An Ascending Auction with Multidimensional Signals *

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Abstract

A single-unit ascending auction in which agents have interdependent values and observe multidimensional signals is studied. The focus is on a symmetric model in which each agent observes two private signals: a signal about an idiosyncratic shock and a noisy signal about a common shock. The challenge is to characterize how the (multidimensional) signals observed by an agent are aggregated into the agent's (one-dimensional) bid; this is solved by projecting the private signals of an agent onto a one-dimensional *equilibrium statistic*. An agent's equilibrium statistic aggregates the agent's private signals while taking into account the feedback effects between agents' bids.

The equilibrium characterization reveals important properties of bidding strategies that do not arise when agents observe one-dimensional signals. In contrast to one-dimensional environments, the ascending auction may have multiple symmetric equilibria that yield different social surpluses. This is the result of a strategic complementarity on how signals are aggregated into an agent's bid. In multidimensional environments, there is a failure of the linkage principle; in fact, a public signal may jointly increase the social surplus and decrease the revenue. This is because changes in the parameter of the model change how the signals are aggregated into an agent's bid.

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1 Introduction

Motivation. Auctions have been extensively studied in economics. It is an empirically relevant and a theoretically rich literature: auctions are commonly used to allocate goods across agents and there is a rich class of models that allows the study of bidding in auctions. One of the critical assumptions in most auction models is that agents observe one-dimensional signals.

The objective of our paper is twofold: (*i*) characterize the equilibrium of an auction in which agents observe multidimensional signals and (*ii*) analyze the differences between auctions with one-dimensional signals and auctions with multidimensional signals. As a byproduct, we provide predictions of auctions that arise only when agents observe multidimensional signals.

Our paper is motivated by the observation that in many environments agents' information is naturally a multidimensional object. As an example, consider the auction of an oil field. Suppose that an agent's valuation of the oil field is determined by the size of the oil field and by the agents' cost of extracting oil. Furthermore, assume that each agent can privately observe his own cost of extracting oil and, additionally, agents observe conditionally independent noisy signals about the size of the oil field. This would be an environment in which agents observe two-dimensional signals. Similarly, in most auction environments agents observe multidimensional signals about their valuation of the good (e.g. timber, procurements, art, and real estate).[1]

There is an important conceptual difference between bidding in an auction with one-dimensional signals and multidimensional signals. If agents observe one-dimensional signals, observing the bid of agent i is informationally equivalent to observing agent i's signal. In contrast, in environments with multidimensional signals, observing the bid of agent i is not informationally equivalent to observing all the signals observed by agent i. Hence, a bid must aggregate an agent's signals. In the oil field example, agent j cannot disentangle whether agent i's low bid is caused by a high cost of extracting oil or by the belief that the oil reservoir is small. The extent to which agent i's bidding is

[1] In timber auctions, agents may differ in their harvesting cost and their estimate about the harvest quality (see Haile (2001) or Athey and Levin (2001)). In highway procurement auctions, bidders are exposed to idiosyncratic cost shocks and common cost shocks (see Somaini (2011) or Hong and Shum (2002)). In art auctions and real estate auctions, agents have a known taste shock and an unknown common shock that can represent the quality of the good or the future resale value.

driven by his private costs or his beliefs about the size of the oil reservoir is critical for agent j to determine his own bidding strategy. After all, agent j's valuation of the oil field is independent of agent i's costs but is affected by agent i's signal about the size of the oil reservoir. This leads to an important difference in the equilibrium bidding.

The conceptual challenge is to take into account the feedback between agents' bids: the way agent i's signals are aggregated into his bid depends on how agent j's signals are aggregated into his bid. For example, in an oil field auction, if costs are positively correlated across agents, then agent i's costs allow to partially disentangle the elements that determine agent j's bid (that is, agent j's costs and his private signal about the size of the oil reservoir). This influences agent i's bid; in fact, as we show, this leads agent i to bid less aggressively on his own private cost. This, in turn, influences agent j's bidding strategy. This problem of information aggregation has impeded the study of auctions with multidimensional signals. We provide an equilibrium characterization that allows to understand this problem of information aggregation.

Model. The model consists of N agents bidding for an indivisible good in an ascending auction. The utility of an agent if he wins the object is determined by a common shock and an idiosyncratic shock. Each agent privately observes his own idiosyncratic shock and, additionally, each agent observes a conditionally independent signal about the common shock. The valuations are log-normally distributed and the signals are normally distributed. We focus on symmetric environments and symmetric equilibria.

The two-dimensional signals contains the elements of a pure common values environment and a pure private values environment.[2] Hence, the only departure from the classic models in the auction literature is the multidimensionality of the information structure. This allows to distill the essential elements of bidding strategies that arise with multidimensional signals. Yet, the solution method extends to any Gaussian information structure, possibly asymmetric.

The focus on an ascending auction and Gaussian signals is helpful to fully characterize a class of equilibria. The assumption of Gaussian signals has been used in the empirical auction literature (see, for example, Hong and Shum (2002)). Hence, this is

[2]If agents observed only their idiosyncratic shock, this would be a classic pure private value environment. If agents observed only the signal on the common shock, this would be a classic pure common value environment.

a natural model to study auctions with multidimensional signals.

Characterization of the Equilibrium. The main result of our paper is the characterization of a class of equilibria in the ascending auction. In the class of equilibria we characterize the drop-out time of an agent is determined by a linear combination of the signals he observes. This linear combination is a sufficient statistic to determine an agent's bid in equilibrium. Since this is not in general a sufficient statistic of all the information observed by an agent, we call this linear combination of signals an *equilibrium statistic*. The equilibrium statistic has two roles: (i) it determines the information that agent n learns from the drop-out time of agent m, and (ii) it is optimal for agent n to use *only* his equilibrium statistic to determine his drop-out time. Of course, the optimality condition of agent n takes into account the information he infers from the drop-out time of other agents.

The equilibrium characterization is tractable because the drop-out time of an agent is determined by a linear combination of the signals he observes: the equilibrium statistic. The linearity arises because expectations with Gaussian signals are linear. Gaussian signals are commonly used in models in which agents have linear best response.[3] However, the ascending auction is not a linear best response game. In fact, at any point in time, the beliefs of an agent about his own valuation are not Gaussian; this is because an agent can only infer a lower bound on the drop-out time of the agents that have not yet dropped out.

The ascending auction plays a critical role in keeping the Bayesian updating within the Gaussian family when we evaluate the equilibrium conditions. In the equilibria we characterize, an agent's drop-out time remains optimal even after observing the drop-out time of all other agents.[4] Consequently, we evaluate the best response conditions using the realized drop-out time of each agent (and not a lower bound). This property of the equilibria in an ascending auction, in conjunction with the Gaussian signals, makes the problem tractable. For example, a first-price auction with Gaussian signals does not preserve the same tractability because it is not possible to evaluate an agent's best response conditions using the realized bids of all other agents.

[3]The classic approach in linear best-response games is to conjecture (and later verify) that there is an equilibrium in which the joint distribution of actions is Gaussian.

[4]Formally, the set of equilibria we characterize form a posterior equilibrium. This is a stronger notion of equilibrium, due to Green and Laffont (1987).

Novel Predictions. The outcome of the auction is ultimately determined by the equilibrium statistic. The analysis of auctions in multidimensional environments is different than in one-dimensional environments because the equilibrium statistic is an endogenous object. This leads to important features of bidding strategies that arise only when gents observe multidimensional signals. These additional elements of bidding strategies result in predictions of the ascending auction that arise only when agents observe multidimensional signals.

In contrast to one-dimensional environments, in multidimensional environments, the ascending auction may have multiple symmetric equilibria.[5] The different equilibria will yield different social surplus and different revenue.[6] The multiplicity of equilibria is caused by a complementarity in the weight agents place on their own idiosyncratic shock in their bidding strategy. This complementarity arise because signals must be aggregated into an agent's bid. In general, with multidimensional signals, there is no straightforward mapping between the distribution of signals and the social surplus.

In one-dimensional environments, public signals do not change the social surplus, and public signals increase the revenue.[7] In contrast, a public signal about the average idiosyncratic shock across agents overturns both of these predictions: (i) the public signal increases the social surplus generated by the auction, and (ii) the public signal may also decrease the revenue.[8] Note that a public signal about the average idiosyncratic shock across agents does not change an agent's expected valuation conditional only on his private information: the valuation of an agent is independent of the idiosyncratic shock of other agent. Nevertheless, this public signal changes an agent's beliefs about the realization of the common shock. The public signal allows an agent to disentangle what component of the bid of other agents is determined by their own idiosyncratic shock or their signal about the common shock. This ultimately changes the equilibrium statistic. More broadly, the comparative statics will be different in one-dimensional environments than in multidimensional environments because any change in the primitives of the model will change the equilibrium statistic. Hence, comparative

[5]Bikhchandani, Haile, and Riley (2002) show that there is a continuum of symmetric equilibria. Nevertheless, the allocation and equilibrium price is the same across equilibria. See Krishna (2009) for a textbook discussion.

[6]We write "revenue" for ex ante expected revenue, and "social surplus" for ex ante expected surplus.

[7]In one-dimensional environments, the auction is efficient, so public signals cannot change the social surplus. The fact that public signals increase the revenue is called the linkage principle. We discus this in Section 5.3.

[8]In fact, if the public signal about the average idiosyncratic shock is precise enough then the auction will be efficient but the revenue will be equal to 0.

statics are mediated by changes in the equilibrium statistic.

Literature Review. The literature on auctions with one-dimensional signals is extensive. A large part of this literature is based on the seminal contribution by Milgrom and Weber (1982). They eloquently describe the assumption as follows:

> "To represent a bidder's information by a single real-valued signal is to make two substantive assumptions. Not only must his signal be a sufficient statistic for all of the information he possesses concerning the value of the object to him, it must also adequately summarize his information concerning the signals received by the other bidders." Milgrom and Weber (1982), p. 1097

The quote provides a clear explanation of what is entailed by assuming one-dimensional signals. This also illustrates the difficulty in characterizing an equilibrium when agents observe multidimensional signals: in general, an agent's bid is not determined simply by his interim expected valuation.[9]

The literature on auctions with multidimensional signals has made progress in two ways. The first way is to make the appropriate assumptions on the distributions of signals such that an agent's bid is determined only by his interim expected valuation. The second way is to provide properties of an auction without the need to characterize the equilibrium bids. The distinguishing feature of our paper is that we do not impose any assumptions on the correlations of signals across agents and we fully characterize a class of equilibria. This allows to study how signals are aggregated into an agent's bid taking into account the feedback effects between the bids of different agents. This ultimately delivers the new predictions about ascending auctions. We now discuss the literature on auctions with multidimensional signals and interdependent valuations.[10]

Wilson (1998) studies an ascending auction with two-dimensional signals. Wilson (1998) assumes that the random variables are log-normally distributed and drawn from a diffuse prior.[11] This can be seen as a particular limit of our model (see Footnote 21).

[9] An agent's interim expected valuation is his expected valuation conditional only on his private signals.

[10] There is a literature that studies multidimensional signals in private value environments (see, for example, Fang and Morris (2006) or Jackson and Swinkels (2005)). This literature is largely based on first-price auctions and aims to understand how multidimensional signals change the bid-shading in a first-price auction. The presence of multidimensional signals in this literature play a different role than in our model. In fact, an ascending auction has an equilibrium in dominant strategies when agents have private values.

[11] The signals are drawn from a diffuse prior, hence, these are not technically random variables, and the updating is not technically done by Bayes' rule. The updating is done using a heuristic linear rule, which is akin of Bayesian updating under log-normal random variables.

In this limit, the bid of an agent is determined only by an agents interim expected valuation. Due to its tractability, which is shared to a great extent by our model, the model studied by Wilson (1998) has been used in empirical work.[12]

Dasgupta and Maskin (2000) study a generalized VCG mechanism. They show that, if agents' signals are independently distributed across agents, then an agent's interim expected valuation delivers a one-dimensional statistic that can be used to characterize the Nash equilibria of the mechanism.[13] Moreover, this can be used in many other mechanisms as long as signals are independently distributed, including an ascending auction and a first-price auction (see, for example, Goeree and Offerman (2003) for an application to auctions).[14]

Jackson (2009) provides an example of an ascending auction in which an equilibrium does not exist. The model studied therein is similar to our model — with a private and a common signal — except the distribution of signals and payoff shocks has a finite support (and hence, non-Gaussian). This shows that existence of an equilibrium is not guaranteed in an auction model with multidimensional signals. The extent to which it is possible to construct equilibria with multidimensional non-Gaussian information structures is still an open question.

Pesendorfer and Swinkels (2000) study a sealed-bid uniform price auction in which there are k goods for sale, each agent has a unit demand and each agent observes two-dimensional signals. They study the limit in which the number of agents grows to infinity. Pesendorfer and Swinkels (2000) are able to provide asymptotic properties of any equilibrium (if this exists) without the need to characterize or prove the existence of an equilibrium.

The paper is organized as follows. Section 2 provides the model. Section 3 studies one-dimensional signals. Section 4 characterizes the equilibrium with two-dimensional signals. Section 5 studies the impact of public signals. Section 6 generalizes the methodology to allow for multidimensional asymmetric signals and other mechanisms. Section 7 concludes. All proofs that are omitted in the main text are collected in the appendix.

[12]See Hong and Shum (2003) for further discussions on the empirical anlysis and use of normal distributions.

[13]An interesting variation of a VCG mechanism for environments in which agents observe multidimensional signals that are not independently distributed is studied by McLean and Postlewaite (2004).

[14]See also Levin, Peck, and Ye (2007).

2 Model

2.1 Payoffs and Information

We study N agents bidding for an indivisible good in an ascending auction. The utility of agent $n \in N$ if he wins the object at price p is given by:

$$u(i_n, c, p) \triangleq \exp(i_n) \cdot \exp(c) - p, \tag{1}$$

where $\exp(\cdot)$ denotes the exponential function, $i_n \in \mathbb{R}$ is an idiosyncratic shock and c is a common shock. If an agent does not win the good he gets a utility equal to 0. To make the notation more compact, we define:

$$v_n \triangleq i_n + c. \tag{2}$$

The payoff shock v_n summarizes the valuation of agent n (note that $\exp(i_n) \cdot \exp(c) = \exp(v_n)$).

The idiosyncratic shocks and the common shock are jointly normally distributed with mean 0 and variance σ_i^2 and σ_c^2 respectively. Assuming that the idiosyncratic and common shock have 0 mean reduces the amount of notation, but it does not have any role in the analysis. The idiosyncratic shocks have a correlation $\rho_i \in (-1/(N-1), 1)$ across agents and are independently distributed of the common shock.[15]

Agent n observes two signals. The first signal agent n observes is a perfectly informative signal about his own idiosyncratic shock i_n. The second signal is a noisy signal about the common shock:

$$s_n \triangleq c + \varepsilon_n, \tag{3}$$

where ε_n is a noise term independent across agents, independent of all other random variables in the model and normally distributed with variance σ_ε^2. The private information of agent n is summarized by the pair of random variables (i_n, s_n). If every agent n observed only signal i_n, this would be a pure private values model. If every agent n observed only signal s_n, this would be a pure common values model.

In a model of an oil field, $\exp(c)$ can be interpreted as the size of the oil field and

[15] The minimum statistically feasible correlation is $-1/(N-1)$. Hence, we do not impose any restrictions on the set of possible feasible correlations beyond the fact that it must be an interior correlation.

$\exp(i_n)$ can be interpreted as the technology of firm n. The total amount of oil that firm n, with technology $\exp(i_n)$, can extract from an oil reserve $\exp(c)$ is equal to $\exp(i_n) \cdot \exp(c)$. Li, Perrigne, and Vuong (2000) use log-additive payoffs (as in (1)) to study Outer Continental Shelf wildcat auctions.

Multiplying the utility function by -1, the model can be interpreted as the procurement of a project, with $\exp(i_n) \cdot \exp(c)$ being cost of delivering the project. $\exp(i_n)$ can be interpreted as the total amount of inputs that bidder n needs to complete the project and $\exp(c)$ can be interpreted as a price index of the inputs needed to complete the project. Hong and Shum (2002) use log-additive payoffs (as in (1)) to study procurements held by the state of New Jersey.

2.2 Ascending Auction

We study an ascending auction.[16] An auctioneer rises the price continuously. At each moment in time, an agent can drop out of the auction, in which case the agent does not pay anything and does not get the object. The last agent to drop out of the auction wins the object and pays the price at which the second to last agent dropped out of the auction.[17] [18] As each drop-out time is associated to a unique price, we often use the words price and drop-out time interchangeably.

The outcome of the ascending auction is described by the order in which each agent drops out and the price at which each agent drops out. The number of agents left in the auction when agent n dropped out of the auction is denoted by a permutation π.[19] For example, the identity of the last agent to drop out of the auction is given by $\pi^{-1}(1)$. The price at which agents drop out of the auction is denoted by $p_1 > .... > p_N$. Hence, for any strategy profile the expected utility of agent n is:

$$\mathbb{E}[\mathbb{1}\left\{\pi^{-1}(1) = n\right\}(\exp(i_n) \cdot \exp(c) - p_2)],$$

[16] We follow Krishna (2009) in the formal description of the ascending auction.

[17] We assume that the auction continues until all agents have dropped out of the auction. The price at which the last agent drops out is obviously payoff irrelevant because he only pays the price at which the second to last agent dropped out of the auction. This allow us to simplify the notation in some parts of the paper because there is always one drop-out time for each agent.

[18] In case of a tie the good is sold to each agent that was last to drop out with equal probability. In equilibrium, there will be no ties and an agent will not be able to tie with another agent by changing his strategy unilaterally. Hence, as it is standard in an ascending auction, the tie breaking rule does not matter.

[19] A permutation is a bijective function $\pi : N \to N$.

where $\mathbb{1}\{\cdot\}$ is the indicator function. We study the symmetric Nash equilibria of the auction.

A symmetric strategy of agent n is a set of functions $\{P_n^k\}_{k\in N}$, with

$$P_n^k : \mathbb{R}^2 \times \mathbb{R}^{N-k} \to \mathbb{R}_+. \tag{4}$$

The function $P_n^k(i, s_n, p_{k+1}, ..., p_N)$ is the drop-out time of agent n, when k agents are left in the auction and the observed drop-out times are $p_N < ... < p_{k+1}$. The function $P_n^k(i, s_n, p_{k+1}, ..., p_N)$ must satisfy:

$$P_n^k(i, s_n, p_{k+1}, ..., p_N) \geq p_{k+1}.$$

That is, agent n cannot drop out of the auction at a price lower than the price at which another agent has already dropped out. Note that we restrict attention to symmetric equilibria in symmetric environments. Hence, it is sufficient to specify the price at which an agent dropped of the auction but the identity of the agent is irrelevant (see Section 6 for a generalization).

3 Benchmark: One-Dimensional Signals

We first study one-dimensional signals. The analysis of one-dimensional environments will be helpful to understand the analysis of two-dimensional environments. The results in this section are either direct corollaries or simple extensions of results that are well known in the literature.

