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Journal of Economic Theory 159 (2015) 891–928

JOURNAL OF Economic Theory

www.elsevier.com/locate/jet

Dynamic screening with limited commitment ☆

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Received 30 September 2013; final version received 16 April 2015; accepted 25 May 2015

Available online 29 May 2015

Abstract

We examine a model of dynamic screening and price discrimination in which the seller has limited commitment power. Two cohorts of anonymous, patient, and risk-neutral buyers arrive over two periods. Buyers in the first cohort arrive in period one, are privately informed about the distribution of their values, and then privately learn the value realizations in period two. Buyers in the second cohort are "last-minute shoppers" that already know their values upon their arrival in period two. The seller can fully commit to a long-term contract with buyers in the first cohort, but cannot commit to the future contractual terms that will be offered to second-cohort buyers. The expected second-cohort contract serves as an endogenous type-dependent outside option for first-cohort buyers, reducing the seller's ability to extract rents via sequential contracts. We derive the seller-optimal equilibrium and show that, when the seller cannot condition on future contractual terms (either explicitly or implicitly), she endogenously generates a commitment to maintaining high future prices by manipulating the timing of contracting.

JEL classification: C73; D82; D86

Keywords: Asymmetric information; Dynamic mechanism design; Limited commitment; Recontracting; Sequential screening; Type-dependent participation

☆ We thank Dirk Bergemann, Ralph Boleslavsky, Juan Dubra, Hao Li, Matt Mitchell, Mallesh Pai, Mike Peters, Vasiliki Skreta, Andrzej Skrzypacz, Jeroen Swinkels, Juuso Toikka, Venky Venkateswaran, Gabor Virag, and participants at various seminars and conferences for their many helpful comments and suggestions. This work has also benefited greatly from the extensive comments of the editor, Alessandro Pavan, as well as three anonymous referees.

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http://dx.doi.org/10.1016/j.jet.2015.05.015

1. Introduction

In many contracting settings, agents have private information that changes over time. Recent advances in dynamic mechanism design have highlighted the benefits of using dynamic contracts in such settings. Different short- and long-term prices, option contracts, and introductory offers are all methods by which a principal can provide incentives for agents to reveal new private information over time; by doing so, a principal is able to make contingent decisions that extract greater rents than those generated by unconditional, static contracts. One of the basic intuitions arising from this literature is that by contracting in the earliest stages of a relationship, when her informational disadvantage is at its smallest, a principal can relax the participation and incentive constraints she faces. Thus, early contracting leads to more effective price discrimination and smaller information rents. This intuition arises in large part from the assumption that the principal is able to determine the timing of contracting. In many settings, however, such an assumption need not be justified: in many markets, agents are "born" or enter the market at different times, and they are frequently able to time their transactions or delay entry into contractual relationships. Moreover, a principal may be unable to prevent such delays and treat different agent cohorts differently.

This strategic delay by agents in the timing of contracting is even more of a concern when the principal has limited commitment power. We have in mind settings in which the principal can commit to fully enforceable long-term contracts that bind (with some restrictions) her bilateral relationship with individual agents, but cannot commit in advance to the contractual terms that may be offered in future periods. This form of limited commitment, in addition to being of natural theoretical interest, also arises in a variety of real-world settings. For instance, consider the market for airline tickets. Each ticket sold for future travel is a long-term contract, complete with a commitment to its provisions for future refundability and exchangeability. The features of tickets that may be sold in the future (including prices, fare classes, and other terms and conditions) are not advertised or made available, nor is there any presumption that an airline is pre-committed to these ticket characteristics. Potential ticket buyers, on the other hand, face uncertainty about their value for traveling at the date in question. They must therefore decide whether to purchase a ticket immediately and take advantage of its option-like features (canceling the ticket if their realized value is low), or instead postpone their purchase in hopes of more advantageous contracting opportunities in the future. Optimal ticketing schemes must take this strategic timing of contracting into consideration, accounting for buyers' option values of postponing purchases and the impact of such behavior on the seller's ability to extract rents from different cohorts of buyers.

