Should Dark Pools Improve Upon Visible Quotes? The Impact of Trade-at Rules[*]

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Abstract

I study a financial market in which investors trade on private information or liquidity needs. Investors can elect to trade through a visible limit order book, or through a dark market where limit orders are hidden. Market makers are present in both markets. The dark market accepts orders from investors and fills them with some probability at a price better than the quote at the visible market—the so-called "trade-at rule". The impact of dark trading on visible market quality and social welfare depends on the trade-at rule. There exists a non-zero trade-at rule (the benchmark) at which visible orders weakly dominate dark orders. When introduced alongside a visible limit order book, a dark market with a large trade-at rule (relative to the benchmark) improves market quality and welfare; a small trade-at rule, however, impacts market quality and social welfare negatively. The availability of dark trading leads to a decline in price efficiency with any trade-at rule. When the trade-at rule is pegged to the midpoint of the bid-ask spread, no liquidity is provided in the dark market, and hence, investors do not use the dark market.

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In recent years, concern has arisen over the impact of dark trading on equity markets. While dark markets are not new, this concern comes as they begin to match the electronic organization of visible markets, and gain a significant share of global trading activity. The CFA Institute estimates that dark markets in the U.S. generate approximately one third of all volume[1]. Because dark markets benefit from visible market transparency, there is concern from regulators that these benefits come at the expense of market quality (see Securities and Exchange Commission (2010)).

Equity trading primarily conducts through two types of markets: visible limit order markets, and dark markets. On visible limit order markets, investors submit limit orders at pre-specified (limit) prices; a trade occurs when a subsequent investor submits a market order. Limit orders are visible to all market participants. In dark markets, available limit orders are hidden, and investors submit market orders to trade against limit orders that may or may not be present in the market. Since investors do not know if a limit order is available to trade against, an investor who submits a market order to the dark suffers execution risk. As compensation for bearing this risk, a dark market order trades at a price better than is offered by the visible market—the so-called "price improvement".

Many dark markets permit orders improve only marginally on posted visible market quotes. While marketplaces generally give first-fill priority to visible orders over dark orders at the same price, marginal price improvement moves the dark order to the front of the queue, according to price-visibility-time priority. Because marginal price improvement effectively eliminates the priority given to visible orders—with minimal benefit to investors—some countries have chosen to require dark limit orders to provide "meaningful price improvement". This price improvement requirement is known as the "trade-at rule". Canada and Australia now have trade-at rules in place that require dark market orders to receive a price improvement of at least one trading increment (i.e., one penny in most major markets).[2]

[1]CFA Institute (2012): "Dark Pools, Internalization, and Equity Market Quality".

[2]The immediate impact of the trade-at rule in Canada was a 50% drop in dark trading volume. Rosenblatt Securities Inc. (2013)

For securities with bid-offer spreads of one or two trading increments, trades must be executed at the midpoint of the spread (see Investment Industry Regulatory Organization of Canada (2011) and Australian Securities and Investments Commission (2012)). By imposing a trade-at rule, regulators suggest that dark markets may have a place in the financial landscape, but only if they offer a significant discount. But how does the trade-at rule impact investor trading decisions? Do dark markets that offer large discounts improve equity market liquidity and investor welfare?

An investor's order placement strategy has three components: their valuation, the price of the order, and its probability of execution. For an investor, the choice of order type focuses on the trade-off between price and execution risk. In a market where liquidity providers ensure the limit order book is full, market orders face lower execution risk than visible limit orders or dark market orders. As compensation, visible limit orders and dark market orders offer better prices. The trade-off between dark market orders and visible limit orders, however, is more complex, as both orders face execution uncertainty. To compensate for execution risk, dark market orders receive price improvement on the prevailing quote, while visible limit orders permit investors to set their own quote. Hence, a dark market's trade-at rule plays a key role in how dark orders compete with visible limit orders. In this paper, I analyze how trade-at rules impact overall market quality and investor welfare. I then discuss my results in the context of trade-at rule requirements by regulators.

I propose a dynamic trading model where investors trade for either informational or liquidity reasons. They may trade at a visible market using either limit orders or market orders, or at a dark market using market orders. Both markets are monitored by uninformed liquidity providers that act as market makers. They possess a monitoring advantage towards interpreting and reacting to market data that they use to ensure that the visible limit order book always contains limit orders on either side of the book and moreover, that the limit orders are priced competitively. Consequently, investors who submit market orders are guaranteed execution at the available quote. Limit orders, however, are subject to execution risk, as they may trade only if the subsequent investor submits the appropriate market order.

In the dark market, the price of an order is derived from the visible market: the prevailing quote, improved by a percentage of the bid-ask spread.[3] Because the price of a dark market order is predetermined, liquidity providers set the frequency at which they submit limit orders to the dark market such that they earn zero expected profits in equilibrium. Moreover, because the quote from the dark market is better than the visible market, liquidity providers do not have the incentive to ensure that the dark market always contains one buy and one sell limit order. As a result, dark market orders are also subject to execution risk. In equilibrium, an investor's order placement strategy depends on their valuation, and the trade-off between an order's price and probability of execution.

I analyze the impact of introducing a dark market alongside a visible limit order market, by first solving the case where all investors are indifferent between visible limit orders and dark market orders. I assume that when indifferent, investors use visible limit orders. In this way, this setting serves as the "visible market only" benchmark. I find that there exists a non-trivial trade-at rule where investors are indifferent to visible limit orders and dark market orders, which I refer to as the "benchmark" trade-at rule. I subsequently solve the model for the cases where both markets are used in equilibrium. Investors use dark market orders in equilibrium when the trade-at rule is above or below the benchmark level. I refer to these trade-at rules as a "large trade-at rule" and a "small trade-at rule", respectively.

I find that when the dark market competes with the visible market by setting a large trade-at rule, market liquidity improves and visible market volume increases. By offering a trading opportunity at a "discount price", investors who would otherwise be pushed out of the market by the costs of visible orders can afford to participate. To offer this discount, liquidity providers set the execution risk of dark market orders higher than the execution risk of visible limit orders. The dark market then attracts investors who submit visible limit orders, but not market orders. This leads to an increase in the relative attractiveness of

[3] For instance, prior to minimum price improvement regulation in Canada, the dark pools MatchNow and Alpha IntraSpread used trade-at rules of $\lambda = 0.2$ and $\lambda = 0.1$, respectively.

visible market orders, thereby increasing volume on the visible market. Thus, dark markets with large trade-at rules result in net volume creation in the visible market instead of net volume migration. However, if the trade-at rule is at the midquote (or better), then no liquidity is provided to the dark market.

Conversely, a dark market that sets a small trade-at rule incentivizes liquidity providers to provide dark limit orders with a greater frequency than in the benchmark case. As a result, dark market orders have lower execution risk than visible limit orders. The result is an increase in quoted spreads, price impact and visible market volume. Because dark market orders have lower execution risk than visible limit orders, the dark market attracts investors away from both visible order types: the visible market order submitters with the lowest valuations, and the visible limit order submitters with the highest valuations. Hence, investors who submit visible market orders have higher average valuations, which, because some investors are informed, implies that visible market order submitters are now more informed on average. This leads to an increase in the trading costs of visible market orders, thus negatively impacting liquidity.

By modelling both informed investors, and uninformed investors with private values, I can study price efficiency and social welfare, respectively. I study price efficiency by measuring the difference between the change in the fundamental value of the security and the price impact of an investor's order placement. I measure welfare in the sense of allocative efficiency, similar to Bessembinder, Hao, and Lemmon (2012): the expected private valuation realized by trade counterparties, discounted by the probability of a trade. I find that price efficiency declines regardless of the level of the trade-at rule offered by the dark market. Social welfare increases with a large trade-at rule, but falls with a small trade-at rule.

My model suggests that the impact of dark market trade-at rules on equity markets is dichotomous: a large trade-at rule impacts investors and visible markets positively, whereas a small trade-at rule has a negative impact. These results suggest that there may be a role for minimum trade-at rule requirements. By restricting the entry of dark markets with a small trade-at rule, (e.g., setting a minimum trade-at rule equal to the benchmark level) dark markets with a negative impact would be eliminated.

Related Literature. The prevalence of dark trading generates a need to understand the impact of dark trading on market quality, price efficiency and investor welfare. Several papers examine the impact of introducing crossing networks alongside visible markets. In these models, dark orders fill at the prevailing visible market quote, or midquote of the spread: Zhu (2013), Hendershott and Mendelson (2000) and Degryse, van Achter, and Wuyts (2009) fill orders at the midquote, whereas Ye (2011) assumes orders fill at the prevailing quote. Buti, Rindi, and Werner (2014) examine both periodic and continuous crossing networks where trades occur at the midquote. The literature on continuous dark pools, however, is relatively new. Malinova (2012) investigates a market where a broker may competitively internalize orders. I contribute to the dark trading literature by studying an increasing common dark pool setup: continuous dark pools where orders may be priced away from the midquote, depending on the pricing rule set by the dark pool. In particular, I investigate how the pricing mechanism (the "trade-at rule") affects visible market quality and price efficiency.

My work is most closely related to Buti, Rindi, and Werner (2014), who also study the impact of competition between a dark pool and a visible limit order book. They model the dark pool as a crossing network that executes trades at the midquote of the visible market spread. In their model, uninformed traders trade either a large (institutional) or a small (retail) order, at a visible limit order book or a dark pool. Buti, Rindi, and Werner (2014) find that exchange orders migrate to the dark pool, but that there is volume creation overall. They predict that this order migration to the dark pool intensifies for stocks with narrower spreads or greater market depth at the prevailing quotes. For periodic-crossing dark pools, they find that all traders benefit from the availability of a periodic-crossing dark pool when stocks are liquid. When stocks are illiquid, however, institutional traders see welfare gains, while retail traders suffer reduced welfare. For dark pools with continuous crossing, welfare effects are amplified.

Complementing their work, I model dark pools as venues with liquidity provision, informed trading, and pricing at discounts of the bid/ask prices according to a trade-at rule. Liquidity in the dark pool is supplied by competitive liquidity providers that submit to both the visible and dark market, and investors submit only market orders to the dark pool. Similarly to Buti, Rindi, and Werner (2014), I find that volume migrates to the dark pool, but in my model additional volume is only created when the trade-at rule demands sufficient improvements over the visible market prevailing quotes. If the trade-at-rule is too loose, volume declines.

Zhu (2013) models traders with correlated information who choose between a visible market and a dark pool. He finds that access to a dark pool improves price discovery, because informed traders concentrate their orders on the visible market. The increased adverse selection on the visible exchange then leads to worsening exchange liquidity. Malinova (2012) studies a model where informed traders submit to an exchange (via a broker), or to a broker who may potentially internalize orders. All brokers are competitive. Malinova (2012) finds that informed trading is concentrated on the visible exchange, but that the size of a trader's order affects the impact of internalization on price efficiency. Ye (2011) models informed trading in the sense of Kyle (1985), and contrary to Zhu (2013), finds that the availability of a dark pool harms price discovery. My model predicts that a dark market negatively impacts price efficiency, regardless of the trade-at rule.

I also contribute to a wider theoretical literature on dark trading. Recent works by Boulatov and George (2013), Buti and Rindi (2012) and Moinas (2011) study dark trading within a limit order market via the use of hidden orders. Boulatov and George (2013) examine the differences in market quality between a fully-displayed limit order book, and a fully-hidden limit order book, where informed trading is modelled in the tradition of Kyle (1985). They find that a fully-hidden limit order book entices informed traders to limit orders, and moreover, the increased competition in liquidity provision lowers transaction costs for uninformed traders, in turn improving market quality. Buti and Rindi (2012) and Moinas (2011) consider a limit order market that permits both visible and hidden orders. Both studies find that traders choose to hide their orders to reduce exposure costs.

My predictions may also explain some of the seemingly contradictory results in the empirical literature. Comerton-Forde and Putniņš (2013) analyze Australian exchange and dark pool data. They find that dark trades contain information, but that those who migrate to the dark are less informed than those traders that remain on the exchange. The increase in informed trading on the exchange worsens liquidity. In the context of my model, their results are consistent, with a dark pool that sets a small trade-at rule. Comerton-Forde and Putniņš (2013) state that a large majority of dark trades execute at the bid-offer quotes, where liquidity providers trade against incoming investor and institutional orders. Similar to Comerton-Forde and Putniņš (2013), Nimalendran and Ray (2012) study U.S crossing networks data and find that dark trades have an informational impact on visible market prices. Consistent with my prediction for a minimum trade-at rule, Larrymore and Murphy (2009) find that the implementation of a minimum trade-at rule for internalized orders improves the quoted spread.[4]

Degryse, de Jong, and van Kervel (2011), Weaver (2011) and Foley, Malinova, and Park (2012) analyze Dutch, U.S. and Canadian data, respectively, and conclude that dark trading negatively impacts market quality. Using U.S. data, Buti, Rindi, and Werner (2012) find that dark trading improves liquidity. In a labratory experiment, Bloomfield, O'Hara, and Saar (2013) conclude that dark trading has no impact on liquidity and price efficiency.

1 Model

I model a financial market where risk-neutral investors enter a market sequentially to trade for either informational or liquidity reasons (as in Glosten and Milgrom (1985)). Investors

[4] Larrymore and Murphy (2009) study the Toronto Stock Exchange Price Improvement Rule, implemented on October 26th, 1998.

have access to a visible limit order book and a dark market, at which (uninformed) professional liquidity providers compete to supply limit orders, similar to Brolley and Malinova (2014). The price at which investors trade in the dark market is given by a "trade-at rule" that improves the prevailing quote at the visible market.[5] The dark market in my model mirrors several types of continuous dark pools. The market effectively operates similar to a dark limit order market where liquidity is provided solely by professional liquidity providers (e.g., Alpha Intraspread). It also reflects the setup of ping destinations (dark pools of liquidity that accept immediate-or-cancel orders; e.g., Citadel, Getco), and internalization pools that procure liquidity from outside liquidity partners (e.g., Sigma X, Crossfinder).

Security. There is a single risky security with an unknown fundamental value. The fundamental value follows a random walk, and at each period t an innovation δ_t to the fundamental value occurs, which is independently and identically distributed by the density f on $[-1, 1]$. f is symmetric around zero. The fundamental value in period t is given by,

$$V_t = \sum_{\tau \le t} \delta_\tau \tag{1}$$

Market Organization. Trading is organized as a market where traders can access a visible limit order book, or a dark market. A trader in period t chooses between submitting an order to the visible market, or the dark market. Traders have the choice between three types of words: a market order on the visible market, a limit order on the visible market, and a market order on the dark market. In period t, I denote the price of the best-priced buy limit order (submitted in period $t-1$) as bid_t; for the analogous sell limit order, I denote the best price as ask_t. Limit orders remain in the book for one period, after which any unfilled limit orders are cancelled.

An order submitted to the dark market fills with some probability, at a pre-specified price. The probability that an order is filled in the dark is determined endogenously by the probability that liquidity is provided to the dark market in the previous period. The dark

[5] Figure 2 in the Appendix diagrammatically illustrates the timing of the model.

market sets the price based on an exogenous trade-at rule, $\lambda \in [0, 1]$, measured as a fraction of the spread, $\mathsf{ask}_t - \mathsf{bid}_t$, in the visible market in period t. For example, a buy order in the dark trades at: $ask_t - \lambda \times (\mathsf{ask}_t - \mathsf{bid}_t)$. By modelling the trade-at rule as an improvement on the visible spread, the model effectively imposes price-visibility-time priority, as is common in industry.

