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Discrimination via Symmetric Auctions†

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Discrimination (for instance, along the lines of race or gender) is often prohibited in auctions. This is legally enforced by preventing the seller from explicitly biasing the rules in favor of bidders from certain groups (for example, by subsidizing their bids). In this paper, we study the efficacy of this policy in the context of a single object: independent private value setting with heterogeneous bidders. We show that restricting the seller to using an anonymous, sealed bid auction format (or, simply, a symmetric auction) imposes virtually no restriction on her ability to discriminate. Our results highlight that the discrepancy between the superficial impartiality of the auction rules and the resulting fairness of the outcome can be extreme. (*JEL* D44, D82)

Consider an auctioneer selling a single object to (ex ante heterogeneous) buyers whose private values are drawn from independent but, possibly different, distributions. Now suppose this seller is restricted to choosing a symmetric sealed bid auction format (henceforth, referred to simply as a symmetric auction). These are auctions in which buyers submit sealed bids (of a single number), the highest bid wins (subject to being greater than a reservation bid), and the payments are determined as an anonymous function of the bids. To what extent can the seller favor some bidders over others by designing an auction in this class? In particular, can he favor ex ante disadvantaged bidders (or "level the playing field") under this constraint? In this paper, we characterize the range of outcomes that a seller can achieve with symmetric auctions. In doing so, we show that symmetry by itself is a very unrestrictive constraint, and we discuss the implications of this for policy.

Formally, we characterize the set of (incentive compatible) direct mechanisms for which there exists a symmetric auction implementation. Here, implementation implies that there exists a symmetric auction format that has an equilibrium with the same ex post allocation (the winner is the same at all value profiles) and interim payments (the expected payment by all values of each buyer is the same). It is

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straightforward to argue that, in order to implement the same ex post allocation, the allocation rule of the direct mechanism essentially pins down the equilibrium bidding function (up to monotone transformations) that must arise in the symmetric auction implementation. Loosely speaking, this is because the bidder that receives the good in the direct mechanism at each profile of values must make the highest bid in the corresponding symmetric auction. The bidding function and the distribution over values then determines a distinct distribution over equilibrium bids for each buyer. The main technical challenge is to show the existence of an anonymous payment rule that, in equilibrium, yields the same expected payments as the direct mechanism. Note that an anonymous payment can yield different expected payments across buyers for the same bid. This is because expectations are taken with respect to the opponents' equilibrium bid distributions (which differ across buyers).[1]

Our main insight is stark: restricting the seller to using a symmetric auction imposes virtually no restriction on her ability to achieve discriminatory outcomes. This is demonstrated by the main result (Theorems 1 and 2), which is a complete characterization of the set of implementable direct mechanisms. We show that *almost any* hierarchical mechanism (defined formally in Section III) has a symmetric implementation. Hierarchical mechanisms are a large class that contain virtually all mechanisms that arise in applied mechanism design (such as the efficient, revenue maximizing mechanism, etc.). Motivated by this, we argue that real limits on the seller's ability to discriminate can only be imposed by further restricting the set of auctions that he can choose from. We discuss a few such restrictions that are easy to describe and enforce in practice.

There are at least two ways to think about the underlying mechanics of our result. First, observe that revenue equivalence does not hold with heterogeneous bidders and, additionally, a symmetric auction may have asymmetric equilibria. The asymmetry sustains itself in equilibrium; even though the *ex post* payments do not depend on the bidders' identities, the *expected* (interim) payment for the same bid might differ across bidders. This is because bidders take expectations over the bids of their competitors, and in an asymmetric equilibrium, the distribution of competing bids differs across bidders. This intuition is well known and is the reason why first and second-price auctions yield different revenues (Kirkegaard 2012). Our insight is that when the seller has full freedom to design the symmetric auction, he can choose one that has the desired (asymmetric) equilibrium.

A second intuition may be useful for readers familiar with the literature on full surplus extraction. There, agents' values are correlated and Crémer and McLean (1988) show that, under certain conditions, the principal can extract the entire surplus in expectation. The result is surprising as even though the agents' values are private, the principal does not need to provide any information rents for the revelation of their private information. Essentially, the principal offers each agent a menu of bets on the realized values of the other agents, from which she must pick one. Since the values are correlated, an agent's beliefs about the conditional distribution

[1] Mathematically, the exercise can be described as follows. Suppose for each $i \in \{1, \dots, n\}$, we are given a function $p_i^d : B_i \to \mathbb{R}$ and a distribution over $B_i \subset \mathbb{R}$. When is it possible to find a function $p : \mathbb{R}^n \to \mathbb{R}^n$ that is permutation invariant such that for all i and $b_i \in B_i$, we have $p_i^d(b_i) = E_{b_{-i}} p_i(b_i, b_{-i})$?

of other agents' values differ based on her own realized value. The side bets are designed so that, for every value, each agent pays in expectation (taken over the distribution of her opponents' values) her full expected surplus when she picks the bet corresponding to truthful reporting. Further, given her conditional beliefs over other agents' values, all other bets offer net negative expected surplus. The insight is akin to "proper scoring rules," which can be used to truthfully elicit an agent's beliefs about a stochastic event. By contrast, in our model, values are independently distributed. However, since bidders are heterogeneous, intuitively, a bidder's *identity* is "correlated" with the values of other bidders. This allows us to construct an analogous menu of bets (within the class of symmetric auctions), which undo the constraints imposed by symmetry.[2]

The focus of this paper is motivated by real world policies. Auctioneers may not be allowed to use formats that are not symmetric as discrimination is often banned. For example, states such as California and Michigan have explicitly changed their laws (Proposition 209 and Proposal 2, respectively) to prohibit favored treatment in government procurement on the basis of race, sex, or ethnicity. One reason for such policies is that auction designs that explicitly favor some groups (such as minorities, women, veterans, etc.) have been successfully challenged in court (for example, in the prominent US Supreme Court case *Adarand Constructors v. Pena 1995*). As a practical matter, the main policy response has been to suggest that auctioneers, especially governments/public bodies, be constrained to using anonymous auction designs. While this is a natural candidate for a policy, its efficacy is not understood. This is especially relevant for practice today, as auction designers often have access to, and the ability to analyze, rich data from past auctions to guide their format design.

Why might an auctioneer want to use a format that is not symmetric? One reason is revenue maximization: the direct mechanism corresponding to the Myerson (1981) optimal auction with heterogeneous bidders is discriminatory. There are several notable instances where it has been suggested that revenue maximization and symmetry of the auction form are fundamentally incompatible desiderata.[3] Some argue that this observation justifies the removal of legal hurdles that prevent discrimination. In the context of international trade, McAfee and McMillan (1989) used the theory of optimal auctions to show that explicitly discriminating amongst suppliers can reduce the costs of procurement. Their aim was to provide an argument against the 1981 Agreement on Government Procurement (in the General Agreement on Tariffs and Trade (GATT)), which set out rules to ensure that domestic and international suppliers were treated equally.[4] Similarly, Ayres and Cramton (1996) suggest that, in government license auctions, subsidizing minority owned or local businesses may actually result in more revenue to the government.[5] We show that, at least from

[2] We further flesh out the connection to Crémer and McLean (1988) in the discussion that follows a formal description of the symmetric payment rule in Section II.

[3] For instance, in Riley and Samuelson (1981), the authors state, "An optimal auction extends the asymmetry of the buyer roles to the allocation rule itself. The assignment of the good and the appropriate buyer payment will depend not only on the list of offers, but also on the identities of the buyers who submit the bids. In short, an optimal auction under asymmetric conditions violates the principle of buyer anonymity."

[4] Such an agreement is also currently present in the World Trade Organization, which has replaced the GATT.

[5] Corns and Schotter (1999) experimentally test these arguments.

a theoretical perspective, such goals can be achieved without explicit discrimination by the auctioneer (Corollary 2).

That being said, there are limits to what the seller can achieve: Corollary 3 qualitatively describes the set of unimplementable direct mechanisms.[6] While non-generic (Corollary 1), these exceptions are economically interesting.[7] An example is the seller facing one (disadvantaged) weak and one strong buyer where the former has the same distribution as the latter with the support shifted down. Suppose the seller wanted to run an "efficient" auction that corrects for the difference. An "ideal" affirmative action policy would be to run a second-price auction where the bid of the weaker buyer was subsidized by a value equal to the difference in the (lower bound) of the supports. While such a format is clearly discriminatory, it follows from Corollary 3 that there is no other symmetric auction that can yield the same outcome.

Finally, we show that certain simple policies (which are easy to implement and enforce) in addition to symmetry can introduce real constraints on the seller. One restriction that we analyze is the prohibition of the seller from either charging or subsidizing losers in the auction (Proposition 1). In the online Appendix, we examine the effects of requiring the payment rule to have other desirable properties such as continuity or monotonicity in the bids. These are all features of most auctions actually employed in practice. We show that each of these additional requirements impose meaningful restrictions on the seller; in particular, she can no longer always maximize revenue.

Alternatively, practical constraints may restrict the seller. For instance, even though the symmetric implementations we construct are individually rational in an interim sense, they need not be individually rational in an ex post sense. Requiring a bidder to sometimes make a very large payment (for some bid realizations) may not be feasible in practice since bidders might be budget constrained or could simply refuse to pay. Once again, the online Appendix shows that when the seller is restricted to using a symmetric auction that is ex post individually rational, she may not be able to maximize revenue.

A. *Related Literature*

Our demonstration of the flexibility of symmetric auctions can be viewed as similar to the revelation principle. The revelation principle states that restricting the seller to direct mechanisms is without loss of generality. Analogously, our characterization shows that restricting the auctioneer to symmetric auction formats does not prevent him from achieving a wide variety of discriminatory goals. In this regard, our results are also similar in spirit to the recent work of Manelli and Vincent (2010) and Gershkov et al. (2013). These authors show that, in the independent private values model, any incentive compatible and individually rational outcome that can

[6] Allocations in nonimplementable mechanisms are to the buyer whose value has the highest "statistical rank" (the cumulative distribution evaluated at that value).

[7] The statistical rank is sometimes used as a criterion in college admissions. For instance, the "Texas Top 10" program guarantees admission to the University of Texas to any Texan high school student in the top ten percent of their class.

be achieved in Bayes-Nash equilibrium can also be achieved (in expectation) in dominant strategies. Thus, as with the case of dominant strategy implementation, the requirement of symmetric implementation is not restrictive in and of itself.

In a recent work, Azrieli and Jain (2015) consider a general mechanism design environment (which allows for interdependent values and correlated types) and, like us, characterize the set of direct mechanisms that have symmetric indirect implementations. The main difference is that they do not restrict the set of indirect mechanisms that the designer can employ. In the independent value auction environment we consider, implementation would be trivial if the seller were allowed to use such abstract mechanisms.[8] Our main insight is that the seller need not resort to such abstract mechanisms as she is essentially unconstrained even when restricted to the commonly employed class of symmetric sealed bid auctions. Moreover, additional constraints (such as continuity or monotonicity of the payments) on sealed bid auctions are easy to describe and enforce. Our results also have practical relevance as the legality of mechanisms that require bidders to explicitly reveal their identities can be easily challenged in court.

Finally, our paper is also related to the literature in market design that explicitly considers fairness. For example, several of the prominent matching mechanisms (for instance, for the deferred acceptance and Boston mechanisms, see Roth and Sotomayor 1992) and allocation mechanisms (for the random priority mechanism, see Abdulkadiroğlu and Sönmez 1998, and for the probabilistic serial mechanism, see Bogomolnaia and Moulin 2001) proposed for use in practice are anonymous. Even among anonymous designs, other notions of fairness such as the "equal treatment of equals" (two agents making the same reports receive the same allocations) and "envy-freeness" (each agent prefers her allocation to that of any other agent) are imposed in addition. Additionally, in the market design for school choice, the fact that existing designs can be "gamed" has been deemed unfair, since different socioeconomic groups may have varying abilities to game the system—see, for example, Pathak and Sönmez (2008), Abdulkadiroğlu et al. (2006). Similarly, some business schools have changed their course allocation procedures to ones that explicitly guarantee fairness (Budish 2011), motivated by the demonstrated unfairness of existing systems (Budish and Cantillon 2010).

I. The Model

We consider an independent private value auction setting. A set $N = \{1, 2, \ldots, n\}$ of risk-neutral buyers or bidders (used interchangeably) compete for a single indivisible object.[9] Buyer $i \in N$ draws a value $v_i \in V_i \equiv [\underline{v}_i, \overline{v}_i]$ independently from a distribution F_i. We assume that F_i is twice continuously differentiable with

[8] Consider a simple example: there are two bidders, i and j, and the seller would like to discriminate against the latter. The seller announces a mechanism in which the bidders are asked to announce both their identity and their value for the good, after which the buyer reporting i is favored. Bidders can always be incentivized to reveal their identity truthfully as misreports can be detected in equilibrium (both would have reported the same identity) and punished. Incentives to report values truthfully can be provided as usual. Further, this mechanism is symmetric since a permutation of messages would lead to the same permutation of outcomes.

[9] Equivalently, our model could be considered as a procurement setting where a firm or government wants a single project to be completed and solicits quotes from contractors, each of whom has an independent private cost.

corresponding density f_i which is strictly positive throughout the support $[\underline{v}_i, \overline{v}_i]$. Note that both V_i and F_i can be different across i, so we allow for ex ante heterogenous bidders. We denote $\mathbf{V} \equiv \times_{j \in N} V_j$ and $\mathbf{V}_{-i} \equiv \times_{j \neq i} V_j$, with $\mathbf{v} \in \mathbf{V}$ and $\mathbf{v}_{-i} \in \mathbf{V}_{-i}$ denoting typical elements of these sets. As with values, we use the notation $F \equiv \prod_{j \in N} F_j$ and $F_{-i} \equiv \prod_{j \neq i} F_j$. We will use similar notation for other vectors and vector-valued functions throughout the paper.

