|\text{disagree}])\,,$$

where $\alpha \equiv \alpha(B,S)$ is an endogenous probability of agreement in state (B,S). Requiring individual rationality by buyers and sellers (that is, assuming that they can opt out from bargaining and obtain their continuation value instead) imposes that $V_\theta^{\mathrm{m}} \geq V_\theta$ for both $\theta \in \{b,s\}$, so $V_\theta^{\mathrm{m}} \in [V_\theta, 1-V_{\bar{\theta}}]$. Consequently, $\alpha = 0$ and $\mathbb{E}[p|\text{disagree}] = 0$ whenever $V_s + V_b > 1$. Our results rely on the fact that α is "high" when $V_b + V_s > 1$, but not on the fact that is 1. Indeed, if there is a cap $\bar{\alpha}$ to the probability of agreement, the meeting frequency λ can be readjusted accordingly.

The previous properties make the arguments used to prove the results in Sections 3 and 4 hold for a generalized bargaining protocol. Indeed, we do not use the particular structure of the bargaining protocol to show Results 3.1-3.4, as well as Corollary 3.1. Now, in Proposition 3.2, $\xi(1,1)$ has to be replaced by the expected fraction of the net surplus captured by a seller when she is alone with one buyer in the market. Finally, the "size of the pie" over which traders bargain in every meeting, $1-V_b-V_s$, can be shown to shrink when bargaining frictions disappear, so the results in Section 4 also hold.

6 Conclusions

We have studied decentralized bargaining in a dynamic thin market. Since each trader has some market power, the outcome of bilateral bargaining not only depends on the current market conditions, but also on the future expectations about them. We have characterized how the trade outcome is affected by the bargaining protocol and the arrival process.

Our results highlight that trade outcomes in thin markets differ from those of large markets in several important dimensions. Even when bargaining frictions are small and traders are homogeneous, a thin market may feature significant trade delay and price dispersion. Delay occurs because traders on different sides of the market assign different relative likelihoods to arrivals or other traders trading before them. This may make the joint surplus they expect from not trading higher than the trade surplus. Additionally, the market exhibits price dispersion, and this arises from the each trader's reservation value from the possibility of waiting to trade in the future depends on the current state. The fact that traders slowly arrive over time makes their reservation value depend on the current market composition in a non-trivial way even when the bargaining frictions are small.

We have identified three main features of the price dynamics in the absence of trade delay. First, market prices drift towards the price of a balanced market. Second, increases on the interest rate generate mean preserving spreads. Finally, in the limit where traders become increasingly patient, the distribution of transaction prices degenerates towards a competitive price. Differently from a big market, such a competitive price is not degenerated, that is, no side of the market obtains all surplus from trade.

Our model can be generalized in multiple directions. One that may be particularly interesting is allowing buyers and sellers to be heterogeneous, both in terms of the quality of their goods and their valuation for them. This would make the analysis much more involved, as it would enlarge the dimensionality of the state of the market. For example, Elliott and Nava (forthcoming) show that, in a model of bargaining in networks, the outcome of the bargaining is stochastic even in the limit when bargaining frictions vanish, as sometimes transactions with low gains from trade are realized in the presence of more beneficial trade opportunities. The analysis of this and other extensions is left to future research.

A Omitted expressions and proofs of the results

A.1 Payoffs and equilibria

In this section, and in the proofs below, we use $\mathscr{B} \equiv \{0,\ldots,\bar{B}\}$ and $\mathscr{S} \equiv \{0,\ldots,\bar{S}\}$ and $\mathscr{B}^* \equiv \mathscr{B}\backslash\{0\}$ and $\mathscr{S}^* \equiv \mathscr{S}\backslash\{0\}$. We fix a strategy for the sellers, (π_s, α_s), and for the buyers, (π_b, α_b), where, for each type $\theta \in \{b, s\}$ and state $(B, S) \in \mathscr{B}^* \times \mathscr{S}^*$, $\pi_\theta(B, S) \in \Delta(\mathbb{R})$ is the distribution of price offers that type-θ traders make, while $\alpha_\theta(\cdot; B, S) : \mathbb{R} \to [0, 1]$ maps each price offer to a probability of acceptance.

The continuation values that a strategy profile $\{(\pi_\theta, \alpha_\theta)\}_{\theta\in\{b,s\}}$ generates are given by equations (3.1), (3.5), with the expected continuation values conditional on being selected in the match given by

$$\begin{aligned}
V_b^{\mathrm{m}}(B,S) \equiv\ & \xi\, \mathbb{E}_{\tilde{p}}\big[\, \alpha_b(\tilde{p})\, (1-\tilde{p}) + (1-\alpha_b(\tilde{p}))\, V_b(B,S) \,\big|\, \pi_s \,\big] \\
& + (1-\xi)\, \mathbb{E}_{\tilde{p}}\big[\, \alpha_s(\tilde{p})\, (1-\tilde{p}) + (1-\alpha_s(\tilde{p}))\, V_b(B,S) \,\big|\, \pi_b \,\big]\ , \qquad \text{(A.1)} \\
V_s^{\mathrm{m}}(B,S) \equiv\ & \xi\, \mathbb{E}_{\tilde{p}}\big[\, \alpha_b(\tilde{p})\, \tilde{p} + (1-\alpha_b(\tilde{p}))\, V_s(B,S) \,\big|\, \pi_s \,\big] \\
& + (1-\xi)\, \mathbb{E}_{\tilde{p}}\big[\, \alpha_s(\tilde{p})\, \tilde{p} + (1-\alpha_s(\tilde{p}))\, V_s(B,S) \,\big|\, \pi_b \,\big] \qquad \text{(A.2)}
\end{aligned}$$