Similar to Foucault (1999), I assume that the security is traded throughout a "trading day" where the trading process ends after period t with probability $(1-\rho) > 0$, at which point the payoff to the asset is realized. Investors and professional liquidity providers observe the history of all transactions on both markets, as well as quotes and cancellations on the visible market. I denote this history up to (but not including) period t as H_t. The public expectation of the security's fundamental value at period t conditional on the public history is denoted by v_t. The structure of the model is common knowledge among all market participants.

Investors. At each period t, a single investor randomly arrives at the market from a continuum of risk-neutral investors. With probability $\mu \in (0, 1)$ the investor privately learns the period t innovation δ_t. Uninformed investors are endowed with liquidity needs, y_t, independently and identically distributed by g on $[-1, 1]$. g is symmetric around zero.[6] Informed investors have no liquidity needs (i.e., $y_t = 0$). All investors observe the history, H_t.

Upon arriving at the market in period t and only then, an investor may submit an order for a single unit (round lot) of the security. I assume that, if an investor is indifferent to submitting an order to the visible market, or an order to the dark market, the investor chooses the visible market.[7] IAn investor leaves the market forever upon the execution or cancellation of their order.

Professional Liquidity Providers. There is a continuum of professional liquidity providers that are always present in the market. They are risk-neutral, do not receive

[6] Assuming that some investors have liquidity needs is common practice in the literature on trading with asymmetric information, to avoid the no-trade result of Milgrom and Stokey (1982).

[7] As dark trading has been historically popular for executing large orders, trading in a single unit may seem unnatural. The CFA Institute notes, however, that order and transaction sizes in today's dark pools are similar to those on visible markets (CFA, 2012). Data from FINRA from July 21-27 2014 finds that for 41 ATSs, the average trade size is 209 shares.

information about the security's fundamental value, and have no liquidity needs. Within any period t, they update their orders in response to market information (i.e. changes to the public history) before the arrival of the next investor.[8] I assume that professional liquidity providers act competitively, and thus earn zero expected profits, conditional on execution. When they post limit orders to the visible market, professional liquidity providers compete in price in the sense of Bertrand. Professional liquidity providers maintain a "full" limit order book at any period t by submitting limit orders to fill any vacancies on either side of the limit order book, at prices $\mathsf{bid}_t^{\mathrm{LP}}$ and $\mathsf{ask}_t^{\mathrm{LP}}$ for buy and sell limit orders, respectively.

Professional liquidity providers also post limit orders to the dark market. However, because prices for dark limit orders are derived from the visible market prices and the trade-at rule, λ, professional liquidity providers do not compete in price in the dark market. Instead, professional liquidity providers compete by choosing the probability with which they provide liquidity to the dark market after observing the period t investors action, such that their expected payoffs are zero. Because the prices in the dark market are set at a discount from the prices they would offer in the visible market, professional liquidity providers submit limit orders to the dark market with a probability less than one. That is, orders sent to the dark market are cheaper for investors than market orders sent to the visible market, but, as a trade-off, they are not guaranteed execution.

Investor Payoffs. I focus on the payoff to an investor who buys at period t. As notational shorthand, period t visible market and limit buy orders are denoted MB_t and LB_t, respectively. Period t dark buy orders are denoted DB_t. An investor's payoff to any order type is the difference between their valuation (their private value y_t, plus their assessment of the security's value) and the price paid, discounted by the execution probability. Investors who abstain from trading receive a payoff of zero. The payoffs to each order type given

[8] See Brolley and Malinova (2014) for a justification of this assumption.

below,

$$\pi^{\text{MB}}_{t,\text{inv}} = y_t + \mathsf{E}[V_t \mid \text{info}_t, H_t] - \mathsf{ask}_t \tag{2}$$

$$\pi^{\text{LB}}_{t,\text{inv}} = \rho \cdot \mathsf{Pr}(\text{fill} \mid \text{info}_t, H_t, \mathsf{bid}^{\text{inv}}_{t+1}) \times \left(y_t + \mathsf{E}[V_{t+1} \mid \text{info}_t, H_t, \text{fill at } \mathsf{bid}^{\text{inv}}_{t+1}] - \mathsf{bid}^{\text{inv}}_{t+1}\right) \tag{3}$$

$$\pi^{\text{DB}}_{t,\text{inv}} = \mathsf{Pr}(\text{fill} \mid \text{info}_t, H_t) \times \left(y_t + \mathsf{E}[V_t \mid \text{info}_t, H_t] - \mathsf{ask}^{\text{DLP}}_t\right) \tag{4}$$

where info_t is the period t investor's information about the innovation δ_t; $\mathsf{Pr}(\text{fill} \mid \text{info}_t, H_t, \mathsf{bid}^{\text{inv}}_{t+1})$ and $\mathsf{Pr}(\text{fill} \mid \text{info}_t, H_t)$ are the respective probabilities that a limit order is filled in the visible market, and that a dark market order is filled in the dark market; $\mathsf{E}[V_{t+1} \mid \text{info}_t, H_t, \text{fill at } \mathsf{bid}^{\text{inv}}_{t+1}]$ is the period t investor's expectation of the fundamental value, conditional on the fill of their limit order. Payoffs from sell orders are analogous.

Professional Liquidity Provider Payoffs. In any period t, professional liquidity providers observe the period t investor's action, and then submit limit orders (if any) to both markets. At period t, a visible buy limit order at price $\mathsf{bid}^{\text{LP}}_{t+1}$ earns the payoff,

$$\begin{aligned}\pi^{\text{LB}}_{t,\text{LP}} = {} & \rho \cdot \mathsf{Pr}(\text{fill} \mid \text{investor action at } t, H_t, \mathsf{bid}^{\text{LP}}_{t+1}) \\ & \times \left(y_t + \mathsf{E}[V_{t+1} \mid \text{investor action at } t, H_t, \text{fill at } \mathsf{bid}^{\text{LP}}_{t+1}] - \mathsf{bid}^{\text{LP}}_{t+1}\right)\end{aligned} \tag{5}$$

Sell order payoffs are similarly defined. Their period t dark limit orders are defined in shorthand as DLB_t for buy orders, and DLS_t for sell orders. Given a trade-at rule λ, professional liquidity providers choose $\mathsf{Pr}(\text{DLS}_t)$ such that $\pi_{DLS} = 0$. This yields the condition,

$$\mathsf{ask}^{\text{DLP}}_{t+1} = \mathsf{E}[V_{t+1} \mid \text{investor action at } t, H_t, \text{fill at } \mathsf{ask}^{\text{DLP}}_{t+1}] \tag{6}$$

The zero profit condition for dark limit buy orders is similar.

2 Equilibrium

I search for a symmetric, stationary perfect Bayesian equilibrium in which the best bid and ask prices at the visible market in period t are competitive with respect to information that is available to professional liquidity providers just prior to the arrival of the period t investor.

2.1 Order Pricing Rules

In equilibrium, professional liquidity providers post competitive limit orders to the visible market. I use $*$ to denote equilibrium prices and orders. Equilibrium bid and ask prices are given by bid_t^* and ask_t^*, respectively. An equilibrium order is an order submitted at or against an equilibrium price: an equilibrium market buy order in period t trades against a limit sell order priced at ask_t^*. Similarly, an equilibrium limit buy order submitted in period t is priced at bid_{t+1}^*. The competitive pricing assumption implies that bid_t^* and ask_t^* are given by:

$$\mathsf{bid}_t^* = \mathsf{E}[V_t \mid H_t, \mathrm{MS}_t^*] = v_t + \mathsf{E}[\delta_t \mid \mathrm{MS}_t^*] \tag{7}$$

$$\mathsf{ask}_t^* = \mathsf{E}[V_t \mid H_t, \mathrm{MB}_t^*] = v_t + \mathsf{E}[\delta_t \mid \mathrm{MB}_t^*] \tag{8}$$

where H_t represents the quote and transaction history up to and including period $t-1$.

To post a competitive limit order in period t for execution in period $t+1$, an investor must compensate the period $t+1$ investor for the fact that the investor who submitted the period t limit order *may* be informed. Thus, they incorporate their expected price impact, $\mathsf{E}[\delta_t \mid \mathrm{LB}_t^*]$, into the limit price they set. Hence, limit orders posted by investors have the following quotes,

$$\mathsf{bid}_{t+1}^* = \mathsf{E}[V_{t+1} \mid H_t, \mathrm{MS}_{t+1}^*, \mathrm{LB}_t^*] = v_t + \mathsf{E}[\delta_t \mid \mathrm{LB}_t^*] + \mathsf{E}[\delta_{t+1} \mid \mathrm{MS}_{t+1}^*] \tag{9}$$

$$\mathsf{ask}_{t+1}^* = \mathsf{E}[V_{t+1} \mid H_t, \mathrm{MB}_{t+1}^*, \mathrm{LS}_t^*] = v_t + \mathsf{E}[\delta_t \mid \mathrm{LS}_t^*] + \mathsf{E}[\delta_{t+1} \mid \mathrm{MB}_{t+1}^*] \tag{10}$$

Prices in the dark market are derived from (a) the quote in the visible market, and, (b) the trade-at rule, λ. Hence, professional liquidity providers do not choose the price of their dark limit orders, and their prices, $\mathsf{bid}_t^{\mathrm{DLP}}(\lambda)$ and $\mathsf{ask}_t^{\mathrm{DLP}}(\lambda)$ are given by,

$$\mathsf{bid}_t^{\mathrm{DLP}*}(\lambda) = \mathsf{bid}_t^* + \lambda \times (\mathsf{ask}_t^* - \mathsf{bid}_t^*) \tag{11}$$

$$\mathsf{ask}_t^{\mathrm{DLP}*}(\lambda) = \mathsf{ask}_t^* - \lambda \times (\mathsf{ask}_t^* - \mathsf{bid}_t^*) \tag{12}$$

where equations 11 simplify to $(1+2\lambda) \times \mathsf{bid}_t^*$ and $(1-2\lambda) \times \mathsf{ask}_t^*$, respectively.

Because quotes in the dark are predetermined, professional liquidity providers that submit orders to the dark market choose the intensity with which they submit limit orders such that they earn zero profits. Hence, liquidity providers choose this intensity such that the expected price impact of selling to an investor in the dark, $\mathsf{E}[\delta_t \mid \mathrm{DB}_t(\mathsf{ask}_t^{\mathrm{DLP}^*})]$, equals the price they must sell for, $\mathsf{ask}_t^{\mathrm{DLP}^*}$; similarly for a dark buy limit order. Hence, the competitive prices in the dark market are given by,

$$\mathsf{bid}_t^{\mathrm{DLP}*}(\lambda) = \mathsf{E}[V_t \mid H_t, \mathrm{DS}_t^*] = v_t + \mathsf{E}[\delta_t \mid \mathrm{DS}_t^*] \tag{13}$$

$$\mathsf{ask}_t^{\mathrm{DLP}*}(\lambda) = \mathsf{E}[V_t \mid H_t, \mathrm{DB}_t^*] = v_t + \mathsf{E}[\delta_t \mid \mathrm{DB}_t^*] \tag{14}$$

It is not immediate that dark market orders have a price impact, even post-trade. Because the market only reveals that a trade has occurred in the dark (and not which side is active or passive), it would be intuitive that dark orders would have no price impact post-trade. However, professional liquidity providers submit limit orders to both dark and visible markets, and upon the fill of an order in the dark, they use this information to update their limit orders on the visible market. In this way, trades in the dark have a post-trade price impact.

Lastly, all investors form a common prior, v_t from H_t. It does not, however, appear in any expectations about future innovations, as they are independent of past price history.

2.2 Investor Decision Rules

An investor will submit an order (to either market) if, conditional on their information *and* on the submission of the order, their expected profits are non-negative. Moreover, conditional deciding to trade, an investor chooses the order type that maximizes their expected profits. An investor abstains from trading if all order types yield negative expected profits. Similar to Brolley and Malinova (2014), I restrict my attention to equilibria where investors submit visible limit orders that cannot be improved upon by professional liquidity providers. That

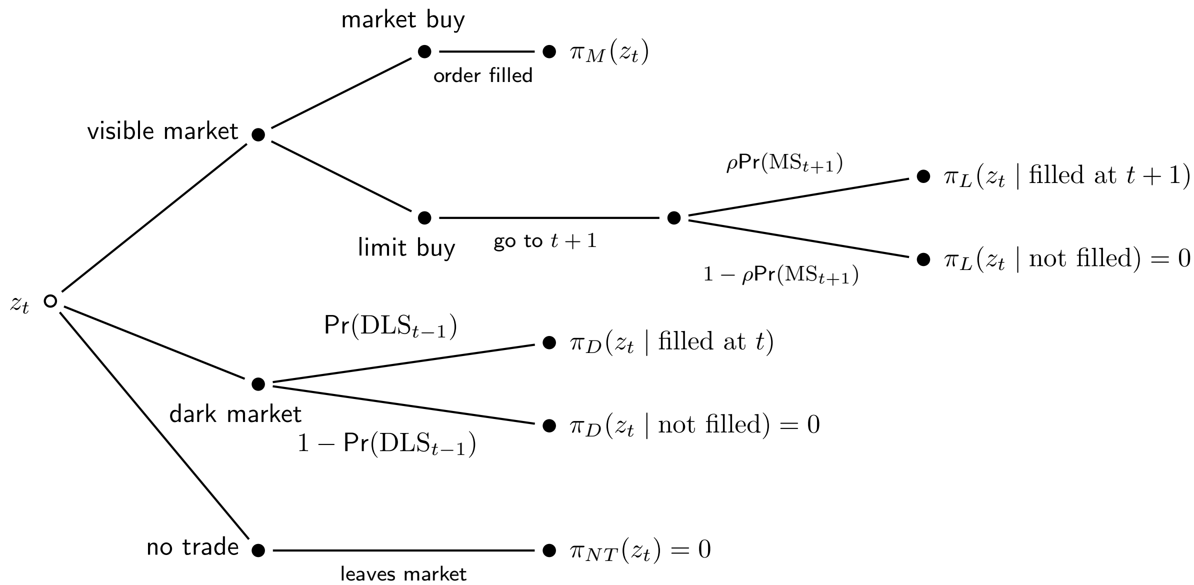

Figure 1: Investor Decision Tree: Given their valuation, z_t, the period t investor chooses to submit an order to the visible market, dark market, or abstain from trading. If the investor sends an order to the visible market, the investor chooses either a market buy, or a limit buy (with the appropriate competitive price). A market order is filled automatically in period t, while a limit order is filled in period $t+1$ with probability $\mathsf{Pr}(\text{MS}_{t+1})$. An order sent to the dark market, is filled with probability $\mathsf{Pr}(\text{DLS}_{t-1})$.

is, where $\mathsf{bid}_t^{\text{inv}} = \mathsf{bid}_t^*$. Orders with non-competitive prices yield the investor negative profits, or an execution probability of zero. I discuss the off-the-equilibrium-path beliefs about non-competitive limit orders in the Appendix.

Investors choose their order based on available public information, plus: (a) their private valuation y_t (if uninformed), or; (b) their knowledge of δ_t, (if informed). Because these valuations enter investor payoff functions identically, I can summarize investor decisions in terms of an investor's "valuation", regardless of type. I denote the period t investor's valuation by, $z_t = y_t + \mathsf{E}[\delta_t \mid \text{info}_t]$. z_t is symmetrically distributed on the interval $[-1, 1]$. Using this notation, I depict the decision tree diagrammatically in Figure 1.

