Online Appendix to "Robustly Optimal Auctions with Unknown Resale Opportunities"*

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Abstract

This supplementary file provides proofs for some auxiliary results in Carroll and Segal (2018), as well as an additional example.

This document supplements the paper of Carroll and Segal (2018) on robustly optimal auctions with unknown resale opportunities. It provides proofs for Propositions 1, 2, and 3 stated in Section 5 of that paper. It also includes a counterexample to show that without any assumptions on the bidders' value distributions, the optimal auction may not be of the ACV form.

To avoid excessive repetition, we assume familiarity with the model from the main paper (stated in Section 2), and all terminology and notation from that paper. Only the propositions to be proved will be restated here. Also, we will frequently need to make reference to equations and results from the main paper. To avoid ambiguity, all sections, equations, results, and figures introduced in this document will be referenced with the prefix "OA"; references without this prefix refer to the main paper.

*See main paper for acknowledgments.

OA-A Proof of Proposition 1

Proposition 1. *Supppose that for each bidder i, there is a family of possible distributions $\mathcal{F}_i \subseteq \Delta([0,1])$, and the designer's robust revenue maximization problem is*

$$\max_{\Gamma}\left(\inf_{v, F_1 \in \mathcal{F}_1, \ldots, F_n \in \mathcal{F}_n} \operatorname{Rev}(\Gamma, v, F_1, \ldots, F_n)\right).$$

Assume that for each i, there is a "lowest" distribution $F_i \in \mathcal{F}_i$ that is first-order stochastically dominated by every other $F_i' \in \mathcal{F}_i$. Assume that $F_1, \ldots, F_n$ have continuous densities and satisfy assumptions (A1)–(A3). Then the robust revenue maximization problem is solved by the ACV auction with reserves $p_1, \ldots, p_n$ constructed in (13)–(14).

To prove this, we will first show the following:

Lemma OA-1. *Let $\bar{\Gamma} = \left\langle \bar{\chi}, \bar{\psi} \right\rangle$ be the optimal ACV auction for distributions $F_1, \ldots, F_n$ as characterized in (13)–(14). If, for each i, distribution F_i' first-order stochastically dominates distribution F_i, then $R_n' \equiv \mathbb{E}_{\tilde{\theta}}^{F_1', \ldots, F_n'}\left[\sum_i \bar{\psi}_i(\tilde{\theta})\right] \geq \mathbb{E}_{\tilde{\theta}}^{F_1, \ldots, F_n}\left[\sum_i \bar{\psi}_i(\tilde{\theta})\right] = \bar{R}_n$.*

Together with Theorem 2, the lemma implies that $\operatorname{Rev}(\bar{\Gamma}, v, F_1', \ldots, F_n') \geq \bar{R}_n$ for any resale procedure v. Since Theorem 3 establishes that any mechanism Γ satisfies $\operatorname{Rev}(\Gamma, \underline{v}, F_1, \ldots, F_n) \leq \bar{R}_n$ under the Bertrand resale procedure $\underline{v}$, the argument of Lemma 1 establishes the proposition.

We prove the lemma by induction on the number n of bidders. For $n = 0$, $R_0' = \bar{R}_0 = 0$. Now, suppose the inductive hypothesis holds for $n-1$ bidders. Let $z(\theta_n) = \mathbb{E}_{\tilde{\theta}_{-n}}^{F_1, \ldots, F_{n-1}}\left[\sum_i \bar{\psi}_i(\theta_n, \tilde{\theta}_{-n})\right]$ and $z'(\theta_n) = \mathbb{E}_{\tilde{\theta}_{-n}}^{F_1', \ldots, F_{n-1}'}\left[\sum_i \bar{\psi}_i(\theta_n, \tilde{\theta}_{-n})\right]$. Observe that:

- $z(\theta_n) \leq z'(\theta_n)$ for each θ_n. Indeed, for $\theta_n < p_n$, $z(\theta_n) = \bar{R}_{n-1} \leq R_{n-1}' = z'(\theta_n)$ by the inductive hypothesis, and otherwise

$$\begin{aligned} z(\theta_n) &= \mathbb{E}_{\tilde{\theta}_{-n}}^{F_1, \ldots, F_{n-1}}\left[\max\left\{p_n, (\theta_n, \tilde{\theta}_{-n})_{(2)}\right\}\right] \\ &\leq \mathbb{E}_{\tilde{\theta}_{-n}}^{F_1', \ldots, F_{n-1}'}\left[\max\left\{p_n, (\theta_n, \tilde{\theta}_{-n})_{(2)}\right\}\right] \\ &= z'(\theta_n). \end{aligned}$$

