OPTIMAL DEFAULTS AND ACTIVE DECISIONS
Online Appendix

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This file is an appendix to Carroll et al. (2008). We provide an analysis of the model of procrastination and optimal 401(k) enrollment regimes introduced in Section 4 of that paper, including proofs of all the results mentioned there. The reader is referred to the main paper for an exposition of the model. Equations and propositions in this appendix are numbered in sequence, continuing the numbering from the main paper.

More extensive analysis can be found in an earlier version of the paper (Carroll et al. 2007). That version allows general values of the long-run discount factor δ and gives some results for arbitrary distributions of the preferred savings rate s over the population; here we maintain the assumptions that $\delta = 1$ and s is uniformly distributed.

A.1 The Employee's Problem

For the first set of results, we do not need the assumption that c is uniformly distributed; we assume only that it has a continuous, positive density f with support $[\underline{c}, \bar{c}]$. We will restrict

L to be strictly positive; clearly the employee never enrolls if $L = 0$.

We are concerned mainly with identifying the stationary equilibria of the game played by the employee's various selves, where a fixed cutoff c^* is used in all periods. As in the main text of the paper, we let ϕ be the expected total utility loss over all periods, measured without β-discounting. ϕ is given by equation (1) of the main text, which we rewrite as

$$\phi = E(c|c < c^*) + L \cdot P(c > c^*)/P(c < c^*). \tag{3}$$

The cutoff c^* will be chosen so that that the present-biased agent is indifferent between incurring transactions cost c^* and continuing to wait, except that c^* is constrained to lie in the interval $[\underline{c}, \bar{c}]$. That is:

$$c^* = \begin{cases} \underline{c} & \text{if } \beta(L+\phi) < \underline{c} \\ \beta(L+\phi) & \text{if } \underline{c} \leq \beta(L+\phi) \leq \bar{c} \\ \bar{c} & \text{if } \beta(L+\phi) > \bar{c}. \end{cases}$$

Proposition 4 *The agent's game has a unique stationary equilibrium:*

- *if $L \geq \bar{c}/\beta - E(c)$, then $c^* = \bar{c}$ (the agent acts immediately no matter what the cost);*
- *otherwise, c^* is the unique solution to the equation*

$$c^* = \beta[L + E(c|c < c^*)P(c < c^*)] + c^* P(c > c^*). \tag{4}$$

Proof. We never have an equilibrium at $c^* = \underline{c}$, because this would lead to $\phi = \infty$, inducing the agent to act. A cutoff of $c^* = \bar{c}$ yields $\phi = E(c)$. This cutoff is an equilibrium if and only if $\beta(L+\phi) \geq \bar{c}$, which is equivalent to $L \geq \bar{c}/\beta - E(c)$. And any given value of c^* in the middle is an equilibrium if and only if $c^* = \beta(L+\phi)$, where ϕ is given by (3). This equation simplifies to $c^* = \beta[E(c|c < c^*) + L/P(c < c^*)]$. Rearranging gives (4).

For each $c^* \in [\underline{c}, \bar{c}]$, let

$$g(c^*) = \beta[L + E(c|c < c^*)P(c < c^*)] + c^*P(c > c^*).$$

To complete the existence-uniqueness proof, we need to check that equation (4), which states that $c^* = g(c^*)$, has exactly one solution on $(\underline{c}, \bar{c})$ if $0 < L < \bar{c}/\beta - E(c)$, and none otherwise.

