Optimal Mechanism with Budget Constraint Bidders*

Alexei Bulatov† Sergei Severinov‡

This draft: November 2016

Abstract

The paper deals with the optimal mechanism design for selling to buyers who have commonly known budget constraints. With unequal budgets, our problem is that of asymmetric optimal mechanism design. We derive and characterize the optimal mechanism. It belongs to one of two classes. When the budget differences are small, the mechanism discriminates only between high-valuation buyers for whom the budget constraint is binding. All low valuations buyers are treated symmetrically despite budget differences. When budget differences are sufficiently large, the optimal mechanism discriminates in favor of buyers with small budgets when the valuations are low, and in favor of buyers with larger budgets when the valuations are high.

*We are grateful to the colleagues at UBC and Higher School of Economics, seminar participants at Bocconi University and European University Institute and the participants of the Meetings for the Advancement of Economic Theory for helpful comments and suggestions. A standard disclaimer applies.

†Higher School of Economics, Moscow; email: aboulatov@gmail.com

‡Vancouver School of Economics, University of British Columbia; email: sseverinov@gmail.com.

1 Introduction

Buyers often face budget constraints that restrict their ability to pay for the goods that they want to purchase. For example, in the keyword search auctions run by the internet search engines such as Google and Bing the advertisers typically have budget limits set by senior management. Most households have limited savings and incomes. Therefore, when they participate in consumer good auctions such as car auctions, eBay auctions, etc., they face budgets limits that may be lower than their valuations for the goods, especially big-ticket items. Budget constraints faced by bidders were an important practical matter in spectrum auctions. Rothkopf (2007) provides an example of a spectrum auction in which a bidder valued the asset at $85 million, but was only able to finance a bid of $65 million, and therefore stopped bidding when the price reached $65 million.

Therefore, it is natural that the economic analysis of trading mechanisms and institutions should take the budget constraints into account. Yet, with some notable exceptions discussed below, the literature on mechanism design and auctions had for most part focussed on the situations where budget constraints are absent.

In this paper we study an environment in which a number of buyers compete for a single good and the seller acts as a mechanism designer. The buyers have private values and commonly known and asymmetric/unequal budgets. We focus on deriving an optimal mechanism for the seller. Importantly, the asymmetry of the budget constraints implies that our problem is that of asymmetric mechanism design, which is significantly more complex than a mechanism in which all participants are ex-ante symmetric. In a symmetric situation -such as, for example, when all budgets are equal and the bidders' valuations are drawn from the same distribution- a mechanism designer has to construct a single allocation profile (probability of trading and transfer function) which is offered to every participant. This affords a significant simplification in the analysis and characterization of the optimal mechanism. Yet, in the asymmetric environment, such as the one we study, the designer has to design ex-ante asymmetric allocation profiles, one for each buyer, and do so in a consistent way.

The optimal mechanism that we derive has a number of interesting and novel qualitative properties. First, we show that it belongs to one of the two classes, depending on the profile of budgets[1] If the budget differences across the bidders are sufficiently small (in the sense made precise below), the optimal mechanism is a so-called "top-auction." It is characterized by a common threshold valuation $\bar{x}^t$ at which the budget constraint of each bidder becomes binding, so any bidder with valuation exceeding $\bar{x}^t$ pays a transfer equal to her budget, and the probability of getting the good is the same for all types of a particular bidder in $[\bar{x}^t, 1]$.

Thus, in the top auction all buyers whose types exceed $\bar{x}^t$ are tied. The tie-breaking rule determining the probabilities with which the good is allocated to different bidders with valuations exceeding $\bar{x}^t$ plays an important role in this mechanism. In fact, these probabilities are the only instrument used by the seller to discriminate between different bidders. A richer bidder with valuation exceeding $\bar{x}^t$ gets the good with a higher probability to compensate her for the higher transfer, equal to her budget, that she pays to the seller.

All the buyers with valuations below $\bar{x}^t$ are treated symmetrically in the "top auction:" each of them gets the good when she has the highest valuation, and pays a transfer derived by the standard envelope result. Thus, the top auction necessarily involves discontinuity in the allocation rule: the probability of getting the good increases discontinuously at $\bar{x}^t$, and the sizes of these jumps are positively correlated with the budget levels.

The reserve price in the top-auction is a function of the threshold valuation $\bar{x}^t$ reflecting the fact that virtual valuations of the buyers depend on $\bar{x}^t$, and is lower than in the standard case. Thus, the inefficiency in the top auction takes two forms. First, there is the inefficiency in choosing between the buyers at the top: since all bidders with valuation above $\bar{x}^t$ are tied, the one with a lower valuation among them may be awarded the good. Second, there is also an inefficiency associated with a positive reserve price. However, this inefficiency is smaller in magnitude than in the standard case due to a lower reserve price.

When the buyers' budgets are sufficiently different, the "top auction" is no longer feasible because the seller can no longer achieve the necessary differentiation between the buyers with different budgets by discriminating only "at the top." In particular, it becomes impossible to allocate the good to the buyers with valuations exceeding the (endogenous) threshold $\bar{x}^t$

[1] We assume that each bidder's budget is sufficiently small so that it becomes binding at higher valuations in the optimal mechanism. An exogenous condition guaranteeing this is that each budget is less than a monopoly price set by a seller facing a single bidder.

in such a way that the budget constraint of every buyer is binding. Therefore, a different kind of mechanism which we call a "budget-handicap auction" and in which the seller uses other instruments of discrimination between the buyers, becomes optimal. Specifically, the seller sets different thresholds for different buyers or groups of buyers. Not surprisingly, richer buyers have higher thresholds. Not all thresholds have to be different: there may be clusters of buyers who share the same threshold. But there is more than one threshold across bidders. The allocation rule between the buyers in the same cluster with valuations above the corresponding threshold follows the same principle as in the "top auction:" the probabilities of awarding the good to them are chosen so that their budget constraints bind.

Importantly, in the budget-handicap auction the seller starts discriminating between the buyers also when they have low valuations. In particular, consider two buyers whose valuations are equal and are below their respective thresholds. Then the buyer with the smaller budget gets the good with a higher probability than the buyer with a higher budget. The buyers with lower budgets also have lower reserve prices than the buyers with higher budgets. Thus, the seller handicaps the buyers with high budgets at lower valuations. This introduces an additional inefficiency into the mechanism compared to the top auction. However, this "handicap" on the high-budget bidders at low valuations creates more competition for them from low-budget bidders. This allows the seller to extract more surplus from higher-budget bidders and increases her profits.

Our main results and, in particular, Theorem 3 provide necessary and sufficient conditions characterizing unique optimal mechanism. Interestingly, as the discussion following Theorem 3 highlights, the optimality conditions for a profile of thresholds are essentially the feasibility conditions ensuring consistency between the allocation probabilities defined optimally for a given profile of thresholds (as specified in Lemma 4) and the binding budget constraints at the thresholds. Theorem 5 builds on these results to specify the conditions under which the "top auction" is optimal. Naturally, the "budget-handicap" auction is optimal in the complementary case, when the conditions of Theorem 5 fail.

A natural question that arises in our model is how the variability of budgets among the bidders affects the seller's profits. It turns out that this question has a simple answer. The seller prefers less budget variability and, with a fixed aggregate budget, she gets the highest expected profits when each bidder has the same budget (Lemma 8).

On the computational side, it is fairly straightforward to check the conditions of Theorem 5 and compute the "top auction" when these conditions hold. The most challenging part in computing the "budget-handicap" auction is determining the "clusters" of bidders who share the same threshold. This problem does not present analytical difficulties as it only involves checking whether the conditions of Theorem 3 hold or not. However, one may have to go through all possible configurations of clusters which is a combinatorial problem that can be solved computationally. We provide an illustration by computing the optimal mechanism with two and three bidders, the latter - under uniform type distribution. The example with three bidders is particularly telling about the budget handicap auction as it shows that every possible configuration of clusters is optimal for a set of budget profiles of a positive measure.

Technically, our paper contains a number of interesting aspects. Among them - the characterization of the virtual values in the optimal mechanism and, in particular, the virtual value for an endogenous "atom" of types above the threshold all of whom get the same allocation (see expression (15)). Another interesting aspect is the uncovered strong connection between these thresholds and the Lagrange multipliers associated with budget constraints. Not only there is a one-to-one relationship between these two sets of variables, as demonstrated by Theorem 2, but also the strong duality property between them ultimately allows us to complete the solution to the optimal mechanism design problem.

In the related literature, the paper closest to ours is Laffont and Robert (1996) who consider a similar environment with commonly known but equal budgets among the bidders. They derive an optimal mechanism which is a special case of our top auction. Their optimal mechanism is symmetric and does not allow to understand what the seller should do when the buyers have different budgets, and hence are asymmetric from the ex-ante point of view. Maskin (2000) studies efficient mechanism design in the same environment as Laffont and Robert (1996). Thus, our results confirm the robustness of the optimal auction of Laffont and Robert (1996) to budget asymmetry, when budget differences are sufficiently small across the bidders. However, a qualitatively different mechanism - the "budget-handicap auction"- becomes optimal when budget differences become large.

Malakhov and Vohra (2008) derive optimal dominant strategy mechanism for two buyers one of whom faces no budget constraint and the other has a known fixed budget, and whose valuations are distributed over a discrete support. Their mechanism is similar to the one that we derive in the extension of our example with two bidders one of whom has a small budget and the budget of the other is larger than the "monopoly" price that the seller would set for a single bidder. Pai and Vohra (2014) study an optimal mechanism when the bidders' budgets and valuations are private information, with identical distributions of budgets and values across the bidders. Hence, unlike in our model, their bidders are ex-ante symmetric. They assume that the budgets and valuations are distributed over a finite support, with a continuous distribution of types considered in an extension.

Thus, an important difference between our paper and Laffont and Robert (1996) and Pai and Vohra (2014) is that we allow for any profile of commonly known budgets and solve an asymmetric mechanism design problem, while these authors focus on equal budgets among the bidders and hence consider a symmetric mechanism design problem. In a symmetric mechanism design one has to derive a single allocation profile (probabilities of trading and transfers) which is offered to each buyer. In contrast, under asymmetric mechanism design which we are dealing with, all allocation profiles are different, and the mechanism designer has to ensure that they are consistent with each other.

In the related literature, Che and Gale (1998) and (1996) were the earlier contributions studying auctions with budget constraints. Che and Gale (1998) compare the performance of first- and second-price auctions when the buyers have privately known budgets and valuations. They conclude that the first-price auction yields higher expected social surplus and expected revenue. Che and Gale (1996) show that the all-pay auction performs better than the first-price auction under common value and private budgets. Che and Gale (2000) explore optimal nonlinear pricing for a buyer with privately known valuation and budget. Hafalir, Ravi and Sayedi (2012) study a Vickrey auction with budget-constraint bidders, and focus on ex-post Nash equilibrium in it. In their framework, the bidders have different and essentially known budgets. Although their mechanism is not optimal, it has good efficiency properties and is "nearly" Pareto optimal.

The rest of the paper is organized as follows. Section 2 develops the model and preliminary results. Section 3 contains the central analysis of the problem. Section 4 presents the optimal mechanisms and their qualitative properties. Section 5 is devoted to the examples. Section 6 concludes. The proofs are relegated to the Appendix.

2 Model and Preliminaries

A single seller wants to sell one unit of the good to n bidders. Bidder $i \in \{1, ..., n\}$ has privately known valuation x_i for the good which is drawn from the common knowledge distribution $F(.)$, with a continuous positive density function $f(.)$. Without loss of generality, we assume that the support of $F(.)$ is $[0, 1]$. Additionally, bidder i has budget m_i, so that his payment in the mechanism can never exceed m_i. The budgets are commonly known and will be assumed to be sufficiently small. Plainly, we focus on the situation in which the range of possible valuations -normalized to $[0, 1]$- is large compared to the limited financial resources of each bidder. As we will see below, a sufficient condition for all budget constraints to be binding in the optimal mechanism is that $\max_i m_i \leq \arg\max p(1 - F(p))$ i.e., the highest budget is below the monopoly price set in a situation with a single buyer. With multiple bidders, competition causes a bidder's budget constraint to be binding even if her budget exceeds this threshold.

We will impose a standard assumption on the distribution $F(.)$:

Assumption 1 *Increasing Hazard rate:*

$$\frac{f(x)}{1 - F(x)} \text{ is increasing in } x \text{ for all } x \in [0, 1] \tag{1}$$

In fact, a weaker assumption that $x - \frac{1-F(x)}{f(x)}$ is increasing is sufficient, and we make the increasing hazard rate assumption mainly for the sake of conformity with the literature. [2]

Bidder i with valuation x_i gets a payoff equal to $x_i q_i - t_i$ if she gets the good with probability q_i and makes a payment t_i to the seller. The seller has zero value for the good, so

[2] Pai and Vohra (2014) claim that a stronger assumption that $f(x)$ is nonincreasing is necessary in this setting because bidder i's virtual valuation s equal to $x - \frac{1-F(x)-\lambda_i}{f(x)}$, where λ_i is a Lagrange multiplier associated with the budget constraint of bidder i. However, as we show below, $\lambda_i \leq 1 - F(x)$. Therefore, the monotonicity of $x - \frac{1-F(x)}{f(x)}$ guarantees the monotonicity of $x - \frac{1-F(x)-\lambda_i}{f(x)}$.

her payoff is simply the sum of the payments that she receives from the buyers, $\sum_{i=1,..,n} t_i$. All the bidders and the seller are risk-neutral and strive to maximize their expected utilities.

The seller has all bargaining power in our environment and acts as a mechanism designer to maximize her expected payoff from the mechanism. Thus, we will focus on the optimal expected revenue-maximizing mechanism.

By the Revelation principle (Myerson 1979) we can restrict attention to direct truthful mechanisms which specify the probabilities of trading and the payments as the functions of the buyers' announced valuations. Accordingly, the mechanism offered by the seller to the buyers can be represented by a tuple $(Q_1(\hat{x}_1, ..., \hat{x}_n), ..., Q_n(\hat{x}_1, ..., \hat{x}_n), T_1(\hat{x}_1, ..., \hat{x}_n), ..., T_n(\hat{x}_1, ..., \hat{x}_n))$, where $(\hat{x}_1, ..., \hat{x}_n)$ is the profile of the valuations (types) announced by the bidders; $Q_i(\hat{x}_n, ..., \hat{x}_n)$ is the probability that the buyer i gets the good and $T_i(\hat{x}_1, ..., \hat{x}_n)$ is the transfer that the buyer i pays to the seller when the profile of types $(\hat{x}_n, ..., \hat{x}_n)$ is announced by the buyers.

Let us also define the expected probability of getting the good and the expected transfer. Specifically, let $q_i(x_i) = \int_{x_{-i} \in [0,1]^{n-1}} Q_i(x_i, x_{-i}) d \prod_{j \neq i} F(x_j)$ be the expected probability that bidder i gets the good when she announces type x_i and all other bidders announce their types truthfully. Also, let $t_i(x_i) = \int_{x_{-i} \in [0,1]^{n-1}} T_i(x_i, x_{-i}) d \prod_{j \neq i} F(x_j)$ be the expected payment by bidder i to the seller when bidder i announces type x_i and all other bidders announce their types truthfully.

The optimal mechanism $(Q_1(x_1, ..., x_n), ..., Q_n(x_1, ..., x_n), T_1(x_1, ..., x_n), ..., T_n(x_1, ..., x_n))$ solves the following seller's maximization problem:

$$\max \sum_{i=1,...,n} \int_{(x_1,...,x_n) \in [0,1]^n} T_i(x_1, ..., x_n) d \prod_{i=1,...,n} f(x_i) \tag{2}$$

subject to the following constraints:

(i) the buyers announce their types truthfully i.e., the following *interim incentive constraints* hold for all $(x_i, \hat{x}_i) \in [0, 1]^2$ and all i:

$$x_i \int_{x_{-i} \in [0,1]^{n-1}} (Q_i(x_i, x_{-i}) - T_i(x_i, x_{-i})) \, d \prod_{j \neq i} F(x_j) \equiv x_i q_i(x_i) - t_i(x_i) \geq$$
$$x_i \int_{x_{-i} \in [0,1]^{n-1}} (Q_i(\hat{x}_i, x_{-i}) - T_i(\hat{x}_i, x_{-i})) \, d \prod_{j \neq i} F(x_j) \equiv x_i q_i(\hat{x}_i) - t_i(\hat{x}_i) \tag{3}$$

(ii) the *individual rationality constraints* hold for all i and $x_i \in [0,1]$:

$$x_i \int_{x_{-i}\in[0,1]^{n-1}} (Q_i(x_i,x_{-i}) - T_i(x_i,x_{-i}))\, d\prod_{j\neq i} F(x_j) \equiv x_i q_i(x_i) - t_i(x_i) \geq 0 \tag{4}$$

(iii) Budget constraint of every bidder type holds i.e.,

$$T_i(x_i,x_{-i}) \leq m_i \quad \text{for all } i,\ x \in [0,1],\ x_{-i} \in [0,1]^{n-1}. \tag{5}$$

(iv) The mechanism is feasible:

$$Q_1(x_1,...,x_n) \geq 0, \tag{6}$$

$$\sum_i Q_i(x_1,...,x_n) \leq 1 \text{ for all } (x_1,...,x_n) \in [0,1]^n. \tag{7}$$

Our first result establishes the existence and uniqueness of the optimal mechanism.

Theorem 1 *There exists an optimal mechanism* $(Q_1(x_1,...,x_n),...,Q_n(x_1,...,x_n),$ $T_1(x_1,...,x_n),...,T_n(x_1,...,x_n))$ *that solves the problem (2) subject to the constraints (3)-(7).*

Proof of Theorem 1: The objective for our maximization problem in (2) is a continuous linear functional in the Hilbert space $L^2([0,1]^n)$. It is easy to see that its admissible set specified by constraints (3)-(7) is convex. Therefore, by Theorem 2.6.1 in Balakrishnan (1993) the solution to our problem exists. The uniqueness almost everywhere follows by standard arguments, in particular, the linearity of the objective and the convexity of the constraints. *Q.E.D.*

Next, let $U_i(x_i)$ denote the net expected payoff of buyer i of type x_i in the optimal mechanism i.e.,

$$U_i(x_i) \equiv q_i(x_i)x_i - t_i(x_i)$$

The following result is standard and is left without proof:

Lemma 1 *The mechanism* $(Q_1(x_1,...,x_n),...,Q_n(x_1,...,x_n),\ t_1(x_1,...,x_n),...,t_n(x_1,...,x_n))$ *is incentive compatible and individually rational if any only if the expected trading probability* $q_i(x_i)$ *is nondecreasing in* x_i *for all* i *and* $x_i \in [0,1]$*, and:*

$$U_i(x_i) = \int_0^{x_i} q_i(s)ds + c_i \ \ \textit{for some}\ \ c_i \in \mathbb{R}_+ \tag{8}$$

The individual rationality requires the constant c_i to be nonnegative. The optimality of the mechanism then implies that the mechanism designer should set $c_i = 0$. We will assume the latter in the sequel and drop c_i altogether from the analysis. Combining the definition of $U_i(x_i)$ with (8) yields the following expression for the expected transfer by bidder i:

$$t_i(x_i) = x_i q_i(x_i) - \int_0^{x_i} q_i(s)ds \tag{9}$$

Consider now the budget constraints. First, we can replace the ex-post budget constraint in (5) with the interim one, $t_i(x_i) \leq m_i$ for all i and x_i. Indeed, the interim budget constraints obviously hold when (5) holds. In the opposite directions, if $t_i(x_i) \leq m_i$ for all i and x_i, then we can ensure that (5) holds by setting $T_i(x_i, x_{-i}) = t_i(x_i)$ for all i and x_i. This modification changes neither the value of the seller's objective nor the incentive or individual rationality constraints, because all of these depend only on the expected transfers $t_i(.)$, but it potentially relaxes the budget constraint in some states of the world since the maximal payment by bidder i weakly decreases.

Next, let us establish the following simple but useful result:

Lemma 2 *Let $\bar{x}_i \in [0,1]$ be defined as follows:*

$$\bar{x}_i = \sup\{x_i \in [0,1] | t_i(x_i) < m_i\} \tag{10}$$

If $\bar{x}_i < 1$, then $t_i(x_i) = m_i$ for all $x_i \in [\bar{x}_i, 1]$

Proof of Lemma 2: Since by Lemma 1 $q_i(x_i)$ must be increasing in x_i, the expected transfer $t_i(x_i)$ must also be increasing in x_i, for otherwise the mechanism will not be incentive compatible. Therefore, if $t_i(x_i) = m_i$, then $t_i(x_i') = m_i$ for all $x_i' \in [x_i, 1]$. *Q.E.D.*

The threshold values $\bar{x}_i$ play an important role in our analysis as the key choice variables which ultimately determine the whole mechanism. Lemma 2 and equation (9) imply that whenever $\bar{x}_i < 1$, the budget constraint condition $t_i(x_i) \leq m_i$ can be replaced with the following two conditions:

$$m_i = \bar{x}_i q_i(\bar{x}_i) - \int_0^{\bar{x}_i} q_i(s)ds \tag{11}$$

$$q_i(x) = q_i(\bar{x}_i) \text{ for all } x_i \in [\bar{x}_i, 1] \tag{12}$$

Applying standard methodology, let us now substitute the expression (9) for the transfers into the objective (2). Then using (12) and integrating by parts yields the modified objective:

$$\sum_{i=1}^{n}\int_0^1 t_i(x_i)dF(x_i) = \sum_{i=1}^{n}\int_0^1 \left(q_i(x_i)x_i - \int_0^{x_i} q_i(x)dx\right)dF(x_i) = \sum_{i=1}^{n}\int_0^1 q_i(x_i)\left(x_i - \frac{1-F(x_i)}{f(x_i)}\right)dF(x_i)$$

$$= \sum_{i=1}^{n}\int_0^{\bar{x}_i} q_i(x_i)\left(x_i - \frac{1-F(x_i)}{f(x_i)}\right)dF(x_i) + \sum_{i=1}^{n}\int_{\bar{x}_i}^1 q_i(\bar{x}_i)\bar{x}_i dF(x_i) \tag{13}$$

By Lemma 1, the incentive constraints (3) and the individual rationality constraints (4) can now be omitted and replaced with the condition that $q_i(x_i)$ is increasing in x_i for all i. Following standard approach, we will consider a relaxed program omitting the latter condition. In our relaxed program we will also omit condition (10) replacing it with condition (11) that the budget constraint is binding for type $\bar{x}_i$. Since we have already imposed constraint (12) on the objective explicitly, this will be sufficient to guarantee that the budget constraint is also binding for any type $x_i \in (\bar{x}_i, 1]$. When $q_i(.)$ is increasing, (11) guarantees that the budget constraint holds for any type $x_i \in [0, \bar{x}_i)$.