With this in mind, the present work studies the role of limited commitment in dynamic screening with strategic agents. We construct a simple two-period model in order to isolate the role of limited commitment in a transparent fashion. Our model features a monopolist that faces two cohorts of buyers that arrive over two periods; all consumption occurs at the end of the second period. Each buyer in the first cohort (which arrives in the first period) initially has private information regarding the distribution from which her private value is drawn, but does not learn the realized value until the second period.[1] Buyers in the second cohort arrive in period two, and already know their private value (which is drawn from a commonly known distribution). We assume that buyers are anonymous, so that the seller is unable to distinguish in period two between

[1] This period-one uncertainty about the final value differentiates our sequential screening framework from the Coasian durable goods framework; see Bulow (1982) for a two-period durable goods model.

first-cohort buyers who postponed contracting and second-cohort buyers who have just arrived to the market. Thus, the seller cannot prevent first-cohort buyers from contracting in period two.

A straightforward strategic tension arises in this setting. Since buyers in the first cohort learn their values over time, the seller has a strong incentive to sequentially screen these buyers using (dynamic) option contracts. The seller would also like to sell to buyers in the second cohort by offering a (static) "last-minute price" contract in period two. We assume that the seller *can* credibly commit to a sequential contract offered in the first period, but that she *cannot* commit at that time—either explicitly with a promise, or implicitly with contingencies in the period-one contract—to the contract she intends to offer in period two. That future contract affects the seller's ability to screen cohort-one buyers and extract rents, however: the period-two contract serves as an endogenous outside option for cohort-one buyers. If the seller can commit to a relatively high period-two price, this outside option becomes less attractive, and the seller's profits from cohort-one buyers increase. With limited commitment, however, competition between the period-one seller and her future self increases the rents left to buyers and reduces the seller's profits.

A simple thought experiment is helpful in illustrating the interplay between the seller's limited commitment and buyers' ability to strategically delay contracting. Suppose that the mass of cohort two is small, and suppose further that the monopoly price that corresponds to this cohort (in isolation) is low. If cohort-one buyers anticipate this low price, then waiting until the second period to contract is a very attractive outside option, and the seller's ability to extract rents in period one is reduced. Note, however, that the small mass of the second cohort implies that their contribution to total profits is also small; therefore, a seller with full commitment power could (relatively costlessly) commit to forgoing profits from the second-period cohort by charging an excessively high second-period price, thereby reducing the option value of strategic delay for cohort-one buyers and increasing overall profits. A seller with limited commitment power, on the other hand, would be unable to carry out the threat to maintain a high price in period two. Because the mass of cohort two is small, however, small changes in the composition of the set of buyers contracting in the second period can have a large impact on the distribution of buyers' values. In particular, the seller has a strong incentive to postpone contracting and encourage delay by some cohort-one buyers in order to generate stronger period-two demand. This delayed contracting by a subset of buyers can generate (via sequential rationality) a higher period-two price and, hence, a commensurately lower period-one outside option—appropriate "management" of demand across the two periods yields the seller some measure of endogenous commitment power.

Our main result identifies the subset of cohort-one buyers that delay contracting to period two in the seller-optimal contract. The seller can achieve the highest period-two price (and hence the greatest reduction in the option value of strategic delay) by delaying buyers with the largest expected values. However, these buyers are precisely those for whom early contracting generates the greatest gains from trade. Meanwhile, delaying buyers with the lowest expected values may actually decrease the period-two price, thereby increasing the option value of strategic delay. The seller's optimal contract therefore trades off these forces by inducing the strategic delay of an interior subset of "intermediate" cohort-one buyers.

This insight serves as a complement to the findings in the literature that long-term contracts can be used by sellers in dynamic environments to increase profits. In particular, our result shows that the *absence* of contracts with some buyers can be a useful tool for changing the composition of the buyer population in future periods and thereby constraining the seller's *own* future behavior. This sheds new light on the role of commitment power in dynamic settings, and on the underlying sources of that commitment.

