Identification in Auction Models with Interdependent Costs*

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Abstract. This paper provides a positive identification result for procurement models with asymmetric bidders, statistically dependent private information, and interdependent costs. When bidders are risk neutral, the model's payoff-relevant primitives are: (i) the joint distribution of private information and (ii) each bidder's full-information expected cost—the expected cost conditional on own and competitors' information. These primitives are nonparametrically identified from the distribution of bids conditional on observable cost shifters under the following four assumptions. First, each bidder's private information can be summarized by a real-valued signal. Second, the joint distribution of bidders' signals does not depend on cost shifters. Third, each bidder's full-information cost depends on own cost shifters but not on competitors'. Fourth, the observed data are generated by the repeated play of the same equilibrium where bidders use monotone pure strategies. I illustrate how the identification argument is useful for estimation using data from Highway Procurements in Michigan. The estimates are used to evaluate policies that reduce the severity of the winner's curse by restricting participation.

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1 Introduction

In procurement auctions with interdependent costs the information about each bidder's cost is scattered among all bidders. Bidders should realize that the result of the auction is informative about competitors' information and that they win the projects deemed too costly or undesirable by other participants. This adverse selection phenomenon is usually referred to as the winner's curse in the auctions literature. The effect of procurement policies on equilibrium behavior and outcomes depends crucially on the nature and extent of the winner's curse and on how bidders adjust their bids in response to it.

While the interdependent costs model has been analyzed extensively in the theoretical literature, the empirical literature has mainly analyzed two polar cases. In one extreme, there is the private costs model where each bidder knows its own costs and there is no room for the winner's curse. In the other extreme, there is the pure common cost model where all bidders would incur in exactly the same cost if they win, but they are uncertain about it. Intermediate cases have received little empirical attention mainly because of the lack of positive identification results.[1] In fact, Laffont and Vuong (1996) provide a negative result. They show that any joint distribution of bids that is rationalizable by an interdependent costs model is also rationalizable by some private costs model. This paper provides a positive identification result using variation in competitive conditions.

One of the main insights of the theoretical analysis of models with interdependent costs is that changes in competitive conditions have different effects on equilibrium bid strategies depending on the true information structure. In models with strong interdependence, the effect of the winner's curse is important and bidders may bid less aggressively in more competitive environments. If interdependence is weak or inexistent, so is the effect of the winner's curse and bidders may bid more aggressively is more competitive environments. Haile, Hong, and Shum (2004) use these insights to develop a test for the null hypothesis of private costs against an alternative of interdependence exploiting variation in the number of participants. In this paper, I assume richer variation in competitive conditions generated by cost shifters that take over a continuum of values, e.g., distance from each bidder's plant to the project location.

The objects to be identified are the primitives of the Bayesian game that represents the procurement. When bidders are risk neutral and the rules of the auction are known, the game is fully defined by (i) the distribution of bidders' private information, and (ii) each bidder's full information cost: a function that returns the expected completion cost conditional on own and competitors' private information. While these payoff-relevant primitives are coarser than the joint distribution of costs and private information—regarded as the true primitive by Athey and Haile (2002, 2007)—they are sufficient to analyze the effects of many relevant policy changes (e.g., rules of the auction,

[1]The theoretical literature includes Athey (2001); Reny and Zamir (2004); McAdams (2003, 2007); Maskin and Riley (2000). Laffont, Ossard, and Vuong (1995); Athey, Levin, and Seira (2011); Roberts and Sweeting (2013); Krasnokutskaya (2011) are examples within the private values/cost paradigm, while Athey and Levin (2001a); Bajari and Hortasu (2003); Hendricks and Porter (1988); Hendricks, Pinkse, and Porter (2003a) are examples within common values/cost paradigm. Hong and Shum (2001) is one of the few empirical papers that estimate a model with interdependent costs.

reserve prices, subsidies) on outcomes such as bidding behavior, project allocation and prices.

The payoff-relevant primitives are nonparametrically identified from the distribution of bids conditional on observable cost shifters under four assumptions on technology and information. First, each bidder's private information can be summarized by a real-valued random variable: a signal. Second, the joint distribution of bidders' signals does not depend on the cost shifters. Third, each bidder's cost shifter affects his own full-information cost but not his competitors'. Fourth, the observed data are generated by the repeated play of the same equilibrium where bidders use monotone pure strategies.

The restrictions on the auction format are that bidders submit simultaneous or sealed bids, the project is awarded to the bidder who submits the lowest bid and the payment that each bidder receives is given by a publicly known function of all the bids submitted to the auction. The description encompasses a wide variety of sealed bid auctions including first-price, second-price and all-pay. It also allows for preferential treatments, bid discounts and subsidies.

The point identification result in the paper requires that observed cost shifters induce enough variation in the observed conditional distribution of bids and that the observed distribution of bids satisfy other conditions that ensure that no information is lost due to equilibrium strategies. If the conditions fail, it will still be possible to obtain some informative nonparametric bounds on the primitives.

The identification result shows that the observed distribution of bids uniquely determines the payoff-relevant primitives of the model without relying on further distributional or functional form assumptions. Even in the simplest model with two bidders, the full-information cost function has three arguments: own signal, competitor's signal and own cost shifter. Nonparametric estimators of this object may suffer from the curse of dimensionality. Applied researchers may employ some convenient parametric and distributional assumptions to address the curse of dimensionality and use variation in cost shifters to estimate the parameters that determine the magnitude of cost interdependences. I illustrate how the identification arguments can be used for estimation in an application to first-price Highway Procurements in Michigan using bidder's distance to the project as the bidder-specific cost shifter. The estimation method is semiparametric and has multiple steps (Hubbard, Li, and Paarsch, 2012; Aryal, Gabrielli, and Vuong, 2014; Campo, Guerre, Perrigne, and Vuong, 2011). The first step consists in estimating the joint distribution of signals semiparametrically as in Hubbard, Li, and Paarsch (2012). The second step uses the bidder's first-order condition to recover the marginal cost that rationalizes each observed bid as in Campo, Perrigne, and Vuong (2003); the marginal cost is the expected full information cost conditional on the event where the observed bid is pivotal, i.e., the set of competitors' signals such that their minimum bid equals the observed bid. The third step consists in estimating competitors' pivotal signals. The fourth and final step uses instrumental variables quantile regression methods (Chernozhukov and Hansen, 2006) to estimate the parameters of the full information costs.

The full information cost estimates point to a structure that lies between the two polar paradigms of private and pure common costs. The effect of competitors' signals is statistically significant but of smaller magnitude than the effect of own signals. To illustrate the economic importance of the estimated degree of interdependence and the implied winner's curse, I perform a series of counterfactuals where the Department of Transportation restricts participation. Restricting participation reduces the severity of the winner's curse and, under some conditions, it may even reduce procurement costs (Matthews, 1984; Hong and Shum, 2002). The estimates suggest that the effect of the winner's curse is strong enough that bidders bid more aggressively when participation is restricted, but not enough to generate cost savings. Even if each individual participant submits a lower bid, the minimum bid still increases as the minimum is taken over a smaller set of bids.

This paper contributes to the literature on identification of auction models. The initial contributions focused on the private values model with independent (Guerre, Perrigne, and Vuong, 2000) or affiliated private information (Li, Perrigne, and Vuong, 2002; Campo, Perrigne, and Vuong, 2003). Laffont and Vuong (1996) showed that the affiliated or interdependent value model is not identified from the joint distribution of bids. Li, Perrigne, and Vuong (2000) studied the conditionally independent private information model and showed identification of the two polar cases of private and pure common values. Fevrier (2008) considered a particular class pure common values model. More recently, the literature has focused on cases where the researcher has access to the distribution of bids conditional on other covariates that introduce some exogenous variation in competitive or informational conditions. This additional source of variation can be used to test the null hypothesis of private values (Haile, Hong, and Shum, 2004; Hortasu and Kastl, 2012), and the null of pure common values (Hendricks, Pinkse, and Porter, 2003b, footnote 2). It can also be used to identify attitudes towards risk (Guerre, Perrigne, and Vuong, 2009), correlated private values in ascending auctions (Aradillas-Lopez, Gandhi, and Quint, 2013), and a selective entry process (Gentry and Li, 2012). In this paper, I show that variation in cost shifters can be used to identify all the payoff-relevant characteristics of the interdependent values model with risk-neutral bidders.

The rest of the paper is organized as follows. Section 2 describes the general interdependent cost framework. Section 3 shows the main identification results. Section 4 presents an application using highway procurement data from Michigan. The last section concludes.

2 The interdependent cost model

2.1 Primitives

An auctioneer procures the completion of a project and runs a sealed-bid auction between n risk-neutral bidders. The cost to bidder i is a random variable C_i and his information is summarized by a signal S_i. I will adhere to the convention of using upper-case and lower-case letters to denote random variables and their realized values, respectively. At the time of the auction, i knows s_i, the realization of his own signal, but is uncertain about the realization of the vector of competitors' signals $S_{-i} = [S_j]_{j \neq i}$ and own future project completion cost C_i. In other words, each bidder knows his own information but does not know his competitors'; moreover, his information only allows him to make an imperfect forecast of his own costs. Denote the full random vector of signals by $S = [S_i]_{i=1}^n$, and the vector of costs by $C = [C_i]_{i=1}^n$.

All bidders have access to the following public information: bidder-specific cost shifters $[x_1, x_2, ..., x_n]$, and a set of observable auction characteristics x_0. All public information is denoted by $x = [x_0, x_1, x_2, ..., x_n] \in X^o \subset \mathbb{R}^m$, $m \geq n+1$. Cost shifters and auction characteristics are not necessarily one-dimensional. For example, the cost shifter of bidder i may include his distance to the project site and publicly observable predictors of his capacity constraints. Auction characteristics x_0 may include publicly available estimates of the cost and duration of the project. Observed heterogeneity x_0 can be conditioned upon and omitted from notation.

The primitives of the model are the joint distribution of costs and signals conditional on public information. Its cumulative distribution function (CDF) is denoted by $F_{C,S|x}$, or F for short. The CDF of i's completion cost given his information at the time of the auction is $F_{C_i|s_i,x}$ and its expectation is $E(C_i|s_i, x)$. If i learns all competitors' signals, the CDF of his costs becomes $F_{C_i|s_{-i},s_i,x}$ with expectation $E(C_i|s_{-i}, s_i, x)$. This expectation is bidder i's full-information expected completion cost or, for short, full-information cost. All these distributions and expectations of costs are uniquely determined by the primitive F.

The identification results in this paper require the following assumptions on bidders' technology and information:

A.1. Cost shifters and signals are independent: $F_{S|x} = F_S$.

A.2. Signals are one-dimensional random variables. The joint distribution F_S, admits a continuously differentiable density function f that satsifies the following regularity condition: for every $I = [\underline{s}_i, \overline{s}_i]$ in the support of S_i, the class of random variables

$$\left\{ Y : Y = \left(\frac{d \log f(S_{-i}|s_i)}{ds_i}^2 \middle| S_i = s_i \right), s_i \in I \right\}$$

is uniformly integrable.

A.3. The full-information cost of bidder i is $c_i(s_{-i}, s_i, x_i)$ which does not depend on x_j for all $j \neq i$, i.e.,

$$c_i(s_{-i}, s_i, x_i) := E(C_i|s_{-i}, s_i, x_i) = E(C_i|s_{-i}, s_i, x), \tag{1}$$

and satisfies the following regularity conditions: for every $I = [\underline{s}_i, \overline{s}_i]$ in the support of S_i and every x_i: $c_i(s_{-i}, \cdot, x_i)$ is continuously differentiable on I, $\sup_{s_i \in I} E\left(\left| \frac{d}{ds_i} c_i(S_{-i}, s_i, x_i) \right| \middle| s_i \right) < \infty$ and $\sup_{s_i \in I} E\left(c_i(S_{-i}, s_i, x_i)^2 |s_i \right) < \infty$.

Assumptions A.1 and A.3 define the conditions that an observable auction characteristic has to satisfy to be considered a cost shifter instead of one of the characteristics in x_0. Assumption A.1 states that the joint distribution of signals do not depend on cost shifters. Assumption A.3 requires that bidders' cost shifters only affect their own full-information cost but not competitors'. For example, if x_i is distance to the project, these assumptions rule out that bidder i has systematically lower or higher costs when j is close to the project. They also rule out the case where i regards the signal of bidder j as more informative when j is close to the project.

Assumption A.2 states that bidders summarize all private information in a single dimensional variable. It rules out the possibility that bidder i receives two signals: one that is informative about his own costs (e.g. own equipment availability) and one that is informative about competitors' information and costs (e.g. conditions in the equipment rental market). The boundedness condition rules out distributions where the support of S_{-i} depends on S_i, e.g, if S_1 and S_2 are perfectly correlated.[2] With the sole purpose of simplifying notation it will be assumed that signals are marginally uniformly distributed, and therefore, the distribution F_S is a copula. This is just a normalization as signals are ordinal.

Most models considered in the single-unit auction literature are special cases of the interdependent cost model. Write $F = F_{C|S,x}F_{S|x}$. Typical models impose conditions on either $F_{C|S,x}$ or $F_{S|x}$. Conditions on $F_{C|S,x}$ determine whether the model is in the private cost paradigm. Conditions on $F_{S|x}$ determine how bidders private information is distributed.

In private costs models each bidder knows his own expected completion cost, so i's full-information cost does not depend on competitors' signals, i.e.,

$$c_i\left(s_{-i}, s_i, x_i\right) = c_i\left(s_i, x_i\right) \text{ for all } s_{-i}, s_i \text{ and } x_i. \tag{2}$$

In the traditional independent private values (costs) model, $F_{C|S,x}$ satisfies (2) and $F_{S|x}$ is the product of the marginal distributions $\left\{F_{S_i|x}\right\}_{i=1}^{n}$ (Milgrom and Weber, 1982, Section 2.1). The affiliated private values model allows signals to be affiliated (Li, Perrigne, and Vuong, 2002).[3]

In pure common costs models the cost of completing the project is common to all bidders. They all share exactly the same full-information cost; $c_i(s_{-i}, s_i) = c_j(s_{-j}, s_j)$. For example, in Rothkopf (1969) and Wilson (1977) bidders have identical completion costs ($C = C_j$ for all j) and receive a noisy signal with distribution $F_{S|C}$. This is a special case of the model in this paper where $F_{C_i|S} = F_{C_j|S}$ for all i, j. Fevrier (2008) shows identification of a Pure Common Values Model under the additional assumption that the support of the distribution of signals changes with the cost. Hendricks and Porter (1988) present and estimate a model where one bidder is privately informed about the common value of the good being auctioned and there are $n-1$ uninformed bidders. In equilibrium, uninformed bidders play mixed strategies that can be purified by a set of signals. In this case, signals are independent and all bidders have the same full information cost that depends only on the informed bidder's signal.

The General Symmetric Model in Milgrom and Weber (1982) assumes that $c_i\left(s_{-i}, s_i, x_i\right)$ is interchangeable in competitors signals, that the function c_i is identical across bidders and that the random variables (C, S) are statistically affiliated. This model includes as special cases the independent private cost and pure common cost models, as well as a range of intermediate models.

The interdependent cost model is more general than the pure common costs and the private

[2] If $f\left(S_{-i}|s_i\right)$ is interpreted as a likelihood of obtaining sample S_{-i} from the distribution of S_{-i} given a single parameter s_i, the regularity condition implies that the Fisher information contained in one sample is finite.

[3] Bidders' signals are affiliated if $f_S(s' \vee s) f_S(s' \wedge s) \geq f_S(s) f_S(s')$ for all s, s', where $\vee$ and $\wedge$ denote the component-wise maximum and minimum, respectively. Affiliation is a stronger notion of positive correlation. See Milgrom and Weber (1982) for more details.

cost paradigm. First, it allows for arbitrary interdependence. Each bidder may have a different full-information cost function which may arbitrarily depend on competitors' signals. Second, private information need not be independent. $F_{S|x}$ is not necessarily the product of the marginals). Third, bidders can be asymmetric. $F_{S|x}$ is not required to be exchangeable in its arguments and $c_i(\cdot)$ may be different than $c_j(\cdot)$). Fourth, it allows for for asymmetries introduced by observable cost shifters which will be key for the identification argument.

Example 1. *Hong and Shum (2001) use Wilson (1998) log-additive model where each bidders log costs is* $\log C_i = A_i + V$*, where* A_i *is a private component of* i*'s costs and* V *is an unknown cost component that is common across bidders. They assume that* $A_i \sim N\left(\bar{a}, \sigma_a^2\right)$ *and* $V \sim N\left(m, \sigma_v^2\right)$*. Each bidder has a noisy signal of its cost* $S_i = \log C_i + E_i$*, where* $E_i = N\left(0, \sigma_e^2\right)$*. This implies that*

$$\log C_i | s \sim N\left(\frac{\sigma_v^2}{\sigma_a^2 + \sigma_e^2 + N\sigma_v^2}\left(\bar{a} + m + \frac{\sigma_e^2}{\sigma_a^2 + \sigma_e^2}\Sigma_{j \neq i} s_i\right) + \alpha s_i, \sigma_e^2 \alpha\right)$$

where

$$\alpha = \frac{1}{N}\left(1 + \frac{(N-1)\,\sigma_a^2}{\sigma_a^2 + \sigma_e^2} - \frac{\sigma_v^2}{\sigma_a^2 + \sigma_e^2 + N\sigma_v^2}\right).$$

The model can be renormalized so that each signal has a marginal uniform distribution.

Example 2. *(Gaussian Information Structure) The log-additive model can be generalized so that bidder* i*'s cost is* $C_i = \psi_{x_i}^{-1}\left(\beta_i' V + A_i\right)$*, where* ψ_{x_i} *is a strictly monotone function that may vary with* x_i*,* V *is a vector of common costs distributed* $N(0, I)$*,* β_i *is a bidder specific vector of weights and* A_i *is a private cost component distributed independently* $N\left(0, \sigma_{ia}^2\right)$*. Each bidder's signal is* $S_i = \psi(C_i) + E_i$ *where* $E_i = N\left(0, \sigma_{ie}^2\right)$*. Notice the special cases of independent private costs when* $\beta_i = 0$ *for all* i*; the affiliated private costs when* $\sigma_{ie}^2 = 0$ *for all* i*; and the pure common values model when* $\beta_i = \beta_j$ *and* $\sigma_{ia}^2 = 0$ *for all* $i \neq j$*.*

In the most general case, the vector of Gaussian signals is $S \sim N(0, B'B + \Sigma_a + \Sigma_e)$*, where* B*'s* i*-th column is* β_i*,* Σ_a *and* Σ_e *are diagonal matrices with* σ_{ia}^2 *and* σ_{ie}^2 *in the* i*-th diagonal entry, respectively.* $\psi_{x_i}(C)\,|S \sim N(\mu S, \Sigma)$ *where*

$$\begin{aligned}
\mu &= I - \Sigma_e\left(B'B + \Sigma_a + \Sigma_e\right)^{-1} \\
\Sigma &= \left(B'B + \Sigma_a\right)\left(\left(B'B + \Sigma_a\right)^{-1} - \left(B'B + \Sigma_a + \Sigma_e\right)^{-1}\right)\left(B'B + \Sigma_a\right).
\end{aligned}$$

Therefore, the full information cost of bidder i *is a function of* $\mu_i S$ *and the* i*-th row of* Σ*. Signals can be renormalized to be marginally uniform.*

2.2 Auction Rules

The identification results apply to auctions where bidders submit simultaneous or sealed bids and the project is awarded to the bidder with the lowest bid. Let $p_i(b_i, b_{-i})$ be the function that determines i's payment given own and competitors' bids, r be the maximum price the auctioneer is willing to pay and $1(b_i \prec b_{-i})$ be an indicator that is one if and only if b_i beats bids b_{-i} given the tie-breaking rule in place.[4] In first-price auctions $p_i(b_i, b_{-i}) = \min(b_i, r)\, 1(b_i \prec b_{-i})$, in second-price auctions $p_i(b_i, b_{-i}) = \min_{j \neq i}(b_j, r)\, 1(b_i \prec b_{-i})$, and in all pay auctions $p_i(b_i, b_{-i}) = \min(b_i, 0) + 1(b_i \prec b_{-i})\, r$. The identification results in this paper only require that the set of discontinuities of $p_i(\cdot, b_{-i})$ is $\cup_{k=1}^{\infty}\{d_k(b_{-i})\}$, where functions $d_k : \mathbb{R}^{n-1} \to \mathbb{R}$ are Borel measurable. First-price, second-price and all-pay auctions satisfy this condition with $d_k(b_{-i}) = \min_{j \neq i} b_j$.

Sealed-bid auctions are modeled as Bayesian games. Each primitive F and each market configuration x define a game in which bidders are players, signals are types and bids are actions. The payoff functions and joint distribution of signals are common knowledge. The payoff function of a risk-neutral bidder is:

$$u_i(b, s, x) = p_i(b_i, b_{-i}) - c_i(s_{-i}, s_i, x_i)\, 1(b_i \prec b_{-i}). \tag{3}$$

The information structure of F is defined as the pair:

$$\left\{\{c_i(s_{-i}, s_i, x_i)\}_{i=1}^{n}, F_S\right\}. \tag{4}$$

This information structure summarizes the payoff-relevant characteristics of the primitive F. The full-information costs and the rules of the auction determine the players' payoffs and the joint distribution of signals is the joint distribution of types. All other characteristics of F are irrelevant for the description of the game. This paper studies identification of the information structure (4). Non payoff-relevant characteristics of F pose no additional restriction on observed behavior and will not be identified.

The information structure is sufficient to compute the equilibrium behavior and outcomes in many counterfactual situations. Any change in the rules of the auction that alters the payment function or the allocation rule can be represented by a Bayesian game with primitives that depend only on the identified information structure (e.g., changes in reserve prices, subsidies to some preferred bidders, participation fees, changes in auction format from first-price to second-price auction, random allocation among the two lowest bidders). As long as the counterfactual situation does not require that bidders make bidding or bargaining decisions after they learn additional information, their payoff functions still depend only on their full-information costs and payment rule. For an example where bidders do make decisions after they learn additional information, consider a counterfactual policy that allows resale or subcontracting after bidders learn their costs. Suppose that at the subcontracting stage there is no private information and that each bidder's publicly known cost is its full-information cost evaluated at the realized signals plus an idiosyncratic ex-post shock. The winner may make a take-it-or-leave-it offer to the competitor that has the lowest ex-post costs. His resale market opportunities are more profitable when the variance of the ex-post shock is larger because the expected minimum competitors' cost is lower. Because bidders should take into account the possibility of subcontracting (or resale as in Haile, 2001), this policy results

[4]The main results of this paper are valid under any tie-breaking rule as long as the winner is selected among those bidders who tied with the lowest bid.

in a more competitive auction environment if the variance of the ex-post shock is large. This variance is not an identified characteristic of the model. Therefore, it is not possible to compute this counterfactual using the identified information structure.

2.3 Observables

I consider situations where the data consist of a sample of independent auctions. In each auction there is a draw of signals and costs from F. Each bidder submits a bid after observing his own private signal and all public information. The researcher observes bids and public information.

Let B_i denote the bid made by i, $H_{B_i|x}$ its CDF conditional on public information, and H the joint distribution of bids conditional on public information (this is a shorthand notation for $H_{B|x}$). If bidder i decides not to participate in an auction, his bid is recorded as infinity; therefore, $\sup_{b_i \in \mathbb{R}} H_{B_i|x}(b_i)$ may be less than one. X^o is the observed support of x, and X_i^o that of x_i. For simplicity, assume that $X^o = X_1^o \times ... \times X_n^o$. It will be useful to define $Q_{B_i|x}$ as the right continuous quantile function of $B_i|x$ and $H_{M_i|B_i=b_i,x}$ as the CDF of $M_i = \min_{j \neq i} B_j$ conditional on $B_i = b_i, X = x$.

If bidder i plays a monotone pure strategy in every market configuration x, then his behavior is described by a bid function $\beta_i(s_i, x)$ that is increasing in its first argument. The set of bid functions $\beta = \{\beta_i\}_{i=1..n}$ describe how each bidder behaves under different private and public information. It is said that β generates H if each marginal distribution of bids $H_{B_i|x}$ is generated by repeated play of the strategy β_i, i.e., if $H_{B_i|x}(b_i) = P(\beta(S_i, x) \leq b_i)$ for all $x \in X^o$, $i \in \{1, .., n\}$ and $b_i \in \mathbb{R}$. Similarly, (β, F) generates $H \in \mathcal{H}$ if the repeated play of strategy profile β given the joint distribution of signals F_S generates the joint distribution of bids H, i.e., if $H_{B|x}(b) = P(\cap_{i=1}^n \{\beta_i(S_i, x) \leq b_i\})$ for all (x, b).

Throughout the paper it will be assumed that the observed data is generated by the repeated play of equilibrium strategies:

A.4. The observables H are generated by (β, F), where F represents the true primitives of the model and β are bid functions that constitute a Bayes Nash equilibrium in monotone pure strategies in every $x \in X^o$.

This assumption does not rule out existence of multiple equilibria, but it requires that in the game indexed by x bidders always play the same equilibrium.[5] The theoretical literature on auctions has derived sufficient conditions for existence of equilibria in monotone pure strategies (Reny and Zamir, 2004; Athey, 2001; McAdams, 2007) depending on the rules of the auction. Reny and Zamir (2004) show that if signals are affiliated, the first-price auction has an monotone equilibrium. These results do not extend to second-price auctions.

[5]Similar assumptions are required to identify and estimate dynamic models (Bajari, Benkard, and Levin, 2007).

3 Identification

Assumption A.4 provides the key links between observables and economic primitives. By monotonicity, bid functions and marginal bid distributions are inverses of each other. In particular, $\beta_i(\cdot, x) = Q_{B_i|x}(\cdot)$ at every continuity point and

$$H_{B|x}(b) = P\left(\cap_{i=1}^{n} \{\beta_i(S_i, x) \leq b_i\} |x\right) = F_{S|x}(s),$$

where $s_i = H_{B_i|x}(b_i)$ for all i. The implications of this expression are threefold. First, Assumption A.1 is testable because it implies that $H_{B|x}(b) = H_{B|x'}(b')$ whenever x, x' and b, b' are such that $H_{B_i|x}(b_i) = H_{B_i|x'}(b'_i)$ for all i. Second, $F_{S|x}(s)$ is identified if there is a vector b such that $s = \left[H_{B_i|x}(b_i)\right]_{i=1}^{n}$. Third, if equilibrium strategies are such that $\beta_i(\cdot, x)$ is constant for an interval $I \subset (0,1)$, then $F_{S|x}(s)$ is not identified whenever s_i is in the interior of I.