The function g is differentiable, with $g'(c^*) = \beta c^* f(c^*)(1 - 1/\beta) + P(c > c^*) \leq P(c > c^*) \leq 1$; equality is possible only if $c^* = \underline{c}$. This implies that $g(c^*) = c^*$ has one root in $(\underline{c}, \bar{c})$ if $g(\underline{c}) > \underline{c}$ and $g(\bar{c}) < \bar{c}$, and no root otherwise. Since $g(\underline{c}) = \beta L + \underline{c}$ and $g(\bar{c}) = \beta(L + E(c))$, the result follows. ■

In Carroll et al. (2007) we also consider the possibility of nonstationary equilibria, which we allow to be time-varying but otherwise history-independent. We show that if

$$\beta > 1 - \min_{c^* \in [\underline{c}, \bar{c}]} \frac{P(c > c^*) + 1}{c^* f(c^*)},$$

then the only equilibrium is the stationary one. If c is uniformly distributed, the condition reduces to $\beta > \underline{c}/\bar{c}$, which holds if the range of transactions costs is wide.

Next, we have an important corollary to Proposition 4:

Corollary 5 *The cutoff c^* of the stationary equilibrium is a continuous function of β and L, and it is differentiable except at $L = \bar{c}/\beta - E(c)$. The same holds for ϕ.*

Proof. Continuity of c^* at the kink $L = \bar{c}/\beta - E(c)$ is straightforward to check. Differentiability of c^* away from the kink is easily checked by writing out $c^* = g(c^*)$ and differentiating implicitly with respect to β and L (on which the value of g depends). Continuity and differentiability of ϕ then follow from equation (3). ■

Next, we prove comparative statics on c^* and ϕ.

Proposition 6 *1. In the region where $c^* < \bar{c}$, c^* is strictly increasing in L and β.*

2. In this region, ϕ is strictly decreasing in β.

3. If $\beta = 1$ then ϕ is weakly increasing in L. However, if $\beta < 1$, then there exist values of L for which $\phi > E(c)$.

4. If β is sufficiently low, then $\phi \geq E(c)$ for all L.

Proof. For part 1, use $c^* = g(c^*)$. If we hold c^* constant but increase either L or β, then $g(c^*)$ increases. Since $g(c^*) - c^*$ is a decreasing function of c^*, the new value c^{**} such that $g(c^{**}) = c^{**}$ must be greater than c^*.

For part 2, first take equation (4) and differentiate implicitly with respect to β. This gives us

$$\frac{\partial c^*}{\partial \beta} = \frac{L + E(c|c < c^*)P(c < c^*)}{1 + (1-\beta)c^* f(c^*) - P(c > c^*)} = \frac{P(c < c^*)c^*/\beta}{P(c < c^*) + (1-\beta)c^* f(c^*)} \leq c^*/\beta,$$

where the second equality comes from rearranging (4) and substituting into the numerator.

Now rearrange the indifference condition for c^* to get $\phi = c^*/\beta - L$, and differentiate with respect to β: $\partial\phi/\partial\beta = (\partial c^*/\partial\beta)/\beta - c^*/\beta^2 \leq 0$. Note that equality holds if and only if $\beta = 1$.

For part 3, note that $\partial\phi/\partial L = (\partial c^*/\partial L)/\beta - 1$. If $\beta = 1$, then we can differentiate (4) implicitly with respect to L to get $\partial c^*/\partial L = 1/P(c < c^*) \geq 1$. Therefore, $\partial\phi/\partial L \geq 0$.

However, if $\beta < 1$, then consider $L = \bar{c} - E(c)$. For this L , at $\beta = 1$ we have $\phi = E(c)$. As β falls, by part 2, ϕ will strictly increase.

Finally, for part 4, pick $\beta < \min_{c^*}[c^*/E(c|c > c^*)]$. We know $c^* > \underline{c}$ because never acting leads to an infinite present value of losses. If $c^* = \bar{c}$ then $\phi = E(c)$. Otherwise, the indifference condition gives $c^* = \beta(L+\phi) \le c^*(L+\phi)/E(c|c>c^*)$, or $E(c|c>c^*) \le L+\phi$. Hence

$$\begin{aligned} \phi &= E(c|c<c^*)P(c<c^*) + (L+\phi)P(c>c^*) \\ &\ge E(c|c<c^*)P(c<c^*) + E(c|c>c^*)P(c>c^*) = E(c). \end{aligned}$$