3.1 Information Structure

We assume agent n observes a one-dimensional signal:

$$s_n' = i_n + b \cdot (c + \varepsilon_n), \tag{5}$$

where $b \in \mathbb{R}_+$ is an exogenous parameter. That is, agent n observes only a linear combination of the two-dimensional signal (i_n, s_n).

The one-dimensional signal (5) provides a parametrized class of information structures that allows to span from pure common values to pure private values. If $b = 0$, then the model is a pure private value auction. The social surplus created will be large and the winner's curse will be low. If $b \to \infty$, then the model is a pure common value auction. The social surplus created will be low and the winner's curse will be high.

The specific form of the signal (in (5)) makes the connections to the model in which agents observe both signals separately more transparent. This class of one-dimensional signals is essentially a particular case of the model studied by Milgrom and Weber (1982).[20] Although we believe (5) provides a natural class of one-dimensional information structures, to the best of our knowledge, there is no paper that studies this specific class of signals except for the case $b = 1$.[21]

3.2 Characterization of Equilibrium with One-Dimensional Signals

We now characterize the equilibrium of the ascending auction. We relabel agents such that the realization of signals satisfy:

$$s'_1 > ... > s'_N.$$

As signals are noisy, we might have that the order over valuations is not preserved. For example, we may have $v_{n+1} > v_n$ (even though by construction $s'_{n+1} \leq s'_n$).

The expectation of v_n assuming that signals $(s'_1, ..., s'_{n-1})$ are equal to s'_n (that is, assuming that all signals higher than s'_n are equal to s'_n) is denoted by:

$$\mathbb{E}[v_n | s'_n, ..., s'_n, s'_{n+1}, ..., s'_N]. \tag{6}$$

For example, if $N = 3$, then $\mathbb{E}[v_2 | s'_2, s'_2, s'_3]$ denotes the expected valuation of the agent with the second highest signal, conditional on the realization of his own signal, the signal of agent 3, and assuming that the realization of agent 1's signal is equal to s'_2.

[20] If $b \leq 1$, then this environment is a particular case of the model studied by Milgrom and Weber (1982). If $b > 1$, then this environment may fail to satisfy all the assumptions in Milgrom and Weber (1982) but their analysis goes through without important changes. For example, if $b > 1$ and $\sigma^2_\varepsilon = 0$, then this information structure would not satisfy a monotonicity assumption in Milgrom and Weber (1982). In particular, in this case the utility of agent n will be decreasing in the realization of the signal of agent m. The failure of this monotonicity condition is "mild enough" that all the analysis in Milgrom and Weber (1982) goes through unchanged.

[21] Hong and Shum (2002) study a model in which the payoff environment is as in (1) and agents observe one-dimensional signals as in (5) with $b = 1$ (see also Hong and Shum (2003)). In Wilson (1998) agents observe two-dimensional signals as in our model. Yet, the shocks are drawn from a diffuse prior (this corresponds to taking the limits $\sigma^2_c \to \infty$, $\sigma^2_i \to \infty$ and $\rho_i \to 1$ at a particular rate). For this reason the model reduces to a one-dimensional signal as in (5) with $b = 1$ (see also Hong and Shum (2002) for a discussion).

Proposition 1 (Equilibrium of Ascending Auction)**.**
The ascending auction with one-dimensional signals as in (5) *has a Nash equilibrium in which agent n's drop-out time is given by:*

$$p_n = \mathbb{E}[\exp(v_n)|s'_n, ..., s'_n, s'_{n+1}..., s'_N]. \tag{7}$$

In equilibrium, agent 1 *gets the good and pays* $p_2 = \mathbb{E}[\exp(v_2)|s'_2, s'_2, s'_3, ..., s'_N]$.

Proposition 1 provides the classic equilibrium characterization found in Milgrom and Weber (1982). This is essentially the unique symmetric equilibrium.[22] In equilibrium the agent with the n-th highest signal drops out of the auction at his expected valuation conditional on the signals observed by the agents that already dropped out of the auction (that is, agents $m > n$) and assuming that the $n-1$ signals that are higher than s'_n are equal to s'_n.

The equilibrium strategies (see (7)) satisfy the following two conditions: (i) agent 1 does not regret winning the good at price p_2, and (ii) every agent $m > 1$ does not regret dropping out of the auction instead of waiting until agent 1 drops out of the auction. Formally, the two conditions are written as follows:

$$\mathbb{E}[\exp(v_1)|s'_1, ..., s'_N] - \mathbb{E}[\exp(v_2)|s'_2, s'_2, ..., s'_N] \geq 0; \tag{8}$$

$$\forall m > 1, \quad \mathbb{E}[\exp(v_m)|s'_1, ..., s'_N] - \mathbb{E}[\exp(v_1)|s'_1, s'_1, s'_2, ..., s'_{m-1}, s'_{m+1}, ..., s'_N] \leq 0. \tag{9}$$

Condition (8) states that the expected valuation of agent 1 conditional on all the signals is greater than the price at which agent 2 drops out of the auction. Hence, agent 1 does not regret winning the good. Condition (9) states that the expected valuation of agent m conditional on all the signals is less than the price at which agent 1 would drop out of the auction if agent m waits until agent 1 drops out of the auction.[23] Hence, agent $m > 1$ does not regret dropping out of the auction, even if he observed the realization of all the signals. This constitutes an important property; the strategy profile (see (7)) would still be a Nash equilibrium even if every agent observed the realization of the

[22] Bikhchandani, Haile, and Riley (2002) show that there is a continuum of symmetric equilibria. Nevertheless, the allocation and equilibrium price is the same across equilibria. See Krishna (2009) for a textbook discussion.

[23] Note that if agent $m > 1$ waits until all other agents drop out of the auction, then he would win the good at price: $\tilde{p}_2 = \mathbb{E}[\exp(v_1)|s'_1, s'_1, s'_2, ..., s'_{m-1}, s'_{m+1}, ..., s'_N]$. This is the expected valuation of agent 1, conditional on the signals of all agents different than agent m, and assuming that agent m observed a signal equal to agent 1.

signals of all other agents.[24]

We show that the social surplus generated by the auction is decreasing in the weight b.

Proposition 2 (Comparative Statics: Social Surplus)**.**
The social surplus $E[\exp(v_1)]$ *is decreasing in* b*.*

Proposition 2 provides an intuitive result. As b becomes larger, the correlation between the drop-out time of agent n and the noise term ϵ_n increase. This leads to inefficiencies that reduce the social surplus. If $b \to 0$, then the drop-out time of an agent is perfectly correlated with his idiosyncratic shock. Hence, the auction is efficient. If $b \to \infty$, then the drop-out time of an agent is perfectly correlated with the noise term. Hence, the allocation of the object is independent of the realization of the idiosyncratic shocks.

4 Characterization of Equilibrium

We now characterize a class of equilibria when agents observe two-dimensional signals (i_n, s_n). The first step of the equilibrium characterization is to project the signals into a one-dimensional object. We call this an equilibrium statistic. We then show that there exists a class of equilibria in which each agent behaves as if he observes only his equilibrium statistic. After we characterize the equilibrium, we provide an intuition on how the equilibrium statistic is determined. As an illustration of the subtle mapping between the information structure and the outcome of the auction, we show that the ascending may have multiple symmetric equilibria that generate different social surpluses.

4.1 Equilibrium Statistic

The fundamental object that allow us to characterize an equilibrium is the equilibrium statistic. This is the projection of signals that determine the drop-out time of agents.

[24] Formally, this is an ex post equilibrium. We discuss this in more detail in Section 6.2.

Definition 1 (Equilibrium Statistic)**.**
The random variables $\{t_n\}_{n\in N}$ *are an equilibrium statistic if there exists* $\beta \in \mathbb{R}$ *such that for all* $n \in N$*:*

$$t_n = i_n + \beta \cdot s_n; \tag{10}$$

$$\mathbb{E}[v_n|i_n, s_n, t_1, ..., t_N] = \mathbb{E}[v_n|t_1, ..., t_N]. \tag{11}$$

An equilibrium statistic is a linear combination of signals that satisfy statistical condition (11). The expected value of v_n conditional on all equilibrium statistics $\{t_n\}_{n\in N}$ is equal to the expected value of v_n conditional on all the equilibrium statistics $\{t_n\}_{n\in N}$ and conditional on (i_n, s_n). In other words, if agent n knows the equilibrium statistic of other agents, then the equilibrium statistic of agent n is a sufficient statistic of both signals observed by agent n to compute the expectation of v_n. Note that the weight β is the same for all agents. This is because we focus on symmetric equilibria, and hence, all agents use the same weight. Throughout the paper, we use t_n to denote an equilibrium statistic.

We characterize the set of equilibrium statistics.

Proposition 3 (Equilibrium Statistic)**.**
A linear combination of signals $t_n = i_n + \beta \cdot s_n$ *is an equilibrium statistic if and only if* β *is a root of the cubic polynomial:*

$$x_3 \cdot \beta^3 + x_2 \cdot \beta^2 + x_1 \cdot \beta + x_0, \text{ with:}$$

$$x_3 = \frac{1}{(1-\rho_i)(1+(N-1)\rho_i)} \frac{(\sigma_\varepsilon^2 + N \cdot \sigma_c^2)}{\sigma_i^2 \sigma_c^2} \ ; \ x_2 = \frac{-1}{(1-\rho_i)\sigma_i^2} \ ; \ x_1 = \frac{\sigma_\varepsilon^2 + \sigma_c^2}{\sigma_\varepsilon^2 \sigma_c^2} \ ; \ x_0 = \frac{-1}{\sigma_\varepsilon^2}. \tag{12}$$

Moreover, all real roots of the polynomial are between 0 and 1.

Proposition 3 shows that the set of equilibrium statistics is determined by a cubic equation. The cubic equation always has at least one real root. We first provide the equilibrium characterization of the ascending auction and later provide an intuition on how the information structure determines the equilibrium statistic.

4.2 Equilibrium Characterization

We show that for every equilibrium statistic there exists a Nash equilibrium in which each agent n behaves as if he observed *only* his equilibrium statistic t_n. The characterization of the equilibrium strategies are analogous to Section 3, but using the equilibrium statistic. It is important to highlight that agents observe two-dimensional signals (i_n, s_c). Hence, the equilibrium statistic is only an auxiliary element that helps characterize a class of equilibria.

Analogous to the analysis of one-dimensional signals, we assume that agents are ordered as follows:

$$t_1 > ... > t_N. \tag{13}$$

If there are multiple equilibrium statistics, then there will be one Nash equilibrium for each equilibrium statistic. Different equilibrium statistics induces a different order (as in (13)), so the Nash equilibrium is described in terms of the order induced by each equilibrium statistic.

Theorem 1 (Symmetric Equilibrium with Multidimensional Signals)**.**
For every equilibrium statistic, there exists a Nash equilibrium in which agent n's drop-out time is given by:

$$p_n = \mathbb{E}[\exp(v_n)|t_n, ..., t_n, t_{n+1}, ..., t_N], \tag{14}$$

In equilibrium, agent 1 *gets the object and pays* $p_2 = \mathbb{E}[\exp(v_2)|t_2, t_2, ..., t_N]$.

Theorem 1 shows that there exists a class of equilibria in which agents project their signals into a one-dimensional statistic using the equilibrium statistic $t_n = i_n + \beta \cdot s_n$. In equilibrium every agent n behaves as if he observed *only* t_n, which is a one-dimensional object.

We prove Theorem 1 in two steps: (i) we provide the equilibrium conditions, and (ii) we show that these conditions are satisfied. The equilibrium conditions are similar to (8) and (9): an agent's drop-out time remains optimal even after observing the realized drop-out times of all agents in the auction.[25] Importantly, agent n can learn the equilibrium statistic of agent m by looking at his drop-out time, but not both signals

[25]Formally, the Nash equilibrium we characterize is also a posterior equilibrium (see Green and Laffont (1987)).

agent m observed separately. Therefore, the optimality condition of agent n's drop-out time takes into account both signals observed by agent n and the equilibrium statistic of other agents $\{t_m\}_{m\neq n}$. We then use the properties of the equilibrium statistic to show that the optimality conditions are satisfied. We do this by reducing the equilibrium conditions in the two-dimensional environment to the same equilibrium conditions that arise in a one-dimensional, but replacing the signals with the equilibrium statistics.

Proof of Theorem 1. We check the following two conditions: (i) agent 1 never regrets winning the object at price p_2 after all agents $m > 1$ drop out of the auction; and (ii) every agent $m > 1$ does not regret dropping out of the auction instead of waiting until all other agents (including agent 1) drop out of the auction. Formally, the conditions that need to be satisfied are the following:

$$\mathbb{E}[\exp(v_1)|i_1, s_1, t_1, ..., t_N] - \mathbb{E}[\exp(v_2)|t_2, t_2, ..., t_N] \geq 0; \tag{15}$$

$$\forall m > 1, \quad \mathbb{E}[\exp(v_m)|i_m, s_m, t_1, ..., t_N] - \mathbb{E}[\exp(v_1)|t_1, t_1, t_2, ..., t_{m-1}, t_{m+1}, ..., t_N] \leq 0. \tag{16}$$

Condition (15) states that the expected valuation of agent 1 conditional on both signals he observes and the information he learns from the drop-out time of other agents is greater than the price at which agent 2 drops out of the auction. Hence, agent 1 does not regret winning the good. Condition (16) states that the expected valuation of agent m conditional on both signals he observes and the information he learns from the drop-out time of other agents is less than the price at which agent 1 would drop out of the auction if agent m waits until agent 1 drops out of the auction. Hence, agent $m > 1$ does not regret dropping out of the auction before agent 1.

We now prove that (15) and (16) are satisfied. Using (11), we note that:

$$\forall n, \quad \mathbb{E}[\exp(v_n)|i_n, s_n, t_1, ..., t_N] = \mathbb{E}[\exp(v_n)|t_1, ..., t_N].$$

Note that in (11) the expectations are taken without the exponential function. Yet, as all random variables are Gaussian, the distribution of v_n conditional on $(i_n, s_n, t_1, ..., t_N)$ is the the same as the distribution of v_n conditional on $(t_1, ..., t_N)$.[26] Hence, if (11) is

[26]For any (x, y) jointly normally distributed, $x|y \sim \mathcal{N}(\mathbb{E}[x|y], \sigma_x^2 - var(\mathbb{E}[x|y]))$. Since $\mathbb{E}[v_n|i_n, s_n, t_1, ..., t_N] = \mathbb{E}[v_n|t_1, ..., t_N]$, we also have that $var(\mathbb{E}[v_n|i_n, s_n, t_1, ..., t_N]) = var(\mathbb{E}[v_n|t_1, ..., t_N])$. Hence, $v_n|(i_n, s_n, t_1, ..., t_N) \sim v_n|(t_1, ..., t_N)$.

satisfied, then (11) is also satisfied for any function of v_n. Hence, (15) and (16) are satisfied if and only if:

$$\mathbb{E}[\exp(v_1)|t_1, ..., t_N] - \mathbb{E}[\exp(v_2)|t_2, t_2, ..., t_N] \geq 0; \tag{17}$$

$$\forall m > 1, \quad \mathbb{E}[\exp(v_m)|t_1, ..., t_N] - \mathbb{E}[\exp(v_1)|t_1, t_1, t_2, ..., t_{m-1}, t_{m+1}, ..., t_N] \leq 0. \tag{18}$$

Note that checking (17) and (18) is equivalent to checking the equilibrium conditions in one-dimensional environments (see (8) and (9)). That is, since in Section 3 we proved that (8) and (9) are satisfied, then (17) and (18) are also satisfied (just replace b with β). ■

In the class of equilibria characterized in Theorem 1, the analysis in Section 3 can be applied with the modification that we need to replace s'_n with t_n (or alternatively, replace b with β). The key element of the characterization that determines the qualitative properties of the equilibrium is the weight that the equilibrium statistic places on the signals about the common shock: namely β. If $\beta \approx 0$, then the outcome of the auction will be efficient and the outcome will resemble a pure private value environment. As β increases, the social surplus decreases and the model resembles more an interdependent value environment. Note that all equilibrium statistics satisfy $\beta \leq 1$. Yet, if $\beta \approx 1$ and the variance of the idiosyncratic shock is small enough (relative to the variance of the common shock and the noise term), then the model will resemble a pure common values model. The natural question that arises is how does the information structure determine the equilibrium statistic.

4.3 Analysis of the Equilibrium Statistic

We now provide an intuition on how β is determined. Analogous to how the Nash equilibrium of any game can be understood by analyzing agents' best response function, we understand how the equilibrium statistic is determined by analyzing how the expectations are determined "out of equilibrium". We fix an exogenous one-dimensional signal:[27]

$$s'_m = s_m + \frac{1}{b} i_m, \tag{19}$$

[27] The signal is as in Section 3. We divided the signal by $1/b$. This obviously makes no difference, but some comparisons will be more transparent.

and define $\gamma_i, \gamma_s, \gamma' \in \mathbb{R}$ implicitly as follows:[28]

$$\mathbb{E}[v_n | i_n, s_n, \{s'_m\}_{m \neq n}] = \gamma_i \cdot i_n + \gamma_s \cdot s_n + \frac{\gamma'}{N-1} \cdot \sum_{m \neq n} s'_m. \tag{20}$$

The weight b is an equilibrium statistic if and only if it satisfies:

$$b = \frac{\gamma_s}{\gamma_i}.$$

We provide an intuition on how the equilibrium statistic is determined by characterizing how (γ_i, γ_s) change with b. We provide a lemma that formalizes how γ_i and γ_s change with b.

Lemma 1 (Best Responses)**.**
The weights (γ_i, γ_s) satisfy:

1. *γ_s is decreasing in b*
2. *If $\rho_i > 0$, then γ_i is strictly quasi-convex in b. Moreover, in the limit $b \to 0$ and $b \to \infty$, $\gamma_i \to 1$*

Lemma 1 illustrates how agent n modifies the weights he places on his private signals if agents $m \neq n$ change the weight he places on his private signals. If agents $m \neq n$ place a larger weight on their signal about the common shock, then agent n will place a smaller weight on his signal about the common shock. On the other hand, the weight that agents $m \neq n$ place on their signal about the common shock has a non-monotonic effect on the weight that agent n places on his idiosyncratic shock. Note that γ_i is decreasing in b (at least in some range of b). This shows that the weight agents place on their idiosyncratic shock exhibits a complementarity: if agent m increases the weight he places on i_m, then agent n will also increase the weight he places on i_n. This is the key intuition for the multiplicity of equilibria we illustrate in the following section. We now provide an intuition for both results in Lemma 1.

Analysis of γ_s. The informativeness of s'_m about c is increasing in b. The amount that agent n relies on his own private signal about c is decreasing in the amount of additional information that agent n has about c. Hence, γ_s is decreasing in b.

[28]Note that by symmetry the weight on all signals $\{s'_m\}_{m \neq n}$ are the same (denoted by γ').