Having solved the relaxed program, we will check that our solution is such that $q_i(.)$ is increasing and is strictly increasing at $\bar{x}_i$ from the left. The latter condition guarantees that (10) holds i.e., $\bar{x}_i$ is the lowest type for whom the budget constraint is binding. This would imply that the solution to the relaxed problem does, in fact, solve the full unrelaxed problem. Finally, we will take care of the feasibility constraints (6) and (7) by imposing them directly on the probabilities of trading.

3 Analysis

Imposing the constraint (11) on the objective (13) yields the following Lagrangian of our relaxed program:

$$\mathcal{L}(Q,\bar{x},\lambda) = \sum_{i=1}^{n}\int_0^{\bar{x}_i} q_i(x_i)\left(x_i - \frac{1-F(x_i)}{f(x_i)}\right)dF(x_i) + \int_{\bar{x}_i}^{1} q_i(\bar{x}_i)\bar{x}_i dF(x_i) - \lambda_i\left(q_i(\bar{x}_i)\bar{x}_i - \int_0^{\bar{x}_i} q_i(x)dx - m_i\right)$$

$$= \sum_{i=1}^{n}\left(\int_0^{\bar{x}_i} q_i(x_i)\left(x_i - \frac{1-\lambda_i-F(x_i)}{f(x_i)}\right)dF(x_i) + \int_{\bar{x}_i}^{1} q_i(\bar{x}_i)\left(\bar{x}_i - \frac{\lambda_i\bar{x}_i}{1-F(\bar{x}_i)}\right)dF(x_i) + \lambda_i m_i\right) \tag{14}$$

where λ_i is a Lagrange multiplier associated with bidder i's budget constraint (11).

Let us define function $\gamma_i(x_i)$ for $i = 1, ..., n$ as follows:

$$\gamma_i(x_i) = \begin{cases} x_i - \frac{1-\lambda_i - F(x_i)}{f(x_i)}, & \text{if} \quad x_i < \overline{x}_i, \\ \overline{x}_i - \frac{\lambda_i \overline{x}_i}{1-F(\overline{x}_i)}, & \text{if} \quad x_i \geq \overline{x}_i. \end{cases} \tag{15}$$

As one can see from (14), $\gamma_i(.)$ plays the role of the virtual value of player i. It depends on Lagrange multiplier λ_i, as well as bidder i's type x_i if $x_i < \bar{x}_i$ and the threshold $\bar{x}_i$ if $x_i \geq \bar{x}_i$.

In the optimal mechanism without budget constraints, bidder i's virtual is $x_i - \frac{1-F(x_i)}{f(x_i)}$, and so the budget constraint causes the latter to change. For a type $x_i \in [0, \overline{x}_i)$, the virtual value becomes larger in proportion to the Lagrange multiplier. Intuitively, this change reflects that higher types above $\bar{x}_i$ face a binding budget constraint and hence the seller's ability to extract their informational rents is limited. Therefore, increasing the probability with which lower types get the good has less of a negative effect on the seller's profits. Further, all types in $[\overline{x}_i, 1]$ experience a binding budget constraint and have the same probability of trading $q(\overline{x}_i)$ and pay the same transfer, m_i. Thus, the set of types $[\overline{x}_i, 1]$ constitute an endogenously chosen atom. Reflecting that all of them have the same virtual value, $\overline{x}_i - \frac{\lambda_i \overline{x}_i}{1-F(\overline{x}_i)}$.

Using (15) and replacing $q_i(x_i)$ with $\int_{x_{-i} \in [0,1]^{n-1}} Q(x_i, x_{-i}) d \prod_{j \neq i} F(x_j)$ in (14), and then changing the order of summation and integration allows us to rewrite (14) as follows:

$$\mathcal{L}(Q, \bar{x}, \lambda) = \int_{(x_1,...,x_n) \in [0,1]^n} \sum_{i=1}^{n} Q_i(x_1, ..., x_n) \gamma_i(x_i) d \prod_{i=1,...,n} F(x_i) + \sum_{i=1}^{n} \lambda_i m_i. \tag{16}$$

Inspection of (16) yields the following Lemma:

Lemma 3 *For any bidder $i \in \{1, 2, ..., n\}$ and any $(x_i, x_{-i}) \in [0, \bar{x}_i]^n$, the optimal $Q_i(.)$ is such that:*

1. $Q_i(x_i, x_{-i}) = 1$ *if* $\gamma_i(x_i) > \max\{0, \max_{j \neq i} \gamma_j(x_j)\}$;

2. $Q_i(x_i, x_{-i}) \in [0, 1]$ *if* $\gamma_i(x_i) = \max\{0, \max_{j \neq i} \gamma_j(x_j)\}$;

3. $Q_i(x_i, x_{-i}) = 0$ *if* $\gamma_i(x_i) < \max\{0, \max_{j \neq i} \gamma_j(x_j)\}$.

4. $\sum_{i=1}^{n} Q_i(x_1, ..., x_n) = 1$ *if* $\min_i \gamma_i(x_i) > 0$.

According to this Lemma, the profile of virtual values $(\gamma_1(x_1), ..., \gamma_n(x_n))$ determines the trading probabilities $(Q_i(x), ..., Q_n(x))$ uniquely except when there are ties i.e., when two or more bidders have the highest virtual value. The ties that have zero probability can be ignored, as the designer can resolve them arbitrarily without affecting her expected profits. For this reason, we can ignore ties that involve a bidder type x_i such that $x_i < \bar{x}_i$.

However, all bidder types x_i, $x_i \geq \bar{x}_i$, have the same virtual value $\gamma_i(\bar{x}_i)$ and essentially constitute an atom of probability $1 - F(\bar{x}_i)$. If $\gamma_i(\bar{x}_i) = \gamma_j(\bar{x}_j)$ for some $j \neq i$, then any bidder type $x_i \in [\bar{x}_i, 1]$ is tied with any bidder type $x_j \in [\bar{x}_j, 1]$. Such a tie has a positive probability. As we will see below, under certain conditions there will, in fact, exist clusters of bidders sharing the same threshold $\bar{x}$ and the same virtual values $\gamma(\bar{x})$, even if they have unequal budgets.[3] Tie-breaking rules between the bidders in a cluster have to be constructed judiciously as part of the optimal design.

Then using Lemma 3 we obtain the following Lemma characterizing the expected probabilities of trading, $q_i(x_i) = \int_{x_{-i}\in[0,1]^{n-1}} Q_i(x_i, x_{-i}) d\prod_{j\neq i} F(x_j)$:

Lemma 4 *If $x_i < \bar{x}_i$ or $x_i = \bar{x}_i$ and $\gamma(\bar{x}_i) \neq \gamma(\bar{x}_j)$ for all j, $j \neq i$, then the expected trading probability $q_i(x_i)$ is given by:*

$$q_i(x_i) = \int_{x_{-i}\in[0,1]^{n-1}:\gamma_i(x_i)>\max\{0,\max_{j\neq i}\gamma_j(x_j)\}} d\prod_{j\neq i} F(x_j) = \mathbb{I}_{\gamma_i(x_i)>0} \prod_{j\neq i} \int_{x_{-i}\in[0,1]^{n-1}:\gamma_i(x_i)>\gamma_j(x_j)} d\prod_{j\neq i} F(x_j) \tag{17}$$

where $\mathbb{I}_{\gamma_i(x_i)>0}$ is an indicator function equal to 1 if $\gamma_i(x_i) > 0$ and zero otherwise.

If $\gamma_i(\bar{x}_i) = \gamma_j(\bar{x}_j)$ for some $j \neq i$, then:

$$\int_{x_{-i}\in[0,1]^{n-1}:\gamma_i(\bar{x}_i)>\max\{0,\max_{j\neq i}\gamma_j(x_j)\}} d\prod_{j\neq i} F(x_j) \leq q_i(\bar{x}_i) \leq \int_{x_{-i}\in[0,1]^{n-1}:\gamma_i(\bar{x}_i)\geq\max\{0,\max_{j\neq i}\gamma_j(x_j)\}} d\prod_{j\neq i} F(x_j) \tag{18}$$

Lemma 3 implies that in the optimal mechanism $\sum_{i=1}^n Q_i(x_1, ..., x_n)\gamma_i(x_i) = \max\{0, \max_i \gamma_i(x_i)\}$ for all $x = (x_1, ..., x_n) \in [0,1]^n$. Using this, we will now proceed to replace the Lagrangian (16) of our relaxed program with the one that only depends on $\bar{x} = (\bar{x}_1, ..., \bar{x}_n)$

[3] A cluster of bidders at threshold $\bar{x}^C$ is defined as $C(\bar{x}^C) \equiv \{i | \bar{x}_i = \bar{x}^C\}$. Plainly, a cluster must have at least two bidders in it.

and $\lambda = (\lambda_1, ..., \lambda_n)$. Particularly, let

$$\mathcal{L}(\bar{x}, \lambda) = \max_{Q:\, 0 \le Q_i(x) \le 1,\, \sum_i Q_i(x) \le 1} \mathcal{L}(Q, \overline{x}, \lambda) = \int_{x \in [0,1]^n} \max\{0, \max_{i=1,\dots,n} \gamma_i(x_i)\} d \prod_i F(x_i) + \sum_{i=1}^{n} \lambda_i m_i. \tag{19}$$

Now we are in a position to prove the following important Theorem. Note that the right-side and left-side limits of $\gamma_i(x_i)$ at $\bar{x}_i$ are given by:

$$\lim_{x_i \downarrow \bar{x}_i} \gamma_i(x_i) = \gamma_i(\bar{x}_i) = \overline{x}_i - \frac{\lambda_i \overline{x}_i}{1 - F(\overline{x}_i)},$$
$$\lim_{x_i \uparrow \bar{x}_i} \gamma_i(x_i) \equiv \gamma_i^-(\bar{x}_i) = \bar{x}_i - \frac{1 - \lambda_i - F(\bar{x}_i)}{f(\bar{x}_i)}. \tag{20}$$

Then we have:

Theorem 2 *1. For all $i \in \{1, ..., n\}$ s.t. $\bar{x}_i \le \bar{x}_j$ for some $j \ne i$, $\lambda_i = \frac{(1-F(\bar{x}_i))^2}{(1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i))}$ or, equivalently, $\gamma_i(x_i)$ is continuous at $x_i = \bar{x}_i$.*

2. For bidder $\hat{i}$ such that $\bar{x}_{\hat{i}} > \bar{x}_j$ for all $j \ne \hat{i}$, we have:

$$\lambda_{\hat{i}} = 1 - F(\overline{x}_{\hat{i}}) - f(\overline{x}_{\hat{i}}) \left(\overline{x}_{\hat{i}} - \max_{j \ne \hat{i}} \gamma_j(\bar{x}_j) \right) \equiv 1 - F(\overline{x}_{\hat{i}}) - f(\overline{x}_{\hat{i}}) \left(\overline{x}_{\hat{i}} - \max_{j \ne i} \left(\frac{\bar{x}_j^2 f(\overline{x}_j)}{1 - F(\overline{x}_j) + \overline{x}_j f(\overline{x}_j)} \right) \right),$$

and so $\gamma_{\hat{i}}(\bar{x}_{\hat{i}}) > \gamma_{\hat{i}}^-(\bar{x}_{\hat{i}}) = \max_{j \ne \hat{i}} \gamma_j(\bar{x}_j)$.

Theorem 2 establishes a one-to-one relationship between the Lagrange multipliers $(\lambda_1, ..., \lambda_n)$ and the profile of the threshold values $(\bar{x}_1, ..., \bar{x}_n)$.

Substituting $\lambda_i = \frac{(1-F(\bar{x}_i))^2}{(1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i))}$ into (15) yields the following for any i s.t. $\bar{x}_i \le \bar{x}_j$ for some j: $\gamma_i(x_i) = x_i - \frac{1 - \frac{(1-F(\bar{x}_i))^2}{(1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i))} - F(x_i)}{f(x_i)}$ for $x_i \le \bar{x}_i$ which is decreasing in $\bar{x}_i$; while $\gamma_i(\bar{x}_i) = \frac{\bar{x}_i^2 f(\bar{x}_i)}{1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i)}$, which is increasing in $\bar{x}_i$.

By part (2) of Theorem 2, for $\hat{i}$ such that $\bar{x}_{\hat{i}} > \max_{j \ne i} \bar{x}_j$ we have: $\gamma_{\hat{i}}(x_{\hat{i}}) = x_{\hat{i}} - \frac{F(\bar{x}_{\hat{i}}) + f(\bar{x}_{\hat{i}})\left(\bar{x}_{\hat{i}} - \max_{j \ne \hat{i}} \gamma_j(\bar{x}_j)\right) - F(x_{\hat{i}})}{f(x_{\hat{i}})}$ for $x_{\hat{i}} \le \bar{x}_{\hat{i}}$, and $\gamma_{\hat{i}}(\bar{x}_{\hat{i}}) > \gamma_{\hat{i}}^-(\bar{x}_{\hat{i}}) = \max_{j \ne \hat{i}} \gamma_j(\bar{x}_j)$. The latter implies that $q_{\hat{i}}(\overline{x}_{\hat{i}}) = 1$.

Now we can verify that the solution to our relaxed program does, in fact, solve the full unrelaxed program:

Lemma 5 *The solution to the Lagrangian problem 19 is such that $\gamma_i(x_i)$ and $q_i(x_i)$ are strictly increasing on $[0, \bar{x}_i]$ for all i. Therefore, the solution to the relaxed program also solves the full unrelaxed program.*

Theorem 2 also allows us to establish the following intuitive relationship between the budgets and thresholds:

Lemma 6 *If $m_i > m_j$ for some $i, j \in \{1, ..., n\}$, then $\bar{x}_i \geq \bar{x}_j$.*

An immediate implication of this Lemma is that bidder $\hat{i}$ with the highest threshold $\bar{x}_{\hat{i}}$ is, in fact, the highest-budget bidder 1.

Now we can also compute the "reservation values" $r_i = \inf\{x_i \in [0,1] | q_i(x_i) > 0\}$. First, by Lemma 4, $r_i = \inf\{x_i \in [0,1] | \gamma_i(x_i) > 0\}$ and hence r_i is implicitly defined by the equation $r_i(\bar{x}_i) - \frac{1-F(r_i(\bar{x}_i))-\lambda_i}{f(r_i(\bar{x}_i))} = 0$. Then by Theorem 2, for $i \in \{2, ..., n\}$, we have:

$$r_i(\bar{x}_i) = \frac{1 - F(r_i(\bar{x}_i)) - \frac{(1-F(\bar{x}_i))^2}{1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i)}}{f(r_i(\bar{x}_i))}. \tag{21}$$

Note that $r_i(\bar{x}_i)$ is well-defined and increasing in $\bar{x}_i$, with $r_i(\bar{x}_i) \in (0, \bar{x}_i)$ by the Increasing Hazard Rate Assumption and because $\frac{(1-F(\bar{x}_i))^2}{1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i)}$ is less than $1 - F(\bar{x}_i)$ and is decreasing in $\bar{x}_i$.

Part (2) of Theorem 2 implies that the reservation value of bidder 1 is a function of $\bar{x}_1$ and $\bar{x}_2 = \max_{j:\, j>1} \bar{x}_j$. So, we denote it by $r_1(\bar{x}_1, \bar{x}_2)$. Particularly, given the value of λ_1 in part 2 of Theorem 2, $r_1(\bar{x}_1, \bar{x}_2)$ can be found as a solution in r_1 to the following equation:

$$r_1 - \frac{F(\bar{x}_1) + f(\bar{x}_1)\left(\bar{x}_1 - \bar{x}_2 + \frac{\bar{x}_2(1-F(\bar{x}_2))}{1-F(\bar{x}_2)+\bar{x}_2 f(\bar{x}_2)}\right) - F(r_1)}{f(r_1)} = 0. \tag{22}$$

Next, to complete the derivation of the optimal mechanism and solve (19), we will use the Lagrangian duality theory (see e.g. Boyd and Vandenberghe (2009) and Bertsekas (2001)). To this end, let $\bar{x}_i(\lambda_i)$ be the inverse of the function $\lambda_i = \frac{(1-F(\bar{x}_i))^2}{(1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i))}$ for $i \geq 2$, which is well-defined since λ_i is monotone decreasing in $\bar{x}_i$. Also, let $\bar{x}_1(\lambda_1, \lambda_2)$ be the solution for $\bar{x}_1$ of the equation in part (2) of Theorem 2. To see that $\bar{x}_1(\lambda_1, \lambda_2)$ is well-defined rewrite this equation as follows:

$$\bar{x}_1 - \frac{1 - F(\bar{x}_1) - \lambda_1}{f(\bar{x}_1)} = \frac{(\bar{x}_2(\lambda_2))^2 f(\bar{x}_2(\lambda_2))}{1 - F(\bar{x}_2(\lambda_2)) + \bar{x}_2(\lambda_2) f(\bar{x}_2(\lambda_2))}$$

The left-hand side of this equation is increasing in $\bar{x}_1$ (by monotone hazard rate property and because $\lambda_1 \leq 1 - F(\bar{x}_1)$) and increasing in λ_1, while its right-hand side is decreasing in λ_2. Therefore, $\bar{x}_1(\lambda_1, \lambda_2)$ is well-defined and is decreasing in both λ_1 and λ_2.

Next, let $g(\lambda)$ be Lagrange dual function (Boyd and Vandenberghe (2009), p. 215) i.e.:

$$g(\lambda) \equiv \mathcal{L}(\lambda, \bar{x}(\lambda)) = \max_{\bar{x}} \mathcal{L}(\lambda, \bar{x}) \tag{23}$$

By Danskin's Theorem (Bertsekas (2001), Ch. 1, p. 131), $g(\lambda)$ is convex and therefore has a unique minimum characterized by the corresponding first-order conditions. Importantly, the next Lemma establishes the strong duality property for our problem implying that its solution is obtained by minimizing $g(\lambda)$.

Lemma 7 *The problem of maximizing (19)* [4] *has strong duality property i.e.,*

$$\max_{\bar{x}} \min_{\lambda} \mathcal{L}(\bar{x}, \lambda) = \min_{\lambda} \max_{\bar{x}} \mathcal{L}(\bar{x}, \lambda)$$

Using Lemma 7 we will now complete the solution to our problem by minimizing the Lagrange dual function $g(\lambda)$. This solution is provided in the next Theorem. To state is, let us introduce the following notation. For any set $J \subseteq \{1, ..., n\}$ s.t. $i \notin J$, let $Prob.[\gamma_i(x_i) > \max_{j \in J} \gamma_j] = \prod_{j \in J} \int_{x_j \in [0,1]:\, \gamma_i(x_i) > \gamma_j(x_j)} dF(x_j)$. Then we have:

Theorem 3 *The optimal profile of threshold values $(\bar{x}_1, ..., \bar{x}_n)$ is unique and is characterized by the following necessary and sufficient conditions.*

If i is such that $\bar{x}_i \neq \bar{x}_j$, $j \neq i$, we have:

$$m_i = \bar{x}_i q_i(\bar{x}_i) - \int_0^{\bar{x}_i} q_i(s) ds \tag{24}$$

i.e., budget constraint (11) holds.

If $\{k_1, ..., k_l\} \subset \{1, ..., n\}$ is such that $\bar{x}_{k_1} = ... = \bar{x}_{k_l} = \bar{x}^c \neq \bar{x}_j$ for any $j \notin \{k_1, ..., k_l\}$,

[4] It is well know that the primal problem of maximizing (19), $\max_{\bar{x}} \mathcal{L}(\bar{x}, \lambda)$, is equivalent to the following one: $\max_{\bar{x}} \min_{\lambda} \mathcal{L}(\bar{x}, \lambda)$.

then:[5,6]

$$\sum_{h\in\{1,...,l\}} m_{k_h} = \bar{x}^c \frac{1-F(\bar{x}^c)^l}{1-F(\bar{x}^c)} Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j\notin\{k_1,...,k_l\}} \gamma_j] - l\int_0^{\bar{x}^c} q_{k_1}(s)ds \tag{25}$$

Also, for all $r \in \{2, ..., l-1\}$ *we have:*

$$\frac{m_{k_1}+...+m_{k_r}}{r} - \frac{m_{k_{r+1}}+...+m_{k_l}}{l-r} \leq$$
$$\bar{x}^c \left(\frac{1-F(\bar{x}^c)^r}{r(1-F(\bar{x}^c))} - F(\bar{x}^c)^r \frac{1-F(\bar{x}^c)^{l-r}}{(l-r)(1-F(\bar{x}^c))} \right) Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j\notin\{k_1,...,k_l\}} \gamma_j] \tag{26}$$

It is important to understand the significance of Theorem 3. The conditions (24)-(26) are the necessary and sufficient conditions for the minimization of the dual Lagrange function $g(\lambda) \equiv \mathcal{L}(\lambda, \bar{x}^*(\lambda))$. Since $g(\lambda)$ is convex, the solution to the system (24)-(26) is unique and completes the derivation of the solution to our optimal mechanism design problem.

Although condition (26) may appear non-transparent, it has a clear an intuitive interpretation. Its' left-hand side is the difference between the average budget of the richest r bidders and the average budget of the poorest $l-r$ bidders in the cluster. For the cluster to exist, this difference cannot be too large, for otherwise all budget constraints cannot hold at the cluster threshold $\bar{x}^c$. Precisely, it cannot exceed the largest possible difference between the average transfers paid by these two groups. The latter difference is equal to the maximal difference between the average expected surpluses of these two groups which is the right-hand side of (26). Indeed, the maximal average surplus of the richest r bidders is equal to the threshold value $\bar{x}^c$ times the maximal average probability of trading in this group. The latter is a product of the probability $Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j\notin\{k_1,...,k_l\}} \gamma_j]$ that no bidder outside the cluster has a virtual value exceeding the virtual value of a cluster member of type $\bar{x}^c$, $\gamma_{k_1}(\bar{x}^c)$, and the average probability that at least one among r bidders has a type

[5]Note that without loss of generality we may assume here that indexes $k_1, ..., k_l$ are ordered according to the budgets i.e., $k_1 < k_2 ... < k_{l-1} < k_l$ and $m_{k_1} \geq m_{k_2} ... \geq m_{k_{l-1}} \geq m_{k_l}$. This is so because when (25) holds for this ordering, it also holds for any alternative ordering.