A *direct mechanism* asks bidders to report their values, and uses these reports to determine allocations and payments. Allocations are determined via an ordered list of functions:

(Direct Allocation)

$$a^d = \left(a_1^d, \ldots, a_n^d\right) \quad \text{where} \quad a_i^d : \mathbf{V} \to [0, 1] \text{ and } \sum_{i=1}^{n} a_i^d(\mathbf{v}) \leq 1.$$

Here, $a_i^d(\mathbf{v})$ is the probability that bidder i wins the auction when the profile of reported types is $\mathbf{v}$. The inequality above reflects the fact that the seller has a single unit to sell, so the probability of allocating it cannot exceed 1 at any profile $\mathbf{v}$. Additionally, this allows for the possibility that the seller may choose to withhold the good. Similarly, payments are determined via an ordered list of functions:

(Direct Payment) $$p^d = \left(p_1^d, \ldots, p_n^d\right) \quad \text{where} \quad p_i^d : \mathbf{V} \to \mathbb{R}.$$

Here, $p_i^d(\mathbf{v})$ is the payment made by bidder i when the profile of reported types is $\mathbf{v}$. Note that, when it is positive, this is a transfer to the seller, and, when it is negative, it is a subsidy from the seller. In addition, the bidder may be required to make payments even when she does not receive the object.

Values are private; that is, buyers do not know the realized valuations of other bidders. Hence, each bidder's expected utility from participating in this mechanism is determined by her expected allocation and payment. For a given direct mechanism $\left(a^d, p^d\right)$, we define *interim* allocations and payments to be the expected allocations and payments conditioned on truthful reporting by all the bidders. Formally, these are given by

(Interim Allocation) $$a_i^d(v_i) \equiv \int_{\mathbf{V}_{-i}} a_i^d(v_i, \mathbf{v}_{-i}) dF_{-i}(\mathbf{v}_{-i})$$

and

(Interim Payment) $$p_i^d(v_i) \equiv \int_{\mathbf{V}_{-i}} p_i^d(v_i, \mathbf{v}_{-i}) dF_{-i}(\mathbf{v}_{-i}).$$

For simplicity, we deliberately abuse notation by denoting interim allocations using the same symbol; the difference is determined by whether the argument is a single value or a value profile.

We make the additional standard assumption that the bidders are risk neutral and that their utilities are quasi-linear in the transfers. Conditional on truthful reporting by the other bidders, the interim expected utility for bidder i with value v_i who announces a value v_i' is simply

$$\text{(Bidder Utility)} \qquad v_i a_i^d(v_i') - p_i^d(v_i').$$

A mechanism $\left(a^d, p^d\right)$ is said to be (Bayesian) *incentive compatible* or simply IC if reporting truthfully is a Bayes-Nash equilibrium, i.e.,

$$\text{(IC)} \quad v_i a_i^d(v_i) - p_i^d(v_i) \geq v_i a_i^d(v_i') - p_i^d(v_i') \quad \forall i \in N, \forall v_i, v_i' \in V_i.$$

Myerson (1981) showed that incentive compatibility implies that the allocation rule a^d pins down the payments p^d up to constants $c_i \in \mathbb{R}$; that is,

$$\text{(Payoff Equivalence)} \qquad p_i^d(v_i) = v_i a_i^d(v_i) - \int_{\underline{v}_i}^{v_i} a_i^d(w)\, dw + c_i.$$

Additionally, a mechanism is said to be *individually rational* or simply IR if truthful reporting leads to a nonnegative payoff, or

$$\text{(IR)} \qquad v_i a_i^d(v_i) - p_i^d(v_i) \geq 0 \quad \forall v_i \in V_i.$$

A. *Symmetric Auctions*

We define a *symmetric auction* as a game with three properties: buyers simultaneously submit real numbers called bids; the winner is the highest bidder over a given reservation bid (ties are broken uniformly); and payments are determined via an anonymous payment function. This is an indirect sealed bid auction mechanism with the additional restriction that allocations and payments depend only on the profile of bids and not the identity of the bidders. Formally, in a symmetric auction, each bidder i chooses a bid $b_i \in \mathbb{R}$, and allocations and payments are determined by functions $a^s : \mathbb{R}^n \to [0, 1]$ and $p^s : \mathbb{R}^n \to \mathbb{R}$, respectively. Bidder i's allocation or simply her probability of winning the item is given by

(Symmetric Auction Allocation)

$$a^s(b_i, \mathbf{b}_{-i}) = \begin{cases} \dfrac{1}{\#\{j \in N : b_j = b_i\}} & \text{when } b_i \geq \max\{\mathbf{b}_{-i}, r\}, \\ 0 & \text{otherwise,} \end{cases}$$

where r is the reservation bid. As with the values, we use $\mathbf{b}$ and $\mathbf{b}_{-i}$ to denote the vector of all bids and the vector of all bids except that of bidder i, respectively.

Bidder i's payment is given by

$$\text{(Symmetric Auction Payment)} \qquad p^s(b_i, \mathbf{b}_{-i}),$$

where p^s is invariant to permutations of $\mathbf{b}_{-i}$ but can depend on the underlying distribution of values $(F_1, \ldots, F_n)$. Notice that, since the allocation and payment rules do not depend on the identity of the bidders, we only need a single function, as opposed to lists of functions, to define these mechanisms. Most commonly used auction formats, such as first-price, second-price, and all-pay auctions are symmetric in this sense.

In a symmetric auction, a pure strategy (henceforth, referred to simply as a strategy) for a bidder i is a mapping

$$\text{(Buyer Strategy)} \qquad \sigma_i : V_i \to \mathbb{R},$$

that specifies the bid corresponding to each possible value. A profile of strategies $\sigma = (\sigma_1, \ldots, \sigma_n)$ constitutes a (Bayesian Nash) equilibrium of the symmetric auction (a^s, p^s) if each buyer's strategy is a best response to the strategies of other buyers. Formally, this requires that, for all $i \in N$ and $v_i \in V_i$, we have

$$\sigma_i(v_i) \in \arg\max_{b \in \mathbb{R}} \int_{\mathbf{V}_{-i}} \left[v_i a^s(b, \sigma_{-i}(\mathbf{v}_{-i})) - p^s(b, \sigma_{-i}(\mathbf{v}_{-i})) \right] dF_{-i}(\mathbf{v}_{-i}).$$

As we discussed in the introduction, we restrict attention to this particular format as it is legal (it maintains buyer privacy by ensuring that they are not forced to reveal their identities via their bids); it allows for flexible design of the payment rules; and commonly used auctions fall in this class. However, a weakness is that we require common knowledge of the underlying value distributions so that, in particular, (unlike a second-price auction) bidders can compute their equilibrium bid. That said, this requirement is imposed in almost all auction theory and, in particular, is necessary for buyers to calculate equilibrium bids even in standard first-price auctions.

We say that an IC and IR direct mechanism (a^d, p^d) is *implemented* by a symmetric auction (a^s, p^s) if there is a pure strategy equilibrium in undominated strategies of the latter mechanism that yields the same allocation and expected payment as the former. Specifically, we say that a direct mechanism is implementable if there exists an undominated equilibrium strategy profile σ such that, for all $\mathbf{v} \in \mathbf{V}$,[10]

$$\text{(1A)} \qquad a_i^d(\mathbf{v}) = a^s(\sigma_i(v_i), \sigma_{-i}(\mathbf{v}_{-i}))$$

and

$$\text{(1B)} \qquad p_i^d(v_i) = \int_{\mathbf{V}_{-i}} p^s(\sigma_i(v_i), \sigma_{-i}(\mathbf{v}_{-i}))\, dF_{-i}(\mathbf{v}_{-i}).$$

In this notion of implementability, we require the equilibrium allocation of the symmetric auction to be identical to the direct mechanism for each profile of values but the payments to be equal in expectation. Additionally, note that we only require the

[10] We use the additional restriction of undominated equilibrium strategies to ensure that our symmetric implementation is not based on "implausible" buyer behavior.

conditions above to hold for one equilibrium of the symmetric auction. Hence, this is a "partial implementation" criterion.[11]

More generally, we say that an IC and IR direct mechanism $\left(a^d, p^d\right)$ is *implementable* if there exists a symmetric auction $\left(a^s, p^s\right)$ that implements it (almost sure and interim implementability are defined analogously). Our main goal is to show that restriction to a symmetric auction format does not constrain the seller, and we do so by explicitly characterizing the set of IC and IR direct mechanisms that are implementable.[12] To make the exposition cleaner, we have deliberately defined implementation only in terms of pure strategies for the bidders. This restriction does not affect any of the results in the paper. We show in the Appendix that allowing for mixed strategies does not expand the set of implementable mechanisms (or the set of implementable mechanisms subject to the additional conditions in Proposition 1 and in the online Appendix).

We also refer to two additional weaker implementation criteria. The first is *almost sure implementation*, which requires (1) to hold almost surely (over the distribution of buyer values). In other words, according to this criterion, the allocations and interim payments are the same except at a measure zero set of values. The second is *interim implementation*, which requires the allocation rule (as with the payment) to be implemented in an expected sense or that $a_i^d(v_i) = \int_{\mathbf{v}_{-i}} a^s\left(\sigma_i(v_i), \sigma_{-i}(\mathbf{v}_{-i})\right) dF_{-i}(\mathbf{v}_{-i})$. The recent work on the equivalence of Bayesian and dominant strategy implementability (Manelli and Vincent 2010, Gershkov et al. 2013) uses an even weaker notion that instead requires the expected utilities (as opposed to interim allocations and payments separately) of the agents to be the same.

II. Example: Implementing the Optimal Auction with Two Buyers

In this section, we explain our approach by describing a symmetric implementation of the revenue maximizing auction when there are two buyers. As we mentioned in the introduction, this is a natural example to demonstrate our techniques as the optimal direct mechanism is discriminatory and it has long been thought that there may not exist a symmetric implementation of the optimal auction. To simplify the example, we additionally assume that the distributions of both buyers satisfy the increasing virtual value property. Formally, this condition requires that, for each buyer $i \in N$, the virtual value

$$\text{(Virtual Value)} \qquad \phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

is increasing in v_i. An implication is that ϕ_i^{-1} is a single valued function.

[11] Given the fact that we allow for very general value distributions, it is perhaps unrealistic to expect a symmetric auction implementation to have a unique equilibrium. Note that even standard formats like the first or second-price auction can have multiple equilibria. This is because our setting is more general than even the fairly unrestrictive conditions required for uniqueness in first-price auctions (Lebrun 2006), and we do not require bidders to bid below their value (Kaplan and Zamir 2015).

[12] Since the additional requirement of IR only involves changing the payment rules by a constant, our characterization results can also be viewed as simply characterizing the set of IC direct mechanisms which are implementable.

We denote the allocation and payment rule of the optimal auction by (a^*, p^*). Recall that in the optimal auction, bidders announce their values and the mechanism awards the good to the bidder who has the highest positive virtual value (it is without loss to assume that ties are broken equally). Hence, when bidders draw their values from different distributions, this direct mechanism is not symmetric, as the allocation rule depends on the bidder specific value distribution.

A natural way to attempt a symmetric implementation of the optimal auction is to construct a payment rule such that it is an equilibrium for both bidders to bid their virtual values. The auction could then allocate the good to the higher bid and have a reservation bid of 0. We denote the set of virtual values of bidder i by

$$B_i \equiv [\phi_i(\underline{v}_i), \phi_i(\overline{v}_i)].$$

The distribution F_i over V_i induces a distribution G_i over the set B_i of virtual values.

We claim that the optimal auction can be implemented if we can construct a payment rule p^s that satisfies

$$p_i^*(v_i) = \int_{B_j} p^s(\phi_i(v_i), b_j)\, dG_j(b_j) \quad \text{for } i \neq j \text{ and all } v_i \in V_i.$$

This is simply a restatement of the implementability requirement where equilibrium strategies of bidding the virtual value have been substituted in. This claim is easy to see:

(i) Suppose that buyer i with value v_i bids $b_i \in B_i$ but $b_i \neq \phi_i(v_i)$. This is equivalent to her reporting a value $\phi_i^{-1}(b_i) \neq v_i$ in the direct mechanism (a^*, p^*), which yields a lower payoff because the optimal auction is IC.

(ii) Suppose that buyer i with value v_i bids $b_i \notin B_i$. This can be detected with positive probability by the auctioneer when the other bidder is bidding truthfully. This is because there will be a positive measure of bids b_j such that $(b_i, b_j) \notin (B_1 \times B_2) \cup (B_2 \times B_1)$. Such off-equilibrium bids can be discouraged by making the payments high enough at these bids.