In addition, note that the optimal contract with limited commitment features an interesting non-monotonicity: in sharp contrast to most optimal contracting results in both the static and dynamic mechanism design literatures, where exclusion typically follows a simple cutoff rule, the set of buyers in our setting that contract in the first period is disjoint. This non-monotonicity reduces the seller's ability to "separate" and discriminate types in the first period. Thus, demand management, though valuable in raising future prices and creating endogenous commitment, entails an additional deadweight loss relative to the full commitment benchmark. Indeed, the potential for endogenous commitment arising from delayed contracting highlights the complications that arise due to limited commitment, but also suggests that studying such models can lead to rich predictions and insights that further our understanding of dynamic contracting in real-world settings.

We also extend our analysis to examine the case where, in addition to their ability to delay contracting, buyers are also free to recontract in the second period. (For example, a traveler may choose to exercise her right to a partial refund of an airplane ticket and purchase an entirely new ticket at a later date.) In such cases, the constraints placed on the seller by her limited ability to commit are exacerbated by the additional power granted to buyers. In particular, the seller cannot exclude buyers from the period-two spot market simply by contracting with them in the initial period. Effectively, this implies that the seller's period-two price factors in the cohort-one buyers with the lowest expected values as they can always be enticed to "trade-in" relatively inefficient contracts. We provide a partial characterization of the seller's optimal contract in this setting, and show that it resembles the optimal contract in the setting without recontracting: in both cases, the period-two price is greater than the cohort-two monopoly price, and the pattern of distortions relative to the full-commitment optimal contract are qualitatively similar. However, the timing of contracting that arises is indeterminate (among others, there exists a seller-optimal equilibrium in which no buyers delay contracting), and the possibility of recontracting leads to a further decrease in the seller's ex ante profits.[2]

The present work contributes to the literature on optimal dynamic mechanism design.[3] This literature focuses on characterizing revenue-maximizing dynamic contracts for a principal facing agents with evolving private information. Typically, the principal is endowed with full commitment power and observes agent arrivals, enabling her to commit to excluding agents that do not contract immediately. Thus, in contrast to our model, all agents receive their (exogenous) reservation utility if they attempt to delay contracting, thereby incentivizing contracting upon arrival. Baron and Besanko (1984) were the first to study such problems and point out the crucial role of the "informativeness" of initial-period private information about future types in determining the optimal distortions away from efficiency used to reduce information rents. More recently, Pavan et al. (2014) derive a dynamic envelope formula that is necessarily satisfied in general dynamic environments, and also identify some sufficient conditions for incentive compatibility.

Our model is most closely related to the now-canonical work of Courty and Li (2000), who demonstrate the utility of sequential screening when buyers' private information may evolve

[2] Recontracting is typically permitted in the sale of airline tickets and hotel rooms. By contrast, for tickets to certain highly demanded events (including some high-profile soccer or rugby matches), buyers can choose the timing of purchase but are only allowed to place a single order and cannot recontract. Note that in all the above mentioned examples, the seller could enforce a no-recontracting policy by requiring identifying information at the time of purchase. Our results suggest that enforcing such a policy could be beneficial.

[3] There is also an extensive literature on efficient dynamic mechanism design. See Bergemann and Said (2011) and Vohra (2012) for surveys of both literatures.

between contracting and consumption.[4] We extend their model in several important ways: we introduce a second cohort of informed buyers who arrive in period two; we allow cohort-one buyers to postpone contracting until the second period; and we relax the assumption that the principal has unlimited commitment power. The first two of these features are shared with the work of Ely et al. (2015), who show that a capacity-constrained seller with full commitment power can benefit from "overbooking" (selling more units than capacity); overdemanded units can then be repurchased from low-value buyers and reallocated. By committing to biasing the reallocation stage away from late-arriving buyers, the seller can incentivize early purchases and increase her profits. Full commitment also plays an important role in Deb (2014) and Garrett (2013), who also consider models where agents' incentives to delay contracting influence the optimal contracts. In related work, Armstrong and Zhou (2014) study a setting where privately informed buyers can either purchase immediately from a seller or incur a search cost to learn their value for an outside option. The seller employs sales tactics such as exploding offers and "buy-now" discounts to manage the timing of purchases; instead of inducing delayed contracting as in our setting, however, their seller uses these option-like contracts to discourage consumer search and induce immediate transactions.