[6] Note that the integrand in the last term of (25) is $q_{k_1}(.)$. Yet, every bidder in the cluster $\{k_1, ..., k_l\}$, to which (25) pertains, has the same threshold $\bar{x}^c$ and hence, by Theorem 2, the same λ and γ, we have $q_{k_1}(x) = ...q_{k_l}(x)$ for all x in the range $[0, \bar{x}^c]$ of this integral.

of at least $\bar{x}^c$, $\frac{1-F(\bar{x}^c)^r}{r(1-F(\bar{x}^c))}$. Similarly, the minimal average surplus of the poorest $l-r$ bidders is equal to the threshold value $\bar{x}^c$ times the minimal average probability of trading in that group, and the latter probability is a product of $Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j]$ and the average probability that at least one among $l-r$ bidders has a type of at least $\bar{x}^c$ and the other r bidders have types below $\bar{x}^c$, $F(\bar{x}^c)^r \frac{1-F(\bar{x}^c)^{l-r}}{(l-r)(1-F(\bar{x}^c))}$.

Theorem 3 does not explicitly identify the probability of trading $q_{k_j}(\bar{x}^c)$ for a buyer k_j in cluster $C(\bar{x}^c)$ with the threshold $\bar{x}^c$. Yet, it is straightforward to compute it. First, given the profile of threshold values satisfying the conditions of Theorem 3, Theorem 2 and Lemma 4 allow us to explicitly compute the trading probability functions $q_i(x_i)$ for all $i = 1, ..., n$ and $x_i < \bar{x}_i$. Then, the probability of trading $q_{k_j}(\bar{x}^c)$ for any $j = 1, ..., l$ in cluster $C(\bar{x}^c) = \{k_1, ..., k_l\}$ is uniquely defined by the budget constraint $m_{k_j} = \bar{x}^c q_{k_j}(\bar{x}^c) - \int_0^{\bar{x}^c} q_{k_j}(s)ds$.

We need to confirm that the vector $(q_{k_1}(\bar{x}^c), ..., q_{k_l}(\bar{x}^c))$ defined in this way is feasible. There are two feasibility conditions that need to be satisfied. The first one is the upper bound condition from Theorem 3 in Border (2007):

$$\sum_{j=1,...,h} q_{k_j}(\bar{x}^c) \leq \frac{1-F(\bar{x}^c)^h}{1-F(\bar{x}^c)} Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{i \notin \{k_1,...,k_l\}} \gamma_i] \quad \text{for all } \ h \in \{1, ..., l\} \tag{27}$$

Intuitively, this condition requires that the probability of assigning the good to any subset of bidders from the cluster does not exceed the probability that a bidder from this subset has value in $[\bar{x}^c, 1]$ and the bidders outside the cluster have lower virtual values than $\gamma_{k_j}(\bar{x}^c)$ for $k \in \{1, ..., l\}$. Although this condition has to hold for every subset of bidders in a cluster of size $h \in \{1, ..., l\}$, it is sufficient that it holds for the subset consisting of the bidders with h highest budgets $k_1, ..., k_h$ since these bidders have higher trading probabilities in the cluster i.e., $q_{k_1}(\bar{x}^c) \geq ... \geq q_{k_l}(\bar{x}^c)$).

The second feasibility condition establishes the lower bound on the probability of assigning the good to any subset of bidders in a cluster. It requires that the good should be assigned for sure to a bidder from this subset if some bidder in this subset has value in $[\bar{x}^c, 1]$, the rest of the bidders in the cluster have values below $\bar{x}^c$, and the bidders outside the cluster have lower virtual values than $\gamma_{k_j}(\bar{x}^c)$. Clearly, it is sufficient that this condition hold for the subsets of every size composed of the bidders with the lowest budgets because they have lower probabilities of trading $q^C_{k_j}((\bar{x}^c))$. Thus, we require:

$$\sum_{j=h,...,l} q_{k_j}(\bar{x}^c) \geq F(\bar{x}^c)^{h-1}\frac{1-F(\bar{x}^c)^{l-h+1}}{1-F(\bar{x}^c)}Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j\notin\{k_1,...,k_l\}}\gamma_j] \quad \text{for all } \; h\in\{1,...,l\} \tag{28}$$

Significantly, the feasibility conditions (27) and (28) are equivalent to the first-order conditions (63) and (64) derived in the proof of Theorem 3 which, in turn, are equivalent to the conditions (24)-(26) in the statement of Theorem 3, as shown in the proof of the Theorem. Hence, the optimality conditions of Theorem 3 are equivalent to the feasibility conditions.

The equivalence of the optimality and the feasibility conditions for the solution to the dual problem $\min_\lambda g(\lambda)$ which we have just confirmed, combined with the uniqueness of the solution imply that there is a unique mechanism satisfying conditions (25) and (26) i.e., a unique feasible mechanism which therefore is optimal. Combining this conclusion with the results of Theorem 2 and Lemma 4, we can state the following:

Corollary 1 *There is a unique profile of threshold values $(\bar{x}_1, ..., \bar{x}_n)$ that satisfies the conditions of Theorem 2, conditions (17) and (18) in Lemma 4, and (24)-(26) in Theorem 3. This profile is a unique solution to the optimal mechanism design problem.*

Corollary 1 summarizes the set of necessary and sufficient conditions which the optimal profile of threshold values $(\bar{x}_1, ..., \bar{x}_n)$ has to satisfy. Our results so far do not provide a method to compute $(\bar{x}_1, ..., \bar{x}_n)$ from these conditions. This task could potentially be computationally intensive. In particular, this concerns computing the clusters of bidders with the same thresholds. Lemma 6 implies that any cluster with more than one bidder has to contain "adjacent" bidders. That is, for a cluster $C(\bar{x}^c) = \{k_1, ..., k_l\}$ we must have $k_{h+1} = k_h + 1$ for all $h \in \{2, ...l\}$. Then the number of potential clusters equals $2^n - 1$ and, in the worst case, one may have to go over all possibilities i.e., all non-trivial subsets of $\{1, ..., n\}$ to find the unique feasible/optimal set of clusters. In the next section, we make progress towards simplifying this problem, characterizing the optimal mechanisms qualitatively and identifying the structure of the clusters.

4 Top and Budget-Handicap Auctions

In this section, we focus on the qualitative properties of the optimal mechanism and demonstrate how these properties depend on the profile of budgets $m_1, ..., m_n$ and, in particular, on the variability of the budgets among the bidders.

Qualitatively, we will distinguish between two kinds of optimal mechanisms. A mechanism of the first kind is called a "top auction." In a top auction all thresholds are equal i.e., $\bar{x}_1 = ... = \bar{x}_n = \bar{x}^t$, so all bidders belong to the same cluster. Therefore, all bidders with valuations below $\bar{x}^t$ are treated symmetrically: every bidder with valuation $x \in [0, \bar{x}^t)$ pays the same transfer and has the same expected probability of trading equal to $F^{n-1}(x)$: she gets the good when she has the highest valuation. But, because the bidders have unequal budgets, the seller discriminates between them "at the top" by treating asymmetrically different bidders with valuations above $\bar{x}^t$: richer high valuation bidders get the good with a higher probability and pay a higher transfer than poorer high valuation bidders. So, we have: $q_1(\bar{x}^t) \geq ... \geq q_n(\bar{x}^t)$, with $q_i(\bar{x}^t) > q_j(\bar{x}^t)$ if and only if $m_i > m_j$.

The mechanisms of the second kind are called "budget-handicap auctions." In a "budget-handicap auction" not all thresholds are equal: the mechanism designer sets different thresholds for different bidders, or for different groups of bidders. There may still exist clusters of bidders with the same threshold, but not all bidders belong to the same cluster. In this mechanism, there are two types of price discrimination. First, a higher-budget bidder with a value above her threshold has a higher probability of trading than a poorer bidder with a value above her respective threshold. So, this type of price discrimination applies to any two bidders with different budgets, irrespectively of whether they belong to the same cluster or different clusters.

The second type of price discrimination works in the opposite direction. A poorer bidder with a low value (below her threshold) has a higher probability of trading and pays a higher transfer than a richer bidder with the same value. This motivates the use of the term "budget-handicap." Higher-budget bidder are handicapped, and lower-budget bidders are given an advantage in the form of a lower reserve price and a higher probability of trading at lower valuations.

As we show below, which mechanism is offered by the designer - a top auction or a budget-handicap auction- is ultimately dictated by the feasibility conditions. The designer offers a top auction whenever it is feasible, namely, when the budget differences across buyers are not too large. However, when these differences are large, price-discrimination only at the "top" is no longer feasible: all budget constraints cannot be made binding at the same threshold by constructing a "lottery" among the bidders with valuations that exceed that threshold. Therefore, different thresholds have to be set across bidders, and the seller has to discriminate between the bidders with low valuations in favor of those with low budgets.

We will start our characterization of the optimal mechanism with the "top auction." First, let us define $\bar{x}^t$ as the unique solution to the following equation:[7]

$$\sum_{i=1,\dots,n} m_i = \bar{x}^t \frac{1-F(\bar{x}^t)^n}{1-F(\bar{x}^t)} - n \int_{r_t : r_t = \frac{1-F(r_t) - \frac{(1-F(\bar{x}^t))^2}{1-F(\bar{x}^t)+\bar{x}^t f(\bar{x}^t)}}{f(r_t)}.}^{\bar{x}^t} F(s)^{n-1} ds \tag{29}$$

We will assume that the budgets are sufficiently small so that $\bar{x}^t < 1$.

Definition 1 *A "top auction" for n bidders with budgets $m_1, ...m_n$, with $m_i \geq m_{i+1}$ for all $i = 1, .., n-1$, is a mechanism with a common threshold $\bar{x}^t = \bar{x}_1 = ...\bar{x}_n$ uniquely solving (29), reservation values $r_1 = ... = r_n = r_t$ defined by $r_t = \frac{1-F(r_t) - \frac{(1-F(\bar{x}^t))^2}{1-F(\bar{x}^t)+\bar{x}^t f(\bar{x}^t)}}{f(r_t)}$, and trading probabilities $q_i(x_i) = F(x_i)^{n-1}$ for all $x_i \in [r, \bar{x}^t)$ and $q_i(\bar{x}^t)$ satisfying:*

$$m_i = \bar{x}^t q_i(\bar{x}^t) - \int_{r_t}^{\bar{x}^t} F(s)^{n-1} ds \tag{30}$$

$$\sum_{i=1,\dots,n} q_i(\bar{x}^t) = \frac{1-F(\bar{x}^t)^n}{1-F(\bar{x}^t)} \tag{31}$$

This Definition is consistent with the definition of $\bar{x}^t$ in (29): summing up (30) over i and substituting (31) into the result yields (29). Note that condition (31) says that with

[7] The solution to (29) is unique because its right-hand side: (i) is increasing in x^t. Indeed, its derivative is equal to $\frac{1-F(\bar{x}^t)^n}{1-F(\bar{x}^t)} + \frac{xf(\bar{x}^t)}{(1-F(\bar{x}^t))^2}(1+(n-1)F(\bar{x}^t)^n - nF(\bar{x}^t)^{n-1}) - nF(\bar{x}^t)^{n-1} + nF(r(\bar{x}^t))^{n-1}\frac{dr(\bar{x}^t)}{d\bar{x}^t}$. It is easy to ascertain that this expression is positive, in particular, because $\frac{dr(\bar{x}^t)}{d\bar{x}^t} > 0$; (ii) is equal to zero when $x^t = 0$; (iii) exceeds $\sum_i m_i$ when $x^t = 1$. The latter holds because by assumption $m_1 \leq 1 - \int_{r: r - \frac{1-F(r)}{f(r)} = 0}^{1} F^{n-1}(x)dx$.

probability 1 the good is allocated to some bidder whose value is at least $\bar{x}^t$, whenever there is at least one such bidder.

Our next result shows that the "top auction" is optimal whenever it is feasible i.e., whenever there exists a profile of trading probabilities "at the top," $(q_1(\bar{x}^t), ..., q_n(\bar{x}^t))$ that satisfies conditions (30) and (31) as well as the familiar reduced-form (interim) implementability conditions (27) and (28) adapted to the current case. This result presented in the next Theorem is a direct consequence of Theorem 3.

Theorem 4 *Suppose that for a profile of bidders with budgets $m_1, ..., m_n$, with $m_i \geq m_{i+1}$ for all $i = 1, ..., n-1$, the threshold $\bar{x}^t$ uniquely solving (29) is such that $\bar{x}^t < 1$.*

The unique optimal mechanism is a "top auction" with a common threshold $\bar{x}^t$ if and only if for every $k = 1, 2, ..., n-1$ we have:

$$\frac{m_1 + ... + m_k}{k} - \frac{m_{k+1} + ... + m_n}{n-k} \leq \bar{x}^t \left(\frac{1 - F(\bar{x}^t)^k}{k(1 - F(\bar{x}^t))} - F(\bar{x}^t)^k \frac{1 - F(\bar{x}^t)^{n-k}}{(n-k)(1 - F(\bar{x}^t))} \right) \tag{32}$$

Condition (32) is equivalent to condition (26) in Theorem 3 for the case of the top auction. As the discussion following that Theorem points out, it says that the difference between the average budget of the richest k bidders and the average budget of the poorest $n-k$ ones does not exceed the maximal difference between the average expected surpluses of these two groups.

The distinguishing feature of the top auction is that it allocates the good efficiently when the buyers' valuations lie in $[r, \bar{x}]$. The only additional inefficiency compared to the standard optimal auction happens at the "top:" when several buyers have valuations above $\bar{x}$, the good is allocated randomly among them, with probabilities increasing in their budgets. So a bidder with a lower value in $[\bar{x}, 1]$ may end up getting the good despite there being another bidder with a higher value in $[\bar{x}, 1]$.

However, when the feasibility condition for the top auction (32) fails, the seller has to use additional tools to discriminate between the bidders and, in particular, set different thresholds for them. Naturally, lower-budget bidders have lower thresholds (see Lemma 6), although there may still exist some clusters of bidders sharing the same threshold. The richer bidders with valuations above their higher thresholds have higher probabilities of trading and pay higher transfers than poorer bidders with valuations above their lower thresholds.

Significantly, there is another type of price discrimination in this second kind of mechanism, the "budget handicap auction:" a poorer bidder with a low value has a higher probability of trading and pays a higher transfer than a richer bidder with the same value. To ascertain that, consider two bidders i and j such that $m_i > m_j$ and $\bar{x}_i > \bar{x}_j$. Recall that by Theorem 2 $\lambda_i < \lambda_j$, and $\gamma_i(x) < \gamma_j(x)$ for $x \in [0, \bar{x}_j]$. Therefore by Lemma 4 $q_i(x) < q_j(x)$ on this interval. Particularly, the reservation values are such that $r_i > r_j$. This handicapping of higher-budget bidders creates a stronger competition for them from lower-budget bidders, and extracts higher transfers from the former when they have high values. This additional type of price discrimination increases inefficiency, but is unavoidable when budget differences are sufficiently large. Formally, we have:

Theorem 5 *Suppose that (32) fails for some k. Then the optimal auction is a "budget handicap auction" which is uniquely defined by a vector of threshold values $(\bar{x}_1, ..., \bar{x}_n)$ s.t. $\bar{x}_i \geq \bar{x}_{i+1}$ for all $i \in \{1, .., n-1\}$, with strict inequality for at least some i.*

By Corollary 1, the vector of thresholds $(\bar{x}_1, ..., \bar{x}_n)$ is uniquely defined by conditions (24)-(26) in Theorem 3, the conditions of Theorem 2, and conditions (17) and (18) in Lemma 4. The probabilities of trading $q_i(x_i)$ are defined by (17) in Lemma 4 for all i and $x_i < \bar{x}_i$, and $q_i(\bar{x}_i)$ is determined by the binding budget constraint of that bidder.

The most challenging part in computing the optimal "budget handicap" auction is to determine which groups of bidders constitute clusters with common thresholds. In the next section we consider several examples and, in particular, provide conditions for the existence of various clusters.

As a final result of this section, we will establish that the seller's expected payoff function is concave in the bidders' budgets and explore the implications of this. Recall that By Lemma 7 (strong duality), the seller's expected profit in the optimal mechanism is given by the minimum of the dual Lagrange function $g(\lambda) = \mathcal{L}(\lambda, \bar{x}(\lambda)) = \int_{x \in [0,1]^n} \max\{0, \max_{i=1,...,n} \gamma_i(x_i)\} dF(x) + \sum_{i=1}^n \lambda_i m_i$. Therefore, the seller's expected payoff written as a function of the vector of budgets $m = (m_1, ..., m_n)$ is given by:

$$\pi(m_1, ..., m_n) = \min_{\lambda} \left\{ \int_{x \in [0,1]^n} \max\{0, \max_{i=1,...,n} \gamma_i(x_i, \lambda)\} dF(x) + \sum_{i=1}^{n} \lambda_i m_i \right\}. \quad (33)$$

Since $\pi(m_1, ..., m_n)$ is a pointwise minimum in λ of a function affine in $(m_1, ..m_n)$, $\pi(m_1, ..., m_n)$ is concave in the vector $(m_1, ..., m_n)$.[8]

Now consider a situation in which the aggregate budget $\sum_i m_i$ is fixed. Then the seller is better off when each bidder has an equal share of the aggregate budget.

Lemma 8 *Suppose that the aggregate budget of all bidders is fixed i.e.* $\sum_i m_i = M$[9]

Then the seller gets a maximal payoff in the optimal mechanism when all bidders' budgets are equal i.e., $m_i = \frac{M}{n}$ *for all* $i = 1, ..., n$*.*

More generally, given the aggregate budget M, the seller's revenue decreases as the variability of budgets increases across the bidders. We leave this assertion without proof, as it is a straightforward extension of the proof of Lemma 8.

5 Examples

In this section we compute the optimal mechanism in several special cases. First we look at the case with two bidders. Then we consider mechanisms under uniform distribution of the bidders' types.

5.1 Two-bidder Mechanism.

Consider a mechanism with two bidders, 1 and 2, with $m_1 \geq m_2$. Suppose that $m_1 \leq \arg\max_p p(1-F(p))$. This condition says that bidder 1's budget is smaller than the price that the seller would set when facing a single bidder, which guarantees that the budget constraint of bidder 1 would be binding even if m_2 is very small. This condition is sufficient but not

---

[8]Note that this is true even if some bidder i's budget constraint is not binding. In this case we have $\lambda_i = 0$ and $\pi(m_1, ..., m_n)$ does not depend on m_i.

[9]To make this result non-trivial M has to be sufficiently small. In particular, we will assume that $M \leq np^m$ where p^m is a monopoly price i.e., $p^m = \arg\max_p F(p)(1-p)$.

necessary for both budget constraints to be binding in the optimal mechanism. In particular, as we will see below, the following condition is sufficient for both budget constraints to bind when the conditions for the top auction hold: $m_1 < 1 - \int_{r':r'=\frac{1-F(r')}{f(r')}}^{1} F(x)dx$.

Next, let $\bar{x}^t$ be the unique solution to:

$$m_1 + m_2 = \bar{x}^t \left(1 + F(\bar{x}^t)\right) - 2 \int_{r:r=\frac{1-F(r)-\frac{(1-F(\bar{x}^t))^2}{1-F(\bar{x}^t)+\bar{x}^t f(\bar{x}^t)}}{f(r)}}^{\bar{x}^t} F(x)dx \tag{34}$$

If $m_1 - m_2 \leq \bar{x}^t(1 - F(\bar{x}^t))$, then conditions (25) and (26) of Theorem 3 hold for $\bar{x}_1 = \bar{x}_2 = \bar{x}^t$, and the optimal mechanism is a "top auction" with threshold $\bar{x}^t$, and the following trading probabilities for $i \in \{1, 2\}$ and $j \neq i$:

$$q_i(x) = \begin{cases} \frac{1+F(\bar{x}^t)}{2} + \frac{m_i - m_j}{2\bar{x}^t}, & \text{if} \quad x \geq \bar{x}^t, \\ F(x) & \text{if} \quad x \in [r, \bar{x}^t), \text{ where } r = \frac{1-F(r)-\frac{(1-F(\bar{x}^t))^2}{1-F(\bar{x}^t)+\bar{x}^t f(\bar{x}^t)}}{f(r)}. \\ 0 & \text{if} \quad x < r. \end{cases}$$

Note that to compute $q_1(\bar{x}^t)$ and $q_2(\bar{x}^t)$ we use the binding budget constraints i.e., $m_i = \bar{x}^t q_i(\bar{x}^t) - \int_r^{\bar{x}^t} F(s)ds$ for $i \in \{1, 2\}$ and equation (34).

For illustration purposes we also provide the ex-post probabilities of trading $Q_i(x_i, x_j)$:

$$Q_i(x_i, x_j) = \begin{cases} \frac{1}{2} + \frac{m_i - m_j}{2\bar{x}(1-F(\bar{x}))} & \text{if} \ (x_i, x_j) \in [\bar{x}^t, 1] \times [\bar{x}^t, 1], \\ 1 & \text{if} \ (x_i, x_j) \in [\bar{x}^t, 1] \times [0, \bar{x}^t), \\ 1 & \text{if} \ x_i \in [r, x^t), \ x_j < x_i, \\ 0 & \text{otherwise} \end{cases}$$

If $m_1 - m_2 > \bar{x}^t(1 - F(\bar{x}^t))$ then condition (26) fails, and the top auction is infeasible. So, by Theorem 3 the optimal mechanism is a "handicap auction" with thresholds that satisfy $\bar{x}_1 > \bar{x}_2$. To solve for this mechanism, first recall that $\gamma_2(x_2) = x_2 - \frac{1-F(x_2)-\frac{(1-F(\bar{x}_2))^2}{(1-F(\bar{x}_2)+\bar{x}_2 f(\bar{x}_2))}}{f(x_2)}$ for $x_2 \in [0, \bar{x}_2)$, $\gamma_1(x_1) = x_1 - \frac{F(\bar{x}_1)+f(\bar{x}_1)\left(\bar{x}_1 - \bar{x}_2 + \frac{\bar{x}_2(1-F(\bar{x}_2))}{1-F(\bar{x}_2)+\bar{x}_2 f(\bar{x}_2)}\right)-F(x_1)}{f(x_1)}$ for $x_1 \in [0, \bar{x}_1)$. Also, $\gamma_2(\bar{x}_2) = \bar{x}_2 - \frac{\bar{x}_2(1-F(\bar{x}_2))}{1-F(\bar{x}_2)+\bar{x}_2 f(\bar{x}_2)} = \gamma_2^-(\bar{x}_2) = \gamma_1^-(\bar{x}_1) < \gamma_1(\bar{x}_1)$. The last inequality implies, in particular, that $q_1(\bar{x}_1) = 1$ and the equality before it implies that $q_2(\bar{x}_2) = F(\bar{x}_1)$.