We now construct such a symmetric payment rule. Since it is easy to discourage bids that lie outside the support of the virtual values, the payment rule is deliberately defined only for equilibrium bid profiles $(b_i, b_j) \in (B_1 \times B_2) \cup (B_2 \times B_1)$. We separately construct the payment for bids that lie in the supports of only one and both virtual value distributions, respectively. In equilibrium, bids $b_i \in B_i \backslash B_j$ are made only by buyer i. Hence, for such bids, we can simply define the payment rule to be the interim payment from the optimal auction or

$$p^s(b_i, b_j) = p_i^*(\phi_i^{-1}(b_i)) \quad \text{when} \quad b_i \in B_i \backslash B_j \text{ and } b_j \in B_j.$$

To construct the payments for bids $b_i \in B_1 \cap B_2$ that lie in the support of both virtual value distributions, we first observe that, for asymmetric buyers $(F_1 \neq F_2)$, there exists a $\hat{b} \in \mathbb{R}$ such that $G_1(\hat{b}) \neq G_2(\hat{b})$. In other words, different value distributions yield different virtual values distributions. Consider the payment rule

$$p^s(b_i, b_j) = \begin{cases} p^u(b_i) & \text{if } b_j \geq \hat{b} \text{ and } b_j \in B_1 \cup B_2 \\ p^l(b_i) & \text{if } b_j < \hat{b} \text{ and } b_j \in B_1 \cup B_2, \end{cases} \tag{2}$$

where

$$p^u(b_i) = \frac{p_1^*(\phi_1^{-1}(b_i))\, G_1(\hat{b}) - p_2^*(\phi_2^{-1}(b_i))\, G_2(\hat{b})}{G_1(\hat{b}) - G_2(\hat{b})} \tag{3}$$

$$p^l(b_i) = \frac{p_2^*(\phi_2^{-1}(b_i))\,[1 - G_2(\hat{b})] - p_1^*(\phi_1^{-1}(b_i))\,[1 - G_1(\hat{b})]}{G_1(\hat{b}) - G_2(\hat{b})}.$$

The amount a buyer pays ex post given this payment rule depends on both bids—her own and the opponent's. More specifically, it depends on the exact amount of her bid, and whether or not her opponent's bid is above or below $\hat{b}$. A bidder i who bids b_i pays an amount $p^u(b_i)$ when her opponent bids higher than $\hat{b}$ and an amount $p^l(b_i)$ when her opponent's bid is lower than $\hat{b}$.

Given the putative equilibrium bidding strategies, this particular auction exhibits a flavor of the side bets of Crémer and McLean (1988) that we alluded to in the introduction. Since the payment of a buyer also depends on whether or not her competitor bids above or below $\hat{b}$, her expected payment depends on the probability she assigns to her competitor bidding above and below $\hat{b}$. Buyer 1 understands that she is competing against buyer 2, and therefore assigns probabilities $1 - G_2(\hat{b})$ and $G_2(\hat{b})$ to each of these events, while buyer 2 assigns probabilities $1 - G_1(\hat{b})$ and $G_1(\hat{b})$. As a result, the expected payments of a buyer, for the same bid, differ according to whether she is bidder 1 (and therefore taking expectations based on the fact that she is competing against bidder 2, whose bid distribution is G_2) or bidder 2.

Further, note that the expected payment of a bidder i who bids $b_i \in B_1 \cap B_2$ when bidder j bids $\phi_j(v_j)$ for all $v_j \in V_j$ is

$$p^u(b_i)\,[1 - G_j(\hat{b})] + p^l(b_i)\, G_j(\hat{b}) = p_i^*(\phi_i^{-1}(b_i)), \tag{4}$$

which is precisely the required payment for implementation.

Notice also that equation (4) can be used to derive the expressions for p^u and p^l. An equivalent matrix representation is the following system for $b_i \in B_1 \cap B_2$:

$$\mathcal{M} \begin{bmatrix} p^u(b_i) \\ p^l(b_i) \end{bmatrix} = \begin{bmatrix} p_1^*\big(\phi_1^{-1}(b_i)\big) \\ p_2^*\big(\phi_2^{-1}(b_i)\big) \end{bmatrix}, \quad \text{where } \mathcal{M} = \begin{bmatrix} 1 - G_2(\hat{b}) & G_2(\hat{b}) \\ 1 - G_1(\hat{b}) & G_1(\hat{b}) \end{bmatrix}. \tag{5}$$

By definition, $G_1(\hat{b}) \neq G_2(\hat{b})$ implies that $\mathcal{M}$ is a full rank matrix. Therefore (5) has a solution for all $b_i \in B_1 \cap B_2$, and p^u, p^l can be obtained by inverting $\mathcal{M}$. Observe that this condition bears some resemblance to the full rank condition required by Crémer and McLean (1988) for full surplus extraction.

In summary, the symmetric payment rule that implements the optimal auction in this example is

$$p^s(b_i, b_j) = \begin{cases} p^u(b_i) & \text{if } b_i \in B_1 \cap B_2,\ b_j \geq \hat{b} \text{ and } b_j \in B_1 \cup B_2 \\ p^l(b_i) & \text{if } b_i \in B_1 \cap B_2,\ b_j < \hat{b} \text{ and } b_j \in B_1 \cup B_2 \\ p_1^*(\phi_1^{-1}(b_i)) & \text{if } b_i \in B_1 \backslash B_2 \text{ and } b_j \in B_2 \\ p_2^*(\phi_2^{-1}(b_i)) & \text{if } b_i \in B_2 \backslash B_1 \text{ and } b_j \in B_1. \end{cases}$$

The following numerical example illustrates this construction.

Example 1: Consider a setting with two buyers. Buyer 1 has a value that is uniformly distributed over $[2, 4]$, while buyer 2's value is uniformly distributed over $[1, 2]$. The seller wants to conduct a symmetric implementation of the optimal auction. In this setting, the virtual value of buyer 1 is $\phi_1(v_1) = 2v_1 - 4$, and the virtual value of buyer 2 is $\phi_2(v_2) = 2v_2 - 2$. Therefore, buyer 1's virtual value (bid) is uniformly distributed over $B_1 \equiv [0, 4]$, while buyer 2's is uniformly distributed over $B_2 \equiv [0, 2]$.

We begin by deriving the interim payments. These can be determined using (Payoff Equivalence) as follows:

$$p_1^*(v_1) = v_1 a_1^*(v_1) - \int_2^{v_1} a_1^*(w)\, dw = v_1 \min\{v_1 - 2, 1\} - \int_2^{v_1} \min\{w - 2, 1\} dw$$

$$= \begin{cases} \frac{v_1^2}{2} - 2 & \textit{for } v_1 \in [2, 3] \\ \frac{5}{2} & \textit{for } v_1 \in (3, 4] \end{cases}$$

and

$$p_2^*(v_2) = v_2 a_2^*(v_2) - \int_1^{v_2} a_2^h(w)\, dw$$

$$= v_2 \left[\frac{v_2 - 1}{2}\right] - \int_1^{v_2} \left[\frac{w - 1}{2}\right] dw = \frac{v_2^2 - 1}{4} \quad \textit{for } v_2 \in [1, 2].$$

Interim payments expressed in terms of bids are

$$(6)\qquad p_1^*\big(\phi_1^{-1}(b_1)\big) = \begin{cases} \frac{b_1^2}{8} + b_1 & \text{for } b_1 \in [0,2] \\ \frac{5}{2} & \text{for } b_1 \in (2,4] \end{cases}$$

and

$$(7)\qquad p_2^*\big(\phi_1^{-1}(b_2)\big) = \frac{b_2^2}{16} + \frac{b_2}{4} \quad \text{for } b_2 \in [0,2].$$

Consider now $\hat{b} = 1$, for which we have that $G_1(\hat{b}) = \frac{1}{4}$ and $G_2(\hat{b}) = \frac{1}{2}$. This choice of $\hat{b}$ yields

$$p^u(b_i) = -\frac{b_i}{2} \quad \text{and} \quad p^l(b_i) = \frac{5b_i}{2} + \frac{b_i^2}{4},$$

from which we can define the symmetric payment rule for equilibrium bids:

$$p^s(b_i, b_j) = \begin{cases} -\frac{b_i}{2} & \text{if } b_i \in [0,2] \text{ and } b_j \in [1,4] \\ \frac{5b_i}{2} + \frac{b_i^2}{4} & \text{if } b_i \in [0,2] \text{ and } b_j \in [0,1) \\ \frac{5}{2} & \text{if } b_i \in (2,4] \text{ and } b_j \in [0,2]. \end{cases}$$

Finally, observe that this choice of $\hat{b}$ implies that there are bids for which the winner could win the auction and, in addition, receive a subsidy (as $p^u(\cdot) < 0$).

Our main result in the next section builds on the intuition in this example. The key difficulty in a symmetric implementation is that the same bid, when made by different bidders, must lead to the appropriate, potentially different interim payments. For this to be the case, the payment rule needs to be designed in a way that utilizes the difference in the distribution of the equilibrium bids of each bidder. In this example, we simply had to charge different amounts depending on whether the opponent's bid was above or below $\hat{b}$. The proof of the main result contains the substantially harder generalization of this construction to n bidders.

III. Implementation via Symmetric Auctions

In this section, we present and discuss the main result—a characterization of implementable IC and IR direct mechanisms. A constructive approach to determining whether a particular direct mechanism is implementable requires first the design of a symmetric auction and then a derivation of its equilibrium. However, deriving equilibria for a given symmetric auction can be a hard task. For instance, it is well known that it is difficult to obtain closed-form solutions for equilibrium bids in the first-price auction for arbitrary distributions. We simplify our task by showing that the set of implementable mechanisms is a subset of the set of hierarchical mechanisms. This allows us to use an argument similar to that of the example in the previous section.

We begin by defining *hierarchical allocation rules*.[13] These are generated by an ordered list $I = (I_1, \ldots, I_n)$ of *index functions* that are nondecreasing mappings $I_i : V_i \to \mathbb{R}$ for $i \in N$. A hierarchical allocation rule is generated from a given list of index functions I as follows

(Hierarchical Allocation)

$$a_i^h(\mathbf{v}) = \begin{cases} \frac{1}{\#\{j \in N : I_j(v_j) = I_i(v_i)\}} & \text{when } I_i(v_i) \geq \max\{I_{-i}(\mathbf{v}_{-i}), 0\} \\ 0 & \text{otherwise.} \end{cases}$$

Each buyer's value is transformed into an index via the index function. The good is then allocated to the buyer with the highest positive index, and ties are broken equally. Restricting allocations to buyers with positive indices is essentially equivalent to setting a reservation bid. Choosing a reserve of 0 for the index functions is without loss of generality, as all bids can always be moved up or down by a constant. In addition, note that index functions can be chosen so that allocations occur above different reservation values across the buyers.

A *hierarchical mechanism* (I, p^h) is an IC and IR mechanism that consists of index functions I and payment functions p^h. The allocation a^h is determined as shown above from the index functions. For the results that follow, we find it convenient to denote a hierarchical mechanism in terms of the index functions I as opposed to the allocation rule a^h. If two lists of index functions I and I' generate the same allocation rule a^h, then it must be that one is a monotone transformation of the other. Formally, if I and I' generate the same allocation a^h, then there exists a monotone function $\Gamma : \mathbb{R} \to \mathbb{R}$ such that $I_i(v_i) = \Gamma(I_i'(v_i))$ for all i and v_i. The particular choice of index functions that correspond to a given allocation a^h does not matter for the statement of any of our results.

Since the index functions are nondecreasing, having a higher value implies a weakly higher probability of winning. This implies that every hierarchical allocation rule a^h has associated IC transfers p^h (pinned down to constants) that yield a hierarchical mechanism. All mechanisms in applied mechanism design that we are aware of fall within the class of hierarchical mechanisms (we provide examples of nonhierarchical mechanisms below). In the efficient Vickrey auction, values serve as indices or $I_i(v_i) = v_i$, and in the optimal auction (with increasing virtual values) the indices are given by the virtual values or $I_i(v_i) = \phi_i(v_i)$. When the virtual values are not increasing, the index functions are simply the "ironed" virtual value functions (Myerson 1981). Alternatively, suppose an auctioneer with affirmative action concerns wants to "subsidize" a historically disadvantaged bidder i over a bidder j where the latter has index $I_j(v_j) = v_j$. The index for bidder i could

[13] This term was introduced by Border (1991).

reflect either a flat subsidy $I_i(v_i) = v_i + s$ (where $s > 0$) or a percentage subsidy $I_i(v_i) = s\,v_i$ (where $s > 1$).

We show below that any implementable mechanism must effectively be a hierarchical mechanism. This allows us to focus on this smaller class of mechanisms, which in turn simplifies the implementation task, as in the previous section. Since the allocation rule of a symmetric auction that implements a hierarchical mechanism must allocate the good to the bidder with the highest index, a natural assumption is to make equilibrium bids correspond to the index values. Then, constructing the symmetric implementation essentially boils down to finding a symmetric payment rule that yields the same interim payments. Given a hierarchical mechanism (I, p^h), the distribution F_i on the set of values V_i induces a distribution G_i on the set of indices or bids

(Bid Space) $$B_i \equiv \{I_i(v_i) \mid v_i \in V_i\}.$$

At times, we will slightly abuse notation and use G_i as both a distribution and a measure. The meaning will be clear depending on whether the argument of G_i is a real number or a set. The notation G_i deliberately suppresses the dependence on the index function I_i; the meaning will always be clear from the context. Since index functions I are not necessarily strictly increasing, the induced distributions G_i may have atoms. Additionally, notice that the set B_i need not be an interval because the index functions I may be discontinuous.

A hierarchical allocation mechanism (I, p^h) can be implemented if we can find a symmetric payment function p^s such that

$$(\star) \qquad p_i^h(v_i) = \int_{\mathbf{B}_{-i}} p^s(I_i(v_i), \mathbf{b}_{-i})\, dG_{-i}(\mathbf{b}_{-i}) \qquad \text{for all } i \in N \text{ and } v_i \in V_i.$$

If such a symmetric payment function exists, it follows that an equilibrium of the symmetric auction with this payment rule will involve each buyer i with value v_i bidding their index $I_i(v_i)$. By construction, such bids generate the required allocation.

The intuition is straightforward and identical to that of the example. Suppose that a bidder with value v_i makes a bid $b_i' \in B_i$ other than her index so $b_i' \neq I_i(v_i)$. Her corresponding allocation and payment would be identical to what she would get by reporting a value $v_i' \in I_i^{-1}(b_i')$, resulting in lower utility as the direct mechanism (I, p^h) is IC.[14] Off-equilibrium bids $b_i' \notin B_i$, which lie outside the bid space, can be punished by requiring high expected payments at these bids.