Since our seller cannot commit in advance to contracts that might be offered in the second period, this paper also ties into the broader literature on dynamic contracting without full commitment. In such settings, the optimal contract can often be implemented by entering into short-term contracts with some types while committing to long-term contracts with other types—see, for instance, Fudenberg and Tirole (1990) or Laffont and Tirole (1990). In these models, leaving some contingencies unspecified and differentiating agents by the timing of contracting is used to address the multiplicity of continuation equilibria, but is not necessary for optimality if multiplicity is not a concern (as in the present work, where the continuation equilibrium is always unique).

2. Model

2.1. Environment

We consider a monopoly seller of some good who faces a continuum of privately informed buyers. We normalize the seller's constant marginal cost of production to zero, and we assume that she faces no capacity constraints on the quantity that she may sell. There are two periods, in each of which a cohort of anonymous and risk-neutral buyers arrives. Anonymity implies that, outside a contractual relationship, the seller is unable to distinguish in period two between buyers from each cohort. Each buyer has single-unit demand, and all consumption occurs at the end of period two. For simplicity, we assume that neither the seller nor the buyers discount the future; note, however, that our results remain unchanged in the presence of a common discount factor.

The first cohort of buyers consists of a unit mass of agents arriving in period one. Each such buyer has a private initial type $\lambda \in \Lambda := [\underline{\lambda}, \bar{\lambda}]$ that is drawn from the distribution F with continuous and strictly positive density f. In period two, each buyer then learns her value $v \in \mathbf{V} := [\underline{v}, \bar{v}]$, where v is drawn from the (conditional) distribution $G(\cdot|\lambda)$ with strictly positive density $g(\cdot|\lambda)$. We assume throughout that all partial derivatives of G and g exist and are bounded.

[4] Boleslavsky and Said (2013) show that sequential screening becomes "progressive" screening when buyers' values are subject to additional (correlated) shocks over time. In contrast, Krähmer and Strausz (2015) show that there is no benefit to sequential screening when buyers have ex post participation constraints or limited (ex post) liability.

The second cohort of buyers consists of a mass $\gamma > 0$ of new entrants arriving in period two. Upon arrival, each such buyer already knows her private value $v \in \mathbf{V}$. Cohort-two values are drawn from the commonly known distribution H with continuous and strictly positive density h.

2.2. Additional assumptions

The following additional assumptions on primitives are maintained throughout what follows.

Assumption 1. *The family of distributions* $\{G(\cdot|\lambda)\}_{\lambda\in\Lambda}$ *satisfies the monotone likelihood ratio property: for all* $\lambda', \lambda'' \in \Lambda$ *with* $\lambda' > \lambda''$, $\frac{g(v|\lambda')}{g(v|\lambda'')}$ *is increasing in* v.

Assumption 2. *The monopoly profit functions* $\pi_\lambda(p) := p(1 - G(p|\lambda))$ *and* $\pi_H(p) := p(1 - H(p))$ *are strictly concave for all* $p \in \mathbf{V}$ *and all* $\lambda \in \Lambda$.

Assumption 3. *Let* $p_H := \arg\max_p\{\pi_H(p)\}$ *denote the cohort-two monopoly price. There exists some* $\hat{\mu} \in (\underline{\lambda}, \bar{\lambda})$ *such that* $p_H = \arg\max_p\{\pi_{\hat{\mu}}(p)\}$.

Assumption 4. *The function* $\max\left\{0, v + \frac{\partial G(v|\lambda)/\partial\lambda}{g(v|\lambda)} \frac{1-F(\lambda)}{f(\lambda)}\right\}$ *is nondecreasing in both* v *and* λ.[5]

Assumption 1 implies first-order stochastic dominance, so that

$$G_\lambda(v|\lambda) := \frac{\partial G(v|\lambda)}{\partial\lambda} \le 0 \text{ for all } v \in \mathbf{V} \text{ and } \lambda \in \Lambda.$$

Thus, buyers with higher initial types expect higher values.