Analysis of γ_i. The analysis of γ_i is more subtle. From the perspective of agent n, i_m is a noise term in s'_m. That is, agent n would like to observe simply s_m. If $\rho_i = 0$, then γ_i is constant in b and equal to 1. This is natural, an agent knows his own idiosyncratic shock and this is independent of the noise in s'_m. Hence, he just places a weight of 1 on this signal. The conceptual difference between $\rho_i = 0$ and $\rho_i \neq 0$ is that in the latter case i_n has an impact on agent n's beliefs about c. When i_n is correlated with i_m, agent n uses i_n to filter out the noise in s'_m. The non-monotonicity of γ_i comes from the fact that i_n is used to filter out part of the noise in s'_m.[29]

The reason that $\gamma_i < 1$ (when $\rho_i > 0$) is that the direct effect of observing a high or a low idiosyncratic shock is offset by updating the beliefs about the common shock in the opposite direction. This can be clearly illustrated in terms of the oil field example.

Suppose agents are bidding for an oil field, the technology shocks are correlated ($\rho_i > 0$), and agent n observes a very high technology shock ($i_n >> 0$). If agent n observes that agent m dropped out early from the auction, then he must infer that agent m observed a very bad signal about the size of the oil field ($s_m << 0$). After all, technology shocks are correlated, and hence agent n expects agent m to also observe a relatively high technology shock. Conversely, if agent n observed a low technology shock, then agent n would not become so pessimistic about the size of the oil field. In this way, the direct effect of observing a high or a low technology shock is offset by updating the beliefs about the size of the oil field in the opposite direction. Hence, conditional on the drop-out time of agent m, agent n's technology shock is not very informative about agent n's preferences. This makes agents bid less aggressively on their technology shock (or equivalently, decreases γ_i). This ultimately reduces the social surplus.

Comparative Statics with Respect to ρ_i As suggested by the discussion, ρ_i plays an important role in determining β. It is possible to show that, if there exists a unique equilibrium, then β is increasing in ρ_i.[30] Hence, if there exists a unique equilibrium, the efficiency of the auction is decreasing in ρ_i.[31] If the idiosyncratic shocks

[29] That is, agent n would like to observe simply s_m. If $b \to \infty$, then agent n can observe s_m directly, and hence, agent n does not need to use i_n to filter out the noise s'_m. Hence, in this case $\gamma_i = 1$. If $b \to 0$, then signal s'_m does not provide any information about c, and hence, agent n does not use s'_m to predict c at all. Again, in this case $\gamma_i = 1$. It is only for intermediate values of b that agent n uses i_n to predict c.

[30] If there exists 3 equilibria, then the comparative statics is reversed in one of the equilibrium.

[31] To be more precise: as ρ_i increases, the correlation between the identity of the winner of the auction and the realization of the idiosyncratic shock of this agent decreases. As $\rho_i \to 1$, the good is allocated independent of the

are independently distributed ($\rho_i = 0$), then there is no complementarity. This implies that there is a unique equilibrium (within the class of equilibria studied in Theorem 1). The formal statements and proofs of the aforementioned results can be found in the online appendix.

4.4 Illustration of the Equilibrium Multiplicity

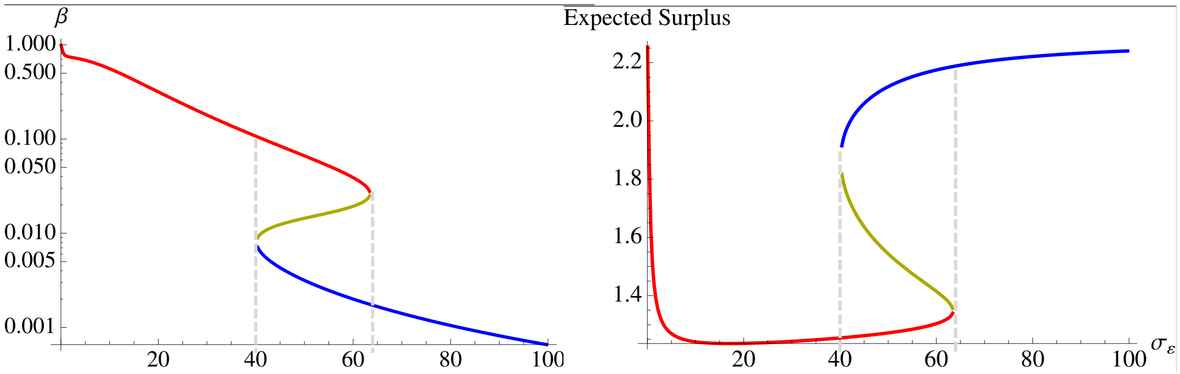
The cubic polynomial that determines the set of equilibrium statistics (see (12)) may have multiple roots. This implies that an ascending auction with multidimensional signals may have multiple symmetric equilibria; a different equilibrium for every root. The multiplicity of equilibria is caused by the complementarity in the weight agents place on their signals (discussed in the previous section).

We illustrate the multiplicity of equilibria in a parametrized example. In Figure 1a we plot the set of equilibrium statistics for different values of the variance of the noise. The different colors in the plot corresponds to the different roots of the cubic polynomial that determines the set of equilibrium statistics (see (12)). We can see that there are values of the noise term for which there are multiple equilibria (e.g. $\sigma_\varepsilon = 50$).

In Figure 1b we plot the expected social surplus generated in the auction corresponding to the equilibrium statistic shown in Figure 1a. There is one equilibrium in which β is small (plotted in blue). This equilibrium will look more like a private value environment: the social surplus generated will be large and the winner's curse will be low. There is one equilibrium in which β is large (plotted in red). This equilibrium will look more like a common value environment: the social surplus generated will be small and the winner's curse will be high. We do not plot the revenue or the buyers' rents, but these are qualitatively similar to the social surplus generated in the auction.

The social surplus generated in the auction is non-monotonic in the size of the noise term (σ_ε^2). This is because two different effects change the social surplus. First, for a fixed β, as σ_ε^2 increases, the correlation between the drop-out time of an agent and the noise term ε_n increases. This decreases the social surplus. On the other hand, as σ_ε^2 increases the weight on s_n decreases (and hence, the weight on the noise term ε_n decreases). Since the weight on the noise term decreases, this decreases the correlation

idiosyncratic shocks. Note that in the limit $\rho_i = 1$ this is a pure common value environment, and hence, any allocation is efficient.



(a) Equilibrium Statistic. (b) Expected Social Surplus ($\mathbb{E}[exp(v_1)]$).

Figure 1: Outcome of ascending auction for $\sigma_c = 5/2$, $\sigma_i = 0.6$, $\rho_i = 3/4$ and $N = 50$.

between the drop-out time of an agent and the noise term ε_n. This increases the social surplus.

If the noise term is too large or too small ($\sigma_\varepsilon^2 \to \infty$ or $\sigma_\varepsilon^2 \to 0$), then there is a unique equilibrium. This is because in both limits the model approaches a private values model. If the variance is too small, then agents know almost perfectly the realization of c just looking at their private information. If the variance is too large, then agents ignore s_n, and hence the model is again a private values model. Note that in both limits the equilibrium is efficient. This is not only true for this parametrized example; always in these extreme cases there is a unique equilibrium (we prove this in the online appendix).

5 Impact of Public Signals

In this section we study how the precision of a public signal affects the social surplus and the revenue generated in the auction. The analysis shows that comparative statics in two-dimensional environments may be different than in one-dimensional environments. This is because in the class of equilibria characterized by Theorem 1 agents behave as if they observed only the equilibrium statistic. Yet, the equilibrium statistic is an endogenous object. This implies that the comparative statics are partially determined by changes in the equilibrium statistic.

5.1 Public Signals

We now study the impact of public information on the equilibrium outcome. In a model with one-dimensional signals, it is natural to consider a public signal about the average valuation across agents. In our environment, the valuation of an agent is determined by two payoff shocks. Hence, it is natural to consider a public signal about the common shock and a public signal about the average idiosyncratic shock.

We assume agents have access to two public signals (in addition to (i_n, s_n)). The first signal provides agents with more information about the common shock:

$$\bar{s}^1 = c + \varepsilon^1, \tag{21}$$

where ε^1 is independent of all random variables defined so far and normally distributed with variance σ_1^2. Signal $\bar{s}^1$ can be interpreted as disclosing additional information about the good (e.g. more information about the size of an oil field). The second public signal provides agents with information about the average idiosyncratic shock:

$$\bar{s}^2 = \frac{1}{N}\sum_{n\in N} i_n + \varepsilon^2, \tag{22}$$

where ε^2 is independent of all random variables defined so far and normally distributed with variance σ_2^2. Signal $\bar{s}^2$ can be interpreted as providing more information about the bidders' characteristics (e.g. more information about the average cost of extracting oil in the industry).[32]

Agent n observes the signals $(i_n, s_n, \bar{s}^1, \bar{s}^2)$. The analysis in Section 4 can be extended in a simple way to accommodate for public signals. The only modification to the analysis is that the public signals must be added as a conditioning variables in the expectations. That is, the definition of an equilibrium statistic (see (11)) must be modified as follows:

$$\mathbb{E}[v_n|i_n, s_n, t_1, ..., t_N, \bar{s}^1, \bar{s}^2] \;\; = \;\; \mathbb{E}[v_n|t_1, ..., t_N, \bar{s}^1, \bar{s}^2]. \tag{23}$$

[32] All the results go through in the same way if instead of having a public signal $\bar{s}^2 = \sum_{n\in N} i_n/N + \varepsilon^2$ each agent n observes $N-1$ private signals on the idiosyncratic shocks of agents $m \neq n$. That is, if agent n observes signals $s_n^m = i_m + \varepsilon_n^m$ for all $m \neq n$.

Additionally, the strategy of agents (see (14)) must be modified as follows:

$$p_n = \mathbb{E}[\exp(v_n)|t_n, ..., t_n, t_{n+1}, ..., t_N, \bar{s}^1, \bar{s}^2]. \tag{24}$$

Clearly, under these two modifications all the analysis in Section 4 remains the same.

5.2 Impact of Public Signals on the Social Surplus

We study the impact of the public signals on the social surplus. The social surplus is equal the expected valuation of the agent who observed the highest equilibrium statistic (that is, $\mathbb{E}[\exp(v_1)]$).

Proposition 4 (Comparative Statics of Public Signals: Social Surplus).
If the ascending auction has a unique equilibrium, then the social surplus is decreasing in σ_2^2 and σ_1^2. In the limit:[33]

$$\lim_{\sigma_2^2 \to 0} \mathbb{E}[\exp(v_1)] = \lim_{\sigma_1^2 \to 0} \mathbb{E}[\exp(v_1)] = \mathbb{E}[\max_{n \in N} \exp(v_n)].$$

Proposition 4 shows that the social surplus increases with the precision of the public signals. In the limit in which one of the public signals is arbitrarily precise, the equilibrium approaches the efficient outcome. Note that for any value of σ_ε^2 the ascending auction would implement the efficient outcome if agents "ignored" signal s_n. Hence, a precise enough public signal reduces the weight that agents place on s_n all the way to 0. If the ascending auction has three equilibria then the social surplus is increasing in the precision of the public signal in the equilibria with the highest and the lowest β, while the comparative static is reversed in the equilibrium with the β in the middle.

The intuition on why the social surplus is decreasing in σ_1^2 is simple. As the public information about c is more precise, an agent needs to place less weight on his private signal s_n to predict c. This implies that the correlation between the drop-out time of an agent and the realization of the noise term ε_n decreases. Hence, the social surplus increases.

[33] We study the ex ante expected social surplus instead of the interim expected social surplus in order to avoid having to take limits of random variables. The statement goes through without the expectations by considering convergence in probability.

The reason that $\bar{s}^2$ changes the social surplus is that it changes how agent n's idiosyncratic shock affects his beliefs about the common shock. As explained in Section 4.3, if $\rho_i > 0$, then the direct effect of observing a high idiosyncratic shock is partially offset by updating the beliefs about the common shock in the opposite direction. This makes agent n bid less aggressively on his idiosyncratic shock, which reduces the social surplus. If there is a public signal about i_m, then the weight on i_n to predict i_m is reduced. Hence, the public signal decreases the correlation between agent n's idiosyncratic shock and agent n's beliefs about the common shock. Hence, agent n trades more aggressively on his idiosyncratic shock, which increases the social surplus.

The equilibrium converges to the efficient outcome as $\bar{s}^2$ becomes arbitrarily precise because in the limit the effects are reversed. Namely, the direct effect of observing a high idiosyncratic shock is reinforced by updating the beliefs about the common shock in the same direction. Hence, agents trade evermore aggressively on their idiosyncratic shocks. This increases the social surplus to the efficient levels. This is essentially the same that happens if idiosyncratic shocks are negatively correlated.

5.3 Impact of Public Information on Revenue

We now study the impact of the public signal about the common shock on the revenue. We denote by $\max^{(2)}\{\cdot\}$ the second order statistic (that is, the second maximum).

Proposition 5 (Public Signal About Common Shock on Revenue)**.**
If the signal about the common shock becomes arbitrarily precise:

$$\lim_{\sigma_1^2 \to 0} \mathbb{E}[p_2] = \mathbb{E}[\max_{n\in N}{}^{(2)} \quad \exp(v_n)].$$

Proposition 5 shows that, as the public signal about c becomes arbitrarily precise ($\sigma_1^2 \to 0$), the revenue approaches the expected second highest valuation. The intuition is that in the limit, agents ignore their private signal s_n. Hence, it is "as if", the only private signal they observe is i_n. Hence, in this limit, it is "as if" agents had private values. We now study the impact of a public signal about the average idiosyncratic shock.

Proposition 6 (Public Signal About Average Idiosyncratic Shock on Revenue)**.** *If the signal about the average idiosyncratic shock becomes arbitrarily precise:*

$$\lim_{\sigma_2^2 \to 0} \mathbb{E}[p_2] = 0.$$

Proposition 6 shows that, as $\bar{s}^2$ becomes arbitrarily precise ($\sigma_2^2 \to 0$), the revenue becomes arbitrarily close to 0. Note that the price is greater or equal than 0 in every realization of the auction. Hence, the price converges in distribution to 0.

We provide an intuition of Proposition 6. To simplify the exposition, suppose $N = 2$. The fundamental component of the analysis is that the different between agent 2's valuation and agent 2's drop-out time is increasing in $\bar{s}^2$. That is,

$$\mathbb{E}[v_2|t_1, t_2, \bar{s}^2] - \mathbb{E}[v_2|t_2, t_2, \bar{s}^2],$$

is increasing in $\bar{s}^2$. This is because $\bar{s}_2$ provides information about i_1, which makes t_1 more informative about the common shock. Hence, agent 2 can place a larger weight on t_1 to predict c. The revenue converges to 0 in the limit because the weight that the expectation $\mathbb{E}[v_2|t_1, t_2, \bar{s}^2]$ places on t_1 diverges to infinity. This is because the expectation is increasing in t_1 and decreasing in $\bar{s}^2$. Hence, the agent places a arbitrarily large weight on both t_1 and $\bar{s}^2$.

To provide a further intuition why the weight that the expectation $\mathbb{E}[v_2|t_1, t_2, \bar{s}^2]$ places on t_1 diverges to infinity consider the following. If $\sigma_2^2 \approx 0$, the equilibrium statistic of agent $n \in \{1, 2\}$ is given by:[34]

$$t_n = i_n + \epsilon \cdot s_n,$$

where $\epsilon \approx 0$ (equal to 0 in the limit). Since $\sigma_2^2 \approx 0$ agent 2 can infer s_1 from t_1 almost perfectly. If $\sigma_2^2 \approx 0$, then the expectation can be approximated as follows:

$$\mathbb{E}[v_2|t_1, t_2, \bar{s}^2] \approx t_2 + \frac{\sigma_c^2}{\sigma_c^2 + \sigma_\varepsilon^2/2} \cdot \frac{1}{\epsilon} \cdot (\frac{t_1 + t_2}{2} - \bar{s}^2). \tag{25}$$

Note that the expectations places a weight of order $1/\epsilon$ on the last term. The second

[34]Proposition 4 implies that in the limit $t_2 \approx i_2$.

term in (25) does not diverge precisely because $\sigma_2^2 \approx 0$, and hence, $((t_1+t_2)/2-\bar{s}^2) \approx \epsilon \cdot (s_1+s_2)/2$.

The revenue is found by replacing t_1 with t_2 (that is, computing $\mathbb{E}[v_2|t_2,t_2,\bar{s}^2]$). Hence, when computing the revenue the term $1/\epsilon$ multiplies $((t_2+t_2)/2-\bar{s}^2) \approx (t_2-t_1)$, which is negative and does not converge to 0. Hence, $\mathbb{E}[v_2|t_2,t_2,\bar{s}^2]$ diverges to $-\infty$ (which yields 0 when taking the exponential function).

In our model, public signals $\bar{s}^1$ and $\bar{s}^2$ have opposite effects on revenue. The signal about the common shock decreases the weight that agent 2 places on t_1 to compute the expected value of v_2. This decreases the difference between agent 2's expected valuation and the time he drops out of the auction (that is, $(\mathbb{E}[v_2|t_1,t_2,\bar{s}^2]-\mathbb{E}[v_2|t_2,t_2,\bar{s}^2])$), and hence, increases the revenue. In contrast, the signal about the average idiosyncratic shock increases the weight that agent 2 places on t_1 to compute the expected value of v_2. This increases the difference between agent 2's expected valuation and the time he drops out of the auction (that is, $(\mathbb{E}[v_2|t_1,t_2,\bar{s}^2]-\mathbb{E}[v_2|t_2,t_2,\bar{s}^2])$), and hence, decreases the revenue.

The previous discussion shows that to evaluate the impact of a public signal on revenue, it is necessary to consider the nature of the public signal. For this, it is also important to fully account for all private signals of agents. To provide a sharper illustration consider the following signal:

$$\tilde{s}_n \triangleq i_n + \tilde{\varepsilon}_n, \tag{26}$$

where $\tilde{\varepsilon}_n$ is a noise term independent of all other random variables in the model, independent across agents and with a small variance ($var(\tilde{\varepsilon}_n) \approx 0$). Additionally, assume that $\rho_i > 0$. It is easy to check that, if agents observe only $\tilde{s}_n$, then the revenue would be strictly increasing in the precision of $\bar{s}^2$ (this is a particular case of Milgrom and Weber (1982)). If agents observe $(\tilde{s}_n, s_n)$, then the revenue would be decreasing in the precision of $\bar{s}^2$.[35] Hence, adding a private signal to the information structure of agents can reverse the effect of a public signal.

Failure of the Linkage Principle.[36] Proposition 6 can be interpreted as a failure

[35] This can be seen by Proposition 6 and a continuity argument. In Section 6 we generalize the model to any Gaussian information structure, and the equilibrium changes continuously in the variance covariance matrix of the information structure.