We now present our main result as two separate theorems.

THEOREM 1: *Suppose that a direct revelation mechanism (a^d, p^d) is implementable. Then, there exists an implementable hierarchical mechanism (I, p^h) such that its implementation is an almost sure implementation of (a^d, p^d).*

[14] Here, $I_i^{-1}(\cdot)$ is the correspondence defined by $I_i^{-1}(b_i) = \{v_i \in V_i \mid I_i(v_i) = b_i\}$.

Theorem 1 says that it is essentially without loss to restrict attention to hierarchical mechanisms. It states that, for any implementable direct mechanism, there is an implementable hierarchical mechanism that almost surely has exactly the same allocation and payments. An implication is that, for any implementation of any nontrivial objective, a principal can restrict attention to hierarchical mechanisms. This result is intuitive. Clearly, a nonhierarchical mechanism cannot have an implementation in pure strategies because, if so, the allocation rule could have been generated by an index rule with indices equal to the equilibrium bids in the symmetric auction. The Appendix contains the argument for mixed strategies.

Theorem 2 provides conditions that characterize the set of implementable hierarchical mechanisms.[15]

THEOREM 2: *A hierarchical mechanism* (I, p^h) *is implementable if and only if, for any pair of distinct buyers* $i, j \in N$ *who have the same distribution of bids* $(G_i = G_j)$, *and any pair of values for these two buyers* $v_i \in V_i$, $v_j \in V_j$ *satisfying* $I_i(v_i) = I_j(v_j)$, *we have that* $p_i^h(v_i) = p_j^h(v_j)$.

We can decompose this into two parts:

(i) Whenever bid distributions G_i differ across the buyers, the condition of the theorem is vacuously satisfied, and therefore, it is possible to construct a payment rule so that $(\star)$ is satisfied. When there are two bidders, a payment rule like the one in the previous section can be used to construct the implementation. The construction for more that two bidders is considerably more complicated and can be found in the Appendix.

(ii) When two buyers are such that the two induced bid distributions are the same, that is $G_i = G_j$, then the interim payments must be the same for any two values (one for each buyer) that correspond to the same index. This is because it is no longer possible to generate different equilibrium expected payments for distinct buyers who make the same bid.

Player i uses the distribution G_{-i} of other players' bids to calculate her expected payment as a function of her bid. The construction in the Appendix shows that there is a way to exploit the differences in distributions G_{-i}, such that for each bidder i, the expected payment when bidding b_i is exactly the interim payment in the original direct revelation mechanism corresponding to values v_i satisfying $I_i(v_i) = b_i$. (The example in Section II demonstrates such a construction for the two bidder case.) Case (i) above says that any asymmetry in the direct mechanism can be undone as long as the bid distributions differ across buyers. Conversely, case (ii) says that differences in the bid distributions are necessary to generate different interim payments for the same bids.

[15] The theorem is actually slightly stronger. The conditions are also necessary and sufficient for (the weaker criterion of) interim implementability of a hierarchical mechanism.

The set of hierarchical mechanisms that do not satisfy the conditions above is small; in fact, the set of implementable mechanisms is generic in a topological sense, formalized below.

For each buyer i, the distribution F_i defines a measure space on V_i. Consider the space of index functions for buyer i, as an L_p space where $1 \leq p \leq \infty$. The space of index functions $I = (I_1, \ldots, I_n)$ is topologized with the product topology and denoted $\mathcal{I}$. Since a finite product of complete normed vector spaces is a Baire space, standard topological notions of genericity are well defined. Recall that a property is said to be *generically satisfied* on a topological space if the set that does not satisfy it is a meager set (or conversely, the set that does satisfy it is a residual set). Further, recall that a set in a topological space is meager if it can be expressed as the union of countably many nowhere dense subsets in that space.

COROLLARY 1: *Generically, on $\mathcal{I}$, $G_i \neq G_j$ for every pair of buyers i and j.*[16]

The intuition and proof for this result are straightforward: two index functions that result in the same distribution over bids can be made different by slightly perturbing them. The fact that a large number of disparate objectives can be achieved either exactly or arbitrarily closely via a symmetric implementation is the main insight of this paper, and it is worth repeating its implications. A policy that simply forces an auctioneer to treat the bids of different buyers similarly need not imply fairness as it does not imply that the resulting outcomes are equal from an ex ante perspective. Careful auction design can allow the seller to achieve a wide variety of discriminatory goals in environments where explicit favoritism is prohibited. In particular, counter to prevailing intuition,[17] the following corollary points out that the seller can always maximize revenue.

COROLLARY 2: *The optimal auction can be implemented.*

It is worth stressing that the corollary above requires no hazard rate assumptions on the value distributions. When the distributions satisfy the increasing virtual value property, it is easy to show that if the bidders are asymmetric, the distribution over virtual values must also be different. When the virtual values are not increasing, then the proof of the corollary shows that if the distributions over the "ironed" virtual values are the same, then the condition of the theorem must be satisfied.

[16] It is possible to restate this corollary to say instead that implementability is a generic property in the space of hierarchical mechanisms (instead of in the space of index functions that do not include payments). We have deliberately chosen not to do so to avoid the distracting technicalities inherent in defining the appropriate topology on the space of hierarchical mechanisms. The complications arise from the fact that the index functions restrict the payments (up to constants) via incentive compatibility, so we cannot simply employ a product topology over index functions and payments.

[17] For instance, in an influential paper, Cantillon (2008) conjectured that bidder asymmetries hurt the auctioneer in any anonymous mechanism after showing that this is not the case in the optimal auction. Corollary 2 answers this conjecture in the negative by showing that the optimal auction can be implemented by an anonymous mechanism.

IV. The Limits to Implementability

The conditions in Theorem 2 were on the distributions of the bid space. The following corollary qualitatively describes the types of hierarchical allocation rules that cannot be implemented.

COROLLARY 3: *Suppose that a hierarchical mechanism (I, p^h) is not implementable. Then there must exist two distinct buyers j and j' such that their index functions can be written*

$$\text{for } i = j, j' : \; I_i(v_i) = \Gamma(F_i(v_i)) \text{ for almost every } v_i \in V_i,$$

f*or some nondecreasing function* $\Gamma(\cdot)$.

The corollary above demonstrates that the only non-implementable hierarchical mechanisms are ones where there are two buyers whose indices corresponding to a value depend solely on the "statistical rank" of that value in the distribution of that buyer's values.

First, note that Corollary 3 does not state that all mechanisms with this property are non-implementable, that is, this condition is necessary but not sufficient. For a mechanism to be non-implementable, two buyers with values that correspond to the same statistical rank must also need to make different interim payments (as mentioned in the remarks following Theorem 2). There are hierarchical mechanisms that satisfy the condition of Corollary 3 but have a symmetric auction implementation. For instance, in the efficient auction with ex ante identical bidders, the good is given to the bidder with the highest value (and hence statistical rank), but it can still be implemented by a symmetric, second-price auction.

The corollary above identifies exactly those hierarchical mechanisms for which the resulting distribution over indices is the same. In particular, two buyers have the same distribution over indices if and only if, for each buyer, the mapping from value to index depends solely on the statistical rank. Why? Since the distribution of indices are identical, this implies that the mass of values who bid below a given index is the same for each of these buyers. Also, hierarchical mechanisms have nondecreasing index rules (to satisfy IC). These two facts in conjunction imply that if the values of these two buyers have the same statistical rank, they must make the same bid.

Interestingly, there are real world examples (in non-auction contexts) where the statistical rank is a criterion that is utilized. A prominent such example is the "Texas Top 10" program, which guarantees admission to the University of Texas to any Texan high school student in the top 10 percent of their class. This policy was instituted after Texas outlawed explicit race-based affirmative action. While race-blind on the surface, this policy is well understood as a second best form of affirmative action, since relatively segregated neighborhoods imply that minority students are concentrated in some high schools.[18]

[18] Chan and Eyster (2003); Fryer, Loury, and Yuret (2008) demonstrate the theoretical extent to which universities can achieve diversity goals via "race-blind" admission criteria.

The following two examples demonstrate why hierarchical mechanisms with allocation rules as in Corollary 3 may not be implementable. In the first example, the good is allocated randomly and in the second, the seller would like to subsidize one of the buyers.

Example 2: There are two buyers. Buyer 1 has a value uniformly distributed on $[0, 1]$. Buyer 2 has a value uniformly distributed on $[0.5, 1]$. The seller assigns the good at random (with equal probability) to each of the two buyers irrespective of their value. Buyer 1 is never asked to pay anything, whereas buyer 2 is always asked to pay 0.25.

Notice that this mechanism is a hierarchical mechanism where each bidder's index function is a constant nonnegative function, or, $I_1(v_1) = I_2(v_2) \geq 0$ *for all* $v_1 \in [0, 1]$ and $v_2 \in [0.5, 1]$. In terms of Corollary 3, Γ in this case is the constant mapping. Here, the bid space just consists of a single point, and distributions G_1, G_2 are degenerate and therefore satisfy $G_1 = G_2$. However, the payments differ. Therefore, this mechanism violates the conditions of Theorem 2. It follows that there is no symmetric implementation of this direct revelation mechanism.

Example 3: Consider an environment where there are two buyers. Buyer 1 has a value v_1 that is uniformly distributed on $[0, 1]$. Buyer 2 has a value v_2, which is uniformly distributed on $[1, 2]$.

Suppose that the seller would like to "subsidize" the bid of buyer 1 by a dollar. Put differently, buyer 2 wins the good if and only if his value exceeds that of buyer 1 by 1. Therefore, for any $v_1 \in [0, 1]$, the interim allocation probabilities are given by

$$a_1^h(v_1) = a_2^h(1 + v_1).$$

The IC and IR payments are chosen to be such that the lowest type of both buyers for whom there is no probability of winning neither make payments nor are paid. This is clearly a hierarchical mechanism with index functions $I_1(v_1) = I_2(v_1 + 1)$, where $I_1(\cdot)$ is strictly increasing on the interval $[0, 1]$. In terms of Corollary 3, Γ in this case is the identity mapping.

Observe that this implies that the distributions over the bid spaces are identical, since G_1 and G_2 are both $U[0, 1]$. Moreover, incentive compatibility pins down payments, and therefore we have

$$p_2^h(v_1 + 1) = p_1^h(v_1) + a_1^h(v_1).$$

For all values $v_1 \in (0, 1]$, therefore, the equation above implies that

$$p_2^h(v_1 + 1) \neq p_1^h(v_1).$$

Since the interim payments differ for values that have the same index and the bid spaces have identical distributions, symmetric payments cannot be constructed to implement this mechanism.

We end this example with a few remarks. First, observe that the argument above does not depend on the fact that the value distributions are uniform: the allocation rule would remain non-implementable (using the identical argument) with any other distribution. What matters is the fact that the buyers have the same distributions with shifted supports and that the seller's goal is to subsidize the weaker bidder to exactly make up the difference. Consequently, note that this mechanism could have been implemented if buyer 2's value distribution was even slightly different than that of buyer 1 as this would imply that $G_1 \neq G_2$. Similarly, implementability could be restored if the seller changed the subsidy amount to any value that did not make up the exact difference.

A. *Inactive Losers*

Policymakers may want to prevent the seller from discriminating to an extent beyond the restrictions of Corollary 3. What then can be done? We argue that one avenue can be to impose certain simple restrictions in addition to symmetry on the auction format. Such additional restrictions can introduce real constraints on the seller. Importantly, such policies are easy to implement and enforce. In what follows, we describe the impact of one such possible restriction: the seller is prohibited from either charging or subsidizing losers in the auction. In the online Appendix, we consider instead requiring the payment rule to have properties such as continuity or monotonicity in the bids. We show that each of these additional requirements impose meaningful restrictions on the seller; in particular, she can no longer always maximize revenue.

An important property of first-price and second-price auctions is that losers neither make nor receive payments. With the notable exception of charity auctions (see, for instance, Goeree et al. 2005), most auctions conducted in the real world have this feature. It is often argued that requiring the loser to pay reduces participation, which is one of the reasons that all-pay auctions are seldom used in practice. Hence, this might be construed as a shortcoming of Theorem 2: the implementation that we construct may not have this property.

A hierarchical mechanism has a symmetric, *inactive losers implementation* (a^s, p^s) if $p^s(b_i, b_j) = 0$ whenever $b_i < b_j$. Note that such an implementation may sometimes require the winner to make payments that are greater than his value and thus may not be ex post IR (we consider the ex post IR requirement in Section 1.3 of online Appendix). We now state a condition that is necessary and sufficient for there to exist such an implementation. For simplicity and brevity, we restrict attention to the case of two bidders as this is sufficient to make our point.

Consider a hierarchical mechanism (I, p^h) that induces distributions G_1 and G_2 on the set of bids. This mechanism satisfies the *inactive losers condition* if, for all $\tilde{b}$ such that there is a constant $\alpha > 0$ for which $G_1(b) = \alpha G_2(b)$ for all $b \leq \tilde{b}$, we have $\alpha p_1^h(v_1) = p_2^h(v_2)$ for all $v_i \in I_i^{-1}(\tilde{b})$.

The necessity of this condition for an inactive losers implementation is intuitive. Consider a bid $\tilde{b}$ for which $G_1(b) = \alpha G_2(b)$ for all $b \leq \tilde{b}$. For any $v_i \in I_i^{-1}(\tilde{b})$, the interim payments for any inactive losers implementation must satisfy

$$p_2^h(v_2) = \int_{b_1 \leq \tilde{b}} p^s(\tilde{b}, b_1)\, dG_1(b_1) = \int_{b_2 \leq \tilde{b}} p^s(\tilde{b}, b_2)\alpha\, dG_2(b_2) = \alpha p_1^h(v_1).$$

The next proposition argues that this condition is also sufficient. The sufficiency follows from a similar construction to that utilized in Theorem 2.