Assumption 2 ensures that the monopoly prices

$$p_\lambda := \arg\max_p\{\pi_\lambda(p)\} \text{ and } p_H := \arg\max_p\{\pi_H(p)\}$$

are well-defined. Since Assumption 1 also implies the hazard rate order, p_λ is increasing in λ. Assumption 2 also guarantees the existence and uniqueness of optimal prices when considering a monopolist selling to a "mixed" population consisting of a subset of cohort-one buyers alongside cohort-two buyers. This assumption also guarantees (via the Theorem of the Maximum) the continuity of these optimal prices. In concert with Assumption 1, this implies that adding a positive measure of sufficiently high-λ cohort-one buyers to a population increases the optimal price, while adding a positive measure of sufficiently low-λ cohort-one buyers decreases that price.

Assumption 3 implies that $p_H = p_{\hat{\mu}} \in (p_{\underline{\lambda}}, p_{\bar{\lambda}})$, so that we can compare (in price-space) the second-cohort monopoly price p_H to monopoly prices for various cohort-one initial types. Note that we do not require $H = G(\cdot|\hat{\mu})$, so the distribution of cohort-two buyers' values need not be identical to that of any type of cohort-one buyer.

[5] Simply assuming that $v + \frac{\partial G(v|\lambda)/\partial\lambda}{g(v|\lambda)} \frac{1-F(\lambda)}{f(\lambda)}$ is *everywhere* increasing in both v and λ is also sufficient for our purposes, but rules out certain natural families of distributions that otherwise satisfy the remaining assumptions.

Finally, we use Assumption 4 to justify a "local" first-order approach to incentive compatibility.[6] It is stronger than the standard log-concavity assumption on the initial distribution of private information; indeed, it is a joint assumption on both the distribution F of initial types λ and the conditional distributions G of values. The assumption is, however, an easily verified condition on primitives that is satisfied in a wide variety of economic environments of interest.

One natural example satisfying these assumptions is the family of power distributions. Note that Assumptions 1 and 2 are satisfied when $G(v|\lambda) = v^\lambda$ for $v \in \mathbf{V} = [0, 1]$ and $\lambda \in [\underline{\lambda}, \bar{\lambda}] \subseteq \mathbb{R}_+$. If we further let $H = G(\cdot|\hat{\mu})$ for any $\hat{\mu} \in (\underline{\lambda}, \bar{\lambda})$, then Assumption 3 is also satisfied. Finally, it is easy to verify that Assumption 4 holds when $\Lambda = [0, 1]$ and $F(\lambda) = \lambda^\rho$ for any $\rho > 0$.

2.3. Contracts

In the absence of full commitment, we cannot appeal directly to the revelation principle. Instead, we must consider mechanisms with more general message spaces. We restrict attention throughout to deterministic mechanisms, with pure message-reporting strategies for buyers.[7]

In the initial period, the seller offers, and fully commits to, a dynamic mechanism to cohort-one buyers. Such a mechanism is a game form $D = \{M_{11}, M_{12}, \tau_{11}, \tau_{12}, a_1\}$, where M_{11} is a set of period-one messages; $M_{12}(m_{11})$ is a set of period-two messages; $\tau_{11}(m_{11})$ is a period-one transfer; $\tau_{12}(m_{11}, m_{12})$ is a period-two transfer; and $a_1(m_{11}, m_{12}) \in \{0, 1\}$ is the eventual allocation in period two. We impose the restriction that there exist $m_{11} \in M_{11}$ and $m_{12} \in M_{12}$ that correspond to non-participation in the dynamic mechanism—buyers are not compelled to participate in the seller's mechanism in the first period, and are also free to exit the mechanism in the second period.