instead of equations (3.2) and (3.2), and the continuation value of the type-θ trader conditional on some other traders being selected in the match given by

$$\begin{aligned}
V_\theta^{\mathrm{a}}(B,S) \equiv\ & \xi\, \mathbb{E}_{\tilde{p}}\big[\, \alpha_b(\tilde{p})\, V_\theta(B-1,S-1) + (1-\alpha_b(\tilde{p}))\, V_\theta(B,S) \,\big|\, \pi_s \,\big] \\
& + (1-\xi)\, \mathbb{E}_{\tilde{p}}\big[\, \alpha_s(\tilde{p})\, V_\theta(B-1,S-1) + (1-\alpha_s(\tilde{p}))\, V_\theta(B,S) \,\big|\, \pi_b \,\big] \qquad \text{(A.3)}
\end{aligned}$$

instead of by equation (3.4); where, to keep notation short, γ and ξ should be interpreted as evaluated at (B, S), and $\alpha_\theta(\tilde{p})$ should be interpreted as $\alpha_\theta(p; B, S)$.

The system of equations has a unique solution by the standard fixed-point argument. Indeed, one can replace V_b by $W_s \equiv 1 - V_b$ and verify that the previous equations can be understood as an operator which maps any pair of functions $(V_s, W_s) : \mathscr{B}^* \times \mathscr{S}^* \to \mathbb{R}^2$ to another pair of similar functions, and that such operator satisfies the sufficient Blackwell conditions for a contraction.

Then, using the principle of optimality, we define $\{(\pi_\theta, \alpha_\theta)\}_{\theta\in\{b,s\}}$ to be a *symmetric Markov perfect* if for each state (B, S) and $\theta \in \{b, s\}$, fixing the continuation values of both types of traders in all other states (obtained solving the previous system of equations), as well as the strategy and continuation value of type $\bar{\theta} \neq \theta$ at (B, S), we have that $(\pi_\theta(B, S), \alpha_\theta(\cdot; B, S))$ maximizes the value of $V_\theta(B, S)$.

A.2 Proofs of the results

Proposition 3.1

Proof. Fix an equilibrium. Standard arguments imply that if there is a positive probability that offers made by a seller are accepted in state (B,S), then the equilibrium probability that such offers are equal to $1-V_b(B,S)$ is one. Similarly, an equilibrium offer by a buyer in state (B,S) is accepted with positive probability in equilibrium if and only if it is equal to $V_s(B,S)$. Since these offers make the receiver of the offer indifferent on accepting it or not, it is without loss of generality (to prove existence) to focus on equilibria where, at state (B,S) and for all $\theta \in \{b,s\}$, buyers offer $V_s(B,S)$ and sellers offer $1-V_b(B,S)$ for sure, and the θ-trader accepts such an offer with some probability $\alpha_\theta(B,S)$. Thus, equations (A.1) and (A.2) can be replaced by equations (3.2) and (3.3). Note that the continuation values of a seller and a buyer only depend on α_b and α_s through

$$\alpha \equiv (1-\xi)\,\alpha_b + \xi\,\alpha_s$$

(see equation (A.3)), with the convention that $\alpha(B,S)=0$ whenever $B=0$ or $S=0$. Hence, equations (3.1)-(3.5) determine the continuation payoffs in an equilibrium.

Fix some $\alpha \in [0,1]^{\mathscr{B}^*\times\mathscr{S}^*}$, interpreted as a putative equilibrium probability of trade. We can compute the equilibrium continuation value by solving equations in (3.1)-(3.5), and let $V_b(\cdot;\alpha)$ and $V_s(\cdot;\alpha)$ denote the corresponding solutions. Note also that a buyer and a seller are indifferent on accepting the equilibrium offer at state (B,S) if and only if $V_b(B,S;\alpha)+V_s(B,S;\alpha)=1$. Hence, there is no $\theta\in\{b,s\}$ such that the θ-trader has a profitable deviation at a given state $(B,S)\in\mathscr{B}^*\times\mathscr{S}^*$ only if

$$\alpha(B,S) \in \begin{cases} \{0\} & \text{if } V_b(B,S;\alpha)+V_s(B,S;\alpha)>1, \\ [0,1] & \text{if } V_b(B,S;\alpha)+V_s(B,S;\alpha)=1, \\ \{1\} & \text{if } V_b(B,S;\alpha)+V_s(B,S;\alpha)<1. \end{cases}$$