PROPOSITION 1: *Suppose that* $n = 2$. *An implementable hierarchical mechanism* (I, p^h) *has an inactive losers implementation if and only if the induced bid distributions* G_1 *and* G_2 *satisfy the inactive losers condition.*

PROOF:

We begin by showing necessity. Suppose the inactive losers condition does not hold. This implies that there is $\tilde{b}$ such that $G_1(b) = \alpha G_2(b)$ for all $b \leq \tilde{b}$ but $\alpha p_1^h(\tilde{b}) \neq p_2^h(\tilde{b})$. By observation it must be the case that $\underline{b}_1 = \underline{b}_2 = \underline{b}$.

If there is an inactive losers implementation $p(b_i, b_j)$, we have the following:

$$p_2^h(\tilde{b}) = \int_{\underline{b}}^{\tilde{b}} p(\tilde{b}, b')\, dG_1(b') = \alpha \int_{\underline{b}}^{\tilde{b}} p(\tilde{b}, b')\, dG_2(b') = \alpha p_1^h(\tilde{b}),$$

which is a contradiction. Notice that the equation above follows from the fact that an inactive loser implementation by definition requires that the losing bidder make or receive no payments.

We now show sufficiency. We construct the payment rule for an arbitrary $\tilde{b}$, for each of the two cases of the inactive losers condition.

Case 1: There is a $\min\{\underline{b}_1, \underline{b}_2\} < b^* < \tilde{b}$ such that $\frac{G_1(\tilde{b})}{G_2(\tilde{b})} \neq \frac{G_1(b^*)}{G_2(b^*)}$. This implies that the matrix

$$\begin{bmatrix} G_1(\tilde{b}) - G_1(b^*) & G_1(b^*) \\ G_2(\tilde{b}) - G_2(b^*) & G_2(b^*) \end{bmatrix}$$

has full rank. This in turn means that following system of equations (8) with variables x, y has a solution:

$$(8) \qquad \begin{aligned} p_1^h(\tilde{b}) &= x[G_2(\tilde{b}) - G_2(b^*)] + yG_2(b^*) \\ p_2^h(\tilde{b}) &= x[G_1(\tilde{b}) - G_1(b^*)] + yG_1(b^*). \end{aligned}$$

Finally, setting

$$p(\tilde{b}, b') = \begin{cases} x & \text{for } b^* \leq b' < \tilde{b} \\ y & \text{for } b' < b^* \\ 0 & \text{otherwise} \end{cases}$$

results in the desired interim payments.

Case 2: There is a constant $\alpha > 0$ such $G_1(b) = \alpha G_2(b)$ for all $\min\{\underline{b}_1, \underline{b}_2\} \leq b \leq \tilde{b}$.

In this case, the condition implies that $\alpha p_1(\tilde{b}) = p_2(\tilde{b})$. We can simply set

$$p(\tilde{b}, b') = \begin{cases} \frac{p_2(\tilde{b})}{G_1(\tilde{b})} & \text{for } b' < \tilde{b} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, in both cases, the corresponding construction results in the desired interim payments and is an inactive losers implementation. ∎

We should point out that the inactive losers condition is generically satisfied in the sense of Corollary 1. Intuitively, this is because any mechanism that does not satisfy it can be converted to one that does by slightly perturbing it at the lower bound of the support of the bid distribution. Put differently, almost all hierarchical mechanisms have an inactive losers implementation. A natural way to further restrict the seller would be to require that the winner, in addition, must always make a positive payment or, put differently, cannot receive subsidies (which is a feature of all commonly used formats). To see the impact of this, observe that it would require the system of equations (8) in the construction above to have a positive solution. Intuitively, in this particular construction, a system that does not have a positive solution cannot to altered to one that does by minor alterations of the bid distribution.

That said, we now revisit Example 1 to show that the inactive losers condition is not economically vacuous in that it can be violated in the optimal auction.

Example 1 (Continued): Recall that buyer 1 has a value that is uniformly distributed over $[2,4]$, while buyer 2's value is uniformly distributed over $[1,2]$ and the seller wants to maximize revenue. The distributions of virtual values are $G_1 \sim U[0,4]$ and $G_2 \sim U[0,2]$, respectively. Therefore, for any $\tilde{b} \in [0,2]$, $G_1(b) = 0.5\,G_2(b)$ for all $b \leq \tilde{b}$.

Now consider $\tilde{b} = 2$. From (6) and (7), the interim payments at this bid are $p_1^*\big(\phi_1^{-1}(2)\big) = \frac{5}{2}$ and $p_2^*\big(\phi_2^{-1}(2)\big) = \frac{3}{4}$. Note that the payment of buyer 1 is not twice that of buyer 2, so the inactive losers condition is not satisfied.

We end this section by observing that practical constraints (without additional explicit restrictions) may restrict the seller's ability to discriminate. For instance, even though the symmetric implementations we construct are individually rational in an interim sense, they need not be individually rational in an ex post sense. Requiring a bidder to sometimes make a very large payment (for some bid realizations) may not be feasible in practice since bidders might be budget constrained or could simply refuse to pay. Once again, the online Appendix shows that when the seller is restricted to using a symmetric auction that is is ex post individually rational, she may not be able to maximize revenue.

V. Concluding Remarks

In this paper, we have introduced the problem of policy design to prevent discrimination in auctions. We have shown that, in theory, the current antidiscrimination policies in place are completely unrestrictive as symmetric auctions can implement virtually all possible seller goals. That said, we have argued that the government can meaningfully restrict sellers by placing additional, easy to enforce restrictions on the set of allowable formats.

We end with a few avenues for future research. Practical auctions often have rules that are "simple" to explain to bidders. Regulators could codify and require auctioneers to use simple formats. Such a simplicity constraint could be an effective way to restrain sellers. Needless to say, one of the challenges in designing policies favoring simple auctions is a formal definition of simplicity.

Finally, auctions conducted in different contexts often have idiosyncratic "quirks" in the formats.[19] Even among auctions that are based around the more standard formats, the seller has freedom to choose supplementary details such as activity rules, bid qualification, etc. Can these be designed to achieve a discriminatory motive and, if so, how can this be detected by regulators?

For instance, in spectrum auctions, the seller decides how to split the bandwidth for sale. It is well understood that, in general, smaller bandwidths are substitutes for incumbents who already own a large amount of spectrum but are complements for new entrants (see, for instance, Binmore and Klemperer 2002). Thus, by his choice, the seller can implicitly favor certain companies. Klemperer (2002) has several accounts of sellers carefully exercising such design flexibility to favor new entrants to the local telecom market. While such motives may be justified, this flexibility can also be used by corrupt bureaucrats to favor preferred companies. Detection of such favoritism is important in countries with high levels of corruption such as India where there has been historical malpractice in government auctions (and, in particular, in the allocation of spectrum).

Appendix: Proofs from Section III

A. *Proof of Theorem 1*

Fix a direct revelation mechanism (a^d, p^d), and suppose it has a symmetric implementation (a^s, p^s) where buyers use strategies σ. We are left to construct a symmetrically implementable hierarchical mechanism with the desired property.

Pure Strategies.—First suppose the mechanism (a^d, s^d) has an implementation in pure strategies. In this case, note that the hierarchical mechanism defined as

$$I_i(v_i) \;=\; \sigma_i(v_i),\; p_i^h(v_i) \;=\; p_i^d(v_i),$$

has a symmetric implementation that (exactly) implements the direct revelation mechanism.

Mixed Strategies.—So now suppose the implementation of the direct revelation mechanism is in mixed strategies. Fix the symmetric auction game. A mixed

[19] An example of a nonstandard design used in practice is the eponymous Amsterdam auction, used for real estate sales there, which awards money to the highest losing bidder (Goeree and Offerman 2004).

strategy equilibrium in this setting is a mapping for each buyer i, $\sigma_i : V_i \to \Delta\mathbb{R}$, that is, buyer i with value v_i randomizes over bids with probability measure $\sigma_i(v_i)$.

A little notation is useful. Given that buyer i's values are distributed according to F_i, and that when he has value v_i, he randomizes over bids with measure $\sigma_i(v_i)$, let G_i^r denote the implied distribution over bids.

Let us denote by $\breve{a}_i(b_i)$ the interim winning probability of buyer i when he bids b_i, with associated interim payment $\breve{p}_i(b_i)$. Note that $\breve{a}_i(b_i)$ is nondecreasing in b_i for all buyers i.

The following observation shows that the bids over which different values of a given buyer randomize are disjoint and ordered.

OBSERVATION 1: *For any buyer i and values $v_i < v_i'$, the support of the distributions of bids by these two values is completely ordered*, *i.e.*,

$$\max\{\breve{a}_i(b_i) : b_i \in \operatorname{supp}(\sigma_i(v_i))\} \leq \min\{\breve{a}_i(b_i) : b_i \in \operatorname{supp}(\sigma_i(v_i'))\}.$$

PROOF:

Firstly, note that if buyer i with value v_i mixes over bids $b_i < b_i'$ with $\breve{a}_i(b_i) < \breve{a}_i(b_i')$, then he must be indifferent between these bids. Therefore, $\breve{p}_i(b_i') - \breve{p}_i(b_i) = v_i(\breve{a}_i(b_i') - \breve{a}_i(b_i))$, implying that v_i' cannot be indifferent between both these bids.

So now suppose buyer i with value $v_i' > v_i$ has an equilibrium bid b_i'' with $\breve{a}_i(b_i'') < \breve{a}_i(b_i')$. Combining the equilibrium constraints that v_i prefers to bid b_i' than b_i'' and that v_i' prefers to bid b_i'' than b_i', we have a contradiction. The observation follows. ∎

OBSERVATION 2: *For any buyer i*, *the set of values $v_i \in V_i$ such that*

$$\exists b_i, b_i' \in \operatorname{supp}(\sigma_i(v_i)) : \breve{a}_i(b_i) \neq \breve{a}_i(b_i')$$

has F_i measure 0.

PROOF:

By Observation 1, we have that the support of distribution of bids for a given buyer is effectively disjoint. Therefore, at most a countable number of values for any buyer can have two bids with different interim probabilities of winning in their support, since the range of $\breve{a}_i(\cdot)$ is $[0, 1]$ and the reals can have an at most countable set of positive length intervals. Since F_i is differentiable, the measure of a countable set of values is 0.

Finally consider the hierarchical mechanism (I, p^h) constructed as follows. Fix a buyer i. Some buyer types may be following properly mixed strategies. These can be separated into two parts.

(i) By Observation 2, there are an at most countable set of buyer types who randomize over different values which result in different probabilities of getting the good. For each of these values, define $I_i(v_i) = b_i$ for some $b_i \in \operatorname{supp}(\sigma_i(v_i))$, with $p_i^h(v_i) = \int_{b_{-i}} p^s(b_i, b_{-i})\, dG_{-i}^r(b_{-i})$.

(ii) Finally the remaining types for any buyer i can be partitioned into intervals, such that all types in each interval receive a constant probability of winning. Any randomization by any value in this interval must be such that for any two distinct bids $b_i < b_i'$, such that $b_i, b_i' \in \text{supp}(\sigma_i(v_i))$, $\breve{a}_i(b_i) = \breve{a}_i(b_i')$. It follows that $\breve{a}_i(\cdot)$ is constant on $[b_i, b_i']$. This implies that there cannot be a positive measure of values of other buyers $-i$ that submit bids in the interval $[b_i, b_i']$. For any value in each such interval, $[\underline{v}, \overline{v}]$,[20] define $I_i(v) = b$ for some b within the smallest interval within $\cup_{v \in [\underline{v}, \overline{v}]} \text{supp}(\sigma_i(v_i))$ which carries all the mass. Further, define $p_i^h(v) = p_i^d(v)$.

For each remaining value v_i, $\sigma_i(v_i)$ is a single point, and we define $I_i(v_i) = \sigma_i(v_i)$, $p_i^h(v_i) = p_i^d(v_i)$.

The hierarchical mechanism (I, p^h) has a symmetric implementation by construction. Further, by construction, it achieves the same ex post allocation and interim payment almost surely (that is, for all buyer values other than the countable set of buyers in point (i) above). Therefore this implementation is an almost sure implementation of the original direct revelation mechanism. ∎

B. *Proof of Theorem 2*

Sufficiency.—At a high level, we generalize the ideas in our construction of the two bidder example in Section II. Recall that our goal is to construct a symmetric auction game that implements a hierarchical mechanism (I, p^h) with corresponding allocation rule a^h.

We construct a symmetric auction game that has a pure strategy Bayes-Nash equilibrium in which buyer i with value v_i reports $I_i(v_i)$. By construction, therefore, the allocation of this mechanism equals a^h. We are left to show that:

(i) as constructed, this auction game implements the desired payments p^h;

(ii) for each buyer i, bidding according to $I_i(\cdot)$ constitutes a Bayes-Nash equilibrium of the construction auction.

Step 1: Preliminaries.—Our goal is to show that we can construct a symmetric $p^s : \mathbb{R}^n \to \mathbb{R}$, such that

$$\text{(A1)} \qquad \forall i, \forall v_i \in V_i, p_i^h(v_i) = \int_{\mathbf{B}_{-i}} p^s(I_i(v_i), \mathbf{b}_{-i})\, dG_{-i}(\mathbf{b}_{-i}).$$

In other words, we need to show that we can construct a p^s such that each buyer i's expected payment, in expectation over candidate equilibrium bids of other buyers, equals $p_i^h(\cdot)$.

[20] Note that while this is written as a closed interval, the interval may be closed, open, or half open, half closed.