Then the trading probabilities in the optimal handicap auction are as follows:

$$q_1(x_1) = \begin{cases} 1, & \text{if} \quad x_1 \geq \bar{x}_1, \\ \int_{s:\gamma_1(x_1)>\gamma_2(s)} dF(s) & \text{if} \quad x_1 \in [r_1, \bar{x}_1), \text{ where } r_1 = \frac{F(\bar{x}_1)+f(\bar{x}_1)\left(\bar{x}_1 - \bar{x}_2 + \frac{\bar{x}_2(1-F(\bar{x}_2))}{1-F(\bar{x}_2)+\bar{x}_2 f(\bar{x}_2)}\right)-F(r_1)}{f(r_1)}. \\ 0 & \text{if} \quad x_1 < r_1. \end{cases}$$

$$q_2(x_2) = \begin{cases} F(\bar{x}_1), & \text{if} \quad x_2 \geq \bar{x}_2, \\ \int_{s:\gamma_2(x_2)>\gamma_1(s)} dF(s) & \text{if} \quad x_2 \in [r_2, \bar{x}_2), \text{ where } r_2 = \frac{1-F(r_2)-\frac{(1-F(\bar{x}_2))^2}{(1-F(\bar{x}_2)+\bar{x}_2 f(\bar{x}_2))}}{f(r_2)}. \\ 0 & \text{if} \quad x_2 < r_2. \end{cases}$$

Finally, $\bar{x}_1$ and $\bar{x}_2$ can be solved for from the budget constraints:

$$m_1 = \bar{x}_1 - \int_0^{\bar{x}_1} q_1(s)ds \tag{35}$$

$$m_2 = \bar{x}_2 F(\bar{x}_1) - \int_0^{\bar{x}_2} q_2(s)ds \tag{36}$$

The solution for $\bar{x}_1$ and $\bar{x}_2$ exists, is unique, and satisfies $1 \geq \bar{x}_1 > \bar{x}_2 > 0$. This follows from the uniqueness of the optimal mechanism and the optimality conditions established above.[10]

This example illustrates the general properties of the 'handicap' auctions. Importantly, bidder 2 with a lower budget has a lower threshold than the richer buyer 1 i.e., $\bar{x}_2 < \bar{x}_1$. This has a number of implications. First, bidder 1's probability of trading jumps discontinuously from $F(\bar{x}_2)$ to 1 as his value reaches the threshold $\bar{x}_1$. In contrast, the probability of trading of the poorer bidder 2 raises continuously to $F(\bar{x}_1)$ as her value reaches the threshold $\bar{x}_2$.

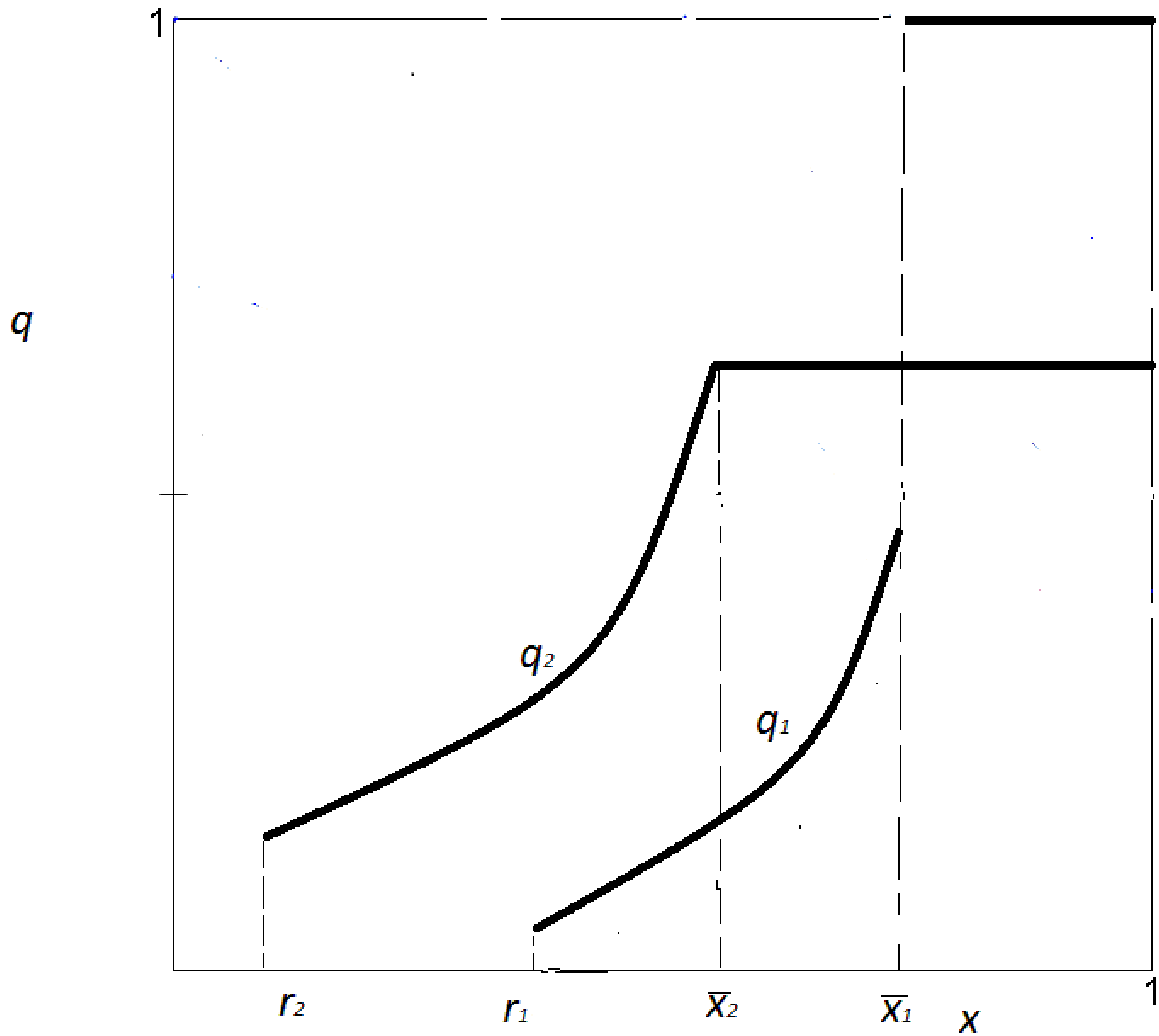
However, the reservation value of the second bidder r_2 is lower than the reservation value r_1 of the first bidder, and for all $x \in [r_2, \bar{x}_2)$ bidder 2 has a higher probability of trading i.e., $q_2(x) > q_1(x)$. This property motivates the term "budget-handicap" auction. A high-budget bidder is handicapped and a lower-budget bidder is given an advantage. The latter is favored over the higher-budget bidder and given the good with a higher probability in a range of low valuations. The seller does so in order to intensify the competition for the higher-budget bidder 1 and extract a higher payment from her. The expected probabilities of trading in this mechanisms are depicted in Figure 1.

5.2 Optimal Mechanism Under the Uniform Type Distribution.

In this section, we assume that types are distributed uniformly over $[0, 1]$. That allows us to compute the optimal mechanism in a closed form in some cases. In particular, the virtual valuations are given by $\gamma_i(x_i) = 2x_i - 2\bar{x}_i + \bar{x}_i^2$ for $i \geq 2$ and $x_i \leq \bar{x}_i$. Hence, the bidders'

[10] For illustration purposes, we also provide a fixed point argument of the existence of the equilibrium for this case in the online Appendix available at www.severinov.com/fixedpointbudget2.

Figure 1: Expected Probabilities of Trading in The Handicap Auction with Two Players ($m_1 > m_2$).



reservation values are $r_i = \bar{x}_i - \frac{\bar{x}_i^2}{2}$ for $i \in \{2, ..., n\}$. For bidder 1, we have: $\gamma_1 = 2x_1 - 2\bar{x}_1 + \bar{x}_2^2$ for $x_1 \leq \bar{x}_1$ and hence $r_1 = \bar{x}_1 - \frac{\bar{x}_2^2}{2}$.

5.2.1 Two Bidders.

When $n = 2$ (two bidders), equation (29) defining the threshold value in the top auction becomes:

$$m_1 + m_2 = \bar{x}^t + \left(\bar{x}^t\right)^2 - \left(\bar{x}^t\right)^3 + \frac{\left(\bar{x}^t\right)^4}{4}$$

Also, the condition (32) for the optimality/feasibility of the top auction simplifies to $m_1 - m_2 \leq \bar{x}^t(1 - \bar{x}^t)$. If this condition holds, then $\bar{x}_1 = \bar{x}_2 = \bar{x}^t$. The expected probabilities of trading are easily computed as follows given that $\gamma_1(x) = \gamma_2(x)$ for $x < \bar{x}^t$ and $r_i = \bar{x}^t - \frac{\left(\bar{x}^t\right)^2}{2}$:

$$q_i(x_i) = \begin{cases} 0, & \text{if} \quad x_i < \bar{x}^t - \frac{\left(\bar{x}^t\right)^2}{2}, \\ x_i & \text{if} \quad x_i \in [\bar{x}^t - \frac{\left(\bar{x}^t\right)^2}{2}, \bar{x}^t), \\ \frac{1+\bar{x}^t}{2} + \frac{m_i - m_j}{2} & \text{if} \quad x_i \geq \bar{x}^t. \end{cases}$$

Note that both $q_1(x)$ and $q_2(x)$ jump discontinuously at $x = \bar{x}^t$, except in the borderline situation where $m_1 - m_2 = \bar{x}^t(1 - \bar{x}^t)$. In the latter case $q_1(x)$ jumps from $F(\bar{x}^t) = \bar{x}^t$ to 1 at $\bar{x}^t$ and $q_2(x)$ remains continuous with $q_2(\bar{x}^t) = F(\bar{x}^t) = \bar{x}^t$.

If $m_1 - m_2 > \bar{x}^t(1 - \bar{x}^t)$, then by Theorem 5 the top auction is infeasible and the optimal mechanism is a"handicap auction" with different thresholds $\bar{x}_1$ and $\bar{x}_2$. To compute it, note the following: $\gamma_2(x_2) = x_2 - \bar{x}_2 + \frac{\bar{x}_2^2}{2}$ for $x_2 \in [0, \bar{x}_2)$, $\gamma_1(x_1) = x_1 - \bar{x}_1 + \frac{\bar{x}_2^2}{2}$ for $x_1 \in [0, \bar{x}_1)$, $\gamma_2(\bar{x}_2) = \gamma_2^-(\bar{x}_2) = \gamma_1^-(\bar{x}_1) = \frac{\bar{x}_2^2}{2} < \gamma_1(\bar{x}_1)$, $r_1 = \bar{x}_1 - \frac{\bar{x}_2^2}{2}$, $r_2 = \bar{x}_2 - \frac{\bar{x}_2^2}{2}$.

Then $\bar{x}_1$ and $\bar{x}_2$ solve the following equations:

$$m_1 = \bar{x}_1 - \int_{r_1}^{\bar{x}_1} \int_{\gamma(x_1) > \gamma(x_2)} dx_2 dx_1 = \bar{x}_1 - \int_{\bar{x}_1 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_1} \int_{x_1 - \bar{x}_1 > x_2 - \bar{x}_2} dx_2 dx_1 =$$

$$\bar{x}_1 - \int_{\bar{x}_1 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_1} \int_0^{x_1 - \bar{x}_1 + \bar{x}_2} dx_2 dx_1 = \bar{x}_1 - \int_{\bar{x}_1 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_1} x_1 - \bar{x}_1 + \bar{x}_2 dx_1 = \bar{x}_1 - \frac{\bar{x}_2^3}{2} + \frac{\bar{x}_2^4}{8} \tag{37}$$

$$m_2 = \bar{x}_2 F(\bar{x}_1) - \int_{r_2}^{\bar{x}_2} \int_{\gamma(x_2) > \gamma(x_1)} dx_1 dx_2 = \bar{x}_2 \bar{x}_1 - \int_{\bar{x}_2 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_2} \int_{x_2 - \bar{x}_2 > x_1 - \bar{x}_1} dx_1 dx_2 =$$

$$\bar{x}_2 \bar{x}_1 - \int_{\bar{x}_2 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_2} x_2 - \bar{x}_2 + \bar{x}_1 dx_2 = \bar{x}_1 \bar{x}_2 - \bar{x}_1 \frac{\bar{x}_2^2}{2} + \frac{\bar{x}_2^4}{8} \tag{38}$$

Note that by Lemma 6, $\bar{x}_1 > \bar{x}_2$ since $m_1 > m_2$.

Under the condition $m_1 \leq \frac{1}{2}$, which we have imposed, the Lagrange multipliers are positive and so both budget constraints are binding. Indeed, recall that by Theorem 2, $\lambda_2 = (1-\bar{x}_2)^2$ and $\lambda_1 = 1 - 2\bar{x}_1 + \bar{x}_2^2$. Thus, $\lambda_1 \geq 0$ if and only if the solutions to (37) and (38) are such that $\bar{x}_1 \leq \frac{1+\bar{x}_2^2}{2}$. Substituting $\bar{x}_1 = \frac{1+\bar{x}_2^2}{2}$ into (37) yields the inequality $m_1 \leq \frac{1+\bar{x}_2^2}{2} - \frac{\bar{x}_2^3}{2} + \frac{\bar{x}_2^4}{8}$, which holds when $m_1 \leq \frac{1}{2}$. So, the solution has to satisfy $\bar{x}_1 \leq \frac{1+\bar{x}_2^2}{2}$.

Next, consider the expected trading probabilities $q_1(x_1)$ and $q_2(x_2)$. As shown above, $q_1(\bar{x}_1) = 1$ and $q_2(\bar{x}_2) = F(\bar{x}_1) = x_1$. The reservation values are given by $r_1 = \bar{x}_1 - \frac{\bar{x}_2^2}{2}$ and $r_2 = \bar{x}_2 - \frac{\bar{x}_2^2}{2}$, and so $q_1(x_1) = 0$ if $x_1 < \bar{x}_1 - \frac{\bar{x}_2^2}{2}$ and $q_2(x_2) = 0$ if $x_2 < \bar{x}_2 - \frac{\bar{x}_2^2}{2}$. Finally, for $x_1 \in [\bar{x}_1 - \frac{\bar{x}_2^2}{2}, \bar{x}_1)$ and $x_2 \in [\bar{x}_2 - \frac{\bar{x}_2^2}{2}, \bar{x}_2)$, we have:

$$q_1(x_1) = \int_{\gamma_1(x_1) > \gamma_2(s)} ds = \int_{x_1 - \bar{x}_1 > s - \bar{x}_2} ds = x_1 - \bar{x}_1 + \bar{x}_2$$
$$q_2(x_2) = \int_{\gamma_2(x_2) > \gamma_1(s)} ds = \int_{x_2 - \bar{x}_2 > s - \bar{x}_1} ds = x_2 - \bar{x}_2 + \bar{x}_1$$

Thus, the higher-budget bidder 1's probability of trading $q_1(x_1)$ increases continuously on the interval $[\bar{x}_1 - \frac{\bar{x}_2^2}{2}, \bar{x}_1]$ and jumps at $\bar{x}_1$ from $\bar{x}_2$ to 1. Bidder 2's probability of trading $q_2(x_2)$ increases continuously on the interval $[\bar{x}_2 - \frac{\bar{x}_2^2}{2}, \bar{x}_2]$ to $\bar{x}_1$ and is flat at $\bar{x}_1$ when $x_2 \geq \bar{x}_2$.

Note that $q_1(x) - q_2(x) = 2(\bar{x}_2 - \bar{x}_1) < 0$ for $x \in [\bar{x}_1 - \frac{\bar{x}_2^2}{2}, \bar{x}_2]$. So buyer 1 is, indeed, handicapped and has a lower probability of trading on the intermediate range of valuations.

We can also compute the seller's expected profits using the formula (13):

$$\sum_{i=1}^{2} \int_0^1 t_i(x_i) dx_i = \sum_{i=1}^{2} \int_0^{\bar{x}_i} q_i(x_i)\,(2x_i - 1)\,dx_i + \sum_{i=1}^{2} \int_{\bar{x}_i}^{1} q_i(\bar{x}_i)\bar{x}_i dx_i =$$
$$\sum_{i=1}^{2} \int_0^{\bar{x}_i} q_i(x_i)\,(2x_i - \bar{x}_i)\,dx_i + \sum_{i=1}^{2} (1-\bar{x}_i)\left(q_i(\bar{x}_i)\bar{x}_i - \int_0^{\bar{x}_i} q_i(x_i) dx_i\right) =$$
$$\int_{\bar{x}_1 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_1} (2x - \bar{x}_1)(x - \bar{x}_1 + \bar{x}_2) dx + \int_{\bar{x}_2 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_2} (2x - \bar{x}_2)(x - \bar{x}_2 + \bar{x}_1) dx + \sum_{i=1,2} m_i(1-\bar{x}_i) =$$
$$\int_{-\frac{\bar{x}_2^2}{2}}^{0} (2y + \bar{x}_1)(y + \bar{x}_2) dy + \int_{-\frac{\bar{x}_2^2}{2}}^{0} (2y + \bar{x}_2)(y + \bar{x}_1) dy + \sum_{i=1,2} m_i(1-\bar{x}_i)$$
$$= \frac{\bar{x}_1 \bar{x}_2^2 + \bar{x}_2^3}{2} + \bar{x}_1 \bar{x}_2^3 - \frac{\bar{x}_2^4}{2}(1 + \bar{x}_1 + \bar{x}_2) + \frac{\bar{x}_2^6}{12} + \sum_{i=1,2} m_i(1-\bar{x}_i)$$

This expression illustrates that the seller's payoff increases in both m_1 and m_2 since $\bar{x}_1$ and $\bar{x}_2$ are increasing in the budgets.

To conclude with this example, let us relax the assumption that $m_1 \leq \frac{1}{2}$ and characterize the optimal mechanism when only bidder 2's budget constraint is binding. Recall from the earlier analysis, and, in particular, part (2) of Theorem 2, that the budget constraint of bidder 1 is not binding in the optimal mechanism if the solution to (37) and (38) is such that $\bar{x}_1 > \frac{1+\bar{x}_2^2}{2}$. Then the Lagrange multipliers are such that $\lambda_1 = 0$ and $\lambda_2 = (1-\bar{x}_2)^2$. Consequently, $\gamma_1(x_1) = 2x_1 - 1$, $\gamma_2(x_2) = 2x_2 - 2\bar{x}_2 + \bar{x}_2^2$, $r_1 = \frac{1}{2}$, and $r_2 = \bar{x}_2 - \frac{\bar{x}_2^2}{2}$. So the probabilities of trading $q_1(.)$ and $q_2(.)$ are easily derived as follows: $q_1(x_1) = 0$ if $x_1 < \frac{1}{2}$; $q_1(x_1) = x_1 - \frac{1}{2} + \bar{x}_2 - \frac{\bar{x}_2^2}{2}$ if $x_1 \in \left[\frac{1}{2}, \frac{1+\bar{x}_2^2}{2}\right)$; and $q_1(x_1) = 1$ if $x_1 \geq \frac{1+\bar{x}_2^2}{2}$, while $q_2(x_2) = 0$ if $x_2 < \bar{x}_2 - \frac{\bar{x}_2^2}{2}$; $q_2(x_2) = x_2 - \bar{x}_2 + \frac{1+\bar{x}_2^2}{2}$ if $x_2 \in \left[\bar{x}_2 - \frac{\bar{x}_2^2}{2}, \bar{x}_2\right)$; and $q_2(\bar{x}_2) = \frac{1+\bar{x}_2^2}{2}$ if $x_2 \geq \bar{x}_2$.

Then equation (38) becomes $m_2 = \frac{1+\bar{x}_2^2}{2}\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right) + \frac{\bar{x}_2^4}{8}$, which uniquely defines $\bar{x}_2$. Note that $\bar{x}_2$ converges to zero when m_2 becomes small. Although the budget constraint of bidder 1 is no longer binding, the right-hand side of (37) with $\bar{x}_1 = \frac{1+\bar{x}_2^2}{2}$ gives the transfer of all types of bidder 1 with value $x_1 \in [\frac{1+\bar{x}_2^2}{2}, 1]$ i.e., $t_1(x_1) = \frac{1+\bar{x}_2^2}{2} - \frac{\bar{x}_2^3}{2} + \frac{\bar{x}_2^4}{8} < m_1$.

Note that if bidder 1's value belongs to $[\frac{1}{2}, \frac{1+\bar{x}_2^2}{2})$, she competes with bidder 2 with bidder 1 handicapped in this competition i.e., $q_1(x) < q_2(x)$ for x in this interval.

5.2.2 Three-Bidder Mechanism Under the Uniform Distribution

First, we characterize the conditions for the optimality of the "top auction" with threshold $\bar{x}^t$. In the top auction, $\gamma_i(x) = 2x - 2\bar{x}^t + (\bar{x}^t)^2$ for $x \in [0, \bar{x}^t]$, $r_i = \bar{x}^t - \frac{(\bar{x}^t)^2}{2}$, $q_i(x) = x^2$ for all $x \in \left[\bar{x}^t - \frac{(\bar{x}^t)^2}{2}, \bar{x}^t\right)$, and $q_i(\bar{x}^t)$ set to satisfy the budget constraint of bidder i for $i \in \{1,2,3\}$. The conditions (29) and (32) simplify to:

$$\begin{aligned}
&\sum_{i=1}^{3} m_i = \bar{x}^t(1+\bar{x}^t) + \left(\bar{x}^t - \frac{(\bar{x}^t)^2}{2}\right)^3 \\
&m_1 - \frac{m_2+m_3}{2} \leq \bar{x}^t\left(1 - \bar{x}^t\frac{1+\bar{x}^t}{2}\right) \\
&\frac{m_1+m_2}{2} - m_3 \leq \bar{x}^t\left(\frac{1+\bar{x}^t}{2} - (\bar{x}^t)^2\right) \qquad (39)
\end{aligned}$$

When the system (39) has a solution, then the optimal mechanism is a top auction with threshold $\bar{x}^t$. If the system (39) does not have a solution, then the optimal mechanism is a "budget-handicap auction" in which at least two bidders have different thresholds. There are three possible kinds of "budget-handicap auctions:"

- "top cluster:" $\bar{x}_1 = \bar{x}_2 > \bar{x}_3$.