Another key implication of Assumption A.4 is that for every signal s_i, $b_i = Q_{B_i|x}(s_i)$ satisfies a best-response condition. This is true even at discontinuity points of $Q_{B_i|x}$ because expected costs are continuous in own signals. In second-price auctions, for example, the first-order condition of bidder i is (See Athey and Haile, 2002):

$$b_i = E\left(C_i | S_i = s_i, \min_{j \neq i} \beta_j(S_j, x) = b_i, x_i\right). \tag{5}$$

The right hand side is the expected cost conditional on submitting a pivotal bid—a bid that ties with at least one competitor for the lowest bid. The event "submitting a pivotal bid" or "tie with a competitor for the lowest bid" is $\left\{s_{-i} : \min_{j \neq i} \beta_j(s_j, x) = b_i\right\}$. This set has Lebesgue measure zero if competitors' bid functions are continuous. Therefore, it is important to define the right hand side of (5) as the limit of the conditional expectation on events with positive probability:

$$b_i = \lim_{\varepsilon \downarrow 0} E\left(C_i | S_i = s_i, S_{-i} \in L_\varepsilon(b_i, x), x_i\right), \tag{6}$$

where $L_\varepsilon(b_i, x) = \left\{s_{-i} : \min_{j \neq i} \beta_j(s_j, x) \in [b_i, b_i + \varepsilon]\right\}$.[6] Each event $L_\varepsilon(b_i, x)$ can be described in terms of the observables. Let $R(b_i, x) = \left\{s_{-i} : s_j \geq H_{B_j|x}(b_i)\right\}$ denote the set of competitors signals for which bidder i wins with bid b_i in configuration x. This set is an $n-1$ dimensional box in the space of signals. Then $L_\varepsilon(b_i, x) = R(b_i, x) \cap R(b_i + \varepsilon, x)^c$, i.e., it is the set for which i wins with bid b_i but loses with bid $b_i + \varepsilon$. If $n = 3$, this set has an L-shape in the unit square.

In first-price auctions the first-order condition becomes:[7]

$$b_i - \frac{\Pr(M_i \geq b_i | B_i = b_i, x)}{\frac{\partial}{\partial m} \Pr(M_i \geq m | B_i = b_i, x)|_{m=b_i}} = \lim_{\varepsilon \downarrow 0} E\left(C_i | S_i = s_i, S_{-i} \in L_\varepsilon(b_i, x), x_i\right). \tag{7}$$

[6] Athey and Haile (2002) describe the symmetric case where the right hand side of (5) becomes $E(C_i | S_i = s_i, \min_{j \neq i} S_j = s_i, x_i)$. This simplification only applies to the symmetric case.

[7] See Campo, Perrigne, and Vuong (2003); Athey and Haile (2002). For ease of exposition, I assume that equilibrium bid functions are differentiable. The formal proof does not use any differentiability assumption. All the results in this paper apply when bid functions are non-differentiable.

Notice that the right hand side of equations 6 and 7 are identical. In both cases, the optimality condition states that the marginal expected revenue of choosing a bid that wins with a marginally larger probability should equal the marginal cost. The marginal expected revenue is identified from observables, while the marginal cost is the expectation of $c_i(S_{-i}, s_i, x_i)$ conditional on $S_{-i} \in L$ where L is a set that can be written in terms of observables.

As noted by Haile, Hong, and Shum (2004), Assumption A.3 and the null hypothesis of private values imply that the marginal costs do not depend on x_{-i} which is a testable prediction. The identification argument in this paper shows how to recover $c_i(s_{-i}, s_i, x_i)$ exploiting rich variation in competitors' cost shifters. Intuitively, x_{-i} is an $(n-1)$-dimensional instrument that identifies the effect of S_{-i}, which has dimension $(n-1)$, on the full-information cost.

It is illustrative to consider a second-price auction between two bidders: 1 and 2. Because there is only one competitor to tie with, the event "tie with a competitor for the lowest bid" has a simple representation in the space of competitors' signals: "$S_2 = H_{B_2|x}(b_1)$", that is, the event where bidder 2 receives signal $H_{B_2|x}(b_1)$ which is the signal that prompts 2 to bid b_1. Bidder 1's best-response can be written as:

$$b_1 = E\left(C_1 | S_1 = H_{B_1|x}(b_1), S_2 = H_{B_2|x}(b_1), x_1\right) = c_1\left(H_{B_1|x}(b_1), H_{B_2|x}(b_1), x_1\right). \tag{8}$$

The identification argument is straightforward: each bid b_1 equals the full information cost of bidder 1 evaluated at $s_1 = H_{B_1|x}(b_1)$, $s_2 = H_{B_2|x}(b_1)$ and x_1, where $x = [x_1, x_2]$. Evaluating this expression for different values of (x_2, b_1) results in the full-information cost evaluated at different pairs of signals while x_1 is held constant. The exclusion restriction A.3 avoids confounding the effect of x_2 on the pairs of signals with a direct effect on costs. If there is a pair (x_2, b_1) such that $s_1 = H_{B_1|[x_1,x_2]}(b_1)$, and $s_2 = H_{B_2|[x_1,x_2]}(b_1)$, then the full-information cost $c_1(s_1, s_2, x_1)$ is identified by b_1.

The identification argument for first-price auctions is very similar. The best-response condition is

$$b_1 - \frac{1 - H_{B_2|B_1,x}(b_1|b_1)}{h_{B_2|B_1,x}(b_1|b_1)} = E\left(C_1 | H_{B_1|x}(b_1), H_{B_2|x}(b_1), x_1\right), \tag{9}$$

where $H_{B_2|B_1,x}(b_1|b_1)$ is the distribution function of B_2 conditional on $B_1 = b_1$ and $X = x = [x_1, x_2]$ evaluated at b_1, and $h_{B_2|B_1,x}(b_1|b_1)$ is its density. The identification argument remains intact. If for the pair (s_1, s_2) there is a pair (x_2, b_1) such that $s_1 = H_{B_1|[x_1,x_2]}(b_1)$, and $s_2 = H_{B_2|[x_1,x_2]}(b_1)$, then the full-information cost $c_1(s_1, s_2, x_1)$ is identified by the left hand side of (9), which is the bid b_1 minus the markup.

If $n > 2$, the event "tie with a competitor for the lowest bid" has a more complex representation because there are more than one competitor to tie with. It takes the form "S_{-i} such that $S_j = s_j$ for some competitor j and $S_k \geq s_k$ for all other competitors". Consider a second price auction with three bidders: 1, 2 and 3. Figure 1 shows the pair of competitors' signals that makes them both bid exactly b_1 along with $L_\epsilon(b_1, x)$, an L-shaped set containing all competitors' signals such that their minimum bid is in $[b_1, b_1 + \varepsilon]$. The right-hand side of (6) denotes the expected cost conditional on this set as $\varepsilon \to 0$. This limit can be written as:

$$\sum_{j=2,3}\left[\frac{h_{B_j|x}(b_1)\, f(s_1,s_j)\, P(S_{5-j}\geq s_{5-j}|s_1,s_j)}{\sum_{k=2,3} h_{B_k|x}(b_1)\, f(s_1,s_k)\, P(S_{5-k}\geq s_{5-k}|s_1,s_k)}\right] E(C_1|s_1,s_j,S_{5-j}\geq s_{5-j},x_1) \qquad (10)$$

where $s_j = H_{B_j|x}(b_1)$ and $h_{B_j|x}$ is the density of $B_j|x$. The term in the square bracket is the probability that bidder i ties with bidder j conditional on tying with at least one competitor. These probabilities are identified from the observed distribution of bids. The objects of interest are the two terms $E(C_1|s_1,s_j,S_{5-j}\geq s_{5-j},x_1)$— the expected cost conditional tying with bidder j while underbidding the other bidder for $j=2,3$. The best-response condition implies that each observed bid equals (10), which results in a single equation for two unknowns. The information provided by a single bid is insufficient to identify these terms. It will be shown that using bids under different values of competitors' cost shifters will generate additional information that identifies the expected costs conditional on winning: $E(C_1|s_1,S_2\geq s_2,S_3\geq s_3,x_1)$.

Consider again Figure 1 and fix $\varepsilon>0$. Keeping s_1 and x_1 constant, find a triplet $[x_2,x_3,t]$ so that $s_1 = H_{B_1|[x_1,x_2,x_3]}(t)$ and $L_\varepsilon(t,[x_1,x_2,x_3])$ stacks on top of the previous L-shaped set. The expected cost conditional on the union of these two sets is equal to a weighted average of the expected cost conditional on each L-shaped set. The weights are given by the probability of each set which are identified from the joint distribution of signals. The expected cost conditional on each L-shaped set can be approximated by observed bid. This process can be repeated to obtain a weighted average over the whole rectangle $\{S_j\geq s_j\}_{j=2,3}$, as shown by Figure 2. Under assumption A.1, signals are independent from cost shifters and this average equals $E(C_1|s_1,S_2\geq s_2,S_3\geq s_3,x_1)$, which is the expected cost conditional on the event where 1 wins the auction. If $\varepsilon\to 0$, the approximation error for each L-shaped set vanishes and the average becomes an integration. The set of points in Figure 2, which are determined by the thickness of the legs of each L-shaped set, becomes a parametric curve over which the integration is performed.

If $E(C_1|s_1,S_2\geq s_2,S_3\geq s_3,x_1)$ is identified around a neighborhood of (s_2,s_3), then its derivative is also identified. Differentiating it with respect to s_2 and s_3:

$$\frac{d^2E(C_1|s_1,S_2\geq s_2,S_3\geq s_3,x_1)\,P(S_2\geq s_2,S_3\geq s_3|s_1)}{ds_2,ds_3} = c_1([s_2,s_3],s_1,x_1)\, f_{S_2,S_3|s_1}(s_2,s_3), \qquad (11)$$

where $f_{S_2,S_3|s_1}$ is the density of competitors' signals conditional on $S_1=s_1$. The full-information cost is obtained dividing the left hand side of (11) by the density of signals, both identified objects.

Theorem 1 formalizes and generalizes this argument allowing for non-differentiable bid functions, $n>3$ bidders, and a general payment function. Theorem 2 in Appendix A generalizes this result further by relaxing the independence assumption A.1 substantially.

Theorem 1. *Under Assumptions A.1-A.4, $c_i(s_{-i},s_i,x_i)$ is identified from H if:*

C.1. Non-vanishing competition: There is a constant $\kappa>0$ such that for all $\delta>0$ and $x\in$

$\{x_i\} \times X^o_{-i}$,

$$\begin{aligned} H_{M_i|B_i=b_i,x}\left(b_i'\right) &= 1 \textit{ or} \\ \frac{H_{M_i|B_i=b_i,x}\left(b_i'\right) - H_{M_i|B_i=b_i,x}\left(b_i\right)}{1 - H_{M_i|B_i=b_i,x}\left(b_i'\right)} &\geq \kappa\delta, \end{aligned}$$

where $b_i = Q_{B_i|x}(s_i)$, $b_i' = Q_{B_i|x}(s_i + \delta)$.

C.2. *Uniformly integrable payments: There is an interval* $I = [\underline{s}_i, \overline{s}_i] \subset (0,1)$ *such that* $\underline{s}_i < s_i < \overline{s}_i$ *and the class of random variables*

$$\left\{Y : Y = \left(p_i\left(b_i, B_{-i}\right)^2 \middle| B_i = b_i', x\right), x \in \{x_i\} \times X^o_{-i}, b_i, b_i' \in \left[Q_{B_i|x}\left(\underline{s}_i\right), Q_{B_i|x}\left(\overline{s}_i\right)\right]\right\}$$

is uniformly integrable.

C.3. *Variation of cost shifters: For all* $\hat{s}_{-i} \geq s_{-i}$*:*

$$\left[H_{B_j|x}\left(Q_{B_i|x}\left(s_i\right)\right)\right]_{j \neq i} = \hat{s}_{-i}$$

for some $x \in \{x_i\} \times X^o_{-i}$.

Conditions C.1-C.3 are imposed on the observables H, and therefore, can be verified. They ensure that the cost shifters generate enough variation in bidding behavior and that equilibrium bid functions preserve all relevant information. C.1 implies that there is no probability mass at any bid that has some positive probability of winning and that for every b_i in the support of $B_i|x$, there is a competitor who also bids in the neighborhood of b_i. C.2 restricts the tails of the distribution of payments. It is trivially satisfied if either $p_i(b_i, \cdot)$ is bounded by a maximum payment or if the support of $B_{-i}|B_i = b_i, x$ is bounded for all $x \in X^o$ and all $b_i \in \left[Q_{B_i|x}\left(\underline{s}_i\right), Q_{B_i|x}\left(\overline{s}_i\right)\right]$. C.3 ensures that cost shifters generate enough variation in the distribution of bids: for any $\hat{s}_{-i} \geq s_{-i}$, it is possible to find $x_{-i} \in X^o_{-i}$ such that the vector of marginal distributions of bids conditional on $[x_i, x_{-i}]$ evaluated at $Q_{B_i|[x_i,x_{-i}]}(s_i)$ equals the vector $\hat{s}_{-i}$.

Proof: The proof consists in approximating

$$\phi_i\left(s_{-i}|s_i, x_i\right) := E\left(C_i \times 1\left(S_{-i} \geq s_{-i}\right)|s_{-i}, x_i\right) = \int_{\{\tau:\tau \geq s_{-i}\}} c_i\left(\tau, s_i, x_i\right) f_{S_{-i}|s_i}(\tau)\, d\tau. \tag{12}$$

using a finite number of pivotal or L-shaped sets as those in Figure 2 and showing that the approximation error converges to zero as the measure of each set shrinks to zero and the number of pivotal sets goes to infinity.

For every $\delta > 0$, construct a sequence of pairs $\{(\tau_t, x_t)\}_{t=1}^T$ so that $\tau_1 = s_{-i}$ and for every t, $x_t \in \{x_i\} \times X^o_{-i}$ is such that $\left[H_{B_j|x_t}\left(Q_{B_i|x_t}\left(s_i\right)\right)\right]_{j \neq i} = \tau_t$, which exists by Condition C.3, and $\tau_{t+1} = \left[H_{B_j|x_t}\left(Q_{B_i|x_t}\left(s_i + \delta\right)\right)\right]_{j \neq i}$. Define $L_t = \{\tau : \tau \geq \tau_t \wedge \tau < \tau_{t+1}\}$. $\{L_t\}_{t=1}^T$ is a collection of disjoint sets that satisfy $\{\tau : \tau \geq s_{-i}\} - \cup_{t=1}^T L_t = \{\tau : \tau \geq \tau_T\}$. By Condition C.1, if we choose $T \geq -(\kappa\delta)^{-1} \log \delta$, then $P\left(S_{-i} \geq \tau_T | s_i\right) \leq \delta$.

Define the expected revenues, costs and profits for bid b_i, signal s_i, and market conditions x as:

$$\begin{aligned} r_i(b_i, s_i, x) &:= E(p_i(b_i, B_{-i}) | s_i, x) && (13) \\ q_i(b_i, s_i, x) &:= E(C_i \times 1(b_i \prec B_{-i}) | s_i, x) && (14) \\ U_i(b_i, s_i, x) &:= r_i(b_i, s_i, x) - q_i(b_i, s_i, x), && (15) \end{aligned}$$

where for all $j \neq i$, $B_j = \beta_j(S_j, x)$.

Lemma 2 in the appendix shows that $r_i(b_i, \cdot, x)$ and $q_i(b_i, \cdot, x)$ are absolutely continuous and differentiable; therefore, the Envelope Theorem 2 in Milgrom and Segal (2002) applies: if β_i is a best-response to competitors' monotone strategies β_{-i}, then

$$U_i(\beta(s_i+\delta), s_i+\delta, x) - U_i(\beta(s_i), s_i, x) = \int_{s_i}^{s_i+\delta} \frac{\partial}{\partial s_i} r_i(\beta_i(\sigma), \sigma, x) - \frac{\partial}{\partial s_i} q_i(\beta_i(\sigma), \sigma, x)\, d\sigma. \quad (16)$$

Continuity of the profit function with respect to own signals and optimality of strategy β implies:

$$U_i(\beta_i(s_i), s_i, x) = \lim_{\sigma \to s_i} U_i(\beta_i(\sigma), s_i, x) = \overline{r}_i\left(Q_{B_i|x}(s_i), s_i, x\right) - \overline{q}_i\left(Q_{B_i|x}(s_i), s_i, x\right), \quad (17)$$

where $\overline{r}_i(b_i, s_i, x) := \lim_{a \downarrow b_i} r_i(a, s_i, x)$, $\overline{q}_i(b_i, s_i, x) := \lim_{a \downarrow b_i} q_i(a, s_i, x) = E(C_i \times 1(b_i < B_{-i}) | s_i, x)$.

Take any $t = 1, .., T$ and let $b_t = Q_{B_i|x_t}(s_i)$ and $b'_t = Q_{B_i|x_t}(s_i + \delta)$. The sets $\{s_{-i} : s_{-i} \geq \tau_t\}$ and $\left\{s_{-i} : b_t < \beta_j(s_j, x_t)\right\}$ coincide up to a set of measure zero and, by Assumptions A.2 and A.3, $\phi_i(\tau_t | s_i, x_i) = \overline{q}_i\left(Q_{B_i|x_t}(s_i), s_i, x_t\right)$. Replacing in (16),

$$\phi_i(\tau_t | s_i, x_i) = \overline{r}_i(b_t, s_i, x_t) - \overline{r}_i\left(b'_t, s_i + \delta, x_t\right) + \int_{s_i}^{s_i+\delta} \frac{\partial}{\partial s_i} r_i(\beta_i(\sigma), \sigma, x_t)\, d\sigma + \phi_i(\tau_{t+1} | s_i, x_i) + \nu_t \quad (18)$$

where

$$\nu_t = \int_{s_i}^{s_i+\delta} \frac{\partial}{\partial s_i} q_i\left(b'_t, \sigma, x_t\right) - \frac{\partial}{\partial s_i} q_i(\beta_i(\sigma), \sigma, x_t)\, d\sigma$$

Replacing recursively in equation 18:

$$\phi_i(s_{-i} | s_i, x_i) = \sum_{t=1}^{T} \left(\overline{r}_i(b_t, s_i, x_t) - \overline{r}_i\left(b'_t, s_i+\delta, x_t\right) + \int_{s_i}^{s_i+\delta} \frac{\partial}{\partial s_i} r_i(\beta_i(\sigma), \sigma, x_t)\, d\sigma \right) + \nu + \eta \quad (19)$$

where $\nu = \sum_{t=1}^{T} \nu_t$ and $\eta = \int_{\{\tau : \tau \geq \tau_T\}} c_i(\tau, s_i, x_i) f_{S_{-i}|s_i}(\tau)\, d\tau$. By integrability of $c_i(\cdot, s_i, x_i)$, $\eta \to 0$ as $\delta \to 0$.

$$\begin{aligned} |\nu| &= \left| \sum_{t=1}^{T} \int_{s_i}^{s_i+\delta} \frac{\partial}{\partial s_i} \left(\int_{\{\tau : \beta_i(\sigma) \prec \beta_{-i}(\tau) \not\prec b'_t\}} f(\tau|\sigma) c_i(\tau, \sigma, x_i)\, d\tau \right) d\sigma \right| \\ &\leq \sum_{t=1}^{T} \int_{s_i}^{s_i+\delta} \int_{L_t} \left| \frac{\partial}{\partial s_i} \left[f(\tau|\sigma) c_i(\tau, \sigma, x_i) \right] \right| d\tau d\sigma \\ &= \int_{s_i}^{s_i+\delta} \int_{\{\tau : \tau \geq s_{-i}\}} \left| \frac{\partial}{\partial s_i} \left[f(\tau|\sigma) c_i(\tau, \sigma, x_i) \right] \right| d\tau d\sigma \\ &\leq \delta \kappa'. \end{aligned}$$

The first line obtains after writting q_i as an integral over competitors' signals. The second line line follows from differentiating under the integral sign, passing the absolute value through the summation and integration operators and integrating over a weakly larger set of τ for each value of σ. Lemma 1 in the Appendix shows that for every I in the interior of the support of S_i,

$$\sup_{s_i \in I} \int_{[0,1]^{n-1}} \left| \frac{\partial}{\partial s_i} \left[f(\tau|s_i) c_i(\tau, s_i, x_i) \right] \right| d\tau < \kappa', \tag{20}$$

for some finite κ'. This result ensures that the step of differentiating under the integral sign is legitimate. The third line follows from the fact that $\{L_t\}_{t=1}^{T} \subset \{\tau : \tau \geq s_{-i}\}$. The last line follows from the bound in (20). As $\delta \to 0$ while I remains fixed, the approximation (19) becomes arbitrarily precise.

The next step of the proof is to show that the terms in brackets in (19) are identified. Lemmas 1 and 3 in Appendix A show that $\frac{\partial}{\partial s_i} r_i(\beta_i(s_i), s_i, x_t)$ is bounded and continuous almost everywhere. Therefore, it is Riemann integrable and its integral over $\sigma \in [s_i, s_i + \delta]$ can be approximated by the Riemann sum $\sum_{k=0}^{K-1} (\overline{r}_i(b_{tk}, s_{k+1}, x_t) - \overline{r}_i(b_{tk}, s_k, x_t))$ where $s_k = s_i + \frac{k\delta}{K}$ and $b_{tk} = Q_{B_i|x_t}(s_k)$. Each of the terms in brackets in the right hand side of (19) becomes:

$$\sum_{k=1}^{K} \left(\overline{r}_i \left(b_{t(k-1)}, s_k, x_t \right) - \overline{r}_i(b_{tk}, s_k, x_t) \right) \tag{21}$$

By Condition C.1, conditioning on s_k is equivalent to conditioning on b_{tk}:

$$\overline{r}_i(b_{tk}, s_k, x_t) = \lim_{a \downarrow b_{tk}} E\left(p_i(a, B_{-i}) | B_i = b_{tk}, x_t\right). \tag{22}$$

Each of the K terms in (21) is identified as the difference in expected payment that bidder i would have received if it bid $b_{t(k-1)}$ instead of b_{tk} and competitors bid according to $H_{B_{-i}|B_i = b_{tk}, x_t}$.

To sum up, for large enough T and K

$$\phi_i(s_{-i}|s_i, x_i) \approx \sum_{t=1}^{T} \sum_{k=1}^{K} \left(\overline{r}_i \left(b_{t(k-1)}, s_k, x_t \right) - \overline{r}_i(b_{tk}, s_k, x_t) \right),$$

with arbitrary precision. Therefore, $\phi_i(\sigma_{-i}|s_i, x_i)$ is identified for all $\sigma_{-i} \geq s_{-i}$. Differentiating,

$$\frac{d^{n-1}\phi_i(s_{-i}|s_i, x_i)}{ds_{-i}} = c_i(s_i, s_{-i}, x_i) f_{S_{-i}|s_i}(s_{-i}). \tag{23}$$

The full-information cost can be recovered dividing by the conditional density of signals which is also identified. $\square$

It may be possible to derive sufficient (and even necessary) conditions on primitives that guarantee existence of equilibria that generate observables satisfying C.1-C.3. These conditions will depend on the particular rules of the auction. For example, Appendix C derives sufficient conditions for first-price auctions with two bidders. It also shows that Condition C.3 requires variation of x_{-i} over a bounded set which means that it is not necessary to take any competitor's cost shifter to infinity to achieve point identification.

The exercise of finding restrictions on primitives that ensure C.1-C.3 is not very useful from an empirical perspective. In any empirical application, the distribution of observables can be estimated and the researcher can test whether C.1-C.3 hold. If they hold, the full-information cost is identified. If they do not hold, some information is lost and some features of the primitives are not identified. Even in this case, it is still possible to construct informative bounds for the expected costs conditional on winning (see Appendix B). In practice, applied researchers may follow a semi-parametric approach where one first obtains an estimate $\hat{H}$ imposing conditions C.1-C.3 and then estimates the parameters of a parsimonious interdependent cost model. This is the approach I follow to obtain estimates from data on highway procurements in Michigan.

4 Winner's Curse in Michigan Highway Procurement

Each year the Michigan Department of Transportation (MDOT) uses first-price sealed-bid auctions to award around 1,050 highway construction and maintenance contracts at a cost of 1.2 billion dollars. On each monthly letting date, 150-200 firms submit a sealed bid for one or more of 50-70 contracts. Firms may participate in as many auctions per letting date as allowed by their work-type and financial prequalification status. The work-type prequalification status of a firm is a list of all the classes of work that the firm can perform. There are 52 work classifications and the typical firm is prequalified for 6-10 of these. The financial rating of a firm is the maximum dollar amount of contracts it can have pending with the MDOT. While the work-type classification is public information, the financial rating of a firm is confidential.

Contracts are advertised for at least 45 days before the letting date, so bidders have detailed information about them. The information about future projects is less precise. The MDOT publishes a 3-month projection of future projects and a 5-year Transportation Improvement Plan, but these are subject to frequent changes and updates.

Prior to submitting a bid, firms download the technical plan and submit a form to become eligible to bid. The MDOT keeps an updated list of both eligible bidders and plan holders that is publicly available on its website. Firms may submit the eligibility form as late as 5:00 p.m. on the day preceding the letting date, and may not appear in the list of eligible bidders prior to the bid submission deadline. Moreover, eligible bidders and plan holders often choose not to bid. As a result, firms are unable to predict with certainty the set of participants in an auction. Thus it is likely that firms base their expectations of competition largely on the location and technical characteristics of the project.

Each contract describes a list of tasks that the contractor has to perform. A task specifies a description and a quantity, e.g., earth excavation, 600 cubic yards. For each task, the MDOT engineer sets a unit price, so that the total estimated cost of the task is the price times the quantity. The engineer's estimate for the contract is the sum of all the tasks' estimated costs. Bidders submit a unit bid for each task, and the total bid is the inner product of unit prices and quantities. The bidder with the lowest total bid wins. If the are no modifications to the contract, the MDOT pays the winner its total bid upon project completion.[8]

Although there is no formal reserve price, the procuring agency has the option to reject all bids if the lowest bid exceeds 110% of the MDOT engineer's estimate. In the case the bids are rejected, the project can be revised and offered in a future letting date. If the agency accepts a bid despite exceeding the estimate, it must justify in writing why the estimate was not correct or why the bids were excessive. From 2001 to 2010, the lowest bid exceeded 110% of the engineer's estimate in 11% of cases and 15% of these were rejected.