[36] The linkage principle states that public signals increase the revenue and ascending auctions yield higher revenue

of the linkage principle.[37] The linkage principle has been shown to fail in other environments.[38] In contrast to the previous literature we show that the linkage principle may fail in natural symmetric environments. This is only due to the multidimensionality of the information structure. Hence, our paper provides a new channel by which the linkage principle may fail.

Failure of Assumptions in Milgrom and Weber (1982). The fact that public signals may decrease the revenue all the way to 0 implies that some of the assumptions in Milgrom and Weber (1982) are not satisfied. In our original model agent n's expected valuation conditional on all signals ($\mathbb{E}[v_n|\{i_m\}_{m\in N},\{s_m\}_{m\in N},\bar{s}^2]$) is non-decreasing in all the conditioning variables and all signals are positively correlated. Hence, all the assumptions in Milgrom and Weber (1982) are satisfied, except for the assumption that private signals are one-dimensional. Yet, we could look at the reduced information structure in which agent n observes only his equilibrium statistic. Under the reduced information structure the expected valuation of agents is decreasing in the realization of $\bar{s}^2$ (see (25)). Hence, the assumptions in Milgrom and Weber (1982) that is not satisfied under the reduced information structure is that the utility of agents is increasing in the realization of the public signals.[39]

6 Extensions

We now discuss how to extend the solution method used in Section 4 to other environments. We first explain how the analysis can be extended to any Gaussian multidimensional information structure, possibly asymmetric. We then explain how the same equilibrium statistic can be used to find a class of Nah equilibrium in other mecha-

than first-price auctions (see Krishna (2009) for a textbook discussion).

[37] Proposition 6 shows that public signals may decrease revenue. Bergemann, Brooks, and Morris (2017) show that the revenue in a first-price auction are bounded away from 0. Hence, Proposition 6 also shows that an ascending auction may yield lower revenue than a first-price auction.

[38] Perry and Reny (1999) show that the linkage principle may fail in multi-unit auctions. The linkage principle has also been shown to fail in environments in which the payoff structure is asymmetric (see Krishna (2009)) and in environments with independent and private values (see Thierry and Stefano (2003)). Axelson and Makarov (2016) shows that the linkage principle fails in common-value auctions when an agent must take an action after winning an object. As in our model, in the model studied by Axelson and Makarov (2016) the bid of an agent does not fully reveal the signal this agent observed, but the reason is that the final payoff of the good is not strictly monotonic in the realization of the signals observed by agents (see also Atakan and Ekmekci (2014)).

[39] Milgrom and Weber (1982) assumes that the utility of agents is increasing in the realization of all signals and signals are positively correlated (strictly speaking, they assume that signals are affiliated, but there is no difference in a Gaussian environment). Of course, it is possible to change the sign of $\bar{s}^2$, in which case the public signal would have a positive impact on agent n's valuation. Yet, in this case the public signal $\bar{s}^2$ would be negatively correlated with t_n, which would break the affiliation property.

nisms. In this section we provide an informal discussion. All the formal results and analysis can be found in the online appendix.

6.1 General Multidimensional Signals

The analysis in Section 4 can be extended in a natural way to any multidimensional Gaussian information structure. Suppose that each agent $n \in N$ observes J signals:

$$\mathbf{s}_n = (s_n^1, ..., s_n^J),$$

where bold fonts denote vectors and superscript denotes the number of the signal. The utility of agent n if he wins the object is equal to:

$$u(v_n) - p,$$

where $v_n \in \mathbb{R}$ is a payoff shock and $u(\cdot)$ is a strictly increasing function. In our baseline model we assumed $u(\cdot) = \exp(\cdot)$. The joint distribution of signals and payoff shocks $(v_1, ..., v_N, \mathbf{s}_1, ..., \mathbf{s}_N)$ is jointly Gaussian, but possibly asymmetrically distributed.

There is a class of equilibria that can be characterized in the same way as we characterized a class of equilibria with symmetric two-dimensional signals. First, we project the signals. For this, we need to find a set of weights $(\boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_N) \in \mathbb{R}^{N \times J}$ such that, for all $n \in N$:

$$\mathbb{E}[v_n | \mathbf{s}_n, \mathbf{s}_1 \cdot \boldsymbol{\beta}_1, ..., \mathbf{s}_N \cdot \boldsymbol{\beta}_N] = \mathbb{E}[v_n | \mathbf{s}_1 \cdot \boldsymbol{\beta}_1, ..., \mathbf{s}_N \cdot \boldsymbol{\beta}_N], \quad (27)$$

where $\boldsymbol{\beta}_n \cdot \mathbf{s}_n$ denotes the dot product. In asymmetric environments the weights of each agent ($\boldsymbol{\beta}_n$) may be different. There always exist weights such that (27) is satisfied.

For each projection of signals:

$$(t_1, ..., t_N) = (\mathbf{s}_1 \cdot \boldsymbol{\beta}_1, ..., \mathbf{s}_N \cdot \boldsymbol{\beta}_N),$$

that satisfy a regularity condition, there exists an equilibrium in which agent $n \in N$ behaves "as if" he observes only the one-dimensional signal t_n. The regularity condition is called the average crossing property. The average crossing property is necessary to guarantee that in asymmetric environments the ascending auction has an ex post equilibrium when agents observe one-dimensional signals.[40] It is simple to check in applications whether the information structure has an equilibrium statistic that satisfies the average crossing condition.[41] Note that the only constraint in the equilibrium characterization comes from characterizing an equilibrium in one-dimensional environments. The projection of signals per se is not a problem, as this always exists. It is worth highlighting that in Section 4 we implicitly checked that an ex post equilibrium exists when agents observe only their equilibrium statistic. We did this by first solving the model in which agents observe only one-dimensional signals in Section 3.

6.2 Other Mechanisms

We now extend the methodology used in Section 4 to find a class of Nash equilibrium when agents observe multidimensional Gaussian signals in a larger class of games. The solution method remains the same. We first project the signals into a one-dimensional equilibrium statistic. We then show that an equilibrium exists in which each agent behaves "as if" he observes only his equilibrium statistic. Importantly, the definition of an equilibrium statistic does not change.

Consider a game with N agents. Agent $n \in N$ takes an action $a_n \in A_n$ and the payoff function is given by:

$$u_n(v_n, a_1, ..., a_N),$$

where $v_n \in \mathbb{R}$ is a payoff shock. Agent n observes J signals $(s_n^1, ..., s_n^J)$ and the joint distribution of all signals and payoff shocks is jointly normally distributed. An equilibrium statistic is defined the same way as in the previous section (as in (27)).

Fix an equilibrium statistic, and consider first an auxiliary game in which agent n only observes a one-dimensional signal equal to his equilibrium statistic $(\boldsymbol{\beta}_n \cdot \mathbf{s}_n)$. Suppose in this auxiliary game, there exists a strategy profile $\{\hat{\alpha}_n\}_{n \in N}$, with $\hat{\alpha}_n : \mathbb{R} \to A_n$, that is an ex post equilibrium.[42] Then in the original game (where agent n observes J signals), the following strategy profile $\{\alpha_n\}_{n \in N}$, with $\alpha_n : \mathbb{R}^J \to A_n$, is a

[40] See Krishna (2009) for a textbook discussion

[41] The same characterization can be applied if we consider an ascending auction with reentry (see the following section for a discussion). Besides being a more realistic model in many applications, allowing for reentry relaxes the conditions under which an ex post equilibrium exists when agents observe one-dimensional signals (see Izmalkov (2001)).

[42] A Nash equilibrium $(a_1, ..., a_N)$ is an ex post equilibrium if agent n's action is optimal even if he knew the realization of the signals of all other agents. See Bergemann and Morris (2005) for a discussion.

Nash equilibrium:[43]

$$\alpha(\mathbf{s}_n) = \hat{\alpha}(\boldsymbol{\beta}_n \cdot \mathbf{s}_n). \tag{28}$$

This is the natural extension of Theorem 1.

The methodology can be extended to games that have an ex post equilibrium when agents observe one-dimensional signals. These mechanisms have the property that the optimality condition can be evaluated using the realized value of all signals when agents observe one-dimensional signals (as in an ascending auction). We briefly provide an overview of some of the mechanisms that have an ex post equilibrium when agents observe one-dimensional signals.

There are classic trading mechanisms that have an ex post equilibrium when agents observe one-dimensional signals. For example, multi-unit ascending auctions (see, for example, Ausubel (2004) or Perry and Reny (2005)) and generalized VCG mechanism (see for example, Dasgupta and Maskin (2000)). Supply function competition in linear-quadratic environments has an ex post equilibria when agents are symmetric (see, for example, Vives (2011)). Additionally, many recent papers study novel mechanisms that have an ex post equilibria when agents observe one-dimensional signals.[44]

The equilibrium statistic can also be used to understand models of competitive equilibrium under asymmetric information. Ganguli and Yang (2009) and Manzano and Vives (2011) study a rational expectations equilibrium in which agents observe two-dimensional signals. Amador and Weill (2010) study a micro-founded macro model with informational externalities. The fixed-point that determines an equilibrium in those models is closely related to the equilibrium statistic defined in our paper. In Heumann (2016) we study the properties of a competitive economy when agents observe multidimensional signals. We use the equilibrium statistic defined in this paper to characterize the equilibrium.

It is worth mentioning some mechanisms that do not have an ex post equilibria when agents observe one-dimensional signals. Two classic examples are a first-price auctions and Cournot competition. In a first-price auction an agent tries to anticipate the bid

[43] As in the ascending auction, this in fact constitutes a posterior equilibrium (see Green and Laffont (1987)).

[44] Ausubel, Crampton, and Milgrom (2006) propose the Combinatorial Clock Auction that is meant to auction many related items. Sannikov and Skrzypacz (2014) study a variation of supply function equilibria in which each agents can condition on the quantity bought by other agents. Kojima and Yamashita (2014) study a variation of a double auction that improves upon the standard double auction along several dimensions. All the mechanisms previously mentioned have an ex post equilibria when agents observe one-dimensional signals.

of other agents in order determine how much he wants to shade his bid. In contrast, in an ascending auction an agent's drop-out time remains optimal, even if he knew the drop-out time of other agents. Hence, an agent does not need to anticipate the drop-out time of other agents. In Cournot competition an agent tries to anticipate the quantity submitted by other agents, as these quantities will ultimately determine the equilibrium price. In contrast, in supply function competition an agent can condition the quantity he buys on the equilibrium price, and hence he does not need to anticipate the demands submitted by other agents. Understanding a first-price auction or Cournot competition when agents observe multidimensional signals requires different techniques than the ones developed in this paper.[45]

7 Conclusions

The auction literature has largely relied on the assumption that agents observe one-dimensional signals. We provided a tractable model of an ascending auction in which agents observe multidimensional signals. The key conceptual contribution is that in multidimensional environments the bid of an agent is determined by an endogenous object, namely, the equilibrium statistic. We showed that in multidimensional environments there may be multiple symmetric equilibria and classic results on the impact of public signals are overturned. These novel predictions are a sharp illustration of two broader points: (*i*) in multidimensional environments there is no simple mapping between the primitives of the model and the outcome of the auction, and (*ii*) in multidimensional environments comparative statics will change with respect to one-dimensional environments. Our paper provides a set of tools that can be used to further understand multidimensional environments and how these environments differ from their one-dimensional counterparts.

There are two important assumptions in our model: (*i*) agents bid in an ascending auction, and (*ii*) signals are normally distributed. Extending our analysis to other auction formats requires developing new techniques. This is because different auction formats provide different incentives for bidders. For example, in a first-price auction

[45] Lambert, Ostrovsky, and Panov (2014) study a static version of a Kyle (1985) trading model under multidimensional Gaussian signals (this is strategically similar to Cournot competition). For the aforementioned reasons, the methodology developed in this paper is not useful to study a trading model as in Lambert, Ostrovsky, and Panov (2014) and vice versa.

agents have the incentive to shade their bid. For this reason, we believe characterizing the equilibrium of a first-price auction with multidimensional signals would yield substantive new insights. Generalizing the analysis to non-Gaussian signals would be an important technical extension. We believe the novel predictions we provided do not hinge on the assumption of Gaussian signals, but allowing for non-Gaussian signals may deliver even further new predictions.

8 Appendix: Proofs

Preamble. We first provide explicit expressions for the expectations with normal random variables. To do this we use the definition of one-dimensional signal in (5).

If y is normally distributed, then:

$$\mathbb{E}[\exp(y)] = \exp(\mathbb{E}[y] + \frac{1}{2}var(y)). \tag{29}$$

This is just the mean of a log-normal random variable. Since $(v_1, ..., v_N, s'_1, ..., s'_N)$ are jointly Gaussian, we have that the distribution of $(v_1, ..., v_N)$ conditional on $(s'_1, ..., s'_N)$ is jointly Gaussian. Hence, using (29):

$$\mathbb{E}[\exp(v_n)|s'_1..., s'_N] \quad = \quad exp(\mathbb{E}[v_n|s'_1, ..., s'_N] + \frac{1}{2} \cdot var(v_n|s'_1, ..., s'_N))$$

Similarly, by replacing $(s'_1, ..., s'_N)$ with $(s'_n, ..., s'_n, s'_{n+1}, ..., s'_N)$ we get

$$\mathbb{E}[\exp(v_n)|s'_n, ..., s'_n, s'_{n+1}..., s'_N] = exp(\mathbb{E}[v_n|s'_n, ..., s'_n, s'_{n+1}..., s'_N] + \frac{var(v_n|s'_1, ..., s'_N)}{2}) \tag{30}$$

Note that $\mathbb{E}[\exp(v_n)|s'_n, ..., s'_n, s'_{n+1}..., s'_N]$ is computed as if the realization of $(s'_1, ..., s'_n)$ is equal to $(s'_n, ..., s'_n)$. Since the conditional variance of normal random variables is constant, we have that $var(v_n|s'_1, ..., s'_N) = var(v_n|s'_n, ..., s'_n, s'_{n+1}..., s'_N)$.

We now explicitly compute the coefficients of the Bayesian updating with the normal random variables. We have that:

$$\mathbb{E}[v_n|s'_1, ..., s'_N] = \kappa \cdot \left(s'_n + \lambda \sum_{m=1}^{N} s'_m \right) \tag{31}$$

with

$$\kappa \triangleq \frac{(1-\rho_i)\sigma_i^2}{(1-\rho_i)\sigma_i^2 + b^2 \cdot \sigma_\varepsilon^2}; \tag{32}$$

$$\lambda \triangleq \frac{1}{N}\left(\frac{((1-\rho_i)+\rho_i \cdot N)\cdot \sigma_i^2 + b \cdot N \cdot \sigma_c^2}{((1-\rho_i)+\rho_i \cdot N)\cdot \sigma_i^2 + b^2(N \cdot \sigma_c^2 + \sigma_\varepsilon^2)} \frac{(1-\rho_i)\sigma_i^2 + b^2 \cdot \sigma_\varepsilon^2}{(1-\rho_i)\sigma_i^2} - 1\right). \tag{33}$$

This is just computing the coefficients of the Bayesian updating. To check the coefficients λ and κ are correctly computed it is sufficient to check that:

$$\forall m \in N, \quad cov(v_n - \mathbb{E}[v_n | s'_1, ..., s'_N], s'_m) = 0, \tag{34}$$

using (31) and the definitions of κ and λ.[46] Finally, note that $\kappa > 0$ and for all $n \in N$:

$$(1+n\cdot\lambda) = \frac{n}{N}\frac{((1-\rho_i)+\rho_i \cdot N)\cdot \sigma_i^2 + b \cdot N \cdot \sigma_c^2}{((1-\rho_i)+\rho_i \cdot N)\cdot \sigma_i^2 + b^2(N \cdot \sigma_c^2 + \sigma_\varepsilon^2)} \frac{(1-\rho_i)\sigma_i^2 + b^2 \cdot \sigma_\varepsilon^2}{(1-\rho_i)\sigma_i^2} + \frac{N-n}{N} > 0.$$

Proof of Proposition 1 The proof is standard in the literature (see, for example, Krishna (2009)). Nevertheless, we provide the proof to simplify the reading and to check all the conditions are satisfied. We check the following three conditions:

1. According to the equilibrium strategies (see (7)) agent $n+1$ drops out of the auction before agent n. This is a necessary condition for an equilibrium as the equilibrium strategy of agent n (according to (7)) conditions on the signals $(s'_{n+1}, ..., s'_N)$. Hence, it is necessary to check that agents with higher signals drop out later in the auction.

   Using (31), we note that:

$$\mathbb{E}[v_{n-1} | s'_{n-1}, ..., s'_{n-1}, s'_n, ..., s'_N] - \mathbb{E}[v_n | s'_n, ..., s'_n, s'_{n+1}, ..., s'_N] = \kappa(1+\lambda\cdot(n-1))(s'_{n-1} - s'_n) > 0.$$

   The equality is using (31), while the inequality comes from the fact that $\kappa > 0$, $(1+(n-1)\lambda) > 0$ (as previously shown), and $(s'_{n-1} - s'_n) > 0$ by construction.

[46] That is, λ and κ solve the following system of equations:

$$\sigma_i^2 + b\sigma_c^2 = \kappa(\sigma_i^2 + b^2(\sigma_c^2 + \sigma_\varepsilon^2) + \lambda(\sigma_i^2 + b^2(\sigma_c^2 + \sigma_\varepsilon^2) + (N-1)(\rho_i \cdot \sigma_i^2 + b^2 \cdot \sigma_c^2)));$$

$$\rho_i \cdot \sigma_i^2 + b\sigma_c^2 = \kappa(\rho_i\sigma_i^2 + b^2 \cdot \sigma_c^2 + \lambda(\sigma_i^2 + b^2(\sigma_c^2 + \sigma_\varepsilon^2) + (N-1)(\rho_i \cdot \sigma_i^2 + b^2 \cdot \sigma_c^2))),$$

which corresponds to (34) for $m = n$ and $m \neq n$ respectively.

Hence, using (30), we have that $\forall n \in \{2, ..., N\}$:

$$\mathbb{E}[\exp(v_{n-1})|s'_{n-1}, ..., s'_{n-1}, s'_n..., s'_N] - \mathbb{E}[\exp(v_n)|s'_n, ..., s'_n, s'_{n+1}..., s'_N] > 0 \quad (35)$$

Hence, agent n drops out of the auction before agent $n-1$.

2. We now check that agent 1 does not regret wining the auction (this is (8)).

   Using (31), we note that:

$$\mathbb{E}[v_1|s'_1, ..., s'_N] - \mathbb{E}[v_2|s'_2, s'_2, ..., s'_N] = \kappa(1+\lambda)(s'_1 - s'_2) > 0.$$

   Clearly the inequality is also be satisfied if we take the exponential of v_1 and v_2. Hence, (8) is satisfied. Hence, agent 1 does not regret wining the auction.

3. We now check that agent $m > 1$ does not regret waiting until agent 1 drops out of the auction (this is (9)).

   Using (31), we note that:

$$\mathbb{E}[v_m|s'_1, ..., s'_N] - \mathbb{E}[v_1|s'_1, s'_1, s'_2, ..., s_{m-1}, s_{m+1}, ..., s'_N] = \kappa(1+\lambda)(s'_m - s'_1) < 0.$$

   The inequality will also be satisfied if we take the exponential of v_m and v_1. Hence, (9) is satisfied. Hence, agent $m > 1$ does not regret waiting until agent 1 drops out of the auction.

