ONLINE APPENDIX FOR "MULTI-DIMENSIONAL SCREENING: BUYER-OPTIMAL LEARNING AND INFORMATIONAL ROBUSTNESS"

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In this online appendix, we prove Corollary 1. For completeness, we first restate the result.

COROLLARY 1. *Suppose the value for each good is iid with distribution $\check{F}$ (so the prior $F = \check{F} \times \cdots \times \check{F}$). No random separate sales mechanism provides the highest revenue guarantee $\pi^\star$.*

To prove this result, we need to define the special class of truncated Pareto distributions. These are defined as

$$H_{\alpha,\beta}(\overline{\theta}) = \begin{cases} 0 & \text{if } \overline{\theta} < \underline{\alpha}, \\ 1 - \alpha/\overline{\theta} & \text{if } \overline{\theta} \in [\alpha,\ \beta), \\ 1 & \text{if } \overline{\theta} \geq \beta, \end{cases} \tag{1}$$

where $\alpha \leq \beta$. Truncated Pareto distributions are supported on $[\alpha, \beta]$, are continuous on (α, β) and have an atom of size $\frac{\alpha}{\beta}$ at the truncation point β. When $\alpha = \beta$, $H_{\alpha,\alpha}$ is the degenerate distribution with an atom of size 1 at α.

We use $\mathcal{H}^P_{F'} = \{H_{\alpha,\beta} \mid F' \succsim H_{\alpha,\beta}\}$ to denote the subset of Pareto distributed signals corresponding to the one-dimensional distribution F' with mean μ'. This set is non-empty because it includes the distribution $H_{\mu',\mu'}$ which corresponds to the completely uninformative signal.

Roesler and Szentes (2017) showed that the class of truncated Pareto distributions can be used to characterize the buyer-optimal outcome for a single good. The truncated Pareto distribution $H_{\alpha,\beta}$ has the property that all pure bundling mechanisms with price $\overline{p}$ in $[\alpha, \beta]$ yield the same profit α. Since their work, the properties of this class of distributions have been exploited in several information design papers. Moreover, we showed that the highest revenue guarantee $\pi^\star$ is exactly the same as the seller's profit in the buyer-optimal outcome. We use this observation to prove the corollary.

In order to prove Corollary 1, we need to establish some properties of Pareto distributed grand bundle signals. These properties are known (although, to our knowledge, have not appeared in print) but we reproduce the entire arguments here to make the complete proof self contained. We use $\pi^\star$ to denote the profit for the seller in a buyer-optimal outcome. Recall that this is also the highest revenue guarantee attained by the robustly optimal mechanism.

LEMMA 1. *For any $\alpha \in [\pi^\star, \overline{\mu}]$, there exists a unique $\beta_{F'}(\alpha) \geq \alpha$ for which $H_{\alpha,\beta_{F'}(\alpha)} \in \mathcal{H}^P_{F'}$. Moreover, $\beta_{F'}(\alpha)$ is continuous and strictly decreasing in $\alpha \in (\pi^\star, \overline{\mu})$.*

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In words, this lemma says that, for all $\alpha \in [\pi^\star, \overline{\mu}]$, there is a Pareto distributed grand bundle signal that has α as the minimum of the support and, moreover, that the maximum of the support is strictly decreasing in α.

PROOF OF LEMMA 1. Take an arbitrary Pareto distribution $H_{\alpha,\beta} \in \Delta(\overline{S})$ with $\alpha \in [\pi^\star, \mu']$ and $\beta \in [\alpha, 1]$ that need not be a grand bundle signal.

First observe, that for $\overline{s} \in [\theta_\ell, \alpha]$, $\int_{\theta_\ell}^{\overline{s}} H_{\alpha,\beta}(x)\, \mathrm{d}x = 0$. Moreover, for $\overline{s} \in [\alpha, \beta)$, it holds that

$$\int_{\theta_\ell}^{\overline{s}} H_{\alpha,\beta}(x)\, \mathrm{d}x = \int_{\alpha}^{\overline{s}} \left(1 - \frac{\alpha}{x}\right)\, \mathrm{d}x = \overline{s} - \alpha - \alpha \log \overline{s} + \alpha \log \alpha.$$

The above expression is increasing in $\overline{s}$ and decreasing in α.