4.1 Why distance matters

This paper will focus on contracts where contractors must be prequalified to perform Hot-Mix-Asphalt (HMA) work. HMA is the pavement material used in 96% of all paved roads and streets in the US. It consists of asphalt or bitumen and mineral aggregate that is heated and mixed in a plant. The mix must be trucked to the project site, laid on the road and compacted while the mix is sufficiently hot (above 275°F/135°C). The temperature of the mix at the time of compaction is key to the quality of the pavement mat. Once the mix falls below 175°F/79°C, it cannot be further compacted, and a poorly compacted mat will age faster. HMA pavement projects are rarely performed during winter for this reason.

Trucking time from the plant to the project location is an important determinant of costs not only because of transport costs, but due to the cooling process. During transport, the surface layer of the mix cools faster than the inner mass. Once the mix is dumped into the paver and laid on the road, these temperature differentials may persist and result in cool spots in the pavement mat that cannot be properly compacted. These problems can be mitigated by incurring additional labor and rental costs, for example, by using a Material Transfer Vehicle to remix on site. Thus firms that own plants located close to the project have lower transportation costs and lower costs associated with excess cooling of asphalt.

4.2 Data

Data for all auctions and bids from 2001 to 2010 are available through the MDOT. For each auction, the data includes the project's description, location, prequalification requirements, the engineer's estimate of the total cost of the project, and the list of participating firms and their bids. To obtain the geographical coordinates for each project location, I match the road names in the description to the database of roads available at the Michigan's Geographical Data Library.

The location of each firm's plants and mineral aggregate quarries were obtained from several sources: the MDOT contractor directory; individual firm searches using OneSource North American Business Browser, Duns & Bradstreet's Hoovers and yellowpages.com; firm websites; and the data

[8]The unit bid for each task becomes relevant if the quantity has to be modified after the award of the contract, e.g., if the contractor needs to excavate 650 cubic yards insted of 600. The payment is adjusted by the unit bid times the difference between the actual and estimated quantities. Bidders' incentives to skew their bids are analyzed in Athey and Levin (2001b) and Bajari, Houghton, and Tadelis (2011).

on firms collected by Einav and Esponda (2008). A firm location was included in the final data set if it appeared in at least two sources or if it was listed explicitly on a firm's website.

Of the 10,522 MDOT auctions that were run from January 2001 to December 2010, 3,851 auctions required the prime contractor or one of the subcontractors to be prequalified to perform work with HMA and in 1925 auctions this is the only prequalification requirement for the prime contractor. Table 1 shows the main descriptive statistics of this set of auctions. The typical auction has three bidders. The median engineer's estimate is around \$633,000, while the median winning bid is around \$586,000. It is convenient to normalize bids with respect to the engineer's estimate. Let the normalized bid be $b = Bid/Engineer - 1$. The median normalized winning bid is 6% below the engineer's estimate. The median participant is located 38 km (26 miles) from the project, while the median winner is only 22 km (14 miles) away. The average "money-left-on-the-table", or how much higher the second lowest bid is relative to the lowest, is about 8%.

It is interesting to observe how normalized bids vary with the number of actual participants. Table 2 shows that the average winning bid is decreasing with the number of participants, but the average bid is not. In a standard symmetric independent private cost model with exogenous participation, both the expected lowest bid and the expected bid should be decreasing in the number of bidders. Of course, in this setting it is likely that the symmetry and exogenous participation assumptions fail.

4.3 Estimation

The primitives of the interdependent cost model are high-dimensional even in the simplest case with only two bidders. In most empirical settings, including the one in this paper, it will be necessary to use some parametric and distributional assumptions to deal with the curse of dimensionality. This section proposes an estimation method that follows the ideas in Guerre, Perrigne, and Vuong (2000) and Campo, Perrigne, and Vuong (2003) of estimating the marginal distributions of bids flexibly in a first step. These estimates are used to estimate a parsimonious Gausian Information Structure as in Example 2. In particular, the joint distribution of signals is assumed to be a Gaussian copula with correlation matrix Σ and the full information cost of bidder i is assumed to be additively separable in cost shifters and signals:

$$c_i\left(s_{-i}, s_i, x_0, x_i\right) = \alpha_{i0}' x_0 + \alpha_{i1} x_i + \alpha_{i2} \sum_{j \neq i} \mu_{ij} \Phi^{-1}\left(s_j\right) + \alpha_{i3} \Phi^{-1}\left(s_i\right) \tag{24}$$

where Φ^{-1} is the inverse of a standard normal cumulative distribution function so that $Z_j = \Phi^{-1}\left(S_j\right)$ is a standard normal and the vector $Z = \left\{\Phi^{-1}\left(S_j\right)\right\}_{j=1}^{n}$ is jointly normal with covariance matrix Σ.

One important implication of this functional form assumption is that the parameters of the full information costs are linked with those of the joint distribution of bids. The parameters μ are such that for all distinct i, j and k: $\mu_{ij}/\mu_{ik} = \tilde{\sigma}_{ij}/\tilde{\sigma}_{ik}$, where $\tilde{\sigma}_{ij}$ is the ijth element of Σ^{-1}. This is particularly convenient for the empirical setting under consideration. The parameters μ will be identified from the correlation in bids across different bidders. The degree of interdependence is summarized by the parameter α_{i2}. This parameters is identified using the exclusion restriction on competitors' cost shifters which, after controlling for own distance, exhibit only one-dimensional variation.[9] With private costs $\alpha_{i2} = 0$, and with interdependent values $\alpha_{i2} > 0$. The Gaussian structure is falsified if $\alpha_{i2} < 0$.

I estimate the model parameters in several steps. In the first step, the parameters of the joint distribution of signals are estimated by simulated maximum likelihood. In the second step, the marginal cost that rationalizes each observed bid is obtained using the approach in Guerre, Perrigne, and Vuong (2000) and Campo, Perrigne, and Vuong (2003). In the third step, the set of pivotal signals is estimated. The fourth and final step estimates the parameters of the full information cost by instrumental variables quantile regression using cost shifters as instruments.

4.3.1 First step: estimation of the joint distribution of signals

Monotonicity of the bid function implies that the copula of bids conditional on the vector of cost shifters is equal to the joint distribution of signals. The conditional copula is estimated semi-parametrically. In the first stage, the univariate conditional marginals are estimated flexibly. In the second stage, the copula parameters are estimated by likelihood methods.

I estimate the univariate conditional marginals using splines. Instead of conditioning on the full vector of competitors' distance, I condition on the geographical coordinates (lat, lon) of the project location. These coordinates fully determine the vector of distances as firms' plants locations are fixed. The marginal distribution of $B_i|lat, lon$ is estimated as:

$$\hat{F}_{B_i|lat,lon}(b) = \sum_{j=1}^{K_{lat}} \sum_{k=1}^{K_{lon}} \hat{c}_{jk} \theta_j (lat) \theta_k (lon)$$

where θ_j and θ_k are cubic B-splines and the coefficients $\hat{c}_{jk}$ solve the following problem:

$$\min_{c \in [0,1]^{K_{lat} \times K_{lon}}} \sum_{t=1}^{T} \left(1(b_{it} \le b) - \sum_{j=1}^{K_{lat}} \sum_{k=1}^{K_{lon}} c_{jk} \theta_j (lat_t) \theta_k (lon_t) \right)^2 .$$

The restriction that $c \in [0,1]^{K_{lat} \times K_{lon}}$ ensures that the predicted probabilities lie between zero and one. The set of splines in each dimension are chosen to cover the full range of locations observed in the data. I use $K_{lat} = K_{lon} = 15$ in the sample which results in one knot every 87 km from east to west and one knot every 91 km from south to north.

For each bidder i and auction t, I obtain a set of coefficients $\hat{c}_{jk}$ and construct $\hat{s}_{it} = \hat{F}_{B_i|lat,lon}(b_{it})$. This is an estimate of the signal that prompted bid b_{it}. If bidder i does not participate in auction t, I obtain $\hat{F}_{B_i|lat,lon}(\bar{b})$, where $\bar{b}$ is the maximum finite normalized bid observed in the data. This is an estimate of the threshold signal where bidder i is indifferent between participating and staying out. Non participation implies that the realized signal was above $\hat{F}_{B_i|lat,lon}(\bar{b})$.

[9] It may be possible to exploit the fact that firms have multiple plants to vary the set of competitors' who may be close, and estimate a more flexible structure than the Gaussian.

The data now consists on a T realizations of a vector of signals that may be truncated due to non-participation. The likelihood of the observed data can be written explicitly for each correlation matrix Σ which is assumed to have a factor structure: $\Sigma = LL' + \Lambda$, where L is a $n \times l$ loading matrix and Λ is a positive diagonal matrix that ensures that the elements on the main diagonal of Σ are all one. Restricting $l < n-1$ reduces the number of free parameters. I estimate the parameters L using a Simulated Maximum Likelihood estimator proposed by Kamakura and Wedel (2001).

4.3.2 Second step: bid inversion

Following Guerre, Perrigne, and Vuong (2000) and Campo, Perrigne, and Vuong (2003), one can estimate the marginal cost that rationalizes each observed bid as the left hand side of (7) that depends on the probability of winning and its derivative. The probability of winning with the observed bid b is estimated as:

$$\hat{P}\left(M_i \geq b | b, lat, lon\right) = \sum_{j=1}^{K_{lat}} \sum_{k=1}^{K_{lon}} \sum_{m=1}^{K_b} \hat{c}_{jkm} \theta_j\left(lat\right) \theta_k\left(lon\right) \theta_m\left(b\right)$$

where θ_j, θ_k and θ_m are the cubic B-splines and the coefficients $\hat{c}_{jkm}$ solve the following problem:

$$\min_{c \in [0,1]^{K_{lat} \times K_{lon} \times K_b}} \sum_{t=1}^{T} \left(1\left(m_{it} \geq b_{it}\right) - \sum_{j=1}^{K_{lat}} \sum_{k=1}^{K_{lon}} \sum_{m=1}^{K_b} c_{jk} \theta_j\left(lat_t\right) \theta_k\left(lon_t\right) \theta_m\left(b_{it}\right) \right)^2 .$$

The restriction on c ensures again that the fitted probabilities lie between zero and one. The derivative is estimated in a similar way except that the $\hat{c}'$ coefficients solve

$$\min_{c \in [0,1]^{K_{lat} \times K_{lon} \times K_b}} \sum_{t=1}^{T} \left(\frac{1}{h} K\left(\frac{m_{it} - b_{it}}{h}\right) - \sum_{j=1}^{K_{lat}} \sum_{k=1}^{K_{lon}} \sum_{m=1}^{K_b} c_{jk} \theta_j\left(lat_t\right) \theta_k\left(lon_t\right) \theta_m\left(b_{it}\right) \right)^2 ,$$

where K is a kernel function and h is a bandwidth. The set of splines for each geographical dimension identical to those in the previous step. I use 5 splines for bids that are chosen so that the knots partition the support of bids in 5 segments of the same length. The location of the knots vary by bidder i. The estimate of the marginal cost that rationalize each observed bid as:

$$m\hat{c}_{it} = b_{it} - \frac{\sum_{j=1}^{K_{lat}} \sum_{k=1}^{K_{lon}} \sum_{m=1}^{K_b} \hat{c}_{jkm} \theta_j\left(lat\right) \theta_k\left(lon\right) \theta_m\left(b\right)}{\max\left(\sum_{j=1}^{K_{lat}} \sum_{k=1}^{K_{lon}} \sum_{m=1}^{K_b} \hat{c}'_{jkm} \theta_j\left(lat\right) \theta_k\left(lon\right) \theta_m\left(b\right), \nu\right)}$$

where ν is a vanishing positive constant that corrects for estimation error in the denominator that may result in very small densities.

4.3.3 Third step: pivotal signals

The signal that makes j tie with bidder i at bid b is estimated by

$$\hat{F}_{B_j | lat, lon}\left(b\right) = \sum_{j=1}^{K_{lat}} \sum_{k=1}^{K_{lon}} \hat{c}_{jk} \theta_j\left(lat\right) \theta_k\left(lon\right)$$

where the coefficients $\hat{c}_{jk}$ solve the following problem:

$$\min_{c\in[0,1]^{K_{lat}\times K_{lon}}} \sum_{t=1}^{T} \left(1\left(b_{jt} \le b\right) - \sum_{j=1}^{K_{lat}} \sum_{k=1}^{K_{lon}} c_{jk}\theta_j\left(lat_t\right)\theta_k\left(lon_t\right) \right)^2 .$$

For each bidder i, competitor j and auction t such that b_{it} is finite, I obtain a set of coefficients $\hat{c}_{jk}$ and construct $\hat{s}_{jt}^{(i)} = \hat{F}_{B_j|lat_t,lon_t}\left(b_{it}\right)$. The set of splines are the same as those used in the previous steps.

I also estimate the probability that bidder i ties with bidder j conditional on tying with at least one competitor. First, I estimate:

$$\tilde{\pi}_{jt}^{(i)} = \sum_{j=1}^{K_{lat}} \sum_{k=1}^{K_{lon}} \sum_{m=1}^{K_b} \hat{c}_{jkm}\theta_j\left(lat_t\right)\theta_k\left(lon_t\right)\theta_m\left(b_{it}\right).$$

where the coefficients $\hat{c}_{jkm}$ solve:

$$\min_{c\in[0,1]^{K_{lat}\times K_{lon}\times K_b}} \sum_{t=1}^{T} \left(\frac{1}{h}K\left(\frac{b_{jt}-b_{it}}{h}\right)\prod_{k\neq i,j} 1\left(b_{kt}\ge b_{it}\right) - \sum_{j=1}^{K_{lat}} \sum_{k=1}^{K_{lon}} \sum_{m=1}^{K_b} c_{jkm}\theta_j\left(lat_t\right)\theta_k\left(lon_t\right)\theta_m\left(b_{it}\right) \right)^2 .$$

This is an estimate of the unconditional probability that j ties for the lowest bid with i. The conditional probability is estimated as

$$\pi_{jt}^{(i)} = \frac{\tilde{\pi}_{jt}^{(i)}}{\sum_{k\neq i}\tilde{\pi}_{kt}^{(i)}}$$

4.3.4 Fourth step: determinant of costs

The last estimation step obtains the parameters of the full information cost function in equation (24). There are two complications that need to be addressed. First, the second step recovers the marginal cost that rationalizes each observed bid not the full information cost on the left hand side of (24). Second, the marginal cost that rationalizes non-participation cannot be recovered unambiguously.

The identification argument shows how high-dimensional variation in cost shifters can be used to recover the full information costs from marginal costs. In the MDOT estimation, I will restrict the information structure to be Gaussian so that one dimensional variation in competitors' cost shifters suffices for to identify the parameter α_{i2} that determines the degree of interdependence. Under the Gaussian information structure:

$$mc_{it} = \alpha_{i0}'x_0 + \alpha_{i1}x_i + \alpha_{i2}k_{it} + \alpha_{i3}z_{it}. \tag{25}$$

where

$$\begin{aligned} k_{it} &= \sum_{j\neq i}\mu_{ij}\left[\lim_{\varepsilon\downarrow 0} E\left(Z_j|s_i, S_{-i}\in L_\varepsilon\left(b_i,x\right)\right)\right] \\ &= \sum_{j\neq i}\mu_{ij}\left(\left[\pi_{jt}^{(i)}z_{jt}^{(i)}\right] + \sum_{k\neq i,j}\pi_{kt}^{(i)}E\left(Z_j|s_{kt}^{(i)}, s_i, \left\{S_m \ge s_{mt}^{(i)}\right\}_{m\neq i,k}\right)\right). \end{aligned}$$

The previous steps obtained estimates of $\pi_{jt}^{(i)}$ and $z_{jt}^{(i)} = \Phi^{-1}\left(s_{jt}^{(i)}\right)$. The expectation of Z_j conditional on $s_{kt}^{(i)}$,s_i, and $\left\{S_m \geq s_{mt}^{(i)}\right\}_{m\neq i,k}$ can be calculated numerically given the estimated joint distribution of signals. These are all the elements needed to obtain an estimate of k_{it}.

Assume for the moment that bidders always participate so that it is always possible to invert the observed bid. It is tempting to estimate equation (25) by ordinary least squares replacing mc_{it} and k_{it} by their estimated counterparts and letting the error term be $\varepsilon_{it} = \alpha_{i3} z_{it}$. One problem with this approach is that k_{it} and ε_{it} are correlated because a higher signal prompts a higher bid which changes the pivotal set. In fact, k_{it} is a function of bidders' own signal s_{it} and the vector of cost shifters $x = (lat, lon)$. The identification result in this paper indicates that competitors' distance to the project can be used as instruments for k.

The censoring problem introduced by non-participation can be addressed by quantile regression approaches.[10] Monotonicity of the bid functions imply that there will be a cutoff signal such that only signals below it prompt finite bids. Denote this unobserved cutoff by $\psi_i\,(lat, lon)$, which stresses that the cutoff is specific to the bidder and to the full vector of cost shifters. Conditional on a market configuration (lat, lon) and a given k, the marginal cost MC_i is just a monotone function of i's signal. The τ-th quantile of MC_i is:

$$Q_\tau\left(MC_i|x_0, x_1, k\right) = \alpha_{i0}' x_0 + \alpha_{i1} x_i + \alpha_{i2} k + \alpha_{i3} \Phi^{-1}(\tau). \tag{26}$$

The probability of obtaining a realization less than $Q_\tau\left(MC_i|x_0, x_1, k\right)$ conditional on participation is $\tau\left[\psi_i\,(lat, lon)\right]^{-1}$. This observation gives rise to the following set of moment conditions:

$$E\left(1\,(B_i < \infty)\left[1\left\{MC_i - \alpha_{i0}' x_0 + \alpha_{i1} x_i + \alpha_{i2} K_i \leq 0\right\} - \tau\left[\psi_i\,(lat, lon)\right]^{-1}\right] X\right) = 0 \tag{27}$$

where the instruments X are functions of (x_0, x_i, x_{-i}). The moment conditions are similar as those implied by quantile regressions and include an adjustment for censoring that was first proposed by Buchinsky and Hahn (1998).

The parameters α can be estimated using the techniques developed for instrumental variables quantile regressions. I use the procedure proposed by Chernozhukov and Hansen (2006) in Remark 5 and Chernozhukov and Hansen (2008) in Comment 3. The OLS projection of k_i on (x_0, x_i, x_{-i}) is used as the instrument for k_i and x_0, x_i are used as instruments for themselves. I construct a grid of values $\{\alpha_j, j = 1, ..., J\}$ and run for each j the τ-Quantile Regression of $MC_i - \alpha_j K_i$ on the set of included instruments (x_0, x_i) and the instrument for k_i to obtain coefficients $\alpha\,(\alpha_j, \tau)$ and $\hat{\gamma}\,(\alpha_j, \tau)$, respectively. I choose $\hat{\alpha}_2\,(\tau)$ as the value among $\{\alpha_j, j = 1, ..., J\}$ that sets $\hat{\gamma}\,(\alpha_j, \tau) = 0$. The estimate of $\hat{\alpha}\,(\tau)$ is given by $\hat{\alpha}\,(\hat{\alpha}_2\,(\tau), \tau)$. I calculate standard errors for the coefficients of interest by bootstrap. I run the same multi-step procedure for 200 bootstrap samples of 1925 auctions each, and report the standard deviation of each parameter across samples.

[10]Bidders may decide not to participate because their maximum expected profits may be negligible or below some bid preparation cost (Samuelson, 1985). Models with information acquisition costs are beyond the scope of this paper.

4.4 Results

I focus on the 19 firms that were observed participating with higher frequency in the final sample. The rest of the firms are grouped together as a fringe 20th bidder and I keep only the lowest of their bids. The results below show the firm-specific parameters estimates for the distribution of signals and full-information costs.

4.4.1 Correlation of signals

I estimated the correlation structure for $l = \{0, 1, 2, 3\}$ factors and used the Akaike Information Criterion to select $l = 2$. [11] Table 3 shows the estimates of the parameters of the loading matrix L and the elements on the diagonal of Λ. It also shows a decomposition of the total signal variance in the l factors and the bidder specific component. The first factor introduces correlation in the signals of bidders 1, 2, 3, 5, 9, 11, 13, 14 and 17. The implied correlation among their signals ranges between 0.17 and 0.29 (see Table 4). This factor explains up to 20% of the signal variance for bidders in this group. It also introduces negative correlation between the signals of this group and that of bidder 4 (between 0 and -0.15). The second factor introduces correlation in signals of bidders 3, 4, 5, 7, 10, 15, 19 and the fringe bidder. This factor explains up to 25% of the signal variance for bidders in this group. Many of the firms in this group also participate in construction projects (which are excluded from the sample) and they may have different technology and capabilities. In particular, Firms 4 and 10 do not own asphalt plants. In paving projects, they have to buy asphalt from other firms.

4.4.2 Full Information Costs

Table 5 shows the estimates of the full information cost for bidders 1 - 12. The included exogenous covariates are a set of dummies for projects farther than 10, 50 and 100 kilometers, a measure of road density in the vicinity of the project that allows for potential differential costs in highly urbanized areas[12] and three constants: one for the 25th percentile of the distribution of costs, one for the difference between the 50th and the 25th percentile and one for the difference between the 75th and the 25th percentile.

The expectation of the common cost components conditional on the set of pivotal signals was constructed following equation (25) and is included as an endogenous variable. I use the set of competitors' distance as instruments and collapse them into a one dimensional variable as suggested

[11] The AIC selects $l = 2$: $AIC_1 - AIC_2 = 120$ and $AIC_3 - AIC_2 = 6.5$. The BIC is inconclusive between $l = 1$ and $l = 2$, but strongly rejects $l = 3$: $AIC_1 - AIC_2 = -0.28$ and $AIC_3 - AIC_2 = 121$.

[12] I used geographical information on all roads available at the Michigan Center for Geographical Information http://www.mcgi.state.mi.us/mgdl/framework/statewide/allroads_mi.zip. I compute each road segment's length and assign a weight based on its classification (Interstates, 3.5; Freeways, 3; Principal Arterials, 2.5; Minor Arterials, 2.2; Major Collectors, 2; Minor Collectors, 1.5 and Local Roads, 1). I compute a the density of roads using a 10 km bandwidth kernel an evaluate it at each project location. This measure was subsequently standardized. Rural areas have a standardized measure of approximately -1, sparse areas such as Holland, MI have 0, denser areas such as Grand Rapids have 1 and Detroit reaches 3.5.

by Chernozhukov and Hansen (2008). This is achieved by projecting the endogenous variable on a space of piecewise linear functions of the vector of distances with kinks at 10, 50 and 100km and using the fitted value as the instrument for the instrumental variable quantile regression.

The estimate of the full information cost for bidder 1 evaluated at the 25th percentile own signal and the 50th percentile competitors' signal for projects within 10km of its plants is 73% of the engineer estimate. For projects in the ranges 10-50km and 50-100km, this cost increases by 2.5 and 15.7 percentage points, respectively. Bidder 1 does not seem to be more efficient in denser areas. The 50th (75th) percentile cost is 8.2 (18.1) percentage points higher than the 25th. Table 7 provides a easy way to compare the estimated effect of own and competitors' signals on the full information cost. A one standard deviation increase in the Gaussian signal received by bidder 1 increases its full information cost by 10 percentage points. The effect of the same increase in a competitor's signal depends on the identity of the competitor. The average effect over competitors is 4 percentage points and the maximum is 7.

The results for other bidders are qualitatively similar but there are a couple of noteworthy differences that justify estimating these cost functions separately. While the private cost hypothesis is rejected for bidders 1 and 2, it cannot be rejected for bidder 3. The coefficient on the common cost for bidder 3 is precisely estimated around zero. The same can be said about bidder 9. Another important difference that while bidders 3, 5 and 10 seem to be more efficient in road-dense areas, bidder 9 seem to be more efficient in less dense areas. Because bidders 13-19 are not observed participating as regularly, their parameters are estimated less precisely. Nonetheless, the point estimates are comparable to those of firms that are observed participating more often.

4.5 The Effect of Competition

I consider cases where participation is restricted to a set of invited bidders. Inviting more bidders will always result in more competitive bidding and lower procurement costs in independent private cost auctions (Bulow and Klemperer, 1996) but the results may be different in affiliated private or in common cost models. Pinkse and Tan (2005) show that affiliation of signals can offset the pro-competitive effect of an additional bidder. This affiliation effect arises because bidders realize that winning in the presence of one more bidder implies that it is more likely that rivals' costs are high (due to affiliation), and that they can profitably increase their markups. Interdependent or common costs models have the additional anti-competitive effect of adverse selection or winner's curse. Each bidder realizes that it wins on an adversely selected sets of states of the world: when competitors have bad signals. Their expected completion cost conditional on winning is thus higher than the unconditional one. Bidding against an additional bidder worsens the selection problem so bidders may react by increasing their bids to account for higher expected costs.

To evaluate the effects of increased competition on bidding behavior and procurement costs, I simulate the effects of a policy that restricts bidder participation in 250 randomly drawn auctions from the 1925 in the sample. I order bidders according to their estimated median full information cost in each auction excluding those that are estimated to participate with less than 0.07 probability.

I assume that only the first N bidders with the lowest median cost will be allowed to participate. The set of invitees will vary across auctions because bidders have different estimated costs depending on the location of the project. Three different set of primitives/models are considered: the common cost (CC) model that uses the primitives estimated in the previous section; the affiliated private cost (APC) model that sets all common cost coefficients α_2 to zero; and the independent private cost (IPC) model that also sets the parameters of the loading matrix to zero.[13] For each model and each $N = \{2, 3, ..., 10\}$, I compute a Bayes Nash Equilibrium using a numerical algorithm described in Appendix C and simulate auction outcomes drawing 250,000 signal realizations from the estimated joint distribution (in APC or CC models) or from independent distributions (in the IPC model).[14]

Figure 3 illustrates the effect of competition under the three models. To construct this figure I averaged the transaction price across the 250,000 signal realizations and averaged over the 250 auctions weighting by the engineer's estimate. I normalized this average it by the duopoly outcome for each model. The figure plots the resulting average transaction price (relative to the duopoly) for each model and number of invitees. The three models predict that inviting an additional bidder reduces procurement costs. The magnitudes are different though. The IPC model has the largest effect followed by the APC model. The CC model has the smallest effect. For example, increasing the number of invitees from 2 to 7 reduces costs by 5.69 percentage points in IPC, 5.15 pp in APC and 4.42 pp in CC. In all three models, the effect of inviting a ninth or a tenth bidder is negligible because this bidder is typically quite inefficient.