■

We will use the following two results later:

Corollary 7 *If $L > \bar{c} - E(c)$, then $\phi(L) \ge E(c)$.*

Proof. Exactly as for part 3 of Proposition 6. ■

Proposition 8 *Suppose $\beta < 1$. As $L \to 0$ (and all other parameters stay constant), $c^* \to \underline{c}$.*

Proof. Rewrite (4) as

$$[c^* - \beta E(c|c<c^*)]P(c<c^*) = \beta L. \tag{5}$$

If c^* is bounded away from $\underline{c}$, then both factors on the left side are bounded away from zero, so the right side is also. Therefore, if the right side tends to zero we must have $c^* \to \underline{c}$. ■

We now return to the assumption that c is uniformly distributed on $[\underline{c}, \bar{c}]$, and study the employee's welfare in more detail, with attention to how ϕ varies with $\Delta = s - d$. Recall that $L = \kappa\Delta^2$, and we write $\Phi(\Delta)$ for the welfare loss ϕ as a function of Δ.

Lemma 1

1. *If $\beta = 1$, then $\Phi(\Delta)$ is weakly increasing in $|\Delta|$.*

2. *If $2 - \bar{c}/\underline{c} < \beta < 1$, then (a) there exist $0 < \Delta_m < \bar{\Delta}$ such that $\Phi(\Delta)$ is increasing on $(0, \Delta_m]$, decreasing on $[\Delta_m, \bar{\Delta}]$, and constant at $E(c) = (\underline{c} + \bar{c})/2$ on $[\bar{\Delta}, \infty]$, and (b) there is at most one value $\Delta_e \in (0, \bar{\Delta})$ such that $\Phi(\Delta_e) = E(c)$.*

3. *If* $\beta \leq 2-\bar{c}/\underline{c}$*, then there exists* $\bar{\Delta}$ *such that* $\Phi(\Delta)$ *is decreasing on* $(0, \bar{\Delta}]$ *and constant at* $E(c)$ *on* $[\bar{\Delta}, \infty]$.

Proof. Part 1 follows from part 3 of Proposition 6, and part 2(b) is immediate from 2(a), so we need only prove parts 2(a) and 3.

It suffices to show the analogous statements for ϕ as a function of L: if $2 - \bar{c}/\underline{c} < \beta < 1$, there exist $0 < L_m < \bar{L}$ such that $\phi(L)$ is increasing on $[0, L_m]$, decreasing on $[L_m, \bar{L}]$, and constant at $E(c)$ on $[\bar{L}, \infty]$; if $\beta \leq 2 - \bar{c}/\underline{c}$ then there exists $\bar{L}$ such that $\phi(L)$ is decreasing on $(0, \bar{L}]$ and constant at $E(c)$ on $[\bar{L}, \infty]$. Then $\Delta_m = \sqrt{L_m/\kappa}$ and $\bar{\Delta} = \sqrt{\bar{L}/\kappa}$. Clearly we should set $\bar{L} = \bar{c}/\beta - E(c)$, since ϕ is constant above this $\bar{L}$.

We wish to compute $d\phi/dL$ for $L < \bar{L}$. To do this, first rewrite equation (3) using the indifference condition $c^* = \beta(L + \phi)$, and rearrange to get

$$\phi = E(c|c < c^*)P(c < c^*) + \frac{c^*}{\beta}P(c > c^*). \tag{6}$$

We can treat this as an expression for ϕ in terms of c^* only. The derivative is

$$\frac{d\phi}{dc^*} = c^* f(c^*) \left(1 - 1/\beta\right) + P(c > c^*)/\beta = \frac{c^*(1 - 2/\beta) + \bar{c}/\beta}{\bar{c} - \underline{c}}.$$

This is decreasing in c^*, zero at $c^* = \bar{c}/(2 - \beta) < \bar{c}$ (hence negative at $c^* = \bar{c}$), and positive at $c^* = \underline{c}$ if and only if $\underline{c} < \bar{c}/(2 - \beta)$, which is equivalent to $\beta > 2 - \bar{c}/\underline{c}$.