To see this assume, for example, that $V_b(B,S;\alpha)+V_s(B,S;\alpha)>1$ and that $\alpha_s(B,S)>0$ (if, instead, $\alpha_s(B,S)<1$ the argument is analogous). If a buyer makes the equilibrium offer (equal to $V_s(B,S;\alpha)$) at state (B,S) he obtains

$$\begin{aligned} &\alpha\,(1-V_s(B,S;\alpha)) + (1-\alpha)\,V_b(B,S;\alpha) \\ &\quad = V_b(B,S;\alpha) - \alpha\left(V_s(B,S;\alpha)+V_b(B,S;\alpha)-1\right) < V_b(B,S;\alpha)\,. \end{aligned}$$

If, instead, he offers $V_s(B,S;\alpha)-\varepsilon$, for some $\varepsilon>0$, the seller rejects the offer for sure, so he obtains $V_b(B,S;\alpha)$, which makes him strictly better off.

To conclude the proof of existence of equilibria we define the correspondence $A : [0,1]^{\mathscr{B}^*\times\mathscr{S}^*} \rightrightarrows [0,1]^{\mathscr{B}^*\times\mathscr{S}^*}$ as follows

$$A(\alpha)(B,S) = \begin{cases} \{0\} & \text{if } V_b(B,S;\alpha) + V_s(B,S;\alpha) > 1, \\ [0,1] & \text{if } V_b(B,S;\alpha) + V_s(B,S;\alpha) = 1, \\ \{1\} & \text{if } V_b(B,S;\alpha) + V_s(B,S;\alpha) < 1. \end{cases}$$

Standard arguments apply to show that $A(\cdot)$ has a closed graph, and that $A(\alpha)$ is, for all $\alpha \in [0,1]$, non-empty and convex. Hence, the existence of equilibria follows from Kakutani's fixed point theorem. $\square$

Proof of Results 3.1-3.4

Proof. The proofs follow from the arguments in the main text. $\square$

Proof Proposition 3.2

Proof. Throughout the proof we fix a sequence $(k_n)_n$ tending to $+\infty$ and, for each n, an equilibrium for the model where $k = k_n$. For each n and fixed state (B,S), we let $V_{\theta,n}$ denote the continuation value of a θ-trader in the n-th equilibrium, for $\theta \in \{b,s\}$, and α_n denote the probability of trade in a meeting.

We now define a function $\tilde{V}_{\theta,n} : \mathscr{B}\times\mathscr{S} \to [0,1]$ interpreted as the payoff of a θ-trader when he/she decides to trade only when the market is balanced. It is obtained solving equations (3.1)-(3.5) (adding tildes to all V's) with the exception that, when $B \neq S$, $\tilde{V}_\theta{}^{\mathrm{m}} = \tilde{V}_\theta$ (no trade when the market is imbalanced) and, when $B = S$, $\tilde{V}_\theta{}^{\mathrm{m}}$ satisfies equations (3.2) and (3.3) replacing all V's with $\tilde{V}$'s (trade for sure when the market is balanced). Note that equation (3.1) can be rewritten, for $\theta = s$ and $S \neq B$ as equation (3.8) (adding tildes to the V's and replacing α by α_n), and $\tilde{V}_{b,n}$ follows an analogous equation (replacing s by b). Thus, defining $W_n(B,S)$ to be equal to $\tilde{V}_{s,n}(B,S)$ when $B \le S$, and to be equal to $\tilde{V}_{b,n}(B,S)$ when $B > S$, it is clear that W_n is approximated by the right hand side of equation (3.7) as $n \to \infty$. Our goal is then to find that $W_n \simeq V_{s,n}$ for all $B \le S$ and $W_n \simeq V_{b,n}$ for all $B > S$.

Fix now a state (B,S) satisfying that $B = S$. In this case we have that, for all $\varepsilon > 0$, there is some n such that $\left|\tilde{V}_{s,n}(B,S) - V_{s,n}(B,S)\right| < \varepsilon$ for all $S \ge 1$. To see this, recall that by Result 3.2 there is immediate trade when the market is balanced. Also, by Result 3.4, $V^{\mathrm{a}}_{s,n} \simeq V_{s,n}$. Hence, as $n \to \infty$, we have

$$\begin{aligned} V_{s,n}(S,S) &\simeq \tfrac{1}{S}\, V_{s,n}(B,S) + \tfrac{S-1}{S}\, V_{s,n}(B-1,S-1) \\ \Rightarrow \quad V_{s,n}(B,S) &\simeq V_{s,n}(B-1,S-1) \simeq \ldots \simeq V_{s,n}(1,1)\ , \end{aligned}$$

where, as in the main text, "$\simeq$" means equal except for terms that go to 0 as n increases. Proceeding similarly, we have that it is also the case that $\tilde{V}_{s,n} \simeq V_{s,n}(1,1)$ because, as n increases, it is increasingly unlikely that an arrival happens before the market clears. Using then the standard analysis in Rubinstein (1982), we have that equation (3.10) holds for both $V_{s,n}$, and also for $\tilde{V}_{s,n}$ replacing $V^{\mathrm{a}}_{s,n}$ by $\tilde{V}^{\mathrm{a}}_{s,n}$. Indeed, for state $(B,S)=(1,1)$ we can write, for all $\theta \in \{b,s\}$,