Because innovations δ_t are time-independent, I can decompose investors' valuations into two parts: the expected value of the security based on the public information, plus z_t,

$$\mathsf{E}[V_t \mid \text{info}_t, H_t] = y_t + \mathsf{E}[\delta_t \mid \text{info}_t] + \mathsf{E}[V_{t-1} \mid H_t] = z_t + \mathsf{E}[V_{t-1} \mid H_t] \tag{15}$$

In addition, when submitting a limit buy order to the visible market, an investor incurs an adverse selection cost of trading with a potentially informed investor in period $t+1$, $\mathsf{E}[\delta_{t+1} \mid H_t, \mathrm{LB}_t^*, \mathrm{MS}_{t+1}^*]$. This cost is identical for informed and uninformed investors, because information about period $\tau \leq t$ innovations does not inform about future innovations. Because, in equilibrium, limit orders are priced competitively by taking advantage of the public information H_t, this adverse selection cost is "priced" into the limit order. I summarize all the costs of submitting an order as "transaction costs", which includes: order prices, and, if trading in the following period (with a visible limit order), the expected adverse selection associated with being picked off.

Hence, the payoff expressions (2)-(4) can be reduced to three components: (a) the investor's valuation, (b) the order's transaction costs, and (c) the order's execution probability. The payoff of any order type, k, can then be characterized by,

$$\pi_t^k = \text{execution probability } \times (z_t - \text{transaction costs}) \tag{16}$$

In a symmetric equilibrium, visible market buy orders are guaranteed execution. Limit buy orders submitted at period t, however, are triggered by incoming market sell orders submitted by investors, and thus, their execution probability is dependent on the likelihood that a market sell order is submitted *and* that the market does not close, $\rho \times \mathsf{Pr}(\mathrm{MS}_{t+1})$. Dark market buy orders at period t trade against limit sell orders posted at period $t-1$ by liquidity providers with probability $\mathsf{Pr}(\mathrm{DLS}_{t-1})$.

2.3 Equilibrium Characterization

I begin by deriving order properties that must hold in equilibrium. These properties, in turn, successively reduce the set of candidate equilibria. Proofs are in the appendix.

Lemma 1 (Buy, Sell or Abstain) *In any symmetric, stationary equilibrium where investors use all order types, investors do not sell if $z_t \geq 0$ and do not buy if $z_t \leq 0$.*

I describe the equilibrium in terms of the marginal investor who is indifferent between two order types. Since, I focus only on investors with $z_t \geq 0$, the term 'order' will refer to a buy order, unless explicitly stated otherwise. Intuitively, investors do not trade in the opposite direction of their valuation. However, some investors with non-zero valuations may abstain from trading because transaction costs are too high.

Lemma 2 (Transaction Costs) *In any symmetric, stationary equilibrium where investors use all order types, an order's transaction costs are equal to its price impact. Moreover, any buy (sell) order used in equilibrium has a positive (negative) price impact.*

When an investor submits an order, their payoff simplifies to expression (16), where the term "transaction costs" aggregates the ask or bid price of the order, and all adverse selection costs they expect to incur. In equilibrium, it is the competitive liquidity provision in the visible and dark markets that results in transaction costs being equal to the price impact of the order. On the visible market, professional liquidity providers compete in price. On the dark market, because prices are predetermined, professional liquidity providers compete by choosing the probability with which they submit orders to the dark market, such that the investors who submit dark market orders have a price impact equal to the price they must offer.

In any order placement strategy, an investor prefers lower execution risk, ceteris paribus. However, to ensure that the investor's payoff is positive, they must have a valuation greater than the price impact of the order. Because investors with the most extreme valuations are the most informed (on average), the price impact of the order with the lowest execution risk will be large enough such that investors below a certain valuation threshold will opt for an order with higher execution risk (but lower price impact). Lemma 3 formalizes this intuition..

Lemma 3 (Execution Risk) *In any symmetric, stationary equilibrium where investors use all order types, investors use a threshold strategy where the indifference thresholds $|z^J|$ are decreasing with the order's execution risk.*

Lemma 3 is similar to Hollifield, Miller, and Sandås (2004), who predict that the more extreme an investor's valuation, the higher the execution probability of the order they submit. They test this prediction empirically, as a monotonicity restriction on investors' order submission strategies, and find support when the choices of buy and sell orders are considered separately.

2.4 Equilibrium Existence

Lemmas 1-3 reduce the set of candidate equilibria to symmetric and stationary equilibria in threshold strategies that are decreasing in execution risk; investors with higher valuations choose order types with lower execution risk. Lemma 3 further implies that investors with the most extreme valuations submit visible market orders, because the execution risk for visible market orders is zero; investors with valuations close to zero abstain from trading, as not trading has the highest execution risk (an execution probability of zero). Thus, two equilibrium candidates remain: (a) a dark order's execution risk is greater than that of a visible limit order, implying that the equilibrium thresholds are $0 \leq z^D \leq z^L \leq z^M \leq 1$, and, (b) a dark order's execution risk is smaller than a visible limit order, implying that the equilibrium thresholds are $0 \leq z^L \leq z^D \leq z^M \leq 1$.

2.4.1 Benchmark Equilibrium

I begin by examining the model from the perspective of *introducing* a dark market alongside a visible limit order market. In the absence of a dark market, there exists an equilibrium with only a visible market, similar to Brolley and Malinova (2014), which I refer to as the benchmark equilibrium.

Theorem 1 (Benchmark Equilibrium) *If investors may only access the visible market, there exist values z^M and z^L, where $0 < z^L < z^M < 1$ that constitute an equilibrium in symmetric, stationary threshold strategies.*

If we introduce a dark market with a trade-at rule λ such that dark market orders are identical in every payoff-relevant aspect to visible limit orders, investors have no incentive to submit dark market orders, in equilibrium, by the assumption that investors choose to send orders to the visible market when they are indifferent between a visible order and a dark order. Thus, the benchmark equilibrium can occur in a financial market with dark orders, if there exists a trade-at rule λ such that dark orders are payoff-equivalent to visible limit orders.

Given that the price offered by the dark market to a buy order in period t is equal to,

$$\mathsf{ask}_t^{\mathrm{DLP}*} = \mathsf{ask}_t^* - \lambda \times (\mathsf{ask}_t^* - \mathsf{bid}_t^*) = (1 - 2\lambda)\mathsf{ask}_t \tag{17}$$

where the second equality follows by symmetry. Hence, the range of prices for dark limit buy orders is $(-\mathsf{ask}_t, \mathsf{ask}_t)$. By Lemma 2, however, in any equilibrium, a dark order has a price impact. If $\lambda \geq 1/2$, then the trade-at rule is set to the midquote or better, implying a price below the public value, v_t. Moreover, liquidity providers face an expected positive adverse selection cost of $\mathsf{E}[\delta_t \mid \mathrm{DB}_t^*]$ when trading with an investor in the dark market (by Lemma 2). Thus, liquidity providers would expect to earn a loss when submitting a dark limit order.

Proposition 1 (No Dark Liquidity Provision) *If $\lambda \geq 1/2$, then in equilibrium,* $\mathsf{Pr}(DLS_{t-1}) = \mathsf{Pr}(DLB_{t-1}) = 0.$

The set of viable prices for dark orders to be used in equilibrium is then reduced to $(0, \mathsf{ask}_t)$. Now consider the investor's problem. There is another trade-at rule for which investors are indifferent between the visible and dark market. This trade-at rule λ equates the transaction costs of submitting a dark market order to that of submitting a visible limit order,

$$(1 - 2\lambda^*) \times \mathsf{E}[\delta_t \mid \mathrm{MB}_t^*] - \mathsf{E}[\delta_t \mid \mathrm{LB}_{t+1}^*] = 0 \tag{18}$$

When the transaction costs of a dark market order and a visible limit order are equal, liquidity providers can only earn zero expected profits on their dark limit orders if the execution risk for dark market orders and visible limit orders are equal. Hence, professional liquidity providers compete in the execution likelihood of dark market orders, $\mathsf{Pr}(\text{DLS}^*)$, so that, in equilibrium, $\mathsf{Pr}(\text{DLS}^*)$ equals the execution likelihood of a visible limit order, $\mathsf{Pr}(\text{MS}^*)$. Then, because investors use visible orders when they are indifferent to dark orders, the dark market is not used in equilibrium for λ^*.

Proposition 2 (No Dark Market) *Given the benchmark equilibrium threshold values z^M and z^L, as determined in Theorem 1, there exists a unique $\lambda^* \in (0, 1/2)$ that solves (18). Moreover, in equilibrium, $\mathsf{Pr}(DLS^*) = \mathsf{Pr}(MS^*)$.*

2.4.2 Dark Orders with a Large Trade-at Rule.

If the trade-at rule is $\lambda^* \leq \lambda < 1/2$ (referred to as a "large trade-at rule"), then dark orders have lower transaction costs than visible limit orders, because expression (18) is negative. Lemma 3 dictates that $0 \leq z^D \leq z^L \leq z^M \leq 1$ can be the only possible equilibrium, if any. The equilibrium conditions for an equilibrium with a large trade-at rule are,

$$z^M - \mathsf{E}[\delta_t \mid \text{MB}_t^*] = \rho \cdot \mathsf{Pr}(\text{MS}_{t+1}^*) \times \left(z^M - \mathsf{E}[\delta_t \mid \text{LB}_t^*]\right) \quad (19)$$

$$\mathsf{Pr}(\text{DLS}_{t-1}) \times \left(z^L - (1-2\lambda) \times \mathsf{E}[\delta_t \mid \text{MB}_t^*]\right) = \rho \cdot \mathsf{Pr}(\text{MS}_{t+1}^*) \times \left(z^L - \mathsf{E}[\delta_t \mid \text{LB}_t^*]\right) \quad (20)$$

$$z^D = (1-2\lambda) \times \mathsf{E}[\delta_t \mid \text{MB}_t^*] \quad (21)$$

$$(1-2\lambda) \times \mathsf{E}[\delta_t \mid \text{MB}_t^*] = \mathsf{E}[\delta_t \mid \text{DB}_t^*] \quad (22)$$

Conditions (19)-(21) represent the following indifference conditions, respectively: *i)* visible market orders and limit orders; *ii)* limit orders and dark orders, and; *iii)* dark orders and abstaining from trade. Finally, condition (22) describes the liquidity provider's zero expected profit condition for dark limit orders. For $\lambda \in (\lambda^*, 1/2)$, there exist z^M, z^L and z^D such that $0 \leq z^D \leq z^L \leq z^M \leq 1$, that solve the system (19)-(22), yielding the following equilibrium.

Theorem 2 (Large Trade-at Rule) *There exist values z^M, z^L and z^D, where $0 \leq z^D \leq z^L \leq z^M \leq 1$, that constitute an equilibrium in symmetric, stationary threshold strategies, if and only if $\lambda \in (\lambda^*, 1/2)$.*

When the trade-at rule is large, dark orders provide substantial price improvement over posted prices, making them attractive relative to limit orders in terms of transaction costs. Because the transaction costs of visible limit orders are determined in equilibrium to ensure that professional liquidity providers break even, professional liquidity providers can only break even with dark orders if they dis-incentivize moderate-valuation investors (who are, on average, more-informed) from trading in the dark, instead attracting only low-valuation investors. To do so, professional liquidity providers offer a lower fill rate ($\mathsf{Pr}(\mathrm{DLS}^*)$) for dark orders relative to limit orders, $\mathsf{Pr}(\mathrm{MS}^*)$.

2.4.3 Dark Orders with a Small Trade-at Rule.

If the dark market has a trade-at rule $\lambda \leq \lambda^*$, (which I refer to as a "small trade-at rule") dark orders have higher transaction costs than visible limit orders, because expression (18) is positive. Lemma 3 dictates that $0 \leq z^L \leq z^D \leq z^M \leq 1$ can be the only equilibrium that exists, if any. The equilibrium conditions for an equilibrium with a small trade-at rule are,

$$\mathsf{Pr}(\mathrm{DLS}_{t-1}) \times \left(z^M - (1-2\lambda) \times \mathsf{E}[\delta_t \mid \mathrm{MB}_t^*]\right) = z^M - \mathsf{E}[\delta_t \mid \mathrm{MB}_t^*] \tag{23}$$

$$\mathsf{Pr}(\mathrm{DLS}_{t-1}) \times \left(z^D - (1-2\lambda) \times \mathsf{E}[\delta_t \mid \mathrm{MB}_t^*]\right) = \rho \cdot \mathsf{Pr}(\mathrm{MS}_{t+1}^*) \times \left(z^D - \mathsf{E}[\delta_t \mid \mathrm{LB}_t^*]\right) \tag{24}$$

$$z^L = \mathsf{E}[\delta_t \mid \mathrm{LB}_t^*] \tag{25}$$

$$(1-2\lambda) \times \mathsf{E}[\delta_t \mid \mathrm{MB}_t^*] = \mathsf{E}[\delta_t \mid \mathrm{DB}_t^*] \tag{26}$$

Similar to the previous section, conditions (23)-(25) represent the following indifference conditions, respectively: *i)* visible market orders and dark orders; *ii)* dark orders and limit orders, and; *iii)* limit orders and abstaining from trade. Condition (26) describes the liquidity provider's zero expected profit condition for dark limit orders. For all $\lambda \in (0, \lambda^*)$, there exist z^M, z^L and z^D such that $0 \leq z^L \leq z^D \leq z^M \leq 1$, that solve the system (23)-(26), yielding the following equilibrium.

Theorem 3 (Small Trade-at Rule) *There exist values z^M, z^L and z^D, where $0 \leq z^L \leq z^D \leq z^M \leq 1$, that constitute an equilibrium in symmetric, stationary threshold strategies, if and only if $\lambda \in (0, \lambda^*)$.*

In an equilibrium with a small trade-at rule, dark orders have higher transaction costs to investors when compared to visible limit orders. For dark orders to be used in equilibrium, professional liquidity providers must compensate investors for these higher transaction costs by ensuring that the fill rate of their dark orders, $\mathsf{Pr}(\text{DLS}^*)$ is greater than the fill rate of visible market orders, $\mathsf{Pr}(\text{MS}^*)$.

3 Impact of Dark Markets on Market Quality

In this section, I analyze the impact of dark trading on market quality and welfare numerically. I do so by introducing a dark market with either a small or large trade-at rule alongisde a visible limit order market. The classification of trade-at rule level is relative to the benchmark trade-at rule, λ^*. In my numerical analysis, I assume the distribution of innovations, $f(\delta)$ and private values $g(y)$ to be uniform. The following subsections focus on aspects of market quality; Section 4 discusses price efficiency and welfare.

3.1 Trading Volume and Market Participation

Trading volume in the context of my model has two components: visible and dark market volume. Visible market volume is measured by the probability that an investor submits either a buy or sell market order to the visible market (these orders are always filled). Orders submitted to the dark market, however, have some level of execution risk. Dark market volume is therefore measured by the probability that an investors submits an order to the dark market, discounted by the likelihood that the order will be filled. Total trading volume at period t is thus measured as,

$$\mathsf{Trading\ Volume}_t = 2 \times (\mathsf{Pr}(\mathrm{MB}_t) + \mathsf{Pr}(\mathrm{DLS}_{t-1}) \times \mathsf{Pr}(\mathrm{DB}_t)) \tag{27}$$

where the factor of 2 accounts for sell orders. Market participation measures the likelihood that an investor who arrives at the market in period t submits an order of *some type* (i.e does not abstain from trading), which is measured by,

$$\mathsf{Market\ Participation}_t = 2 \times (\mathsf{Pr}(\mathrm{MB}_t) + \mathsf{Pr}(\mathrm{DB}_t) + \mathsf{Pr}(\mathrm{LB}_t)) \tag{28}$$

I summarize the numerical findings graphically in Figure 4.