Now viewing ϕ and c^* as functions of L, $d\phi/dL = d\phi/dc^* \cdot dc^*/dL$, and c^* is increasing in L (for $L < \bar{L}$) by Proposition 6. So $d\phi/dL$ has the same sign as $d\phi/dc^*$, which was computed in the previous paragraph. It follows that if $\beta > 2 - \bar{c}/\underline{c}$ then ϕ is increasing in L up until the (unique) value for which $c^* = \bar{c}/(2-\beta)$ (this value exists by Corollary 5 and Proposition 8), and decreasing for higher values of L; if $\beta \leq 2 - \bar{c}/\underline{c}$ then ϕ is always decreasing in L. ■

It is straightforward to calculate the values of $\bar{\Delta}$, Δ_m, and Δ_e (if it exists) explicitly.

We present these values here; since we will not actually need to use them, we omit the computations:

$$\bar{\Delta} = \sqrt{\frac{\bar{c}/\beta - (\bar{c} + \underline{c})/2}{\kappa}}, \ \Delta_m = \sqrt{\frac{(\bar{c} - \beta\underline{c})(\bar{c} - (2-\beta)\underline{c})}{2\beta(2-\beta)(\bar{c} - \underline{c})\kappa}}, \ \Delta_e = \sqrt{\frac{\beta\bar{c} - (2-\beta)\underline{c}}{2(2-\beta)\kappa}}.$$

Now, we need a technical detail: when $2 - \bar{c}/\underline{c} < \beta < 1$, so that the function Φ has humps, the outer portion of the humps is steeper than the inner portion.

Lemma 9 *For* $0 < \Delta < \bar{\Delta}$, $d^3\Phi/d\Delta^3 < 0$.

Proof. Write out the condition (4) for c^* explicitly, using the uniform distribution. This condition is a quadratic equation for c^*, whose constant term is a linear function of L and whose other coefficients are independent of L. So c^* can be written in the form $\sqrt{A+BL}+C$, where A, B, and C are constants.[31] Next, $\phi = c^*/\beta - L = \left(\sqrt{A+BL}+C\right)/\beta - L$, where $A + BL \geq 0$ on the relevant range.

Finally, using $L = \kappa\Delta^2$, we see we can write $\Phi(\Delta)$ in the form $\left(\sqrt{A+B\kappa\Delta^2}+C\right)/\beta - \kappa\Delta^2$. By taking $\Delta \to 0$, we see $A \geq 0$. It is straightforward to check that the third derivative of this function is $-3AB^2\kappa^2\Delta/[\beta(A+B\kappa\Delta^2)^{5/2}] < 0$. (We can check that A is strictly greater than zero: by differentiating (4) implicitly we can check that $dc^*/dL \not\to \infty$ as $L \to 0$, whereas $A = 0$ would imply $dc^*/dL \to \infty$ as $L \to 0$.) ■

Proposition 10 *If* $0 < \Delta_1 < \Delta_2 < \bar{\Delta}$ *and* $\Phi(\Delta_1) = \Phi(\Delta_2)$, *then* $\Phi'(\Delta_1) + \Phi'(\Delta_2) < 0$. *(Note that* Δ_1 *is on the inside of the hump and* Δ_2 *is on the outside.)*

Proof. Let $\Psi(\Delta) = d\Phi/d\Delta$. By Lemma 9, Ψ is strictly concave. Jensen's inequality implies that the average value of Ψ on the interval $[\Delta_1, \Delta_2]$ is greater than the average of its values