$$V_{\theta,n} = \frac{k_n\,\ell}{k_n\,\ell+\gamma+r}\,(\xi_\theta\,(1-V_{\bar{\theta},n}) + (1-\xi_\theta)\,V_{\theta,n}) + \frac{\gamma}{k_n\,\ell+\gamma+r}\,V^{\mathrm{a}}_{\theta,n}$$

where $\xi_b = 1-\xi$ and $\xi_s - \xi$. The previous equations coincide with the equations for the continuation payoffs in a two-player Rubinstein bargaining where the value from not trading for a type-θ trader is $V^{\mathrm{a}}_{\theta,n}$. Solving for $V_{b,n}$ and $V_{s,n}$, it is easy to show that

$$\lim_{n\to\infty} \left| \frac{r}{\gamma+r}\,\xi_\theta + \frac{\gamma}{\gamma+r}\,V^{\mathrm{a}}_{\theta,n} - V_{\theta,n} \right| = 0\ .$$

For each n, let $D_{s,n}$ denote the maximum distance between $V_{s,n}$ and $\tilde{V}_{s,n}$ among all states where $B \le S$, and let $D_{b,n}$ denote the maximum distance between $V_{b,n}$ and $\tilde{V}_{b,n}$ among all states where $B > S$. Let $D_n \equiv \max\{D_{b,n}, D_{s,n}\}$ and assume it is achieved at some state (B_n, S_n). We also assume, for the sake of contradiction, that $\limsup_{n\to\infty} D_n > 0$ (otherwise the result holds) and, without loss of generality and for simplicity (considering a subsequence if necessary), assume that $\lim_{n\to\infty} D_n > 0$. Assume finally, taking a subsequence if necessary, that the sequence $(B_n, S_n)_n$ is constant at some state (B,S) such that $B \le S$ (the other case is analogous).[12] We can then write, for each n,

$$D_n \le \frac{\frac{1}{S_n}\,k_n\,\ell}{k_n\,\ell+\gamma+r}\,|\tilde{V}^{\mathrm{m}}_{s,n} - V^{\mathrm{m}}_{s,n}| + \frac{\frac{S_n-1}{S_n}\,k_n\,\ell}{k_n\,\ell+\gamma+r}\,D_n + \frac{\gamma}{k_n\,\ell+\gamma+r}\,D_n\ .$$

If $B=S$ then

$$D_n \le \frac{k_n\,\ell+\gamma}{k_n\,\ell+\gamma+r}\,D_n\ ,$$

but this implies that $D_n = 0$ for all n, a contradiction. Assume then that $B<S$, so $\tilde{V}^{\mathrm{m}}_{s,n} = \tilde{V}_{s,n}$ and we can write

$$D_n \le \frac{\frac{1}{S_n}\,k_n\,\ell}{\frac{1}{S_n}\,k_n\,\ell+r}\,|\tilde{V}_{s,n} - V^{\mathrm{m}}_{s,n}|\ .$$

There are three cases:

1. Assume first that there is a subsequence indexed by $(n_i)_i$ such that, for each i, there is trade delay at state (B,S) in the n_i-th equilibrium. In this case, $V^{\mathrm{m}}_{s,n_i} = V_{s,n_i}$. Nevertheless, this implies

   $$D_{n_i} \le \frac{k_{n_i}\,\ell+\gamma}{k_{n_i}\,\ell+\gamma+r}\,D_{n_i} \;\;\Rightarrow\;\; D_{n_i} = 0\ .$$

   This contradicts that $\lim_{n\to\infty} D_n > 0$.

[12] Given that the number of states is finite, it is always possible to find a constant subsequence.

2. Assume now that $B = 0$. In this case we have

$$D_n \simeq \tfrac{\gamma}{\gamma+r} D_n \;\;\Rightarrow\;\; D_n \simeq 0\,.$$

This is, again, a contradiction.

3. We then have that, without loss of generality, we can assume $S > B > 0$ and that, in the n-th equilibrium, there is trade for sure at state (B,S). Assume, taking a subsequence if necessary, that $\alpha_n(B',S')\,k_n$ tends to some $\bar{\delta}(B',S') \in [0,+\infty]$ for all states (B',S') (with the convention that $\alpha_n(B',S') = 0$ when $B' = 0$ or $S' = 0$). We let $m \leq \bar{S}$ denote the minimal natural number such that $\bar{\delta}(B-m,S-m) \neq \infty$ (note that $m > 0$ since there is trade for sure at state (B,S) and $S > B > 0$). Then, as $n \to \infty$,

$$\tilde{V}_{s,n}(B,S) \simeq \tilde{V}_{s,n}(B-m,S-m)\,.$$

Similarly, by Result 3.4, we have that, $V_{s,n} \simeq 1 - V_{b,n}$, and therefore

$$V_{s,n}(B,S) \simeq \tfrac{1}{S}\, V_{s,n}(B,S) + \tfrac{S-1}{S}\, V_{s,n}(B-1,S-1)\,.$$