For $\overline{s} \in [\beta, \theta_h]$, we obtain

$$\begin{aligned} \textstyle\int_0^{\overline{s}} H_{\alpha,\beta}(x)\, \mathrm{d}x = \int_0^{\beta} H_{\alpha,\beta}(x)\, \mathrm{d}x + \int_{\beta}^{\overline{s}} H_{\alpha,\beta}(x)\, \mathrm{d}x \quad &= (\beta - \alpha - \alpha \log \beta + \alpha \log \alpha) + (\overline{s} - \beta) \\ &= \overline{s} - \alpha - \alpha \log \beta + \alpha \log \alpha, \end{aligned} \tag{2}$$

which is increasing in $\overline{s}$ and decreasing in both α and β.

Now note that for the Pareto distributed signal $H_{\pi^\star,\beta_{F'}(\pi^\star)}$ in the buyer-optimal outcome, $\beta_{F'}(\pi^\star)$ is unique because the mean of the Pareto distribution $H_{\alpha,\beta}$ changes in location β of the atom.

Because $H_{\pi^\star,\beta_{F'}(\pi^\star)}$ is a Pareto distributed grand bundle signal it must have the same mean as F' so

$$\int_{\theta_\ell}^{\theta_h} \left(1 - H_{\pi^\star,\beta_{F'}(\pi^\star)}(x)\right)\, \mathrm{d}x = \int_{\theta_\ell}^{\theta_h} (1 - F'(x))\, \mathrm{d}x = \mu'.$$

Because $\int_{\theta_\ell}^{\theta_h} H_{\alpha,\beta}(x)\, \mathrm{d}x$ is decreasing in both α and β, this in turn implies that

$$\int_{\theta_\ell}^{\theta_h} (1 - H_{\alpha,1}(x))\, \mathrm{d}x \geq \int_{\theta_\ell}^{\theta_h} \left(1 - H_{\pi^\star,\beta_{F'}(\pi^\star)}(x)\right)\, \mathrm{d}x = \int_{\theta_\ell}^{\theta_h} (1 - F'(x))\, \mathrm{d}x = \mu'.$$

Moreover, note that

$$\int_{\theta_\ell}^{\theta_h} (1 - H_{\alpha,\alpha}(x))\, \mathrm{d}x = \alpha \leq \mu'.$$

From (2), we know that $\int_{\theta_\ell}^{\theta_h} (1 - H_{\alpha,\beta}(x))\, \mathrm{d}x$ is continuous in β. Therefore, by the intermediate-value theorem, there must be a $\beta_{F'}(\alpha)$ that is the solution to

$$\begin{aligned} \mu' = \int_{\theta_\ell}^{\theta_h} (1 - H_{\alpha,\beta}(x))\, \mathrm{d}x =& 1 - (\beta - \alpha - \alpha \log \beta + \alpha \log \alpha + (1 - \beta)) \\ =& \alpha + \alpha \log \beta - \alpha \log \alpha \end{aligned}$$

and we define

$$\beta_{F'}(\alpha) = \alpha\ \exp\left(\frac{\mu' - \alpha}{\alpha}\right).$$

Observe that $\beta_{F'}(\alpha)$ is continuous and decreasing over the interval $\alpha \in [\pi^\star, \mu']$.