[31] The explicit expression is $c^* = [\underline{c} + \sqrt{(1-\beta)^2\underline{c}^2 + 2\beta(2-\beta)(\bar{c}-\underline{c})L}]/(2-\beta)$. One can check that the lower root of the quadratic is less than $\underline{c}$.

at the two endpoints. But $\int_{\Delta_1}^{\Delta_2} \Psi(\Delta)\, d\Delta = \Phi(\Delta_2) - \Phi(\Delta_1) = 0$, so the average value of Ψ on the interval is zero. Hence, the average of the values of Ψ at Δ_1 and Δ_2 must be negative. ■

A.2 Socially Optimal Default Policies

Now we are ready to consider the social planner's problem. As in Figure IX in the main paper, we will allow only $\beta, \underline{s}, \bar{s}$ to vary; all other exogenous parameters will be considered fixed from now on.

Lemma 11 *If β is sufficiently close to 1, then active decisions are never optimal.*

Proof. Let $\bar{\Delta}$ be the value given by Lemma 1, so that $\Phi(\Delta) = E(c)$ when $\Delta > \bar{\Delta}$. When $\beta = 1$, we know that

$$\int_{-\bar{\Delta}}^{\bar{\Delta}} [\Phi(\Delta) - E(c)]\, d\Delta < 0, \tag{7}$$

since $\Phi(\Delta)$ is always $\leq E(c)$, with strict inequality when Δ is near zero. Using Corollary 5, this integral is continuous in β. So for β close to 1, the integral remains less than 0.

Consider any such β. We will show that

$$\int_{-r}^{r} [\Phi(\Delta) - E(c)]\, d\Delta < 0, \tag{8}$$

where $r = (\bar{s} - \underline{s})/2$. This will imply that setting a default of $d = (\underline{s} + \bar{s})/2$ is better than active decisions, since the former yields welfare cost $\int_{-r}^{r} \Phi(\Delta)\, d\Delta$ and the latter yields $2rE(c)$.

If $r > \bar{\Delta}$ then (8) is immediate from (7), because $\Phi(\Delta) = E(c)$ when $|\Delta| > \bar{\Delta}$. Next, we know (8) holds when $r = \bar{\Delta}$. The derivative of the left-hand side with respect to r is $2[\Phi(r) - E(c)]$, which is positive when $r \geq \Delta_e$ (when Δ_e exists; or for all $r \geq 0$ otherwise).

So as we lower r, (8) will continue to hold as long as $r \geq \Delta_e$. Finally, if $r \leq \bar{\Delta}$ and $\Phi(r) < E(c)$, then Lemma 1 implies that $\Phi(\Delta) < E(c)$ for $|\Delta| \leq r$, and it follows directly that the left-hand side of (8) is negative. ■

Lemma 12 *If β is sufficiently low, then active decisions are always optimal.*

Proof. Immediate from part 4 of Proposition 6: that proposition says active decisions are best for each individual employee (except for the measure-zero subset who are already at their optimum savings rate), so they are also socially optimal. ■

Lemma 13 *If active decisions are optimal at some parameter values, then when β is decreased (and all other parameters unchanged), active decisions are still optimal.*

Proof. Proposition 6, part 2 implies that, for any given Δ, $\Phi(\Delta)$ cannot decrease when β decreases. Integrating over Δ, we see that the welfare loss from any given default d can only increase when β decreases, so no such d can become better than active decisions. ■

Now we are finally prepared to prove the classification of optimal defaults.

Proposition 2 *If $\beta < 1$, then the optimal default is one of the following three types:*

- *the center default $d = (\underline{s} + \bar{s})/2$;*
- *an offset default, such that $\underline{s} - d = -\Delta_e$ and $\bar{s} - d \geq \bar{\Delta}$ (or its symmetric equivalent, $\bar{s} - d = \Delta_e$ and $\underline{s} - d \leq -\bar{\Delta}$);*
- *active decisions, which correspond to any d with $\underline{s} - d \geq \bar{\Delta}$ or $\bar{s} - d \leq -\bar{\Delta}$.*

Proof. For the purposes of this proof, we redefine $\Phi(0)$ to equal $\lim_{\Delta \to 0} \Phi(\Delta)$. This makes Φ continuous everywhere and (by Corollary 5) differentiable except at $\Delta = 0, \pm\bar{\Delta}$.