Hence, the equilibrium strategies constitute an ex post equilibrium. ■

Proof of Proposition 2. In order to characterize how the ex ante expected social surplus is determined by $b \in \mathbb{R}$, we first provide an orthogonal decomposition of signals and payoff shocks. This is also used later in the rest of the proofs. We define:

$$\bar{v} \triangleq \frac{1}{N} \sum_{n \in N} v_n \;\; ; \;\; \Delta v_n \triangleq v_n - \bar{v} \;\; ; \;\; \bar{s}' \triangleq \frac{1}{N} \sum_{n \in N} s'_n \;\; ; \;\; \Delta s'_n \triangleq s'_n - \bar{s}'. \quad (36)$$

Variables with an over-bar correspond to the average of the variable over all agents. Variables preceded by a Δ correspond to the difference between a variable and the average variable. We refer to variables that have an over-bar as the common component of a random variable and a variables preceded by a Δ as the orthogonal component of a random variable. For example, $\bar{v}$ is the common component of v_n while Δv_n is the orthogonal component of v_n. Importantly, the common component of a random variable is always independent of the orthogonal component of a random variable. For example, $cov(\Delta v_n, \bar{s}') = 0$, which implies independence of the two random variables in our Gaussian environment.[47]

By construction agent 1 wins the good. Hence, the expected social surplus given the realization of the signals is given by:

$$S(s'_1, ..., s'_N) \triangleq \mathbb{E}[\exp(v_1)|s'_1, ..., s'_N].$$

We can write the expected social surplus as follows:

$$\mathbb{E}[S(s'_1, ..., s'_N)] = \mathbb{E}[\mathbb{E}[\exp(v_1)|s'_1, ..., s'_N]] = \mathbb{E}[\mathbb{E}[exp(\bar{v}) \cdot \exp(\Delta v_1)|\bar{s}', \Delta s'_1, ..., \Delta s'_N]]$$

Since the common component of the random variables are independent of the orthogonal component of the random variables, we have:

$$\begin{aligned} \mathbb{E}[S(s'_1, ..., s'_N)] &= \mathbb{E}[\mathbb{E}[exp(\bar{v})|\bar{s}'] \cdot \mathbb{E}[\exp(\Delta v_1)|\Delta s'_1, ..., \Delta s'_N]] \\ &= \mathbb{E}[\mathbb{E}[\exp(\bar{v})|\bar{s}']] \cdot \mathbb{E}[\mathbb{E}[\exp(\Delta v_1)|\Delta s'_1, ..., \Delta s'_N]]. \end{aligned}$$

Using the law of iterated expectations: $\mathbb{E}[\mathbb{E}[\exp(\bar{v})|\bar{s}']] = \mathbb{E}[\exp(\bar{v})] = exp(\frac{1}{2}\sigma^2_{\bar{v}})$. Hence,

$$\mathbb{E}[S(s'_1, ..., s'_N)] = exp(\frac{1}{2}\sigma^2_{\bar{v}}) \times \mathbb{E}[\mathbb{E}[\exp(\Delta v_1)|\Delta s'_1, ..., \Delta s'_N]]. \quad (37)$$

Since the equilibrium is efficient, we have that:

$$\mathbb{E}[\mathbb{E}[\exp(\Delta v_1)|\Delta s'_1, ..., \Delta s'_N]] =$$

$$\mathbb{E}[\max\{\mathbb{E}[\exp(\Delta v_1)|\Delta s'_1, ..., \Delta s'_N], ..., \mathbb{E}[\exp(\Delta v_N)|\Delta s'_1, ..., \Delta s'_N]\}]$$

[47]To check this, note that by construction $\sum_{n \in N} \Delta v_n = 0$. By symmetry, for all $n, m \in N$, $cov(\Delta v_n, \bar{s}') = cov(\Delta v_m, \bar{s}')$. By the collinearty of the covariance: $\sum_{n \in N} cov(\Delta v_n, \bar{s}') = 0$. Hence, we must clearly have that $cov(\Delta v_n, \bar{s}') = 0$. The argument can be obviously repeated for the common and orthogonal component of any random variables.

Note that $\Delta v_n = \Delta i_n$ and hence:

$$\Delta s'_n = \Delta v_n + \beta \cdot \Delta \varepsilon_n.$$

Clearly, if β increases then $(\Delta s'_1, ..., \Delta s'_N)$ becomes Blackwell less informative about $(\Delta v_1, ..., \Delta v_N)$. Hence, $\mathbb{E}[\mathbb{E}[\exp(\Delta v_1)|\Delta s'_1, ..., \Delta s'_N]]$ is decreasing in β. Hence, we prove the result.■

Proof of Proposition 3 We prove the result in two steps.

(Step 1.) We first prove that $t_n = i_n + \beta \cdot s_n$ is an equilibrium statistic if and only if:

$$cov(v_n - \mathbb{E}[v_n|t_1, ..., t_N], i_n) = 0, \tag{38}$$

and $\beta \neq 0$.

Only If. Clearly $\beta = 0$ is not an equilibrium statistic (simply note that $\mathbb{E}[v_n|i_1, ..., i_N] \neq \mathbb{E}[v_n|s_n, i_1, ..., i_N]$) and by the construction of the expectation:

$$cov(v_n - \mathbb{E}[v_n|i_n, s_n, t_1, ..., t_N], i_n) = 0, \tag{39}$$

Hence, if $\mathbb{E}[v_n|i_n, s_n, t_1, ..., t_N] = \mathbb{E}[v_n|t_1, ..., t_N]$, then (38) must be satisfied. Hence, we prove the "only if" direction.

If. Note that by the construction of the expectation,

$$\forall m \in N, \quad cov(v_n - \mathbb{E}[v_n|t_1, ..., t_N], t_m) = 0. \tag{40}$$

By the collinearity of the expectation, if (38) is satisfied and (40) is satisfied with $\beta \neq 0$, then it is also the case that:

$$cov(v_n - \mathbb{E}[v_n|t_1, ..., t_N], s_n) = 0. \tag{41}$$

This is because s_n is a linear combination of t_n and i_n. Hence, if (38) is satisfied, then by construction (40) and (41) are satisfied. Hence, the covariance of $(i_n, s_n, t_1, ..., t_N)$ with $(v_n - \mathbb{E}[v_n|t_1, ..., t_N])$ is equal to 0. Hence, it must be the case that:

$$\mathbb{E}[v_n|t_1, ..., t_N] = \mathbb{E}[v_n|i_n, s_n, t_1, ..., t_N] \tag{42}$$

Hence, we prove the "if" part.

Step 2. We now prove that β satisfies (38) if and only if β solves the cubic polynomial (12). It is clear that: $cov(v_n, i_n) = \sigma_i^2$. Using (31):

$$cov(\mathbb{E}[v_n|t_1, ..., t_N], i_n) = \kappa((\lambda+1)\sigma_i^2 + \lambda(N-1)\rho_i\sigma_i^2).$$

Hence, we can re-write (38) as follows:

$$1 - \kappa((\lambda+1) + \lambda(N-1)\rho_i) = 0.$$

Multiplying both sides by:

$$-\frac{\left(\beta^2\sigma_\epsilon^2 - (\rho_i-1)\sigma_i^2\right)\left(\beta^2\left(N\sigma_c^2+\sigma_\epsilon^2\right) + \sigma_i^2((N-1)\rho_i+1)\right)}{\beta(\rho_i-1)\sigma_c^2\sigma_i^2\sigma_\epsilon^2((N-1)\rho_i+1)},$$

We get the cubic polynomial (12). Hence, we prove the result.■

Proof of Theorem 1 We proved this in the main text. ■

Proof of Lemma 1 In order to write the expectations in terms of conditionally independent signals, we define:

$$\hat{s} \triangleq \frac{1}{N-1}\sum_{m\neq n}(s'_m - \frac{\rho_i \cdot i_n}{b}) = c + \frac{1}{N-1}\sum_{m\neq n}\left(\varepsilon_m + \frac{1}{b}(i_m - \rho_i \cdot i_n)\right). \quad (43)$$

Note that $\hat{s}$ is independent of i_n. Additionally, by symmetry, $\hat{s}$ is a sufficient statistic of $\{s'_m\}_{m\neq n}$ to predict v_n. Expectation (20) can be written as follows:

$$\mathbb{E}[v_n|i_n, s_n, \{s'_m\}_{m\neq n}] = i_n + \mathbb{E}[c|s_n, \hat{s}], \quad (44)$$

where $(s_n, \hat{s})$ are conditionally independent signals of c. Using standard formulas of expectations with Gaussian random variables:

$$\gamma_s = \frac{1/\sigma_\varepsilon^2}{1/\sigma_\varepsilon^2 + 1/\sigma_c^2 + 1/var(\hat{s}|c)}\ ;\ \gamma' = \frac{1/var(\hat{s}|c)}{1/\sigma_\varepsilon^2 + 1/\sigma_c^2 + 1/var(\hat{s}|c)}.$$

Using (43), it is easy to check that:

$$\begin{aligned} var(\hat{s}|c) &= var\left(\frac{1}{N-1}\sum_{m\neq n}\left(\varepsilon_m + \frac{1}{b}(i_m - \rho_i \cdot i_n)\right)\right) \\ &= \frac{1}{N-1}\left(\sigma_\varepsilon^2 + \frac{1}{b^2}(1-\rho_i^2)\sigma_i^2 + \frac{1}{b^2}(N-2)(\rho_i - \rho_i^2)\sigma_i^2\right) \end{aligned}$$

Replacing and simplifying terms, we get that:

$$\gamma_s = \frac{(b^2\sigma_\varepsilon^2\sigma_c^2 + (1-\rho_i)\sigma_i^2\sigma_c^2((N-1)\rho_i + 1))}{b^2\sigma_\varepsilon^2(N\sigma_c^2 + \sigma_\varepsilon^2) + (1-\rho_i)\sigma_i^2((N-1)\rho_i + 1)(\sigma_c^2 + \sigma_\varepsilon^2)};$$

$$\gamma' = \left(\frac{b^2(N-1)\sigma_c^2\sigma_\varepsilon^2}{b^2\sigma_\varepsilon^2(N\sigma_c^2 + \sigma_\varepsilon^2) + (1-\rho_i)\sigma_i^2((N-1)\rho_i + 1)(\sigma_c^2 + \sigma_\varepsilon^2)}\right).$$

Finally, we have that:

$$\gamma_i = 1 - \frac{\rho_i \cdot \gamma'}{b}.$$

This comes directly from that fact that γ_i is equal to 1 plus the weight on i that comes from the prediction of c. Yet, from (43) it is clear that the weight on i from the prediction of c is $-\rho_i \cdot \gamma'/b$. Hence,

$$\gamma_i = 1 - \frac{\rho_i \cdot \gamma'}{b} = 1 - b \cdot \frac{(N-1)\rho_i\sigma_c^2\sigma_\varepsilon^2}{b^2\sigma_\varepsilon^2(N\sigma_c^2 + \sigma_\varepsilon^2) + (1-\rho_i)\sigma_i^2((N-1)\rho_i + 1)(\sigma_c^2 + \sigma_\varepsilon^2)}.$$

It is easy to check that γ_s is decreasing in b and in the limits $b \to 0$ and $b \to \infty$ we get $\sigma_c^2/(\sigma_c^2 + \sigma_\varepsilon^2)$ and $\sigma_c^2/(N\sigma_c^2 + \sigma_\varepsilon^2)$ respectively. Similarly, it is easy to check that $\gamma_i < 1$ and if $b \to 0$ or $b \to \infty$, then $\gamma_i \to 1$. It is also possible to check that γ_i is quasi-convex in b.■

Proof of Proposition 4. Before we provide the proof, we make some observations.

Remark 1. We use the notation defined in (36), and we extend the definition to all other random variables. That is, variables with an over-bar correspond to the average of the variable over all agents. Variables preceded by a Δ correspond to the difference between a variable and the average variable. We also note that:

$$\sigma_{\bar{v}}^2 = \sigma_v^2(\frac{1 + (N-1)\rho_v}{N}) \text{ and } \sigma_{\Delta v}^2 = \sigma_v^2\frac{(N-1)(1-\rho_v)}{N},$$

where ρ_v is the correlation of the payoff shocks across agents. Similarly, for any random variable, the variances of the common and orthogonal components are determined by the correlation of the random variable across agents in the same way. Multiplying all terms in the cubic polynomial (12) by $N/(N-1)$, (12) can be written in terms of the common and the orthogonal component of a random variable as follows:

$$x_3 = \frac{(\sigma^2_{\Delta i} + \sigma^2_{\bar{\imath}})(\sigma^2_{\bar{\varepsilon}} + \sigma^2_c)}{\sigma^2_{\Delta i}\sigma^2_{\bar{\imath}}\sigma^2_c} \; ; \; x_2 = \frac{-1}{\sigma^2_{\Delta i}} \; ; \; x_1 = \frac{\sigma^2_{\Delta\varepsilon} + \sigma^2_{\bar{\varepsilon}} + \sigma^2_c}{\sigma^2_{\Delta\varepsilon}\sigma^2_c} \; ; \; x_0 = \frac{-1}{\sigma^2_{\Delta\varepsilon}}. \tag{45}$$

Remark 2. We now show that in the model with public signals, a linear combination of signals $t_n = i_n + \beta \cdot s_n$ is an equilibrium statistic if and only if β is a root of the cubic polynomial $x_3 \cdot \beta^3 + x_2 \cdot \beta^2 + x_1 \cdot \beta + x_0$, with:

$$x_3 = \frac{(\sigma^2_{\Delta i} + \sigma^2_{\bar{\imath}'})(\sigma^2_{\bar{\varepsilon}} + \sigma^2_{c'})}{\sigma^2_{\Delta i}\sigma^2_{\bar{\imath}'}\sigma^2_{c'}} \; ; \; x_2 = \frac{-1}{\sigma^2_{\Delta i}} \; ; \; x_1 = \frac{\sigma^2_{\Delta\varepsilon} + \sigma^2_{\bar{\varepsilon}} + \sigma^2_{c'}}{\sigma^2_{\Delta\varepsilon}\sigma^2_{c'}} \; ; \; x_0 = \frac{-1}{\sigma^2_{\Delta\varepsilon}}, \tag{46}$$

where:

$$\sigma^2_{\bar{\imath}'} = \sigma^2_{\bar{\imath}} - \frac{\sigma^4_{\bar{\imath}}}{\sigma^2_{\bar{\imath}} + \sigma^2_2} \text{ and } \sigma^2_{c'} = \sigma^2_c - \frac{\sigma^4_c}{\sigma^2_c + \sigma^2_1}. \tag{47}$$

That is, the analysis of the equilibrium with public signals is equivalent to redefining the variances of the common shocks and the common component of the idiosyncratic shock.

To prove this define:

$$i'_n \triangleq i_n - \mathbb{E}[i_n|\bar{s}^2] \text{ and } c' \triangleq c - \mathbb{E}[c|\bar{s}^1].$$

Note that $\mathbb{E}[i_n|\bar{s}^2]$ is the same across agents, and hence, $\Delta i_n = \Delta i'_n$. That is, public signals do not change the idiosyncratic component of a random variables. The variance of $\bar{\imath}'$ and $\bar{c}'$ are given by (47). Analogously, define:

$$s'_n \triangleq c' + \varepsilon_n \quad \text{ and } \quad t'_n \triangleq i'_n + \beta \cdot (c' + \varepsilon_n). \tag{48}$$

Note that for the purpose of this proof s'_n is defined differently than in (5). All variables with a prime are orthogonal to $(\bar{s}^1, \bar{s}^2)$. Hence, we have that the linear combination of signals $t'_n = i'_n + \beta \cdot s'_n$ is an equilibrium statistic if and only if:

$$\mathbb{E}[i'_n + c'|t'_1, ..., t'_N] = \mathbb{E}[i'_n + c'|i'_n, s'_n, t'_1, ..., t'_N].$$

Hence, we can use the characterization of an equilibrium statistic in Proposition 3, but using the variables with primes. This corresponds to changing the variance of $\bar{\imath}'$ and c' according to (47).

Remark 3. Analogous to the proof of Proposition 2, the expected social surplus can be written as follows:

$$\mathbb{E}[S(s_1, ..., s_N)] = \exp(\frac{1}{2}var(\bar{v})) \times \mathbb{E}[\mathbb{E}[\exp(\Delta v_1)|\Delta t_1, ..., \Delta t_N]]. \tag{49}$$

where:

$$\Delta t_n = \Delta i_n + \beta \cdot \Delta\varepsilon_n.$$

As in Proposition 2, it is easy to check that the expected social surplus is decreasing in β and if $\beta \to 0$, then the equilibrium approaches the efficient outcome.

Main Step. Define the polynomial:

$$q(\beta, \sigma_2^2, \sigma_1^2) \triangleq x_3 \cdot \beta^3 + x_2 \cdot \beta^2 + x_1 \cdot \beta + x_0, \tag{50}$$

with x_3, x_2, x_1, x_0 defined in (46) (note that these coefficients depend on (σ_2^2, σ_1^2)). Let $\beta^*(\sigma_2^2, \sigma_1^2)$ be a root of (50), and let this root be unique. It is easy to check that:

$$\frac{\partial \beta^*(\sigma_2^2, \sigma_1^2)}{\partial \sigma_2^2} = -\frac{\frac{\partial q(\beta^*(\sigma_2^2,\sigma_1^2),\sigma_2^2,\sigma_1^2)}{\partial \sigma_2^2}}{\frac{\partial q(\beta^*(\sigma_2^2,\sigma_1^2),\sigma_2^2,\sigma_1^2)}{\partial \beta}}, \tag{51}$$

and similar for the derivative with respect to σ_1^2. It is easy to check that, $q(\beta, \sigma_2^2, \sigma_1^2)$ is decreasing in $\sigma_{\bar{\imath}'}^2$ and $\sigma_{c'}^2$ (hence, also decreasing in σ_2^2 and σ_1^2 respectively). Hence, the numerator of (51) is negative. If $q(\beta, \sigma_2^2, \sigma_1^2)$ has a unique root then $q(\beta, \sigma_2^2, \sigma_1^2)$ is increasing at this root. Hence, the denominator of (51) is positive. Hence, if $q(\beta, \sigma_2^2, \sigma_1^2)$ has a unique root, then this root is increasing in σ_2^2 and σ_1^2. This implies that the social surplus is decreasing σ_1^2 and ${\sigma_2}^2$.

For the limit, note that in the limit $\sigma_{\bar{\imath}'}^2 \to 0$ or $\sigma_{c'}^2 \to 0$ every root of the polynomial

$q(\beta)$ must converge to 0. Hence, the social surplus is equal to the social surplus of the efficient outcome. Hence, we prove the result. ■

Proof of Proposition 5 The proof is similar to the proof of Proposition 4. We use all the definitions and arguments therein, and extend them to show the results on revenue. We first provide some additional observations.