- "lower cluster:" $\bar{x}_1 > \bar{x}_2 = \bar{x}_3$.

- "no clusters:" $\bar{x}_1 > \bar{x}_2 > \bar{x}_3$.

Below, we will derive conditions for these three kinds of the "budget-handicap auction."

5.2.3 Top cluster

Since $\bar{x}_1 = \bar{x}_2$ in the top cluster, we will simplify the notation and let $\bar{x}_1$ denote the threshold of bidders 1 and 2 in the rest of this subsection. So, we have $\bar{x}_1 > \bar{x}_3$, $\gamma_1(x) = \gamma_2(x) = 2x - 2\bar{x}_1 + \bar{x}_1^2$ for $x < \bar{x}_1$, $\gamma_1(\bar{x}_1) = \gamma_2(\bar{x}_1) = \bar{x}_1^2$; $\gamma_3(x) = 2x - 2\bar{x}_3 + \bar{x}_3^2$ for $x < \bar{x}_3$, $\gamma_3(\bar{x}_3) = \bar{x}_3^2$. The bidders' reservation values are given by $r_1 = r_2 = \bar{x}_1 - \frac{\bar{x}_1^2}{2}$, $r_3 = \bar{x}_3 - \frac{\bar{x}_3^2}{2}$.

Then by Lemma 4 for $i \in \{1, 2\}$, $q_i(x) = 0$ for $x < \bar{x}_1 - \frac{\bar{x}_1^2}{2}$, $q_i(x) = x(x - \bar{x}_1 + \frac{\bar{x}_1^2}{2} + \bar{x}_3 - \frac{\bar{x}_3^2}{2})$ for $x \in (\bar{x}_1 - \frac{\bar{x}_1^2}{2}, \bar{x}_1 - \frac{\bar{x}_1^2}{2} + \frac{\bar{x}_3^2}{2}]$, and $q_i(x) = x$ for $x \in (\bar{x}_1 - \frac{\bar{x}_1^2}{2} + \frac{\bar{x}_3^2}{2}, \bar{x}_1)$. The values of $q_1(\bar{x}_1)$ and $q_2(\bar{x}_1)$ are determined by the budget constraints of bidders 1 and 2.

For bidder 3, we have $q_3(x) = 0$ for $x < \bar{x}_3 - \frac{\bar{x}_3^2}{2}$, $q_3(x) = \left(x - \bar{x}_3 + \frac{\bar{x}_3^2}{2} + \bar{x}_1 - \frac{\bar{x}_1^2}{2}\right)^2$ for $x \in (\bar{x}_3 - \frac{\bar{x}_3^2}{2}, \bar{x}_3)$, and $q_3(\bar{x}) = \left(\frac{\bar{x}_3^2}{2} + \bar{x}_1 - \frac{\bar{x}_1^2}{2}\right)^2$.

Note that while $q_3(x)$ is continuous everywhere above r_3, $q_1(x)$ and $q_2(x)$ experience two jumps. First, there is a jump at $\bar{x}_1 - \frac{\bar{x}_1^2}{2} + \frac{\bar{x}_3^2}{2}$, as bidders 1 and 2 with values above this level no longer face the competition from bidder 3 because $\gamma_1(\bar{x}_1 - \frac{\bar{x}_1^2}{2} + \frac{\bar{x}_3^2}{2}) = \gamma_3(\bar{x}_3)$. The second jump happens at the threshold $\bar{x}_1$, since $\lim_{x \to \bar{x}-} q_1(x) + q_2(x) = 2\bar{x} < 1 + \bar{x} = q_1(\bar{x}) + q_2(\bar{x})$.

By Theorem 3 (conditions (24)-(26)), the budget-handicap auction with a top cluster is optimal if the following system of two equations and one inequality has a solution:

$$m_3 = \bar{x}_3 q_3(\bar{x}_3) - \int_{\bar{x}_3 - \frac{\bar{x}_3^2}{2}}^{\bar{x}_3} q_3(x_3) dx_3 \tag{40}$$

$$m_1 + m_2 = (1 + \bar{x}_1) - 2 \int_{\bar{x}_1 - \frac{\bar{x}_1^2}{2}}^{\bar{x}_1} q_1(x_1) dx_1 \tag{41}$$

$$m_1 - m_2 \leq \bar{x}_1(1 - \bar{x}_1).$$

Using the expressions for $q_i(x)$, $i \in \{1,2,3\}$ in (40) and (41) yields:

$$m_3 = \bar{x}_3\left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}\right)^2 - \int_{\bar{x}_3 - \frac{\bar{x}_3^2}{2}}^{\bar{x}_3}\left(s - \bar{x}_3 + \bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}\right)^2 ds = \bar{x}_3\left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}\right)^2 - \frac{\left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}\right)^3}{3} + \frac{\left(\bar{x}_1 - \frac{\bar{x}_1^2}{2}\right)^3}{3} = -\frac{\bar{x}_3^6}{24} + \frac{\bar{x}_3^5}{4} + \bar{x}_3^3\left(1 - \frac{\bar{x}_3}{4}\right)\left(\bar{x}_1 - \frac{\bar{x}_1^2}{2}\right) + \left(\bar{x}_3 - \frac{\bar{x}_3^2}{2}\right)\left(\bar{x}_1 - \frac{\bar{x}_1^2}{2}\right)^2 \tag{42}$$

$$m_1 + m_2 = \bar{x}_1(1+\bar{x}_1) - 2\int_{\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}}^{\bar{x}_1} y dy - 2\int_{\bar{x}_1 - \frac{\bar{x}_1^2}{2}}^{\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}} y\left(y - \bar{x}_1 + \bar{x}_3 + \frac{\bar{x}_1^2}{2} - \frac{\bar{x}_3^2}{2}\right))dy$$
$$= \bar{x}_1(1+\bar{x}_1) + \frac{\bar{x}_3^4}{4}\left(1 - \bar{x}_3 + \frac{\bar{x}_3^2}{6}\right) - \bar{x}_1^3\left(1 - \frac{\bar{x}_1}{4}\right) + \left(\bar{x}_1 - \frac{\bar{x}_1^2}{2}\right)\bar{x}_3^2\left(1 - \frac{\bar{x}_3}{2}\right)^2 \tag{43}$$

Equations (42) and (43) implicitly define $\bar{x}_1$ and $\bar{x}_3$ and can be easily solved numerically. If the solution is such that $m_1 - m_2 \le \bar{x}_1(1-\bar{x}_1)$, then the optimal mechanism is a handicap auction with a “top cluster.”

5.2.4 Lower cluster

Next, consider the case of the “lower cluster” with $\bar{x}_1 > \bar{x}_2 = \bar{x}_3$. To simplify the presentation, we let $\bar{x}_2$ denote the threshold of bidders 2 and 3 and drop $\bar{x}_3$ from the notation.

Then we have: $\gamma_1(x_1) = 2x - 2\bar{x}_1 + \bar{x}_2^2$ for $x_1 < \bar{x}_1$, $\gamma_1(\bar{x}_1) > \gamma_1^-(\bar{x}_1) = \frac{\bar{x}_2^2}{2}$, $\gamma_2(x) = \gamma_3(x) = 2x - 2\bar{x}_2 + \bar{x}_2^2$ for $x < \bar{x}_2$, $\gamma_2(\bar{x}_2) = \gamma_3(\bar{x}_2) = \bar{x}_2^2$. The reservation values are $r_1 = \bar{x}_1 - \frac{\bar{x}_2^2}{2}$ and $r_2 = r_3 = \bar{x}_2 - \frac{\bar{x}_2^2}{2}$.

The probabilities of trading are given by: $q_1(x_1) = 0$ for $x_1 < \bar{x}_1 - \frac{\bar{x}_2^2}{2}$, $q_1(x_1) = (x_1 - \bar{x}_1 + \bar{x}_2)^2$ for $x_1 \in \left[\bar{x}_1 - \frac{\bar{x}_2^2}{2}, \bar{x}_1\right)$, $q_1(\bar{x}_1) = 1$. For $i \in \{2,3\}$, $q_i(x) = 0$ for $x < \bar{x}_2 - \frac{\bar{x}_2^2}{2}$, and $q_i(x) = x\left(x - \bar{x}_2 + \bar{x}_1\right)$ for $x \in \left[\bar{x}_2 - \frac{\bar{x}_2^2}{2}, \bar{x}_2\right)$. Finally, $q_2(\bar{x}_2)$ and $q_3(\bar{x}_2)$ are determined by the budget constraints of bidders 2 and 3, correspondingly.

By Theorem 3, condition (24) must hold for bidder 1 and conditions (25) and (26) must hold for bidders 2 and 3 i.e.:

$$m_1 = \bar{x}_1 - \int_{\bar{x}_1 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_1} (s - \bar{x}_1 + \bar{x}_2)^2 \, ds = \bar{x}_1 - \frac{\bar{x}_2^3}{3} + \frac{\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)^3}{3}$$

$$= \bar{x}_1 - \frac{\bar{x}_2^2}{6}\left(\bar{x}_2^2 + \bar{x}_2\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right) + \left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)^2\right) = \bar{x}_1 - \frac{\bar{x}_2^4}{2}\left(1 - \frac{\bar{x}_2}{2} + \frac{\bar{x}_2^2}{12}\right) \tag{44}$$

$$m_2 + m_3 = \bar{x}_2\bar{x}_1(1 + \bar{x}_2) - 2\int_{\bar{x}_2 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_2} s\,(s - \bar{x}_2 + \bar{x}_1)\, ds = \bar{x}_1\bar{x}_2(1 + \bar{x}_2) - \frac{2\bar{x}_2^3}{3} + \frac{2\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)^3}{3}$$

$$- (\bar{x}_1 - \bar{x}_2)\left(\bar{x}_2^2 - \left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)^2\right) = \bar{x}_1\bar{x}_2(1 + \bar{x}_2) + \frac{\bar{x}_2^5}{4}\left(1 - \frac{\bar{x}_2}{3}\right) - \bar{x}_2^3\bar{x}_1\left(1 - \frac{\bar{x}_2}{4}\right) \tag{45}$$

$$m_2 - m_3 \le \bar{x}_2(1 - \bar{x}_2)\bar{x}_1 \tag{46}$$

Equations (44) and (45) implicitly define $\bar{x}_1$ and $\bar{x}_2$. If the solution satisfies (46), the optimal mechanism is the handicap auction with the lower cluster and thresholds $\bar{x}_1$ and $\bar{x}_2 = \bar{x}_3$.

5.2.5 No Clusters.

Finally, we consider the case with no clusters i.e., $\bar{x}_1 > \bar{x}_2 > \bar{x}_3$.

In this case, $\gamma_1(x_1) = 2x - 2\bar{x}_1 + \bar{x}_2^2$ for $x_1 < \bar{x}_1$, $\gamma_1(\bar{x}_1) > \gamma_1^-(\bar{x}_1) = \frac{\bar{x}_2^2}{2}$, $\gamma_2(x) = 2x - 2\bar{x}_2 + \bar{x}_2^2$ for $x < \bar{x}_2$, $\gamma_2(\bar{x}_2) = \bar{x}_2^2$, $\gamma_3(x) = 2x - 2\bar{x}_3 + \bar{x}_3^3$ for $x < \bar{x}_3$, $\gamma_3(\bar{x}_3) = \bar{x}_3^2$. The reservation values are $r_1 = \bar{x}_1 - \frac{\bar{x}_2^2}{2}$, $r_2 = \bar{x}_2 - \frac{\bar{x}_2^2}{2}$, and $r_3 = \bar{x}_3 - \frac{\bar{x}_3^2}{2}$.

Therefore, the probabilities of trading of bidder 1 are as follows: $q_1(x) = 0$ for $x < \bar{x}_1 - \frac{\bar{x}_2^2}{2}$, $q_1(x) = (x - \bar{x}_1 + \bar{x}_2)\left(x - \bar{x}_1 + \bar{x}_3 + \frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2}\right)$ for $x \in \left[\bar{x}_1 - \frac{\bar{x}_2^2}{2}, \bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right]$, $q_1(x) = x - \bar{x}_1 + \bar{x}_2$ for $x \in \left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}, \bar{x}_1\right)$, and $q_1(\bar{x}_1) = 1$.

For bidder 2, $q_2(x) = 0$ for $x < \bar{x}_2 - \frac{\bar{x}_2^2}{2}$, $q_2(x) = (x - \bar{x}_2 + \bar{x}_1)\left(x - \bar{x}_2 + \bar{x}_3 + \frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2}\right)$ for $x \in \left[\bar{x}_2 - \frac{\bar{x}_2^2}{2}, \bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right]$, $q_2(x) = x - \bar{x}_2 + \bar{x}_1$ for $x \in \left(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}, \bar{x}_2\right)$, $q_2(\bar{x}_2) = \bar{x}_1$.

Finally, for bidder 3, $q_3(x) = 0$ for $x < \bar{x}_3 - \frac{\bar{x}_3^2}{2}$, $q_3(x) = \left(x - \bar{x}_3 + \bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)\left(x - \bar{x}_3 + \bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)$ for $x \in \left[\bar{x}_3 - \frac{\bar{x}_3^2}{2}, \bar{x}_3\right)$, and $q_3(\bar{x}_3) = (\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2})(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2})$.

By Theorem 3, in the "no cluster" case the necessary and sufficient conditions characterizing the optimal thresholds $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ are the budget constraints (24) i.e., $m_i = \bar{x}_i q_i(\bar{x}_i) - \int_{r_i}^{\bar{x}_i} q_i(s)ds$ for $i = 1, 2, 3$. If the solution to this system of three equations exists and is such that $1 \ge \bar{x}_1 > \bar{x}_2 > \bar{x}_3 \ge 0$, then we have an optimal mechanism with no clusters.

In the rest of this subsection, we will exhibit the system of three equations $m_i = \bar{x}_i q_i(\bar{x}_i) - \int_{r_i}^{\bar{x}_i} q_i(s)ds$ for $i = 1, 2, 3$ explicitly using the expressions for $q_i(.)$ above and then replace it with a simpler system. First, consider $i = 1$. We have:

$$m_1 = \bar{x}_1 - \int_{\bar{x}_1 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}} (x - \bar{x}_1 + \bar{x}_2) \left(x - \bar{x}_1 + \bar{x}_3 + \frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2} \right) dx - \int_{\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}}^{\bar{x}_1} x - \bar{x}_1 + \bar{x}_2 ds =$$
$$\bar{x}_1 - \frac{\left(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)^3}{3} + \frac{\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)^3}{3} + \frac{\left(\bar{x}_2 - \bar{x}_3 - \frac{\bar{x}_2^2}{2} + \frac{\bar{x}_3^2}{2}\right)}{2} \left(\left(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2} \right)^2 - \left(\bar{x}_2 - \frac{\bar{x}_2^2}{2} \right)^2 \right)$$
$$- \frac{\bar{x}_2^2}{2} + \frac{\left(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)^2}{2} = \bar{x}_1 + \frac{\bar{x}_3^4}{8} \left(1 - \bar{x}_3 + \frac{\bar{x}_3^2}{6} \right) - \frac{\bar{x}_2^3}{2} \left(1 - \frac{\bar{x}_2}{4} \right) + \left(\bar{x}_2 - \frac{\bar{x}_2^2}{2} \right) \frac{\bar{x}_3^2}{2} \left(1 - \frac{\bar{x}_3}{2} \right)^2 \tag{47}$$

Second, using the expressions for $q_2(.)$ and $q_3(.)$ derived above, we obtain:

$$m_2 = \bar{x}_2 \bar{x}_1 - \int_{\bar{x}_2 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}} (x - \bar{x}_2 + \bar{x}_1) \left(x - \bar{x}_2 + \bar{x}_3 + \frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2} \right) dx - \int_{\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}}^{\bar{x}_2} x - \bar{x}_2 + \bar{x}_1 ds \tag{48}$$

$$m_3 = \bar{x}_3 \left(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2} \right) \left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2} \right) - \int_{\bar{x}_3 - \frac{\bar{x}_3^2}{2}}^{\bar{x}_3} (x - \bar{x}_3 + \bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2})(x - \bar{x}_3 + \bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}) dx \tag{49}$$

Next, we replace (48) and (49) with the equations for $m_1 - m_2$ and $m_2 - m_3$ as follows. First, subtracting (48) from (47) we obtain:

$$m_1 - m_2 = \bar{x}_1(1 - \bar{x}_2) + \int_{\bar{x}_2 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}} (\bar{x}_1 - \bar{x}_2)(x - \bar{x}_2 + \bar{x}_3 + \frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2}) dx + \int_{\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}}^{\bar{x}_2} \bar{x}_1 - \bar{x}_2 ds$$
$$= \bar{x}_1(1 - \bar{x}_2) + \frac{\bar{x}_1 - \bar{x}_2}{2} \left(\bar{x}_2^2 - \left(\bar{x}_3 - \frac{\bar{x}_3^2}{2} \right)^2 \right). \tag{50}$$

Finally, we perform a change of variable of integration in the second term of (48) to $y =$

$x - \bar{x}_2 + \frac{\bar{x}_2^2}{2} + \bar{x}_3 - \frac{\bar{x}_3^2}{2}$ and subtract (49) from the result to obtain:

$$m_2 - m_3 = \bar{x}_1\bar{x}_2 - \frac{\bar{x}_1^2}{2} + \frac{\left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)^2}{2} - \bar{x}_3\left(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)\left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)$$
$$+ \int_{\bar{x}_3 - \frac{\bar{x}_3^2}{2}}^{\bar{x}_3}\left(x - \bar{x}_3 + \bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2} - \bar{x}_3 + \frac{\bar{x}_3^2}{2}\right)dx =$$
$$\bar{x}_1\bar{x}_2 + (\bar{x}_2\bar{x}_3 - \bar{x}_1(1-\bar{x}_3))\frac{\bar{x}_2^2 - \bar{x}_3^2}{2} + \left(\frac{1}{2} - \bar{x}_3\right)\left(\frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2}\right)^2 + \frac{\bar{x}_3^2}{2}\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2} - \bar{x}_3 + \frac{\bar{x}_3^2}{2}\right)\left(\bar{x}_1 + \frac{\bar{x}_3^2}{4} - \frac{\bar{x}_2^2}{2}\right) \tag{51}$$

To conclude, when the solution to the system (47), (50) and (51) satisfies $\bar{x}_1 > \bar{x}_2 > \bar{x}_3$, we have no clusters in the optimal mechanism.

6 Conclusions

In this paper, we have derived an optimal mechanism for a seller facing bidders who have commonly known and unequal budgets. We have shown that when the differences between the budgets are not too large, the seller uses a "top auction" mechanism in which all bidders are treated symmetrically when their valuation do not exceed a certain threshold valuation. At that threshold all budget constraints become binding, and the richer bidders are given the good with a higher probability.

When the differences between the budgets are sufficiently large, then the seller uses a "budget-handicap" auction in which the valuation thresholds at which budget constraints become binding differ across the bidders. Budget-handicap auction also discriminates between the bidders at low valuations favoring low-budget bidders, who have higher probabilities of trading at low valuations and lower reserve prices. The seller does so to create a stronger competition for higher-budget bidders and extract more surplus from them. The latter result can be interpreted as providing justification for favoring smaller or minority-owned businesses in public procurement and other allocation mechanisms, such as spectrum auctions.

Our mechanisms have the nature of an all-pay auction, since a bidder always pays her bid. It would be interesting to consider a modification of our set-up and consider mechanisms in which a bidder pays only when (s)he gets the good. We leave this issue for future research.

Another interesting qualitative property of the optimal mechanism emerges from our analysis of the two bidder case. There, we show that when one bidder has a significantly larger budget than the other, a mechanism with "buy-it-now" features is optimal. Precisely, a rich bidder is given an option to either participate in an auction, where she competes with the not-so-rich bidder, or to purchase the good immediately at a higher price. Generalizing this result to a more general set-up with many bidders is another extension which we leave for future research.

7 Appendix

Proof of Theorem 2: The proof of the Theorem consists of two parts. In Part I, we explore the first-order conditions of the optimization problem (19) to restrict the set of possible values of the profile of $\{\gamma^-(\bar{x}_i), \gamma(\bar{x}_i)\}_{i=1}^n$. In Part II, we use the results of part I to establish the statements of the Theorem.

Part I. First, note that by the definition of $\gamma_i(.)$ in (15), it is continuous at $x_i = \bar{x}_i$ if and only if $\lambda_i = \frac{(1-F(\bar{x}_i))^2}{(1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i))}$.

To characterize optimal $\bar{x}_i$ we derive the first-order conditions involving derivatives of the Lagrangian $\mathcal{L}$ in (19). Although it may not possess a derivative with respect to x_i at $x_i = \bar{x}_i$ because its second term contains a max operator, it does however, possess left- and right- derivatives at $\bar{x}_i$ which we denote by $\frac{\partial_- \mathcal{L}}{\partial \bar{x}_i}$ and $\frac{\partial_+ \mathcal{L}}{\partial \bar{x}_i}$, respectively. If $\bar{x}_i$ is interior in $(0, 1)$, then we have the following conditions for optimality: $\frac{\partial_+ \mathcal{L}}{\partial \bar{x}_i}$ is nonpositive and $\frac{\partial_- \mathcal{L}}{\partial \bar{x}_i}$ is nonnegative. If $\bar{x}_i = 1$ then $\frac{\partial_- \mathcal{L}}{\partial \bar{x}_i} \geq 0$. Note that $\bar{x}_i = 0$ cannot be optimal because in this case $t_i(x_i) = q_i(x_i) = 0$ for all $x_i \in [0, 1]$.

Now let us compute the left- and right-derivatives, $\frac{\partial_- \mathcal{L}}{\partial \bar{x}_i}$ and $\frac{\partial_+ \mathcal{L}}{\partial \bar{x}_i}$. Using the notation in (20) we have:

$$\frac{\partial_+ \mathcal{L}}{\partial \bar{x}_i} = f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1}} \left(\max\{0, \gamma_i^-(\bar{x}_i), \max_{j \neq i} \gamma_j(x_j)\} - \max\{0, \gamma_i(\bar{x}_i), \max_{j \neq i} \gamma_j(x_j)\} \right) dF(x_{-i}) +$$
$$\int_{x \in [0,1]^n} \frac{\partial_+ \max\{0, \max_{j=1,\ldots,n} \gamma_j(x_j)\}}{\partial \bar{x}_i} dF(x) \qquad (52)$$

The first term in (52) arises because the range of integration in (19) over x_i includes the point $\bar{x}_i$ at which the integrand may be discontinuous. The second term comes from differentiating the integrand of $\mathcal{L}$.