The optimal default is the d that minimizes $\int_{\underline{s}}^{\bar{s}} \Phi(s - d)\, ds$. This integral is continuous in d and constant (at $(\bar{s} - \underline{s})E(c)$) when $|d|$ is large, so the minimum exists. By differentiating with respect to d, we get the first-order condition $\Phi(\underline{s}-d)-\Phi(\bar{s}-d)=0$, which any optimum must satisfy. The second-order condition is $-\Phi'(\underline{s}-d)+\Phi'(\bar{s}-d)\geq 0$, which must also hold if both derivatives are defined.

Consider now the common value $\Phi(\bar{s}-d)=\Phi(\underline{s}-d)$ that emerges from the first-order condition. We use Lemma 1 to draw conclusions about where $\bar{s}-d$ and $\underline{s}-d$ are located for a given common value:

- If the common value is $< E(c)$, then there are only two points $\pm\Delta$ where Φ takes on this value. So $\bar{s}-d=-(\underline{s}-d)$, and we have a center default.

- If the common value is $> E(c)$, then we may again have just two points where Φ takes on this value. At most, there are four such points, of the form $\pm\Delta_1, \pm\Delta_2$, where $0<\Delta_1<\Delta_m<\Delta_2<\bar{\Delta}$. If $\underline{s}-d, \bar{s}-d$ are equal to $\pm\Delta_1$ then we have a center default. Otherwise, assume $\bar{s}-d=\Delta_2$. (By symmetry, the argument is equivalent when $\underline{s}-d=-\Delta_2$.)

  If $\underline{s}-d=-\Delta_2$ we have a center default. If $\underline{s}-d=-\Delta_1$, then $-\Phi'(\underline{s}-d)+\Phi'(\bar{s}-d)=\Phi'(\Delta_1)+\Phi'(\Delta_2)<0$ by Proposition 10. This violates the second-order condition. And if $\underline{s}-d=\Delta_1$ then the entire interval $[\underline{s}-d, \bar{s}-d]$ lies within the hump, so that $\Phi(s-d)>E(c)$ for each s in the interval, and this default is strictly worse than active decision.

- If the common value is equal to $E(c)$, then the possible values for the two endpoints are $\pm\Delta_e$ and $\pm\Delta$ for any $\Delta\geq\bar{\Delta}$. If the endpoints are $\pm\Delta_e$ then we have a center default. Otherwise, without loss of generality, assume $\bar{s}-d\geq\bar{\Delta}$.

  If $\underline{s}-d\leq-\bar{\Delta}$ then the interval $[\underline{s}-d, \bar{s}-d]$ contains both humps and the valley between them. We can increase d until the upper endpoint $\bar{s}-d$ hits Δ_e, thus eliminating one hump and replacing it with a plateau. This changes the integral of Φ by $\int_{\Delta_e}^{\bar{\Delta}} [E(c) - \Phi(\Delta)]\, d\Delta < 0$. Thus, total social loss is decreased, so the original d was not optimal.

If $\underline{s} - d = -\Delta_e$ then we have an offset default. And if $\underline{s} - d = \Delta_e$ then the interval contains one hump and a plateau, which is again strictly inferior to active decisions.