Remark 1. The coefficients λ and κ (defined in (32) and (33)) can be re-written in terms of the common and orthogonal components of the random variables as follows:

$$\kappa = \frac{\sigma_{\Delta i}^2}{\sigma_{\Delta i}^2 + \beta^2 \cdot \sigma_{\Delta\varepsilon}^2}; \tag{52}$$

$$\lambda = \frac{1}{N}\left(\frac{\sigma_{\bar{i}}^2 + \beta \cdot \sigma_c^2}{\sigma_{\bar{i}}^2 + \beta^2(\sigma_c^2 + \sigma_{\bar{\varepsilon}}^2)} \frac{\sigma_{\Delta i}^2 + \beta^2 \cdot \sigma_{\Delta\varepsilon}^2}{\sigma_{\Delta i}^2} - 1\right). \tag{53}$$

Remark 2. Using the definition of s'_n in (48), the expectations can be written as follows:

$$\mathbb{E}[v_n|s'_1, ..., s'_N, \bar{s}^1, \bar{s}^2] = \kappa' \cdot \left(s'_n + \lambda' \sum_{m=1}^{N} s'_m\right) + \mathbb{E}[v_n|\bar{s}^1, \bar{s}^2], \tag{54}$$

where κ' and λ' are defined as follows:

$$\kappa' \triangleq \frac{\sigma_{\Delta i}^2}{\sigma_{\Delta i}^2 + \beta^2 \cdot \sigma_{\Delta\varepsilon}^2}; \tag{55}$$

$$\lambda' \triangleq \frac{1}{N}\left(\frac{\sigma_{\bar{i}'}^2 + \beta \cdot \sigma_{c'}^2}{\sigma_{\bar{i}'}^2 + \beta^2(\cdot\sigma_{c'}^2 + \sigma_{\bar{\varepsilon}}^2)} \frac{\sigma_{\Delta i}^2 + \beta^2 \cdot \sigma_{\Delta\varepsilon}^2}{\sigma_{\Delta i}^2} - 1\right), \tag{56}$$

where $\sigma_{\bar{i}'}^2$ and $\sigma_{c'}^2$ are defined in (47).

Remark 3. The price paid in the auction is equal to $\mathbb{E}[\exp(v_2)|t_2, t_2, ..., t_N, \bar{s}^1, \bar{s}^2]$. Using (30) and (54) we have that:

$$\begin{aligned} p_2 &= \mathbb{E}[\exp(v_2)|t_2, t_2, ..., t_N, \bar{s}^2, \bar{s}^1] \\ &= \exp(\mathbb{E}[v_2|t_1, ..., t_N] - \kappa'\lambda'(t_1 - t_2) + \frac{1}{2}var(v_2|t_1, ..., t_N, \bar{s}^2, \bar{s}^1)) \\ &= \exp(-\kappa' \cdot \lambda' \cdot (t_1 - t_2)) \times \mathbb{E}[\exp(v_2)|t_1, ..., t_N, \bar{s}^2, \bar{s}^1] \end{aligned} \tag{57}$$

Main Step. We now provide the main part of the proof. As shown in Proposition 4, in the limit $\sigma_1^2 \to 0$ we have that $\beta \to 0$ and $\sigma_{c'}^2 \to 0$ (note that $\sigma_{\bar{i}'}^2$ is strictly above 0 in the limit $\sigma_1^2 \to 0$). In this case $\lambda' \cdot \kappa' \to 0$ and $t_n \to i'_n$. Hence, using (57), in the limit:

$$\lim_{\sigma_1^2 \to 0} \mathbb{E}[p_2] = \mathbb{E}[\mathbb{E}[\exp(v_2)|i_1, ..., i_N, \bar{s}^2, \bar{s}^1]],$$

where v_2 is the valuation of the agent that observed the second maximum over $(t_1, ..., t_N)$. Yet, since $\beta \to 0$, this is the same as the agent that has the second maximum over $(v_1, ..., v_N)$. Hence, we prove the result.■

Proof of Proposition 6 The proof is similar to the proof of Proposition 5. We use all the definitions and arguments therein, and extend them to show the results on revenue. We now prove that:

$$\lim_{\sigma_2^2 \to 0} p_2 = 0.$$

In the limit $\sigma_2^2 \to 0$ we have that $\beta \to 0$ and $\sigma_{\bar{\imath}} \to 0$. In this limit we have that $\lambda' \to \infty$ and $\kappa' \to 1$ (see (55) and (56)). Note that $t_1 - t_2$ is greater than 0 always because agents are relabelled such that this is satisfied. Additionally, in the limit, $t_1 - t_2 \to i_1 - i_2$. Hence, $(t_1 - t_2)$ has positive variance. Also, clearly $\mathbb{E}[\exp(v_2)|t_1, ..., t_N, \bar{s}^2, \bar{s}^1]$ is finite as this is the expected valuation of the agent with the second highest equilibrium statistic. Hence, we have that:

$$\lim_{\sigma_2^2 \to 0} \mathbb{E}[\exp(-\kappa' \cdot \lambda' \cdot (t_1 - t_2)) \times \mathbb{E}[\exp(v_2)|t_1, t_2, ..., t_N, \bar{s}^2, \bar{s}^1]] = 0,$$

because $\lambda' \to \infty$ and $(t_1 - t_2)$ has positive variance. Hence, we prove the result. ■

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Online Appendix to: An Ascending Auction with Multidimensional Signals

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The online appendix contains complementary results to the main paper. It is organized as follows. Section A provides additional results on the baseline model (with two-dimensional signals). Section B shows how to characterize a class of Nash equilibrium in an ascending auction for any Gaussian information structure. Section C shows how to characterize a class of Nash equilibrium in a broader class of mechanisms when agents observe multidimensional Gaussian signals. Section D provides the proofs of the results in the online appendix.

A Online Appendix: Additional Results on Baseline Model

In this section we provide additional results of our baseline model. That is, throughout this section we continue to assume a model as in Section 2. We provide conditions that guarantee the existence of a unique equilibrium and we provide additional comparative statics.

A.1 Uniqueness of Equilibrium

We now study when there is a unique equilibrium (within the class of equilibria studied in Theorem 1). We begin by providing conditions under which the cubic equation (12) has a unique root.

Proposition 7 (Multiplicity of Equilibria).
The auction has a unique (multiple) equilibrium within the class of equilibria studied in Theorem 1 if: $18x_3x_2x_1x_0 - 4x_2^3x_0 + x_2^2x_1^2 - 4x_3x_1^3 - 27x_3^2x_0^2 < 0(> 0)$ *(with* x_3, x_2, x_1, x_0 *defined in* (12)*).*[48]

Proposition 7 is a characterization of the environments in which the cubic polynomial (12) has multiple roots. We use Proposition 7 to derive corollaries that are easier to interpret. We show that the following conditions are necessary for the multiplicity of equilibria: (i) the idiosyncratic shocks are positively correlated; and (ii) the noise therm in signal s_n is large, but not too large.

(i) Correlated idiosyncratic shocks. Multiple equilibria arise only if idiosyncratic shocks are positively correlated ($corr(i_n, i_m) > 0$).

Corollary 1 (Correlation in Idiosyncratic Shocks).
If the idiosyncratic shocks are independently distributed ($\rho_i = 0$ *), then the ascending auction has a unique equilibrium (within the class of equilibria studied in Theorem 1).*

Corollary 1 shows that correlated idiosyncratic shocks is a necessary condition to find multiple equilibria in the ascending auction. If the idiosyncratic shocks are independently distributed, then the beliefs of agent n about c are independent of i_n. Hence, there is no complementarity in the weight agents place on i_n.

[48] The case $18x_3x_2x_1x_0 - 4x_2^3x_0 + x_2^2x_1^2 - 4x_3x_1^3 - 27x_3^2x_0^2 = 0$ must be considered independently. If $18x_3x_2x_1x_0 - 4b^3d + x_2^2x_1^2 - 4x_3x_1^3 - 27x_3^2x_0^2 = 0$, then there is a unique equilibrium if and only if $x_2 = 3x_3x_1$.

***(ii)* Intermediate size of the noise terms.** We show that for large enough σ_ε^2 or small enough σ_ε^2 there is a unique equilibrium.

Corollary 2 (Uniqueness of Equilibrium)**.**
If either $\sigma_\varepsilon^2 \to 0$ or $\sigma_\varepsilon^2 \to \infty$, then there exists a unique equilibrium (within the class of equilibria studied in Theorem 1).

Corollary 2 shows that if the noise term is large enough or small enough, then there exist a unique equilibrium. If $\sigma_\varepsilon^2 \to 0$, then agents have complete information about the common shock. This limit corresponds to a private value environment, and hence there is a unique equilibrium. If $\sigma_\varepsilon^2 \to \infty$, then agent n ignore s_n. This limit also corresponds to a private value environment. Corollary 2 shows that multiple equilibria only arise for an intermediate level of noise. This does not provide any intuition on the magnitudes that are necessary to consider.

Multiple equilibria arise when signal s_n is noisy enough such that agent n does not learn "too much" about c from s_n. Additionally, signal s_n must be precise enough such that the collection of all signals $(s_1, ..., s_N)$ is informative about c. In a loose sense, this can be stated as follows:

$$corr(s_n, c) \approx 0 \text{ and } corr(s_n, c) \approx 1. \tag{58}$$

If (58) is satisfied, when agent n observes only s_n he cannot make a precise prediction of c. Yet, if agent n observed $(s_1, ..., s_N)$, then he would be able to make a precise prediction of c.

As the number of agents increases, the possibility that (58) is satisfied for very noisy signals increases. This is because $corr(\sum_{n\in N} s_n, c)$ converges to 1 as $N \to \infty$. Hence, by taking the limits $N \to \infty$ and $\sigma_\varepsilon^2 \to \infty$ at the right rates, $corr(\sum_{n\in N} s_n, c)$ converges to 1 and $corr(s_n, c)$ converges to 0. This happens when N grows faster than σ_ε^2.

Corollary 3 (Multiplicity of Equilibria)**.**
If $\sigma_\varepsilon^2 \to \infty,\ N \to \infty$ (with N diverging faster than σ_ε^2), then there are multiple equilibria if and only if:

$$\sigma_c^2 \geq \frac{4(1-\rho_i)}{\rho_i}\sigma_i^2. \tag{59}$$

Corollary 3 shows that it is possible to have multiple equilibria, even in the limit $\sigma_\varepsilon^2 \to \infty$ (as long as N grows faster than σ_ε^2). This shows that the model has an interesting discontinuity as we approach large markets ($N \to \infty$). In the limit $N \to \infty$ and $\sigma_\varepsilon^2 \to \infty$, the model does not approach a model of private values. This is because if the number of agents grows faster than σ_ε^2, then the collection of the signals observed by all the agents is still informative about c.

A.2 Comparative Statics with Respect to ρ_i.

Finally, we provide the comparative statics with respect to ρ_i.

Proposition 8 (Comparative Statics with Respect to ρ)**.**
If there is a unique equilibrium statistic, then β is increasing in ρ_i.

Proposition 8 shows that β decreases with the correlation of the idiosyncratic shocks. This implies that the efficiency of the auction is decreasing in the correlation of the idiosyncratic shocks across agents.

B Online Appendix: General Multidimensional Signals

We extend the methodology developed in Section 4 to allow for general Gaussian signals. The idea remains the same as in Section 4. That is, we first compute an equilibrium statistic, and then compute the equilibrium as if agents observe only the equilibrium statistic.

B.1 Payoff Environment and Information Structure

As in Section 2, we study N agents bidding for an indivisible good in an ascending auction. The utility of agent $n \in N$ if he wins the object at price p is given by:

$$u(v_n, p) \triangleq \exp(v_n) - p, \tag{60}$$

where $v_n \in \mathbb{R}$ is a payoff shock. If an agent does not win the good he gets a utility equal to 0. Agent n observes J signals:

$$\mathbf{s}_n = (s_n^1, ..., s_n^J),$$

where bold fonts denote vectors. The joint distribution of signals and payoff shocks is jointly Gaussian (possibly asymmetrically distributed). That is, $(v_1, ..., v_N, \mathbf{s}_1, ..., \mathbf{s}_N)$ is jointly normally distributed. The description of the auction remains the same as Section 2 (although we do not restrict attention to equilibria in which agents use symmetric strategies).

B.2 One-Dimensional Signals

We begin by studying one-dimensional signals. If agents observe one-dimensional signals, and the average crossing condition is satisfied, then the ascending auction has an ex post equilibrium that is efficient (see Krishna (2003)). The average crossing condition is defined as follows.

Definition 2 (Average Crossing Condition)**.**
The one-dimensional information structure $(s_1, ..., s_N, v_1, ..., v_N)$ *satisfies the average crossing condition if for all* $\mathscr{A} \subset \{1, ..., N\}$*, and for all* $n, m \in \mathscr{A}$ *with* $n \neq m$*:*

$$0 < \frac{\partial \mathbb{E}[v_n | s_1, ..., s_N]}{\partial s_m} \leq \frac{1}{|\mathscr{A}|} \sum_{h \in N} \frac{\partial \mathbb{E}[v_h | s_1, ..., s_N]}{\partial s_m}$$

The average crossing condition guarantees that the impact of agent n's signal on agent m's valuation is not to high. The comparison is done with respect to the average impact that agent n's signal has on any group of agents that contains n.

To characterize the equilibrium we assume that agents are ordered as follows:

$$\mathbb{E}[v_n|s_1, ..., s_N] > ... > \mathbb{E}[v_N|s_1, ..., s_N]. \tag{61}$$

That is, we assume that agents are ordered according to their expected valuation conditional on the signals of all agents. We define $\tilde{s}_1 \in \mathbb{R}$ as follows:

$$\begin{aligned} \tilde{s}_1 \triangleq \quad & \arg\min_{s' \in \mathbb{R}} && \mathbb{E}[v_1|s', s_2, ..., s_N] \\ & \text{subject to} \quad \forall n \in N, && \mathbb{E}[v_1|s', s_2, ..., s_N] \geq \mathbb{E}[v_n|s', s_2, ..., s_N] \end{aligned} \tag{62}$$

$\tilde{s}_1$ is the signal that yields the lowest expected payoff shock to agent 1, but keeping the expected payoff shock of agent 1 above the expected payoff shocks of other agents.

Proposition 9 (Equilibrium for One-Dimensional Signals)**.**
The ascending auction with one-dimensional signals has a Nash equilibrium in which agent 1 *wins the object and pays a price:*

$$p_2 = \mathbb{E}[v_1|\tilde{s}_1, s_2, ..., s_N]. \tag{63}$$

The ascending auction has an equilibrium in which the agent with the highest expected valuation wins the object. The price paid for the object is the expected valuation of the winner of the object, but evaluated at the minimum signal this agent could have observed and still win the good.

B.3 Equilibrium Statistic

As in section 4 the characterization of the equilibrium relies on projecting the signals an agent observes into a one-dimensional statistic. In equilibrium, agents behave as if they observed only their equilibrium statistic. The only difference in the characterization of the equilibrium statistic is that we need to allow for higher dimensional objects for the weights. Since different signals generally receive different weights, it is convenient to work with vectors. We denote the dot product between $\boldsymbol{\beta} \in \mathbb{R}^J$ and $\mathbf{s}_n \in \mathbb{R}^J$ by $\boldsymbol{\beta} \cdot \mathbf{s}_n$.

We define an equilibrium statistic for general Gaussian information structures.

Definition 3 (Equilibrium Statistic).
The random variables $\{t_n\}_{n \in N}$ *are an equilibrium statistic if there exists* $(\boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_N) \in \mathbb{R}^{N \times J}$ *such that for all* $n \in N$*:*

$$t_n = \boldsymbol{\beta}_n \cdot \mathbf{s}_n; \tag{64}$$

$$\mathbb{E}[v_n | t_1, ..., t_N] = \mathbb{E}[v_n | \mathbf{s}_n, t_1, ..., t_N]. \tag{65}$$

The definition of an equilibrium statistic is the natural extension of Definition 1, but allowing for general J-dimensional signals. Note that the weights on the signals of agent n ($\boldsymbol{\beta}_n$), may be different than the weights on the signals of agent m ($\boldsymbol{\beta}_m$). The equilibrium statistic is the fundamental object that allows us to characterize the equilibrium in multidimensional environments.

In order to prove that an equilibrium statistic exists, we assume that every agent n observes an additional signal (label $J+1$):

$$s_n^{J+1} = \epsilon_n, \tag{66}$$

where ϵ_n is normally distributed with mean equal to 0 and variance equal to 1, independent across agents and independent of all other random variables in the model. That is, each agent observes an additional signal that is only noise. We call this the *augmented* information structure. We prove that an equilibrium statistic exists

Proposition 10 (Existence).
If the variance covariance matrix $var(\mathbf{s}_1, ..., \mathbf{s}_N)$ *has full rank, then the augmented information structure has an equilibrium statistic exists. If the information structure is symmetric, then there exists a symmetric equilibrium statistic.*[49]

Proposition 10 guarantees the existence of equilibrium statistic for generic information structures.[50] The additional signal (66) is used to guarantee the existence of an equilibrium statistic in the cases in which agent n observes signals that contain no information about v_n. Hence, in most natural example it is not necessary to consider the additional signal s_n^{J+1} to guarantee the existence of an equilibrium statistic. In

[49]For symmetric information structures it is possible to prove the existence of an equilibrium statistic without the need to consider the augmented information structure.
[50]The uniqueness of the equilibrium statistic is clearly not guaranteed (see Section 4.4).

the proof of Proposition 10 we explain in more detail under what circumstances the additional signal s_n^{J+1} could be needed.

Finding the set of equilibrium statistics ($\{\boldsymbol{\beta}_n\}_{n\in N}$) can be found by using a "guess-and-verify" method and checking that the coefficients satisfy (65). In symmetric environments, this reduces to finding the roots of a polynomial of order $2 \cdot J - 1$. In asymmetric environments, this reduces to solving a multilinear system of equations.

B.4 Equilibrium with Multidimensional Signals

We now characterize a class of equilibria when agents observe multidimensional signals. We fix an equilibrium statistic $(t_1, ..., t_N)$. Agents are ordered as follows:

$$\mathbb{E}[v_1|t_1, ..., t_N] > ... > \mathbb{E}[v_N|t_1, ..., t_N]. \tag{67}$$

That is, we assume that agents are ordered according to their expected valuation conditional on the equilibrium statistic of all agents. We define $\tilde{t}_1 \in \mathbb{R}$ as follows:

$$\begin{aligned} \tilde{t}_1 \triangleq \quad & \arg\min_{t'\in\mathbb{R}} & & \mathbb{E}[v_1|t', t_2, ..., t_N] \\ & \text{subject to} & \forall n \in N, \quad & \mathbb{E}[v_1|t', t_2, ..., t_N] \geq \mathbb{E}[v_n|t', t_2, ..., t_N] \end{aligned} \tag{68}$$

$\tilde{t}_1$ is the analogous of $\tilde{s}_n$, but using the equilibrium statistic.