We will now argue that $H_{\alpha,\beta_{F'}(\alpha)} \in \mathcal{H}^P_{F'}$ for all $\alpha \in [\pi^\star, \mu']$. Observe that

$$H_{\alpha,\beta_{F'}(\alpha)}(\overline{s}) \leq H_{\pi^\star,\beta_{F'}(\pi^\star)}(\overline{s}) \quad \text{for} \quad \overline{s} < \beta_{F'}(\alpha) \quad \text{and} \quad H_{\alpha,\beta_{F'}(\alpha)}(\overline{s}) = 1 \geq H_{\pi^\star,\beta_{F'}(\pi^\star)}(\overline{s}) \quad \text{for} \quad \overline{s} \geq \beta_{F'}(\alpha)$$

or, in words, that $H_{\alpha,\beta_{F'}(\alpha)}$ crosses $H_{\pi^\star,\beta_{F'}(\pi^\star)}$ once from below. Since $\int_{\theta_\ell}^{\theta_h} H_{\alpha,\beta_{F'}(\alpha)}(x)\,\mathrm{d}x = \int_{\theta_\ell}^{\theta_h} H_{\pi^\star,\beta_{F'}(\pi^\star)}(x)\,\mathrm{d}x$ by construction, this implies that the function $\bar{s} \mapsto \int_{\theta_\ell}^{\bar{s}} H_{\alpha,\beta_{F'}(\alpha)}(x)\,\mathrm{d}x$ must lie below the function $\bar{s} \mapsto \int_{\theta_\ell}^{\bar{s}} H_{\pi^\star,\beta_{F'}(\pi^\star)}(x)\,\mathrm{d}x$ for $\bar{s} \in [\theta_\ell, \theta_h]$ or equivalently that

$$\int_{\theta_\ell}^{\bar{s}} H_{\alpha,\beta_{F'}(\alpha)}(x)\,\mathrm{d}x \le \int_{\theta_\ell}^{\bar{s}} H_{\pi^\star,\beta_{F'}(\pi^\star)}(x)\,\mathrm{d}x \le \int_{\theta_\ell}^{\bar{s}} F'(x)\,\mathrm{d}x \quad \text{for all} \quad \bar{s} \in [\theta_\ell, \theta_h].$$

This shows that $H_{\alpha,\beta_{F'}(\alpha)} \in \mathcal{H}^P_{F'}$ and completes the proof of this part. ■

The next lemma establishes a property relates the buyer-optimal grand bundle signal to the prior type distribution. Let $H_{\pi^\star,\beta_{F'}(\pi^\star)} \in \mathcal{H}^P_{F'}$ be the buyer-optimal Pareto distributed grand bundle signal.

LEMMA 2. *Then, there exists an $\bar{s} \in (\theta_\ell, \theta_h)$ such that*

$$\int_{\theta_\ell}^{\bar{s}} H_{\pi^\star,\beta_{F'}(\pi^\star)}(x)\,dx = \int_{\theta_\ell}^{\bar{s}} F'(x)\,dx.$$

PROOF. Suppose, for contradiction that

$$\int_{\theta_\ell}^{\bar{s}} H_{\pi^\star,\beta_{F'}(\pi^\star)}(x)\,\mathrm{d}x < \int_{\theta_\ell}^{\bar{s}} F'(x)\,\mathrm{d}x \quad \text{for all} \quad \bar{s} \in (\theta_\ell, \theta_h). \tag{3}$$

We first argue this implies $\beta_{F'}(\pi^\star) < 1$. Because $\int_{\theta_\ell}^{\theta_h} H_{\pi^\star,\beta_{F'}(\pi^\star)}(x)\,\mathrm{d}x = \int_{\theta_\ell}^{\theta_h} F'(x)\,\mathrm{d}x$, the above inequality (3) implies that $\int_{\bar{s}}^{\theta_h} H_{\pi^\star,\beta_{F'}(\pi^\star)}(x)\,\mathrm{d}x > \int_{\bar{s}}^{\theta_h} F'(x)\,\mathrm{d}x$ for all $\bar{s} \in (\theta_\ell, \theta_h)$.

If $\beta_{F'}(\pi^\star) = \theta_h$, then $H_{\pi^\star,\beta_{F'}(\pi^\star)}(\theta_h) = F'(\theta_h) = 1$ and $H_{\pi^\star,\beta_{F'}(\pi^\star)}(\bar{s}) < F'(\bar{s})$ for some neighborhood $\bar{s} \in (\theta_h - \varepsilon, \theta_h)$, $\varepsilon > 0$ because F' is continuous but $H_{\pi^\star,\beta_{F'}(\pi^\star)}$ has an atom at θ_h. For any $\bar{s}$ in this neighborhood, we would have $\int_{\bar{s}}^{\theta_h} H_{\pi^\star,\beta_{F'}(\pi^\star)}(x)\,\mathrm{d}x < \int_{\bar{s}}^{\theta_h} F'(x)\,\mathrm{d}x$ which is a contradiction.