■

Proposition 3 *Fix κ, $\underline{c}$, and $\bar{c}$. Then there exist values $0 < \beta_{ac} < \beta_{oc} < 1$, and a function $w : (\beta_{ac}, 1] \to (0, \infty]$, with the following properties:*

1. *for $\beta \leq \beta_{ac}$, active decisions are always optimal;*
2. *for $\beta_{ac} < \beta < \beta_{oc}$, active decisions are optimal when $\bar{s} - \underline{s} > w(\beta)$ and a center default is optimal when $\bar{s} - \underline{s} < w(\beta)$;*
3. *for $\beta_{oc} < \beta < 1$, an offset default is optimal when $\bar{s} - \underline{s} > w(\beta)$ and a center default is optimal when $\bar{s} - \underline{s} < w(\beta)$;*
4. *w is increasing on $(\beta_{ac}, \beta_{oc}]$.*

Proof. First, fix any $\beta < 1$. Define $w(\beta)$ to be the supremum of values $\bar{s} - \underline{s}$ for which a center default is optimal, or 0 if a center default is never optimal. This supremum is finite; for example, if $\bar{s} - \underline{s} > 2\bar{\Delta}$, then a center default implies that the interval $[\underline{s} - d, \bar{s} - d]$ contains both humps of the function Φ, and this is strictly worse than the offset default, which contains only one of the humps.

Now suppose a center default is optimal for some width $\bar{s} - \underline{s}$. We claim it is also optimal for all narrower widths. First we check that a center default remains better than active decisions. We are given

$$\int_{-r}^{r} \Phi(\Delta)\, d\Delta \leq 2rE(c) \tag{9}$$

for $r = (\bar{s} - \underline{s})/2$, and we want to show it remains true for all lower r. This is exactly the argument we gave in the proof of Lemma 11.

Next we check that a center default remains better than an offset default (if the offset default is defined): if $\bar{s} - \underline{s} \geq \Delta_e + \bar{\Delta}$ and

$$\int_{-r}^{r} \Phi(\Delta)\, d\Delta \leq \int_{-\Delta_e}^{2r-\Delta_e} \Phi(\Delta)\, d\Delta \tag{10}$$

for $r = (\bar{s} - \underline{s})/2$, then it also holds for all lower $r \geq (\Delta_e + \bar{\Delta})/2$. This is an analogous argument: for all r in the relevant range, the derivative of the left side of (10) is $2\Phi(r) \geq 2E(c)$, while the derivative of the right side is $2\Phi(2r - \Delta_e) = 2E(c)$. So as r decreases, the left side decreases faster than the right side, and (10) remains true.

This shows that if a center default is optimal, it remains optimal when the range of s is narrowed. Therefore, a center default is optimal whenever $\bar{s} - \underline{s} < w(\beta)$. By definition, a center default is not optimal for $\bar{s} - \underline{s} > w(\beta)$, so the optimal default here is either offset or active decisions.

A similar argument to the above shows that if active decisions are better than an offset default, then this remains the case when the width $\bar{s} - \underline{s}$ is increased, and similarly if an offset default is preferred to active decisions. (The marginal employee is in the plateau of the function $\Phi(\Delta)$ in both cases and is thus indifferent between the two policies.) So for fixed β, either active decisions are best for all $\bar{s} - \underline{s} > w(\beta)$, or offset defaults are best for all $\bar{s} - \underline{s} > w(\beta)$. Call β an "active value" or an "offset value" accordingly. Lemma 13 implies that if some β is an active value, then all lower β are also active values.

By Lemma 12, for sufficiently low β, active decisions are optimal for all $\bar{s}$ and $\underline{s}$. Define β_{ac} to be the supremum of all such β. Part 1 of Proposition 3 immediately holds. Conversely, by Lemma 11, when β is high enough, active decisions are never optimal. Define β_{oc} to be the supremum of all β such that active decisions are optimal for some width $\bar{s} - \underline{s}$. Thus we have $0 < \beta_{ac} \leq \beta_{oc} < 1$; moreover, all $\beta < \beta_{oc}$ are active values, and all $\beta > \beta_{oc}$ are offset values. We have now proven parts 2 and 3 of the proposition. We still need to check that $\beta_{ac} < \beta_{oc}$ strictly. When $\beta = \beta_{oc}$ and $\bar{s} - \underline{s} > w(\beta)$, the planner must be indifferent between offset defaults and active decisions; that is,

$$\int_{-\Delta_e}^{\bar{\Delta}} \Phi(\Delta)\, d\Delta = (\bar{\Delta} + \Delta_e) E(c). \tag{11}$$