Theorem 2 (Equilibrium for Multidimensional Signals)**.**
If the equilibrium statistic $\{t_n\}_{n\in N}$ satisfies the average crossing condition, then the ascending auction has a Nash equilibrium in which agent 1 wins the object and pays a price equal to:

$$p_2 = \mathbb{E}[v_1|\tilde{t}_1, t_2, ..., t_N].$$

Theorem 2 characterizes a class of equilibria in which agents behave "as if" they observe only one-dimensional signals. This is the natural extension of Theorem 1. The characterization requires that the equilibrium statistic satisfies the average crossing condition. In applications it is easy to check whether the average crossing condition is satisfied.

The same characterization can be applied if we consider an ascending auction with reentry (see the following section for a formal argument). Besides being a more realistic model in many applications, allowing for reentry relaxes the conditions under which an ex post equilibrium exists when agents observe one-dimensional signals (see Izmalkov (2001)).

C Online Appendix: Other Mechanisms

We now extend the methodology to find a class of Nash equilibrium when agents observe multidimensional Gaussian signals in a larger class of games. The solution method remains the same. We first project the signals into a one-dimensional equilibrium statistic. We then show that an equilibrium exists in which agents behave "as if" agents observe only their equilibrium statistic. Importantly, the definition of an equilibrium statistic does not change.

C.1 General Games

We consider a game with N agents. Agent $n \in N$ takes action $a_n \in A_n$, where A_n is assumed to be a metric space. The payoff of agent $n \in \{1, ..., N\}$ depends on the realization of his payoff shock $v_n \in \mathbb{R}$ and the action taken by all agents. The payoff of agent n is denote by:

$$u_n(v_n, a_1, ..., a_N).$$

As before, vectors are denoted in bold font. A profile of actions is denoted by:

$$\mathbf{a} \triangleq (a_1, ..., a_N).$$

We denote by $(a_n', \mathbf{a}_{-n})$ the action profile:

$$(a_n', \mathbf{a}_{-n}) = (a_1, ..., a_{n-1}, a_n', a_{n+1}, ..., a_N).$$

We keep the information structure the same as in Section B. The definition of an equilibrium statistic is the same as in Definition 3.

We distinguish between the payoff environment and the information structure. This is because we want to compute the set of equilibria for a fixed payoff environment, but under different information structures. The actions available to each agent and the utility functions are called the payoff environment and are denoted by P. The joint distribution of signals and payoff shocks is the information structure and is denoted by $\mathscr{I}$. The game is defined by the payoff environment and the information structure $(P, \mathscr{I})$. Given an equilibrium statistic $(t_1, ..., t_N) \in \mathbb{R}^N$, the information structure in which agent n observes *only* t_n is called the reduced form information structure and is denoted by $\hat{\mathscr{I}}$.

In game $(P, \mathscr{I})$, a strategy profile for agent n is defined by a function $\alpha_n : \mathbb{R}^J \to A_n$. In game $(P, \hat{\mathscr{I}})$ a strategy for agent n is a functions $\hat{\alpha}_n : \mathbb{R} \to A_n$. We denote by $(\boldsymbol{\alpha}(\mathbf{s}))$ the strategy profile:

$$(\boldsymbol{\alpha}(\mathbf{s})) \triangleq (\alpha_1(\mathbf{s}_1), ..., \alpha_N(\mathbf{s}_N)).$$

We denote by $(a'_n, \boldsymbol{\alpha}_{-n}(\mathbf{s}_{-n}))$ the strategy profile in which all agents play according to $(\boldsymbol{\alpha}(\mathbf{s}))$ except for agent n, and agent n takes action a'_n for all realizations of the signals he observes. That is,

$$(a'_n, \boldsymbol{\alpha}_{-n}(\mathbf{s}_{-n})) \triangleq (\alpha_1(\mathbf{s}_1), ..., \alpha_{n-1}(\mathbf{s}_{n-1}), a'_n, \alpha_{n+1}(\mathbf{s}_{n+1}), ..., \alpha_N(\mathbf{s}_N)).$$

C.2 Solution Concepts

In order to provide our results, it is convenient to work with stronger solution concepts than Nash equilibrium. This allow us to provide sharper results. We define posterior equilibrium.

Definition 4 (Posterior Equilibrium)**.**
A strategy profile $(\alpha_1, ..., \alpha_N)$ *forms a posterior equilibrium if for all agents* $n \in N$, *for all signals realizations* $(\mathbf{s}_1, ..., \mathbf{s}_N) \in \mathbb{R}^J$, *and for all actions* $a'_n \in A_n$*:*

$$\mathbb{E}[u_n(v_n, \boldsymbol{\alpha}(\mathbf{s}))|\mathbf{s}_n, \boldsymbol{\alpha}(\mathbf{s})] \geq \mathbb{E}[u_n(v_n, (a'_n, \boldsymbol{\alpha}_{-n}(\mathbf{s}_{-n})))|\mathbf{s}_n, \boldsymbol{\alpha}(\mathbf{s})]. \quad (69)$$

In a posterior equilibrium, the strategy of agent n remains optimal even if he knew the actions taken by all other agents. The definition of posterior equilibrium is due to Green and Laffont (1987). In contrast to a Nash equilibrium, the information set with respect to which the action needs to be optimal is augmented. The action taken by agent n remains optimal even if he knew the action taken by other agents. It is transparent to see that, if a strategy profile is an posterior equilibrium, then it is also a Nash equilibrium.

It is convenient to compare posterior equilibrium with ex post equilibrium. A strategy profile $(\alpha_1, ..., \alpha_N)$ forms an ex post equilibrium if for all agents $n \in N$, for all signals realizations $(\mathbf{s}_1, ..., \mathbf{s}_N) \in \mathbb{R}^J$, and for all actions $a'_n \in A_n$:

$$\mathbb{E}[u_n(v_n, \boldsymbol{\alpha}(\mathbf{s}))|\mathbf{s}_1, ..., \mathbf{s}_N] \geq \mathbb{E}[u_n(v_n, (a'_n, \boldsymbol{\alpha}_{-n}(\mathbf{s}_{-n})))|\mathbf{s}_1, ..., \mathbf{s}_N]. \tag{70}$$

In an ex post equilibrium, agent n's action is optimal even if he knew the realization of the signals of all other agents. The definition of ex post equilibrium is standard in the literature.[51]

The difference between posterior equilibria and ex post equilibria is the amount of information with respect to which a strategies is optimal. That is, the difference lies in the conditioning variables in (70) and (69). The action taken by agent n is less informative than the signals agent n observes. Hence, if an equilibrium is an ex post equilibrium, then it is also a posterior equilibrium.

C.3 General Characterization of Equilibria

We now show how to compute a class of posterior equilibria in game $(P, \mathscr{I})$. We do this by providing an equivalence between ex post equilibria in game $(P, \hat{\mathscr{I}})$ and posterior equilibria in game $(P, \mathscr{I})$.

Theorem 3 (Equivalence)**.**
If $(\boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N) \in \mathbb{R}^N$ is an equilibrium statistic and strategies profile $\{\hat{\alpha}_n\}_{n\in N}$ is an ex post equilibrium in game $(P, \hat{\mathscr{I}})$, then the strategy profile $\{\alpha_n\}_{n\in N}$ defined as follows:

$$\alpha_n(\mathbf{s}_n) = \hat{\alpha}_n(\boldsymbol{\beta}_n \cdot \mathbf{s}_n), \tag{71}$$

is a posterior equilibrium in game $(P, \mathscr{I})$.

Proposition 3 shows that equilibria can be computed using a two step procedure. The first step is to find the one-dimensional equilibrium statistic using (65). The second

[51] Ex post equilibrium has been studied by many papers in different contexts (see, for example, Bergemann and Morris (2005) for a discussion).

step is to compute a posterior equilibrium as if agents observed only the equilibrium statistic.

In order to characterize a posterior equilibrium when agents observe multidimensional signals, the mechanism must have an ex post equilibrium when agents observe only the equilibrium statistic. Yet, the equilibrium statistic is an endogenous object. So, to apply the methodology in our paper, it is necessary that the mechanism has an ex post equilibrium for a broad class of one-dimensional signals. For example, if a *direct revelation mechanism* has an ex post equilibrium under a specific one-dimensional signal then, in general, this same mechanism (that is, this same mapping from messages to outcomes) will no longer have an ex post equilibrium under a different joint distribution of signals and payoff shocks. Hence, the methodology can be applied to a large class of indirect mechanisms that have an ex post equilibrium, regardless of the precise description of the information structure. If a mechanism has an ex post equilibrium for some one-dimensional information structure but not for other one-dimensional information structures, then it is necessary to check whether the mechanism has an ex post equilibrium when agents observe only the equilibrium statistic.

D Online Appendix: Proofs of Results in Online Appendix

Proof of Proposition 7 It is a standard property of cubic polynomials that they have a unique root if and only if their discriminant is greater than 0. For (12) this reduces to the condition in Proposition 7. Hence, we prove the result.■

Proof of Corollary 1 We show this by proving that if $\rho_i = 0$, then the discriminant of the polynomial (12) is negative (see Proposition 7). If $\rho_i = 0$, we have that:

$$x_3 = \frac{(\sigma_\varepsilon^2 + N \cdot \sigma_c^2)}{\sigma_i^2 \sigma_c^2} \; ; \; x_2 = \frac{-1}{\sigma_i^2} \; ; \; x_1 = \frac{\sigma_\varepsilon^2 + \sigma_c^2}{\sigma_\varepsilon^2 \sigma_c^2} \; ; \; x_0 = \frac{-1}{\sigma_\varepsilon^2}. \tag{72}$$

Replacing this into $18abcd - 4b^3 d + b^2 c^2 - 4ac^3 - 27a^2 d^2$ we get:

$$\frac{1}{\sigma_\varepsilon^6 \sigma_i^6 \sigma_c^8} \Bigg(- 4\sigma_\varepsilon^8 \sigma_i^4 + \sigma_\varepsilon^2 \sigma_i^2 \sigma_c^6 \left(\left(-27N^2 + 18N + 1 \right) \sigma_c^2 - 4(3N+1)\sigma_i^2 \right)$$

$$-4\sigma_\varepsilon^6\sigma_i^2\sigma_c^2\left((N+3)\sigma_i^2+2\sigma_c^2\right)-4\sigma_\varepsilon^4\sigma_c^4\left((9N-5)\sigma_i^2\sigma_c^2+3(N+1)\sigma_i^4+\sigma_c^4\right)-4N\sigma_i^4\sigma_c^8\Big)$$

It is easy to check all terms are negative for $N \geq 1$, and hence the ascending auction has a unique equilibrium.■

Proof of Corollary 2 We prove the result using the characterization in Proposition 7. In the limit $\sigma_\varepsilon^2 \to 0$, we have that $x_0 \to \infty$ and $x_1 \to \infty$. Clearly the term that dominates is $-4x_1^3x_3$, and hence, in the limit the discriminant is negative. Hence, the cubic polynomial has a unique root. In the limit $\sigma_\varepsilon^2 \to \infty$ we have that $x_3 \to \infty$. Clearly the term that dominates is $-27x_3^2x_0^2$, and hence, in the limit the discriminant is negative. Hence, the cubic polynomial has a unique root.

Proof of Corollary 3 We prove the result using the characterization in Proposition 7. We take the limit $\sigma_\varepsilon^2 \to \infty$, $N \to \infty$ (with N diverging faster than σ_ε^2). In the limit $N \to \infty$:

$$x_3 = \frac{1}{(1-\rho_i)\rho_i}\frac{1}{\sigma_i^2}$$

Considering this value for x_3 we have that:

$$\begin{aligned}\lim_{\sigma_\varepsilon^2\to\infty} \quad & 18x_3x_2x_1x_0 - 4x_2^3x_0 + x_2^2x_1^2 - 4x_3x_1^3 - 27x_3^2x_0^2 \\ = \quad & \frac{-4(1-\rho_i)^2\cdot\sigma_i^4 - 4(1-\rho_i)\rho_i\cdot\sigma_i^4 + \rho_i\cdot\sigma_i^2\sigma_c^2}{(1-\rho_i)\rho_i\cdot\sigma_i^4\sigma_c^6}. \end{aligned} \tag{73}$$

Clearly (73) is positive if and only if:

$$\sigma_c^2 \geq \frac{4(1-\rho_i)}{\rho_i}\sigma_i^2$$

By re-arranging terms we get the result.■

Proof of Proposition 8 Define the polynomial:

$$q(\beta,\rho_i) \triangleq x_3\cdot\beta^3 + x_2\cdot\beta^2 + x_1\cdot\beta + x_0, \tag{74}$$

with x_3, x_2, x_1, x_0 defined in (12) (note that these coefficients depend on ρ_i). Let $\beta^*(\rho_i)$

be the unique root of (74). It is easy to check that:

$$\frac{\partial \beta^*(\rho_i)}{\partial \rho_i} = -\frac{\frac{\partial q(\beta^*(\rho_i),\rho_i)}{\rho_i}}{\frac{\partial q(\beta^*(\rho_i),\rho_i)}{\beta}}. \tag{75}$$

If $q(\rho_i)$ has a unique root then $q(\rho_i)$ is increasing at this root (hence, the denominator of (75) is positive). It is possible to check that, $q(\rho_i)$ is decreasing in ρ_i (to be checked later). Hence, the numerator of (75) is negative. Hence, if $q(\beta, \rho_i)$ has a unique root, then this root is increasing in ρ_i.

Proof that $q(\beta, \rho_i)$ is decreasing in ρ_i. We first provide bounds on the value of the root $\beta^*(\rho_i)$. It is easy to check that:

$$\text{If } \beta > \frac{(1+(N-1)\rho_i)\sigma_c^2}{(\sigma_\varepsilon^2 + N\sigma_c^2)}, \text{ then } x_3 \cdot \beta^3 + x_2\beta^2 > 0.$$

On the other hand,

$$\text{If } \beta > \frac{\sigma_c^2}{\sigma_\varepsilon^2 + \sigma_c^2}, \text{ then } x_1 \cdot \beta + x_0 > 0.$$

Hence, the root must satisfy that:

$$\beta^*(\rho_i) \leq \max\{\frac{(1+(N-1)\rho_i)\sigma_c^2}{(\sigma_\varepsilon^2 + N\sigma_c^2)}, \frac{\sigma_c^2}{\sigma_\varepsilon^2 + \sigma_c^2}\}.$$

To check that $q(\beta, \rho_i)$ is decreasing in ρ_i note that:

$$\frac{\partial q(\beta, \rho_i)}{\partial \rho_i} = \beta^2 (\frac{(-(N-1)(1-\rho_i) + (1+(N-1)\rho_i))}{(1-\rho_i)^2(1+(N-1)\rho_i)^2} \frac{(\sigma_\varepsilon^2 + N\cdot\sigma_c^2)}{\sigma_i^2\sigma_c^2}\beta - \frac{1}{(1-\rho_i)^2\sigma_i^2}). \tag{76}$$

If (76) is negative when we evaluated at:

$$\beta = \frac{(1+(N-1)\rho_i)\sigma_c^2}{(\sigma_\varepsilon^2 + N\sigma_c^2)} \text{ and } \beta = \frac{\sigma_c^2}{\sigma_\varepsilon^2 + \sigma_c^2}, \tag{77}$$

then it will continue to be negative at the root β^*. This is because the term inside the parenthesis in (76) is an affine function of β with a negative intercept. Hence, the term inside the parenthesis is either an increasing function of β or it is negative for all positive values of β. Hence, if the term inside the parenthesis is negative when we evaluated at (77) then it will continue to be negative at the root β^* (which is smaller than both terms in (77)).

If we replace $\beta = \dfrac{(1+(N-1)\rho_i)\sigma_c^2}{(\sigma_\varepsilon^2 + N\sigma_c^2)}$ in (76) and simplify terms, we get:

$$-\beta^2 \frac{N-1}{(1-\rho_i)\sigma_i^2((N-1)\rho_i+1)} < 0$$

If we replace $\beta = \dfrac{(1+(N-1)\rho_i)\sigma_c^2}{(\sigma_\varepsilon^2 + N\sigma_c^2)}$ in (76) and simplify terms, we get:

$$-\beta^2 \frac{(N-1)\left((N-1)\left(\rho_i^2\sigma_\epsilon^2 + (\rho_i-1)^2\sigma_c^2\right) + \sigma_\epsilon^2\right)}{(\rho_i-1)^2\sigma_i^2((N-1)\rho_i+1)^2\left(\sigma_c^{22} + \sigma_\epsilon^2\right)} < 0$$

Hence, (76) is negative when we evaluated at the root $\beta^*(\rho_i)$. Hence, $q(\rho_i)$ is decreasing in ρ_i. Hence, we prove the result. ■

Proof of Proposition 9 Krishna (2003) shows that the ascending auction has an efficient ex post equilibrium (see Theorem 2 therein). Second we prove that the price paid in the ex post efficient equilibrium is (63). Yet, this is standard in the literature (see for example Ausubel (1999)). Finally, Perry and Reny (1999) provides a revenue equivalence theorem for ex post equilibria. That is, if two mechanisms implement the same allocation as an ex post equilibrium, then the payments must be the same. Hence, (63) must also be the payment in the outcome of the ascending auction. Hence, we prove the result. ■

Proof of Proposition 10 Before we provide the proof it is convenient to provide a brief explanation what is the difficulty of the proof. An equilibrium statistic is defined by the fixed point (65). Hence, to prove existence one would like to directly use a fixed point theorem (e.g. Brouwer's fixed-point theorem). The expectation of agent n is continuous in the weights that agent m places on his signal $\boldsymbol{\beta}_m$, except at 0. That is, at the point $\boldsymbol{\beta}_m = (0, ..., 0)$ the expectation $\mathbb{E}[v_n|\mathbf{s}_n, \boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N]$ is discontinuous in $\boldsymbol{\beta}_m$. Hence, the whole difficulty of the proof is to circumvent this point of discontinuity.

The proof proceeds in two steps. We first define a perturbed payoff environment and show that an equilibrium statistic exists in this perturbed environment. In the perturbed payoff environment an agent will always place a positive weight on one signal, and hence the point $\boldsymbol{\beta}_m = (0, ..., 0)$ will not be in the range of the possible weights we consider. This circumvents the discontinuity. In the second step we show that an equilibrium statistic exists in the original environment by taking a sequence of perturbed environments that converges to the original one.

(Step 1) Recall that the augmented information structure is equal to the original information structure, but adding a signal (labeled $J+1$) to each agent:

$$s_n^{J+1} = \epsilon_n, \tag{78}$$

where $\{\epsilon_n\}_{n\in N}$ have a variance equal to 1, independent across agents and independent of all other random variables in the model. We denote by $\mathbf{s}'_n$ the set of all original signals observed by n plus s_n^{J+1}. That is. $\mathbf{s}'_n \triangleq (\mathbf{s}_n, s_n^{J+1})$. Note that $\mathbf{s}'_n$ is different than s_n as defined in (5). We incur in this abuse of notation because there is no ambiguity on the use of the notation and this allow us to keep the notation simpler.