We now use the fact that $\beta_{F'}(\pi^\star) < \theta_h$ to argue that we can find a Pareto distributed signal $H_{\alpha,\beta_{F'}(\alpha)} \in \mathcal{H}^P_{F'}$ with $\alpha < \pi^\star$. This would provide the requisite contradiction because it would contradict the optimality of $\pi^\star$.

Let

$$\gamma = \min_{\bar{s} \in \left[\frac{\pi^\star}{2}, \frac{1+\beta_{F'}(\pi^\star)}{2}\right]} \left[\int_{\theta_\ell}^{\bar{s}} F'(x) - \int_{\theta_\ell}^{\bar{s}} H_{\pi^\star,\beta_{F'}(\pi^\star)}(x)\,\mathrm{d}x\right]$$

and observe that $\gamma > 0$ because the term in the square brackets is a continuous function that is positive (by (3)) in the interval $\bar{s} \in \left[\frac{\pi^\star}{2}, \frac{1+\beta_{F'}(\pi^\star)}{2}\right]$.

Now consider an $\alpha = \pi^\star - \varepsilon$ with $\varepsilon > 0$ such that $\frac{\pi^\star}{2} < \alpha < \beta_{F'}(\alpha) < \frac{1+\beta_{F'}(\pi^\star)}{2}$. Observe that

$$H_{\alpha,\beta_{F'}(\alpha)}(\bar{s}) - H_{\pi^\star,\beta_{F'}(\pi^\star)}(\bar{s}) = \begin{cases} 0 & \text{if } \bar{s} \in [\theta_\ell, \alpha], \\ 1 - \frac{\alpha}{\bar{s}} & \text{if } \bar{s} \in (\alpha, \pi^\star], \\ \frac{\pi^\star - \alpha}{\bar{s}} & \text{if } \bar{s} \in (\pi^\star, \beta_{F'}(\pi^\star)), \\ -\frac{\alpha}{\bar{s}} & \text{if } \bar{s} \in [\beta_{F'}(\pi^\star), \beta_{F'}(\alpha)), \\ 0 & \text{if } \bar{s} \in [\beta_{F'}(\alpha), \theta_h]. \end{cases}$$

Now observe that $\int_{\theta_\ell}^{\bar{s}} \left[H_{\alpha,\beta_{F'}(\alpha)}(x) - H_{\pi^\star,\beta_{F'}(\pi^\star)}(x)\right]\,\mathrm{d}x$ takes its highest value at $\bar{s} = \beta_{F'}(\pi^\star)$ because the function in the brackets is non-negative for $\bar{s} < \beta_{F'}(\pi^\star)$ but non-positive for $\bar{s} \ge \beta_{F'}(\pi^\star)$.