- $z(\theta_n)$ is a nondecreasing function. For this, note that $z(\theta_n)$ is independent of θ_n when $\theta_n < p_n$ and nondecreasing in θ_n when $\theta_n > p_n$. But as θ_n moves across p_n, $z(\theta_n)$ jumps from $\bar{R}_{n-1}$ to $R_{n-1}(p_n) + p_n \prod_{j<n} F_j(p_n)$. (The argument is the same as the argument for (18), the first-order condition for the optimal reserve p_k.) Using (18) (which was shown to hold in Lemma 2), we get

$$R_{n-1}(p_n) + p_n \prod_{j<n} F_j(p_n) - \bar{R}_{n-1} = (p_n - \nu_n(p_n)) \prod_{j<n} F_j(p_n) \geq 0$$

  which means that the jump in z is upward, as needed.

Using the two observations, we see that

$$R_n' = \mathbb{E}_{\tilde{\theta}_n}^{F_n'}\left[z'(\tilde{\theta}_n)\right] \geq \mathbb{E}_{\tilde{\theta}_n}^{F_n'}\left[z(\tilde{\theta}_n)\right] \geq \mathbb{E}_{\tilde{\theta}_n}^{F_n}\left[z(\tilde{\theta}_n)\right] = \bar{R}_n,$$

establishing the inductive hypothesis for n.

OA-B Proof of Proposition 2

Proposition 2. *Suppose the value distributions of the n bidders are regular. Under our worst-case resale procedure, any auction has a lower expected revenue than the Vickrey auction with one added bidder $n+1$, provided that the added bidder's value distribution first-order stochastically dominates the distributions of all other bidders.*

For convenience, let $\tilde{\theta}_1', \ldots, \tilde{\theta}_n'$ be n random variables, with distributions $F_1, \ldots, F_n$, independent of $\tilde{\theta}_1, \ldots, \tilde{\theta}_{n+1}$. We continue to write θ for the profile $(\theta_1, \ldots, \theta_n)$ (*excluding* θ_{n+1}). Also, for brevity we let $i^* = i^*(\theta)$, and $\tilde{i}^*$ for the corresponding random variable.

The expected revenue from the optimal auction on the original n bidders is at most the revenue from the relaxed problem, by maximizing (12), which is simply

$$\mathbb{E}_{\tilde{\theta}}[\max\{\nu_{\tilde{i}^*}(\tilde{\theta}_{\tilde{i}^*}), 0\}]$$

So it suffices to show this latter is bounded above by the revenue from Vickrey with bidders $1, \ldots, n+1$.

We have

$$
\begin{aligned}
\mathbb{E}\left[\max\{\nu_{\tilde{i}^*}(\tilde{\theta}_{\tilde{i}^*}), 0\}\right] &\leq \mathbb{E}\left[\max\{\nu_{\tilde{i}^*}(\tilde{\theta}_{\tilde{i}^*}), \nu_{\tilde{i}^*}(\tilde{\theta}'_{\tilde{i}^*})\}\right] \\
&= \mathbb{E}\left[\nu_{\tilde{i}^*}(\tilde{\theta}_{\tilde{i}^*}) + \mathbf{1}_{\nu_{\tilde{i}^*}(\tilde{\theta}'_{\tilde{i}^*}) \geq \nu_{\tilde{i}^*}(\tilde{\theta}_{\tilde{i}^*})} \left(\nu_{\tilde{i}^*}(\tilde{\theta}'_{\tilde{i}^*}) - \nu_{\tilde{i}^*}(\tilde{\theta}_{\tilde{i}^*})\right)\right] \\
&= \mathbb{E}\left[\nu_{\tilde{i}^*}(\tilde{\theta}_{\tilde{i}^*}) + \mathbf{1}_{\tilde{\theta}'_{\tilde{i}^*} \geq \tilde{\theta}_{\tilde{i}^*}} \left(\nu_{\tilde{i}^*}(\tilde{\theta}'_{\tilde{i}^*}) - \nu_{\tilde{i}^*}(\tilde{\theta}_{\tilde{i}^*})\right)\right] \\
&= \mathbb{E}\left[\nu_{\tilde{i}^*}(\tilde{\theta}_{\tilde{i}^*}) + \mathbf{1}_{\tilde{\theta}'_{\tilde{i}^*} \geq \tilde{\theta}_{\tilde{i}^*}} \left(\tilde{\theta}_{\tilde{i}^*} - \nu_{\tilde{i}^*}(\tilde{\theta}_{\tilde{i}^*})\right)\right] \\
&\leq \mathbb{E}\left[\nu_{\tilde{i}^*}(\tilde{\theta}_{\tilde{i}^*}) + \mathbf{1}_{\tilde{\theta}_{n+1} \geq \tilde{\theta}_{\tilde{i}^*}} \left(\tilde{\theta}_{\tilde{i}^*} - \nu_{\tilde{i}^*}(\tilde{\theta}_{\tilde{i}^*})\right)\right] \\
&= \mathbb{E}\left[\mathbf{1}_{\tilde{\theta}_{n+1} < \tilde{\theta}_{\tilde{i}^*}} \nu_{\tilde{i}^*}(\tilde{\theta}_{\tilde{i}^*}) + \mathbf{1}_{\tilde{\theta}_{n+1} \geq \tilde{\theta}_{\tilde{i}^*}} \tilde{\theta}_{\tilde{i}^*}\right].
\end{aligned}
$$