Numerical Observation 1 (Volume and Market Participation) *If a dark market with a large trade-at rule is introduced alongside a visible market, total volume and market participation increases; a dark market with a small trade-at rule is decreasing in total volume, and non-monotonic in market participation.*

Numerically, I find that when a dark market competes with the visible market by setting a large trade-at rule, trading volume and market participation improves. Intuitively, by offering an order type with lower transaction costs than visible limit orders, participation increases from investors who would otherwise abstain from the market. As visible limit order submitters with lower valuations switch to the cheaper dark market, limit orders themselves become more expensive. This effect ripples to higher-valuation limit order submitters who now find visible market orders more desirable, further increasing trading volume. While such a move increases the price of both visible market orders and dark orders, thereby counteracting these changes to some extent, the numerical results suggest that the former effects dominate. A dark market that competes by offering a small trade-at rule (which in turn leads to lower execution risk than visible limit orders) has mixed results for volume and market participation.

By separating volume into visible and dark market volume, I can also address whether volumes migrates from visible to dark markets, or whether dark markets facilitate trades that otherwise would not have occurred. See Figure 4 for a graphical illustration.

Numerical Observation 2 (Volume Creation vs. Volume Migration) *If a dark market with a large trade-at rule is introduced alongside an visible market, the dark market creates (net) new volume; a small trade-at rule results in volume migrating to the dark market.*

3.2 Bid-Ask Spread and Price Impact

I also consider the impact of a dark market on visible market liquidity [9]. The results are illustrated graphically in Figure 5.

Numerical Observation 3 (Quoted Spread / Price Impact) *If a dark market with a large trade-at rule is introduced alongside a visible market, the quoted spread tightens (market order price impact declines); a dark market with a small trade-at rule widens the quoted spread (market order price impact increases).*

Here, a dark market with a large trade-at rule adds to the market by 'making' increasingly attractive (i.e., inexpensive) liquidity for low valuation investors. Moreover, limit order submitters (who have lower valuations compared to visible market order submitters) switch to visible market orders, thereby tightening the spread and reducing price impact. Increasing the attractiveness of a dark market with a small trade-at rule through lower execution risk, on the other hand, 'takes' from the visible market: market order submitters migrate to the dark market, increasing the average informativeness of visible market orders, thus widening the spread.

[9] In this framework, the bid-ask spread is synonymous with the price impact of a market order, as the spread measures the adverse selection costs due to information of a trade on the visible market, which occurs when a visible market order is placed. For better illustration, I use the quoted half-spread measure (symmetry implies that the bid-ask spread is equal to twice the quoted half-spread)

4 Price Efficiency and Social Welfare

In this section, I describe how dark market trade-at rules impact price efficiency and social welfare. I analyze price efficiency by measuring the difference between the conditional expected innovation and the investor's price impact in a given period. For social welfare, I use a measure of allocative efficiency. That is, I measure how trade-at rules improve gains from trade for uninformed investors. Because information-based trading is a zero-sum game, there are no additional gains from trade to be made by improving their trading prospects. Instead, I look at how these trade-at rules impact the ability of uninformed investors to realize their private valuation.

4.1 Price Efficiency

Price impact measures the post-trade impact on the public expectation of the security. However, because of noise from uninformed investors, the change in the public expectation does not coincide with the change in the true value of the security (the innovation). Thus, one can think of price efficiency in this framework as the ex ante difference, between the conditional expected innovation's value and the investor's expected price impact (per period). More formally (for buy orders),

$$\begin{aligned}\int_0^1 \frac{\delta f(\delta)}{2}\mathrm{d}\delta - \mathsf{E}[\delta \mid \mathsf{buy}] \quad = \quad & \int_{MB} \frac{\delta - ((1-\mu) + \mu f(\delta)) \times \mathsf{E}[\delta \mid \mathrm{MB}]}{2}\mathrm{d}\delta \\ & + \int_{LB} \frac{\delta - ((1-\mu) + \mu f(\delta)) \times \mathsf{E}[\delta \mid \mathrm{LB}]}{2}\mathrm{d}\delta \\ & + \int_{DB} \frac{\delta - \mathsf{Pr}(\mathrm{DLS}) \times ((1-\mu) + \mu f(\delta)) \times \mathsf{E}[\delta \mid \mathrm{DB}]}{2}\mathrm{d}\delta \end{aligned} \qquad (29)$$

How does the introduction of a dark market affect price efficiency, and does the effect depend on the pricing of dark orders? I find that, regardless of the trade-at rule, (i.e., λ) the introduction of dark orders decreases price efficiency. The numerical observation below summarizes these findings, and Figure 6 presents them graphically.[10]

Numerical Observation 4 (Price Efficiency) *Price efficiency always decreases with the introduction of dark market trading.*

As the general usage of dark orders increases, price efficiency falls because investors migrate from visible orders, to an order that only impacts prices when it is filled. Moreover, in both cases, the mass of investors that migrate to the dark market is larger than those who would participate only in a market with dark market trading (they previously made no contribution to price efficiency). Therefore, price efficiency declines.

4.2 Social Welfare

Uninformed investors earn gains from realizing their private valuation. Following the work of Bessembinder, Hao, and Lemmon (2012), I use private valuations to define social welfare as a measure of allocative efficiency. If a transaction occurs in period t, then the social welfare gain is given by the private valuation of a buyer minus the private valuation of a seller.

A trade occurs when the period t investor submits either a market order to the visible market, or the investor submits an order to the dark market *and* the order gets filled. By symmetry, total expected welfare per period is twice this amount. To assess the expected gains from trade in period t, I measure the expected private valuation of the investor submitting the market order and the expected welfare of the counter-party, discounted by the probability of an order being filled. Thus, the total expected welfare in period t is given by,

$$\begin{aligned} W_t &= 2 \cdot \mathsf{Pr}(\mathrm{MB}_t) \cdot (\mathsf{E}[y_t \mid H_t, \mathrm{MB}_t] - \mathsf{Pr}(\mathrm{LS}_{t-1}) \cdot \mathsf{E}[y_{t-1} \mid H_{t-1}, \mathrm{MB}_t, \mathrm{LS}_{t-1}]) \\ &\quad +2 \cdot \mathsf{Pr}(\mathrm{DB}_t) \cdot \mathsf{Pr}(\mathrm{DLS}_{t-1}) \cdot \mathsf{E}[y_t \mid H_t, \mathrm{DB}_t] \end{aligned} \tag{30}$$

Expression (30) describes total expected welfare per period as the expected private value realized in period t, conditional on a visible or dark trade, and discounted by the likeli-

[10] Figure 6 uses zero as the informationally efficient benchmark, and larger deviations from zero, correspond with less informationally efficient prices.

hood of each trade type. The first term describes the (discounted) total expected welfare of a visible market trade: the expected private valuation from the market order submitter, and the expected private valuation of the limit order submitter (which is non-zero in expectation for the investor, but zero for a professional liquidity provider). The second term describes the (discounted) total expected welfare of a dark market trade. The factors of two in expression (30) account for the symmetry of buyer and seller trades.

I find that introducing dark market trading alongside the visible market has the following effect on expected social welfare, $W(t)$. (See also Figure 7).

Numerical Observation 5 (Social Welfare) *If a dark market with a large trade-at rule is introduced alongside a visible market, social welfare improves; a dark market with a small trade-at rule reduces social welfare.*

A dark market that offers a large trade-at rule (i.e., smaller transaction costs than a visible limit order) has a positive effect on welfare. It does so by encourages greater use of visible market orders from investors who use visible limit orders (implying more frequent trades). This occurs because the transaction costs of a limit order increases when the low-valuation limit order submitters migrate to the dark market. Moreover, the dark market attracts investors who would otherwise not trade in a purely visible market. Hence, private valuations are realized more frequently per order (more visible market orders) and more frequently overall (higher market participation). This result suggests that when dark orders have low costs, it is socially beneficial to have professional liquidity providers as the specialists in passive orders, and for investors to assume the role of liquidity takers (whether on the visible or dark market).

I find that not all dark markets are welfare-improving, however. A dark market with a small trade-at rule leads to fewer visible market orders in equilibrium. Then, despite more limit order submitters now migrating to the dark market (where execution risk is lower), the likelihood that an investor's private valuation is realized, declines. While more investors are able to participate in the market because of the availability of dark market trading, the decline in visible market volume exceeds the increase in market participation, leading to the decline in social welfare.

5 Policy Implications and Empirical Predictions

In sections 3 and 4, my model predicts that the impact of dark trading on visible market quality and social welfare depends on the dark market trade-at rule, and as such, a minimum trade-at rule has the potential to improve welfare and visible market quality. A minimum trade-at rule of $\lambda = \lambda^*$ would make it unprofitable for a dark market to attract moderate valuation investors with a small trade-at rule (which implies low execution risk). However, it would be profitable for the dark market to attract low valuation investors with a large trade-at rule, a dark market that would contribute positively to both market quality and welfare.

The minimum trade-at rule discussion has important implications for equity markets in Canada and Australia. On October 15th, 2012, the Investment Industry Regulatory Organization of Canada (IIROC) implemented a trade-at rule that requires most dark orders to provide a meaningful price improvement of one trading increment (or in the case of a one tick spread, half the spread).[11] Australia followed suit with a similar rule on May 26th, 2013. Since many liquid securities in global equity markets operate at a spread of one or two ticks, the minimum trade-at rule effectively implies dark orders must trade at midpoint. The security that I model is reflective of this type of security, as investors trade in single units (i.e., never walk the book), and professional liquidity providers ensure that the book is always full.

My model predicts that a minimum trade-at rule requiring dark pools to execute orders at midpoint would eliminate all dark trading from the market. At midpoint, liquidity providers vanish from the dark market. It is indeterminable from this model whether a market without a minimum trade-at rule would lead to a dark market that harms market quality and welfare

[11] See Investment Industry Regulatory Organization of Canada (2011)

or one that improves both. However, setting a less-restrictive minimum trade-at rule would lead liquidity providers to provide only the amount of liquidity that results in the dark market being beneficial to market quality and welfare.

For securities where market makers are present but spreads are wider than one or two ticks, a fixed tick size trade-at rule does not restrict the pricing behavior of dark markets the way it restricts more liquid securities, as the fixed trade-at rule is smaller as a fraction of the spread. In those cases, the larger spread weakens the rule, allowing for dark markets with high-price-for-high-fill regimes to participate. In this way, a fixed tick trade-at rule potentially widens the liquidity gap between more and less liquid stocks. Securities whose spreads imply that the effective minimum trade-at rule as a percentage of the spread is $\lambda > \lambda^*$ (more liquid stocks) allow only spread-tightening dark markets to participate. Securities with spreads that imply $\lambda < \lambda^*$ (less liquid stocks) would allow both spread-tightening and spread-widening dark markets to enter the market.

6 Concluding Remarks

In this paper, I study how trade-at rules in dark markets impacts market quality and investor welfare in equity markets. I construct a model where both informed and uninformed investors can access a dark market and a visible limit order market. Moreover, the pricing decision for limit orders is simplified, as both markets are monitored by professional liquidity providers that ensure competitive limit order pricing.

The main result is that the trade-at rule in dark markets matters, the effects of which depend on whether the trade-at rule is large or small. A dark market that competes with a visible market by offering a large trade-at rule improves market quality and social welfare. These dark markets trade-off their discount pricing with higher execution risk. Because they offer lower transaction costs than visible limit orders (when price impact and pick-off risk are combined), the dark market attracts investors from both the pool of investors that submit limit orders, as well as investors who would otherwise abstain from trading. By setting a small trade-at rule such that the price of a dark order is higher than a visible limit order, professional liquidity providers supply more liquidity to the dark market than in the large trade-at rule case. In equilibrium, the execution risk in the dark is lower than with limit orders, and this leads to order migration from investors with moderate valuations. Investors migrate from both the pools of visible market orders submitters and visible limit orders. Consequently, market quality and social welfare decline. In terms of price efficiency, I find that introducing a dark market alongside a visible market results in lower price efficiency, irrespective of the trade-at rule.

These results have implications for minimum trade-at rule regulation. My model predicts that the impact of dark trading on visible market quality and social welfare depends on the trade-at rule of the dark market, and as such, a minimum trade-at rule has the potential to improve welfare and visible market quality. A minimum trade-at rule of equal to the benchmark trade-at rule would prevent dark markets from attracting investors with moderate valuations, as liquidity providers would not be willing to provide liquidity with that level of intensity (it would be unprofitable). However, it would be profitable for the dark market to attract low valuation investors by setting a large trade-at rule, an as such, the dark market would contribute positively to both market quality and welfare.

The minimum trade-at rule discussion has important implications for equity markets in Canada and Australia. On October 15th, 2012, the Investment Industry Regulatory Organization of Canada (IIROC) implemented a minimum trade-at rule that requires most dark orders to provide a meaningful price improvement of one trading increment (or in the case of a one tick spread, half the spread).[12] Australia followed suit with a similar rule on May 26th, 2013. Because many liquid securities in global equity markets operate at a spread of one or two ticks, the minimum trade-at rule effectively implies dark orders must trade at midpoint. My model predicts that a minimum trade-at rule that requires orders to be filled

[12] See Investment Industry Regulatory Organization of Canada (2011)

at the midpoint of the spread would eliminate all dark trading from the market, as liquidity providers would vanish from the dark market.

For securities where spreads are wider than one or two ticks, the fixed-tick trade-at rule imposed by IIROC and ASIC does not restrict the pricing behavior of dark markets in the same way, as orders are not required to fill at midpoint. In this way, a fixed-tick price improvement rule opens the possibility for dark markets to worsen the liquidity of relatively illiquid stocks. If these illiquid securities have spreads are not too wide, then the presence of a dark market improves liquidity. To remedy this issue, a trade-at rule that pegs the trade-at rule to a percentage of the spread would be more equitable across securities of different liquidity levels.

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A Appendix

The appendix contains all proofs and figures not presented in the text.

A.1 Proofs: Lemmas

Proof. (Lemma 1).

I prove this lemma by showing that, in equilibrium, for every type of sell order, there is a buy order that is preferred by investors with valuations $z_t \geq 0$. It then follows that investors non-negative valuations do not use sell orders in any equilibrium with symmetric and stationary strategies. Time subscripts are dropped, as I focus on stationary equilibria.

Let investor j have a valuation equal to $z_j \geq 0$. An investor j prefers a market buy order to a market sell order if,

$$\pi_{\text{MB}}(z_j) \geq \pi_{\text{MS}}(z_j) \iff z_j \geq \frac{\mathsf{E}[\delta \mid \text{MB}^*] + \mathsf{E}[\delta \mid \text{MS}^*]}{2} = 0 \tag{31}$$

where the last equality follows by symmetry. Thus, an investor does not use market sell orders if their valuation is positive.

An investor j prefers a limit buy order to a limit sell order if,

$$\pi_{\text{LB}}(z_j) \geq \pi_{\text{LS}}(z_j) \iff z_j \geq \frac{\mathsf{Pr}(\text{MS}^*)\mathsf{E}[\delta \mid \text{LB}^*] + \mathsf{Pr}(\text{MB}^*)\mathsf{E}[\delta \mid \text{LS}^*]}{\mathsf{Pr}(\text{MB}^*) + \mathsf{Pr}(\text{MS}^*)} = 0 \tag{32}$$

where the last equality follows by symmetry. Thus, an investor does not use limit sell orders if their valuation is positive.