Step 2: Full Rank Events.—We say that an event $E \subseteq \mathbb{R}^{n-1}$ is *symmetric* if

$$\text{for every permutation } \rho : \{1, 2, \ldots, n-1\} \to \{1, 2, \ldots, n-1\},$$

$$(b_1, b_2, \ldots, b_{n-1}) \in E \Rightarrow (b_{\rho(1)}, b_{\rho(2)}, \ldots, b_{\rho(n-1)}) \in E.$$

We start with a simple observation.

OBSERVATION 3: *Consider $k \leq n$ symmetric events $E_1, E_2, \ldots, E_k \subseteq \mathbb{R}^{n-1}$ and define the $k \times k$ matrix*

$$\mathcal{M} \equiv [G_{-i}(E_j)]_{i,j=1}^{k}.$$

If matrix $\mathcal{M}$ is full rank, then there exists a symmetric payment rule p^s such that

$$\text{(A2)} \quad \forall i = 1, \ldots, k, \forall v_i \in V_i : p_i^h(v_i) = \int_{\mathbf{B}_{-i}} p^s(I_i(v_i), \mathbf{b}_{-i})\, dG_{-i}(\mathbf{b}_{-i}).$$

In particular, if $k = n$, then there exists a payment rule p^s that satisfies (A1).

PROOF:

Define the payment rule this way: there are k numbers associated with each bid $b \in \mathbb{R}$, denote the jth number $\pi_j(b)$. Suppose a bid b is made by a buyer, and other buyers make the profile of bids $\mathbf{b}_{-}$. For each event E_j that occurs among other buyers' bids, i.e., $\mathbf{b}_{-} \in E_j$, the buyer is asked to pay $\pi_j(b)$. Formally, the payment function is defined as

$$\text{(A3)} \qquad p^s(b, \mathbf{b}_{-}) = \sum_{j=1}^{k} \pi_j(b)\, \chi_{\{\mathbf{b}_{-} \in E_j\}},$$

where χ is the characteristic function. Note that since each of the E_j's are symmetric (by assumption), the payment rule defined thus is symmetric as well.

Given this definition of p^s, the expected payment made by buyer i bidding $b_i \in B_i$ when all other buyers are bidding according to their candidate equilibrium strategies is

$$\int_{\mathbf{B}_{-i}} p^s(b_i, \mathbf{b}_{-i})\, dG_{-i}(\mathbf{b}_{-i})$$

$$= \int_{\mathbf{B}_{-i}} \left(\sum_{j=1}^{k} \pi_j(b_i)\, \chi_{\{\mathbf{b}_{-i} \in E_j\}} \right) dG_{-i}(\mathbf{b}_{-i})$$

$$= \sum_{j=1}^{k} \pi_j(b_i)\, G_{-i}(E_j).$$

By the full rank assumption, for any $b \in \mathbb{R}$, there exists a solution $\pi(b) \in \mathbb{R}^k$ to the system of equations:

$$\mathcal{M}\pi(b) = \tilde{p}(b), \tag{A4}$$

$$\text{where } \tilde{p}(b) = [\tilde{p}_1(b), \ldots, \tilde{p}_k(b)]^T;$$

$$\tilde{p}_i(b) = \begin{cases} p_i^h(I_i^{-1}(b)) & \text{if } b \in B_i \\ \max_{i \in N}\{\overline{v}_i\} & \text{otherwise.} \end{cases} \tag{A5}$$

Therefore, the payment rule defined using $\pi(\cdot)$ that satisfies this system of equations satisfies (A2). When $k = n$, the constructed system satisfies (A1).

Theorem 3 shows that there always exist such events. ∎

THEOREM 3: *For any $n > 1$ and any $k \leq n$ such that $G_1, G_2, \ldots, G_k$ are all pairwise distinct, there exist symmetric events $E_1, \ldots, E_k \subseteq \mathbb{R}^{n-1}$ such that the $(k \times k)$ matrix $\mathcal{M} = [G_{-i}(E_j)]_{i,j=1}^k$ has full rank.*

A proof of the theorem is deferred to subsection F in this Appendix.

Step 3: Matching Payments.—First, consider the case where $G_i \neq G_{i'}$ for all $i \neq i'$. Then, by Theorem 3, there exist symmetric events $E_1, E_2, \ldots, E_n \subseteq \mathbb{R}^{n-1}$ such that the $n \times n$ matrix $\mathcal{M} = [G_{-i}(E_j)]$ is full rank. Therefore, by Observation 3, we can construct a symmetric payment rule p^s that matches the desired interim payment rule p^h when all buyers make their candidate equilibrium bids, i.e., satisfies (A1).

Now to consider the other case, i.e., there exist i, i' such that $G_i = G_{i'}$. Note that if $G_i = G_{i'}$ for some $i \neq i'$, then $G_{-i} = G_{-i'}$.

We define N_U as the set of "distributionally unique buyers." Formally, for any induced distribution over bids, G defines $N_G = \{i \in N : G_i = G\}$. Now we can define $N_U = \cup_{i \in N}\{\min\{N_{G_i}\}\}$. In other words, N_U is the largest subset of N such that for any distinct $i, i' \in N_U$, $G_i \neq G_{i'}$. Renumber the buyers so that the first $|N_U|$ buyers are distributionally unique. By Theorem 3, we can construct full row rank events for these buyers. We are then done, because by assumption, if $G_i = G_{i'}$ we have that $p_i^h(I_i^{-1}(b)) = p_{i'}^h(I_{i'}^{-1}(b))$.

Step 4: Equilibrium.—We have already shown that if each buyer followed the candidate equilibrium strategy, the desired payment rule p^h is implemented. We are left to show that following the candidate strategy (i.e., that buyer i with value v_i bids $I_i(v_i)$) is a Bayes-Nash equilibrium of the game.

Consider buyer i, with value v_i. His candidate equilibrium bid is $b_i = I_i(v_i)$. Let us divide possible deviations into two types:

(i) buyer i bids $b_i' \in B_i$;

(ii) buyer i bids $b_i' \notin B_i$.

Since the original mechanism (a^h, p^h) is Bayes Incentive Compatible, it should be clear that deviations of type (i) cannot be profitable. If player i with value v_i deviates to some other $b_i' = I_i(v_i')$, then assuming all other players are playing their equilibrium strategies, player i will win the good with probability $a_i^h(v_i')$ and make an expected payment of $p_i^h(v_i')$. Incentive compatibility of the original direct revelation mechanism guarantees that:

$$v_i a_i^h(v_i) - p_i^h(v_i) \geq v_i a_i^h(v_i') - p_i^h(v_i').$$

By construction (A4, A5), deviations of type (ii) will require the buyer to make an expected payment of $\max_{i \in N}\{\overline{v}_i\}$ and hence, such deviations cannot be profitable.

Therefore, our candidate equilibrium strategies constitute a Bayes-Nash equilibrium of the symmetric auction game we constructed, concluding our proof of sufficiency. ∎

Necessity.—We now show that our condition is necessary for there to exist a symmetric implementation. Let us consider a hierarchical allocation rule with index functions $I_1, \ldots, I_n$ such that for buyers 1 and 2, $G_1 = G_2$.

Firstly, note that any other index function I' that implements the same allocation rule must be a strictly monotone transform of I. Therefore the resulting distributions will be such that $G_1' = G_2'$. It is therefore without loss to only check whether there exists an implementation corresponding to the "original" index rule I.

Pick v_1, v_2 such that $I_1(v_1) = I_2(v_2)$, and $p_1^h(v_1) \neq p_2^h(v_2)$. Note that $a_1^h(v_1) = a_2^h(v_2)$ since $G_1 = G_2 \Rightarrow G_{-1} = G_{-2}$.

Recall that a symmetric implementation in pure strategies is a symmetric payment rule p^s, such that for all buyers i and all valuations v_i in V_i,

$$p_i^h(v_i) = \int_{\mathbf{B}_{-i}} p^s(I_i(v_i), \mathbf{b}_{-i})\, dG_{-i}(\mathbf{b}_{-i}).$$

Since $G_1 = G_2$, the product distributions G_{-1} and G_{-2} are also the same. Therefore, for any b,

$$\int_{\mathbf{B}_{-1}} p^s(b, \mathbf{b}_{-1})\, dG_{-1}(\mathbf{b}_{-1}) = \int_{\mathbf{B}_{-2}} p^s(b, \mathbf{b}_{-2})\, dG_{-2}(\mathbf{b}_{-2}).$$

For $b = I_1^{-1}(v_1) \left(= I_2^{-1}(v_2)\right)$, we have the required contradiction.

Mixed Strategies.—We now argue that allowing for mixed strategies does not expand the set of implementable mechanisms.

Consider a mixed implementation such that the resulting distribution over bids of buyer i is G_i^r. It follows from Observation 2 that $G_1 = G_2$ implies $G_1^r = G_2^r$. As a result, $G_{-1}^r = G_{-2}^r$.

Suppose a hierarchical mechanism (I, p^h) cannot be implemented in pure strategies. Then without loss of generality, $G_1 = G_2$ and there are values v_1 and v_2 such that condition (ii) of Theorem 2 is violated. For interim implementation in mixed strategies, we must have that

$$a_i^h(v_i) = \int_{B_i} \breve{a}_i(b_i)\, d\sigma_i(v_i)(b_i);$$

$$p_i^h(v_i) = \int_{B_i} \breve{p}_i(b_i)\, d\sigma_i(v_i)(b_i).$$

Now suppose (without loss of generality) that there is a mixed strategy symmetric implementation of the case where $G_1 = G_2$, $p_1^h(v_1) > p_2^h(v_2)$ and $I_1(v_1) = I_2(v_2)$. Then, buyer 1 with value v_1, strictly prefers the strategy $\sigma_2(v_2)$ over $\sigma_1(v_1)$ (since $a_1^h(v_1) = a_2^h(v_2)$ by assumption), contradicting the assumption that these strategies constitute an equilibrium. ∎

C. *Proof of Corollary 1*

For each buyer i, consider the index function I_i as a point in the L_1 space (the same argument works with any L_p norm) with respect to the measure space defined by measure F_i on V_i. The space I is topologized by the product topology. Since this is finite product of complete normed vector spaces, it is a Baire space, and therefore standard topological notions of genericity are well defined.

To see the desired result, first note that by Corollary 3, the condition of Theorem 2 is violated only if there exists a nondecreasing function Γ and bidders $j, j' \in N$ such that for any $i = j, j'$, $I_i(v_i) = \Gamma(F_i(v_i))$ for almost all $v_i \in V_i$. Consider the set E_S defined on sets $S \subseteq N$, which is a subset of the set of index rules, defined as:

$$E_S = \{I : \text{ for all } i \in S, I_i(v_i) = \Gamma(F_i(v_i)) \text{ almost everywhere}\}.$$

We show that $E := \cup_{S \subset N} E_S$ is a meager set.

First note that E is closed since it is the finite union of closed sets. Each set E_S is closed since the limit of any sequence of Is that violates condition of Corollary 3 for the subset S will also violate this condition for S.

Thus, if we can show that E has a nonempty interior, it will be nowhere dense and we are done. The following lemma delivers this result.

LEMMA 1: *Consider any hierarchical mechanism $I \in E$. Then, for any $\epsilon > 0$, there is an index rule I' which satisfies condition (i) such that for each buyer i,*

$$\int_{V_i} |I_i(v) - I_i'(v)|\, dF_i(v) \leq \epsilon.$$

PROOF:

For any ϵ small, consider an increasing function $X_j^\epsilon : V_j \to [0, \epsilon]$, such that $F_j(\{v_j | X_j^\epsilon(v_j) \neq 0\}) \neq 0$. Define $I_j' = I_j + X_j^\epsilon$. Clearly one can select X_j^ϵ for each j such that $I' \notin E$. ∎

D. *Proof of Corollary 2*

Recall from Myerson (1981) that if the function $\phi_i(v_i)$, defined as

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)},$$

is nondecreasing in v_i, then the allocation rule for the optimal auction is defined by the hierarchical allocation rule with the index rule ϕ_i for buyer i. If ϕ_i is not nondecreasing, then the optimal allocation rule is given by the "ironed" virtual value $\overline{\phi}_i$. Let G_i be the distribution over bids of buyer i induced by $\overline{\phi}_i$.

The following simple lemma shows if two buyers (without loss of generality 1 and 2) have the same distribution of (possibly ironed) virtual values, then the two buyers also have the same function mapping value into virtual value. Therefore, the hierarchical allocation rule implementing the optimal auction either induces different distributions of virtual values, or if not, then this lemma shows it satisfies the condition of Theorem 2. The corollary follows.

LEMMA 2: *Suppose two buyers are such that* $G_1 = G_2$. *Then* $V_1 = V_2$ *and* $\overline{\phi}_1 = \overline{\phi}_2$.

PROOF:

Define $v_i(b) = \overline{\phi}_i^{-1}(b)$ for $b \in B_1$. Since $\overline{\phi}_i(\cdot)$ need not be strictly increasing, it follows that $\overline{\phi}_i^{-1}(\cdot)$ is a correspondence. Define $\underline{v}_i(b) = \inf \overline{\phi}_i^{-1}(b)$ and $\overline{v}_i(b) = \sup \overline{\phi}_i^{-1}(b)$.