Thus, $V_{s,n}(B_n,S_n) \simeq V_{s,n}(B_n{-}1,S_n{-}1)$ and, proceeding iteratively, $V_{s,n}(B_n,S_n) \simeq V_{s,n}(B_n{-}m,S_n{-}m)$. If $B - m > 0$ then we have

$$\begin{aligned} D_n &\simeq \left|V_{s,n}(B-m,S-m) - \tilde{V}_{s,n}(B-m,S-m)\right| \\ &\precsim \underbrace{\frac{\frac{1}{S-m}\bar{\delta}}{\bar{\delta}+\gamma+r}\, D_n + \frac{\frac{S-m-1}{S-m}\bar{\delta}}{\bar{\delta}+\gamma+r}\, D_n + \frac{\gamma}{\bar{\delta}+\gamma+r}\, D_n \simeq \frac{\bar{\delta}+\gamma}{\bar{\delta}+\gamma+r} D_n}_{=(*)}\,, \end{aligned}$$

where $\bar{\delta}$ and γ are evaluated at $(B-m,S-m)$, and where "$\precsim$" means that the left hand side is lower than the right hand side plus terms that go to 0 as n increases.[13] Thus, $D_n \simeq 0$, which is a clear contradiction. Therefore, it is the case that $B-m = 0$, so we have

$$D_n \simeq \tfrac{\gamma}{\gamma+r} D_n \;\;\Rightarrow\;\; D_n \simeq 0$$

where γ is evaluated at $(0,S-B)$, but this is again a contradiction.

□

[13] Formally, the equation should be interpreted as meaning that, for all $\varepsilon > 0$, there is an $\bar{n}$ such that if $n > \bar{n}$ then $D_n \leq (*) + \varepsilon$ in all equilibria.

Proof Corollary 3.1

Proof. We first show the second equality in equation (3.12). We do this for the case $S_t > B_t$, and the other case is analogous. Note that, under the risk-neutral measure, we have

$$\begin{aligned}\tilde{\mathbb{E}}_t[V_{s,t+\Delta}] &\simeq \big(1-\big(k\,\ell_t+\gamma_t\big)\,\Delta\big)\,V_{s,t} + \overbrace{\tfrac{1}{S_t}\,k\,\ell_t\,\Delta\,V_{s,t} + \tfrac{S_t-1}{S_t}\,k\,\ell_t\,\Delta\,V^{\mathrm{o}}_{s,t} + \gamma_t\,\Delta\,V^{\mathrm{a}}_{s,t}}^{\equiv(*)} + o(\Delta) \\ &\simeq (1+\Delta r)\,V_{s,t} - \Delta\underbrace{\Big(\big(\tfrac{S_t-1}{S_t}\,k\,\ell_t+\gamma_t+r\big)\big)\,V_{s,t} - \tfrac{S_t-1}{S_t}\,k\,\ell_t\,V^{\mathrm{o}}_{s,t} - \gamma_t\,V^{\mathrm{a}}_{s,t}\Big)}_{\equiv(**)} + o(\Delta)\ ,\end{aligned}$$

where functions with a subindex t refer to variables evaluated at the state at time t, (B_t, S_t). Note that the term $(**)$ shrinks to 0 when as k gets large. Indeed, this follows from equation (3.1) for $\theta = s$ replacing V^{a}_s by V_s (since the seller is indifferent on trading or not) and the fact that, as it is shown in the proof of Proposition 3.2, $\tilde{V}_s$ is close to V_s when k is large.

To obtain the first inequality in equation (3.12), we write

$$\mathbb{E}_t[V_{s,t+\Delta}] = \big(1-\big(k\,\ell_t+\gamma_t\big)\,\Delta\big)\,V_{s,t} + \overbrace{\tfrac{1}{S_t}\,k\,\ell_t\,\Delta\,V^{\mathrm{o}}_{s,t} + \tfrac{S_t-1}{S_t}\,k\,\ell_t\,\Delta\,V^{\mathrm{o}}_{s,t} + \gamma_t\,\Delta\,V^{\mathrm{a}}_{s,t}}^{(***)} + o(\Delta)\ ,$$

where note that the main difference with respect the expression for $\tilde{\mathbb{E}}_t[V_{s,t+\Delta}]$ is now that the probability that the state changes due to a transaction is now $k\,\ell_t$ instead of $\frac{S-1}{S}\,k\,\ell_t$. Hence, the term $(***)$ in the previous expression is weakly higher than the term $(*)$ because, from Result 3.3, we have $V_{s,t} \le V^{\mathrm{o}}_{s,t}$ whenever there is trade delay, while $V_{s,t} \simeq V^{\mathrm{o}}_{s,t}$ otherwise. □

Proof of Proposition 4.1

Proof. We assume, for the sake of contradiction, that there is a sequence $(k_n)_n$, a corresponding sequence of equilibria and a sequence of states (B_n, S_n) such that, in the n-th equilibrium, equilibrium offers are rejected with positive probability. For each state (B, S), we use $V_{b,n}$ and $V_{s,n}$ to denote the continuation values of buyers and sellers in the n-th equilibrium, and α_n to denote the probability of acceptance of an equilibrium offer (so $\alpha_n(B_n, S_n) < 1$ for all n).