Figure 4 shows the effect on the bid submitted by the bidder invited first which is the most efficient ex-ante. I constructed this figure using the same procedure as above but using the bid of the first invitee instead of the transaction price. While both IPC and APC models predict that this bidder will bid more aggressively when faced with more competition, the CC model predicts the opposite. This bidder reacts to increased competition by bidding higher as it faces a more severe winner's curse.

To illustrate the magnitudes of the different effects at play, I decompose the total effect of competition on CC models in four components. The Competitive Effect is the reduction in the average bid of the first invitee in the IPC model. The Affiliation Effect is the difference in this average bid reduction between the APC and the IPC models. The Winner's Curse Effect is the difference in this average bid reduction between the CC and the APC models. The Sampling Effect is the difference between the reduction of the average bid of the first invitee and the reduction in procurement costs in the CC model.

Table 8 shows the results of this decomposition. The first panel compares the outcomes of inviting $N = 3, ..., 10$ relative to inviting only 2. The competitive effect has cost-saving effects that are increasing in the number of invitees. They range from 2.24 to 3.5 percentage points. Affiliation

[13] A more precise name for the APC model is Correlated Private Costs. The estimated joint distribution of signals has some negative correlations which implies that the signals are correlated but not affiliated.

[14] In cases where there is multiple equilibria, the numerical algorithm can be thought of selecting a particular equilibrium among many possible according to some pre-specified rules. See Appendix C for details.

has an offsetting effect of 0.54 - 0.69 pp. The Winner's Curse has a stronger offsetting effect of 1.07-3.63 pp. These three effects combined make the overall response of the first invitee to be increasing in the number of bidders for any $N > 3$. The sampling effect has a strong cost-saving effect of 2.31-5.25 pp that ensure a total reduction in procurement costs between 2.9 and 4.4 pp. The second panel compares the outcomes of inviting N bidders relative to $N - 1$. It shows that the magnitude of these effects decays rapidly as more and less efficient bidders are invited.

5 Conclusion

I provide a positive identification result for the payoff-relevant characteristics of the interdependent costs model. When bidders are risk neutral, these characteristics are the joint distribution of signals and each bidder's full-information costs. They are sufficient to analyze the effects of most policy changes (e.g., rules of the auction, reserve prices, subsidies) on outcomes such as bidding behavior, project allocation and prices. They are not sufficient to analyze counterfactuals where the timing of the auction changes so that bidders are required to make decisions after some additional uncertainty in the model is resolved.

The result applies to auctions where bidders submit simultaneous or sealed bids, the project is awarded to the bidder who submits the lowest bid and the payment that each bidder receives is given by a publicly known function of all the bids submitted to the auctions. The first-price, second-price and all-pay sealed-bid auctions satisfy these conditions.

The identification result holds under the following assumptions. Each bidder's private information can be summarized by a real-valued signal. The joint distribution of bidders' signals is independent from cost shifters. Each bidder's cost shifter affects his own full-information cost but not his competitors'. The observed data is generated by the repeated play of the same equilibrium where bidders use monotone pure strategies.

The full-information cost is nonparametrically identified provided that the some verifiable conditions on observables hold. Some auction rules are more conducive to these conditions than others. For example, Appendix C shows that they are generally satisfied in a first-price auction model with two bidders. Even if these verifiable conditions fail in the observed data, it is still possible to bound the full-information cost function as described in Appendix B.

I propose an estimation method that follows the indirect approaches proposed in the literature on estimation of independent private values and extend them to obtain estimates of the parameters that describe the primitives of the interdependent cost model. I apply the estimator to bidding data from the Michigan Department of Transportation and find that the estimated full-information cost is increasing with distance to the project and with competitors' signals. Despite being statistically significant, the effect of competitors' signals on expected costs is weaker than the effect of own signals suggesting that the model is closer to pure private costs than to symmetric pure common costs. Policies that restrict the number of participants to ameliorate the winner's curse are successful at inducing more aggressive bidding among participants but fail at reducing overall procurement costs. In other words, the effect of the winner's curse is strong enough to make each bidder bid lower when there is less competition, but not enough to compensate the procurer for the times when a bidder that was not allowed to participate would have won the auction.

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APPENDIX

A Point Identification

This Appendix collects intermediate results used to prove Theorem 1 and the statement and proof of Theorem 2 that generalizes it.

A.1 Intermediate Results for Theorem 1

Consider a model where Assumptions A.1, A.2 and A.3 hold. Fix x_i and interval I such that Condition C.2 holds.

Lemma 1. *For every $x \in x_i \times X_{-i}^o$ and $b_i \in \left[Q_{B_i|x}\left(\underline{s}_i\right), Q_{B_i|x}\left(\overline{s}_i\right)\right]$,*

$$\sup_{s_i \in I} \int_{[0,1]^{n-1}} \left| \frac{d}{ds_i} \left[c_i\left(\tau, s_i, x_i\right) f\left(\tau|s_i\right)\right]\right| d\tau < \infty$$

and

$$\sup_{s_i \in I^\circ} \int_{[0,1]^{n-1}} \left| p_i\left(b_i, \beta_{-i}\left(\tau, x\right)\right) \frac{d}{ds_i} f\left(\tau|s_i\right)\right| d\tau < \infty.$$

Proof: Consider the payment function first.

$$\begin{aligned} & \int_{[0,1]^{n-1}} \left| p_i\left(b_i, \beta_{-i}\left(\tau, x\right)\right) \frac{d}{ds_i} f\left(\tau|s_i\right)\right| d\tau \\ \leq \;& E\left(\left(\frac{d \log f\left(S_{-i}|s_i\right)}{ds_i}\right)^2 \middle| s_i\right)^{1/2} E\left(p_i\left(b_i, B_{-i}\right)^2 |s_i, x_i\right)^{1/2}, \end{aligned}$$

where the inequality follows after dividing and multiplying by $f\left(\tau|s_i\right)$ and using the Cauchy–Schwarz inequality. Assumption A.2 and Condition C.1 imply that this term is uniformly bounded in s_i in the interior of I.

Consider the cost function:

$$\begin{aligned} & \int_{[0,1]^{n-1}} \left| \frac{d}{ds_i} \left[c_i\left(\tau, s_i, x_i\right) f\left(\tau|s_i\right)\right]\right| d\tau \\ \leq \;& \int_{[0,1]^{n-1}} \left| \frac{d}{ds_i} c_i\left(\tau, s_i, x_i\right)\right| f\left(\tau|s_i\right) d\tau + \int_{[0,1]^{n-1}} \left| c_i\left(\tau, s_i, x\right) \frac{d}{ds_i} f\left(\tau|s_i\right)\right| d\tau \\ \leq \;& E\left(\left|\frac{d}{ds_i} c_i\left(S_{-i}, s_i, x_i\right)\right| \middle| s_i\right) + E\left(\left(\frac{d \log f\left(S_{-i}|s_i\right)}{ds_i}\right)^2 \middle| s_i\right)^{1/2} E\left(c_i\left(S_{-i}, s_i, x\right)^2 |s_i, x_i\right)^{1/2} \end{aligned}$$

The first inequality follows from the chain rule and differentiability of the cost and density functions. The second inequality uses the Cauchy–Schwarz inequality. Assumptions A.2 and A.3 imply that both terms are uniformly bounded in $s_i \in I$. □

Lemma 2. *$q_i(b_i,\cdot,x)$ and $r_i(b_i,\cdot,x)$ are absolutely continuous and differentiable in I. Their derivatives are $\int_{1\left(b_i\prec\beta_{-i}(\tau|x)\right)}\left|\frac{d}{ds_i}\left[c_i(\tau,s_i,x_i)f(\tau|s_i)\right]\right|d\tau$ and $\int_{[0,1]^{n-1}}\left|p_i\left(b_i,\beta_{-i}(\tau,x)\right)\frac{d}{ds_i}f(\tau|s_i)\right|d\tau$.*

Proof: Assumptions A.2, A.3 and Condition C.1 imply that:

- $c_i(\tau,s_i,x_i)f(\tau|s_i)$ and $p_i\left(b_i,\beta_{-i}(\tau,x)\right)f(\tau|s_i)$ are integrable over τ for all $s_i\in I$.
- $c_i(\tau,s_i,x_i)f(\tau|s_i)$ and $p_i\left(b_i,\beta_{-i}(\tau,x)\right)f(\tau|s_i)$ are continuously differentiable (and absolutely continuous) functions of s_i for all $s_i\in I$.

Lemma 1 implies:

- $\frac{d}{ds_i}\left[c_i(\tau,s_i,x_i)f(\tau|s_i)\right]$ and $p\left(b_i,\beta_{-i}(\tau,x)\right)\frac{d}{ds_i}f(\tau|s_i)$ are integrable (and locally integrable) over τ for all $s_i\in I$.

These conditions suffice to show that it is legitimate to differentiate under the integral sign. Therefore, $\frac{d}{ds_i}r_i(b_i,s_i,x)$ and $\frac{d}{ds_i}q_i(b_i,s_i,x)$ exist for all $s_i\in I$. Absolute continuity of $c_i(\tau,s_i,x_i)f(\tau|s_i)$ and $p\left(b_i,\beta_{-i}(\tau,x)\right)f(\tau|s_i)$ with respect to $s_i\in I$ imply that $r_i(b_i,\cdot,x)$ and $q_i(b_i,\cdot,x)$ satisfy Fundamental Theorem of Calculus, and therefore, are absolutely continuous. $\square$

Lemma 3. *The function $v(s_i)=\frac{\partial}{\partial s_i}r_i(\beta_i(s_i),s_i,x)$ has, at most, countably many discontinuities in I.*

Proof: Let D_0 be the subset of I where $\beta_i(\cdot,x)$ is discontinuous, by monotonicity it has countably many elements. Let C_0 be the open subset of I where $\beta_i(\cdot,x)$ is constant, i.e., $s\in C_0$ if and only $s\in(s',s'')\subset I$ and $\beta_i(s',x)=\beta_i(s'',x)=\beta_i(s'',x)$. For each set of competitors' bids b_{-i}, the set of discontinuities of the payment function is given by $\cup_{k=1}^{\infty}\{d_k(b_{-i})\}$. The functions $d_k\left(\beta_{-i}(\cdot,x)\right):(0,1)^{n-1}\to\mathbb{R}$ are Lebesgue measurable because the functions d_k are Borel measurable and β_j are monotone. $\mu\left(\left\{\tau:d_k\left(\beta_{-i}(\tau,x)\right)<t\right\}\right)$ is monotone in t and has countably many discontinuities. Construct the following countable set: $D_k=\left\{s\in I-C_0:\mu\left(\tau:d_k\left(\beta_{-i}(\tau,x)\right)=\beta_i(s,x)\right)>0\right\}$. The rest of the proof shows that $v(s_i)$ is continuous for all $s_i\in B=I-\cup_{k=0}^{\infty}D_k$.

Consider a sequence$\{s_t\}_{t=1}^{\infty}$ such that $s_t\to s_i$ for some $s_i\in B$. Define two (measurable) functions $[0,1]^n\to\mathbb{R}$: $Y(\tau)=\frac{d}{ds}f(\tau|s_i)$ and $Z(\tau)=p\left(b_i,\beta_{-i}(\tau,x)\right)$ for $b_i=\beta_i(s_i,x)$; and two families of measurable functions: $\mathcal{Y}=\left\{Y_t:Y_t(\tau)=\frac{d}{ds}f(\tau|s_t)\right\}$, and $\mathcal{Z}=\left\{Z_t:Z_t(\tau)=p\left(b_t,\beta_{-i}(\tau,x)\right)\right\}$ for $b_t=\beta(s_t,x)$.

- $\{Y_t\}_t$ converges in measure to Y:
$$\lim_{t\to\infty}\mu(|Y_t-Y|>\varepsilon)=\lim_{t\to\infty}\mu\left(\left\{\tau:\left|\frac{d}{ds_i}f(\tau|s_t)-\frac{d}{ds_i}f(\tau|s_i)\right|>\varepsilon\right\}\right)=0$$
by continuous differentiability of f (Assumption A.2).
- $\{Z_t\}_t$ converges in measure to Z:
$$\lim_{t\to\infty}\mu(|Z_t-Z|>\varepsilon)=\lim_{t\to\infty}\mu\left(\left\{\tau:\left|p\left(b_t,\beta_{-i}(\tau,x)\right)-p\left(b_i,\beta_{-i}(\tau,x)\right)\right|>\varepsilon\right\}\right)=0$$
because either $s_i\in C_0$, in which case $b_t=b_i$ for all t greater than some t^*, or $b_t\to b_i$ and $p\left(\cdot,\beta_{-i}(\tau,x)\right)$ is continuous at b_i for almost all $\tau\in[0,1]^{n-1}$.

- Therefore, $Z_t Y_t \to ZY$ in measure.
- $\{Z_t Y_t\}$ is a uniformly integrable function. Consider any $A \subset [0,1]^{n-1}$:

$$\begin{aligned}\int_A Z_t Y_t d\mu &= p\left(b_t, \beta_{-i}(\tau, x)\right) \frac{d}{ds} \ln f(\tau|s_t) f(\tau|s_t) d\tau \\ &\leq \sqrt{\int_A \left|p\left(b_t, \beta_{-i}(\tau, x)\right)\right|^2 f(\tau|s_t) d\tau \times \int_A \left|\frac{d}{ds} \ln f(\tau|s_t)\right|^2 f(\tau|s_t) d\tau}\end{aligned}$$

Choose any $\varepsilon > 0$. By uniformly square integrability of $\left\{p\left(b_t, \beta_{-i}(\tau, x)\right)\right\}$ and $\left\{\frac{d}{ds} \ln f(\tau|s_t)\right\}$ there are $\delta_p > 0$ and $\delta_s > 0$ and such that for all A and t, $\mu(A) < \delta_p$ and $\mu(A) < \delta_s$ imply that each of the two factors under the radical sign is less than ε. Take δ equal to the minimum of δ_p and δ_s. For all $\mu(A) < \delta$ and t, $\int_A Z_t Y_t d\mu < \varepsilon$.

The Vitali Convergence Theorem implies:

$$\int Z_t Y_t d\mu = \frac{\partial}{\partial s_i} r_i(\beta_i(s_t), s_t, x) \to \int ZY d\mu = \frac{\partial}{\partial s_i} r_i(\beta_i(s_i), s_i, x).$$

$\square$

A.2 Relaxing the assumption of independence

The independence Assumption A.1 can be relaxed significantly. One can assume local independence at the cost of strengthening Condition C.3.

A'.1. Cost shifters and signals are locally independent: There is a finite partition of X^o denoted by $\{X_1^o, X_2^o, ..., X_P^o\}$ such that for all $p \in \{1, ...P\}$, $\{x, x_p\} \in X_p^o$ implies $F_{S|x} = F_{S|x_p}$.

A'.2. Signals are one-dimensional random variables. For every $p \in \{1, ...P\}$, the joint distribution $F_{S|x_p}$, admits a continuously differentiable density function f_p that satisfies the following regularity conditions:

(a) For every $I = [\underline{s}_i, \overline{s}_i]$ in the support of S_i, the class of random variables

$$\left\{Y : Y = \left(\left.\frac{d \log f(S_{-i}|s_i, x_p)}{ds_i}^2 \right| s_i, x_p\right), s_i \in I\right\}$$

is uniformly integrable.

(b) For any $p, p' \in \{1, ...P\}$, the random variables $S|x_p$ and $S|x_{p'}$ have the same support denoted by $\mathcal{S}$.

A'.3. The full-information cost of bidder i is $c_i(s_{-i}, s_i, x_i)$ which does not depend on x_j for all $j \neq i$, i.e.,

$$c_i(s_{-i}, s_i, x_i) := E(C_i|s_i, s_{-i}, x_i) = E(C_i|s_i, s_{-i}, x), \tag{28}$$

and satisfies the following regularity conditions: for every $p \in \{1, ...P\}$, every signal s_i in the interior of the support of S_i and every x_i: $c_i(s_{-i}, \cdot, x_i)$ is continuously differentiable, $E\left(\left|\frac{d}{ds_i} c_i(S_{-i}, s_i, x_i)\right| \middle| s_i, x_p\right) < \infty$ and $E\left(c_i(S_{-i}, s_i, x_i)^2 | s_i, x_p\right) < \infty$.

The Assumptions in the main paper hold when $P=1$. Allowing for $P>1$ relaxes the assumption of independence.

Theorem 2. *Under Assumptions A'.1-A'.3 and A.4, $c_i(s_{-i}, s_i, x_i)$ is identified from H if:*

C'.1. Non-vanishing competition: There is a constant $\kappa>0$ such that for all $\delta>0$ and $x \in \{x_i\} \times X^o_{-i}$,

$$\begin{aligned} H_{M_i|B_i=b_i,x}\left(b'_i\right) &= 1 \text{ or} \\ \frac{H_{M_i|B_i=b_i,x}\left(b'_i\right) - H_{M_i|B_i=b_i,x}\left(b'_i\right)}{1-H_{M_i|B_i=b_i,x}\left(b'_i\right)} &\geq \kappa\delta, \end{aligned}$$

where $b_i = Q_{B_i|x}(s_i)$, $b'_i = Q_{B_i|x}(s_i+\delta)$.

C'.2. Uniformly integrable payments: There is an interval $I=[\underline{s}_i, \overline{s}_i] \subset (0,1)$ such that $\underline{s}_i < s_i < \overline{s}_i$ and the class of random variables

$$\left\{Y : Y = \left(p_i\left(b_i, B_{-i}\right)^2 \middle| B_i = b_i, x\right), x \in \{x_i\} \times X^o_{-i}, b_i \in \left[Q_{B_i|x}\left(\underline{s}_i\right), Q_{B_i|x}\left(\overline{s}_i\right)\right]\right\}$$

is uniformly integrable.

C'.3. Continuity and sufficient variation of cost shifters: For all $\hat{s}_{-i} \geq s_{-i}$, there exist $p \in \{1,...P\}$ and $\varepsilon>0$ such that for all s'_{-i} in an ε-neighborhood of $\hat{s}_{-i}$

$$\left[H_{B_j|x}\left(Q_{B_i|x}\left(s_i\right)\right)\right]_{j\neq i} = s'_{-i}$$

for some $x \in \{x_i\} \times X^o_{-i} \cap X^o_p$.

Proof: Suppose that there are two different full-information cost functions consistent with observable H. Let c^1_i and c^2_i be these two functions and

$$\phi^m_i\left(s_{-i}|s_i, x_i, p\right) := \int_{\{\tau:\tau\geq s_{-i}\}} c^m_i\left(\tau, s_i, x_i\right) f\left(\tau|s_i, x_p\right) d\tau$$

for $m=1,2$. Let Ψ_p be the support of $\psi(\cdot|s_i, x_i, p) := \left(c^1_i(\tau, s_i, x_i) - c^2_i(\tau, s_i, x_i)\right) f(\tau|s_i, x_p)$. By the same support Assumption in A'.2, $\Psi := cl\left(\left\{\tau \geq s_{-i} : c^1_i(\tau, s_i, x_i) \neq c^2_i(\tau, s_i, x_i), (\tau, s_i) \in \mathcal{S}^\circ\right\}\right) = \Psi_p$ for all p. Define $\overline{s}_{-i} = \arg\max_{s_{-i}\in\Psi} \sum_{j\neq i} s_j$. By Condition C.3, there are $\varepsilon>0$ and $p \in \{1,...P\}$ such that for all $s'_{-i} \in \Delta := \{\tau : \max_{j\neq i}|\tau_j - \overline{s}_j| < \varepsilon\}$

$$\left[H_{B_j|x}\left(Q_{B_i|x}\left(s_i\right)\right)\right]_{j\neq i} = s'_{-i}$$

for some $x \in \{x_i\} \times X^o_{-i} \cap X_p$. Restricting the space of competitors' signals to Δ and the set of cost shifters to X^o_p, the Assumptions and Conditions of Theorem 1 hold. Therefore, for all $\tilde{s}_{-i} \in \tilde{\Delta} := \left\{\tau : \max_{j\neq i}|\tau_j - \overline{s}_j| < \varepsilon\left(n-2\right)^{-1}\right\}$, it is possible to approximate to arbitrary precision:

$$\phi^m_i\left(\tilde{s}_{-i}|s_i, x_i, p\right) \approx \sum_{t=1}^{T}\sum_{k=1}^{K}\left(\overline{r}_i\left(b_{t(k-1)}, s_k, x_t\right) - \overline{r}_i\left(b_{tk}, s_k, x_t\right)\right) + \phi^m_i\left(s'_{-i}|s_i, x_i, p\right),$$

with $\max_{j\neq i} s'_j \geq \overline{s}_j + \varepsilon$ and $\sum_{j\neq i} s'_j > \sum_{j\neq i} \overline{s}_j$. It follows that $\{\tau : \tau \geq s'_{-i}\} \cap \Psi = \emptyset$, $\phi_i^1\left(s'_{-i}|s_i,x_i,p\right) - \phi_i^2\left(s'_{-i}|s_i,x_i,p\right) = 0$ and

$$\psi\left(\tilde{s}_{-i}|s_i,x_i,p\right) = \phi_i^1\left(\tilde{s}_{-i}|s_i,x_i,p\right) - \phi_i^2\left(\tilde{s}_{-i}|s_i,x_i,p\right) = 0.$$

This is true for all $\tilde{s}_{-i} \in \tilde{\Delta}$ contradicting $\overline{s}_{-i} \in \Psi$. The support of Ψ is empty. Therefore, for any $\tau \geq s_{-i}$ such that $(\tau, s_i) \in \mathcal{S}$, $c_i^1(\tau,s_i,x_i) = c_i^2(\tau,s_i,x_i)$. □

B Partial Identification

The optimality conditions can be used to derive bounds on average costs even when the conditions on observables in Theorem 1 do not hold. This section derives bounds for $\phi_i(s_{-i}|s_i,x_i)$ fixing (s_{-i},s_i,x_i). The additional assumption that allows for partial identification is that $c_i(\tau,s_i,x_i) \geq \underline{c}$ for all $\tau \geq s_{-i}$ and some finite $\underline{c}$.

It will be useful to define $\underline{H}_{B_j|x}$ as the left-continuous CDF of $B_j|x$, $\underline{Q}_{B_i|x}(s_i)$ as the left-continuous quantile function of $B_i|x$, $\underline{r}_i(b_i,s_i,x) := \lim_{a\uparrow b_i} r_i(a,s_i,x)$ and $\underline{q}_i(b_i,s_i,x) := \lim_{a\uparrow b_i} q_i(a,s_i,x)$. Therefore:

$$\begin{aligned}
\underline{q}_i(b_i,s_i,x) &= \phi_i\left(\left[\underline{H}_{B_j|x}(b_i)\right]_{j\neq i}|s_i,x_i\right), \text{ and} \\
\overline{q}_i(b_i,s_i,x) &= \phi_i\left(\left[H_{B_j|x}(b_i)\right]_{j\neq i}|s_i,x_i\right).
\end{aligned}$$

B.1 Upper-bound

For any $x \in \{x_i\} \times X_{-i}^o$ and $\delta \geq 0$, define the pivotal set $L_+(x,\delta)$ as:

$$L_+(x,\delta) := \left\{\tau \in [0,1]^{n-1} : \left[\underline{H}_{B_j|x}\left(\underline{Q}_{B_i|x}(s_i)\right)\right]_{j\neq i} \leq \tau < \left[H_{B_j|x}\left(Q_{B_i|x}(s_i+\delta)\right)\right]_{j\neq i}\right\}.$$

Let $\mathcal{L}_+$ be a collection of pivotal sets defined as:

$$\mathcal{L}_+ := \left\{\begin{array}{c} L : L = L_+(x,\delta), \\ x \in \{x_i\} \times X_{-i}^o, \delta \geq 0, \\ \forall\varepsilon > 0, Q_{B_i|x}(s_i - \varepsilon) < Q_{B_i|x}(s_i) < Q_{B_i|x}(s_i+\varepsilon) \end{array}\right\}$$

Let $\mathcal{F}_+$ denote the collection of all the subsets of $\mathcal{L}_+$ whose union includes the set $\{\tau : \tau \geq s_{-i}\}$; formally,

$$\mathcal{F}_+ := \left\{\begin{array}{c} \{L_t\}_{t=1}^T : \forall t \in \{1,..T\}, L_t \in \mathcal{L}_+; \\ \{\tau : \tau \geq s_{-i}\} \subset \cup_{t=1}^T L_t \end{array}\right\}.$$

Take any (x_t,δ_t). The optimality condition

$$\lim_{\sigma\downarrow s_i+\delta_t} U_i(\beta(\sigma),s_i,x_t) - \lim_{\sigma\uparrow s_i} U_i(\beta(\sigma),s_i,x_t) \leq 0$$

implies:

$$\underline{q}_i \left(\underline{Q}_{B_i|x_t}(s_i), s_i, x_t\right) - \overline{q}_i \left(Q_{B_i|x_t}(s_i+\delta_t), s_i, x_t\right) \le \underline{r}_i \left(\underline{Q}_{B_i|x_t}(s_i), s_i, x_t\right) - \overline{r}_i \left(Q_{B_i|x_t}(s_i+\delta_t), s_i, x_t\right).$$

It follows that:

$$\phi_i(s_{-i}|s_i,x_i) \le \inf_{\{L_t\}_{t=1}^T \in \mathcal{F}_+} \left[\begin{array}{c} \sum_{t=1}^T \underline{r}\left(\underline{Q}_{B_i|x_t}(s_i), s_i, x_t\right) - \overline{r}_i \left(Q_{B_i|x_t}(s_i+\delta_t), s_i, x_t\right) \\ +\left(\Pr(S_{-i} \ge s_{-i}|s_i) - \sum_{t=1}^T \Pr(L_t|s_i)\right)\underline{c} \end{array} \right],$$

where $L_t = L_+(x_t, \delta_t)$.