Since $\overline{\phi}_i$ is nondecreasing, it follows that

$$G_i(b) = F_i(\overline{v}_i(b)).$$

There can be at most a countable number of pooling intervals in $\overline{\phi}_i$ (see Myerson 1981, section 6). Each of these pooling intervals correspond to an atom in G_i. We denote the set of atomic bids by $\mathcal{B}_i \subseteq B_i$; denote by β_{in} the bid that corresponds to the nth atom in G_i; and the size of the atom is denoted by

$$\varsigma_{in} = F_i(\overline{v}_i(\beta_{in})) - F_i(\underline{v}_i(\beta_{in})).$$

Since $\overline{\phi}_i$ is differentiable everywhere else, therefore so is $v_i(\cdot)$ whenever it is a singleton. For any $b \in B_i \backslash \mathcal{B}_i$, differentiating we have that

$$g_i(b) = f_i(v_i(b))\, v_i'(b).$$

For any $b \in B_i \backslash \mathcal{B}_i$, we know that $\phi_i(v_i(b)) = \overline{\phi}_i(v_i(b))$, and therefore by the definition of ϕ_i

$$v_i(b) - \frac{1 - F_i(v_i(b))}{f_i(v_i(b))} = b$$

$$\text{(A6)} \qquad \Rightarrow v_i(b) - \frac{1 - G_i(b)}{g_i(b)} v_i'(b) = b.$$

OBSERVATION 4: *Consider any interval* $[\underline{b}, \overline{b}]$ *in the support of G such that there are no atoms in this interval. Further, suppose* $v_1(\overline{b}) = v_2(\overline{b})$. *Then,* $v_1(b) = v_2(b)$ *for every* $b \in [\underline{b}, \overline{b}]$.

PROOF:

From (A6), we know that for each $b \in [\underline{b}, \overline{b}]$, and $i = 1, 2$,

$$v_i(b) - \frac{1 - G_i(b)}{g_i(b)} v_i'(b) = b$$

$$\Rightarrow v_1(b) \lesseqqgtr v_2(b) \Leftrightarrow v_1'(b) \lesseqqgtr v_2'(b),$$

where the implication follows from the fact that $G_1 = G_2$. Therefore, if $v_1(b) \neq v_2(b)$ for some $b \in [\underline{b}, \overline{b}]$, it cannot be that $v_1(\overline{b}) = v_2(\overline{b})$.

For any $\beta_{in} \in \mathcal{B}_i$, the "ironed" virtual value pools all buyers in $[\underline{v}_i(b), \overline{v}_i(b)]$. Therefore,

$$\text{(A7)} \qquad \beta_{in} = \frac{\int_{\underline{v}_i(\beta_{in})}^{\overline{v}_i(\beta_{in})} \phi_i(v) f_i(v)\, dv}{F_i(\overline{v}_i(\beta_{in})) - F_i(\underline{v}_i(\beta_{in}))}$$

$$= \underline{v}_i(\beta_{in}) - (\overline{v}_i(\beta_{in}) - \underline{v}_i(\beta_{in})) \frac{1 - F_i(\overline{v}_i(\beta_{in}))}{F_i(\overline{v}_i(\beta_{in})) - F_i(\underline{v}_i(\beta_{in}))}$$

$$= \underline{v}_i(\beta_{in}) - (\overline{v}_i(\beta_{in}) - \underline{v}_i(\beta_{in})) \frac{1 - G_i(\beta_{in})}{\varsigma_{in}}.$$

Since $G_1 = G_2$, both have the same (at most countable set of) atoms—we denote the set of atoms $\mathcal{B}$ with generic element β_n of "size" ς_n.

OBSERVATION 5: *Consider any atom* $\beta_n \in \mathcal{B}$ *of size* ς_n, *and suppose that* $\overline{v}_1(\beta_n) = \overline{v}_2(\beta_n)$. *Then, we have that* $\underline{v}_1(\beta_n) = \underline{v}_2(\beta_n)$, *i.e.*, $v_1(\beta_n) = v_2(\beta_n)$.

PROOF:

By (A7), we have for $i = 1, 2$

$$\beta_n = \underline{v}_i(\beta_n) - (\overline{v}_i(\beta_n) - \underline{v}_i(\beta_n))\frac{1 - G_i(\beta_n)}{\varsigma_n}$$

$$= \underline{v}_i(\beta_n)\left(1 + \frac{1 - G_i(\beta_n)}{\varsigma_n}\right) - \overline{v}_i(\beta_n).$$

Therefore, if $\overline{v}_1(\beta_n) = \overline{v}_2(\beta_n)$, then $\underline{v}_1(\beta_n) = \underline{v}_2(\beta_n)$.

Finally, letting $\overline{b}$ be the upper bound of the support of $G_1 (= G_2)$, note that by definition:

$$v_1(\overline{b}) = v_2(\overline{b}) = \overline{b}.$$

The fact that $v_1(\cdot) = v_2(\cdot)$ now follows from this initial condition and Observations 4 and 5. Therefore, $G_1 = G_2 \Rightarrow \overline{\phi}_1 = \overline{\phi}_2$. ∎

E. *Proof of Corollary 3*

Without loss of generality, consider only buyers 1 and 2. Since the auction does not have a symmetric implementation, it must be the case that $G_1 = G_2$. First, consider the case that index functions I_1 and I_2 are continuous.

Suppose v_1, v_2 are such that $F_1(v_1) = F_2(v_2)$, but $I_1(v_1) > I_2(v_2)$—if no such v_1, v_2 exists, we are done. Define

$$v_1' = \max\{v \in V_1 : I_1(v) = I_2(v_2)\}.$$

By continuity of I_1, v_1' exists. By monotonicity of I_1, $v_1' < v_1$. By assumption, $G_1(I_1(v_1')) = G_2(I_2(v_2))$. Combining, we have that

$$F_1(v_1) > F_1(v_1') = G_1(I_1(v_1')) = G_2(I_2(v_2)) \geq F_2(v_2),$$

implying that $F_1(v_1) > F_2(v_2)$. This contradicts our assumption that $F_1(v_1) = F_2(v_2)$.

Now suppose I_1 and I_2 are not necessarily continuous. The common support must lie on an at most countable collection of intervals and at most countable atoms. For any point in the interior of any interval in the support of G_1, and any atom, the argument above shows that

$$\text{for } i = 1, 2: \ I_i(v_i) = \Gamma(F_i(v_i)),$$

for any v such that $I_1(v_1)$ is in the interior of an interval in the support of G_1 or an atom on G_1. This leaves only measure 0 end points of the intervals, of which there are an at most countable set. These correspond to discontinuities in the index rules, which are also at most countable. ∎

F. *Full Rank Events*

A little more notation will be useful. We say that an event $E \subseteq \mathbb{R}^{n-1}$ is of type l if there exists a $\beta \in \mathbb{R}$ such that E is the event "l randomly chosen buyers out of the $n-1$ have bids of β or less." For any number k, let $[k] \equiv \{1, 2, \ldots, k\}$. For any set K, $|K| = k$, $l \leq k$, define

$$\binom{K}{l} \equiv \{X : X \subseteq K, |X| = l\},$$

that is, the set of all subsets of K of cardinality exactly l.

By definition, if E is an event of type l with corresponding β, then

$$G_{-i}(E) = \frac{l!(n-1-l)!}{(n-1)!} \sum_{M \in \binom{N \setminus i}{l}} \prod_{j \in M} G_j(\beta). \tag{A8}$$

We also allow for an event of type l to have a random cutoff $\tilde{\beta} \in \Delta\mathbb{R}$. This corresponds to the event that there are l randomly chosen buyers out of the $n-1$ and each of them has a bid less than an i.i.d. realization of the random variable $\tilde{\beta}$. Denote by $G_j(\tilde{\beta})$ the probability that a draw according to G_j is less than or equal to the random variable $\tilde{\beta}$.

Note that if we have an event E of type l with corresponding cutoff $\tilde{\beta}$,

$$G_{-i}(E) = \frac{l!(n-1-l)!}{(n-1)!} \sum_{M \in \binom{N \setminus i}{l}} \prod_{j \in M} G_j(\tilde{\beta}). \tag{A9}$$

Recall the theorem:

THEOREM 1: *For any $n > 1$ and any $k \leq n$ such that $G_1, G_2, \ldots, G_k$ are all pairwise distinct, there exist symmetric events $E_1, \ldots, E_k \subseteq \mathbb{R}^{n-1}$ such that the $(k \times k)$ matrix $\mathcal{M} = [G_{-i}(E_j)]_{i,j=1}^k$ has full rank.*

PROOF:

Fix the number of buyers $n > 1$. We will prove the lemma by induction on k.

Base Case.—$k = 2$. Since $G_1 \neq G_2$, pick $\beta^\star \in \mathbb{R}$ such that $G_1(\beta^\star) \neq G_2(\beta^\star)$. Now pick E_1 to be an event of type 1 with cutoff $\beta^\star$, and $E_2 = \mathbb{R}^{n-1} \setminus E_1$. The corresponding matrix is

$$\mathcal{M} = \begin{bmatrix} \frac{1}{n-1}\sum_{i \neq 1} G_i(\beta^\star) & 1 - \frac{1}{n-1}\sum_{i \neq 1} G_i(\beta^\star) \\ \frac{1}{n-1}\sum_{i \neq 2} G_i(\beta^\star) & 1 - \frac{1}{n-1}\sum_{i \neq 2} G_i(\beta^\star) \end{bmatrix}.$$

By observation, this is full rank.

Inductive Hypothesis.—Suppose this is true for all $k \leq \hat{k}$ for some $\hat{k} < n$.

Inductive Step.—We will show this is true for $k = \hat{k}+1$. By the inductive hypothesis, we have events $E_1, \ldots, E_{\hat{k}} \subseteq \mathbb{R}^{n-1}$ such that

$$\mathcal{M} = [G_{-i}(E_j)]_{i,j=1}^{\hat{k}} \text{ is full rank.}$$

We need to show that we can find a $E_{\hat{k}+1}$ such that

$$\mathcal{M}' = [G_{-i}(E_j)]_{i,j=1}^{\hat{k}+1} \text{ is full rank.}$$

Note that since $\mathcal{M}$ is full rank, there exists a unique row-vector $\alpha \in \mathbb{R}^{\hat{k}}$ such that:

$$\alpha\mathcal{M} = \left[G_{-(\hat{k}+1)}(E_1), G_{-(\hat{k}+1)}(E_2), \ldots, G_{-(\hat{k}+1)}(E_{\hat{k}})\right].$$

If it is not the case that

$$\sum_{i=1}^{\hat{k}} \alpha_i = 1,$$

then we are already done. To see, this note that we can select $E_{\hat{k}+1} = \mathbb{R}^{n-1}$. With this selection, $\mathcal{M}'$ will be full rank, since $G_{-i}(\mathbb{R}^{n-1}) = 1$ for all i by definition, and therefore

$$\sum_{i=1}^{\hat{k}} \alpha_i G_{-i}(\mathbb{R}^{n-1}) \neq G_{-(\hat{k}+1)}(\mathbb{R}^{n-1}).$$

We will now proceed to prove that there exists an event such that M' is full rank. In particular, we will show that either $\mathbb{R}^{n-1}$ suffices or there must exist an event of type 1 to $\hat{k}$. So suppose that for any event $E_{\hat{k}+1}$ of type 1, the matrix $\mathcal{M}'$ is not full rank. For any event of type 1 with corresponding cutoff β, by (A8)

$$G_{-i}(E_{\hat{k}+1}) = \frac{1}{n-1} \sum_{j=1, j\neq i}^{n} G_j(\beta).$$

Since by assumption no such event $E_{\hat{k}+1}$ results in a full rank matrix, we have that for all $E_{\hat{k}+1}$ of type 1 with corresponding β,

$$G_{-(\hat{k}+1)}(E_{\hat{k}+1}) = \sum_{i=1}^{\hat{k}} \alpha_i G_{-i}(E_{\hat{k}+1})$$

$$\Rightarrow \forall \beta \in \mathbb{R},\ G_{(\hat{k}+1)}(\beta) = \sum_{i=1}^{\hat{k}} \alpha_i G_i(\beta).$$

As notational shorthand, we will write this as

$$G_{\hat{k}+1} = \sum_{i=1}^{\hat{k}} \alpha_i G_i .$$

CLAIM 1: *Suppose* $\hat{l} \leq \hat{k}$ *is such that for all* $l = 1, \ldots, \hat{l}$, *selecting* $E_{\hat{k}+1}$ *from events of types* 1 *to* $\hat{l}$ *cannot make* $\mathcal{M}'$ *full rank. Then, for all* $l = 1, \ldots, \hat{l}$:

$$(G_{\hat{k}+1})^l = \sum_{i=1}^{\hat{k}} \alpha_i (G_i)^l, \tag{A10}$$

and

$$\sum_{M \in \binom{[\hat{k}]}{l}} \Big(1 - \sum_{i \in M} \alpha_i\Big) \prod_{i \in M} G_i = \Bigg(\sum_{M \in \binom{[\hat{k}]}{l}} \prod_{i \in M} G_i\Bigg) - G_{\hat{k}+1} \sum_{M \in \binom{[\hat{k}]}{l-1}} \Big(1 - \sum_{i \in M} \alpha_i\Big) \prod_{i \in M} G_i . \tag{A11}$$

Recall that (A10) is notational shorthand for

$$\forall \tilde{\beta} \in \Delta \mathbb{R} : \ \big(G_{\hat{k}+1}(\tilde{\beta})\big)^l = \sum_{i=1}^{\hat{k}} \alpha_i \big(G_i(\tilde{\beta})\big)^l .$$

PROOF OF CLAIM 1:

We prove this claim by induction on $\hat{l}$. While the base case $\hat{l} = 1$ is true by observation, to build intuition we will now prove it for the case of $\hat{l} = 2$. Since by assumption no event of type 2 produces a full rank matrix, it must be that for every event E of type 2,

$$G_{-(\hat{k}+1)}(E) = \sum_{i=1}^{\hat{k}} \alpha_i G_{-i}(E).$$

Substituting in from (A8), and canceling terms, we have

$$\begin{aligned}
\sum_{M \in \binom{[k]}{2}} \prod_{i \in M} G_i &= \sum_{q=1}^{\hat{k}} \alpha_q \Bigg(\sum_{M \in \binom{[\hat{k}+1] \setminus q}{2}} \prod_{i \in M} G_i \Bigg) \\
&= \sum_{q=1}^{\hat{k}} \alpha_q \Bigg(\sum_{M \in \binom{[\hat{k}] \setminus q}{2}} \prod_{i \in M} G_i + G_{\hat{k}+1} \sum_{i=1, i \neq q}^{\hat{k}} G_i \Bigg) \\
&= \sum_{M \in \binom{[\hat{k}]}{2}} \Big(1 - \sum_{i \in M} \alpha_i\Big) \prod_{i \in M} G_i + G_{\hat{k}+1} \sum_{i=1}^{\hat{k}} (1 - \alpha_i) G_i ,
\end{aligned}$$

as $\sum_{i=1}^{\hat{k}} \alpha_i = 1$.