In period two, the seller offers, and fully commits to, a *static* mechanism $S = \{M_{22}, \tau_{22}, a_2\}$, where M_{22} is a set of possible messages; $\tau_{22}(m_{22})$ is a transfer; and $a_2(m_{22}) \in \{0, 1\}$ is the resulting allocation. When recontracting is not permitted, S is offered to cohort-two buyers and cohort-one buyers that chose not to participate in D. When recontracting is allowed, cohort-one buyers participating in D may choose to exit that mechanism and then participate in S. (Note that buyers are free to exercise their right to not purchase *within* D—by sending a message $m_{12} \in a_1^{-1}(0|m_{11})$ that maximizes the period-two transfer $\tau_{12}(m_{11}, m_{12})$ from the seller—and still go on to participate in S; this allows a buyer to, for instance, claim any refunds available from D and apply them towards purchasing in S.) Under both regimes, the fact that the mechanism S is offered to cohort-one buyers not participating in D corresponds to our anonymity assumption: the seller is unable to distinguish between these cohort-one buyers and newly arrived cohort-two buyers.

Strategy profiles are defined in the standard way: they are a choice of an action at each information set. Similarly, beliefs at each information set are defined in the usual way. These jointly generate outcomes: allocations and payments conditional on buyers' types λ and values v. Thus,

[6] Analogous conditions were first imposed by Baron and Besanko (1984) and Besanko (1985). Pavan et al. (2014) develop sufficient conditions guaranteeing the validity of the first-order approach in dynamic environments. Absent such conditions, a local approach may not yield global incentive compatibility—see Battaglini and Lamba (2015).

[7] The restriction to deterministic mechanisms is *with* loss of generality. The various assumptions we impose are too weak to ensure that deterministic contracts are optimal—even in the simplest case of full commitment with only a single cohort, Courty and Li (2000) show that the principal can reduce information rents by "fine-tuning" contracts using randomization. A general analysis that allows for stochastic contracts is beyond the scope of the present work. A key difficulty lies in the absence of natural conditions that yield a tractable characterization of incentive compatible stochastic contracts in this dynamic multidimensional environment.

if a buyer contracts in the initial period, the *within-contract* outcomes (that is, outcomes when contracting in the first period and remaining in a contract through the end of period two) are

$$\{p_{11}(\lambda), p_{12}(v,\lambda), q_1(v,\lambda)\}_{v\in\mathbf{V},\lambda\in\Lambda},$$

where p_{11} is the period-one payment; p_{12} is the period-two payment; and q_1 is the allocation. For buyers that enter into a period-two contract, the resulting second-period outcomes are denoted by

$$\{p_{22}(v), q_2(v)\}_{v\in\mathbf{V}},$$

where p_{22} is the period-two payment and q_2 is the allocation.[8]

Working directly with the underlying mechanisms D and S is intractable, as the set of possible contracts is large and unwieldy. Instead, we adapt the approach of Riley and Zeckhauser (1983) and Skreta (2006) and search for optimal outcomes, with the additional restriction that these outcomes are implementable in a perfect Bayesian equilibrium of the "full" underlying game. Thus, our analysis proceeds *as if* the seller uses a direct revelation mechanism for cohort-one buyers, taking into account the constraints imposed by sequential rationality.

We must therefore account for the potential for strategic delay by cohort-one buyers. Therefore, the period-one mechanism includes a participation decision $x_1 : \Lambda \to [0,1]$, where $x_1(\lambda)$ denotes the probability with which type-λ buyers contract immediately, and $(1 - x_1(\lambda))$ is the probability that a type-λ buyer delays contracting until the second period. Since there is a continuum of buyers, these probabilities do not generate any aggregate uncertainty about the set of buyers who ultimately delay contracting, and so $x_1(\lambda)$ and $(1 - x_1(\lambda))$ also correspond to the fractions of type-λ buyers that contract immediately or delay until the second period, respectively. In addition, we let $x_2(v,\lambda) \in [0,1]$ denote the (conditional) probability with which a type-λ buyer with value $v \in \mathbf{V}$ chooses to remain in a period-one contract that they previously entered, while $(1 - x_2(v,\lambda))$ is the probability with which they recontract and participate in the seller's period-two mechanism. (When recontracting is not permitted, we simply set $x_2(v,\lambda) = 1$ for all $\lambda \in \Lambda$ and $v \in \mathbf{V}$.)