We justify these calculations as follows:

- The first line holds because, conditional on θ, the random variable $\nu_{i^*}(\tilde{\theta}'_{i^*})$ has mean zero, and the function $\max\{\nu_{i^*}(\theta_{i^*}), \cdot\}$ is convex.
- The third line holds because each ν_i is an increasing function.
- The fourth line holds by conditioning on θ, and observing that $\int_z^1 \nu_i(\theta'_i) f_i(\theta'_i)\, d\theta'_i = \int_z^1 z\, f_i(\theta'_i)\, d\theta'_i$, and taking $i = i^*$, $z = \theta_{i^*}$.
- The fifth line holds because, conditional on θ, the event $\tilde{\theta}_{n+1} \geq \theta_{i^*}$ is at least as likely as $\tilde{\theta}'_{i^*} \geq \theta_{i^*}$ (by first-order stochastic dominance).

Finally, the last line is simply the expected revenue from the Vickrey auction on bidders $1, \ldots, n+1$: the first term inside the expectation reflects the revenue from bidders $1, \ldots, n$ (expressed in terms of virtual values as in (12)), and the last term is the revenue from bidder $n+1$.

OA-C Proof of Proposition 3

Proposition 3.

(a) *If all n bidders' distributions are regular and $F_i(r_j) \leq c < 1$ for all i, j, then the difference between the expected revenue from the optimal auction under worst-case resale and the expected revenue from the Vickrey auction is at most $n \cdot c^{n-1}$.*

(b) *Let F, $\hat{F}$ be regular distributions with respective twice-differentiable densities f, $\hat{f}$ such that $f(1), \hat{f}(1) > 0$ and $f'(1)/f(1) < \hat{f}'(1)/\hat{f}(1)$. There is a constant $c' > 0$ with the following property: if there are $n/2$ bidders drawn from distribution F and $n/2$ from distribution $\hat{F}$, the expected revenue from the optimal Myerson auction (without resale) exceeds that of the Vickrey auction by at least c'/n^3.*

Part (a): The revenue from the optimal auction under worst-case resale is bounded above by the relaxed solution from Subsection 4.2. So it suffices to show that the revenue from this relaxed solution exceeds the Vickrey revenue by at most $n \cdot c^{n-1}$. The former can be calculated by assuming the relaxed solution is implemented using threshold payments. Since each auction always yields revenue at most 1, the difference between the two auctions' expected revenue is at most the probability of realizing a type profile θ for which the relaxed solution yields higher revenue ex-post than Vickrey. This can happen only when there is at most one bidder i with $\theta_i > r_i$; this in turn implies there is at most one i with $\theta_i > \max_j r_j$. For each i, the event that $\theta_{i'} \leq \max_j r_j$ for all $i' \neq i$ has probability at most c^{n-1}. Since there are n possible choices of i, the bound $n \cdot c^{n-1}$ follows.

Part (b): Let $\nu(\theta_i), \hat{\nu}(\theta_i)$ be the virtual value functions corresponding to the two distributions. Using the given condition, one can check that $\nu(1) = \hat{\nu}(1)$, $\nu'(1) = \hat{\nu}'(1)$, while $\nu''(1) > \hat{\nu}''(1)$. Hence, $\nu(\theta_i) - \hat{\nu}(\theta_i)$ is quadratic in $1 - \theta_i$, for θ_i near 1. Consequently, by taking a small enough $\varepsilon > 0$, we can assume that

$$\nu(\theta_i) - \hat{\nu}(\hat{\theta}_i) > \varepsilon(1 - \theta_i)^2 \tag{OA-1}$$

whenever

$$1 - \varepsilon < \theta_i < \hat{\theta}_i < \theta_i + \varepsilon^2(1 - \theta_i)^2.$$

Fix such an ε.