B.2 Lower-bound

For any $x \in \{x_i\} \times X_{-i}^o$ and $\delta \ge 0$, define the pivotal set $L_-(x,\delta)$ as:

$$L_-(x,\delta) := \left\{\tau \in [0,1]^{n-1} : \left[\underline{H}_{B_j|x}\left(\underline{Q}_{B_i|x}(s_i-\delta)\right)\right]_{j\ne i} \le \tau < \left[H_{B_j|x}\left(Q_{B_i|x}(s_i)\right)\right]_{j\ne i}\right\}$$

Let $\mathcal{L}_-$ be a collection of pivotal sets defined as:

$$\mathcal{L}_- = \left\{ \begin{array}{c} L : L = L_-(x,\delta), \\ x \in \{x_i\} \times X_{-i}^o, \delta \ge 0, \\ \forall \varepsilon > 0, Q_{B_i|x}(s_i-\varepsilon) < Q_{B_i|x}(s_i) < Q_{B_i|x}(s_i+\varepsilon) \end{array} \right\}$$

Let $\mathcal{F}_-$ denote the collection of all the subsets of $\mathcal{L}_+$ composed of disjoint subsets of $\{\tau : \tau \ge s_{-i}\}$; formally,

$$\mathcal{F}_- = \left\{ \begin{array}{c} \{L_t\}_{t=1}^T : \forall t \in \{1,..T\}, L_t \in \mathcal{L}_-, \\ L_t \subset \{\tau : \tau \ge s_{-i}\}, \\ \forall t' \in \{1,..T\} \backslash t, L_t \cap L_{t'} = \emptyset \end{array} \right\}.$$

Take any (x_t, δ_t). The optimality condition

$$\lim_{\sigma \downarrow s_i} U_i(\beta(\sigma), s_i, x_t) - \lim_{\sigma \uparrow s_i - \delta} U_i(\beta(\sigma), s_i, x_t) \ge 0$$

implies:

$$\underline{q}_i \left(\underline{Q}_{B_i|x}(s_i-\delta), s_i, x_t\right) - \overline{q}_i \left(Q_{B_i|x}(s_i), s_i, x_t\right) \ge \underline{r}_i \left(\underline{Q}_{B_i|x}(s_i-\delta), s_i, x_t\right) - \overline{r}_i \left(Q_{B_i|x}(s_i), s_i, x_t\right)$$

It follows that:

$$\phi_i(s_{-i}|s_i,x_i) \ge \sup_{\{L_t\}_{t=1}^T \in \mathcal{F}_-} \left[\begin{array}{c} \sum_{t=1}^T \underline{r}_i\left(\underline{Q}_{B_i|x_t}(s_i-\delta_t), s_i, x_t\right) - \overline{r}_i \left(Q_{B_i|x_t}(s_i), s_i, x_t\right) \\ +\left(\Pr(S_{-i} \ge s_{-i}|s_i) - \sum_{t=1}^T \Pr(L_t|s_i)\right)\underline{c} \end{array} \right],$$

where $L_t = L_-(x_t, \delta_t)$.

C First Price Auctions

This section focuses on first-price sealed-bid auctions, an auction format that is widely used in practice and has attracted considerable theoretical interest. The first section discusses whether equilibrium strategies typically generate data that satisfy the monotonicity condition in Assumption A.4 and the conditions in Theorem 1. The second section presents an algorithm to compute equilibrium strategies.

C.1 Properties of Equilibrium Bid Strategies

This section presents sufficient restrictions on primitives for existence of equilibria that satisfy Assumption A.4 and generate observables that satisfy the conditions in Theorem 1. The goal of these restrictions is to show that the identification result is far from being vacuous, and that conditions in Theorem 1 should hold in a large class of models. Because these restrictions should not be assumed a-priori to obtain identification, I will not seek for maximum generality. Instead, the conditions below were chosen to simplify proofs and use existent results in the literature.

FPA.1. Signals are affiliated: $f_S(s' \vee s) f_S(s' \wedge s) \geq f_S(s) f_S(s')$ for all s, s', where $\vee$ and $\wedge$ denote the component-wise maximum and minimum, respectively.

FPA.2. Bounded density: there are positive $\underline{f}, \overline{f}$ such that $\underline{f} < f(s) < \overline{f}$ for all $s \in [0,1]^n$.

FPA.3. Common Values: there is a constant $\kappa > 0$ such that for all j, k and x_j: $\frac{\partial c_j(s_j, s_{-j}, x_j)}{\partial s_k} > \kappa$.

FPA.4. Full-information costs are additively separable in cost shifters and cost shifters are scalars: $E(C_i|S, x_i) = c_i(S) + x_i$.[15]

FPA.5. Bounded full-information costs: for all j, $\overline{c}_j := c_i(1, ..., 1)$, $\underline{c}_j := E(c_i(S)|S_j = 0, S_{-j} \geq [0, .., 0])$ and $c_i(0, ..., 0)$ are all finite.

FPA.6. Support of cost shifters: $X^o_{-i} = \prod_{j \neq i} X^o_j$ where $X^o_j = \left[x_i - (\overline{c}_j - \underline{c}_i) - h, x_i + \left(\overline{c}_i - \underline{c}_j\right) + h\right]$ and

$$\frac{\max_k \left(\overline{c}_j - \underline{c}_j\right)}{1 - \max_k \Pr\left(S_{-k} \geq s_{-k} | S_k = 1\right)} < h < \infty.$$

Monotonicity

Reny and Zamir (2004) show that if signals are statistically affiliated, there exists an equilibrium in monotone pure strategies. McAdams (2007) shows that affiliation also implies that every mixed-strategy equilibrium is outcome-equivalent to a monotone pure strategy equilibrium. These results

[15] More generally, the vector of cost shifters for bidder i can be decomposed into: $x_i = \left[x_i^{(1)}, x_i^{(2)}\right]$, where $x_i^{(1)}$ is a scalar that enters separably into the full-information cost: $E(C_i|S, x_i) = c_i\left(S, x_i^{(2)}\right) + k_i\left(x_i^{(1)}\right)$ for some monotonic function k_i. The original condition holds after conditioning all the analysis on a particular realization of $\{x_i^{(2)}\}_{i=1..n}$ and renormalizing the scalar cost shifters so that $x_i = k_i\left(x_i^{(1)}\right)$.

justify the practice of restricting attention to monotone strategies. Therefore, a restriction that F_S exhibits affiliation suffices to ensure existence of equilibrium bid functions that satisfy Assumption A.4. Of course, existence of such equilibrium does not preclude existence of other equilibria that violate Assumption A.4. Therefore, the assumption that the observed data is generated by the repeated play of equilibrium monotone pure strategies may not be relaxed.

Maskin and Riley (2000) prove that the support of the winning bid is a convex interval and that there are no atoms in the interior of its distribution. This implies that there exist an equilibrium in pure strategies that are strictly monotone except at the highest bid that has some positive probability of winning. Moreover, in the interior of the set of serious bids, the marginal distribution of bids is continuous.

Condition C.1

The existence proof in Reny and Zamir (2004) constructs a sequence of auction games where bidders are restricted to select bids from a finite set and show that for each game there is an equilibrium in monotone pure strategies. As the grid of available bids becomes dense in the real line, there exists a subsequence of equilibrium bid functions that converges to a set of monotone bid functions defined over the real line. They also restrict bidders with signals higher than $1-\varepsilon$ to bid infinity—not to participate—and they allow $\varepsilon \to 0$ as the number of available grid of bids becomes dense. They show that the limiting bid functions are an equilibrium of the game where all types of bidders are allowed to bid any real number. For any equilibrium with a finite number of bids, all bids (except non-participation) have positive probability of winning. Therefore, bidders profits conditional on winning should be nonnegative. By continuity of the full-information costs, this property also applies to the limit strategies that constitute an equilibrium of the unrestricted game.

Proposition 3 in Maskin and Riley (2000) ensures that the support of the distribution of the winning bid is an interval $[b_*, b^*]$ and its CDF is continuous in $[b_*, b^*]$. The following Lemma uses restrictions FPA.1-FPA.5 to characterize further the distribution of $W = \min_{i\in\{1,..,n\}} B_i$ under equilibrium strategies where bidders make non-negative profits conditional on winning. Let H_W be the CDF of W conditional on a fixed x which is omitted from notation.

Lemma 4. *Under restrictions FPA.1-FPA.5, equilibrium bid functions are such that (i) every bid $b \in [b_*, b^*]$ belongs to the support of bids of at least two bidders, (ii) $H_W(b_*) = 0$, (iii) $H_W(b^*) = 1$, (iv) H_W is continuous and strictly increasing in $[b_*, b^*]$, (v) $b^* \le \bar{c}_j + x_j$ for all j such that $\bar{c}_j + x_j > \min_k \bar{c}_k + x_k$, (vi) $b_* \ge \underline{c}_j + x_j$ for all j such that $\underline{c}_j + x_j > \min_k \underline{c}_k + x_k$ and (vii) $0 \le b^* - b_* \le \max_j \bar{c}_j - \underline{c}_j$*

Proof: Part (i): Take any $b \in [b_*, b^*)$ and suppose that it belongs to the support of only bidder i. The type of bidder i that bids in the neighborhood of b can deviate to some $b' > b$ and win in exactly the same states of the world but earn higher profits. Therefore, b has to belong to the support of at least two bidders. Now suppose that b^* belongs to the support of bids of at most one bidder. There is a positive ϵ such that every $b \in (b^* - \epsilon, b^*)$ does not belong to the support of at least $n-1$ bidders. This contradicts that $[b_*, b^*)$ belongs to the support of at least two bidders.

Part (ii): Suppose that $H_W(b_*) > 0$. There is at least one bidder who submits bid b_* with positive probability. Part (i) and FPA.3 imply that there is a $\varepsilon > 0$ and another bidder who bids $b_* + \varepsilon$ who can profitably deviate bidding $b_* - \varepsilon$.

Part (iii): Suppose that there are two bidders that bid b^* with positive probability and all bidders with higher priority stay out whenever they receive signals above $\overline{s}_k$. Formally, $\exists i, j$ such that $\beta(s_i) = \beta(s_j) = b^*$ for all $s_i \in [\underline{s}_i, \overline{s}_i]$ and $s_j \in [\underline{s}_j, \overline{s}_j]$, where $\underline{s}_i < \overline{s}_i$ and $\underline{s}_j < \overline{s}_j$. For all $s_i \in [\underline{s}_i, \overline{s}_i]$ bidder i could discontinuously increase its probability of winning by reducing its bid by ε. His expected costs conditional on winning will be weakly lower because the set of competitors signals is now slightly better. The fact that this bidder chooses not to reduce his bid implies that $b^* \leq E(C_i|s_i, S_{-i} \geq \overline{s}_{-i}, x)$, for all $s_i \in [\underline{s}_i, \overline{s}_i]$. However, because strategies are the limit of a sequence of strategies where each bid has a positive probability of winning, $b^* = E(C_i|s_i, S_{-i} \geq \overline{s}_{-i}, x)$ for all $s_i \in [\underline{s}_i, \overline{s}_i]$, which contradicts strict monotonicity of s_i. Therefore, there is at most one bidder that bids b^* with positive probability.

Suppose $H_W(b^*) < 1$. If i bids b^* with positive probability, he can deviate to a slightly higher bid and win in exactly the same (positive probability) set of realizations of competitors' signals and receive a higher payment. Instead, if there is no bidder that bids b^* with positive probability, then for every $b' > b^*$ there is an ε and a bidder i such that when i bids $b - \varepsilon$, he has a profitable deviation to b' that results in an arbitrarily small change in the (positive probability) set of realizations of competitors' signals where he wins and a discrete increase in the payment he receives in that set. Thus, $H_W(b^*) < 1$ leads to a contradiction.

Part (iv): Maskin and Riley (2000) show that H_W is continuous and strictly increasing in $[b_*, b^*)$. I have to show only that $\lim_{b \uparrow b^*} H_W(b) = 1$, i.e., that there is no atom at b^*. Suppose that there is an atom at b^*. By the same arguments used to prove part (iii), there is at most one bidder who bids b^* with positive probability. There has to be a bidder j who bids between b^* and $b^* + \epsilon$; otherwise, the bidder who bids b^* has a profitable deviation to a slightly higher bid. Bidder j makes non-negative profits conditional on winning when he bids $b^* + \epsilon$. Bidding $b^* - \epsilon$ instead would increase the probability of winning and improve discretely the states of the world where it wins. There exist an ϵ for which this deviation is profitable. This contradicts the existence of an atom at b^*.

Part (v): For all j, let $\overline{s}_j = H_{B_j}(b^*)$. Parts (iii) and (iv) show that $\overline{s}_j = 1$ for some j. If $b^* - E(C_i|\overline{s}_i, S_{-i} \geq \overline{s}_{-i}, x) > 0$ then $\overline{s}_i = 1$. Otherwise, there has to be a bidder $j \neq i$ such that $\overline{s}_j = 1$ which implies that bidder i should bid below b^* when it receives signal $\overline{s}_i + \varepsilon$ for some $\varepsilon > 0$ contradicting the definition of $\overline{s}_i$. Suppose that there are two or more bidders with $b^* - E(C_i|\overline{s}_i, S_{-i} \geq \overline{s}_{-i}, x) > 0$; therefore, $\overline{s}_i = 1$ and b^* is in the support of B_i because otherwise b^* would not be in the support of W. Each of these bidders has a profitable deviation towards a lower bid that ensures them positive profits and positive probability of winning. Therefore, for all but one bidder:

$$b^* \leq E(C_i|\overline{s}_i, S_{-i} \geq \overline{s}_{-i}, x) \leq \overline{c}_i + x_i.$$

Part (vi): Let $\underline{b}_j$ be the lower bound in the support of B_j. Because this bid is preferred to staying out of the auction: $\underline{b}_j \geq \underline{c}_j + x_j$. Let i be the bidder with the second lowest $\underline{c}_i + x_i$. For at least $n-1$ bidders, $\underline{b}_j \geq \underline{c}_i + x_i$. The remaining bidder would not bid below $\underline{c}_i + x_i$ because bidding higher results in the same winning probability and higher profits. Therefore, $b_* \geq \underline{c}_j + x_j$ for all j such that $\underline{c}_j + x_j > \min_k \underline{c}_k + x_k$.

Part (vii): Suppose $b^* - b_* > \max_j \bar{c}_j - \underline{c}_j$. By Part (v), there are at least $n-1$ bidders with $\bar{c}_i + x_i - \left(\max_j \bar{c}_j - \underline{c}_j\right) > b_*$, which implies $\underline{c}_i + x_i > b_*$, and contradicts Part (vi). $\square$

Lemma 5. *Under restrictions FPA.1-FPA.6, equilibrium bid functions generate observables that satisfy Condition C.1.*

Proof: Let $b = Q_{B_i|x}(s_i)$ and $b' = Q_{B_i|x}(s_i + \delta)$ and $s_{-i}(b) = \left[H_{B_j|x}(b)\right]_{j \neq i}$. By optimality of b,

$$b'\left(1 - H_{M_i|B_i=b,x}(b')\right) - \phi\left(s_{-i}(b'), s_i, x_i\right) \leq b\left(1 - H_{M_i|B_i=b,x}(b)\right) - \phi\left(s_{-i}(b), s_i, x_i\right).$$

Therefore,

$$\frac{(b' - b)\left(1 - H_{M_i|B_i=b,x}(b')\right)}{\left(H_{M_i|B_i=b,x}(b') - H_{M_i|B_i=b,x}(b)\right)} \leq b - E\left(C_i|S_{-i} \in L_{b'-b}(b,x), s_i\right) \leq \bar{\Pi}$$

where $\bar{\Pi}$ is the maximum profit conditional on winning in equilibrium. Boundedness of b^*, X^o_{-i} and full-information costs ensure existence of a finite $\bar{\Pi}$. Rearranging:

$$\frac{\left(H_{M_i|B_i=b,x}(b') - H_{M_i|B_i=b,x}(b)\right)}{\left(1 - H_{M_i|B_i=b,x}(b')\right)} \geq \frac{(b' - b)}{\bar{\Pi}}.$$

If $H_{M_i|B_i=b,x}(b') = 1$, Condition C.1 is satisfied. The rest of the proof consists in finding a lower bound for $(b' - b)$ when $H_{M_i|B_i=b,x}(b') < 1$.

If $H_{M_i|B_i=b,x}(b') < 1$, b' is in the support of bids of some competitor $j \neq i$. By optimality of b',

$$b'\left(1 - H_{M_j|B_j=b',x}(b')\right) - \phi\left(s_{-j}(b'), s'_j, x_j\right) \geq b\left(1 - H_{M_j|B_j=b',x}(b)\right) - \phi\left(s_{-j}(b), s'_j, x_j\right).$$

Solving for $(b' - b)$ and replacing ϕ:

$$\begin{aligned}
b' - b &\geq \frac{\int_{s_{-j} \leq \tau \ngeq s'_{-j}} \left(b' - c_j\left(\tau, s'_j, x_j\right)\right) f\left(\tau, s'_j\right) d\tau}{\left(1 - H_{M_j|B_j=b',x}(b)\right)} \\
&\geq \frac{\int_{s_i}^{s_i+\delta} \int_{\left\{\tau \geq s'_{-ij}\right\}} \left(b' - c_j\left([\tau, \delta], s'_j, x_j\right)\right) f\left(\tau, s'_j\right) d\tau d\delta}{\left(1 - H_{M_j|B_j=b',x}(b)\right)} \\
&\geq \frac{P\left(S_{-ij} \geq s'_{-ij}, S_i \in [s_i, s_i + \delta] \,|s'_j\right)}{P\left(S_{-j} \geq s_{-j}|s'_j\right)} \left[\int_{s_i+\delta}^{1} \psi(\sigma)\, g_1(\sigma)\, d\sigma - \int_{s_i}^{s_i+\delta} \psi(\sigma)\, g_2(\sigma)\, d\sigma\right].
\end{aligned}$$

The second line follows after writing the integration as a sum of integrals over sets $\left\{s_{-i} \leq S_{-i} \leq s'_{-i}\right\}$ and $A_k = \left\{S_{-ik} \geq s'_{-ik}, S_i \in [s_k, s'_k]\right\}$ for all $k \neq j$, and noting that b' is weakly greater than

$\int_{A_k}\left(b'-c_j\left(\tau,s'_j,x_j\right)\right)f\left(\tau,s'_j\right)d\tau$ for all k. The third line obtains replacing b' by the lesser magnitude $E\left(C_j|S_{-j}\geq s'_{-j},s'_j,x\right)$ and defining g_1 as the density of S_i conditional on $S_{-j}\geq s'_{-j}$ and $S_j=s'_j$, g_2 as the density of S_i conditional on $S_{-ij}\geq s'_{-ij}$, $S_i\in[s_i,s_i+\delta]$ and $S_j=s'_j$, and:

$$\psi(\sigma):=E\left(C_j|S_{-ij}\geq s'_{-ij},S_i=\sigma,s'_j,x\right).$$

The first factor is bounded by $\frac{\underline{f}\delta}{\overline{f}(1-s_i)}$. The second factor can be bounded adding and subtracting $\psi(s_i+\delta)$ and using the following implication of Assumption FPA.3: $|\psi(\sigma')-\psi(\sigma)|\geq\kappa|\sigma'-\sigma|$. Therefore,

$$b'-b\geq\delta\kappa\frac{\underline{f}}{\overline{f}(1-s_i)}\left[\int_{s_i+\delta}^{1}(\sigma-s_i-\delta)\,g_1(\sigma)\,d\sigma-\int_{s_i}^{s_i+\delta}(s_i+\delta-\sigma)\,g_2(\sigma)\,d\sigma\right].$$

The expression in square brackets is minimized when the densities g_1 and g_2 take large values around $s_i+\delta$ and small values elsewhere. Minimizing it with respect to g_1 and g_2 subject to the restrictions imposed by the bounds in f results in a lower bound of: $\left[\sqrt{\overline{f}}+\sqrt{\underline{f}}\right]^{-1}\sqrt{\underline{f}}\,(1-s_i)$. Therefore,

$$\frac{\left(H_{M_i|B_i=b,x}(b')-H_{M_i|B_i=b,x}(b)\right)}{\left(1-H_{M_i|B_i=b,x}(b')\right)}\geq\frac{\kappa}{\overline{\Pi}}\frac{\underline{f}}{\overline{f}}\frac{\sqrt{\underline{f}}}{\sqrt{\overline{f}}+\sqrt{\underline{f}}}\delta.$$

$\square$.

Condition C.2

Equilibrium payments are bounded below by zero. Lemma C.1 shows that for a given $x\in X^o_{-i}$, the maximum payment in equilibrium, b^*, is bounded above by $\max_j\overline{c}_j+x_j$. Because X^o_{-i} is bounded, the maximum equilibrium payment over all $x\in X^o_{-i}$ is also bounded above. Boundedness of equilibrium payments over all $x\in X^o_{-i}$ sufficies for Condition C.2.

Condition C.3

The following Lemma shows that in first-price auctions variation in cost shifters generate sufficient variation in equilibrium bids. Moreover, it shows that if $n=2$, there are equilibrium strategies that generate bids satisfying Condition C.3.

Lemma 6. *Under restrictions FPA.1-FPA.6, for all $\hat{s}$ such that $\hat{s}_{-i}\geq s_{-i}$ and $\hat{s}_i=s_i$, there is a bid t, cost shifters $x\in x_i\times X^o_{-i}$ and equilibrium strategies $\left\{\beta_j(\cdot,x)\right\}_{j=1}^n$ such that $\beta_j(\tau,x)\leq t$ if and only if $\tau\leq\hat{s}_j$. Moreover, if $n=2$, this selection of equilibrium strategies generate a distribution of bids that satisfies Condition C.3.*

The proof uses the discrete bid model in Athey (2001) and Reny and Zamir (2004) to show that there exist an equilibrium in monotone pure strategies and a vector of cost shifters where every bidder j, bids below t if and only if it receives a signal lower than $\hat{s}_j$. This result follows from the Kakutani's fixed point theorem. As the grid of permissible bids becomes dense, the sequence of fixed points converges to a vector of cost shifters and a set of monotone strategies that constitute an equilibrium of the continuous bid auction where $\beta_j(\tau, x) \leq t$ if and only if $\tau \leq \hat{s}_j$. The rest of this section describes the discrete bid model, constructs the fixed point argument, and shows existence of the equilibrium of the continuous bid auction. The section concludes with the proof of the Lemma.

A discrete bid model. Let $\mathcal{A}_i = \{a_0 < a_1 < ... < a_M\}$ be the sets of available bids to bidder i. Let $[\underline{s}_i, \overline{s}_i] \subset [0,1]$ be a subset of i's signals. A monotone pure strategy $\beta_i : [\underline{s}_i, \overline{s}_i] \to \mathcal{A}_i$ can be represented a step function (See Athey, 2001) that describes the points in $[\underline{s}_i, \overline{s}_i]$ at which β_i jumps. The behavior of i at the jump points is inconsequential. Let $\Sigma_i = \left\{t \in [\underline{s}_i, \overline{s}_i]^M \,|\underline{s}_i \leq t_1 \leq t_2 \leq ... \leq t_M \leq \overline{s}_i\right\}$; $t \in \Sigma_i$ represents β_i if $t_m = \inf\{\sigma|\beta_i(\sigma) \geq a_m\}$.