Taking a subsequence if necessary, assume that, for each state (B, S) the continuation values $V_{b,n}(B, S)$ and $V_{s,n}(B, S)$ converge to some values $V_b(B, S)$ and $V_s(B, S)$, and the matching rates $\alpha_n\,k_n\,\ell$ converge to some value $\delta \in [0, +\infty]$. We further assume, taking again a subsequence if necessary, that for each state (B, S) with $B, S > 0$, either $\alpha_n < 1$ for all n, or $\alpha_n = 1$ for all n.

We first focus on characterizing the limit continuation value of seller, V_s, for states (B, S) is such that $0 \le B \le S$. The equations are given by:

1. Consider first the case where (B,S) is such that $S > B \geq 1$ and $\alpha_n < 1$ for all n. Using equation (3.1) we have that the limit continuation value for a seller, V_s, satisfies

$$\delta\,(V_s - V_s(B-1,S-1)) = -r\,\tfrac{B\,S}{S-B}\,.$$

   It is then clear that there is no state where $\delta = 0$, that is, where the trade rate becomes arbitrarily small as k increases. (The logic for this result is analogous to that of Result 3.1.) Using this, we can use again equation (3.1) to obtain

$$V_s = \tfrac{\gamma}{\gamma+r}\,V_s^{\mathrm{a}} + \tfrac{r}{\gamma+r}\,\tfrac{B\,(S-1)}{S-B} = V_s(B-1,S-1) - \tfrac{r}{\delta}\,\tfrac{B\,S}{S-B}\,. \tag{A.4}$$

   Note that the second equality implies that, as indicated in Result 3.3, traders on the long side of the market gain from other's transactions in states where is trade delay, $V_s(B-1,S-1) > V_s$.

2. Consider now the case where (B,S) is such that $B,S > 0$ and $\alpha_n = 1$ for all n. In this case, if $S > 1$, we have

$$V_s = V_s(B-1,S-1)\,.$$

   Note that, by Result 3.2, this is the case for states with $B = S$.

3. Finally, for states where $B = 0$ we have

$$V_s = \tfrac{\gamma}{\gamma+r}\,V_s^{\mathrm{a}}\,.$$

Let, for each state (B,S) with $0 \leq B < S$, $\Delta \equiv V_s(0,S-B) - V_s$, and $\Delta = 0$ for each state (B,S) with $B = S$. Since, when $B \geq 1$, we have $V_s(B-1,S-1) \geq V_s$, it is the case that $\Delta \geq 0$ for all states. Let (B,S) be a state which maximizes $\Delta(B',S')$ among all states with $B' \leq S'$ and assume, for the sake of contradiction, that $\Delta \equiv \Delta(B,S) > 0$ (so necessarily $0 < B < S$). If there are multiple states with this property, assume that (B,S) is such that S is minimal among all of them. Assume first that (B,S) is such that $\alpha_n = 1$ for all n. In this case, since $V_s(B-1,S-1) = V_s$, we have

$$\Delta = V_s(0,S-B) - V_s(B-1,S-1) = \Delta(B-1,S-1)\,.$$

This contradicts the assumption that (B,S) is a state with a minimal number of sellers among those which maximize Δ. Then, it is necessarily the case that (B,S) is such that $\alpha_n < 1$ for all n. In this case, we have that, using equation (A.4),

$$\Delta = \tfrac{\gamma_b}{\gamma+r}\,(\overbrace{V_s(1,S-B) - V_s(B+1,S)}^{\leq\Delta(B+1,S)}) + \tfrac{\gamma_s}{\gamma+r}\,(\overbrace{V_s(0,S-B+1) - V_s(B,S+1)}^{=\Delta(B,S+1)}) + \tfrac{r}{\gamma+r}\,\tfrac{B\,(S-1)}{S-B} \tag{A.5}$$

$$< \tfrac{\gamma}{\gamma+r}\,\Delta\,, \tag{A.6}$$

where the inequality holds because $V_s(1,S-B) \le V_s(0,S-B-1)$ and $\frac{B(S-1)}{S-B} > 0$. This is a clear contradiction. Therefore, we have that, for all states (B,S), $V_s = V_s(0,S-B)$ and that $\alpha_n(B,S) = 1$ if n is high enough.