Let $i^{**}(\theta)$ denote the bidder with the highest *virtual* value at the type profile θ. Since the revenue from an auction is given by the expected virtual value of the winning bidder, in comparing the revenue from the Myerson auction (which awards the good to $i^{**}(\theta)$ whenever his virtual value is positive) against the Vickrey auction (which awards to $i^{*}(\theta)$), it suffices to show that the virtual value of bidder $i^{**}(\theta)$ exceeds that of $i^{*}(\theta)$ by at least c'/n^3 on average.

For any realization of θ, let $\theta_{i^*[F]}$ denote the highest of the $n/2$ values drawn from distribution F, and $\theta_{i^*[\hat{F}]}$ similarly. Write $\xi = 1 - \theta_{i^*[F]}$, and similarly $\hat{\xi}$.

Consider the probability that

(i) $\xi \in (1/n - \varepsilon/n, 1/n)$; and moreover

(ii) $\xi - \hat{\xi} \in (0, \varepsilon^2\xi^2)$.

By standard extreme value theory, as $n \to \infty$, the distributions of $n\xi$ and $n\hat{\xi}$ converge to exponential distributions (see e.g. Falk, Hüsler, and Reiss (2010), Theorem 2.1.1(ii) with $\beta = -1$). Event (i) is equivalent to $n\xi \in (1 - \varepsilon, 1)$, hence its probability is bounded below by a positive constant as $n \to \infty$. Similarly, one can write event (ii) as $n\hat{\xi} \in (n\xi - \varepsilon^2(n\xi)^2/n, n\xi)$; so conditional on (i), its probability is bounded below by a constant times $1/n$. And when both (i) and (ii) hold, the bidder with highest value is $i^*[\hat{F}]$, while the one with highest virtual value is $i^*[F]$, and (OA-1) implies that their virtual values differ by an amount on the order of $1/n^2$. The assertion follows.

OA-D Non-Optimality of ACV Auction Without Regularity

This final section gives an example to show that, if we make no regularity assumptions on the distributions, then the robust-revenue-maximizing auction may not be of the ACV form. Moreover, this example is symmetric. (We do not know the optimal auction in this case; we simply show that one can do better than the best ACV auction.)

Consider a distribution F on $[0, 1]$, and two values $r < s$, satisfying the following properties:

- The one-buyer revenue function R_1 has two maxima, at r and s.
- There are just three points where the virtual value $\nu(\theta)$ is zero, namely r, s, and one in between (a local minimum of the revenue function).
- $F(r) > 1/2$.

Clearly this can be constructed.

Suppose there are two bidders, with both values distributed according to F. We claim that among all ACV auctions, the unique optimal auction is the symmetric auction with $p_1 = p_2 = s$, yielding revenue $R_2(s)$. Indeed, first suppose in negation that an optimal ACV auction has $p_2 < p_1$. Then local optimality with respect to p_1 implies that either $p_1 = r$ or $p_1 = s$. In the former case, we cannot satisfy the first-order condition (18) for local optimality of $p_2 < p_1 = r$, since its left-hand side is negative while its right-hand side is positive. In the latter case, note that raising bidder 2's reserve price from p_2 to $p_1 = s$ would raise the expected revenue by

$$F(p_2)[R_1(s) - R_1(p_2)] + 2\int_{p_2}^{s} [R_1(s) - R_1(\theta)] f(\theta) d\theta, \tag{OA-2}$$

where the first term accounts for the change in expected revenue on bidder 2 when $\theta_1 < p_2$, and the second term accounts for the change in expected revenue on bidder 2 when $\theta_1 \in [p_2, s]$ and for the symmetric change in expected revenue on bidder 1 when $\theta_2 \in [p_2, s]$. Both terms are strictly positive since $R_1(\theta) < R_1(s)$ for all $\theta < s$, $\theta \neq r$. Thus, the symmetric Vickrey auction with reserve price s would be a strict improvement.

Next, note that the Vickrey auction with symmetric reserve price $p > s$ cannot be optimal, since the first-order condition for the reserve price to maximize $R_2(p)$ is $2F(p)f(p)\nu(p) = 0$ implying $\nu(p) = 0$. Finally, for the symmetric Vickrey auction with reserve price $p < s$, raising the reserve price to s would raise the expected revenue by

$$2F(p)[R_1(s) - R_1(p)] + 2\int_{p}^{s} [R_1(s) - R_1(\theta)] f(\theta) d\theta, \tag{OA-3}$$

with the first term accounting for the change in expected revenue on bidder 2 when $\theta_1 < p$, and symmetrically on bidder 1 when $\theta_2 < p$, and the second term accounting for the change in expected revenue on bidder 2 when $\theta_1 \in [p, s]$, and symmetrically on bidder 1 when $\theta_2 \in [p, s]$. Again, both terms are strictly positive since $R_1(\theta) < R_1(s)$ for all $\theta < s$, $\theta \neq r$. Thus, the Vickrey auction with reserve price s is the unique optimal ACV auction.