An investor j prefers sending a buy order over a sell order to the dark if,

$$\pi_{\text{DB}}(z_j) \geq \pi_{\text{DS}}(z_j) \iff z_j \geq (1-2\lambda)\frac{\mathsf{Pr}(\text{DLS}^*)\mathsf{E}[\delta \mid \text{MB}^*] + \mathsf{Pr}(\text{DLB}^*)\mathsf{E}[\delta \mid \text{MS}^*]}{\mathsf{Pr}(\text{DLB}^*) + \mathsf{Pr}(\text{DLS}^*)} = 0 \tag{33}$$

where the last equality follows by symmetry. Thus, an investor does not send sell orders to the dark if their valuation is positive.

The argument for investors not using buy orders when $z_j \leq 0$ follows by symmetry. ■

Proof. (Lemma 2).

Recall that the general payoff function is given by,

$$\text{execution probability} \times (z_t - \text{transaction costs})$$

What I will show is that, for each buy order type, the transaction costs are equal to the price impact, and that the price impact is positive.

The lemma holds trivially for a market buy order, as the investor's payoff (2) is

$$\begin{aligned} v_t + z_t - \mathsf{ask}_t &= v_t + z_t - v_t - \mathsf{E}[\delta_t \mid \text{MB}_t] & (34) \\ &= z_t - \mathsf{E}[\delta_t \mid \text{MB}_t] & (35) \end{aligned}$$

and hence the price impact, $\mathsf{E}[\delta_t \mid \text{MB}_t]$ is equal to the transaction costs of a market buy order.

For limit orders, simplifying the investor's payoff function (3) yields,

$$\begin{aligned} \pi_{LB} &= \rho \mathsf{Pr}(\text{MS}^*)(\mathsf{E}[V_{t+1} \mid \text{LB}_t, \text{MS}_{t+1}] - \mathsf{bid}_{t+1}) & (36) \\ &= \rho \mathsf{Pr}(\text{MS}^*)(v_t + z_t + \mathsf{E}[\delta_{t+1} \mid \text{LB}_t, \text{MS}_{t+1}] - v_t - \mathsf{E}[\delta_t \mid \text{LB}_t] - \mathsf{E}[\delta_{t+1} \mid \text{LB}_t, \text{MS}_{t+1}]) \\ &= \rho \mathsf{Pr}(\text{MS}^*)(z_t - \mathsf{E}[\delta_t \mid \text{LB}_t]) \end{aligned}$$

and hence the price impact, $\mathsf{E}[\delta_t \mid \text{LB}_t]$ is equal to the transaction costs of a limit buy order. The price impact of sending an order to the dark market requires the equilibrium condition of liquidity providers to the dark market. In equilibrium, the price of an order filled in the dark at period t, $(1 - 2\lambda) \times \mathsf{ask}_t$, is equal to $\mathsf{E}[V_t \mid \text{DB}_t, H_{t-1}]$, which simplifies to,

$$\begin{aligned} \mathsf{ask}_t - \lambda \times (\mathsf{ask}_t - \mathsf{bid}_t) &= \mathsf{E}[V_t \mid \text{DB}_t, H_t] & (37) \\ \iff v_t + (1 - 2\lambda) \times \mathsf{E}[\delta_t \mid \text{MB}_t] &= v_t + \mathsf{E}[\delta_t \mid \text{DB}_t] & (38) \end{aligned}$$

I can now write the investor's payoff to a dark order (4) as,

$$\begin{aligned}\pi_{DB} &= \mathsf{Pr}(\text{DLS}^*)(v_t + z_t - v_t - (1-2\lambda) \times \mathsf{E}[\delta_t \mid \text{MB}_t]) \\ &= \mathsf{Pr}(\text{DLS}^*)(z_t - \mathsf{E}[\delta_t \mid \text{DB}_t])\end{aligned} \tag{39}$$

which implies that the (post-trade) price impact of a dark order, $\mathsf{E}[\delta_t \mid \text{DB}_t]$ is equal to the transaction costs of a dark buy order.

Finally, by Lemma 1, only investors with $z_t \geq 0$ submit buy orders. Thus, $\mathsf{E}[\delta_t \mid \mathsf{buy}_t] \geq 0$. The argument for sell orders is symmetric. ■

Proof. (Lemma 3).

Let γ_I denote the execution probability of an order type, I, and let p_I denote I's transaction costs. I will show that in any symmetric, stationary equilibrium where all order types are used, if $\gamma_J \geq \gamma_K$, then investors use a threshold strategy where $z^J \geq z^K$ for all p_J, p_K.

Suppose there is an order type J such that $\gamma_J \geq \gamma_K$ and $p_J < p_K$. Let order type K be used by some investor, t, in equilibrium (investor t earns non-zero profits). Then, the profit of investor t from using order type K is,

$$\pi_K^t = \gamma_K \times (z_t - p_K) < \gamma_J \times (z_t - p_J) = \pi_J^t \tag{40}$$

for all z_t. Thus, for an order to be used in equilibrium, it must be that it dominates another order type in either execution probability or transaction costs.

Thus, we must have that for three order types such that $\gamma_I \geq \gamma_J \geq \gamma_K \geq 0$, it must be the case that $p_I \geq p_J \geq p_K \geq 0$.

This also implies that,

$$\gamma_I p_I \geq \gamma_J p_J \geq \gamma_K p_K \geq 0 \tag{41}$$

Because the general payoff function for an order type i is of the form,

$$\pi_i = \gamma_i \times (z_t - p_i)$$

we can see that the function is linear in z_t, and that the order with the highest execution probability also has the lowest intercept, $-\gamma_i p_i$. The linearity of π_i in z_t implies that since $\gamma_I \geq \gamma_J \geq \gamma_K \geq 0$, if the profit function of order I crosses the profit functions of J and K at some z_I, it remains above them for all $z_t > z_I$.

Then, the relation in 41 implies that for I to be used in equilibrium, it must cross J and K. Thus, investors with $z_t \geq z_I$ use order I.Likewise, if J crosses K, it is above K for all $z_t \geq z_J$. Then, for J to be used in equilibrium, π_J must cross π_K before π_I crosses π_J. Thus, $z_J \leq z_I$. Lastly, π_K crosses the no-trade threshold, $\pi_{NT} = 0$ before π_J and π_I, but at a positive $z_t = \gamma_K p_K$. Hence, $z_K > 0$. ■

A.2 Proofs: Existence Theorems and Propositions

This section contains the proofs of the existence theorems and proposition 1. I conduct the existence proofs in similar steps. I select a single threshold to be the reference threshold, showing that all other thresholds exist and are unique for all values of the reference threshold, making use of the intermediate value theorem, Lemma (3), and the implicit function theorem. Then, using the intermediate value theorem, I show that there exists a value of the reference threshold that such that the equilibrium exists. As a remark on notation, I drop the time subscripts in all proofs, because of the stationarity condition.

Preliminaries. In the proofs to follow, the conditional expectation over the innovation, δ, plays a prominent role. The expectation over δ, conditional on $a \leq z \leq b$ is given by,

$$\mathsf{E}[\delta \mid a \leq z \leq b] = \frac{\mu \int_a^b \delta f(\delta) d\delta}{\mu \int_a^b f(\delta) d\delta + (1-\mu)(b-a)} \tag{42}$$

where z represents the informed investors and uninformed investors who have valuations in $[a, b]$. For example, a market buy would have $[a, b] = [z^M, 1]$. Expression (42) is continuous: it is the quotient of two continuous functions (f is continuous on $\delta \in [0, 1]$) where the denominator is never zero, and by l'Hopital's rule,

$$\lim_{b\to a} \mathsf{E}[\delta \mid a \le z \le b] = \frac{\mu \cdot a f(a)}{\mu f(a) + (1-\mu)} \neq 0 \quad (43)$$

$$\lim_{a\to b} \mathsf{E}[\delta \mid a \le z \le b] = \frac{\mu \cdot b f(b)}{\mu f(b) + (1-\mu)} \neq 0 \quad (44)$$

the limits exist for all $\mu \in (0,1)$. Moreover, because the denominator of (42) is continuously differentiable, then so is (42).

Proof. (Theorem 1).

To prove existence of a symmetric, stationary equilibrium in threshold strategies of this form, I first prove the existence of an equilibrium in an environment without a dark market, and then show that introducing a dark market that satisfies $\lambda = \lambda^*$ to trading on the visible market has no effect on the equilibrium thresholds.

The market order and limit orders indifference conditions are given by,

$$z^M - \mathsf{E}[\delta \mid \mathrm{MB}^*] = \rho \mathsf{Pr}(\mathrm{MS}^*) \times (z^M - \mathsf{E}[\delta \mid \mathrm{LB}^*]), \quad (45)$$

$$z^L = \mathsf{E}[\delta \mid \mathrm{LB}^*] \quad (46)$$

where an investor submits a market buy over a limit buy as long as $z_t \ge z^M$, submits a limit buy if $z^M > z_t \ge z^L$, and abstains from trading otherwise. To show existence of a threshold equilibrium, I need to show existence of thresholds z^M and z^L.

I proceed in 3 steps. In step 1, I show that for any given $z^M \in [0,1]$ there exists a unique z^L that solves (46). In Step 2, I show that there exists a z^M that solves (45).

Step 1: Existence and Uniqueness of $z_*^L(z^M)$

Take the function $Z^l = z^L - \mathsf{E}[\delta_t \mid \mathrm{LB}_t]$. To show that Z^l only crosses zero once from below on $z^L \in [0, z^M]$, first note that,

$$Z^l(0) = 0 - \mathsf{E}[\delta \mid \mathrm{LB}^*] < 0 \tag{47}$$

$$Z^l(z^M) = z^M - \mathsf{E}[\delta \mid \mathrm{LB}^*] > z^M - \mathsf{E}[\delta \mid \mathrm{MB}^*] \geq 0 \tag{48}$$

where the last inequality in (48) follows from the participation constraint of market order users. Thus, z^{L*} exists. Let $\mathsf{E}[\delta \mid \mathrm{LB}^*] = \mathsf{num}_L/\mathsf{pr}_L$. Then, differentiating Z^l by z^{L*},

$$\begin{aligned} Z^l_{z^{L*}} &= 1 - \frac{\partial \mathsf{E}[\delta \mid \mathrm{LB}^*]}{\partial z^{L*}} \\ &= 1 - \frac{\mu g(z^{L*}) z^{L*} \mathsf{pr}_L + (\mu g(z^{L*}) + (1-\mu)\mathsf{num}_L}{\mathsf{pr}_L^2} \end{aligned} \tag{49}$$

In equilibrium, $z^{L*} = \mathsf{E}[\delta \mid \mathrm{LB}^*]$, so (49) can be simplified to,

$$Z^l_{z^{L*}} = 1 - \frac{(1-\mu)\mathsf{E}[\delta \mid \mathrm{LB}^*]}{\mathsf{pr}_L^2} \tag{50}$$

I want to show that the term $\frac{(1-\mu)\mathsf{E}[\delta|\mathrm{LB}^*]}{\mathsf{pr}_L^2} < 1$. I do so using the following property of z^{L*}.

$$\begin{aligned} z^{L*} &= \frac{\mu \int_{z^{L*}}^{z^M} \delta f(\delta) d\delta}{\mu \int_{z^{L*}}^{z^M} f(\delta) d\delta + (1-\mu)(z^M - z^{L*})} \\ &= \frac{\int_{z^{L*}}^{z^M} \delta f(\delta) d\delta}{\int_{z^{L*}}^{z^M} f(\delta) d\delta} \cdot \frac{\mu \int_{z^{L*}}^{z^M} f(\delta) d\delta}{\mu \int_{z^{L*}}^{z^M} f(\delta) d\delta + (1-\mu)(z^M - z^{L*})} \\ &> z^M \times \frac{\mu \int_{z^{L*}}^{z^M} f(\delta) d\delta}{\mu \int_{z^{L*}}^{z^M} f(\delta) d\delta + (1-\mu)(z^M - z^{L*})} \end{aligned} \tag{51}$$

Then, by adding z^M to both sides, and rearranging, I have,

$$\begin{aligned} z^M - z^{L*} &< z^M \times \left(1 - \frac{\mu \int_{z^{L*}}^{z^M} f(\delta) d\delta}{\mu \int_{z^{L*}}^{z^M} f(\delta) d\delta + (1-\mu)(z^M - z^{L*})}\right) \\ &= z^M \times \frac{(1-\mu)(z^M - z^{L*})}{\mathsf{pr}_L} \\ &\iff \mathsf{pr}_L > z^M \times (1-\mu) > z^{L*} \times (1-\mu) \end{aligned} \tag{52}$$

where the last line in (52) implies that $\frac{\partial \mathsf{E}[\delta|\mathrm{LB}^*]}{\partial z^{L*}} < 1$ for all z^M, and thus $Z^l_{z^{L*}} > 0$. I have thus shown that Z^l only crosses 0 once on $[0, z^M]$, for any z^M. Thus, there exists a unique

z^{L*} that solves the indifference equation for the marginal limit buy order submitter, for all $z^M \in [0, 1]$.

Step 2: Existence of z^M

With a similar approach, I show that for the function Z^m crosses zero from below.

$$Z^m(0) = 0 - \mathsf{E}[\delta \mid \text{MB}^*] - \rho \mathsf{Pr}(\text{MS}^*) \times (0 - 0) < 0 \tag{53}$$

$$Z^m(1) = 1 - \mathsf{E}[\delta \mid \text{MB}^*] - 0 \times (1 - z^{L*}) > 0 \tag{54}$$

Because Z^m is continuous in z^M, the intermediate value theorem implies the existence of a $z^M \in [0, 1]$ that crosses zero from below for all $z^{L*}(z^M)$. This holds for z^{L*} by the implicit function theorem, as Z^l is continuous for all z^M, and $Z^l_{z^{L*}} \neq 0$. Therefore, an equilibrium exists in threshold strategies with z^M and z^L such that $z^M \geq z^L$. ■

Proof. (Proposition 1).

The expected payoff for liquidity providers sending sell limit orders to the dark market (conditional on execution) satisfies,

$$\pi_{DLS} = (1 - 2\lambda) \times \mathsf{E}[\delta \mid \text{MB}^*] - \mathsf{E}[\delta \mid \text{DB}^*] \tag{55}$$

By Lemma 1, $\mathsf{E}[\delta \mid \text{DB}^*] > 0$ and $\mathsf{E}[\delta \mid \text{MB}^*] > 0$. Then, for all $\lambda \geq 1/2$, $(1 - 2\lambda) \leq 0$, implying that the premium paid to a professional liquidity provider is $(1 - 2\lambda) \times \mathsf{E}[\delta \mid \text{MB}^*] \leq 0$. Therefore, any dark trade earns the liquidity provider a negative payoff. Thus, in equilibrium, no limit orders are sent to the dark market (i.e., $\mathsf{Pr}(\text{DLS}) = \mathsf{Pr}(\text{DLB}^*) = 0$).

■

Proof. (Proposition 2). Consider a market where investors can send an order to the visible market, or a dark market that offers a trade-at rule λ such that,

$$(1 - 2\lambda^*) \times \mathsf{E}[\delta \mid \text{MB}^*] = \mathsf{E}[\delta \mid \text{LB}^*] \tag{56}$$

To show that such a trade-at rule exists, consider the expression,

$$Z^{\lambda} = (1-2\lambda) \times \mathsf{E}[\delta \mid \text{MB}^*] - \mathsf{E}[\delta \mid \text{LB}^*] \tag{57}$$

Now evaluate expression (57) at the endpoints $\lambda = \{0, 1\}$. We then have,

$$\lambda = 0 \iff Z^{\lambda} = \mathsf{E}[\delta \mid \text{MB}^*] - \mathsf{E}[\delta \mid \text{LB}^*] > 0$$
$$\lambda = 1 \iff Z^{\lambda} = -\mathsf{E}[\delta \mid \text{MB}^*] - \mathsf{E}[\delta \mid \text{LB}^*] < 0$$

which implies, by the intermediate value theorem, that there exists a λ^* such that (57) holds for all z^{L*} and z^{M*}. Moreover, $Z^{\lambda}_{\lambda^*} = -2 \times \mathsf{E}[\delta \mid \text{MB}^*] < 0$, λ^* is unique.