Evaluating the difference at this point, we get

$$\begin{aligned}&\int_{\theta_\ell}^{\beta_{F'}(\pi^\star)}\left[H_{\alpha,\beta_{F'}(\alpha)}(x)-H_{\pi^\star,\beta_{F'}(\pi^\star)}(x)\right]\,\mathrm{d}x\\=&\int_{\theta_\ell}^{\alpha}\left[H_{\alpha,\beta_{F'}(\alpha)}(x)-H_{\pi^\star,\beta_{F'}(\pi^\star)}(x)\right]\,\mathrm{d}x+\int_{\alpha}^{\pi^\star}\left[H_{\alpha,\beta_{F'}(\alpha)}(x)-H_{\pi^\star,\beta_{F'}(\pi^\star)}(x)\right]\,\mathrm{d}x\\&+\int_{\pi^\star}^{\beta_{F'}(\pi^\star)}\left[H_{\alpha,\beta_{F'}(\alpha)}(x)-H_{\pi^\star,\beta_{F'}(\pi^\star)}(x)\right]\,\mathrm{d}x\\=&\int_{\alpha}^{\pi^\star}\left[1-\frac{\alpha}{x}\right]\,\mathrm{d}x+\int_{\pi^\star}^{\beta_{F'}(\pi^\star)}\left[\frac{\pi^\star-\alpha}{x}\right]\,\mathrm{d}x\\\leq&\int_{\alpha}^{\pi^\star}\left[\frac{\varepsilon}{\alpha}\right]\,\mathrm{d}x+\int_{\pi^\star}^{\beta_{F'}(\pi^\star)}\left[\frac{\varepsilon}{\alpha}\right]\,\mathrm{d}x\\=&\frac{\varepsilon}{\alpha}\left(\pi^\star-\alpha+\beta_{F'}(\pi^\star)-\pi^\star\right)=\frac{\varepsilon}{\alpha}\left(\beta_{F'}(\pi^\star)-\alpha\right).\end{aligned}$$

Therefore, for small enough $0<\varepsilon<\frac{\alpha}{\beta_{F'}(\pi^\star)-\alpha}\gamma$, we have

$$\int_{\theta_\ell}^{\bar{s}}H_{\alpha,\beta_{F'}(\alpha)}(x)\,\mathrm{d}x\leq\int_{\theta_\ell}^{\bar{s}}H_{\pi^\star,\beta_{F'}(\pi^\star)}(x)\,\mathrm{d}x+\gamma\leq\int_{\theta_\ell}^{\bar{s}}F'(x)\,\mathrm{d}x\quad\text{for all}\quad\bar{s}\in\left[\frac{\pi^\star}{2},\frac{1+\beta_{F'}(\pi^\star)}{2}\right],$$

$$\int_{\theta_\ell}^{\bar{s}}H_{\alpha,\beta_{F'}(\alpha)}(x)\,\mathrm{d}x=\int_{\theta_\ell}^{\bar{s}}H_{\pi^\star,\beta_{F'}(\pi^\star)}(x)\,\mathrm{d}x=0\quad\text{for all}\quad\bar{s}\in\left[\theta_\ell,\frac{\pi^\star}{2}\right)\text{ and}$$

$$\int_{\theta_\ell}^{\bar{s}}H_{\alpha,\beta_{F'}(\alpha)}(x)\,\mathrm{d}x=\int_{\theta_\ell}^{\bar{s}}H_{\pi^\star,\beta_{F'}(\pi^\star)}(x)\,\mathrm{d}x=\bar{s}-\mu'\quad\text{for all}\quad\bar{s}\in\left(\frac{1+\beta_{F'}(\pi^\star)}{2},\theta_h\right].$$

This implies that $H_{\alpha,\beta_{F'}(\alpha)}\in\mathcal{H}^P_{F'}$ and we have the requisite contradiction for the optimality of $H_{\pi^\star,\beta_{F'}(\pi^\star)}$ which completes the proof of the lemma. ■

Let $n=2$ and $F=\check{F}\times\check{F}$ where $\check{F}$ has mean $\check{\mu}$. We use $\check{\pi}^\star=\min\{\alpha\mid H_{\alpha,\beta_{F'}(\alpha)}\in\mathcal{H}^P_{\check{F}}\}$ and $\pi^\star=\min\{\alpha\mid H_{\alpha,\beta_{F'}(\alpha)}\in\mathcal{H}^P_{\overline{F}}\}$ to denote the profit of a seller with a single good in the buyer-optimal outcome when the prior distributions are $\check{F}$ and $\overline{F}$ respectively.