Note that the left-derivative $\frac{\partial_- \mathcal{L}}{\partial \bar{x}_i}$ is obtained from (52) by replacing the right-sided derivative $\frac{\partial_+ \max\{0, \max_{j:j\neq i} \gamma_j(x_j)\}}{\partial \bar{x}_i}$ with the left-sided derivative $\frac{\partial_- \max\{0, \max_{j:j\neq i} \gamma_j(x_j)\}}{\partial \bar{x}_i}$ in the integrand of the second term+

We will need to consider two cases: (a) $\lambda_i > \frac{(1-F(\bar{x}_i))^2}{1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i)}$; (b) $\lambda_i < \frac{(1-F(\bar{x}_i))^2}{1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i)}$.

First, let us focus on the case (a). Then $\gamma_i^-(\overline{x}_i) > \max\{0, \gamma_i(\overline{x}_i)\}$. Without loss if generality we can focus on the situations when $\gamma_i(\bar{x}_i) \geq 0$, since otherwise the first term in (52) is strictly positive and the second term is zero, and so $\frac{\partial_+ \mathcal{L}}{\partial \bar{x}_i} > 0$, which cannot hold at the optimum. So, without loss, we consider that case (a) is where $\gamma_i^-(\overline{x}_i) > \gamma_i(\overline{x}_i)$. Then the first term in (52) can be computed as follows:

$$f(\bar{x}_i) \int_{x_{-i}\in[0,1]^{n-1}:\max_{j\neq i}\gamma_j(x_j)<\gamma_i^-(\bar{x}_i)} \left(\gamma_i^-(\bar{x}_i) - \max\{0, \gamma_i(\bar{x}_i), \max_{j\neq i} \gamma_j(x_j)\}\right) dF(x_{-i}) =$$
$$f(\bar{x}_i) \int_{x_{-i}\in[0,1]^{n-1}:\max_{j\neq i}\gamma_j(x_j)<\gamma_i(\bar{x}_i)} \gamma_i^-(\bar{x}_i) - \gamma_i(\bar{x}_i) dF(x_{-i}) +$$
$$f(\bar{x}_i) \int_{x_{-i}\in[0,1]^{n-1}:\gamma_i(\bar{x}_i)\leq\max_{j\neq i}\gamma_j(x_j)<\gamma_i^-(\bar{x}_i)} \left(\gamma_i^-(\bar{x}_i) - \max_{j\neq i} \gamma_j(x_j)\right) dF(x_{-i}) \tag{53}$$

To simplify (53), note that by(15) we have $\frac{\partial \gamma_j(x_j)}{\partial \bar{x}_i} = 0$ for $j \neq i$, and

$$\frac{\partial_+ \gamma_i(x_i)}{\partial \overline{x}_i} = \begin{cases} 0, & \text{if } x_i < \overline{x}_i, \\ 1 - \frac{\lambda_i}{(1-F(\overline{x}_i))^2} \left(1 - F\left(\overline{x}_i\right) + \overline{x}_i f\left(\overline{x}_i\right)\right), & \text{if } x_i \geq \bar{x}_i, \end{cases} \tag{54}$$

In the current case (a) we have $\lambda_i > \frac{(1-F(\bar{x}_i))^2}{1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i)}$, and so $\frac{\partial_+ \gamma_i(\bar{x}_i)}{\partial \bar{x}_i} < 0$. Therefore, the second term in (52) is equal to:

$$\int_{x\in[0,1]^n} \frac{\partial_+ \max\{0, \max_{j=1,\ldots,n} \gamma_j(x_j)\}}{\partial \bar{x}_i} dF(x) = \int_{x:\ x_i\in[\bar{x}_i,1]:\max_{j\neq i}\gamma_j(x_j)<\gamma_i(\bar{x}_i)} \frac{\partial_+ \gamma_i(\bar{x}_i)}{\partial \overline{x}_i} dF(x) =$$
$$= (1 - F(\bar{x}_i)) \int_{x_{-i}:\max_{j\neq i}\gamma_j(x_j)<\gamma_i(\bar{x}_i)} \left(1 - \frac{\lambda_i}{\left(1 - F\left(\overline{x}_i\right)\right)^2} \left(1 - F\left(\overline{x}_i\right) + \overline{x}_i f\left(\overline{x}_i\right)\right)\right) dF(x) =$$
$$f(\bar{x}_i) \int_{x_{-i}\in[0,1]^{n-1}:\max_{j\neq i}\gamma_j(x_j)<\gamma_i(\bar{x}_i)} \gamma_i(\bar{x}_i) - \gamma_i^-(\bar{x}_i) dF(x_{-i}) \tag{55}$$

Summing (53) and (55) yields:

$$\frac{\partial_+ \mathcal{L}}{\partial \bar{x}_i} = f(\bar{x}_i) \int_{x_{-i}\in[0,1]^{n-1}:\gamma_i(\bar{x}_i)\leq\max_{j\neq i}\gamma_j(x_j)<\gamma_i^-(\bar{x}_i)} \left(\gamma_i^-(\bar{x}_i) - \max_{j\neq i} \gamma_j(x_j)\right) dF(x_{-i}) \tag{56}$$

From (56) it follows that $\frac{\partial_+ \mathcal{L}}{\partial \bar{x}_i} > 0$ if the set $\{x_{-i} \in [0,1]^{n-1} : \gamma_i(\bar{x}_i) \leq \max_{j\neq i} \gamma_j(x_j) < \gamma_i^-(\bar{x}_i)\}$ has a positive measure. We refer to this situation as subcase (a)-(i). Obviously, subcase (a)-(i) can be ruled out as part of the solution as it would be optimal to raise $\bar{x}_i$.

Also, note that subcase (a)-(i) is equivalent to the following: either $\gamma_i(\bar{x}_i) < \min\{\gamma_j^-(\bar{x}_j), \gamma_i^-(\bar{x}_i)\}$ for some $j \neq i$, or $\gamma_i(\bar{x}_i) \leq \gamma_j(\bar{x}_j) < \gamma_i^-(\bar{x}_i)$ for some $j \neq i$ s.t. $\bar{x}_j < 1$. Thus, there are two remaining subcases of case (a). In subcase (a)-(ii), $\max_{j\neq i} \gamma_j^-(\bar{x}_j) \leq \gamma_i(\bar{x}_i) < \gamma_i^-(\bar{x}_i)$ and $\max_{j\neq i} \gamma_j(\bar{x}_j) < \gamma_i(\bar{x}_i)$. In subcase (a)-(iii) $\max_{j\neq i} \gamma_j^-(\bar{x}_j) \leq \gamma_i(\bar{x}_i) < \gamma_i^-(\bar{x}_i) \leq \max_{j\neq i} \gamma_j(\bar{x}_j)$ and $\gamma_j(\bar{x}_j) \notin [\gamma_i(\bar{x}_i), \gamma_i^-(\bar{x}_i))$ for all $j \neq i$. This, in subcase (a)-(iii) there exists a bidder h with $\bar{x}_h < 1$ such that $\gamma_h^-(\bar{x}_h) \leq \gamma_i(\bar{x}_i) < \gamma_i^-(\bar{x}_i) \leq \gamma_h(\bar{x}_h)$.[11] Based on this condition, we will rule out subcase (a)-(iii) below when we consider Case (b).

Now, let us consider Case (b) where $\lambda_i < \frac{(1-F(\bar{x}_i))^2}{1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i)}$, and so $\gamma_i(\bar{x}_i) > \max\{0, \gamma_i^-(\bar{x}_i)\}$ and $\frac{\partial_+ \gamma_i(\bar{x}_i)}{\partial \bar{x}_i} > 0$. Then the first term in (52) is equal to:

$$f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1} : \max_{j\neq i} \gamma_j(x_j) \leq \gamma_i(\bar{x}_i)} \left(\max\{0, \gamma_i^-(\bar{x}_i), \max_{j\neq i} \gamma_j(x_j)\} - \gamma_i(\bar{x}_i) \right) dF(x_{-i}) \tag{57}$$

Next, since $\frac{\partial_+ \gamma_i(\bar{x}_i)}{\partial \bar{x}_i} > 0$, the second term in (52) is equal to:

$$\int_{x \in [0,1]^n} \frac{\partial_+ \max\{0, \max_{j=1,\ldots,n} \gamma_j(x_j)\}}{\partial \bar{x}_i} dF(x) = \int_{x:\, x_i \in [\bar{x}_i, 1]; \max_{j\neq i} \gamma_j(x_j) \leq \gamma_i(\bar{x}_i)} \frac{\partial_+ \gamma_i(\bar{x}_i)}{\partial \bar{x}_i} dF(x) =$$

$$= (1 - F(\bar{x}_i)) \int_{x_{-i} : \max_{j\neq i} \gamma_j(x_j) \leq \gamma_i(\bar{x}_i)} \frac{\partial_+ \gamma_i(\bar{x}_i)}{\partial \bar{x}_i} dF(x) =$$

$$= (1 - F(\bar{x}_i)) \int_{x_{-i} : \max_{j\neq i} \gamma_j(x_j) \leq \gamma_i(\bar{x}_i)} \left(1 - \frac{\lambda_i}{(1 - F(\bar{x}_i))^2} (1 - F(\bar{x}_i) + \bar{x}_i f(\bar{x}_i)) \right) dF(x) =$$

$$f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1} : \max_{j\neq i} \gamma_j(x_j) \leq \gamma_i(\bar{x}_i)} \gamma_i(\bar{x}_i) - \gamma_i^-(\bar{x}_i) dF(x_{-i}) \tag{58}$$

Combining (57) and (58) yields:

$$\frac{\partial_+ \mathcal{L}}{\partial \bar{x}_i} = f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1} : \max_{j\neq i} \gamma_j(x_j) \leq \gamma_i(\bar{x}_i)} \max\{0, \gamma_i^-(\bar{x}_i), \max_{j\neq i} \gamma_j(x_j)\} - \gamma_i^-(\bar{x}_i) dF(x_{-i}) \tag{59}$$

Now we can rule out the following subcase (b)-(i): $\min\{\gamma_j^-(\bar{x}_j), \gamma_i(\bar{x}_i)\} > \gamma_i^-(\bar{x}_i)$ or $\gamma_i(\bar{x}_i) \geq \gamma_j(\bar{x}_j) > \gamma_i^-(\bar{x}_i)$ for some j, $j \neq i$. In this subcase, (59) implies that $\frac{\partial_+ \mathcal{L}}{\partial \bar{x}_i} > 0$ which cannot hold at the optimal $\bar{x}_i$.

[11] This immediately implies that $\bar{x}_h < 1$ for otherwise we would have $\gamma_h^-(\bar{x}_h) \geq \gamma_h(\bar{x}_h)$. Therefore, in this subcase and any subcase with $\gamma_h^-(\bar{x}_h) < \gamma_h(\bar{x}_h)$ for some h, so we can drop the qualifier $\bar{x}_h < 1$.

Note that ruling out subcase (b)-(i) also rules out subcase (a)-(iii) as the latter is encompassed by the first alternative of case (b)-(i) i.e., $\min\{\gamma_j^-(\bar{x}_j), \gamma_i(\bar{x}_i)\} > \gamma_i^-(\bar{x}_i)$ for some $j \neq i$. To see this, recall that subcase (a)-(iii) implies the existence of a bidder h such that $\gamma_h(\bar{x}_h) \geq \gamma_i^-(\bar{x}_i) > \gamma_i(\bar{x}_i) \geq \gamma_h^-(\bar{x}_h)$. Now, simply relabel the bidders i and h in the latter condition as j and i, respectively, to rewrite it as follows: $\gamma_i(\bar{x}_i) \geq \gamma_j^-(\bar{x}_j) > \gamma_j(\bar{x}_j) \geq \gamma_i^-(\bar{x}_i)$, which makes it obvious that (a)-(iii) is encompassed by (b)-(i).

There remain two other subcases of Case (b). In subcase (b)-(ii) we have: $\max_{j\neq i} \max\{\gamma_j^-(\bar{x}_j), \gamma_j(\bar{x}_j)\} \leq \gamma_i^-(\bar{x}_i) < \gamma_i(\bar{x}_i)$. This case will not be ruled out.

In subcase (b)-(iii) we have: $\max_{j\neq i} \gamma_j^-(\bar{x}_j) \leq \gamma_i^-(\bar{x}_i) < \gamma_i(\bar{x}_i) < \max_{j\neq i} \gamma_j(\bar{x}_j)$ and $\gamma_j(\bar{x}_j) \notin (\gamma_i^-(\bar{x}_i), \gamma_i(\bar{x}_i)]$ for all $j \neq i$. In this subcase, there exists k, $k \neq i$, s.t. $\gamma_k(\bar{x}_k) > \gamma_i(\bar{x}_i) > \gamma_i^-(\bar{x}_i) \geq \gamma_k^-(\bar{x}_k)$. We can rewrite this conditions as follows by relabelling k and i as i as j, respectively: $\gamma_i(\bar{x}_i) > \gamma_j(\bar{x}_j) > \gamma_j^-(\bar{x}_j) \geq \gamma_i^-(\bar{x}_i)$. The latter condition is encompassed by subcase (b)-(i) which we have ruled out. Therefore, subcase (b)-(iii) is also ruled out. This completes Part I of the proof.

Part II. In this part, we will use the results of Part I, in particular, the impossibility of the subcases (a)-(i), (a)-(iii), (b)-(i) and (b)-(iii), to establish the statements of the Theorem.

Let us start by proving statement 1 of the Theorem. So suppose that there exists $h \neq i$ such that $\bar{x}_h \geq \bar{x}_i$. Let us show that $\gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i)$.

If $\gamma_i^-(\bar{x}_i) < \gamma_h^-(\bar{x}_h)$, then since subcases (a)-(i) and (b)-(i) are ruled out, we must have $\gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i)$.

Next, suppose that $\gamma_i^-(\bar{x}_i) > \gamma_h^-(\bar{x}_h)$. Then again, since subcases (a)-(i) and (b)-(i) have been ruled out, we must have $\gamma_h^-(\bar{x}_h) = \gamma_h(\bar{x}_h) = \bar{x}_h - \frac{\bar{x}_h(1-F(\bar{x}_h))}{1-F(\bar{x}_h)+\bar{x}_h f(\bar{x}_h)}$ or, equivalently, $\lambda_h = \frac{(1-F(\bar{x}_h))^2}{(1-F(\bar{x}_h)+\bar{x}_h f(\bar{x}_h))}$. The inequalities $\bar{x}_h \geq \bar{x}_i$, and $\gamma_i^-(\bar{x}_i) > \gamma_h^-(\bar{x}_h)$ together imply that $\lambda_i > \frac{(1-F(\bar{x}_i))^2}{(1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i))}$. Using the latter inequality and $\bar{x}_h > \bar{x}_i$ we obtain that

$$\gamma_i(\bar{x}_i) < \bar{x}_i - \frac{\bar{x}_i(1-F(\bar{x}_i))}{1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i)} \leq \bar{x}_h - \frac{\bar{x}_h(1-F(\bar{x}_h))}{1-F(\bar{x}_h)+\bar{x}_h f(\bar{x}_h)} = \gamma_h^-(\bar{x}_h) = \gamma_h(\bar{x}_h).$$

But this configuration belongs to case (a)-(i) which had been ruled out.

It remains to consider the case $\gamma_i^-(\bar{x}_i) = \gamma_h^-(\bar{x}_h)$. As cases (a)-(i) and (b)-(i) have been ruled out, it follows that $\min\{\gamma_i(\bar{x}_i), \gamma_h(\bar{x}_h)\} = \gamma_i^-(\bar{x}_i) = \gamma_h^-(\bar{x}_h)$. So, to complete the proof that $\gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i)$, we only need to rule out $\gamma_i(\bar{x}_i) > \gamma_h(\bar{x}_h) = \gamma_i^-(\bar{x}_i) = \gamma_h^-(\bar{x}_h)$. To argue by contradiction, suppose that this is the case. Then we have $\gamma_h^-(\bar{x}_h) = \gamma_h(\bar{x}_h) = \bar{x}_h - \frac{\bar{x}_h(1-F(\bar{x}_h))}{1-F(\bar{x}_h)+\bar{x}_h f(\bar{x}_h)}$ or, equivalently, $\lambda_h = \frac{(1-F(\bar{x}_h))^2}{(1-F(\bar{x}_h)+\bar{x}_h f(\bar{x}_h))}$. Note the inequality $\bar{x}_h \geq \bar{x}_i$ together with $\gamma_i^-(\bar{x}_i) = \gamma_h^-(\bar{x}_h)$ imply that $\lambda_i \geq \frac{(1-F(\bar{x}_i))^2}{(1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i))}$. As we have shown above, the latter inequality and $\bar{x}_h \geq \bar{x}_i$ imply that

$$\gamma_i(\bar{x}_i) \leq \bar{x}_i - \frac{\bar{x}_i(1-F(\bar{x}_i))}{1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i)} \leq \bar{x}_h - \frac{\bar{x}_h(1-F(\bar{x}_h))}{1-F(\bar{x}_h)+\bar{x}_h f(\bar{x}_h)} = \gamma_h^-(\bar{x}_h) = \gamma_h(\bar{x}_h).$$

But this contradicts our assumption that $\gamma_i(\bar{x}_i) > \gamma_h(\bar{x}_h)$.

Finally, suppose that there exists a bidder $\hat{i}$ such that $\bar{x}_{\hat{i}} > \bar{x}_j$ for all $j \neq \hat{i}$. By part (i) of the Theorem, it follows that $\gamma_j(\bar{x}_j) = \gamma_j^-(\bar{x}_j)$ for all $j \neq \hat{i}$. Also, we must have $\min\{\gamma_{\hat{i}}(\bar{x}_{\hat{i}}), \gamma_{\hat{i}}^-(\bar{x}_{\hat{i}})\} \geq \gamma_j(\bar{x}_j) = \gamma_j^-(\bar{x}_j)$ for any $j \neq \hat{i}$. For, if this inequality does not hold then subcases (a)-(i) and (b)-(i) imply that $\gamma_j(\bar{x}_j) = \gamma_j^-(\bar{x}_j) = \bar{x}_j - \frac{\bar{x}_j(1-F(\bar{x}_j))}{1-F(\bar{x}_j)+\bar{x}_j f(\bar{x}_j)} > \gamma_{\hat{i}}(\bar{x}_{\hat{i}}) = \gamma_{\hat{i}}^-(\bar{x}_{\hat{i}}) = \bar{x}_{\hat{i}} - \frac{\bar{x}_{\hat{i}}(1-F(\bar{x}_{\hat{i}}))}{1-F(\bar{x}_{\hat{i}})+\bar{x}_{\hat{i}} f(\bar{x}_{\hat{i}})}$, and the latter inequality cannot hold because $\bar{x}_{\hat{i}} > \bar{x}_j$. We must also have $\max\{\gamma_{\hat{i}}(\bar{x}_{\hat{i}}), \gamma_{\hat{i}}^-(\bar{x}_{\hat{i}})\} > \gamma_j(\bar{x}_j) = \gamma_j^-(\bar{x}_j)$ for any $j \neq \hat{i}$. Otherwise we would have $\gamma_{\hat{i}}(\bar{x}_{\hat{i}}) = \gamma_{\hat{i}}^-(\bar{x}_{\hat{i}}) = \gamma_{\hat{i}}(\bar{x}_{\hat{i}}) = \gamma_{\hat{i}}^-(\bar{x}_{\hat{i}}) = \bar{x}_{\hat{i}} - \frac{\bar{x}_{\hat{i}}(1-F(\bar{x}_{\hat{i}}))}{1-F(\bar{x}_{\hat{i}})+\bar{x}_{\hat{i}} f(\bar{x}_{\hat{i}})} = \gamma_j(\bar{x}_j) = \gamma_j^-(\bar{x}_j) = \bar{x}_j - \frac{\bar{x}_j(1-F(\bar{x}_j))}{1-F(\bar{x}_j)+\bar{x}_j f(\bar{x}_j)}$. But the latter cannot hold because $\bar{x}_{\hat{i}} > \bar{x}_j$ for all $j \neq \hat{i}$. Also, note that the configuration $\gamma_{\hat{i}}^-(\bar{x}_{\hat{i}}) > \gamma_{\hat{i}}(\bar{x}_{\hat{i}}) = \gamma_j(\bar{x}_j) = \gamma_j^-(\bar{x}_j)$ for some j is ruled out as it belongs to subcase (a)-(i).

Thus, we either have $\gamma_{\hat{i}}(\bar{x}_{\hat{i}}) > \gamma_{\hat{i}}^-(\bar{x}_{\hat{i}}) = \gamma_j(\bar{x}_j) = \gamma_j^-(\bar{x}_j)$ for some j or $\min\{\gamma_{\hat{i}}(\bar{x}_{\hat{i}}), \gamma_{\hat{i}}^-(\bar{x}_{\hat{i}})\} > \gamma_j(\bar{x}_j) = \gamma_j^-(\bar{x}_j)$ for any $j \neq i$, which are both encompassed by subcases (a)-(ii) and (b)-(ii).

In either case, by Lemma 3 we have $q_{\hat{i}}(x_{\hat{i}}) = 1$ for all $x_{\hat{i}} > \breve{x}_{\hat{i}}$ where $\breve{x}_{\hat{i}}$ is such that $\breve{x}_{\hat{i}} \leq \bar{x}_i$ and satisfies $\breve{x}_{\hat{i}} - \frac{1-\lambda_{\hat{i}}-F(\breve{x}_{\hat{i}})}{f(\breve{x}_{\hat{i}})} = \max_{j \neq i} \gamma_j(\bar{x}_j)$.

Lemma 3 establishes that the mechanism is completely determined by the profile of the functions (virtual values) $\gamma_j(x_j)$ defined in (15) for $j \in \{1, ..., n\}$ and $x_j \in [0,1]$ and which by definition depend only on the profile of the thresholds $\bar{x}_j$ and Lagrange multipliers λ_j for $j \in \{1, .., n\}$. So, to complete the proof of the Theorem, let us show that the two profiles satisfying the already established optimal properties, $(\{\bar{x}_j, \lambda_j\}_{j \neq \hat{i}}, \bar{x}_{\hat{i}}, \lambda_{\hat{i}})$ and $(\{\bar{x}_j, \lambda_j\}_{j \neq \hat{i}}, \breve{x}_{\hat{i}}, \lambda_{\hat{i}})$, induce the same probabilities of trading $(q_1(.), ..., q_n(.))$. . Note that the only difference between the two profiles is that the threshold value of bidder $\hat{i}$ it taken to be $\breve{x}_{\hat{i}}$ in the latter, instead of $\bar{x}_{\hat{i}}$ in the former.