We now prove that p exists satisfying the conditions in the statement. We use $p(\cdot)$ to denote the solution of equations (4.1)-(4.3) (which can be proven to be unique using standard fixed-point arguments similar to those in Section A.1). It is then clear that $V_s(B,S) \simeq p(S-B)$ for all states (B,S). Furthermore, for each $\bar{N} \ge 0$, one can rewrite equation (4.4) for all $N \ge \bar{N}$ as

$$p(N) = \mathbb{E}[e^{-r\,\bar{\tau}}|N_0 = N]\,p(\bar{N})$$

where $\bar{\tau}$ is the stochastic time it takes the net supply to reach $\bar{N}$ for the first time. It is then clear that $p(\cdot)$ is decreasing in $\{0,...,\bar{S}\}$. A similar argument shows that it is also the case that $p(\cdot)$ is decreasing in $\{-\bar{B},...,0\}$. Extending $p(\cdot)$ to the integers by setting $p(N) = 0$ for all $N > \bar{S}$ and $p(N) = 1$ for all $N < -\bar{B}$ completes the proof. □

Proof of Corollary 4.1

Proof. Let F the ergodic distribution of N, so the expected price under such a distribution is

$$\mathbb{E}[p(\tilde{N})|F] = \sum_{N\in\mathbb{Z}} F(\{N\})\,p(N)\ .$$

It is also the case that

$$\begin{aligned}\lim_{t\to\infty}\mathbb{E}[p_t] = \mathbb{E}[p(\tilde{N})|F] &= \lim_{t\to\infty}\mathbb{E}\Big[\int_t^\infty e^{-r\,(s-t)}\,(\mathbb{I}_{N_s<0}+x\,\mathbb{I}_{N_s=0})\,r\,\mathrm{d}s\Big]\\ &= \mathbb{E}\big[I_{\tilde{N}<0}+x\,\mathbb{I}_{\tilde{N}=0}\big|F\big] = F(-\mathbb{N}) + \xi(1,1)\,F(\{0\})\ ,\end{aligned}$$

where $-\mathbb{N}$ is the set of strictly negative integers. This proves that the ergodic mean of the market price is independent of r.

Consider an increase on the discount rate from r_1 to r_2, with $r_1 < r_2$, and let $p^{r_i}(\cdot)$ denote the market price function for each r_i, $i = 1,2$. Assume that $p^{r_1}(0) \ge p^{r_2}(0)$ (the reverse case is analogous). In this case, for all $N > 0$ the price $p^{r_1}(N) > p^{r_2}(N)$. Indeed, using τ_0 to denote the (stochastic) time it takes for the market to become balanced (which is independent of r) and using equation (4.4), we can write

$$p^{r_1}(N) = \mathbb{E}\big[\,e^{-r_1\,\tau_0}\,\big|\,N_0 = N\big]\,p^{r_1}(0) > \mathbb{E}\big[\,e^{-r_2\,\tau_0}\,\big|\,N_0 = N\big]\,p^{r_2}(0) = p^{r_2}(N)\ . \tag{A.7}$$

Since the mean of the market price under the ergodic distribution is independent of the discount rate, there must be some $N < 0$ such that $p^{r_1}(N) < p^{r_2}(N)$. Let $\bar{N}$ be the maximum satisfying this property. Notice that equation (4.4) can be rewritten, for any $N \le \bar{N} < 0$ and $i \in \{1,2\}$, as

$$p^{r_i}(N) = 1 - \mathbb{E}\big[\, e^{-r_i \bar{\tau}} \,\big|\, N_0 = N \big]\, (1 - p^{r_i}(\bar{N}))$$

where $\bar{\tau}$ is the first time where $N_t = \bar{N}$. It is then clear, using equation (A.7) and $p^{r_1}(\bar{N}) < p^{r_2}(\bar{N})$, that for all $N \le \bar{N}$ we have $p^{r_1}(N) < p^{r_2}(N)$. Thus, in fact, $\bar{N}$ is such that

$$p^{r_1}(N) \ge p^{r_2}(N) \text{ for all } N > \bar{N} \text{ and } p^{r_1}(N) < p^{r_2}(N) \text{ for all } N \le \bar{N}\,.$$

This property (and the fact that the ergodic distribution of N is independent of the discount rate) ensures that the distribution of $p^{r_2}(N)$ is a mean-preserving spread of $p^{r_1}(N)$. □

Proof of Corollary 4.2

Proof. If the ergodic distribution F does not have 0 in its support, then the support is either entirely contained in $\mathbb{N}$ or in $-\mathbb{N}$. In this case, it is clear from equation (4.4) that either $\lim_{r\to 0} p(N) = 0$ for all N (if the support of F is a subset of $\mathbb{N}$) or $\lim_{r\to 0} p(N) = 1$ for all N (if the support of F is a subset of $-\mathbb{N}$). Assume then that the support of F contains 0.