Consider a payoff environment in which for all $n \in N$, the payoff shock of agent n is equal to:

$$v'_n \triangleq v_n + \frac{1}{k} s_n^{J+1},$$

where $k \in \mathbb{N}$. Note that we are adding a shock to v_n, and each agent knows the realization of his additional shock.

We denote by $\mathscr{R} \subset \mathbb{R}^{J+1}$, the set of all vectors that have a $1/k$ in the component $J+1$ and have a norm smaller or equal than some big enough $M \subset \mathbb{R}$, where M is big enough to satisfy a condition specified later. Clearly, $\mathscr{R}$ is a convex and compact subset of $\mathbb{R}^{J+1}$.

We define function $f_n : \mathscr{R}^{N-1} \to \mathscr{R}$ as the weights agent n places on his own signals when he knows $(\boldsymbol{\beta'}_1 \cdot \mathbf{s}_1{}', ..., \boldsymbol{\beta'}_{n-1} \cdot \mathbf{s}_{n-1}{}', \boldsymbol{\beta}_{n+1}{}' \cdot \mathbf{s}_{n+1}{}', ..., \boldsymbol{\beta'}_N \cdot \mathbf{s}'_N)$. That is, $f_n(\boldsymbol{\beta'}_1, ..., \boldsymbol{\beta'}_{n-1}, \boldsymbol{\beta}_{n+1}{}', ..., \boldsymbol{\beta'}_N)$ is defined implicitly as follows:

$$\mathbb{E}[v'_n | \mathbf{s}'_n, \boldsymbol{\beta'}_1 \cdot \mathbf{s}'_1, ..., \boldsymbol{\beta'}_N \cdot \mathbf{s}'_N] \triangleq f_n(\boldsymbol{\beta'}_1, ..., \boldsymbol{\beta'}_{n-1}, \boldsymbol{\beta}_{n+1}{}', ..., \boldsymbol{\beta'}_N) \cdot \mathbf{s}'_n + \sum_{m\neq n} \gamma_m \cdot \boldsymbol{\beta'}_m \cdot \cdot \mathbf{s}'_m, \tag{79}$$

for some $\{\gamma_m\}_{m\neq n} \in \mathbb{R}^{N-1}$. There are several things to note.

(i) Since the original signals have full rank, the expectation in (79) is uniquely defined. That is, $f_n(\boldsymbol{\beta'}_1, ..., \boldsymbol{\beta'}_{n-1}, \boldsymbol{\beta}_{n+1}{}', ..., \boldsymbol{\beta'}_N)$ is a function (and not a correspondance).

(ii) Since $\{s_n^{J+1}\}_{n\in N}$ is independent of all other random variables in the model, agent n always places a weight equal to $1/k$ on s_n^{J+1}. This guarantees that $f_n(\boldsymbol{\beta'}_1, ..., \boldsymbol{\beta'}_{n-1}, \boldsymbol{\beta}_{n+1}', ..., \boldsymbol{\beta'}_N)$ is always equal to $1/k$ in the coordinate $J+1$.

(iii) We can bound the weights agent n places on his own signals. This is because the variance of the expectation is always smaller than the variance of the original random variable. That is, it is possible to find M large enough such that if $||\boldsymbol{\beta}_n|| > M$, then

$$var\Bigg(\boldsymbol{\beta}_n \cdot \mathbf{s}_n' + \sum_{m\neq n} \gamma_m \cdot \boldsymbol{\beta'}_m \cdot \cdot \mathbf{s}_m'\Bigg) > var\Bigg(v_n'\Bigg),$$

for any $\{\gamma_m\}_{m\neq n}$. This is because $var(\mathbf{s}_1, ..., var(\mathbf{s}_N)$ has full rank. This guarantees that $f_n(\boldsymbol{\beta'}_1, ..., \boldsymbol{\beta'}_{n-1}, \boldsymbol{\beta}_{n+1}', ..., \boldsymbol{\beta'}_N)$ has a norm smaller or equal than M.

(iv) the points (ii) and (iii) guarantee that the range of $f_n(\boldsymbol{\beta'}_1, ..., \boldsymbol{\beta'}_{n-1}, \boldsymbol{\beta}_{n+1}', ..., \boldsymbol{\beta'}_N)$ is equal to $\mathscr{R}$.

(v) Since the variance covariance matrix of signals has full rank, and the component $J+1$ is always equal to $1/k$, the function $f_n(\boldsymbol{\beta'}_1, ..., \boldsymbol{\beta'}_{n-1}, \boldsymbol{\beta}_{n+1}', ..., \boldsymbol{\beta'}_N)$ is continuous in $\mathscr{R}^{N-1}$. That is, the expectation would be discontinuous at the point $(0, ..., 0)$, but we are not considering this vector in the domain of the function. This is because the domain of the function is $\mathscr{R}$, which has $1/k$ in the component $J+1$.

We now define function $\mathbf{f} : \mathscr{R}^N \to \mathscr{R}^N$ as folllows:

$$\mathbf{f}(\boldsymbol{\beta'}_1, ..., \boldsymbol{\beta'}_N) \triangleq (f_1(\boldsymbol{\beta'}_2, ..., \boldsymbol{\beta'}_N), ..., f_N(\boldsymbol{\beta'}_1, ..., \boldsymbol{\beta'}_{N-1})).$$

By the previous argument, $\mathbf{f}$ is a continuous function, with a domain equal to its range, and defined on a compact domain. By Brouwer's fixed-point theorem $\mathbf{f}$ has a fixed point. By construction, the fixed point $(\boldsymbol{\beta}_1^k, ..., \boldsymbol{\beta}_N^k)$ satisfies:

$$\mathbb{E}[v_n'|\mathbf{s}_n', \boldsymbol{\beta}_1^k \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N^k \cdot \mathbf{s}_N] = \mathbb{E}[v_n'|\boldsymbol{\beta}_1^k \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N^k \cdot \mathbf{s}_N].$$

Hence, $(\boldsymbol{\beta}_1^k, ..., \boldsymbol{\beta}_N^k)$ is an equilibrium statistic under the perturbed payoff shock.

(Step 2) We denote by $(\boldsymbol{\beta}_1^k, ..., \boldsymbol{\beta}_N^k)$ the equilibrium statistic of the perturbed environment. We normalize the equilibrium statistic such that for all $n \in N$, $||\boldsymbol{\beta}_n^k|| = 1$ (remember the $J+1$ component is equal to $1/k$, so the norm is never 0). Since $(\boldsymbol{\beta}_1^k, ..., \boldsymbol{\beta}_N^k)$ is defined on a compact set, it has a convergent subsequence. By re-labeling the series, we can assume that $(\boldsymbol{\beta}_1^k, ..., \boldsymbol{\beta}_N^k)$ is a convergent sequence, and define the limit $(\boldsymbol{\beta}_1^\infty, ..., \boldsymbol{\beta}_N^\infty)$. Note that by taking the limit $k \to \infty$, we are not changing the joint distribution of signals and payoff shocks $(v_1, ..., v_N, \mathbf{s}_1, ..., \mathbf{s}_N)$. By changing k we only change the impact that ϵ_n has on v_n'.

We have considered equilibrium statistics such that for all $n \in N$, $||\boldsymbol{\beta}_n^k|| = 1$, and the joint distribution of signals stays constant as we take the limit. Hence,

$$\lim_{k\to\infty} \mathbb{E}[v_n | \mathbf{s}_n', \boldsymbol{\beta}_1^k \cdot \mathbf{s}_1', ..., \boldsymbol{\beta}_N^k \cdot \mathbf{s}_N'] = \mathbb{E}[v_n | \mathbf{s}_n', \boldsymbol{\beta}_1^\infty \cdot \mathbf{s}_1', ..., \boldsymbol{\beta}_N^\infty \cdot \mathbf{s}_N']; \tag{80}$$

$$\lim_{k\to\infty} \mathbb{E}[v_n | \boldsymbol{\beta}_1^k \cdot \mathbf{s}_1', ..., \boldsymbol{\beta}_N^k \cdot \mathbf{s}_N'] = \mathbb{E}[v_n | \boldsymbol{\beta}_1^\infty \cdot \mathbf{s}_1', ..., \boldsymbol{\beta}_N^\infty \cdot \mathbf{s}_N']. \tag{81}$$

That is, the informational content of $\boldsymbol{\beta}_n^k \cdot \mathbf{s}_n'$ changes continuously in the limit $k \to \infty$. Hence, the expectations must change continuously. By construction of the equilibrium static:

$$\mathbb{E}[v_n + \frac{1}{k}\epsilon_n | \mathbf{s}_n', \boldsymbol{\beta}_1^k \cdot \mathbf{s}_1', ..., \boldsymbol{\beta}_N^k \cdot \mathbf{s}_N'] = \mathbb{E}[v_n + \frac{1}{k}\epsilon_n | \boldsymbol{\beta}_1^k \cdot \mathbf{s}_1', ..., \boldsymbol{\beta}_N^k \cdot \mathbf{s}_N'],$$

then we must also have that:

$$\lim_{k\to\infty} var\bigg(\mathbb{E}[v_n | \mathbf{s}_n', \boldsymbol{\beta}_1^k \cdot \mathbf{s}_1', ..., \boldsymbol{\beta}_N^k \cdot \mathbf{s}_N'] - \mathbb{E}[v_n | \boldsymbol{\beta}_1^k \cdot \mathbf{s}_1', ..., \boldsymbol{\beta}_N^k \cdot \mathbf{s}_N']\bigg) = 0. \tag{82}$$

This implies that:

$$\mathbb{E}[v_n | \boldsymbol{\beta}_1^\infty \cdot \mathbf{s}_1', ..., \boldsymbol{\beta}_N^\infty \cdot \mathbf{s}_N'] = \mathbb{E}[v_n | \mathbf{s}_n', \boldsymbol{\beta}_1^\infty \cdot \mathbf{s}_1', ..., \boldsymbol{\beta}_N^\infty \cdot \mathbf{s}_N'].$$

Hence, $(\boldsymbol{\beta}_1^\infty \cdot \mathbf{s}_1', ..., \boldsymbol{\beta}_N^\infty \cdot \mathbf{s}_N')$ is an equilibrium statistic of the environment in which agents observe an additional signal s_n^{J+1} that is pure noise.

Note that when taking the expectation $\mathbb{E}[v_n | \mathbf{s}_n', \boldsymbol{\beta}_1^\infty \cdot \mathbf{s}_1', ..., \boldsymbol{\beta}_N^\infty \cdot \mathbf{s}_N']$, an agent always places 0 weight on s_n^{J+1} (as this is pure noise in the limit). Hence, if the equilibrium statistic $\boldsymbol{\beta}_n^\infty \cdot \mathbf{s}_n'$ places non-zero weight on s_n^{J+1}, this means that the expectation

$\mathbb{E}[v_n|\boldsymbol{\beta}_n^\infty \cdot \mathbf{s}'_1, ..., \boldsymbol{\beta}_N^\infty \cdot \mathbf{s}'_N]$ places 0 weight on $\boldsymbol{\beta}_1^\infty \cdot \mathbf{s}'_n$. Hence, if the equilibrium statistic $\boldsymbol{\beta}_n^\infty \cdot \mathbf{s}'_n$ places non-zero weight on s_n^{J+1}, this means an agent ignores his own signals in equilibrium. In most natural applications this obviously is not satisfied, and hence the weight on s_n^{J+1} is 0.

(Symmetric Information Structures.) Clearly, if the information structure is symmetric, then we can repeat the argument using symmetric equilibrium statistic. That is, instead of considering $\boldsymbol{\beta}_{-n} \triangleq (\boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_{n-1}, \boldsymbol{\beta}_{n+1}, ..., , \boldsymbol{\beta}_N)$, one needs to consider $\boldsymbol{\beta}_{-n}$ in which $\boldsymbol{\beta}_m = \boldsymbol{\beta}_k$, for all $m, k \neq n$.
Hence, we prove the result.■

Proof of Theorem 2 This is a direct corollary of Theorem 3 and the fact that the equilibrium characterized in Proposition 9 is an ex post equilibrium. ■

Proof of Theorem 3. It is clear that for any equilibrium statistic, the joint distribution of the random variables $(v_1, ..., v_N, \mathbf{s}_1, ..., \mathbf{s}_N, t_1, ..., t_N)$ is jointly normally distributed (this is just a linear combination of Gaussian signals). We first provide the main steps of the proof and then explain each step in detail. If $\hat{\alpha}_n : \mathbb{R} \to A_n$ is an ex post equilibrium of game $(P, \hat{\mathscr{I}})$, then:

$$\Rightarrow \quad \forall n \in N, \forall \boldsymbol{t} \in \mathbb{R}^N,\ \forall a'_n \in A_n, \quad \mathbb{E}[u_n(\hat{\alpha}(t_n), \hat{\alpha}(t_{-n}), v_n)|\boldsymbol{t}] \geq \mathbb{E}[u_n(a'_n, \hat{\alpha}(t_{-n}), v_n)|\boldsymbol{t}] \tag{83}$$

$$\Rightarrow \quad \forall n \in N, \forall \boldsymbol{t} \in \mathbb{R}^N,\ \forall \mathbf{s}_n \in \mathbb{R}^J,\ \forall a'_n \in A_n, \quad \mathbb{E}[u_n(\hat{\alpha}(t_n), \hat{\alpha}(t_{-n}), v_n)|\boldsymbol{t}, \mathbf{s}_n] \geq \mathbb{E}[u_n(a'_n, \hat{\alpha}(t_{-n}), v_n)|\boldsymbol{t}, \mathbf{s}_n] \tag{84}$$

$$\Rightarrow \quad \forall n \in N, \forall \boldsymbol{t} \in \mathbb{R}^N,\ \forall \mathbf{s}_n \in \mathbb{R}^J,\ \forall a'_n \in A_n, \quad \mathbb{E}[u_n(\hat{\alpha}(t_n), \hat{\alpha}(t_{-n}), v_n)|\mathbf{s}_n, \hat{\alpha}_1(t_1), ..., \hat{\alpha}_N(t_N)] \geq \mathbb{E}[u_n(a'_n, \hat{\alpha}(t_{-n}), v_n)|\mathbf{s}_n, \hat{\alpha}_1(t_1), ..., \hat{\alpha}_N(t_N)] \tag{85}$$

$$\Rightarrow \alpha^* : \mathbb{R}^J \to M, \text{ with } \alpha^*(\mathbf{s}_n) = \hat{\alpha}(t_n) = \hat{\alpha}(\boldsymbol{\beta}_n \cdot \mathbf{s}_n) \text{ is a posterior equilibrium of game } G \tag{86}$$

Step (83) This is by definition of ex post equilibria in game $(P, \hat{\mathscr{I}})$.

Step (84) First, note that the expectations are over random variable v_n. Hence, we need to prove that:

$$\forall \boldsymbol{t} \in \mathbb{R}^N,\ \forall \mathbf{s}_n \in \mathbb{R}^J,\ v_n|_{\boldsymbol{t}} = v_n|_{\boldsymbol{t}, \mathbf{s}_n}.$$

That is, the distribution of v_n conditional on $\boldsymbol{t}$ is the same same as the conditional distribution of v_n conditional on $\mathbf{t}$ and $\mathbf{s}_n$. As the random variables are normally distributed, it suffices to prove that:

$$\forall \boldsymbol{t} \in \mathbb{R}^N,\ \forall \mathbf{s}_n \in \mathbb{R}^J,\ \mathbb{E}[v_n|\boldsymbol{t}] = \mathbb{E}[v_n|\boldsymbol{t}, \mathbf{s}_n]; \tag{87}$$

$$\forall \boldsymbol{t} \in \mathbb{R}^N,\ \forall \mathbf{s}_n \in \mathbb{R}^J, var(v_n|\boldsymbol{t}){=}var(v_n|\boldsymbol{t}, \mathbf{s}_n) \tag{88}$$

(87) is true by the definition of an equilibrium statistic. (88) is true because the variables are jointly Gaussian and hence:

$$var(v_n|\boldsymbol{t}) = var(v_n) - var(\mathbb{E}[v_n|\boldsymbol{t}]){=}var(v_n) - var(\mathbb{E}[v_n|\boldsymbol{t}, \mathbf{s}_n]) = var(v_n|\boldsymbol{t}, \mathbf{s}_n)$$

Step (85) Note that $\hat{\alpha}_n(t_n)$ is measurable with respect to t_n. Hence,

$$\mathbb{E}[u_n(a_n', \hat{\alpha}(t_{-n}), v_n)|\boldsymbol{t}, \mathbf{s}_n] = \mathbb{E}[u_n(a_n', \hat{\alpha}(t_{-n}), v_n)|\mathbf{s}_n, \boldsymbol{t}, \hat{\alpha}_1(t_1), ..., \hat{\alpha}_N(t_N)]; \tag{89}$$

$$\mathbb{E}[u_n(\hat{\alpha}(t_n), \hat{\alpha}(t_{-n}), v_n)|\boldsymbol{t}, \mathbf{s}_n] = \mathbb{E}[u_n(\hat{\alpha}(t_n), \hat{\alpha}(t_{-n}), v_n)|\mathbf{s}_n, \boldsymbol{t}, \hat{\alpha}_1(t_1), ..., \hat{\alpha}_N(t_N)]. \tag{90}$$

That is, we can add $\hat{\alpha}_n(t_n)$ as conditioning variable. Hence, we can write (84) as follows:

$$\mathbb{E}[u_n(\hat{\alpha}(t_n), \hat{\alpha}(t_{-n}), v_n)|\mathbf{s}_n, \boldsymbol{t}, \hat{\alpha}_1(t_1), ..., \hat{\alpha}_N(t_N)] \geq \mathbb{E}[u_n(a_n', \hat{\alpha}(t_{-n}), v_n)|\mathbf{s}_n, \boldsymbol{t}, \hat{\alpha}_1(t_1), ..., \hat{\alpha}_N(t_N)].$$

Taking expectation of the previous equation conditional on $(\mathbf{s}_n, \hat{\alpha}_1(t_1), ..., \hat{\alpha}_N(t_N))$ and using the law of iterated expectations:

$$\mathbb{E}[u_n(\hat{\alpha}(t_n), \hat{\alpha}(t_{-n}), v_n)|\mathbf{s}_n, \hat{\alpha}_1(t_1), ..., \hat{\alpha}_N(t_N)] \geq \mathbb{E}[u_n(a_n', \hat{\alpha}(t_{-n}), v_n)|\mathbf{s}_n, \hat{\alpha}_1(t_1), ..., \hat{\alpha}_N(t_N)].$$

Hence, we prove the step.

Step (86) Using that $\boldsymbol{\beta}_n \cdot \mathbf{s}_n = t_n$, we get the definition of posterior equilibria.

Hence, we prove the result.■

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