Indeed, note that changing bidder $\hat{i}$'s threshold to $\breve{x}_{\hat{i}}$ from $\bar{x}_{\hat{i}}$ does not affect the virtual value of any player $j \neq \hat{i}$ and also does not affect the virtual value of player $\hat{i}$ of type $x_{\hat{i}} \in [0, \breve{x}_{\hat{i}})$. However, the virtual value of player $\hat{i}$ of type $x_{\hat{i}} \in [\breve{x}_{\hat{i}}, \bar{x}_{\hat{i}})$ changes from $x_{\hat{i}} - \frac{1-\lambda_{\hat{i}}-F(x_{\hat{i}})}{f(x_{\hat{i}})}$ to $\breve{x}_{\hat{i}} - \frac{\lambda_{\hat{i}}\breve{x}_{\hat{i}}}{1-F(\breve{x}_{\hat{i}})}$, while the virtual value of player $\hat{i}$ of type $x_{\hat{i}} \in [\bar{x}_{\hat{i}}, 1]$ changes from $\bar{x}_{\hat{i}} - \frac{\lambda_{\hat{i}}\bar{x}_{\hat{i}}}{1-F(\bar{x}_{\hat{i}})}$ to $\breve{x}_{\hat{i}} - \frac{\lambda_{\hat{i}}\breve{x}_{\hat{i}}}{1-F(\breve{x}_{\hat{i}})}$.

Recall that when player $\hat{i}$'s threshold is equal to $\bar{x}_{\hat{i}}$, we have $q_{\hat{i}}(x_{\hat{i}}) = 1$ for all $x_{\hat{i}} > \breve{x}_{\hat{i}}$. So to complete the proof, we only need to show that the new virtual value of any type $x_{\hat{i}}$ s.t. $x_i > \breve{x}_i$ is strictly greater than $\max_{j\neq\hat{i}} \gamma_j(\bar{x}_j)$ i.e., $\breve{x}_{\hat{i}} - \frac{\lambda_{\hat{i}}\breve{x}_{\hat{i}}}{1-F(\breve{x}_{\hat{i}})} > \max_{j\neq i} \gamma_j(\bar{x}_j)$.

The argument depends on the value of $\lambda_{\hat{i}}$. If $\lambda_{\hat{i}} \geq \frac{(1-F(\bar{x}_{\hat{i}}))^2}{1-F(\bar{x}_{\hat{i}})+\bar{x}_{\hat{i}}f(\bar{x}_{\hat{i}})}$, then for all $x_{\hat{i}} \leq \bar{x}_{\hat{i}}$ we have: $\frac{\partial\left(x_{\hat{i}} - \frac{\lambda_{\hat{i}}x_{\hat{i}}}{1-F(x_{\hat{i}})}\right)}{\partial x_{\hat{i}}} = 1 - \frac{\lambda_{\hat{i}}}{\left(1-F\left(x_{\hat{i}}\right)\right)^2}\left(1 - F\left(x_{\hat{i}}\right) + x_{\hat{i}} f\left(x_{\hat{i}}\right)\right) \leq 1 - \frac{(1-F(\bar{x}_{\hat{i}}))^2\left(1-F\left(x_{\hat{i}}\right)+x_{\hat{i}}f\left(x_{\hat{i}}\right)\right)}{\left(1-F\left(x_{\hat{i}}\right)\right)^2\left(1-F(\bar{x}_{\hat{i}})+\bar{x}_{\hat{i}}f(\bar{x}_{\hat{i}})\right)} < 0$, where the last inequality relies on the increasing hazard rate assumption. Therefore, $\hat{x}_{\hat{i}} - \frac{\lambda_{\hat{i}}\hat{x}_{\hat{i}}}{1-F(\hat{x}_{\hat{i}})} > \bar{x}_{\hat{i}} - \frac{\lambda_i\bar{x}_{\hat{i}}}{1-F(\bar{x}_{\hat{i}})} \geq \max_{j\neq\hat{i}} \gamma_j(\bar{x}_j)$, as required.

Finally, if $\lambda_{\hat{i}} < \frac{(1-F(\bar{x}_{\hat{i}}))^2}{1-F(\bar{x}_{\hat{i}})+\bar{x}_{\hat{i}}f(\bar{x}_{\hat{i}})}$, then the fact that $\breve{x}_{\hat{i}} < \bar{x}_{\hat{i}}$ and the increasing hazard rate assumption imply that $\lambda_{\hat{i}} < \frac{(1-F(\breve{x}_{\hat{i}}))^2}{1-F(\breve{x}_{\hat{i}})+\breve{x}_{\hat{i}}f(\breve{x}_{\hat{i}})}$. The last inequality is equivalent to $\frac{1-\lambda-F(\breve{x}_{\hat{i}})}{f(\breve{x}_{\hat{i}})} > \frac{\lambda\breve{x}_{\hat{i}}}{1-F(\breve{x}_{\hat{i}})}$, which again implies the desired result, as we have: $\breve{x}_{\hat{i}} - \frac{\lambda_{\hat{i}}\breve{x}_{\hat{i}}}{1-F(\breve{x}_{\hat{i}})} > \bar{x}_{\hat{i}} - \frac{\lambda_{\hat{i}}\bar{x}_{\hat{i}}}{1-F(\bar{x}_{\hat{i}})} \geq \max_{j\neq i} \gamma_j(\bar{x}_j)$.

Finally, note that the solution with the threshold value $\breve{x}_{\hat{i}}$ for player $\hat{i}$ is the appropriate one because it satisfies $\breve{x}_{\hat{i}} = \inf\{x_{\hat{i}} | t_{\hat{i}}(x_{\hat{i}}) = m_{\hat{i}}\}$. *Q.E.D.*

Proof of Lemma 5: By Theorem 2, λ_i satisfies $0 < \lambda_i < 1 - F(\bar{x}_i)$ for all i. Since $\gamma_i(x_i) = x_i - \frac{1-\lambda-F(x_i)}{f(x_i}$ for $x_i < \bar{x}_i$, it is immediate that $\gamma_i'(x_i) > 0$ if $f'(x_i) \geq 0$. If $f'(x_i) < 0$, then $\gamma_i'(x_i) > \frac{d\left(x_i - \frac{1-F(x_i)}{f(x_i)}\right)}{dx_i} \geq 0$. The last inequality holds by the monotone hazard rate property. So, $\gamma_i(x_i)$ is increasing in x_i, and since by Lemma 4 $q_i(x_i)$ is increasing in $\gamma_i(x_i)$, it follows that $q_i(x_i)$ is increasing in x_i. Finally, $q_i(\bar{x}_i) > q_i(x_i)$ for all $x_i < \bar{x}_i$ because $\gamma_i(\bar{x}_i) > \gamma_i(x_i)$. *Q.E.D.*

Proof of Lemma 6: To prove the Lemma we argue by contradiction, so suppose that $\bar{x}_j > \bar{x}_i$. Recall that the binding budget constraints of types $\bar{x}_i$ and $\bar{x}_j$ imply the following: $m_i = \bar{x}_i q_i(\bar{x}_i) - \int_0^{\bar{x}_i} q_i(s)ds$ and $m_j = \bar{x}_j q_j(\bar{x}_j) - \int_0^{\bar{x}_j} q_j(s)ds$. Using the above equations we have:

$$m_j = \bar{x}_j q_j(\bar{x}_j) - \int_0^{\bar{x}_j} q_j(s)ds = \bar{x}_i q_j(\bar{x}_j) + \int_{\bar{x}_i}^{\bar{x}_j} (q_j(\bar{x}_j) - q_j(s))ds - \int_0^{\bar{x}_i} q_j(s)ds \geq \bar{x}_i q_j(\bar{x}_j) - \int_0^{\bar{x}_i} q_j(s)ds \tag{60}$$

Note that the inequality in (60) follows from $\int_{\bar{x}_i}^{\bar{x}_j} (q_j(\bar{x}_j) - q_j(s))ds \geq 0$, and the latter holds because $q_j(s)$ is nondecreasing in s. So we will have established a contradiction to $m_i > m_j$ if we can show both of the following: (a) $q_j(\bar{x}_j) \geq q_i(\bar{x}_i)$; (b) $q_j(s) \leq q_i(s)$ for all $s \in [0, \bar{x}_i]$. Now, (a) holds because, as established in Theorem 2, $\bar{x}_j > \bar{x}_i$ implies that $\gamma_j(\bar{x}_j) > \gamma_i(\bar{x}_i)$. In turn, the latter implies that $q_j(\bar{x}_j) > q_i(\bar{x}_i)$ by Lemma 4.

Finally, to establish that $q_j(s) \leq q_i(s)$ for all $s \in [0, \bar{x}_i]$, note that by Theorem 2, $\bar{x}_j > \bar{x}_i$ implies that $\lambda_j < \lambda_i$. Therefore, $\gamma_j(x) < \gamma_i(x)$, and hence by Lemma 4 $q_j(x) \leq q_i(x)$ for all $x \in [0, \bar{x}_i]$. *Q.E.D.*

Proof of Lemma 7: It is well-known (see e.g. Proposition 1.3.7, page 76, Chapter 1 in Bertsekas (2001)) that the strong duality property holds and (x^*, λ^*) is the solution to both the primal problem, $\max_x \min_\lambda \mathcal{L}(\bar{x}, \lambda)$, and the dual problem, $\min_\lambda \max_x \mathcal{L}(\bar{x}, \lambda)$, if and only if (x^*, λ^*) is a saddle point of the Lagrangian (19) i.e.,

$$\mathcal{L}(x, \lambda^*) \leq \mathcal{L}(x^*, \lambda^*) \leq \mathcal{L}(x^*, \lambda) \tag{61}$$

To establish the existence of a saddle point, we will make use of the Lagrange dual function $g(\lambda) \equiv \max_{\bar{x} \in [0,1]^n} \mathcal{L}(\lambda, \bar{x})$. The solution $\bar{x}(\lambda)$ to the problem $\max_{\bar{x} \in [0,1]^n} \mathcal{L}(\lambda, \bar{x})$ is characterized in Theorem 2. In particular, $\bar{x}(\lambda)$ is the inverse of the function $\lambda(\bar{x})$ given in the statement of Theorem 2. So, $g(\lambda) = \mathcal{L}(\lambda, \bar{x}(\lambda))$.

By Danskin's Theorem (Bertsekas (2001), Ch. 1, p. 131), the Lagrange dual function $g(\lambda)$ is convex and hence has a unique minimizer which we denote by λ^*. Define $x^* = \bar{x}(\lambda^*)$. Let us show that the saddle-point property (61) holds for the pair (x^*, λ^*).

Since $x^* = \bar{x}(\lambda^*)$, the first inequality in (61), $\mathcal{L}(x, \lambda^*) \leq \mathcal{L}(x^*, \lambda^*)$, holds for all $x \in [0, 1]^n$ by Theorem 2.

To show that $\mathcal{L}(\bar{x}(\lambda^*), \lambda^*) \leq \mathcal{L}(\bar{x}(\lambda^*), \lambda)$ we start by arguing that $\mathcal{L}(\bar{x}, \lambda)$ is convex in λ for fixed $\bar{x}$. To see this, note that by definition (15), the virtual value function

$\gamma_i(x_i)$ is linear in λ_i. Since $\max\{0, \max_i\{\gamma_i(x_i)\}\}$ is convex in $(\gamma_1(x_1), ..., \gamma_n(x_n))$, it follows that $\max\{0, \max_i\{\gamma_i(x_i)\}\}$ is also convex in $(\lambda_1, ..., \lambda_n)$. The integration operator over x preserves convexity of the integrand $\max\{0, \max_i\{\gamma_i(x_i)\}\} \prod_i f(x_i)$ in the parameters $(\lambda_1, ..., \lambda_n)$. Therefore, the Lagrangian $\mathcal{L}(\overline{x}, \lambda)$ is convex in $(\lambda_1, ..., \lambda_n)$.

The convexity of $\mathcal{L}(\bar{x}(\lambda^*), \lambda)$ in λ implies that it has a unique minimum which can be found as a unique solution to the first-order conditions $\frac{\partial \mathcal{L}(\bar{x}(\lambda^*), \lambda+\epsilon h)}{\partial h}_{\epsilon=0} \geq 0$ for all $h \in \mathbf{R}^n$. Again by Danskin's Theorem (Bertsekas (2001), Ch. 1, p. 131), $\frac{\partial \mathcal{L}(\bar{x}(\lambda), \lambda+\epsilon h)}{\partial h}_{\epsilon=0} = \frac{\partial g(\lambda+\epsilon h)}{\partial h}_{\epsilon=0}$ for all λ, h. Since by definition $\lambda^* = \arg\min_\lambda g(\lambda)$, we have $\frac{\partial g(\lambda^*+\epsilon h)}{\partial h}_{\epsilon=0} \geq 0$. So, $\lambda^* = \arg\min_\lambda \mathcal{L}(\bar{x}(\lambda^*), \lambda)$, and hence the second inequality in (61) holds for (x^*, λ^*). This completes the proof that (x^*, λ^*) is a saddle point. *Q.E.D.*

Proof of Theorem 3: Since our problem has strong duality property, its solution can be obtained by minimizing the dual Lagrange function $g(\lambda) \equiv \mathcal{L}(\lambda, \bar{x}^*(\lambda))$ with respect to λ.

Since the function $g(\lambda)$ is convex in the vector λ (by Danskin's Theorem (Bertsekas (2001), Ch. 1, p.131; see also Boyd and Vandenberghe (2009), p. 216), and so the minimum of $g(\lambda)$ is unique and is attained at such λ where the the first-order conditions $g'(\lambda; h) \equiv \frac{\partial \mathcal{L}(\lambda+\epsilon h, \bar{x}(\lambda))}{\partial \epsilon}_{\epsilon=0} \geq 0$ hold for any "direction" $h \in \mathbf{R}^n$. In the rest of the proof we focus on these first-order conditions.

To begin with, consider i such that $\bar{x}_i \neq \bar{x}_j$ for any $j \neq i$. In this case, the only variation h in the vector λ that we need to consider to characterize the optimal λ_i involves a change in λ_i only. So we have the following regular first-order condition:

$$\frac{\partial g(\lambda)}{\partial \lambda_i} = m_i - \bar{x}_i \int_{x_{-i} \in [0,1]^{n-1} : \gamma_i(\bar{x}_i) > \max_{j \neq i} \gamma_j(x_j)} d \prod_{j \neq i} F(x_j)$$
$$+ \int_0^{\bar{x}_i} \int_{x_{-i} \in [0,1]^{n-1} : \gamma_i(s) > \max\{0, \max_{j \neq i} \gamma_j(x_j)\}} d \prod_{j \neq i} F(x_j) ds = m_i - \bar{x}_i q_i(\bar{x}_i) + \int_0^{\bar{x}_i} q_i(s) ds \qquad (62)$$

The second equality in (62) holds by Lemma 4. Namely, for almost all $s \in [0, \bar{x}_i)$, the set of x_{-i} such that $\gamma_i(s) = \max_{j \neq i} \gamma_j(x_j)$ has measure zero. The same is true for $s = \bar{x}_i$ since $\bar{x}_i \neq \bar{x}_j$ for any $j \neq i$. Therefore by Lemma 4 for all $s \in [0, \bar{x}_i]$, $q_i(s) = \int_{x_{-i} \in [0,1]^{n-1} : \gamma_i(s) \geq \max\{0, \max_{j \neq i} \gamma_j(x_j)\}} dF(x_{-i}) = \int_{x_{-i} \in [0,1]^{n-1} : \gamma_i(s) > \max\{0, \max_{j \neq i} \gamma_j(x_j)\}} dF(x_{-i})$. Substituting this into (62) yields (24).

Next suppose that there is a "cluster" $\{k_1, ..., k_l\} \subset \{1, ..., n\}$, with $l \in \{2, ..., n\}$, such that $\bar{x}_{k_1} = ... = \bar{x}_{k_l} = \bar{x}^c \neq \bar{x}_j$ for any $j \notin \{k_1, ..., k_l\}$. Since the threshold $\bar{x}^c$, the corresponding λ^c and the set of bidders in the cluster are all chosen optimally, no variation from it should decrease the value of the Lagrange dual function $g(\lambda)$. Formally, we have to consider all variations of the vector λ of the form $\epsilon \times \mathbb{I}_J$ where $J \in \{k_1, ..., k_l\}$ is some subset of bidders in the "cluster" and $\mathbb{I}_J$ is an n-vector with entries corresponding to bidders in J equal to 1 and the rest of the entries equal to zero. Then the following first-order conditions have to hold for any J: $\frac{\partial g(\lambda+\epsilon\mathbb{I}_J)}{\partial \epsilon}_{\epsilon=0+} \geq 0$ and $\frac{\partial g(\lambda+\epsilon\mathbb{I}_J)}{\partial \epsilon}_{\epsilon=0-} \leq 0$. Although there are $2^l - 1$ such subsets $J \in \{k_1, ..., k_l\}$, it will be sufficient to establish only $2l$ of such first-order conditions, as we will see below.

So, let $J = \{k'_1, ..., k'_r\} \subset \{k_1, ..., k_l\}$. Then we have: $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_J)}{\partial \epsilon}_{|\epsilon=0+} =$

$$
\begin{aligned}
&\sum_{h=1,\dots,r} m_{k'_h} + \int_{x:\, \max_{h\in\{1,\dots,r\}} \gamma_{k'_h}(x_{k'_h}) > \max\{0, \max_{j\notin\{k'_1,\dots,k'_r\}} \gamma_j(x_j)\}} \frac{\partial \max_{h=1,\dots,r} \gamma_{k'_h}(x_{k'_h})}{\partial \lambda_{k'_h}}_{|\lambda_{k'_h}=\lambda^c} d\prod_i F(x_i) \\
&= \sum_{h=1,\dots,r} \left(m_{k'_h} + \int_0^{\bar{x}^c} \int_{x_{-k'_h}: \gamma_{k'_h}(s) > \max\{0, \max_{j\neq k'_h} \gamma_j(x_j)\}} \frac{\partial \gamma_{k'_h}(s)}{\partial \lambda_{k'_h}}_{|\lambda_{k'_h}=\lambda^c} f(s) d\prod_{j\neq k'_h} F(x_j) ds \right) \\
&+ F(\bar{x}^c)^{l-r} \frac{1-F(\bar{x}^c)^r}{1-F(\bar{x}^c)} \int_{\bar{x}^c}^1 \int_{x_{-k_1\dots -k_l}\in[0,1]^{n-l}: \gamma_{k'_1}(\bar{x}^c) > \max_{j\notin\{k_1,\dots,k_l\}} \gamma_j(x_j))} \frac{\partial \gamma_{k'_1}(s)}{\partial \lambda_{k'_1}}_{|\lambda_{k'_1}=\lambda^c} f(s) d\prod_{j\notin\{x_{k_1},\dots,x_{k_l}\}} F(x_j) ds = \\
&= \sum_{h=1,\dots,r} m_{k'_h} + \sum_{h=1,\dots,r} \int_0^{\bar{x}^c} \int_{x_{-k'_h}\in[0,1]^{n-1}: \gamma_{k'_h}(s) > \max\{0, \max_{j\neq k'_h} \gamma_j(x_j)\}} d\prod_{j\neq k'_h} F(x_j) ds \\
&- \bar{x}^c F(\bar{x}^c)^{l-r} \frac{1-F(\bar{x}^c)^r}{1-F(\bar{x}^c)} \int_{x_{-k_1\dots -k_l}\in[0,1]^{n-l}: \gamma_{k'_1}(\bar{x}^c) > \max_{j\notin\{k_1,\dots,k_l\}} \gamma_j(x_j)} d\prod_{j\notin\{x_{k_1},\dots,x_{k_l}\}} F(x_j) ds \\
&= \sum_{h=1,\dots,r} m_{k'_h} + r\int_0^{\bar{x}^c} q_{k'_1}(s) ds - \bar{x}^c F(\bar{x}^c)^{l-r} \frac{1-F(\bar{x}^c)^r}{1-F(\bar{x}^c)} Prob[\gamma_{k'_1}(\bar{x}^c) > \max_{j\notin\{k_1,\dots,k_l\}} \gamma_j(x_j)] \qquad (63)
\end{aligned}
$$

Note that the first equality in (63) holds by definition.

The factor $1-F(\bar{x}^c)^r$ in the last term after the second equality reflects conditioning on the event that at least one of the bidders in $J = \{k'_1, ..., k'_r\}$ has a value above $\bar{x}^c$, and the factor $F(\bar{x}^c)^{l-r}$ reflects conditioning on the event that the bidders in $C(\bar{x}^c) \setminus J$ have values below $\bar{x}^c$. We use $\gamma_{k'_1}(s)$ as the integrand in this term, because $\gamma_{k'_1}(s) = \gamma_{k'_h}(s)$ for all $h \in \{1, ..., r\}$.

To obtain the third equality we use the definition (15) and, in particular, $\frac{\partial \gamma_{k'_1}(s)}{\partial \lambda_{k'_1}} = \frac{1}{f(\bar{x}^c)}$ if $s < \bar{x}^c$ and $\frac{\partial \gamma_{k'_1}(s)}{\partial \lambda_{k'_1}} = -\frac{\bar{x}^c}{1-F(\bar{x}^c)}$ if $s > \bar{x}^c$. The final equality uses Lemma 4.