LEMMA 3. *For $n=2$ and $F=\check{F}\times\check{F}$, we have $2\check{\pi}^\star<\pi^\star$.*

PROOF. From Lemma 1, $H_{\check{\pi}^\star,\beta_{\check{F}}(\check{\pi}^\star)}\in\mathcal{H}^P_{\check{F}}$. Then, we must have

$$\check{\pi}^\star\log\beta_{\check{F}}(\check{\pi}^\star)-\check{\pi}^\star\log\check{\pi}^\star+\check{\pi}^\star=\check{\mu}\tag{4}$$

and

$$x-\check{\pi}^\star-\check{\pi}^\star\log x+\check{\pi}^\star\log\check{\pi}^\star\leq\int_{\theta_\ell}^{x}\check{F}(v)dv,\tag{5}$$

for all $x\in[\check{\pi}^\star,\beta_{F'}(\check{\pi}^\star)]$. Lemma 2 shows that inequality (5) must bind for some $\tilde{x}\in(\check{\pi}^\star,\beta_{\check{F}}(\check{\pi}^\star))$. So

$$\int_{\theta_\ell}^{\tilde{x}}H_{\check{\pi}^\star,\beta_{\check{F}}(\check{\pi}^\star)}(s)ds=\tilde{x}-\check{\pi}^\star-\check{\pi}^\star\log\tilde{x}+\check{\pi}^\star\log\check{\pi}^\star=\int_{\theta_\ell}^{\tilde{x}}\check{F}(v)dv.$$

So suppose, as a contradiction, that $2\check{\pi}^\star = \pi^\star$. Lemma 1 implies that $H_{2\check{\pi}^\star,\beta_{\overline{F}}(2\check{\pi}^\star)} \in \mathcal{H}^P_{\overline{F}}$ which in turn implies $\beta_{\overline{F}}(2\check{\pi}^\star) = 2\beta_{\check{F}}(\check{\pi}^\star)$ because the mean of $H_{2\check{\pi}^\star,\beta_{\overline{F}}(2\check{\pi}^\star)}$ must be $2\check{\mu}$ or

$$\begin{aligned}
&2\check{\pi}^\star \log \beta_{\overline{F}}(2\check{\pi}^\star) - 2\check{\pi}^\star \log(2\check{\pi}^\star) + 2\check{\pi}^\star = 2\check{\mu} \\
\Longrightarrow\ & 2(\check{\pi}^\star \log \beta_{\check{F}}(\check{\pi}^\star) - \check{\pi}^\star \log \check{\pi}^\star + \check{\pi}^\star) + 2\check{\pi}^\star \log \beta_{\overline{F}}(2\check{\pi}^\star) - 2\check{\pi}^\star \log \beta_{\check{F}}(\check{\pi}^\star) - 2\check{\pi}^\star \log 2 = 2\check{\mu} \\
\Longrightarrow\ & \beta_{\overline{F}}(2\check{\pi}^\star) = 2\beta_{\check{F}}(\check{\pi}^\star),
\end{aligned}$$

where the last implication follows from (4).

We will establish the contradiction by arguing that $H_{2\check{\pi}^\star,2\beta_{\check{F}}(\check{\pi}^\star)} \notin \mathcal{H}^P_{\overline{F}}$. First, observe that $2\tilde{x} < 2\beta_{\check{F}}(\check{\pi}^\star) = \beta_{\overline{F}}(2\check{\pi}^\star)$ and so

$$\begin{aligned}
\int_{\theta_\ell}^{2\tilde{x}} H_{2\check{\pi}^\star,2\beta_{\check{F}}(\check{\pi}^\star)}(s)ds =& 2\tilde{x} - 2\check{\pi}^\star - 2\check{\pi}^\star \log 2\tilde{x} + 2\check{\pi}^\star \log 2\check{\pi}^\star \\
=& 2(\tilde{x} - \check{\pi}^\star - \check{\pi}^\star \log \tilde{x} + \check{\pi}^\star \log \check{\pi}^\star) \\
=& 2\int_{\theta_\ell}^{\tilde{x}} \check{F}(v)dv.
\end{aligned}$$

Then, note that $\overline{F}$ satisfies

$$\overline{F}(\bar{v}) = \int_{\theta_\ell}^{\bar{v}} \check{F}(y)\check{f}(\bar{v}-y)d\bar{v} = \frac{d}{d\bar{v}} \int_{\theta_\ell}^{\bar{v}} \check{F}(y)\check{F}(\bar{v}-y)dy,$$

for $v \in [2\theta_\ell, 2\theta_h]$ and, so for any $x \in [2\theta_\ell, 2\theta_h]$, we have

$$\int_{\theta_\ell}^{x} \overline{F}(\bar{v})d\bar{v} = \int_{\theta_\ell}^{x} \check{F}(y)\check{F}(x-y)dy.$$