For any N_0, we have

$$\lim_{r\to 0} p(N_0) = \lim_{t\to\infty} p(N_t) = \Pr(N<0|\tilde{F}) + \xi(1,1)\, \Pr(N=0|\tilde{F})\,.$$

As we see, the distribution of prices degenerates, as $r \to 0$, to the term on the right hand side of the previous expression, which proves our result. □

Proof of Propositon 4.2

Proof. The proof is analogous to the Proposition 4.1 until equation (A.5). Now, in equation (A.5), the arrival rate of type-θ traders into the market are $\gamma_\theta \equiv \gamma_\theta(B,S)$ in state (B,S), and $\gamma_\theta(0,S-B)$ in state $(0,S-B)$, which are potentially different. Then, equation (A.5) is now replaced by

$$\begin{aligned}
\Delta &= \tfrac{\gamma_b}{\gamma+r} V_s(B+1,S) + \tfrac{\gamma_s}{\gamma+r} V_s(B,S+1) + \tfrac{r}{\gamma+r}\, \tfrac{B\,(S-1)}{S-B} \\
&\quad - \Big(\tfrac{\gamma_b(0,S-B)}{\gamma(0,S-B)+r} V_s(1,S-B) + \tfrac{\gamma_s(0,S-B)}{\gamma(0,S-B)+r} V_s(0,S-B+1) \Big) \\
&\ge \tfrac{\gamma}{\gamma+r}\, \Delta \\
&\quad + \underbrace{\Big(\tfrac{\gamma_b}{\gamma+r} - \tfrac{\gamma_b(0,S-B)}{\gamma(0,S-B)+r} \Big) V_s(0,S-B-1) + \Big(\tfrac{\gamma_s}{\gamma+r} - \tfrac{\gamma_s(0,S-B)}{\gamma(0,S-B)+r} \Big) V_s(0,S-B+1) + \tfrac{r}{\gamma+r}\, \tfrac{B\,(S-1)}{S-B}}_{=(*)}
\end{aligned}$$

Hence, a sufficient condition for the statement to hold is that the term $(*)$ in the previous equation is positive. Using Condition 2 we have that, for each $\theta \in \{s, b\}$,

$$\frac{\gamma_\theta}{\gamma+r} \geq \frac{\gamma_\theta(0,S-B)}{\gamma(0,S-B)+r} - \frac{r}{\gamma+r}\frac{B}{2} \, .$$

So

$$(*) \geq -\frac{r}{\gamma+r} B + \frac{r}{\gamma+r}\frac{B\,(S-1)}{S-B} > 0 \, .$$

Thus, Condition 2 is sufficient to guarantee that, if k is high enough, there is no equilibrium with trade delay. □

References

Atakan, A. E., 2006, "Assortative matching with explicit search costs," *Econometrica*, 74(3), 667–680.

Burdett, K., and M. G. Coles, 1997, "Steady state price distributions in a noisy search equilibrium," *Journal of Economic Theory*, 72(1), 1–32.

Cai, H., 2000, "Delay in multilateral bargaining under complete information," *Journal of Economic Theory*, 93(2), 260–276.

Coles, M. G., and A. Muthoo, 1998, "Strategic Bargaining and Competitive Bidding in a Dynamic Market Equilibrium," *Review of Economic Studies*, 65(2), 235–60.

Condorelli, D., A. Galeotti, and L. Renou, 2016, "Bilateral trading in networks," *The Review of Economic Studies*, 84(1), 82–105.

Diamond, P. A., 1971, "A model of price adjustment," *Journal of Economic Theory*, 3(2), 156–168.

Elliott, M., and F. Nava, forthcoming, "Decentralized Bargaining: Efficiency and the Core," working paper.

Gale, D., 1987, "Limit theorems for markets with sequential bargaining," *Journal of Economic Theory*, 43(1), 20–54.

——— , 2000, *Strategic foundations of general equilibrium: dynamic matching and bargaining games*. Cambridge University Press.

Lauermann, S., 2012, "Asymmetric information in bilateral trade and in markets: An inversion result," *Journal of Economic Theory*, 147(5), 1969–1997.

———, 2013, "Dynamic matching and bargaining games: A general approach," *The American Economic Review*, 103(2), 663–689.

Lauermann, S., W. Merzyn, and G. Virág, 2017, "Learning and Price Discovery in a Search Market," working paper.

Lauermann, S., and A. Wolinsky, 2016, "Search with adverse selection," *Econometrica*, 84(1), 243–315.

Manea, M., 2011, "Bargaining in stationary networks," *The American Economic Review*, 101(5), 2042–2080.

Osborne, M. J., and A. Rubinstein, 1990, *Bargaining and markets*. Academic press.

Rubinstein, A., 1982, "Perfect equilibrium in a bargaining model," *Econometrica: Journal of the Econometric Society*, pp. 97–109.

Rubinstein, A., and A. Wolinsky, 1985, "Equilibrium in a market with sequential bargaining," *Econometrica: Journal of the Econometric Society*, pp. 1133–1150.

Said, M., 2011, "Sequential auctions with randomly arriving buyers," *Games and Economic Behavior*, 73(1), 236–243.

Satterthwaite, M., and A. Shneyerov, 2007, "Dynamic Matching, Two-Sided Incomplete Information, and Participation Costs: Existence and Convergence to Perfect Competition," *Econometrica*, 75(1), 155–200.

Shimer, R., and L. Smith, 2000, "Assortative matching and search," *Econometrica*, 68(2), 343–369.

Talamàs, E., 2016, "Prices and Efficiency in Networked Markets," working paper.

Taylor, C. R., 1995, "The Long Side of the Market and the Short End of the Stick: Bargaining Power and Price Formation in Buyers', Sellers', and Balanced Markets," *The Quarterly Journal of Economics*, 110(3), 837–55.