By observation, therefore, we have (A11) for $\hat{l} = 2$. Substituting in that $\sum_i \alpha_i G_i = G_{\hat{k}+1}$, we have

$$\sum_{M \in \binom{[k]}{2}} \prod_{i \in M} G_i = \sum_{M \in \binom{[\hat{k}]}{2}} \Big(1 - \sum_{i \in M} \alpha_i\Big) \prod_{i \in M} G_i + \left(\sum_{i=1}^{\hat{k}} \alpha_i G_i\right) \sum_{i=1}^{\hat{k}} G_i - (G_{\hat{k}+1})^2$$

$$\Rightarrow 0 = \sum_{M \in \binom{[\hat{k}]}{2}} \Big(-\sum_{i \in M} \alpha_i\Big) \prod_{i \in M} G_i + \left(\sum_{i=1}^{\hat{k}} \alpha_i G_i\right) \sum_{i=1}^{\hat{k}} G_i - (G_{\hat{k}+1})^2.$$

Canceling terms, we have

$$0 = \sum_{i=1}^{\hat{k}} \alpha_i (G_i)^2 - (G_{\hat{k}+1})^2$$

$$\Rightarrow (G_{\hat{k}+1})^2 = \sum_{i=1}^{\hat{k}} \alpha_i (G_i)^2,$$

as desired.

For our inductive hypothesis, assume that (A10) and (A11) are true for all $l \leq \hat{l} - 1$, and now suppose that no event of type $\hat{l}$ can make matrix $\mathcal{M}'$ full rank. We are therefore left to show (A10) and (A11) for $l = \hat{l}$. It therefore must be that for any event E of type $\hat{l}$,

$$G_{-(\hat{k}+1)}(E) = \sum_{i=1}^{\hat{k}} \alpha_i G_{-i}(E).$$

Substituting in from (A8), and canceling terms, we have

$$\sum_{M \in \binom{[k]}{\hat{l}}} \prod_{i \in M} G_i = \sum_{q=1}^{\hat{k}} \alpha_q \left(\sum_{M \in \binom{[\hat{k}+1] \setminus q}{\hat{l}}} \prod_{i \in M} G_i \right)$$

$$= \sum_{q=1}^{\hat{k}} \alpha_q \left(\sum_{M \in \binom{[\hat{k}] \setminus q}{\hat{l}}} \prod_{i \in M} G_i + G_{\hat{k}+1} \sum_{M \in \binom{[\hat{k}] \setminus q}{\hat{l}-1}} \prod_{i \in M} G_i \right)$$

$$= \sum_{M \in \binom{[\hat{k}]}{\hat{l}}} \Big(1 - \sum_{i \in M} \alpha_i\Big) \prod_{i \in M} G_i + G_{\hat{k}+1} \sum_{M \in \binom{[\hat{k}]}{\hat{l}-1}} \Big(1 - \sum_{i \in M} \alpha_i\Big) \prod_{i \in M} G_i,$$

as $\sum_{i=1}^{\hat{k}} \alpha_i = 1$.

Therefore, we have (A11) as desired for $\hat{l}$. Rearranging, we have

$$\sum_{M \in \binom{[\hat{k}]}{\hat{l}}} \Big(\sum_{i \in M} \alpha_i \Big) \prod_{i \in M} G_i = G_{\hat{k}+1} \left(\sum_{M \in \binom{[\hat{k}]}{\hat{l}-1}} \Big(1 - \sum_{i \in M} \alpha_i \Big) \prod_{i \in M} G_i \right).$$

Substituting the term in the parentheses on the right-hand side from (A11) for $\hat{l} - 1$,

$$\sum_{M \in \binom{[\hat{k}]}{\hat{l}}} \Big(\sum_{i \in M} \alpha_i \Big) \prod_{i \in M} G_i \;=\; G_{\hat{k}+1} \left(\left(\sum_{M \in \binom{[\hat{k}]}{l-1}} \prod_{i \in M} G_i \right) - G_{\hat{k}+1} \sum_{M \in \binom{[\hat{k}]}{l-2}} \Big(1 - \sum_{i \in M} \alpha_i \Big) \prod_{i \in M} G_i \right).$$

Proceeding inductively and collecting terms, we have

$$\sum_{M \in \binom{[\hat{k}]}{\hat{l}}} \Big(\sum_{i \in M} \alpha_i \Big) \prod_{i \in M} G_i \;=\; G_{\hat{k}+1} \left(\sum_{s=0}^{\hat{l}-2} (-1)^s (G_{\hat{k}+1})^s \sum_{M \in \binom{[\hat{k}]}{\hat{l}-1-s}} \prod_{i \in M} G_i \;+\; (-1)^{\hat{l}-1} (G_{\hat{k}+1})^{\hat{l}-1} \right)$$

$$\Rightarrow \sum_{M \in \binom{[\hat{k}]}{\hat{l}}} \Big(\sum_{i \in M} \alpha_i \Big) \prod_{i \in M} G_i \;=\; (-1)^{\hat{l}-1} (G_{\hat{k}+1})^{\hat{l}} + G_{\hat{k}+1} \sum_{M \in \binom{[\hat{k}]}{\hat{l}-1}} \prod_{i \in M} G_i$$

$$+\; G_{\hat{k}+1} \left(\sum_{s=1}^{\hat{l}-2} (-1)^s (G_{\hat{k}+1})^s \sum_{M \in \binom{[\hat{k}]}{\hat{l}-1-s}} \prod_{i \in M} G_i \right)$$

$$\Rightarrow 0 = (-1)^{\hat{l}-1} (G_{\hat{k}+1})^{\hat{l}} + \sum_{M \in \binom{[\hat{k}]}{\hat{l}-1}} \Big(\sum_{i \in M} \alpha_i G_i \Big) \prod_{i \in M} G_i$$

$$+\; G_{\hat{k}+1} \left(\sum_{s=1}^{\hat{l}-2} (-1)^s (G_{\hat{k}+1})^s \sum_{M \in \binom{[\hat{k}]}{\hat{l}-1-s}} \prod_{i \in M} G_i \right)$$

$$\Rightarrow 0 = (-1)^{\hat{l}-1} (G_{\hat{k}+1})^{\hat{l}} + \sum_{M \in \binom{[\hat{k}]}{\hat{l}-1}} \Big(\sum_{i \in M} \alpha_i G_i \Big) \prod_{i \in M} G_i - (G_{\hat{k}+1})^2 \sum_{M \in \binom{[\hat{k}]}{\hat{l}-2}} \prod_{i \in M} G_i$$

$$+\; G_{\hat{k}+1} \left(\sum_{s=2}^{\hat{l}-2} (-1)^s (G_{\hat{k}+1})^s \sum_{M \in \binom{[\hat{k}]}{\hat{l}-1-s}} \prod_{i \in M} G_i \right).$$

Substituting in from (A10) for $l = 2$

$$\Rightarrow 0 = (-1)^{\hat{l}-1}(G_{\hat{k}+1})^{\hat{l}} + \sum_{M \in \binom{[\hat{k}]}{\hat{l}-1}} \left(\sum_{i \in M} \alpha_i G_i\right) \prod_{i \in M} G_i - \left(\sum_{i=1}^{\hat{k}} \alpha_i (G_i)^2\right) \sum_{M \in \binom{[\hat{k}]}{\hat{l}-2}} \prod_{i \in M} G_i$$

$$+ \; G_{\hat{k}+1}\left(\sum_{s=2}^{\hat{l}-2} (-1)^s (G_{\hat{k}+1})^s \sum_{M \in \binom{[\hat{k}]}{\hat{l}-1-s}} \prod_{i \in M} G_i\right).$$

Canceling terms

$$\Rightarrow 0 = (-1)^{\hat{l}-1}(G_{\hat{k}+1})^{\hat{l}} - \sum_{M \in \binom{[\hat{k}]}{\hat{l}-2}} \left(\sum_{i \in M} \alpha_i \; (G_i)^2\right) \prod_{i \in M} G_i$$

$$+ \; G_{\hat{k}+1}\left(\sum_{s=2}^{\hat{l}-2} (-1)^s (G_{\hat{k}+1})^s \sum_{M \in \binom{[\hat{k}]}{\hat{l}-1-s}} \prod_{i \in M} G_i\right).$$

Continuing to open out the summation and cancel terms, we have, as desired,

$$(G_{\hat{k}+1})^{\hat{l}} \; = \; \sum_{i=1}^{\hat{k}} \alpha_i (G_i)^{\hat{l}}.$$

This concludes the proof of the claim. ∎

Having shown Claim 1, we now show that there exist an event of type 1 to $\hat{k}$ such that the matrix $\mathcal{M}'$ has full rank. To see this, assume the converse. Then, by (A10) we have that

$$\forall l = 1, \ldots, \hat{k}, \; (G_{\hat{k}+1})^l \; = \; \sum_{i=1}^{\hat{k}} \alpha_i (G_i)^l,$$

and further, we know by our previous arguments that

$$1 \; = \; \sum_{i=1}^{\hat{k}} \alpha_i.$$

We can rewrite these together as

$$\forall l \; = \; 0, \ldots, \hat{k}, \;\; (G_{\hat{k}+1})^l \; = \; \sum_{i=1}^{\hat{k}} \alpha_i (G_i)^l.$$

We now have $\hat{k} + 1$ functional equations, but only $\hat{k}$ variables (α). Since the distributions are different, it should be intuitive that this system of equations cannot have a solution.

CLAIM 2: *Suppose the distributions G_1 to $G_{\hat{k}+1}$ are pairwise different. Then,*

$$\text{(A12)} \qquad \exists \tilde{\beta} \in \Delta\mathbb{R} \text{ such that } G_1(\tilde{\beta}) \text{ to } G_{\hat{k}+1}(\tilde{\beta}) \text{ are all different.}$$

PROOF:

Consider the subset of $\mathbb{R}^{\hat{k}+1}$ defined as

$$S \equiv \left\{(a_1, a_2, \ldots, a_{\hat{k}+1}) : \exists \tilde{\beta} \in \Delta\mathbb{R} \text{ such that } a_j = G_j(\tilde{\beta}) \text{ for } j = 1, \ldots, \hat{k}+1\right\}.$$

Further, for every j, j', define $X_{j,j'} \subseteq \mathbb{R}^{\hat{k}+1}$:

$$X_{j,j'} \equiv \{(a_1, a_2, \ldots, a_{\hat{k}+1}) : \ a_j = a_{j'}\}.$$

Note that each $X_{j,j'}$ is a $\hat{k}$ dimensional subspace of $\mathbb{R}^{\hat{k}+1}$.

By definition S is convex. Since the distributions are pairwise different, for every j, j' there exists $\beta \in \mathbb{R}$ such that $G_j(\beta) \neq G_{j'}(\beta)$. Therefore, for each j, j', $S \not\subseteq X_{j,j'}$. Further, note that $X \equiv \cup_{j \neq j'} X_{j,j'}$ is not convex, so $S \not\subseteq X$, and therefore we have our desired result.

Note that by Claim 2, possibly by adding a little weight on a low β such that $G_j(\beta) = 0$ for all j, we have that there exists $\tilde{\beta} \in \Delta\mathbb{R}$ such that all $G_1(\tilde{\beta})$ to $G_{\hat{k}+1}(\tilde{\beta})$ are pairwise different, and also different from 1.

Therefore, for this $\tilde{\beta}$, there must exist a solution to:

$$\forall l = 0, \ldots, \hat{k}, \left(G_{\hat{k}+1}(\tilde{\beta})\right)^l = \sum_{i=1}^{\hat{k}} \alpha_i \left(G_i(\tilde{\beta})\right)^l.$$

Taking the appropriate Farkas alternative, therefore, for the previous system to have a solution, here there must exist a nonzero solution $\nu \in \mathbb{R}^{\hat{k}+1}$ to:[21]

$$\forall i = 1, \ldots, \hat{k}+1, \sum_{l=0}^{\hat{k}} \nu_l \left(G_i(\tilde{\beta})\right)^l = 0.$$

But note that this suggests there are $\hat{k}+1$ distinct roots of the $\hat{k}$ degree polynomial

$$\sum_{l=0}^{\hat{k}} \nu_l x^l,$$

which is impossible. Therefore, there is no solution to:

$$\forall l = 0, \ldots, \hat{k}, \left(G_{\hat{k}+1}(\tilde{\beta})\right)^l = \sum_{i=1}^{\hat{k}} \alpha_i \left(G_i(\tilde{\beta})\right)^l,$$

concluding our proof. ∎

[21] The Farkas lemma states that either the system $Cx = d$ has a solution or $yC = 0$, $yd > 0$ has a solution but never both. For the latter system to have no solution, it must be that for every nonzero y such that $yC = 0$, it is the case that $yd = 0$. This is the version we stated.

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