Then, given λ^* as in (57), the level of liquidity provision to the dark market equal to $\mathsf{Pr}(\text{DLS}) = \rho\mathsf{Pr}(\text{MS}^*)$ implies that $\pi_L = \pi_D$, for all investors. Thus, no investor deviates from the equilibrium actions determined by the z^M and z^L thresholds above, as investors stay with the incumbent visible market if they are indifferent between the visible market and the dark market (by assumption). Moreover, given z^M and z^L and λ^*, liquidity providers earn zero profits submitting limit orders to the dark market. Thus, they have no incentive to deviate from the liquidity provision strategy $\mathsf{Pr}(\text{DLS}) = \rho\mathsf{Pr}(\text{MS}^*)$. ■

Proof. (Theorem 2).

Similar to the proof of Theorem 1, I proceed by showing the existence and uniqueness of z^{L*}, z^{D*} and $\mathsf{Pr}(\text{DLS}^*)$ for all z^M, and then show that z^{M*} exists, for all $\lambda \in (\lambda^*, 1/2)$. I do so using the functions below, derived from indifference conditions (19)-(22)),

$$Z^m = z^M - \mathsf{E}[\delta \mid \text{MB}^*] - \rho \cdot \mathsf{Pr}(\text{MS}^*) \times \left(z^M - \mathsf{E}[\delta \mid \text{LB}^*]\right) \tag{58}$$
$$Z^{\gamma} = \rho \cdot \mathsf{Pr}(\text{MS}^*) \times \left(z^L - \mathsf{E}[\delta \mid \text{LB}^*]\right) - \mathsf{Pr}(\text{DLS}) \times \left(z^L - (1-2\lambda) \times \mathsf{E}[\delta \mid \text{MB}^*]\right) \tag{59}$$
$$Z^d = z^D - (1-2\lambda) \times \mathsf{E}[\delta \mid \text{MB}^*] \tag{60}$$
$$Z^l = (1-2\lambda) \times \mathsf{E}[\delta \mid \text{MB}^*] - \mathsf{E}[\delta \mid \text{DB}^*] \tag{61}$$

Moreover, the condition $\lambda^* < \lambda < 1/2$ implies that $(1-2\lambda) \times \mathsf{E}[\delta \mid \text{MB}^*] < \mathsf{E}[\delta \mid \text{LB}^*]$. Steps 1-4 show the existence of an equilibrium where $0 < z^D < z^L < z^M < 1$ for $\lambda \in (\lambda^*, 1/2)$. In step 5, I show that an equilibrium does not exist when $\lambda \in (0, \lambda^*)$.

Step 1: Existence and Uniqueness of $z^{D*}(z^M)$

I first show that there exists a unique $z^D \in [0, z^L]$ that solves (60) for all $z^M \in [0, 1]$. Evaluating Z^d at the end points of z^D,

$$z^D = 0 \;\Rightarrow\; Z^d = 0 - (1-2\lambda) \times \mathsf{E}[\delta \mid \mathrm{MB}^*] < 0$$
$$z^D = z^L \;\Rightarrow\; Z^d = z^L - (1-2\lambda) \times \mathsf{E}[\delta \mid \mathrm{MB}^*] > 0$$

where the last inequality holds because in equilibrium, $z^L > \mathsf{E}[\delta_t \mid \mathrm{LB}_t] > (1-2\lambda)\mathsf{E}[\delta \mid \mathrm{MB}^*]$ for all $\lambda \in (\lambda^*, 1/2)$. Thus by the intermediate value theorem, z^{D*} exists. Then, because $Z^d_{z^{D*}} = 1 > 0$, the function Z^d only ever crosses zero from below, and thus it can only do so once, implying that z^{D*} is unique for all z^M.

Step 2: Existence and Uniqueness of $z^{L*}(z^M)$

I now show that there exists a unique $z^L \in [z^{D*}, z^M]$ that solves (61) for all $z^M \in [0, 1]$. Preliminarily, I show two important monotonicity results. Differentiating $\mathsf{E}[\delta \mid \mathrm{DB}^*]$ by z^{L*},

$$\frac{\partial \mathsf{E}[\delta \mid \mathrm{DB}^*]}{\partial z^{L*}} = -\frac{\mu f(z^{L*})z^{L*} - (\mu f(z^{L*}) + (1-\mu)) \times \mathsf{E}[\delta \mid \mathrm{DB}^*]}{\mathsf{Pr}(\mathrm{DB})^2} \tag{62}$$

which is positive for all z^M by the following argument. Rearranging expression (62),

$$\begin{aligned} \mathsf{E}[\delta \mid \mathrm{DB}^*] &< \frac{\mu f(z^{L*})z^{L*}}{\mu f(z^{L*}) + (1-\mu)} \\ \iff \mathsf{E}[\delta \mid \mathrm{DB}^*] &< \lim_{z^D \to z^{L*}} \frac{\mu \int_{z^{D*}}^{z^{L*}} \delta f(\delta) d\delta}{\mu \int_{z^{D*}}^{z^{L*}} f(\delta) d\delta + (1-\mu)(z^{L*} - z^{D*})} \end{aligned} \tag{63}$$

The right-hand side of (63) is the upper limit of $\mathsf{E}[\delta \mid \mathrm{DB}^*]$. For all $f(\delta) \geq 0$, the result is immediate. If $f(\delta) < 0$, then there exists a point at which the density function for the innovation falls below the uniform distribution of the private values. If this point occurs within (z^{D*}, z^{L*}), then it is possible that,

$$\mathsf{E}[\delta \mid \text{DB}^*] > \lim_{z^D \to z^{L*}} \frac{\mu \int_{z^{D*}}^{z^{L*}} \delta f(\delta) d\delta}{\mu \int_{z^{D*}}^{z^{L*}} f(\delta) d\delta + (1-\mu)(z^{L*} - z^{D*})}$$

But this implies that $\mathsf{E}[\delta \mid \text{DB}^*] > \mathsf{E}[\delta \mid \text{LB}^*] > \mathsf{E}[\delta \mid \text{MB}^*]$. In equilibrium, however, it must be the case that $\mathsf{E}[\delta \mid \text{LB}^*] < \mathsf{E}[\delta \mid \text{MB}^*]$, a contradiction. Thus, (63) must hold. A similar argument applies for $\frac{\partial \mathsf{E}[\delta \mid \text{DB}^*]}{\partial z^{D*}} > 0$.

To show that z^{L*} exists, I evaluate Z^l at the end points of z^L,

$$\begin{aligned} z^L = z^{D*} \quad &\Rightarrow \quad Z^l = z^{D*} - \mathsf{E}[\delta \mid z^{D*} \le z \le z^{D*}] \\ &= \quad z^{D*} \times \left(1 - \frac{\mu f(z^{D*})}{\mu f(z^{D*}) + (1-\mu)}\right) > 0 \\ z^L = z^M \quad &\Rightarrow \quad Z^l = (1-2\lambda) \times \mathsf{E}[\delta \mid \text{MB}^*] - \mathsf{E}[\delta \mid z^{D*} \le z \le z^M] \end{aligned} \tag{64}$$

the sign for the expression of $Z^l(z^L = z^M)$ depends on the value of $\lambda^* \in (\lambda^*, 1/2)$. For $\lambda = 1/2$, $Z^l = -\mathsf{E}[\delta \mid \text{DB}^*] < 0$. However, if $\lambda = \lambda^* (= \mathsf{E}[\delta \mid \text{LB}^*])$,

$$Z^l(z^L = z^M) = \mathsf{E}[\delta \mid z \in (z^L = z^M, z^M)] - \mathsf{E}[\delta \mid z \in (z^{D*}, z^M)] > 0 \tag{65}$$

since $(\partial \mathsf{E}[\delta \mid \text{DB}^*]/\partial z^{D*}) > 0$ on $[0, z^L = z^M]$. Then, because $(1-2\lambda) \times \mathsf{E}[\delta \mid \text{MB}^*]$ is decreasing in λ, there must exist a $\bar{\lambda} > \lambda^*$ such that $Z^l(z^L = z^M) < 0$ for all $\lambda \in (\bar{\lambda}, 1/2)$, implying that z^{L*} exists.

Finally, $Z^l_{z^{L*}} = -\frac{\partial \mathsf{E}[\delta \mid \text{DB}^*]}{\partial z^{L*}} < 0$, and thus the function Z^l only ever crosses zero from above, implying that it can only cross zero once. Thus, I have shown that z^{L*} is unique for all z^M, for all $\lambda \in (\bar{\lambda}, 1/2)$.

Step 3: Existence and Uniqueness of $\mathsf{Pr}(\mathbf{DLS}^*)(z^M)$

I now show that there exists a unique $\mathsf{Pr}(\text{DLS}^*) \in [z^{D*}, z^M]$ that solves (59) for all $z^M \in [0, 1]$.

$$\mathsf{Pr}(\text{DLS}^*) = 0 \;\;\Rightarrow\;\; Z^\gamma = \rho\mathsf{Pr}(\text{MS}^*) \cdot \left(z^{L*} - \mathsf{E}[\delta \mid \text{LB}^*]\right) > 0$$

$$\mathsf{Pr}(\text{DLS}^*) = \mathsf{Pr}(\text{MS}^*) \;\;\Rightarrow\;\; Z^\gamma = \rho\mathsf{Pr}(\text{MS}^*) \cdot ((1-2\lambda)\mathsf{E}[\delta \mid \text{MB}^*] - \mathsf{E}[\delta \mid \text{LB}^*]) < 0$$

where the last inequality holds by the fact that in equilibrium, $\mathsf{E}[\delta_t \mid \text{LB}_t] > (1-2\lambda)\mathsf{E}[\delta_t \mid \text{MB}_t]$ for all $\lambda \in (\bar{\lambda}, 1/2)$. Thus, $\mathsf{Pr}(\text{DLS}^*)$ exists. Then, because,

$$Z^\gamma_{\mathsf{Pr}(\text{DLS}^*)} = -\left(z^{L*} - (1-2\lambda) \times \mathsf{E}[\delta \mid \text{MB}^*]\right) < 0$$

the function Z^γ only ever crosses zero from below. Since it can only do so once, we have that $\mathsf{Pr}(\text{DLS}^*)$ is unique for all z^M.

Step 4: Existence of z^M

Lastly, I show that there exists a $z^M \in [0,1]$ that solves expression (58).

$$z^M = 0 \;\;\Rightarrow\;\; Z^m = -\mathsf{E}[\delta \mid \text{MB}^*] - \rho \cdot \mathsf{Pr}(\text{MS}^*) \times 0 < 0$$

$$z^M = 1 \;\;\Rightarrow\;\; Z^m = 1 - \mathsf{E}[\delta \mid \text{MB}^*] - \rho \cdot 0 \times (1 - \mathsf{E}[\delta \mid \text{LB}^*]) > 0$$

The last inequality arises from the fact that at $z^M = 1$, the measure of market order submitters is zero, and thus, $\mathsf{Pr}(\text{MS}^*) = 0$. Hence, there exists a $z^M \in [0,1]$ that solves expression (58) given z^{L*}, z^{D*} and $\mathsf{Pr}(\text{DLS}^*)$, for all $\lambda \in (\bar{\lambda}, 1/2)$.

Step 5: No Equilibrium when $\lambda \in (0, \lambda^*)$

If $\lambda \in (0, \lambda^*)$, then $(1-2\lambda)\mathsf{E}[\delta \mid \text{MB}^*] > \mathsf{E}[\delta \mid \text{LB}^*]$. If $\rho\mathsf{Pr}(\text{MS}^*) > \mathsf{Pr}(\text{DLS}^*)$, submitting a limit order strictly dominates sending an order to the dark, as a limit order yields both lower execution risk and lower transaction costs. Hence trading with the dark market is never an equilibrium strategy for any investor, which violates the supposition that $0 < z^D < z^L$.

If $\rho\mathsf{Pr}(\text{MS}^*) < \mathsf{Pr}(\text{DLS}^*)$, then Lemma (3) implies that $z^D > z^L$ must hold. Hence, for all λ such that $(0, \lambda^*)$, there can be no equilibrium where $0 < z^D < z^L < z^M < 1$. ■

Proof. (Theorem 3).

Similar to the proof of Theorem 2, I proceed by showing the existence and uniqueness of z^{L*}, z^{D*} and $\mathsf{Pr}(\text{DLS}^*)$ for all z^M, and then show that z^{M*} exists for all $\lambda \in (0, \lambda^*)$. I do so using the functions below, derived from indifference conditions (23)-(26),

$$\begin{aligned}
Z^m &= z^M - \mathsf{E}[\delta \mid \text{MB}^*] - \mathsf{Pr}(\text{DLS}) \times \left(z^M - (1-2\lambda) \times \mathsf{E}[\delta \mid \text{MB}^*]\right) && (66)\\
Z^\gamma &= \mathsf{Pr}(\text{DLS}) \times \left(z^D - (1-2\lambda) \times \mathsf{E}[\delta \mid \text{MB}^*]\right) \\
&\quad -\rho \cdot \mathsf{Pr}(\text{MS}^*) \times \left(z^D - \mathsf{E}[\delta \mid \text{LB}^*]\right) && (67)\\
Z^l &= z^L - \mathsf{E}[\delta \mid \text{LB}^*] && (68)\\
Z^d &= (1-2\lambda) \times \mathsf{E}[\delta \mid \text{MB}^*] - \mathsf{E}[\delta \mid \text{DB}^*] && (69)
\end{aligned}$$

Further, the condition $0 < \lambda < \lambda^*$ implies that $(1-2\lambda) \times \mathsf{E}[\delta \mid \text{MB}^*] > \mathsf{E}[\delta \mid \text{LB}^*]$. In step 5, I show that an equilibrium does not exist when $\lambda \in (\lambda^*, 1/2)$.

Step 1: Existence and Uniqueness of $z^{L*}(z^M)$

I now show that there exists a unique $z^L \in [0, z^D]$ that solves (69) for all $z^M \in [0, 1]$.

$$\begin{aligned}
z^L = 0 &\;\Rightarrow\; Z^l = 0 - \mathsf{E}[\delta \mid \text{LB}^*] < 0\\
z^L = z^D &\;\Rightarrow\; Z^l = z^D - z^D \times \frac{\mu f(z^D)}{\mu f(z^D) + (1-\mu)} > z^D \times \frac{(1-\mu)}{\mu f(z^D) + (1-\mu)} > 0
\end{aligned}$$

Thus, z^{L*} exists. Then, from step 1 of the proof of Theorem 1,

$$Z^l_{z^{L*}} = 1 - \frac{\partial \mathsf{E}[\delta \mid \text{LB}^*]}{\partial z^{L*}} > 0$$

we have that the function Z^l only ever crosses zero from below, and thus it can only do so once, implying that z^{L*} is unique for all z^D.