Now, we can perturb F slightly to make r be the unique maximum of R_1, while holding fixed the stationary points of F and still having both (OA-2) and (OA-3) strictly positive, so that the Vickrey auction with reserve s is still the unique optimal ACV auction. From now on we consider this perturbed F.

Note that the optimal ACV auction is now not described by (13)–(14) (which requires, in particular, $p_1 = r$). But more importantly, for this distribution F, the seller can strictly improve upon the optimal ACV auction by the auction using the following allocation rule (and threshold prices):

- If $\theta_1 > \theta_2$, then buyer 1 receives the good either if $\theta_1 \geq \max\{\theta_2, s\}$, or if $\theta_2 \leq r \leq \theta_1$.
- If $\theta_2 > \theta_1$, the conditions for allocating to buyer 2 are symmetric.

Figure OA-1 shows the allocation rule, with the regions where the good is allocated to bidder 1, bidder 2, and not at all (0).

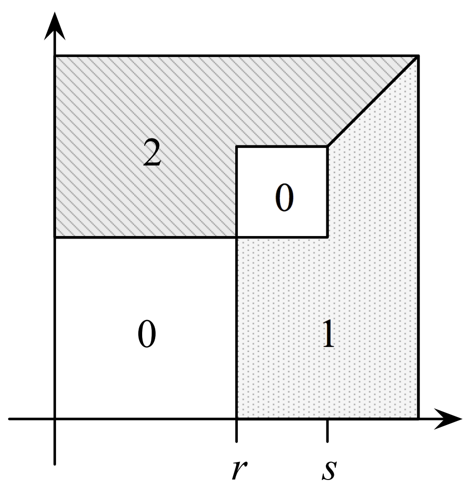

Figure OA-1: Allocation rule for alternative auction that outperforms ACV.

Note that if bidders tell the truth, the revenue from this auction differs from that of the optimal ACV auction in two cases: when $\theta_1 < r$, then the expected revenue from bidder 2 is $R_1(r)$ instead of $R_1(s)$, and similarly for the expected revenue from bidder 1 when $\theta_2 < r$. Thus, the revenue of this new auction exceeds that of the optimal ACV auction by

$$2F(r)(R_1(r) - R_1(s)) > 0.$$

So indeed this auction does strictly better.

It remains to check that truthful bidding in this auction is (Bayesian) incentive-compatible, for any resale procedure. By symmetry, it suffices to check deviations by bidder 1. Overbidding is not profitable since it can cause bidder 1 to win only in situations where the price he pays is higher than both his actual value and bidder 2's value. What about underbidding? Downward deviations to any bid above s can never be profitable, even with resale; the argument is the same as in the ACV auction. Downward deviations from a true value above s to a bid between r and s cannot be better than bidding exactly s, since the only change is that when $\theta_2 \in (r, s)$, the good now goes unsold instead of being sold to bidder 1. And downward deviations from a bid in (r, s) to a bid in (r, s) have no effect. Likewise, downward deviations from below r to below r have no effect.

Finally, consider downward deviations from a true value $\theta_1 \geq r$ to a bid below r. These deviations can potentially be beneficial ex post, because they may result in bidder 2 winning the good (and then bidder 1 maybe buying it from him) when $\theta_2 \in (r, s)$. However, the gain in any such instance is at most $\theta_1 - \theta_2 \leq \theta_1 - r$, so the total expected value of these gains is at most $(F(s) - F(r))(\theta_1 - r)$. Meanwhile, there is an expected loss of $F(r)(\theta_1 - r)$, resulting from the fact that whenever $\theta_2 < r$, the outcome changes from bidder 1 winning the good at price r to no sale. Since $F(r) > 1/2$, clearly $F(r) > F(s) - F(r)$ which implies that the loss from deviation outweighs the gain.

Note that this counterexample also shows that Ausubel and Cramton's (1999) "monotonicity in aggregate" is not actually a necessary condition for incentive-compatibility with resale, as long as agents have no information about each other's values at the time of the auction.

References

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[3] Falk, M., J. Hüsler, and R.-D. Reiss (2010), *Laws of Small Numbers: Extremes and Rare Events* (third edition). Basel: Birkhäuser.