This indifference follows from the definition of β_{oc} and the fact that both sides of the equation are continuous in β. But (11) implies that $\Phi(\Delta) < E(c)$ for some Δ, since the integral includes a hump where $\Phi(\Delta) > E(c)$. This implies that active decisions are not optimal when $\bar{s} - \underline{s}$ is sufficiently small, and this remains true when β is decreased slightly. Hence, β can be decreased slightly from β_{oc} and still be $\geq \beta_{ac}$, proving $\beta_{ac} < \beta_{oc}$.

Finally, Lemma 13 implies that the set of values $\bar{s} - \underline{s}$ for which active decisions are optimal can only grow when β decreases. Part 4 of the proposition follows. ■

The last bit of our analysis is the case $\beta = 1$. The function $\Phi(\Delta)$ now consists of a valley with no humps; the optimal default always takes as much advantage of this valley as possible.

Proposition 14 *Assume $\beta = 1$. Then*

- *if $\bar{s} - \underline{s} \leq 2\bar{\Delta}$, then a center default $d = (\bar{s} + \underline{s})/2$ is the unique optimum;*
- *otherwise, the set of optimal defaults consists of all $d \in [\underline{s} + \bar{\Delta}, \bar{s} - \bar{\Delta}]$.*

(The value of $\bar{\Delta}$ simplifies to $\sqrt{(\bar{c} - \underline{c})/2\kappa}$ when $\beta = 1$.)

Proof. We carry out the same analysis as in Proposition 2. When $\beta = 1$, Lemma 1 tells us that Φ is increasing in $|\Delta|$ for $|\Delta| \leq \bar{\Delta}$, and then becomes flat at $E(c)$ for $|\Delta| \geq \bar{\Delta}$. We want to identify the d that minimizes $\int_{\underline{s}}^{\bar{s}} \Phi(s - d)\, ds$.

The first-order condition is $\Phi(\underline{s} - d) = \Phi(\bar{s} - d)$. If this common value is $< E(c)$, then by monotonicity, there are only two values $\pm\Delta$ at which Φ takes on this value. Hence, $\underline{s} - d = -(\bar{s} - d)$, and we have a center default.

Otherwise, the common value is $E(c)$, which is $\Phi(\Delta)$ for all $\Delta \leq -\bar{\Delta}$ or $\Delta \geq \bar{\Delta}$. By symmetry we may assume $\bar{s} - d \geq \bar{\Delta}$. If $\underline{s} - d \geq \bar{\Delta}$ then we have an active decision regime, which cannot be optimal (by Lemma 11). Hence $\underline{s} - d \leq -\bar{\Delta}$. Then the social welfare integral is equal to

$$\int_{-\bar{\Delta}}^{\bar{\Delta}} \Phi(\Delta)\, d\Delta + (\bar{s} - \underline{s} - 2\bar{\Delta})E(c),$$

and this value is independent of the choice of default d as long as $\underline{s} - d \leq -\bar{\Delta}$ and $\bar{s} - d \geq \bar{\Delta}$; that is, $d \in [\underline{s} + \bar{\Delta}, \bar{s} - \bar{\Delta}]$. So if one such d is optimal, all of them are.

All possible optima are of one of these two types. The first type of optimum only exists for $\bar{s} - \underline{s} < 2\bar{\Delta}$, while the second type only exists for $\bar{s} - \underline{s} \geq 2\bar{\Delta}$. The proposition follows immediately. ■

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