When competitors are restricted to select bids from a discrete set and employ monotone strategies $\left\{\beta_j\right\}_{j\neq i}$, their strategies can be represented by $T_{-i} \in \Sigma_{-i} = \prod_{j\neq i} \Sigma_j$, where $T_{-i} = \{t_j\}_{j\neq i}$ and $t_j \in \Sigma_j$. Similarly, $T \in \Sigma = \prod_{i=1}^n \Sigma_i$ represents the strategies of all bidders. The event where i wins with bid b given competitors strategies represented by T_{-i} will be denoted by: $\eta_i(b|T_{-i})$. The utility of bidder i when competitors use strategies represented by T_{-i} will be denoted by $U_i(b_i, s_i, x_i|T_{-i})$. This notation stresses that competitors are bidding from a discrete set of bids. Define bidder i's best response correspondence when restricted to choose from the set of bids $\mathcal{A}$ as:

$$b_i^*(s_i, x_i, T_{-i}, \mathcal{A}) = \arg\max_{b\in\mathcal{A}} U_i(b, s_i, x_i|T_{-i}).$$

Define the subset of Σ_i that represents monotone best response $b_i^*(s_i, x_i, T_{-i}, \mathcal{A})$ as:

$$T_i^{BR}(x_i, T_{-i}, \mathcal{A}, [\underline{s}_i, \overline{s}_i]) = \{t \in \Sigma_i : \forall s_i \in [\underline{s}_i, \overline{s}_i], t_m < s_i < t_{m+1} \implies a_m \in b_i^*(s_i, x_i, T_{-i}, \mathcal{A})\}.$$

The following results are used in the proof:

Lemma 7. *If $E(C_i|s, x_i)$ is nondecreasing in x_i then $b_i^*(s_i, x_i, T_{-i}, \mathcal{A})$ is nondecreasing in the strong set order in x_i.*

Proof: Consider $b' > b$ and $x_i' > x_i$. Let $\pi = P(S_{-i} \geq \eta_i(b|T_{-i})|s_i)$ denote the probability of the event where i wins with bid b given competitors strategies represented by T_{-i}. Define π' analogously for bid b'

$$\begin{aligned} & U_i\left(b', s_i, x_i'|T_{-i}\right) - U_i\left(b, s_i, x_i'|T_{-i}\right) \\ = \;& b'\pi' - b\pi + \int_{1(\eta_i(b|T_{-i}) \leq \tau \ngeq \eta_i(b'|T_{-i}))} E\left(C_i|s_i, S_i = \tau, x_i'\right) f_{S_{-i}|s_i}(\tau)\, d\tau \\ \geq \;& b'\pi' - b\pi + \int_{1(\eta_i(b|T_{-i}) \leq \tau \ngeq \eta_i(b'|T_{-i}))} E\left(C_i|s_i, S_i = \tau, x_i\right) f_{S_{-i}|s_i}(\tau)\, d\tau \\ \geq \;& U_i\left(b', s_i, x_i|T_{-i}\right) - U_i\left(b, s_i, x_i|T_{-i}\right) \end{aligned}$$

The function $U_i(b, s_i, x_i|T_{-i})$ exhibits increasing differences in (b, x_i); therefore $b_i^*(s_i, x_i, T_{-i}, \mathcal{A})$ is nondecreasing in the strong set order in x_i (by Topkis Theorem). □

Lemma 8. *Because $U_i(b, s_i, x_i|T_{-i})$ is continuous in (s_i, x_i, T_{-i}), the graph of $b_i^*(s_i, x_i, T_{-i}, \mathcal{A})$ as a function of (s_i, x_i, T_{-i})*[16] *is closed for any $\mathcal{A}$.*

Proof: Consider a sequence $\left(b^k, s^k, x_i^k, T_{-i}^k\right)$ that converges to (b, s_i, x_i, T_{-i}) such that $b^k \in b_i^*\left(s^k, x_i^k, T^k, \mathcal{A}\right)$. There is a K, such that for all $k > K$, $b^k = b$ and $U_i\left(b, s^k, x_i^k|T_{-i}^k\right) \geq U_i\left(a, s^k, x_i^k|T_{-i}^k\right)$ for all $a \in \mathcal{A}$. By continuity, it follows that $U_i(b, s_i, x_i|T_{-i}) \geq U_i(a, s_i, x_i|T_{-i})$; thus, $b \in b_i^*(s_i, x_i, T_{-i}, \mathcal{A})$. □

Lemma 9. *If the graph of $b_i^*(s_i, x_i, T_{-i}, \mathcal{A})$ as a function of (s_i, x_i, T_{-i}) is closed for all $s_i \in [\underline{s}_i, \overline{s}_i]$, then the graph $T_i^{BR}(x_i, T_{-i}, \mathcal{A}, [\underline{s}_i', \overline{s}_i'])$ as a function of $(x_i, T_{-i}, \underline{s}_i', \overline{s}_i')$ is also closed for all $[\underline{s}_i', \overline{s}_i']$ strictly included in $[\underline{s}_i, \overline{s}_i]$.*

Proof: Consider a sequence $\left(\underline{s}_i^k, \overline{s}_i^k, x_i^k, T_{-i}^k, t^k\right)$ that converges to $(\underline{s}_i, \overline{s}_i, x_i, T_{-i}, t)$ such that $t^k \in T_i^{BR}\left(x_i^k|T_{-i}^k, \mathcal{A}, \left[\underline{s}_i^k, \overline{s}_i^k\right]\right)$ for all k. Consider signal $s_i \in [\underline{s}_i, \overline{s}_i]$ such that $t_m < s_i < t_{m+1}$ for some $m \in \{0, ..., M\}$. Because t_m^k and t_{m+1}^k converge to t_m and t_{m+1}, there is a K such that $\forall k > K$, $t_m^k < s_i < t_{m+1}^k$, and thus $a_m \in b_i^*\left(s_i, x_i^k, T_{-i}^k, \mathcal{A}\right)$. Because b_i^* has a closed graph, $a_m \in b_i^*(s_i, x_i, T_{-i}, \mathcal{A})$. This argument is very similar to that in the proof of Lemma 3 in Athey (2001). □

Lemma 10. *If signals are affiliated; $E(C_i|s, x_i)$ is bounded, nondecreasing in s_{-i} and strictly increasing in s_i; and ties are precluded: $U_i(b', s_i, x_i|T_{-i}) \geq 0$, $U_i(b', s_i, x_i|T_{-i}) \geq U_i(b, s_i, x_i, T_{-i})$, $(s_i' - s_i)(b' - b) > 0$ imply $U_i(b', s_i', x_i|T_{-i}) \geq U_i(b, s_i', x_i|T_{-i})$.*

Proof: Consider Assumptions A.1 in Reny and Zamir (2004). A.1.i is satisfied by boundedness and continuity conditions on $E(C_i|s, x)$, A.1.ii by boundedness and risk neutrality, A.1.iii by monotonicity assumptions on the effect of s on $E(C_i|s, x_i)$ and A.1.iv by risk neutrality. Affiliation and assumptions on the joint density functions ensure that Assumption A.2 also holds. The result holds by Proposition 2.3 in that paper. □

Lemma 11. *Suppose that signals are affiliated and that $E(C_i|s, x_i)$ is bounded, nondecreasing in s_{-i} and strictly increasing in s_i. If $\forall s_i \in [\underline{s}_i, \overline{s}_i]$, $\exists a \in \mathcal{A}$ such that $U_i(a, s_i, x_i|T_{-i}) \geq 0$ then $b_i^*(s_i, x_i, T_{-i}, \mathcal{A})$ is nondecreasing in the strong set order with respect to $s_i \in [\underline{s}_i, \overline{s}_i]$.*

Proof: Let $\underline{s}_i \leq s_i < s_i' \leq \overline{s}_i$, $b \in b_i^*(s_i, x_i, T_{-i}, \mathcal{A})$ and $b' \in b_i^*(s_i', x_i, T_{-i}, \mathcal{A})$. Let $a, a' \in \mathcal{A}$ such that $U_i(a', s_i, x_i|T_{-i}) \geq 0$ and $U_i(a', s_i', x_i|T_{-i}) \geq 0$. Suppose that $b > b'$. Notice that $(s_i' - s_i)(b' - b) < 0$. Because $b \in b_i^*(s_i, x_i, T_{-i}, \mathcal{A})$

$$U_i(b, s_i, x_i|T_{-i}) \geq U_i\left(b', s_i, x_i|T_{-i}\right), \text{ and } U_i(b, s_i, x_i|T_{-i}) \geq U_i(a, s_i, x_i|T_{-i}) \geq 0.$$

Lemma 10 implies that $U_i(b, s_i', x_i|T_{-i}) \geq U_i(b', s_i', x_i|T_{-i})$ (b in this proof takes the place of b' in the lemma statement and vice versa). Thus $b \in b_i^*(s_i', x_i, T_{-i}, \mathcal{A})$. Similarly, because $b' \in b_i^*(s_i', x_i, T_{-i}, \mathcal{A})$

$$U_i\left(b', s_i', x_i, T_{-i}\right) \geq U_i\left(b, s_i', x_i, T_{-i}\right) \text{ and } U_i\left(b', s_i', x_i, T_{-i}\right) \geq U_i\left(a', s_i', x_i, T_{-i}\right) \geq 0$$

[16] For economy of notation, I will refer to the graph $\{s_i, x_i, T_{-i}, b \in [0,1] \times X_i \times \Sigma_{-i} \times \mathbb{R} : b \in b_i^*(s_i, x_i, T_{-i}, \mathcal{A})\}$ as the graph of $b_i^*(s_i, x_i, T_{-i}, \mathcal{A})$ as a function of (s_i, x_i, T_{-i}).

Lemma 10 implies that $U_i(b', s_i, x_i|T_{-i}) \geq U_i(b, s_i, x_i|T_{-i})$ (s_i in this proof takes the place of s_i' in the lemma statement and vice versa). Thus $b' \in b_i^*(s_i, x_i, T_{-i}, \mathcal{A})$. It has been shown that for any $s_i < s_i'$ in $[\underline{s}_i, \overline{s}_i]$, $b \in b_i^*(s_i, x_i, T_{-i}, \mathcal{A})$ and $b' \in b_i^*(s_i', x_i, T_{-i}, \mathcal{A})$ implies that $\max\{b, b'\} \in b_i^*(s_i', x_i, T_{-i}, \mathcal{A})$, $\min\{b, b'\} \in b_i^*(s_i, x_i, T_{-i}, \mathcal{A})$. $b_i^*(s_i, x_i, T_{-i}, \mathcal{A})$ is nondecreasing in s_i in the strong set order for $s_i \in [\underline{s}_i, \overline{s}_i]$. $\square$

Fixed Point Assume that $\mathcal{A} = \mathcal{A}_i = \{a_0 < a_1 < ... < a_M\}$ for all i and that ties are broken using a priority rule. $a_M = \infty$ is equivalent to nonparticipation and $U_i(a_M, s_i, x_i, T_{-i}) = 0$ for all (s_i, x_i, T_{-i}). Fix $s \in [0,1]^n$ and $\varepsilon \in [0, 1 - \max s]$. Bidders must bid a_M when they receive a signal above $1 - \varepsilon$. (Athey (2001) and Reny and Zamir (2004) use the same device). $\mathcal{A}$ will be omitted from the notation in b_i^* and T_i^{BR}. Similarly, when the set of signals $[\underline{s}_i, \overline{s}_i]$ is $[0, 1-\varepsilon]$ it will be omitted from the notation in T_i^{BR}. Subsets $\mathcal{B} \subset \mathcal{A}$ and $[\underline{s}_i, \overline{s}_i] \subset [0,1]$ will not be omitted.

Define the following correspondence:

$$\begin{aligned}
b_i^+(s_i, x_i, T_{-i}) &= b_i^*(s_i, x_i, T_{-i}) \text{ for all } x_i \in (\underline{x}_i, \overline{x}_i) \\
&= b_i^*(s_i, \underline{x}_i, T_{-i}) \cup \{b \in \mathcal{A} : b \leq \min b_i^*(s_i, \underline{x}_i, T_{-i})\} \text{ for } x_i = \underline{x}_i \\
&= b_i^*(s_i, \overline{x}_i, T_{-i}) \cup \{b \in \mathcal{A} : b \geq \max b_i^*(s_i, \overline{x}_i, T_{-i})\} \text{ for } x_i = \overline{x}_i
\end{aligned}$$

b_i^+ is an extension of the best response correspondence that includes all high bids when $x_i = \overline{x}_i$ and all low bids when $x_i = \underline{x}_i$. b_i^+ inherits the properties of b_i^*. If the graph of $b_i^*(s_i, x_i, T_{-i})$ as a function of (x_i, T_i) is closed, the graph of $b_i^+(s_i, x_i, T_{-i})$ is also closed. If $b_i^*(s_i, x_i, T_{-i})$ is non decreasing in the strong set order in s_i, $b_i^+(s_i, x_i, T_{-i})$ is also nondecreasing.

The goal is to find a set of monotone strategies $T \in \Sigma$ and vector $x \in X$, such that T represents a set of strategies that constitute an equilibrium of the game and $t_{i,\tilde{m}} \leq s_i \leq t_{i,\tilde{m}+1}$ for all i. If $S = s$ is realized under conditions x, all bidders bid $a_{\tilde{m}}$.[17] Let $\mathcal{B}_{\tilde{m}}^- = \{a_0, a_1, ..., a_{\tilde{m}}\} \cup \{a_M\}$, and $\mathcal{B}_{\tilde{m}}^+ = \{a_{\tilde{m}}, a_{\tilde{m}+1}, ..., a_M\}$.

$$\Gamma_i(x_i, T_{-i}) = \left\{ \begin{array}{c} (w_i, y_i) \in [\underline{x}_i, \overline{x}_i] \times \Sigma_i : \\ \exists q : \{y_{i,1}, .., y_{i,\tilde{m}}, q\} \in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^-, [0, s_i]\right), \\ \{y_{i,\tilde{m}+1}, .., y_{i,M}\} \in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^+, [s_i, 1-\varepsilon]\right), \text{ and} \\ \min b_i^+(s_i, w_i, T_{-i}) \leq a_{\tilde{m}} \leq \max b_i^+(s_i, w_i, T_{-i}) \end{array} \right\}. \tag{29}$$

Γ_i is a correspondence that maps elements of $[\underline{x}_i, \overline{x}_i] \times \Sigma_{-i}$ to subsets of $[\underline{x}_i, \overline{x}_i] \times \Sigma_i$. Let

$$\Gamma = \{\Gamma_1, ..., \Gamma_n\}. \tag{30}$$

Γ is a correspondence that maps elements of $X \times \Sigma$ onto subsets of the same set. The following set of Lemmas shows that the conditions to apply the Kakutani Fixed point theorem hold. Lemma 15 states the properties of a fixed point of Γ.

Lemma 12. Γ *is not empty.*

[17] The only exception is when $t_{i,\tilde{m}} = t_{i,\tilde{m}+1}$. In this case, bidder i bids strictly below (above) $a_{\tilde{m}}$ for all signals below (above) s_i.

Proof: By Assumption FPA.1 and Lemma 11, $b_i^*\left(\sigma_i, x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^-\right)$ is nondecreasing in the strong set order with respect to $\sigma_i \in [0, s_i]$. It follows that $T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^-, [0, s_i]\right)$ is not empty. By the same argument $T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^+, [s_i, 1-\varepsilon]\right)$ is not empty either. Let $y_i = \{y_1, .., y_M\} \in \Sigma_i$, where $\{y_1, .., y_{\tilde{m}}, q\} \in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^-, [0, s_i]\right)$ and $\{y_{\tilde{m}+1}, .., y_M\} \in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^+, [s_i, 1]\right)$. Now the focus is on finding an appropriate w_i. $b_i^+\left(s_i, x_i, T_{-i}\right)$ is nondecreasing in the strong set order with respect to x_i; moreover, it is not empty and has a closed graph. If $a_{\tilde{m}} \leq \max b_i^*\left(s_i, \underline{x}_i, T_{-i}\right)$, then $\left(\underline{x}_i, y_i\right) \in \Gamma_i\left(x_i, T_{-i}\right)$. If $a_{\tilde{m}} \geq \min b_i^*\left(s_i, \overline{x}_i, T_{-i}\right)$, then $\left(\overline{x}_i, y_i\right) \in \Gamma_i\left(x_i, T_{-i}\right)$. If $\max b_i^*\left(s_i, \underline{x}_i, T_{-i}\right) < a_{\tilde{m}} < \min b_i^*\left(s_i, \overline{x}_i, T_{-i}\right)$, consider $w_i^1 = 0.5\left(\underline{x}_i + \overline{x}_i\right)$. If $\min b_i^*\left(s_i, w_i^1, T_{-i}\right) \leq a_{\tilde{m}} \leq \max b_i^*\left(s_i, w_i^1, T_{-i}\right)$, then $\left(w_i^1, y_i\right) \in \Gamma_i\left(x_i, T_{-i}\right)$. Instead, if $a_{\tilde{m}} < \min b_i^*\left(s_i, w_i^1, T_{-i}\right)$, let $w_i^2 = 0.5\left(\underline{x}_i + w_i^1\right)$ while if $\max b_i^*\left(s_i, w_i^1, T_{-i}\right) < a_{\tilde{m}}$, $w_i^2 = 0.5\left(w_i^1 + \overline{x}_i\right)$. Repeat this procedure for w_i^2. Either this procedure eventually reaches some k such that $\min b_i^*\left(s_i, w_i^k, T_{-i}\right) \leq a_{\tilde{m}} \leq \max b_i^*\left(s_i, w_i^k, T_{-i}\right)$ and $\left(w_i^k, y_i\right) \in \Gamma_i\left(x_i, T_{-i}\right)$ or w_i^k converges to w_i. For all k such that $w_i < w_i^k$, $a_{\tilde{m}} < \min b_i^*\left(s_i, w_i^k, T_{-i}\right)$ whereas for all $w_i^k < w_i$, $\max b_i^*\left(s_i, w_i^k, T_{-i}\right) < a_{\tilde{m}}$. Let $\left\{w_i^{k_q}\right\}_q$ denote the subsequence such that $w_i < w_i^{k_q}$ for all q and $\left\{w_i^{k_r}\right\}_r$ denote the subsequence where $w_i^{k_r} < w_i$ for all r. By monotonicity in the strong set order $\max b_i^*\left(s_i, w_i^{k_q}, T_{-i}\right)$ converges to b^+ and $\min b_i^*\left(s_i, w_i^{k_r}, T_{-i}\right)$ converges to b^-, where $b^- < a_{\tilde{m}} < b^+$. Because b_i^* has a closed graph, then $b^+ \in b_i^*\left(s_i, w_i, T_{-i}\right)$ and $b^- \in b_i^*\left(s_i, w_i, T_{-i}\right)$. It follows that $\left(w_i, y_i\right) \in \Gamma_i\left(x_i, T_{-i}\right)$. Let $w = \{w_i\}_{i=1}^n$, and $Y = \{y_i\}_{i=1}^n$ such that $\left(w_i, y_i\right) \in \Gamma_i\left(x_i, T_{-i}\right)$, then $(w, Y) \in \Gamma(x, T)$. □

Lemma 13. Γ *has a closed graph.*

Proof: Consider a sequence $\left(x^k, T^k, w^k, Y^k\right)$ that converges to (x, T, w, Y) such that $\left(w^k, Y^k\right) \in \Gamma\left(x^k, T^k\right)$ for all k. Consider bidder i. For all k, $\left(w_i^k, y_i^k\right) \in \Gamma_i\left(x_i^k, T_{-i}^k\right)$, and there is a q^k such that $\left\{y_{i1}^k, ..., y_{i\tilde{m}}^k, q^k\right\} \in T_i^{BR}\left(x_i^k | T_{-i}^k, \mathcal{B}_{\tilde{m}}^-, [0, s_i]\right)$. Take a subsequence of q^k that converges to q. $U_i\left(b, s_i, x_i | T_{-i}\right)$ is continuous in $\left(x_i, T_{-i}\right)$. By Lemmas 8 and 9, $\{y_{i1}, ..., y_{i\tilde{m}}, q\} \in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^-, [0, s_i]\right)$. By the same argument, $\{y_{i\tilde{m}+1}, .., y_{iM}\} \in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^+, [s_i, 1-\varepsilon]\right)$. For all k, $\min b_i^+\left(s_i, w_i^k, T_{-i}^k\right) \leq a_{\tilde{m}} \leq \max b_i^+\left(s_i, w_i^k, T_{-i}^k\right)$. A subsequence of $\min b_i^+\left(s_i, w_i^k, T_{-i}^k\right)$ converges to b^- and a subsequence of $\max b_i^+\left(s_i, w_i^k, T_{-i}^k\right)$ converges to b^+, where $b^- \leq a_{\tilde{m}} \leq b^+$. Because $b_i^+\left(s_i, w_i, T_{-i}\right)$ has a closed graph, $b^-, b^+ \in b_i^+\left(s_i, w_i, T_{-i}\right)$, which implies that $\min b_i^+\left(s_i, w_i, T_{-i}\right) \leq a_m \leq \max b_i^+\left(s_i, w_i, T_{-i}\right)$. It follows that $\left(w_i, y_i\right) \in \Gamma\left(x_i, T_{-i}\right)$ for each i, which implies that $(w, Y) \in \Gamma(x, T)$. □

Lemma 14. Γ *is convex.*

Proof: Take $\left(w_i, y_i\right)$,$\left(w_i', y_i'\right) \in \Gamma_i\left(x_i, T_{-i}\right)$. Let $y_i'' = \lambda y_i + (1-\lambda) y_i'$ and $w_i'' = \lambda w_i + (1-\lambda) w_i'$ for some $\lambda \in [0, 1]$.

$$\begin{aligned}
\{y_{i1}, ..., y_{i\tilde{m}}, q\}, \left\{y_{i1}', ..., y_{i\tilde{m}}', q'\right\} &\in T_i^{BR}\left(x_i | T_{-i}, \mathcal{B}_{\tilde{m}}^-, [0, s_i]\right) \\
\{y_{i\tilde{m}}, .., y_{iM}\}, \left\{y_{i\tilde{m}}', .., y_{iM}'\right\} &\in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^+, [s_i, 1-\varepsilon]\right)
\end{aligned}$$

By Lemma 11, $b_i^*\left(\sigma_i, x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^-\right)$ and $b_i^*\left(\sigma_i, x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^+\right)$ are nondecreasing in the strong set order with respect to $\sigma_i \in [0, s_i]$ and $\sigma_i \in [s_i, 1]$, respectively. Lemma 2 in Athey (2001) ensures that both $T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^-, [0, s_i]\right)$ and $T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^+, [s_i, 1-\varepsilon]\right)$ are convex. Let $q'' = \lambda q + (1-\lambda) q'$, it follows that

$$\begin{aligned}\left\{y_{i1}'', ..., y_{i\tilde{m}}'', q''\right\} &\in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^-, [0, s_i]\right) \\ \left\{y_{i\tilde{m}+1}'', .., y_{iM}''\right\} &\in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^+, [s_i, 1-\varepsilon]\right)\end{aligned}$$

Without loss, assume that $w_i < w_i'' < w_i'$. It is known that

$$\begin{aligned}\min b_i^+\left(s_i, w_i, T_{-i}\right) &\leq a_{\tilde{m}} \leq \max b_i^+\left(s_i, w_i, T_{-i}\right) \\ \min b_i^+\left(s_i, w_i', T_{-i}\right) &\leq a_{\tilde{m}} \leq \max b_i^+\left(s_i, w_i', T_{-i}\right).\end{aligned}$$

By Lemma 7, $b_i^*\left(s_i, w_i, T_{-i}\right) \leq_s b_i^*\left(s_i, w_i'', T_{-i}\right) \leq_s b_i^*\left(s_i, w_i', T_{-i}\right)$. It follows that $b_i^+\left(s_i, w_i, T_{-i}\right) \leq_s b_i^+\left(s_i, w_i'', T_{-i}\right) \leq_s b_i^+\left(s_i, w_i', T_{-i}\right)$, which implies:

$$\begin{aligned}\min b_i^+\left(s_i, w_i'', T_{-i}\right) &\leq \min b_i^+\left(s_i, w_i', T_{-i}\right) \leq a_{\tilde{m}} \\ a_{\tilde{m}} &\leq \max b_i^+\left(s_i, w_i, T_{-i}\right) \leq \max b_i^+\left(s_i, w_i'', T_{-i}\right)\end{aligned}$$

Therefore, $\left(w_i'', y_i''\right) \in \Gamma_i\left(x_i, T_{-i}\right)$. Repeating the argument for all i, it follows that Γ is convex. $\square$

Lemma 15. *Γ has a fixed point. If (x, T) is a fixed point of Γ then $t_{i\tilde{m}} \leq s_i \leq t_{i\tilde{m}+1}$ for all i. Moreover, (i) if $x_i = \underline{x}_i$ then $a_{\tilde{m}} \leq \max b_i^*\left(s_i, \underline{x}_i, T_{-i}\right)$; (ii) $a_{\tilde{m}} \leq \max b_i^*\left(s_i, x_i, T_{-i}\right)$ implies that for any $\sigma_i > s_i$ such that $t_{im} < \sigma_i < t_{im+1}$, $a_m \in b_i^*\left(\sigma_i, x_i, T_{-i}\right)$; (iii) if $x_i = \overline{x}_i$ then $\min b_i^*\left(s_i, \overline{x}_i, T_{-i}\right) \leq a_{\tilde{m}}$; (iv) $\min b_i^*\left(s_i, x_i, T_{-i}\right) \leq a_{\tilde{m}}$ implies that for any $\sigma_i < s_i$ such that $t_{im} < \sigma_i < t_{im+1}$, $a_m \in b_i^*\left(\sigma_i, x_i, T_{-i}\right)$; (v) if $x_i \notin \left\{\underline{x}_i, \overline{x}_i\right\}$ then $\min b_i^*\left(s_i, x_i, T_{-i}\right) \leq a_{\tilde{m}} \leq \max b_i^*\left(s_i, x_i, T_{-i}\right)$ and $t_i \in T_i^{BR}\left(x_i | T_{-i}\right)$.*

Proof: Existence follows from Kakutani's Fixed Point Theorem. $t_{i\tilde{m}} \leq s_i \leq t_{i\tilde{m}+1}$ because $\left\{t_{i1}, .., t_{i\tilde{m}}, q\right\} \in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^-, [0, s_i]\right)$ and $\left\{t_{i\tilde{m}+1}, .., t_{iM}\right\} \in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^+, [s_i, 1]\right)$. (i) If $x_i = \underline{x}_i$, then $a_{\tilde{m}} \leq \max b_i^*\left(s_i, \underline{x}_i, T_{-i}\right)$. (ii) Notice the implications of $a_{\tilde{m}} \leq \max b_i^*\left(s_i, x_i, T_{-i}\right)$. Because b_i^* is monotone in the strong order in the signal, $m > \tilde{m}$, $\sigma > s_i$ and $a_m \in b_i^*\left(\sigma, x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^+\right)$ imply $a_m \in b_i^*\left(\sigma, x_i, T_{-i}\right)$. Therefore, for any $\sigma > s_i$ such that $t_{im} < \sigma < t_{im+1}$, $a_m \in b_i^*\left(\sigma, x_i, T_{-i}\right)$. (iii) If $x_i = \overline{x}_i$, $\min b_i^*\left(s_i, \overline{x}_i, T_{-i}\right) \leq a_{\tilde{m}}$. (iv) Notice the implications of $\min b_i^*\left(s_i, x_i, T_{-i}\right) \leq a_{\tilde{m}}$. Because b_i^* is monotone in the strong order in the signal, $m < \tilde{m}$, $\sigma < s_i$ and $a_m \in b_i^*\left(\sigma, x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^-\right)$ imply $a_m \in b_i^*\left(\sigma, x_i, T_{-i}\right)$. Therefore, for any $\sigma < s_i$ and $m < q$ such that $t_{im} < \sigma < t_{im+1}$, $a_m \in b_i^*\left(\sigma, x_i, T_{-i}\right)$. It remains to show that if $q < \sigma < s_i$ then $a_{\tilde{m}} \in b_i^*\left(\sigma, x_i, T_{-i}\right)$. Suppose that $a_{\tilde{m}} \notin b_i^*\left(\sigma, x_i, T_{-i}\right)$. This implies that there is a $m' < \tilde{m}$, such that $a_{m'} \in b_i^*\left(\sigma, x_i, T_{-i}\right)$ and that $a_M \in b_i^*\left(\sigma, x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^-\right)$. It follows that $U_i\left(a_{m'}, \sigma, x_i, T_{-i}\right) = 0$. Because all bids have positive probability of winning, $a_{m'} - E\left(C_i | s_i, S_{-i} \geq S_{-i} \geq \eta_i\left(a_{m'} | T_{-i}\right), x_i\right) = 0$ which implies that $U_i\left(a_{m'}, \sigma + \varepsilon, x_i, T_{-i}\right) < 0$ and that $a_{m'} \notin b_i^*\left(\sigma + \varepsilon, x_i, T_{-i}\right)$. There is no $m' < \tilde{m}$ such that $a_{m'} \in b_i^*\left(\sigma, x_i, T_{-i}\right)$ for all σ in any subset of $[q, s_i]$. It follows that either $q = s_i$ or that $a_{\tilde{m}} \in b_i^*\left(\sigma, x_i, T_{-i}\right)$ for all $q < \sigma < s_i$. (v) If $x_i \notin \left\{\underline{x}_i, \overline{x}_i\right\}$: $\min b_i^*\left(s_i, x_i, T_{-i}\right) \leq a_{\tilde{m}} \leq \max b_i^*\left(s_i, x_i, T_{-i}\right)$. The results in (ii) and (iv) apply. Therefore, $t_i \in T_i^{BR}\left(x_i, T_{-i}\right)$. $\square$

Lemma 16. Γ *has a closed graph with respect to* s *and* ε. *The fixed point also has a closed graph.*

Proof: Consider a sequence $\left(x^k, T^k, w^k, Y^k, s^k, \varepsilon^k\right) \to (x, T, w, Y, s, \varepsilon)$ where $\left(x^k, T^k\right) = \Gamma\left(w^k, Y^k | s^k, \varepsilon^k\right)$ for all k. Consider bidder i. For all k, $\left(x_i^k, t_i^k\right) \in \Gamma_i\left(w_i^k, Y_{-i}^k | s^k, \varepsilon^k\right)$ and there is a q^k such that $\left\{y_{i1}^k, ..., y_{i\tilde{m}}^k, q^k\right\} \in T_i^{BR}\left(x_i^k | T_{-i}^k, \mathcal{B}_{\tilde{m}}^-, \left[0, s_i^k\right]\right)$. Take a subsequence of q^k that converges to q. By Lemmas 8 and 9, $\{y_{i1}, ..., y_{i\tilde{m}}, q\} \in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^-, [0, s_i]\right)$. By the same argument, $\{y_{i\tilde{m}+1}, .., y_{iM}\} \in T_i^{BR}\left(x_i, T_{-i}, \mathcal{B}_{\tilde{m}}^+, [s_i, 1-\varepsilon]\right)$. For all k, $\min b_i^+\left(s_i^k, w_i^k, T_{-i}^k\right) \leq a_{\tilde{m}} \leq \max b_i^+\left(s_i^k, w_i^k, T_{-i}^k\right)$. A subsequence of $\min b_i^+\left(s_i^k, w_i^k, T_{-i}^k\right)$ converges to b^- and a subsequence of $\max b_i^+\left(s_i^k, w_i^k, T_{-i}^k\right)$ converges to b^+, where $b^- \leq a_{\tilde{m}} \leq b^+$. Because $b_i^+(s_i, w_i, T_{-i})$ has a closed graph, $b^-, b^+ \in b_i^+(s_i, w_i, T_{-i})$, which implies that $\min b_i^+(s_i, w_i, T_{-i}) \leq a_m \leq \max b_i^+(s_i, w_i, T_{-i})$. It follows that $(w_i, y_i) \in \Gamma(x_i, T_{-i} | s_i, \varepsilon)$ for each i, which implies that $(w, Y) \in \Gamma(x, T | s, \varepsilon)$. This proves the first part.