By a similar computation, we obtain: $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_J)}{\partial\epsilon}_{|\epsilon=0-}=$

$$\sum_{h=1,...,r} m_{k'_h} + \int_{x:\ \max_{h\in\{1,...,r\}}\gamma_{k'_h}(x_{k'_h})\geq\max\{0,\max_{j\notin\{k'_1,...,k'_r\}}\gamma_j(x_j)\}} \frac{\partial \max_{h=1,...,r}\gamma_{k'_h}(x_{k'_h})}{\partial\lambda_{k'_h}}_{|\lambda_{k'_h}=\lambda^c} d\prod_i F(x_i)$$

$$= \sum_{h=1,...,r}\left(m_{k'_h} + \int_0^{\bar{x}^c}\int_{x_{-k'_h}:\gamma_{k'_h}(s)\geq\max\{0,\max_{j\neq k'_h}\gamma_j(x_j)\}} \frac{\partial\gamma_{k'_h}(s)}{\partial\lambda_{k'_h}}_{|\lambda_{k'_h}=\lambda^c} f(s)d\prod_{j\neq k'_h}F(x_j)ds\right)$$

$$+\frac{1-F(\bar{x}^c)^r}{1-F(\bar{x}^c)}\int_{\bar{x}^c}^1\int_{x_{-k_1...-k_l}\in[0,1]^{n-l}:\gamma_{k'_1}(\bar{x}^c)\geq\max_{j\notin\{k_1,...,k_l\}}\gamma_j(x_j))} \frac{\partial\gamma_{k'_1}(s)}{\partial\lambda_{k'_1}}_{|\lambda_{k'_1}=\lambda^c} f(s)d\prod_{j\notin\{x_{k_1},...,x_{k_l}\}}F(x_j)ds =$$

$$= \sum_{h=1,...,r} m_{k'_h} + \sum_{h=1,...,r}\int_0^{\bar{x}^c}\int_{x_{-k'_h}\in[0,1]^{n-1}:\gamma_{k'_h}(s)\geq\max\{0,\max_{j\neq k'_h}\gamma_j(x_j)\}} d\prod_{j\neq k'_h}F(x_j)ds$$

$$-\bar{x}^c\frac{1-F(\bar{x}^c)^r}{1-F(\bar{x}^c)}\int_{x_{-k_1...-k_l}\in[0,1]^{n-l}:\gamma_{k'_1}(\bar{x}^c)\geq\max_{j\notin\{k_1,...,k_l\}}\gamma_j(x_j)} d\prod_{j\notin\{x_{k_1},...,x_{k_l}\}}F(x_j)ds$$

$$= \sum_{h=1,...,r} m_{k'_h} + r\int_0^{\bar{x}^c} q_{k'_1}(s)ds - \bar{x}^c\frac{1-F(\bar{x}^c)^r}{1-F(\bar{x}^c)}Prob[\gamma_{k'_1}(\bar{x}^c) > \max_{j\notin\{k_1,...,k_l\}}\gamma_j(x_j)] \tag{64}$$

The only difference between $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_J)}{\partial\epsilon}_{|\epsilon=0-}$ in (64) and $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_J)}{\partial\epsilon}_{|\epsilon=0+}$ in (63) is that the factor $F(\bar{x}^c)^{l-r}$ in the very last term of $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_J)}{\partial\epsilon}_{|\epsilon=0+}$ does not appear in the corresponding term of $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_J)}{\partial\epsilon}_{|\epsilon=0-}$ in (64). This is due to the fact that a negative variation ($\epsilon<0$) in λ^c does increase the value of $\gamma_{k'_h}(x)$ for $x>\bar{x}^c$, and so $\max_{h\in\{1,...,l\}}\gamma_{k_h}(x)$ changes irrespective of whether the maximal value among the $l-r$ bidders in the cluster $C(\bar{x}^c)$ who are not in set J is above or below $\bar{x}^c$. On the other hand, a positive variation ($\epsilon>0$) in λ^c does decrease the value of $\gamma_{k'_h}(x)$ for $x>\bar{x}^c$, and so the $\max_{h\in\{1,...,l\}}\gamma_{k_h}(x)$ changes only if the maximal value among the other $l-r$ bidders in the cluster is below $\bar{x}^c$. The latter occurs with probability $F(\bar{x}^c)^{l-r}$, the factor in the very last term of $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_J)}{\partial\epsilon}_{|\epsilon=0+}$.

Note that the last equality in both (63) and (64) uses the expression for $q_{k'_h}(x)$ in (17) in Lemma 4. Namely, for almost all $s\in[0,\bar{x}^c)$, the set of $x_{-k'_h}$ such that $\gamma_{k'_h}(s)=\max_{j\neq k'_h}\gamma_j(x_j)$ has measure zero. Therefore, for all $h\in\{1,...,r\}$ we have: $q_{k'_h}(s)=q_{k'_1}(s)=$
$\int_{x_{-k'_h}\in[0,1]^{n-1}:\gamma_{k'_h}(s)\geq\max\{0,\max_{j\neq k'_h}\gamma_j(x_j)\}} dF(x_{-k'_h}) = \int_{x_{-k'_h}\in[0,1]^{n-1}:\gamma_{k'_h}(s)>\max\{0,\max_{j\neq k'-h}\gamma_j(x_j)\}} dF(x_{-k'_h})$.
Likewise, $Prob[\gamma_{k'_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j(x_j)] = Prob[\gamma_{k'_1}(\bar{x}^c)\geq\max_{j\notin\{k_1,...,k_l\}}\gamma_j(x_j)]$.

Next, we need to consider all possible subsets J of the cluster $C(\bar{x}^c)=\{k_1,...,k_l\}$, since the first-order conditions $\frac{\partial g(\lambda+\epsilon\mathbb{I}_J)}{\partial\epsilon}_{\epsilon=0+}\geq 0$ and $\frac{\partial g(\lambda+\epsilon\mathbb{I}_J)}{\partial\epsilon}_{\epsilon=0-}\leq 0$ have to hold for any J.

First, suppose that $J=\{k_1,...,k_l\}$. In this case (63) and (64) are equal to each other and so we have:

$$\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_{\{k_1,...,k_l\}}))}{\partial\epsilon}_{|\epsilon=0+}=\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_{\{k_1,...,k_l\}}))}{\partial\epsilon}_{|\epsilon=0-}=$$
$$\sum_{h=1,...,l}m_{k_h}+l\int_0^{\bar{x}^c}q_{k_1}(s)ds-\bar{x}^c\frac{1-F(\bar{x}^c)^l}{1-F(\bar{x}^c)}Prob[\gamma_{k_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j(x_j)]=0$$

This yields equation (25).

To obtain (26), note that by (63) $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_{\{k_{r+1},...,k_l\}}))}{\partial\epsilon}_{|\epsilon=0+}\geq 0$ can be rewritten as:

$$\frac{m_{k_{r+1}}+...+m_{k_l}}{l-r}-\bar{x}^c\frac{F(\bar{x}^c)^r}{l-r}\frac{1-F(\bar{x}^c)^{l-r}}{1-F(\bar{x}^c)}Prob[\gamma_{k_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j(x_j)]\geq-\int_0^{\bar{x}^c}q_{k_1}(s)ds,$$

while by (64) $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_{\{k_1,...,k_r\}}))}{\partial\epsilon}_{|\epsilon=0-}\leq 0$ can be rewritten as follows:

$$\frac{m_{k_1}+...+m_{k_r}}{r}-\bar{x}^c\frac{1}{r}\frac{1-F(\bar{x}^c)^r}{1-F(\bar{x}^c)}Prob[\gamma_{k_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j(x_j)]\leq-\int_0^{\bar{x}^c}q_{k_1}(s)ds.$$

Combining the last two inequalities yields (26) for any $r\in\{2,...,l-1\}$.

To complete the proof let us show that (25) and (26) together imply (63) and (64). Let us fix the size $\#J=r$ of J. By inspection of (63), if $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_J))}{\partial\epsilon}_{|\epsilon=0+}\geq 0$ for $J=\{k_{l-r+1},...,k_l\}$ i.e., J including r lowest-budget bidders from $C(\bar{x}^c)$, then this condition also holds for any other J of size r. Likewise, by inspection of (64), if $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_J))}{\partial\epsilon}_{|\epsilon=0-}\leq 0$ for $J=\{k_1,...,k_r\}$ i.e., J including r highest-budget bidders from $C(\bar{x}^c)$, then this condition also holds for any other J of size r.

Therefore, it is sufficient to show that (63) holds for the subsets J of $C(\bar{x}^c)$ of size $r\in\{2,...,l-1\}$ consisting of the lowest-budget bidders i.e. for $J=\{k_{l-r+1},...,k_l\}$,. Similarly, it is sufficient to show that (64) holds for the subsets J consisting of $l-r$ bidders $k_1,...,k_{l-r}$ who have the highest budgets in the cluster $k_1,...,k_l$.

To obtain (63), we use (25) and (26) to yield:

$$\frac{\left(m_{k_{l-r+1}}+...+m_{k_l}\right)l}{l-r}\geq-\frac{\bar{x}^c}{1-F(\bar{x}^c)}\left(1-F(\bar{x}^c)^r-rF(\bar{x}^c)^r\frac{1-F(\bar{x}^c)^{l-r}}{l-r}\right)Prob.[\gamma_{k_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j]$$
$$+\sum_{j=1,...,l}m_{k_j}=-\frac{\bar{x}^c}{1-F(\bar{x}^c)}\left(1-F(\bar{x}^c)^r-rF(\bar{x}^c)^r\frac{1-F(\bar{x}^c)^{l-r}}{l-r}\right)Prob.[\gamma_{k_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j]$$
$$+\bar{x}^c\frac{1-F(\bar{x}^c)^l}{1-F(\bar{x}^c)}Prob.[\gamma_{k_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j]-l\int_0^{\bar{x}^c}q_{k_1}(s)ds=$$
$$=\frac{l}{l-r}\bar{x}^cF(\bar{x}^c)^r\frac{1-F(\bar{x}^c)^{l-r}}{1-F(\bar{x}^c)}Prob.[\gamma_{k_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j]-l\int_0^{\bar{x}^c}q_{k_1}(s)ds\qquad(65)$$

The inequality in (65) holds by (26), the first equality holds by (25), the second equality holds by rearrangement and implies that (63) holds.

Similarly, we have:

$$\frac{(m_{k_1}+...+m_{k_r})\,l}{r} \le \bar{x}^c\left((l-r)\frac{1-F(\bar{x}^c)^r}{(1-F(\bar{x}^c))\,r} - \frac{F(\bar{x}^c)^r\left(1-F(\bar{x}^c)^{l-r}\right)}{(1-F(\bar{x}^c))}\right)Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j\notin\{k_1,...,k_l\}}\gamma_j]$$
$$+\sum_{j=1,...,l} m_{k_j} = \frac{\bar{x}^c}{1-F(\bar{x}^c)}\left((l-r)\frac{1-F(\bar{x}^c)^r}{r} - F(\bar{x}^c)^r\left(1-F(\bar{x}^c)^{l-r}\right)\right)Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j\notin\{k_1,...,k_l\}}\gamma_j]$$
$$+\bar{x}^c\frac{1-F(\bar{x}^c)^l}{1-F(\bar{x}^c)}Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j\notin\{k_1,...,k_l\}}\gamma_j] - l\int_0^{\bar{x}^c} q_{k_1}(s)ds =$$
$$= \frac{l}{r}\bar{x}^c\frac{1-F(\bar{x}^c)^r}{1-F(\bar{x}^c)}Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j\notin\{k_1,...,k_l\}}\gamma_j] - l\int_0^{\bar{x}^c} q_{k_1}(s)ds \quad (66)$$

The inequality in (66) holds by (26). The first equality holds by (25). The second equality holds by rearrangement and implies that (64) holds. *Q.E.D.*

Proof of Theorem 4: **"Only if" Part (Necessity):** Suppose that the optimal mechanism is a top auction with threshold $\bar{x}_1 = ...\bar{x}_n = \bar{x}^t$ and reservation value r_t. By Theorem 3, the top auction has to satisfy condition (26).

By Definition 1, in the top auction we have: $q_i(s) = 0$ for $x_i < r_t$, $q_i(x_i) = F^{n-1}(x_i)$ for $x_i \in [r_t, \bar{x}^t)$, and (29), (30) and (31) hold. Substituting this into (26) we obtain (32).

"If" Part (Sufficiency): Suppose that condition (32) holds for all $k \in \{1, ..., n-1\}$ and $\bar{x}^t$ defined by (29). By inspection (29) is equivalent to (25) and (32) is equivalent to (26) in Theorem 3 when $\bar{x}^c = \bar{x}^t$ and the number of bidders in the cluster l is equal to n. Since conditions (25) and (26) are necessary and sufficient for the optimality of a mechanism, we conclude that the top auction is optimal. *Q.E.D.*

Proof of Theorem 5:

Theorem 2 shows that the optimal mechanism is uniquely defined by the vector of thresholds $(\bar{x}_1, ..., \bar{x}_n)$. By Theorem 4, the failure of (32) implies that we cannot have $\bar{x}_1 = ... = \bar{x}_n$ in the optimal mechanism. By Lemma 6, $\bar{x}_1 \ge \bar{x}_2 \ge ... \ge \bar{x}_n$, which establishes the ordering of the thresholds and the fact that $\bar{x}_i > \bar{x}_{i+1}$ for some i.

By Theorem 3, this vector of thresholds is uniquely defined by conditions (24), (25) and (26) in which the probabilities of trading $q_i(x_i)$ are given by (17) in Lemma 4 for $x_i \in [0, \bar{x}_i]$

when $\bar{x}_i \neq \bar{x}_j$ for all $j \neq i$, and for $x_i \in [0, \bar{x}_i)$ when $\bar{x}_i = \bar{x}_j$ for some $j \neq i$, with $\gamma_i(x)$ given by (15) and λ_i characterized in Theorem 2. *Q.E.D.*

Proof of Lemma 8: Since the bidder's valuations are identically distributed, the seller's revenue function $\pi(m_1, ..., m_n)$ is exchangeable i.e. $\pi(m_1, ..., m_n) = \pi(P(m_1, ..., m_n))$ where $P(m_1, ..., m_n)$ is a permutation of $(m_1, ..., m_n)$. Let the set of the permutations of $(m_1, ..., m_n)$ be denoted by PM^m. Its cardinality (the total number of permutations) is equal to $n!$.

Fixing some budget profile $(m_1, ..., m_n)$ such that $\sum_i m_i = M$, by concavity of $\pi(.)$ we obtain:

$$\pi\left(\frac{1}{M}, ..., \frac{1}{M}\right) \geq \sum_{P \in PM^m} \frac{\pi(P)}{\#PM^m} = \pi(m_1, ..., m_n)$$

Q.E.D.

References

Balakrishnan, A.V. (1993), "Applied Functional Analysis," Oxford University Press.

Bertsekas, D. (2001), "Convexity, Duality, and Lagrange Multipliers," mimeo, MIT.

Border, K.C. (2007), "Reduced Form Auctions Revisited," *Economic Theory,* 31, 167181.

Boyd, S. and L. Vandenberghe (2009), "Convex Optimization," Cambridge University Press.

Burkett, J. (2016), "Optimally Constraining a Bidder Using a Simple Budget," *Theoretical Economics,* 11(1), 133-155.

Che, Y-K. and I. Gale, (1996), "Expected Revenue of All-pay Auctions and First-Price Sealed-Bid Auctions With Budget Constraints," *Economics Letters,* 50, 373-379.

Che, Y-K. and I. Gale, (1998), "Standard Auctions with Financially Constrained Bidders," *Review of Economic Studies,* 65(1), 1-21.

Che, Y-K. and I. Gale, (2000), "The Optimal Mechanism for Selling to a Budget-Constrained Buyer," *Journal of Economic Theory,* 92(2), 198-233.

Che, Y-K., I. Gale, and J.Kim(2013), "The Optimal Mechanism for Selling to a Budget-Constrained Buyer," *Review of Economic Studies,* 80 ,73107

Deneckere, R. and S. Severinov (2009), "Multi-Dimensional Screening: A Solution to a Class of Problems, " mimeo, University of British Columbia.

Hafalir,I., R. Ravi and A.Sayedi (2012), "A Near Pareto Optimal Auction With Budget Constraints," *Games and Economic Behavior*, 74, 699-708.

Laffont, J-J., and J. Robert (1996), "Optimal Auction With Financially Constrained Buyers," *Economics Letters*, 52, 181-186.

Malakhov, A. and R. Vohra, (2008), "Optimal Auctions for Asymmetrically Budget Constrained Bidders," *Review of Economic Design,* 12(4), 245-257.

Maskin, E., (2000), "Auctions, Development, and Privatization: Efficient Auctions with Liquidity-Constrained Buyers," *European Economic Review,* 44, 667-681.

Myerson, R. B. (1979), "Incentive Compatibility and the Bargaining Problem," Econometrica, 47, 61-73.

Myerson, R. B. (1981), "Optimal Auction Design," Mathematics of Operations Research, 6(1), 58-73.

Pai, M. and R. Vohra, (2014), "Optimal Auctions with Financially Constrained Buyers," *Journal of Economic Theory,* 150, 383-425.

Pitchik, C. (2009), "Budget-Constrained Sequential Auctions With Incomplete Information," *Games and Economic Behavior,* 66(2), 928-949.

Rothkopf, M (2007), "Thirteen Reasons Why the Vickrey-Clarke-Groves Process Is Not Practical," *Operations Research*, 55(2), 191197

Online Appendix

Fixed-Point Proof of the Existence of a Budget-Handicap Auction Mechanism with Two Bidders.

Let us show that the system of two equations (35) and (36) has a solution $(\bar{x}_1, \bar{x}_2)$. First, let $\hat{x}_1$ be the unique solution in x to:

$$m_1 = x - \int_{r:r=\frac{\left(1-F(r)+\frac{(1-F(x))^2}{1-F(x)+xf(x)}\right)}{f(r)}}^{x} F(s)ds \tag{67}$$

Such $\hat{x}_1$ exists, is unique and lies in $(\bar{x}^t, 1)$. This follows from three facts: (i) the right-hand side of (67) is increasing in x, (ii) equation (34) and $m_1 - m_2 > \bar{x}(1 - F(\bar{x}))$ imply that $m_1 > \bar{x}^t - \int_{r:r=\frac{1-F(r)-\frac{(1-F(\bar{x}))^2}{1-F(\bar{x}^t)+\bar{x}^t f(\bar{x}^t)}}{f(r)}}^{\bar{x}^t} F(x)dx$; and (iii) $m_1 < 1 - \int_{r':r'=\frac{1-F(r')}{f(r')}}^{1} F(x)dx$ by asssumption.

Next, let us construct a mapping $(\tilde{x}_1(.), \tilde{x}_2)(.)) : \{(x_1, x_2)|x_1 \in [m_1, \hat{x}_1],\ x_2 \in [m_2, x_1]\} \mapsto \{(x_1, x_2)|x_1 \in [m_1, \hat{x}_1],\ x_2 \in [m_2, x_1]\}$ as follows:

$$\tilde{x}_1(x_1, x_2) = m_1 + \int_0^{x_1} \int_{y-\frac{F(\bar{x}_1)+f(\bar{x}_1)\left(\bar{x}_1-\bar{x}_2+\frac{\bar{x}_2(1-F(\bar{x}_2))}{1-F(\bar{x}_2)+\bar{x}_2 f(\bar{x}_2)}\right)-F(y)}{f(y)} \geq \max\left\{0, s-\frac{1-F(s)-\frac{(1-F(x_2))^2}{(1-F(x_2)+x_2 f(x_2))}}{f(s)}\right\}} dF(\tag{68}$$

$\tilde{x}_2(x_1, x_2) = \min\{\tilde{x}_1(x_1, x_2), z(x_1, x_2)\}$ where

$$z(x_1, x_2) = \frac{1}{F(x_1)} m_2 + \int_0^{x_2} \int_{s-\frac{1-F(s)-\frac{(1-F(x_2))^2}{(1-F(x_2)+x_2 f(x_2))}}{f(s)} \geq \max\left\{0, y-\frac{F(\bar{x}_1)+f(\bar{x}_1)\left(\bar{x}_1-\bar{x}_2+\frac{\bar{x}_2(1-F(\bar{x}_2))}{1-F(\bar{x}_2)+\bar{x}_2 f(\bar{x}_2)}\right)-F(y)}{f(y)}\right.}} \tag{69}$$

The right-hand side of (68) is increasing in x_2 and, since $x_2 \leq x_1$, does not exceed $\hat{x}_1$ defined in (67). Hence, $\tilde{x}_1(x_1, x_2) \in [m_1, \hat{x}_1]$ for all $(x_1, x_2) \in [m_1, \hat{x}_1] \times [m_2, x_1]$. Also, by (69), $\tilde{x}_2(x_1, x_2) \in [m_2, \hat{x}_1(x_1, x_2)]$ for all $(x_1, x_2) \in [m_1, \hat{x}_1] \times [m_2, x_1]$. Since the right-hand sides of (68) and (69) are continuous in (x_1, x_2), by Brower's fixed point theorem there exists $(\bar{x}_1, \bar{x}_2)$ such that $\bar{x}_1 = \tilde{x}_1(\bar{x}_1, \bar{x}_2)$, $\bar{x}_2 = \tilde{x}_2(\bar{x}_1, \bar{x}_2)$.

Finally, let us show that $\bar{x}_2 < \bar{x}_1$ and so $\bar{x}_2 = z(\bar{x}_1, \bar{x}_2)$. Suppose otherwise i.e., $\bar{x}_2 = \bar{x}_1$. In this case, $\bar{x}_1$ has to solve (67) for x, so we obtain $\bar{x}_2 = \bar{x}_1 = \hat{x}_1$. Then (69) becomes

$$m_2 = z(\hat{x}_1, \hat{x}_1)F(\hat{x}_1) - \int_{r:r=\frac{1-F(r)\frac{(1-F(\hat{x}_1))^2}{1-F(\hat{x}_1)+\hat{x}_1 f(\hat{x}_1)}}{f(r)}}^{\hat{x}_1} F(s)ds \tag{70}$$

Using equation (70), let us show that $z(\hat{x}_1, \hat{x}_1) < \hat{x}_1$. To see this, first note that $m_2 < \bar{x}^t F(\bar{x}^t) - \int_{r:r=\frac{1-F(r)-\frac{(1-F(\bar{x}^t))^2}{1-F(\bar{x}^t)+\bar{x}^t f(\bar{x}^t)}}{f(r)}}^{\bar{x}^t} F(x)dx$. This follow from $m_1 - m_2 > \bar{x}^t(1 - F(\bar{x}^t))$ and the fact that $\bar{x}^t$ satisfies (34). Further, because $\hat{x}_1 > \bar{x}^t$ and $xF(x) - \int_{r:r=\frac{1-F(r)-\frac{(1-F(x))^2}{1-F(x)+xf(x)}}{f(r)}}^{x} F(s)ds$ is increasing in x, we obtain:

$$m_2 < \hat{x}_1 F(\hat{x}_1) - \int_{r:r=\frac{1-F(r)-\frac{(1-F(\hat{x}_1))^2}{1-F(\hat{x}_1)+\hat{x}_1 f(\hat{x}_1)}}{f(r)}}^{\hat{x}_1} F(s)ds.$$

The last inequality implies that $z(\hat{x}_1, \hat{x}_1) < \hat{x}_1$. So, $(\hat{x}_1, \hat{x}_1)$ is not a fixed point, and hence the fixed point solving the system of equations (35) and (36) satisfies $\bar{x}_2 < \bar{x}_1$ and so $\bar{x}_2 = z(\bar{x}_1, \bar{x}_2)$. *Q.E.D.*