Now consider the value $2\tilde{x}$. We must have

$$\begin{aligned}
2\int_{\theta_\ell}^{\tilde{x}} \check{F}(v)dv = \int_{\theta_\ell}^{2\tilde{x}} H_{2\check{\pi}^\star,2\beta_{\check{F}}(\check{\pi}^\star)}(s)ds \leq& \int_{\theta_\ell}^{2\tilde{x}} \overline{F}(\bar{v})d\bar{v} \\
=& \int_{\theta_\ell}^{2\tilde{x}} \check{F}(y)\check{F}(2\tilde{x}-y)dy \\
=& \int_{\theta_\ell}^{\tilde{x}} \check{F}(y)\check{F}(2\tilde{x}-y)dy + \int_{\tilde{x}}^{2\tilde{x}} \check{F}(y)\check{F}(2\tilde{x}-y)dy \\
<& \int_{\theta_\ell}^{\tilde{x}} \check{F}(y)dy + \int_{\tilde{x}}^{2\tilde{x}} \check{F}(2\tilde{x}-y)dy \\
=& 2\int_{\theta_\ell}^{\tilde{x}} \check{F}(y)dy
\end{aligned}$$

which provides the necessary contradiction. The weak inequality follows from $H_{2\check{\pi}^\star,2\beta_{\check{F}}(\check{\pi}^\star)} \in \mathcal{H}^P_{\overline{F}}$ and the strict inequality follows from the fact that $\tilde{x} \in (\theta_\ell, \theta_h)$ and that the distribution $\overline{F}$ has full support. ■

In what remains, we denote a separate sales mechanism simply by a joint distribution $\mathcal{P} \in \Delta(\mathbb{R}^n)$ of prices. $\mathcal{P}_i$ denotes the marginal distribution of prices for good $i \in N$. We abuse notation and use $\Pi(\mathcal{P}, G)$ to denote the profit from separate sales mechanism $\mathcal{P}$ and signal $G \in \mathcal{G}$. Similarly, $\Pi_i(\mathcal{P}_i, G_i)$ denotes the profit from the sale of good i alone when the prices for good i are distributed according to $\mathcal{P}_i$ and G_i is the distribution over posterior estimates of good i values.

We are now in a position to complete the proof of Corollary 1.

PROOF OF COROLLARY 1. The highest revenue guarantee from a separate sales mechanism is achieved by solving

$$\sup_{\mathcal{P}\in\Delta(\mathbb{R}^n)} \inf_{G\in\mathcal{G}} \Pi(\mathcal{P},G) = \sum_{i=1}^{n} \sup_{\mathcal{P}_i\in\Delta(\mathbb{R})} \inf_{\check{F}\succsim G_i} \Pi_i(\mathcal{P}_i,G_i) = n\check{\pi}^\star$$

where the first equality follows from the facts that (i) for all $G \in \mathcal{G}$ and $i \in N$, the marginal distribution over good i posterior estimates satisfies $\check{F} \succsim G_i$ and (ii) for every distribution $\check{F} \succsim G_i$ over the goods $i = 1,\ldots,n$, we have $G_1 \times \cdots \times G_n \in \mathcal{G}$.

Clearly, we must have $n\check{\pi}^\star \leq \pi^\star$ since the seller can always do at least weakly better when she is not restricted to the subclass of separate sales mechanisms. But Lemma 3 shows that this inequality is strict for $n = 2$. Therefore, Theorem 4 implies that this inequality must also be strict for all $n > 2$. This completes the proof. ■

REFERENCES

ROESLER, A.-K., AND B. SZENTES (2017): “Buyer-Optimal Learning and Monopoly Pricing,” *American Economic Review*, 107(7), 2072–80.