Step 2: Existence and Uniqueness of $z^{D*}(z^M)$

I first show that there exists a unique $z^D \in [z^{L*}, z^M]$ that solves (68) for all $z^M \in [0, 1]$. Evaluating Z^d at the end points of z^D,

$$z^D = z^{L*} \;\Rightarrow\; Z^d = (1-2\lambda) \times \mathsf{E}[\delta \mid \text{MB}^*] - \mathsf{E}[\delta \mid z^{L*} \le z \le z^M] \ge 0 \tag{70}$$
$$z^D = z^M \;\Rightarrow\; Z^d = (1-2\lambda) \times \mathsf{E}[\delta \mid \text{MB}^*] - \mathsf{E}[\delta \mid z^M \le z \le z^M] \le 0$$

where the inequalities hold simultaneously for some $\lambda \in (0, \lambda^*)$. To see this, note that $Z^d = \mathsf{E}[\delta \mid \text{MB}^*] - \mathsf{E}[\delta \mid \text{DB}] > 0$ for $\lambda = 0$, and that at $\lambda = \lambda^*$,

$$(1-2\lambda) \times \mathsf{E}[\delta \mid \text{MB}^*] = \mathsf{E}[\delta \mid \text{LB}^*] < \mathsf{E}[\delta \mid \text{DB}^*]$$

Then, because

$$\mathsf{E}[\delta \mid z^M \le z \le z^M] > \mathsf{E}[\delta \mid z^{L*} \le z \le z^M]$$

there must exist a (subset of) $\lambda \in (0, \lambda^*)$ such that both inequalities in (70) are strict. Thus, z^{D*} exists. Then, differentiating Z^d by z^{D*},

$$Z^d_{z^{D*}} \;=\; -\frac{\partial \mathsf{E}[\delta \mid \text{DB}^*]}{\partial z^{D*}} = -\frac{(\mu f(z^{D*}) + (1-\mu)) \times \mathsf{E}[\delta \mid \text{DB}^*] - \mu f(z^{D*}) z^{D*}}{\mathsf{Pr}(\text{DB})^2} \tag{71}$$

which is negative for all z^M by an argument similar to step 2 of the proof of Theorem 2. Rearranging expression (71), $Z^d_{z^{D*}}$ is negative if the following holds:

$$\mathsf{E}[\delta \mid \text{DB}^*] \;>\; \frac{\mu f(z^{D*}) z^{D*}}{\mu f(z^{D*}) + (1-\mu)}$$
$$\iff \mathsf{E}[\delta \mid \text{DB}^*] \;>\; \lim_{z^M \to z^{D*}} \frac{\mu \int_{z^{D*}}^{z^M} \delta f(\delta) d\delta}{\mu \int_{z^{D*}}^{z^M} f(\delta) d\delta + (1-\mu)(z^M - z^{D*})} \tag{72}$$

The right-hand side of (72) is the lower limit of $\mathsf{E}[\delta \mid \text{DB}^*]$. For all $f'(\delta) \ge 0$, the result is immediate. If $f'(\delta) < 0$, then there exists a point at which the density function for the innovation falls below the uniform distribution of the private values. If this point occurs within (z^{D*}, z^M), then it is possible that,

$$\mathsf{E}[\delta \mid \text{LB}^*] < \mathsf{E}[\delta \mid \text{DB}^*] < \lim_{z^M \to z^{D*}} \frac{\mu \int_{z^{D*}}^{z^M} \delta f(\delta) d\delta}{\mu \int_{z^{D*}}^{z^M} f(\delta) d\delta + (1-\mu)(z^M - z^{D*})}$$

But this implies that $\mathsf{E}[\delta \mid \text{DB}^*] > \mathsf{E}[\delta \mid \text{MB}^*]$. In equilibrium, however, it must be the case that $\mathsf{E}[\delta \mid \text{DB}^*] < \mathsf{E}[\delta \mid \text{MB}^*]$, a contradiction.

Thus, (72) must hold, implying that $Z^d < 0$ for all z^M, Hence, the function Z^d only ever crosses zero from above, and thus it can only do so once, implying that z^{D*} is unique for all z^M, and all $\lambda \in (0, \lambda^*)$ such that (70) is satisfied.

Step 3: Existence and Uniqueness of $\mathsf{Pr}(\mathbf{DLS}^*)(z^M)$

I now show that there exists a unique $\mathsf{Pr}(\text{DLS}^*) \in [\mathsf{Pr}(\text{MS}^*), 1]$ that solves (66) for all $z^M \in [0, 1]$. At $\mathsf{Pr}(\text{DLS}^*) = \rho\mathsf{Pr}(\text{MS}^*)$,

$$Z^\gamma = \rho\mathsf{Pr}(\text{MS}^*) \times (\mathsf{E}[\delta \mid \text{LB}^*] - (1 - 2\lambda)\mathsf{E}[\delta \mid \text{MB}^*]) < 0$$

Now, let $\mathsf{Pr}(\text{DLS}^*) = 1$. In equilibrium, because $(1 - 2\lambda) \times \mathsf{E}[\delta \mid \text{MB}^*] < \mathsf{E}[\delta \mid \text{MB}^*]$, all investors prefer dark market orders to visible market orders. Thus, $\mathsf{Pr}(\text{MS}^*) = 0$. Then, we have that:

$$\begin{aligned} Z^\gamma &= (1 - \rho\mathsf{Pr}(\text{MS}^*)) \cdot z^{D*} - (1 - 2\lambda) \times \mathsf{E}[\delta \mid \text{MB}^*] + \rho\mathsf{Pr}(\text{MS}^*) \cdot \mathsf{E}[\delta \mid \text{LB}^*] \\ &> z^D - (1 - 2\lambda) \times \mathsf{E}[\delta \mid \text{DB}^*] > 0 \end{aligned}$$

Thus by the intermediate value theorem, $\mathsf{Pr}(\text{DLS}^*)$ exists. Then, differentiating Z^γ by $\mathsf{Pr}(\text{DLS}^*)$, we have,

$$Z^\gamma_{\mathsf{Pr}(\text{DLS}^*)} = \left(z^{D*} - (1 - 2\lambda) \times \mathsf{E}[\delta \mid \text{MB}^*]\right) > 0$$

which implies that the function Z^γ only ever crosses zero from below, and hence can only cross once). Thus, $\mathsf{Pr}(\text{DLS}^*)$ is unique for all z^M, and $\lambda \in (0, \lambda^*)$ such that (70) is satisfied.

Step 4: Existence of z^M

Lastly, I show that there exists a $z^M \in [0, 1]$ that solves (66).

$$\begin{aligned} z^M = 0 \;&\Rightarrow\; Z^m = -\mathsf{E}[\delta \mid \text{MB}^*] \times (1 - \mathsf{Pr}(\text{DLS}^*) \cdot (1 - 2\lambda)) < 0 \\ z^M = 1 \;&\Rightarrow\; Z^m = 1 - \mathsf{Pr}(\text{DLS}^*) - \mathsf{E}[\delta \mid \text{MB}^*] \times (1 - \mathsf{Pr}(\text{DLS}^*) \cdot (1 - 2\lambda)) > 0 \end{aligned}$$

The last inequality arises from the fact that at $z^M = 1$, the measure of market order submitters is zero, and thus, $\mathsf{Pr}(\text{MS}^*) = 0$. $\mathsf{Pr}(\text{DLS}^*) = 0$ by condition (67) evaluated at $z^M = 1$. Hence, z^M exists for all $\lambda \in (0, \lambda^*)$ such that (70) is satisfied.

Step 5: No Equilibrium when $\lambda \in (\lambda^*, 1/2)$

If $\lambda \in (\lambda^*, 1/2)$, then $\mathsf{E}[\delta \mid \text{LB}^*] > (1 - 2\lambda)\mathsf{E}[\delta \mid \text{MB}^*]$. If $\rho\mathsf{Pr}(\text{MS}^*) < \mathsf{Pr}(\text{DLS}^*)$, sending an order to the dark market strictly dominates limit orders, as an order in the dark market yields both lower execution risk and lower transaction costs. Hence, limit orders are not used in equilibrium, which violates the supposition that $z^L < z^D$ (i.e., there is some investor that uses limit orders).

For the second case, Lemma (3) implies that if $\rho\mathsf{Pr}(\text{MS}^*) > \mathsf{Pr}(\text{DLS}^*)$, then $z^L > z^D$ must hold. Hence, for all λ such that $\mathsf{E}[\delta_t \mid \text{LB}_t] \geq (1 - 2\lambda)\mathsf{E}[\delta \mid \text{MB}^*]$ holds, there can be no equilibrium where $0 < z^L < z^D < z^M < 1$. ■

A.3 Out-of-Equilibrium Limit Orders and Beliefs

The equilibrium concept I employ in this paper perfect Bayesian equilibrium. On-the-equilibrium-path, investors submit limit orders with competitive limit prices. However, I require an appropriate set of out-of-equilibrium beliefs to ensure that competitive limit prices strategically dominate any off-equilibrium-path deviations in the limit price. Intuitively, any limit order that is posted at a price worse than the competitive equilibrium price is strategically dominated by the competitive price, as professional liquidity providers react to the non-competitive order by undercutting it. For non-competitive limit orders that undercut the competitive price (i.e., a price inside the competitive spread), however, it is not immediate that the competitive price strategically dominates this set of prices.

Perfect Bayesian equilibrium prescribes that investors and professional liquidity providers update their beliefs by Bayes rule, whenever possible, but it does not place any restrictions on the beliefs of market participants when they encounter an out-of-equilibrium action.

To support competitive prices in equilibrium, I assume (similar to Brolley and Malinova (2014)) that if a limit buy order is posted at a price different to the competitive equilibrium bid price bid^*_{t+1}, then market participants hold the following beliefs regarding this investor's knowledge of the period t innovation δ_t.

If a limit buy order is posted at a price $\widehat{\mathsf{bid}} < \mathsf{bid}^*_{t+1}$, then market participants assume that this investor followed the equilibrium threshold strategy, but "made a mistake" when pricing his orders. A professional liquidity provider then updates his expectation about δ_t to the equilibrium value and posts a buy limit order at bid^*_{t+1}. The original investor's limit order then executes with zero probability.

If a limit buy order is posted at a price $\widehat{\mathsf{bid}} > \mathsf{bid}^*_{t+1}$, then market participants believe that this order stems from an investor with a sufficiently high valuation (e.g., $z_t = 1$) and update their expectations about δ_t to $\mathsf{E}[\delta_t \mid \widehat{\mathsf{bid}}]$ accordingly.The new posterior expectation of V_t equals to $p_{t-1} + \mathsf{E}[\delta_t \mid \widehat{\mathsf{bid}}]$. A professional liquidity provider is then willing to post a bid price $\mathsf{bid}^{**}_{t+1} \leq p_{t-1} + \mathsf{E}[\delta_t \mid \widehat{\mathsf{bid}}] + \mathsf{E}[\delta_{t+1} \mid \mathrm{MS}_{t+1}]$. With the out-of-the-equilibrium belief of $\delta_t = 1$ and with the bid-ask spread< 1, a limit order with the new price bid^{**}_{t+1} outbids any limit buy order that yields investors positive expected profits.

The beliefs upon an out-of-equilibrium sell order are symmetric. The above out-of-equilibrium beliefs ensure that no investor deviates from his equilibrium strategy. I emphasize that these beliefs and actions do *not* materialize in equilibrium. Instead, they can be loosely thought of as a "threat" to ensure that investors do not deviate from their prescribed equilibrium strategies.

Figure 2: Entry and Order Submission Timeline

This figure illustrates the timing of events upon the arrival of an investor at an arbitrary period, t, until their departure from the market. Value y_t is the private valuation of the period t investor and δ_t is the innovation to the security's fundamental value in period t.

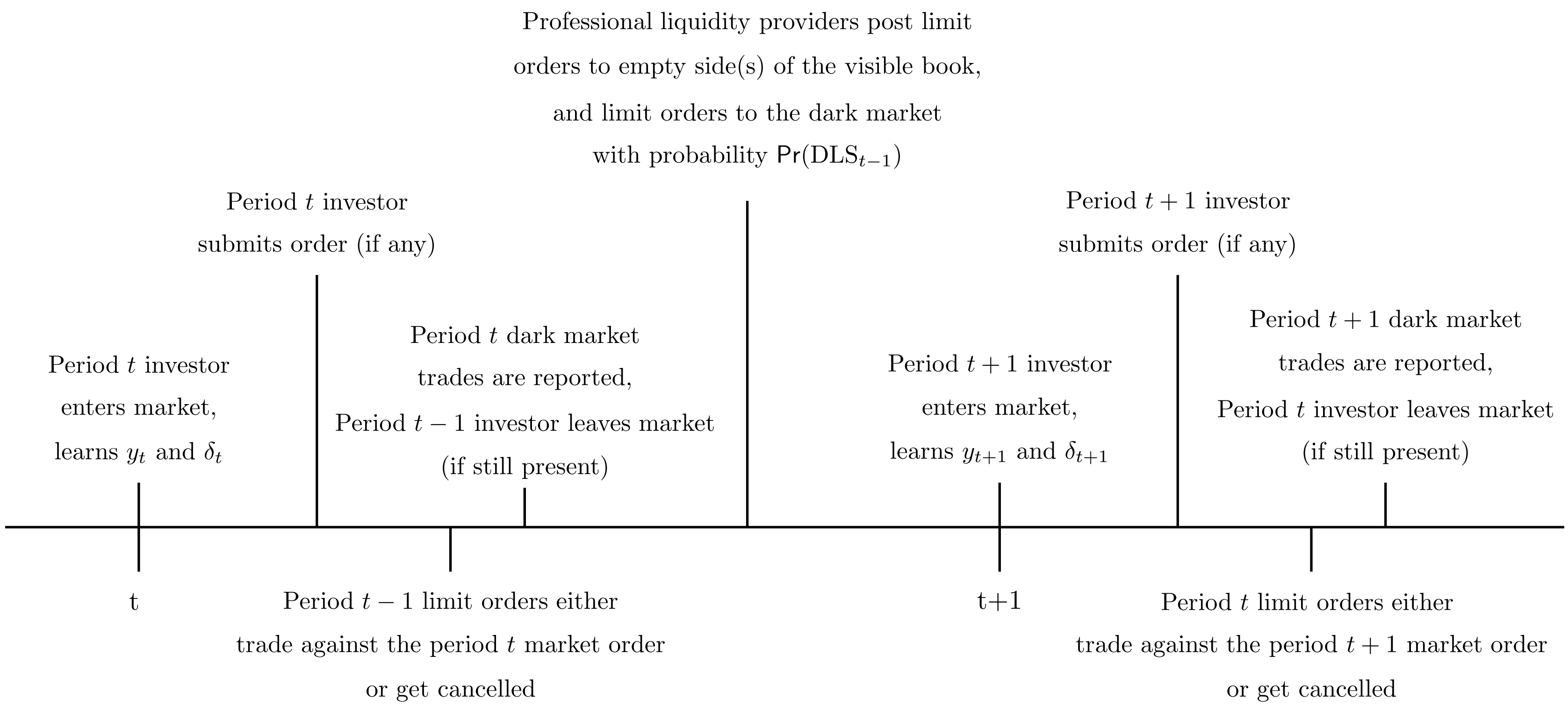

Figure 3: Equilibrium Thresholds

The panels below depict equilibrium valuation thresholds z^M, z^L and z^D as a function of the trade-at rule ($\lambda < \lambda^*$ on the left, $\lambda > \lambda^*$ on the right). A vertical dashed line marks λ^*; the horizontal dashed line indicates the visible market only benchmark value. Parameter $\mu = 0.5$. Results for other values of μ are qualitatively similar.