Consider a sequence $(x^n, T^n, s^n, \varepsilon^n) \to (x, T, s, \varepsilon)$ where $(x^n, T^n) \in \Gamma(x^n, T^n | s^n, \varepsilon^n)$ for all n. Because Γ has a closed graph in (s, ε), then $(x, T) \in \Gamma(x, T | s, \varepsilon)$. This proves the second part. $\square$

Equilibrium Strategies Let Γ^ε be the mapping defined in (29) and (30) some $\varepsilon > 0$, s^ε such that $s_i^\varepsilon = \min(s_i, 1 - 2\varepsilon)$, and $\mathcal{A} = \{a_0, a_1, ..., a_{M-1}, a_M\}$ such that $a_{\tilde{m}} \in \mathcal{A}$. Let $(x^\varepsilon, T^\varepsilon)$ be a fixed point of Γ^ε and (x, T) be the limit of some subsequence of $\{(x^\varepsilon, T^\varepsilon)\}_\varepsilon$ as $\varepsilon \to 0$. Denote $\pi_i = \Pr(S_{-i} \geq s_{-i} | S_i = 1)$. The following Lemma shows that T is an equilibrium of the discrete auction game.

Lemma 17. *If there is a* $\Delta > 0$ *such that for every* i*: (i)* $\bar{x}_i \geq a_{\tilde{m}} + \Delta - \underline{c}_i$ *and* $\underline{x}_i \leq a_{\tilde{m}} - \max_{j \neq i}\left(\bar{c}_j - \underline{c}_j + 3\Delta\right)(1 - \pi_i)^{-1} - \bar{c}_i$*, (ii)* $a_0 \leq \min_i(\underline{c}_i + \underline{x}_i)$*, and* $\max_i(\bar{c}_i + \overline{x}_i) \leq a_{M-1} < a_M = \infty$*, (iii)* $a_m - a_{m-1} < \Delta$ *for all* $m \in \{1, 2, ..., M-1\}$*, and (iv)* $a_{\tilde{m}} < a_{M-n}$*; then* $t_i \in T_i^{BR}(x_i | T_{-i})$ *and* $t_{ia_M} = 1$ *for all* i.

Proof: By Lemma 16, $(x, T) \in \Gamma^0(x, T)$. For all $\varepsilon > 0$, every type of bidder makes strictly positive profits bidding $a_{M-1} \geq \max_i(\bar{c}_i + \overline{x}_i)$. Bidding a_M is suboptimal: $t_{iM}^\varepsilon = t_{iM} = 1$ for all i.

Suppose that $x_i \notin \{\underline{x}_i, \overline{x}_i\}$. Lemma 15 implies that $\min b_i^*(s_i, x_i, T_{-i}) \leq a_{\tilde{m}} \leq \max b_i^*(s_i, x_i, T_{-i})$, and $t_i \in T_i^{BR}(x_i | T_{-i})$.

Suppose that $x_i = \overline{x}_i$. Take any $m < \tilde{m}$.

$$U_i(a_m, s_i, \bar{x}_i | T_{-i}) = [a_m - E(c_i(S) | s_i, S_{-i} \in S_{-i} \geq \eta_i(a_m | T_{-i})) - \bar{x}_i] P(S_{-i} \geq \eta_i(a_m | T_{-i}) | s_i) \leq 0$$

Therefore, $\min b_i^*(s_i, \bar{x}_i, T_{-i}) \geq a_{\tilde{m}}$. Lemma 15 ensures that $\min b_i^*(s_i, \bar{x}_i, T_{-i}) = a_{\tilde{m}}$, and $t_i \in T_i^{BR}(x_i | T_{-i})$.

Suppose that $x_i = \underline{x}_i$. By Lemma 15, $a_{\tilde{m}} \leq \max b_i^*(s_i, \bar{x}_i, T_{-i})$. Suppose that $t_i \notin T_i^{BR}(x_i | T_{-i})$, and therefore, $a_{\tilde{m}} < \min b_i^*(s_i, \bar{x}_i, T_{-i})$.

A similar argument to that used to prove Part (v) in Lemma C.1 shows that there is at most one bidder i with $\bar{c}_i + x_i < a_{\tilde{m}} - \Delta$. Following that proof, let a_k be the maximum bid with a strictly positive probability of winning and $\overline{s}_j = t_{j(k+1)}$. Note that $\tilde{m} \leq k \leq M + 1$. If $a_k - E(C_i | \overline{s}_i, S_{-i} \geq \eta(a_k | T_{-i}), x) > 0$ then $\overline{s}_i = 1$. Otherwise, there has to be a bidder j such that $\overline{s}_j = 1$ which implies that bidder i should bid strictly below a_{k+1} when it receives signal $\overline{s}_i + \varepsilon$ for some $\varepsilon > 0$ contradicting the definition of $\overline{s}_i$. Suppose that there are two or more bidders with $a_k - E\left(C_i|\overline{s}_i, S_{-i} \geq \eta\left(a_k|T_{-i}\right), x\right) > \Delta$; therefore, $\overline{s}_i = 1$ and $t_{ik} < 1$. One of these bidders has a profitable deviation towards a_{k-1} which ensures positive probability of winning and positive profits. Therefore, for all but one bidder:

$$a_{\tilde{m}} \leq a_k \leq E\left(C_i|\overline{s}_i, S_{-i} \geq \eta\left(a_k|T_{-i}\right), x\right) + \Delta \leq \overline{c}_i + x_i + \Delta.$$

By the definition of $\underline{x}_i$, $\underline{x}_i + \bar{c}_i < a_{\tilde{m}} - \Delta \leq a_k - \Delta$. For all $j \neq i$: $x_j > \underline{x}_j$ and $\bar{c}_j + x_j \geq a_k - \Delta > a_{\tilde{m}} - \Delta$. Consider two bids, $\underline{b}$ and $\overline{b} \in \mathcal{A}$, such that: $\underline{b}$ wins with probability one and $\overline{b} \in b_i^*\left(s_i, \bar{x}_i, T_{-i}\right)$. Because $x_i + \bar{c}_i + \Delta < a_{\tilde{m}} < \min b_i^*\left(s_i, \bar{x}_i, T_{-i}\right) \leq \overline{b}$, i makes positive profits when bidding $\overline{b}$ and has to win with positive probability. Otherwise, it has a profitable deviation to $a_{\tilde{m}-1}$. By optimality of $\overline{b}$,

$$\underline{b} \leq \bar{b}\Pr\left(S_{-i} \geq \eta\left(\bar{b}\right)|s_i\right) + \left(1 - \Pr\left(S_{-i} \geq \eta\left(\bar{b}\right)|s_i\right)\right) E\left(C_i|s_i, S_{-i} \ngeq \eta\left(\bar{b}\right), x|s_i\right).$$

Using $\pi_i \geq \Pr\left(S_{-i} \geq \eta\left(\bar{b}\right)|s_i\right) > 0$ and $\underline{x}_i + \bar{c}_i < a_{\tilde{m}} < \min b_i^*\left(s_i, \bar{x}_i, T_{-i}\right) \leq \bar{b}$,

$$\underline{b} \leq \bar{b}\pi_i + \left(1 - \pi_i\right)\left(\bar{c}_i + \underline{x}_i\right).$$

Replacing $\underline{x}_i$:

$$\underline{b} + \max_{j \neq i}\left(\bar{c}_j - \underline{c}_j\right) + 3\Delta \leq \bar{b}\pi_i + \left(1 - \pi_i\right) a_{\tilde{m}} \leq \bar{b}.$$

Bidders $j \neq i$ are best-responding so they never bid below $\underline{c}_j + x_j$. Because $\underline{b}$ is any bid in $\mathcal{A}$ that wins with probability one it is possible to pick it so that $\underline{b} + \Delta \geq \min_j \underline{c}_j + x_j$:

$$\min_{j \neq i}\left(\bar{c}_j + x_j\right) + 2\Delta \leq \bar{b}$$

Because for all $j \neq i$: $x_j > \underline{x}_j$ and $\bar{c}_j + x_j \geq a_k - \Delta \geq \bar{b} - \Delta$. Replacing $\bar{b}$ in the expression above leads to a contradiction. Therefore, $t_i \in T_i^{BR}\left(x_i|T_{-i}\right)$. □

The following Lemma shows existence of equilibria in the continuous bid auction.

Lemma 18. *If for every i, $\bar{x}_i > t - \underline{c}_i$ and $\underline{x}_i < t - \max_{j\neq i}\left(\bar{c}_j - \underline{c}_j\right)\left(1 - \pi_i\right)^{-1} - \bar{c}_i$, there exist a vector of cost shifters $x \in [\underline{x}_1, \overline{x}_1] \times ... \times [\underline{x}_n, \overline{x}_n]$ and an equilibrium in monotone pure strategies where $\sigma < s_j < \sigma'$ implies $\hat{\beta}_j(\sigma) < t$ and $\hat{\beta}_j(\sigma') > t$.*

Proof: Let $\bar{\Delta} = \min\left\{\bar{x}_i - t + \underline{c}_i, \left(t - \underline{x}_i - \bar{c}_i\right)\left(1 - \pi_i\right)/3 - \max_{j\neq i}\left(\bar{c}_j - \underline{c}_j\right)/3\right\}$. For all $\Delta < \bar{\Delta}$, construct a grid $\{a_0, a_1, ..., a_{M-1}, a_M\}$, with $a_0 = \min_i\left(\underline{c}_i + \underline{x}_i\right)$, $a_{M-1} = \max_i\left(\bar{c}_i + \overline{x}_i\right), a_M = \infty$, and $a_m - a_{m-1} = \Delta/2$ for all $m \in \{0, 1, ...\tilde{m} - 1, \tilde{m} + 2, ..., M\}$ and $a_{\tilde{m}} = t$, $a_{\tilde{m}+1} - a_{\tilde{m}-1} = \Delta/2$. This grid satisfies the conditions of Lemma 16. Let $\left(x^\Delta, T^\Delta\right)$ be a the limit of some subsequence of fixed points with $\varepsilon \to 0$ corresponding to this grid. $x^\Delta \in [\underline{x}_1, \overline{x}_1] \times ... \times [\underline{x}_n, \overline{x}_n]$, $t^\Delta_{j\tilde{m}} \leq s_j \leq t^\Delta_{j\tilde{m}+1}$ and T^Δ is an equilibrium of the discrete game that represents strategies β^Δ. By Helley's Selection Theorem and by compactness of X, there exist a subsequence of $\left\{\left(x^\Delta, \beta^\Delta\right)\right\}_\Delta$ that converges to $\left(\hat{x}, \hat{\beta}\right)$ as $\Delta \to 0$, where $\hat{x} \in X$ and $\hat{\beta}$ is a set of n monotone functions.

I need to show that $\hat{\beta}$ is an equilibrium of the unrestricted game when cost shifters are $\hat{x}$. The proof of the second part of Theorem 2.1 in Reny and Zamir (2004) would apply without modifications were it not for the different treatment of the restriction that bidders with signals above $1-\varepsilon$ must bid ∞. They assume that $\varepsilon \to 0$ as $\Delta \to 0$, I assume that for each Δ, $\left(x^{\Delta}, T^{\Delta}\right)$ is the limit of $\left(x^{\Delta,\varepsilon}, T^{\Delta,\varepsilon}\right)$ as $\varepsilon \to 0$. This technical detail only changes the proof of (A.4) in that paper.

Take any j, pick any σ_j such that $\hat{\beta}_j(\sigma_j + v) < \infty$ for some $v > 0$. For all $\Delta < \bar{\Delta}$, Lemma 17 shows that $\beta_j^{\Delta}\left(\sigma_j, x^{\Delta}\right) < \infty$. Because $\beta_j^{\Delta}\left(\sigma_j, x^{\Delta}\right)$ is a limit of strategies where all bids win with positive probability:

$$\begin{aligned}
0 &\leq \beta_j^{\Delta}\left(\sigma_j, x^{\Delta}\right) - E\left(C_j|\sigma_j, S_{-j} \geq \eta_j\left(\beta_j^{\Delta}\left(\sigma_j, x^{\Delta}\right)|T_{-j}^{\Delta}\right), x^{\Delta}\right) \\
&\leq \beta_j^{\Delta}\left(\sigma_j, x^{\Delta}\right) - E\left(C_j|\sigma_j, S_{-j} \geq \eta_j\left(\hat{\beta}_j(\sigma_j, \hat{x}) - \delta|T_{-j}^{\Delta}\right), x^{\Delta}\right) \\
&\to \hat{\beta}(\sigma_j, \hat{x}) - E\left(C_j|\sigma_j n, S_{-j} \geq \eta_j\left(\hat{\beta}_j(\sigma_j, \hat{x})|\hat{\beta}_{-j}\right), \hat{x}\right).
\end{aligned}$$

The second inequality follows for any $\delta > 0$ for a sufficiently low Δ because the marginal cost is lower than the inframarginal. Take the limit as $\Delta \to 0$ for each δ such that $\hat{\beta}_j(\sigma_j) - \delta$ is not an atom in the distribution of $M_j = \min_{k \neq j} \hat{\beta}_k(S_j, \hat{x})$. Take the limit as δ goes to zero to obtain last line. This is (A.4) in Reny and Zamir (2004). The rest of the proof there shows that $\hat{\beta}$ is an equilibrium of the first-price auction game when cost shifters are $\hat{x}$ and bidders are allowed to bid any value between $\min_i(\underline{c}_i + \underline{x}_i)$ and $\max_i(\overline{c}_i + \overline{x}_i)$.

It remains to show that in equilibrium the tie-curve passes through the vector of signals s. For each q, $t_{j\tilde{m}}^{\Delta} \leq s_j \leq t_{j\tilde{m}+1}^{\Delta}$. Take any $\sigma < s_j$, $\beta_j^{\Delta}\left(\sigma, x^{\Delta}\right) \leq t$; take any $\sigma' > s_j$, $\beta_j^{\Delta}\left(\sigma', x^{\Delta}\right) \geq t$. It follows that $\hat{\beta}_j(\sigma, \hat{x}) \leq t$ and $\hat{\beta}_j(\sigma', \hat{x}) \geq t$. □

Proof of Lemma 6 Take any $\hat{s} \in (0,1)^n$ such that $\hat{s}_i = s_i$ and $\hat{s}_{-i} \geq s_{-i}$. Let $\mathcal{X}_{\varepsilon} = \prod_j \left[\underline{x}_j, \overline{x}_j\right]$ where

$$\begin{aligned}
\overline{x}_j &= \overline{c}_i - \underline{c}_j + \varepsilon \\
\underline{x}_j &= \overline{c}_i - \frac{\max_{k \neq j}(\overline{c}_k - \underline{c}_k)}{1 - \Pr(S_{-j} \geq \hat{s}_{-j}|S_j = 1)} - \overline{c}_j - \varepsilon.
\end{aligned}$$

There is an $\varepsilon > 0$ such that $\tilde{x} \in \mathcal{X}_{\varepsilon}$ implies that $\tilde{x}_{-i} - \tilde{x}_i + x_i \in X_{-i}^o$:

$$\begin{aligned}
\tilde{x}_j - \tilde{x}_i + x_i &\geq x_i - (\overline{c}_j - \underline{c}_i) - \frac{\max_{k \neq j}(\overline{c}_k - \underline{c}_k)}{1 - \Pr(S_{-j} \geq \hat{s}_{-j}|S_j = 1)} - 2\varepsilon, \\
\tilde{x}_j - \tilde{x}_i + x_i &\leq x_i + \overline{c}_i - \underline{c}_j + \frac{\max_{k \neq j}(\overline{c}_k - \underline{c}_k)}{1 - \Pr(S_{-j} \geq \hat{s}_{-j}|S_j = 1)} + 2\varepsilon.
\end{aligned}$$

Lemma 18 shows that there exist a vector of cost shifters $\tilde{x} \in \mathcal{X}_{\varepsilon}$ and an equilibrium in monotone pure strategies where for all j, $\sigma < s_j < \sigma'$ implies $\hat{\beta}_j(\sigma, \tilde{x}) < \overline{c}_i$ and $\hat{\beta}_j(\sigma', \tilde{x}) > \overline{c}_i$. Cost shifters $x_{-i} = \tilde{x}_{-i} - \tilde{x}_i + x_i \in X_{-i}^o$ are such that strategies β defined as $\beta_j(\cdot, [x_i, x_{-i}]) := \hat{\beta}_j(\cdot, \tilde{x}) - \tilde{x}_i + x_i$ constitute an equilibrium in monotone pure strategies for the continuous auction game with market conditions $[x_i, x_{-i}]$. Moreover, $\sigma < \hat{s}_j < \sigma'$ implies $\beta_j(\sigma, [x_i, x_{-i}]) \leq \overline{c}_i - \tilde{x}_i + x_i$ and

$\beta_j(\sigma', [x_i, x_{-i}]) \geq \bar{c}_i - \tilde{x}_i + x_i$ for all j. Letting $t = \bar{c}_i - \tilde{x}_i + x_i$ and noting that equilibrium strategies are strictly increasing and whenever there is a jump discontinuity, bidders are indifferent between the two limiting bids implies that $\beta_j(\tau, x) \leq t$ if and only if $\tau \leq \hat{s}_j$.

If $n = 2$, the equilibrium bid strategies are continuous, $\beta_i(\cdot, x) = Q_{B_j|x}(\cdot)$, and Condition C.3 holds. □

C.2 Computation of an Equilibrium in Monotone Pure Strategies

The equilibrium inverse bid functions are calculated using a numerical algorithm that is similar to that in Gayle and Richard (2008). The main difference with their approach is that I allow for interdependent costs and correlated signals. Although the computation time grows fast with the number of bidders, auctions with less than 10 bidders are solved within 5 hours. The algorithm solves the system of differential equations implied by bidders' first-order conditions. Following Gayle and Richard (2008), I guess the initial conditions and, after the system is solved forward, I verify if the terminal conditions are consistent with equilibrium bidding behavior.

C.2.1 Technology and information

Each bidder full information cost is:

$$c_i(s_{-i}, s_i, x_0, x_i) = \hat{\alpha}'_{i0}x_0 + \hat{\alpha}_{i1}x_i + \hat{\alpha}_{i2}\sum_{j\neq i}\hat{\mu}_{ij}\Phi^{-1}(s_j) + \hat{\alpha}_{i3}\Phi^{-1}(s_i) \tag{31}$$

where $\hat{\alpha}$ denote the estimated parameters. The distribution of signals is truncated to avoid numerical problems: instead of using $\Phi^{-1}(S_i)$, I use $\Phi^{-1}((S_i - 0.5)\,0.999 + 0.5)$. The joint copula of signals is still assumed to be Gaussian with correlation matrix equal to $\hat{L}\hat{L}' + \hat{\Lambda}$, where $\hat{L}$ is the estimated loading matrix and $\hat{\Lambda}$ is a diagonal with i-th element equal to $1 - \hat{L}_{i1}^2 - \hat{L}_{i2}^2$. The (rescaled) signals $\Phi^{-1}(S)$ are assumed to be jointly multivariate normal with covariance matrix $\hat{\Sigma}$.

C.2.2 System of differential equations

Bidder i's problem is

$$\max_b bP(S \geq s_{-i}(b)\,|s_i) - \phi(s_{-i}(b)\,|s_i, x_i), \tag{32}$$

where $s_{-i}(b)$ is equal to $\left[H_{B_j|x}(b)\right]_{j\neq i}$. The first-order condition is

$$P_i + b\nabla P_i s'_{-i}(b) - \nabla\phi_i s'_{-i}(b) = 0, \tag{33}$$

where ∇P_i and $\nabla\phi_i$ are the gradients of $P_i = P(S_{-i} \geq s_{-i}(b)|s_i)$ and $\phi(s_{-i}(b)\,|s_i, x_i)$ with respect to s_{-i}. s'_{-i} is the vector of derivatives of each element of s_{-i} with respect to b. P_i and b are scalars, ∇P_i and $\nabla\phi_i$ are $1 \times (n-1)$ vectors, and $s'_{-i}(b)$ is an $(n-1) \times 1$ vector.

If all bidders' first-order conditions are considered together:

$$P = Ms', \tag{34}$$

where P is an $n \times 1$ vector with typical element: P_i, M is an $n \times n$ matrix with zeros in the main diagonal and typical row: $\nabla\phi_i - b\nabla P_i$, and s' is an $n \times 1$ vector.

C.2.3 Algorithm

The algorithm makes an initial guess on the lowest bid that firms are willing to submit when they receive signal $s = 0$. So the following variables are initialized: $b^{(0)}$, and $s^{(0)} = [0, 0, ..., 0]'$. The state variables are the scalar $b^{(t)}$ and the vector $s^{(t)}$.

At step t, each bidder's perceived probability of winning $P_i^{(t)} = P\left(S_{-i} \geq s_{-i}^{(t)} | s_i^{(t)}\right)$, expected cost $\phi\left(s_{-i}^{(t)} | s_i^{(t)}, x_i\right)$ and the gradients ∇P_i and $\nabla \phi_i$ are calculated by numerical integration . A bidder is "active" if it finds it profitable to win at the current state— if $b^{(t)} P_i^{(t)} - \phi\left(s_i^{(t)}, s_{-i}^{(t)}\right) \geq 0$. Bidders that do not satisfy this condition are "inactive" and their signal state in the next step is: $s_j^{(t+1)} = s_j^{(t)}$. Construct vector $P^{(t)}$ and matrix $M^{(t)}$ with all "active" bidders and compute $s' = \left(M^{(t)}\right)^{-1} P^{(t)}$. If all s' are positive, set $s_i^{(t+1)} = s_i^{(t)} + s_i' \Delta b$ and $b^{(t+1)} = b^{(t)} + \Delta b$, where Δb is a predetermined bid step. Some elements of the resulting s' may be negative. It means that some firms would prefer to bid higher even if their profits are positive at the current bid. I discuss how to obtain a subset of "active and willing" bidders from the set of "active" bidders below. For the moment assumethat $s' > 0$.

The simulation stops when all but one bidders have nonpositive expected profits, when a bidder reaches $s_i = 1$, or when the system diverges. For low initial $b^{(0)}$, all but one bidders reach their zero profit conditions at low signals, the resulting bidding strategies are consistent with a reserve price equal to $b^{(T)}$, where T is the terminal step. As the initial $b^{(0)}$ increases, the terminal $b^{(T)}$ and signals $s^{(t)}$ also increase. Eventually, the terminal $s^{(t)}$ is such that $s_i = 1$ for some bidder. The resulting bidding strategies are consistent with a nonbinding reserve price. If $b^{(0)}$ is slightly higher, the system diverges and the resulting strategies are not consistent with any equilibrium bidding behavior. The system diverges when all bidders expected profits increase with t. To obtain the results in Section 4.5, I find the initial $b^{(0)}$ consistent with an equilibrium with the highest non-binding reserve price, i.e., the highest reserve price for which the project is procured with probability one.

C.2.4 "Active and willing" bidders

In a general interdependent asymmetric model the equilibrium bidding strategies may be such that the support of bids is different across bidders. For example, suppose that there are three bidders A, B and C such that A's and B's costs are distributed on $[\underline{c}, \overline{c}]$ while C's costs are distributed on $[\underline{c}^*, \overline{c}]$, for $\underline{c} < \underline{c}^*$. The equilibrium bids can be such that A and B bid $\underline{b} < \underline{c}^*$ when their costs are $\underline{c}$, but C never bids $\underline{b}$. The optimal bid of C when his costs are $\underline{c}^*$ are above $\underline{c}^*$; as a result, there is a range of bids for which bidder C has positive profits, but finds it unprofitable to submit such a bid—he is "active but unwilling". The first-order condition of "active and unwilling" bidders have to be positive, i.e. they would increase their expected profit by bidding higher. It follows that if k_i is a slack variable,

$$P_i - k_i + b \nabla P s'_{-i}(b) - \nabla \phi s'_{-i}(b) = 0, \tag{35}$$

where $k_i = 0$ if i is willing to submit bid b, and $k_i > 0$ otherwise. The system of equations for all bidders can be written as:

$$P - k = Ms'. \tag{36}$$

where $k_i = 0$ and $s'_i > 0$ for willing bidders, and $k_i > 0$ and $s'_i = 0$ for unwilling bidders. Therefore, the problem of finding all "willing" bidders is the problem of choosing a set of indices J such that:

$$\begin{bmatrix} M_s & 0 \\ M_{sn} & I \end{bmatrix}^{-1} P = \begin{bmatrix} s_{j \in J} \\ k_{j \notin J} \end{bmatrix} \geq 0, \tag{37}$$

where M_s is a square matrix $\#J \times \#J$ that contains element m_{ij} only if $i, j \in J$, while M_{sn} is a $(n - \#J) \times \#J$ matrix that contains element m_{ij} only if $i \notin J$ but $j \in J$. This is a combinatorial problem that can be solved by a brute force approach if there are only a few bidders. Instead, I consider the following algorithm: denote $D_p = diag(P)$ and let $\tilde{k} = D_p k$ and $\tilde{M} = D_p^{-1}$. Equation (36) becomes:

$$1 - \tilde{k} = \tilde{M}s. \tag{38}$$

I find the Perron–Frobenius eigenvector of $\tilde{M}$ and try J equal to the indices of its largest elements. I try first with the first two, then the first three largest elements and so on. This algorithm guides the brute force approach and finds the right set of "willing" bidders faster.