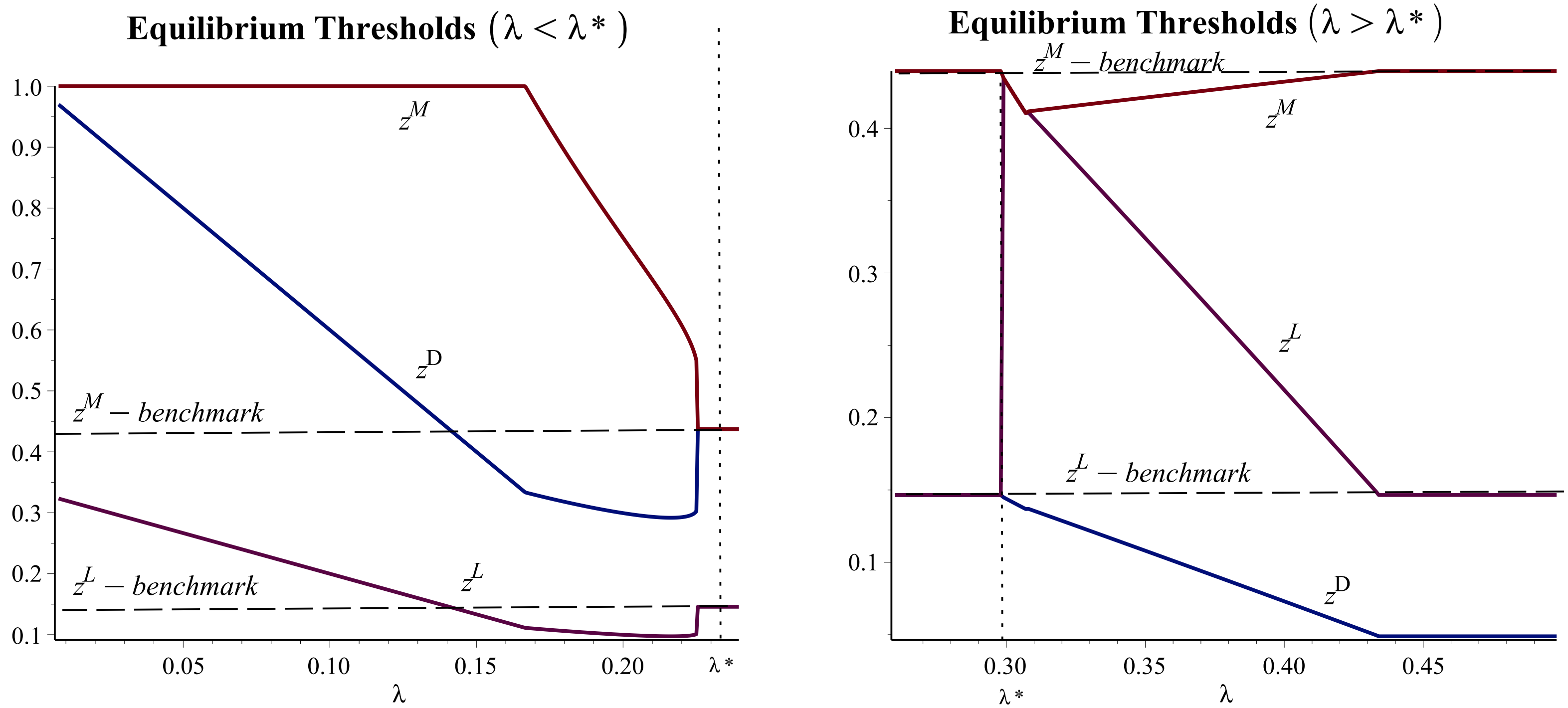

Figure 4: Volume, Market Participation

The panels below depict volume (visible market and total), and market participation as a function of the trade-at rule ($\lambda < \lambda^*$ on the left, $\lambda > \lambda^*$ on the right). A vertical dashed line marks λ^*; the horizontal dashed line indicates the visible market only benchmark value. Parameter $\mu = 0.5$. Results for other values of μ are qualitatively similar.

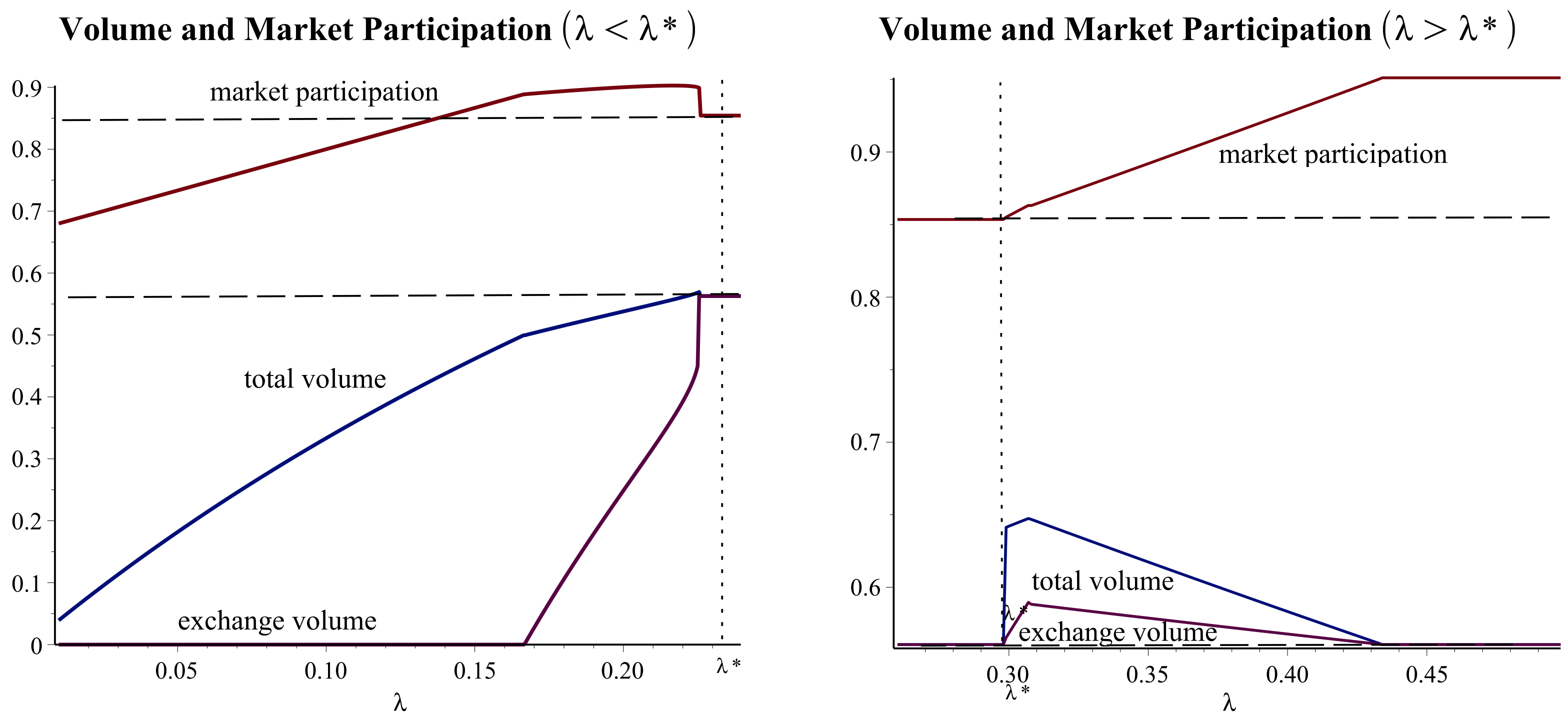

Figure 5: Quoted Half-Spread

The panels below depict the quoted half-spread (also price impact) as a function of the trade-at rule ($\lambda < \lambda^*$ on the left, $\lambda > \lambda^*$ on the right). A vertical dashed line marks λ^*; the horizontal dashed line indicates the visible market only benchmark value. Parameter $\mu = 0.5$. Results for other values of μ are qualitatively similar.

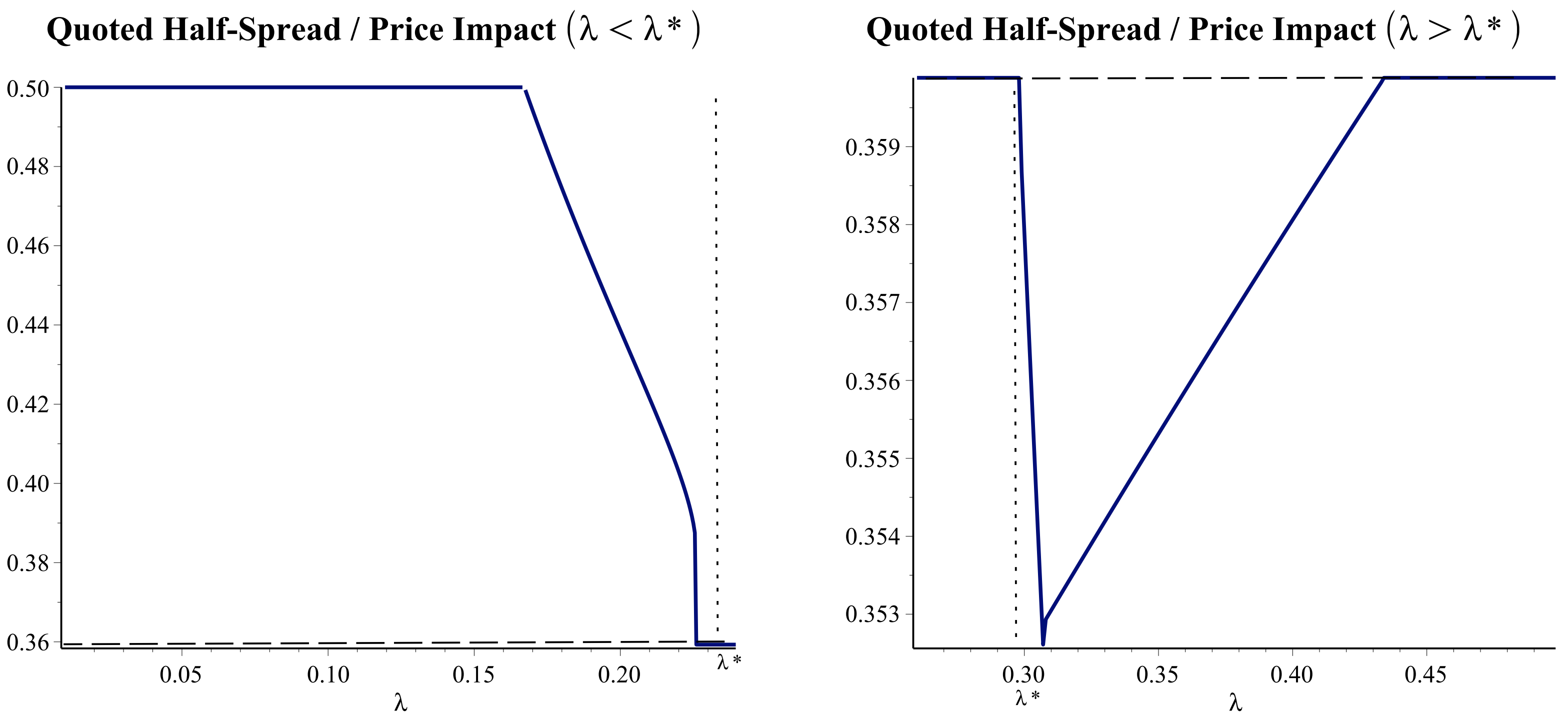

Figure 6: Informational Efficiency

The panels below depict informational efficiency as a function of the trade-at rule ($\lambda < \lambda^*$ on the left, $\lambda > \lambda^*$ on the right). Higher values than the benchmark are less efficient. A vertical dashed line marks λ^*; the horizontal dashed line indicates the visible market only benchmark value. Parameter $\mu = 0.5$. Results for other values of μ are qualitatively similar.

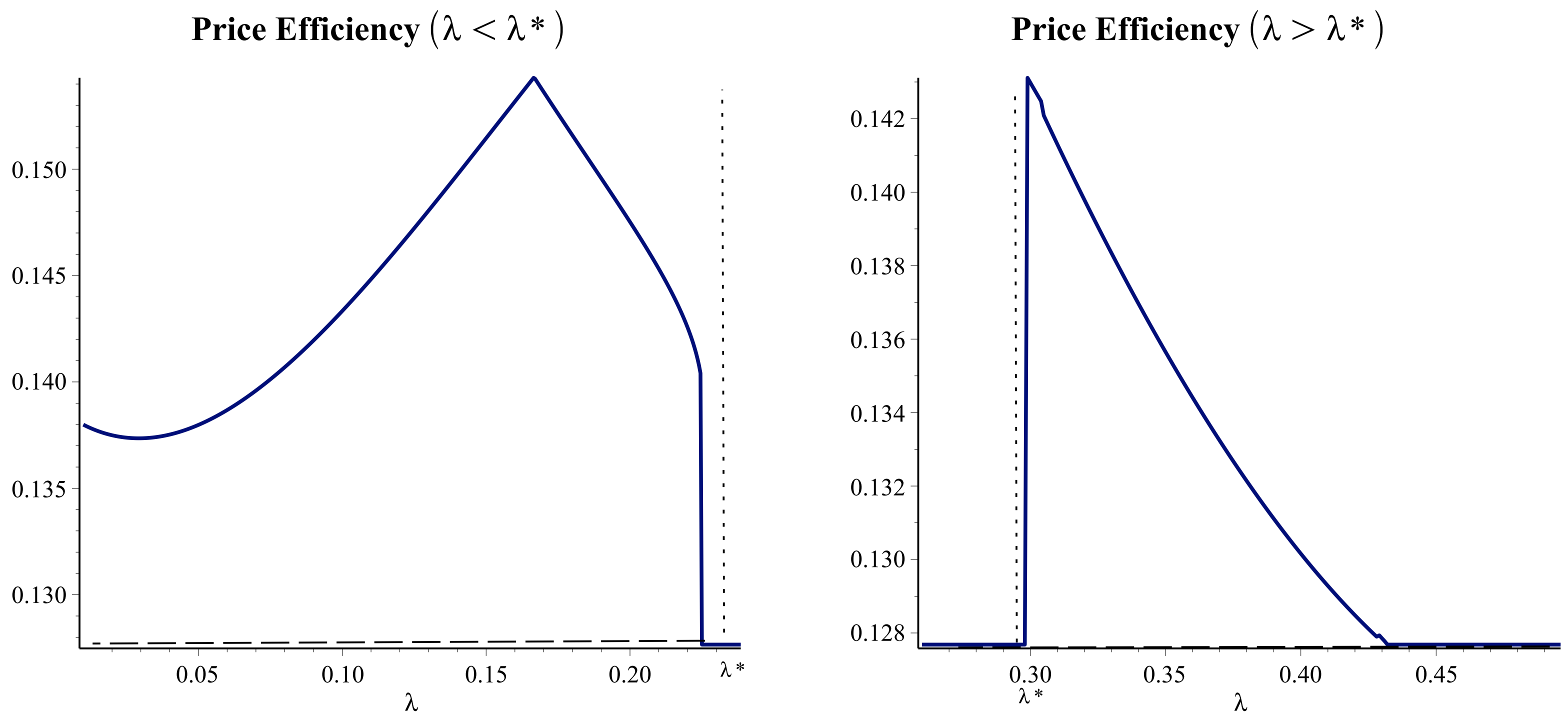

Figure 7: Total Expected Welfare

The panels below depict total expected welfare as a function of the trade-at rule ($\lambda < \lambda^*$ on the left, $\lambda > \lambda^*$ on the right). A vertical dashed line marks λ^*; the horizontal dashed line indicates the visible market only benchmark value. Parameter $\mu = 0.5$. Results for other values of μ are qualitatively similar.