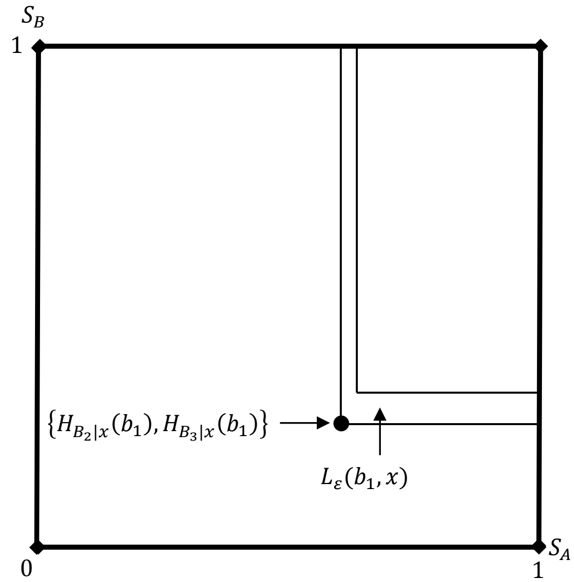


Figure 1: This figure shows the pair of competitors' signals that make them both bid exactly b_1 along with $L_\epsilon(b_1, x)$, an L-shaped set containing all competitors' signals such that their minimum bid is in $[b_1, b_1 + \varepsilon]$. The bidder's first-order optimality condition identifies the expected cost conditional on this set.

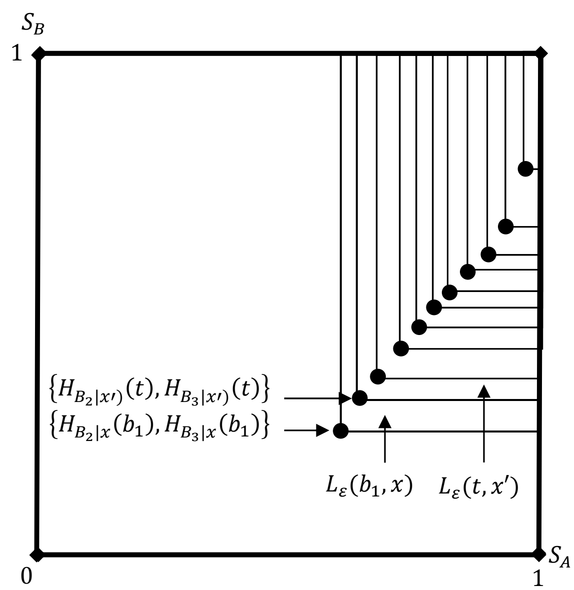
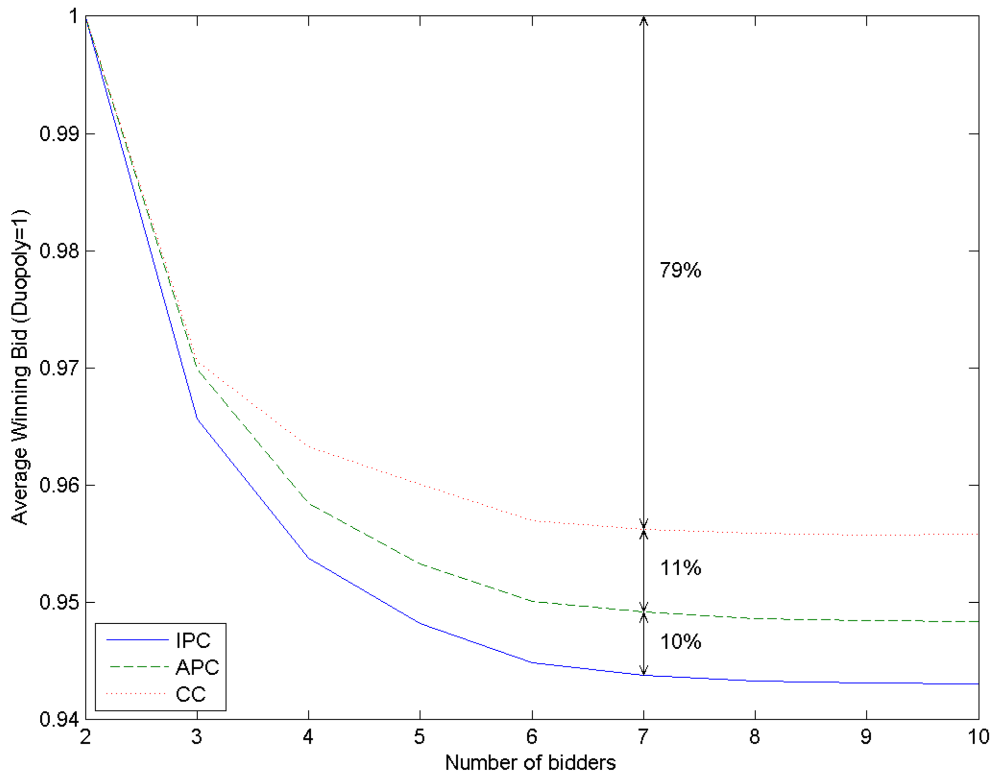


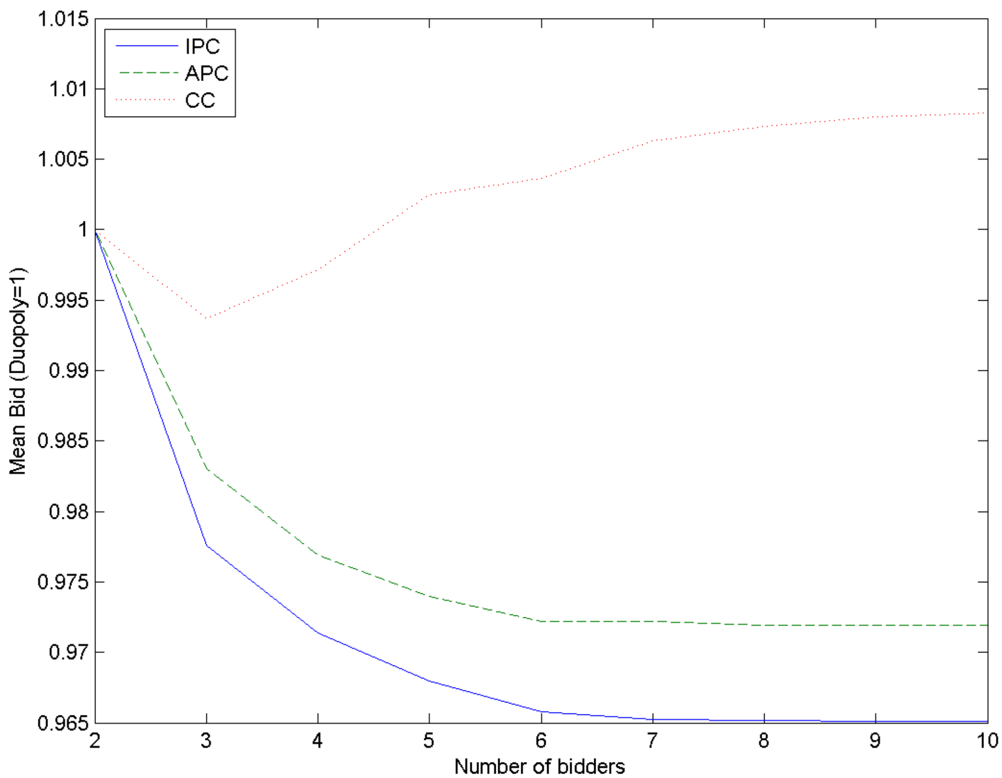
Figure 2: Variation in $[x_2, x_3]$ is used to find a different L-shaped set that stacks on top of $L_\varepsilon(b_1, x)$ holding s_1 and x_1 constant. This set is $L_\varepsilon(t, x')$ with $x' = [x_1, x_2, x_3]$. The expected cost over the union of $L_\varepsilon(b_1, x)$ and $L_\varepsilon(t, x')$ is equal to a probability-weighted average of the expected cost over each L-shaped set. These weights are identified from the joint distribution of signals. This process can be repeated to obtain a probability-weighted average over the whole rectangle $\{S_j \geq s_j\}_{j=2,3}$.

Figure 3: The Effects of Competition on Procurement Costs



Procurement costs relative to the duopoly case in the Independent Private Costs (IPC), Affiliated Private Costs (APC), and Common Costs (CC) models. Average over 250 randomly selected auctions and 100,000 random realization of signals.

Figure 4: The Effects of Competition on Bidding Behavior



Median Bid relative to the duopoly case in the Independent Private Costs (IPC), Affiliated Private Costs (APC), and Common Costs (CC) models. Average over 250 randomly selected auctions.

Table 1: Descriptive Statistics. Engineer's estimate, bids and distances.

Variable	N	Mean	Sd	P5	Median	P95
Engineer's estimate ($000)	1,925	1,145	1,783	119	633	3,899
Lowest bid ($000)	1,925	1,090	1,771	111	586	3,737
Participants	1,925	3.04	1.3	2	3	6
(2nd Lowest/Lowest bid-1) x 100	1,852	8.2	8.7	0.5	5.9	23.4
(Lowest/engineer-1) x 100	1,925	-6	12.5	-25.9	-6.8	15.2
Distance of Winner (km)	1,816	32	40	2	22	91
Distance of Bidder (km)	18,778	51	50	4	38	138

P stands for percentile. 2nd Lowest: the second lowest bid. engineer: engineer's estimate. There were 73 auctions with only one bid and 109 won by a firm for which I did not find any verifiable location.

Table 2: Number of Participants and Normalized Bids

Participants	Auctions	Winning Bid		Average Bid	
		mean	s.e.m	mean	s.e.m
1	73	1.2	1.0	1.2	1.0
2	658	-5.0	0.5	0.0	0.6
3	727	-6.1	0.4	2.1	0.5
4	247	-6.5	0.8	3.3	0.9
5	116	-9.9	1.0	-0.9	1.0
6	54	-10.6	1.9	0.3	2.2
7	28	-9.4	2.3	1.5	2.3
8	14	-11.1	3.6	0.0	4.1
9	6	-15.2	5.1	-5.0	4.9
10	2	-17.9	3.8	-3.3	8.4

Bids were normalized by the engineer's estimate: (Bid/engineer's estimate-1)$\times$100. The average winning bid and average submitted bid are tabulated by the number of participants in the auction. For example, there were 742 auctions with only two participants. In these auctions the normalized winning bid was on average 4.1% below the engineer's estimate, while the average submitted bid was 1.1% above. While the average winning bid decreases with the number of participants, the average submitted bid does not. s.e.m stands for standard error of the mean.

Table 3: Correlation of Signals

	Factor 1			Factor 2			Signal Decomposition		
	Coef.		s.e.	Coef.		s.e.	Factor 1	Factor 2	Individual
Bidder 1	0.410	***	0.032	0.173	***	0.065	0.17	0.03	0.80
Bidder 2	0.449	***	0.044	0.125		0.092	0.20	0.02	0.78
Bidder 3	0.434	***	0.057	0.218	***	0.071	0.19	0.05	0.76
Bidder 4	-0.356	***	0.056	0.322	***	0.086	0.13	0.10	0.77
Bidder 5	0.346	***	0.043	0.291	***	0.050	0.12	0.08	0.80
Bidder 6	0.221		0.14	0.428	*	0.223	0.05	0.18	0.77
Bidder 7	0.140		0.16	0.458	**	0.203	0.02	0.21	0.77
Bidder 8	0.071		0.134	-0.328	***	0.120	0.01	0.11	0.89
Bidder 9	0.370	*	0.209	-0.021		0.261	0.14	0.00	0.86
Bidder 10	0.022		0.097	0.455	***	0.033	0.00	0.21	0.79
Bidder 11	0.382	***	0.065	0.215		0.138	0.15	0.05	0.81
Bidder 12	0.037		0.205	0.099		0.153	0.00	0.01	0.99
Bidder 13	0.351	***	0.09	0.107		0.145	0.12	0.01	0.87
Bidder 14	0.298	***	0.077	0.173		0.108	0.09	0.03	0.88
Bidder 15	0.363	***	0.087	0.204	**	0.103	0.13	0.04	0.83
Bidder 16	0.126		0.24	0.294		0.231	0.02	0.09	0.90
Bidder 17	0.225	***	0.078	0.026		0.110	0.05	0.00	0.95
Bidder 18	-0.444		0.271	0.279		0.437	0.20	0.08	0.73
Bidder 19	-0.075		0.258	0.475	**	0.198	0.01	0.23	0.77
Bidder 20				0.504	***	0.023		0.25	0.75

Loading matrix estimates obtained by Simulated Maximum Likelihood following Kamakura and Wedel (2001). Standard Errors obtained from 200 bootstrap samples. Each bidder's signal can be written as $Z_i = \hat{L}_{i1} Factor_1 + \hat{L}_{i2} Factor_2 + Individual_i$, where $Factor_j$, $j = 1, 2$ and $Individual_i$ are a jointly independent normal random variables with variances $1, 1$ and $1 - \hat{L}_{i1}^2 - \hat{L}_{i2}^2$, respectively. The total variance of Z_i can be decomposed in three terms: $\hat{L}_{i1}^2$, $\hat{L}_{i2}^2$ and $1 - \hat{L}_{i1}^2 - \hat{L}_{i2}^2$.

Table 4: Correlation of Signals

	Bidder 1	Bidder 2	Bidder 3	Bidder 4	Bidder 5	Bidder 6	Bidder 7	Bidder 8	Bidder 9	Bidder 10	Bidder 11	Bidder 12	Bidder 13	Bidder 14	Bidder 15	Bidder 16	Bidder 17	Bidder 18	Bidder 19
Bidder 2	0.259***																		
Bidder 3	0.275***	0.287***																	
Bidder 4	-0.115**	-0.154***	-0.110**																
Bidder 5	0.240***	0.243***	0.274***	-0.038															
Bidder 6	0.210***	0.197***	0.247***	0.077	0.257***														
Bidder 7	0.174***	0.155**	0.209***	0.126	0.232***	0.295***													
Bidder 8	-0.033	-0.011	-0.05	-0.158*	-0.084	-0.151	-0.17												
Bidder 9	0.178***	0.199***	0.192***	-0.17	0.147**	0.089	0.052	0.038											
Bidder 10	0.110**	0.085	0.140***	0.178***	0.176***	0.256	0.271*	-0.177**	-0.002										
Bidder 11	0.241***	0.250***	0.271***	-0.085	0.243***	0.224***	0.193***	-0.052	0.164*	0.133									
Bidder 12	0.036	0.033	0.043	0.021	0.047	0.058	0.058	-0.032	0.013	0.052	0.04								
Bidder 13	0.195***	0.208***	0.216***	-0.111	0.184***	0.151**	0.12	-0.012	0.148***	0.068	0.188***	0.026							
Bidder 14	0.181***	0.187***	0.204***	-0.061	0.183***	0.170***	0.147***	-0.041	0.122*	0.102	0.179***	0.03	0.141***						
Bidder 15	0.226***	0.235***	0.255***	-0.08	0.228***	0.210***	0.181***	-0.048	0.154***	0.125*	0.224***	0.037	0.177***	0.169***					
Bidder 16	0.121	0.112	0.144	0.06	0.153	0.185	0.183	-0.098	0.046	0.162	0.131	0.036	0.086	0.1	0.123				
Bidder 17	0.111***	0.121***	0.122***	-0.084	0.099**	0.072	0.051	0.008	0.092**	0.02	0.105**	0.011	0.091**	0.079**	0.099**	0.039			
Bidder 18	-0.175**	-0.218**	-0.177**	0.332	-0.095	0.029	0.088	-0.153	-0.215***	0.155	-0.143	0.013	-0.159	-0.105	-0.135*	0.032	-0.112		
Bidder 19	0.065	0.033	0.092	0.233	0.143*	0.243*	0.269**	-0.195	-0.047	0.275*	0.093	0.051	0.03	0.073	0.087	0.157	-0.005	0.222	
Bidder 20	0.113**	0.083	0.145***	0.214***	0.190***	0.285*	0.304**	-0.203**	-0.013	0.298***	0.14	0.058	0.067	0.108	0.131*	0.181	0.016	0.191	0.316**

Correlation Matrix derived from the factor structure shown in Table 3. Diagonal elements are all equal to one.

Table 5: Full Information Cost Estimates

Variable	Bidder 1		Bidder 2		Bidder 3		Bidder 4	
	Coeff	Std-Err	Coeff	Std-Err	Coeff	Std-Err	Coeff	Std-Err
Const p25	0.730 ***	0.031	0.782 ***	0.020	0.860 ***	0.025	0.537 *	0.324
Dist > 10	0.025 **	0.011	0.038 **	0.019	0.068 ***	0.023	0.162	0.150
Dist > 50	0.157 ***	0.024	0.105 ***	0.021	0.399 ***	0.112	0.111	0.176
Dist > 100	0.363 **	0.168	0.240 ***	0.067	-0.021	0.132	-0.110	0.175
Road Density	-0.008	0.016	0.010	0.031	-0.032 ***	0.007	0.101	0.321
Const p50-p25	0.082 ***	0.020	0.054 ***	0.017	0.101 ***	0.031	0.205	0.137
Const p75-p25	0.181 ***	0.053	0.126 ***	0.032	0.238 ***	0.068	0.910 *	0.498
Common Costs	0.500 ***	0.173	0.350 ***	0.113	-0.025	0.096	0.000	0.232
N	1096		936		411		72	

Variable	Bidder 5		Bidder 6		Bidder 7		Bidder 8	
	Coeff	Std-Err	Coeff	Std-Err	Coeff	Std-Err	Coeff	Std-Err
Const p25	0.957 ***	0.021	0.927 ***	0.122	0.836 ***	0.133	0.763 ***	0.136
Dist > 10	0.016	0.017	0.085 **	0.039	0.020	0.042	0.051	0.052
Dist > 50	-0.045	0.159	0.127 ***	0.032	0.034	0.037	0.067	0.091
Dist > 100			0.422	0.302	0.056	0.052		
Road Density	-0.021 **	0.009	0.265	0.169	-0.042	0.159	0.022	0.061
Const p50-p25	0.082 ***	0.018	0.101	0.076	0.051 *	0.027	0.169 *	0.095
Const p75-p25	0.186 ***	0.034	0.243	0.152	0.155 ***	0.045	0.422 *	0.253
Common Costs	0.175 ***	0.050	0.800 *	0.471	0.525 ***	0.189	0.950	0.676
N	399		374		249		280	

Variable	Bidder 9		Bidder 10		Bidder 11		Bidder 12	
	Coeff	Std-Err	Coeff	Std-Err	Coeff	Std-Err	Coeff	Std-Err
Const p25	0.919 ***	0.045	0.940 ***	0.068	0.799 ***	0.083	0.833 ***	0.039
Dist > 10	0.096 **	0.045	0.114 *	0.059	-0.013	0.029	-0.003	0.036
Dist > 50	0.207 ***	0.045	-0.001	0.048	0.061	0.054	0.052	0.110
Dist > 100	0.142	0.121	0.421 *	0.221			-0.012	0.182
Road Density	0.288 ***	0.083	-0.034 **	0.016	-0.229 *	0.139	-0.008	0.089
Const p50-p25	0.111 ***	0.026	0.047 **	0.020	0.070 ***	0.024	0.054	0.043
Const p75-p25	0.278 ***	0.063	0.145	0.099	0.134 ***	0.044	0.154	0.106
Common Costs	0.050	0.077	0.325 *	0.180	0.500 ***	0.175	0.525	0.330
N	273		122		148		162	

Estimates of the full information cost parameters for each bidder. The estimates were obtained by IVQR using three different quantiles and restricting all coefficients except the constant from varying across quantiles. The chosen quantiles where 0.25, 0.50 and 0.75. For bidders 4, 8 and 10, I used quantiles 0.25Q, 0.5Q and 0.75Q where Q is the maximum estimated probability of participation in any given auction (0.26,0.69 and 0.41, respectively). Standard errors were obtained by bootstrap.

Table 6: Full Information Cost Estimates

Variable	Bidder 13		Bidder 14		Bidder 15		Bidder 16	
	Coeff	Std-Err	Coeff	Std-Err	Coeff	Std-Err	Coeff	Std-Err
Const p25	1.146 ***	0.158	0.971 ***	0.093	0.903 ***	0.045	0.981 ***	0.104
Dist > 10	0.010	0.046	0.022	0.065	0.004	0.054	-0.055	0.066
Dist > 50	-0.033	0.051	0.013	0.105	0.393 ***	0.085	0.035	0.071
Dist > 100	0.068	0.229	0.176	0.207			0.192	0.317
Road Density	0.617 **	0.240	-0.022	0.035	-0.130 ***	0.038	-0.066	0.079
Const p50-p25	0.063	0.043	0.126 **	0.051	0.094 ***	0.028	0.036	0.043
Const p75-p25	0.127	0.094	0.279	0.172	0.237 ***	0.059	0.101	0.205
Common Costs	0.250 *	0.133	0.075	0.109	0.025	0.114	0.450	0.327
Q	1.00		0.60		0.64		0.44	
N	151		85		129		60	

Variable	Bidder 17		Bidder 18		Bidder 19	
	Coeff	Std-Err	Coeff	Std-Err	Coeff	Std-Err
Const p25	0.759 ***	0.075	0.356	0.488	0.885 ***	0.074
Dist > 10	0.055	0.036	0.025	0.140	0.078	0.066
Dist > 50	0.052	0.082	0.049	0.132	0.218 **	0.097
Dist > 100			-0.080	0.174	-0.123	0.105
Road Density	0.028	0.140	-0.513	0.578	0.019	0.101
Const p50-p25	0.077 ***	0.025	0.091	0.057	0.060	0.043
Const p75-p25	0.137 **	0.066	0.197 *	0.119	0.225	0.148
Common Costs	0.400 **	0.164	0.150	0.219	0.300 *	0.155
Q	0.96		0.37		0.34	
N	106		61		62	

Estimates of the full information cost parameters for each bidder. The estimates were obtained by IVQR using three different quantiles and restricting all coefficients except the constant from varying across quantiles. The chosen quantiles where 0.25,0.50 and 0.75. For bidders 14-19, I used quantiles 0.25Q, 0.5Q and 0.75Q where Q is the maximum estimated probability of participation in any given auction. Standard errors were obtained by bootstrap.

Table 7: Private and Common Cost Components

	Own Signal		Competitor (mean)		Competitor (max)	
	Coeff	Std-Err	Coeff	Std-Err	Coeff	Std-Err
Bidder 1	0.10 ***	0.02	0.04 ***	0.01	0.07 ***	0.03
Bidder 2	0.10 ***	0.02	0.03 ***	0.01	0.06 ***	0.02
Bidder 3	0.13 ***	0.05	0.00	0.01	0.00	0.01
Bidder 4	0.60 *	0.33	0.00	0.11	0.00	0.35
Bidder 5	0.13 ***	0.02	0.01 ***	0.00	0.02 ***	0.01
Bidder 6	0.16	0.11	0.04 *	0.02	0.09 **	0.04
Bidder 7	0.12 ***	0.03	0.03 ***	0.01	0.07 ***	0.02
Bidder 8	0.23	0.15	0.05	0.07	0.15	0.20
Bidder 9	0.18 ***	0.04	0.01	0.03	0.02	0.10
Bidder 10	0.10 **	0.04	0.02 ***	0.01	0.05 ***	0.02
Bidder 11	0.10 ***	0.03	0.03 ***	0.01	0.06 ***	0.02
Bidder 12	0.07	0.06	0.03	0.04	0.06	0.09
Bidder 13	0.09	0.06	0.02 **	0.01	0.04 *	0.02
Bidder 14	0.25 ***	0.10	0.01	0.01	0.01	0.02
Bidder 15	0.20 ***	0.05	0.00	0.01	0.00	0.02
Bidder 16	0.07	0.09	0.03	0.02	0.05	0.05
Bidder 17	0.12 ***	0.04	0.04	0.11	0.09	0.32
Bidder 18	0.23 *	0.12	0.07	0.08	0.20	0.21
Bidder 19	0.15	0.10	0.02 **	0.01	0.06 **	0.03

Effect of a one standard deviation increase of own and competitors' Gaussian costs on each bidder's full information cost. The effect of an increase in competitor's signal depends on its identity. I report the mean and maximum effects over competitors.

Table 8: The Effects of Competition on Bidding Behavior

Effects	3 vs 2	4 vs 2	5 vs 2	6 vs 2	7 vs 2	8 vs 2	9 vs 2	10 vs 2
Competitive	2.24	2.86	3.20	3.42	3.47	3.48	3.49	3.49
Affiliation	-0.54	-0.55	-0.60	-0.65	-0.69	-0.68	-0.68	-0.69
Winner's Curse	-1.07	-2.03	-2.85	-3.14	-3.41	-3.54	-3.61	-3.63
Sampling	2.31	3.39	4.24	4.66	5.00	5.14	5.23	5.25
Total	2.94	3.67	3.99	4.30	4.37	4.41	4.42	4.42

Incremental Effects	3 vs 2	4 vs 3	5 vs 4	6 vs 5	7 vs 6	8 vs 7	9 vs 8	10 vs 9
Competitive	2.24	0.64	0.36	0.23	0.05	0.01	0.01	0.00
Affiliation	-0.54	-0.01	-0.06	-0.04	-0.05	0.02	0.00	-0.01
Winner's Curse	-1.07	-0.99	-0.86	-0.30	-0.28	-0.14	-0.08	-0.02
Sampling	2.31	1.11	0.88	0.44	0.35	0.15	0.08	0.02
Total	2.94	0.75	0.33	0.33	0.08	0.04	0.01	0.00

Decomposition of the cost-saving effect of inviting more bidders. The top panel compares inviting $N = 3, ..., 10$ relative to inviting only 2 bidders. The bottom panel compares inviting $N = 3, ..., 10$ relative to inviting $N - 1$ bidder. All units are normalized so that 100 equals the average procurement costs in the baseline case (2 and $N - 1$ bidders in the top and bottom panels, respectively).