We solve for the optimal contract by letting the seller choose x_1 and x_2, but require that these choices be consistent with rational behavior on the buyers' part. This parallels the approach of Jullien (2000), who characterizes the optimal contract for an environment with exogenous type-dependent participation constraints. In our setting, however, these constraints are endogenously determined, so excluding some buyers has an additional impact on the seller's problem.[9]

It is important to emphasize that we have restricted the seller by ruling out first-period contracts that condition on the seller's second-period contract. (In particular, no elements of the period-two mechanism S are arguments of any elements of the period-one mechanism D.) Instead, the contracts we study are *bilateral* contracts that govern the relationship between the seller and an *individual* buyer and are independent of *other* buyers' choices or behavior. A seller can, of course, generate additional commitment by offering such contracts. A simple example is a contract that, in period one, promises prohibitively large payments to buyers if the period-two contract deviates from that offered by a seller with full commitment power. In this case,

[8] While the period-two mechanism S may *attempt* to discriminate across cohorts or initial-period types λ, this dimension of private information is payoff-irrelevant in period two; therefore, the resulting outcomes depend only on values.

[9] Mierendorff (2014) examines a similar issue in a dynamic auction setting where long-lived buyers can pretend to be impatient. The resulting incentive constraint serves as an endogenously determined participation constraint.

the resulting outcome corresponds to the full-commitment optimal contracts we describe in Section 4.1. Another method of generating additional (but less than full) commitment power is to offer "coupons" that grant additional (type-dependent) discounts on the period-two price in the event of recontracting. (Such a scheme is not permitted in our setting, as it implicitly requires the period-one contract's payment rules to depend on actions taken in the period-two mechanism.) Such coupons induce the seller to avoid lowering prices due to the additional cost of buyers cashing in their coupons.[10]

2.4. *Buyer payoffs and constraints*

With these preliminaries in hand, we can recursively define the buyers' payoffs. We begin with payoffs in the second period. Define

$$U_{12}(v,\lambda) := q_1(v,\lambda)v - p_{12}(v,\lambda)$$

to be the payoffs from continuing in a period-one contract in period two, and let

$$U_{22}(v) := q_2(v)v - p_{22}(v)$$

be the payoffs from entering a period-two contract. In addition, we define the payoff of a buyer who recontracts in period two as

$$\widetilde{U}_{12}(v,\lambda) := U_{22}(v) + \tilde{p}_{12}(\lambda),$$
$$\text{where } \tilde{p}_{12}(\lambda) := \max\left\{0, \max_{v'}\left\{-p_{12}(v',\lambda)\middle| q_1(v',\lambda)=0\right\}\right\}$$

is the largest transfer a buyer can claim while exiting the period-one contract *without* purchasing the good.[11] Thus, the continuation payoff of a buyer who entered into a contract in period one is

$$V_{12}(v,\lambda) := x_2(v,\lambda)U_{12}(v,\lambda) + (1-x_2(v,\lambda))\widetilde{U}_{12}(v,\lambda).$$

We have a similar set of payoff functions for period one. The expected payoff of a cohort-one buyer who enters into a contract in period one is

$$U_{11}(\lambda) := -p_{11}(\lambda) + \int_{\mathbf{V}} V_{12}(v,\lambda)dG(v|\lambda),$$

while the expected payoff of a cohort-one buyer who delays contracting until the second period is

$$\widetilde{U}_{11}(\lambda) := \int_{\mathbf{V}} U_{22}(v)dG(v|\lambda).$$

[10] "Most-favored-nation" clauses or "best-price" guarantees induce similar effects: since the seller can fully commit to the terms of the period-one contract, these clauses allow the seller to pre-commit to some terms of the period-two contract and implicitly endow her with additional long-term commitment power. See Board (2008) and Butz (1990) for analyses of such guarantees in dynamic durable-goods monopoly models.

[11] Recall from Section 2.3 that, in addition to simply opting out in period two, buyers participating in the seller's period-one mechanism are also free to exercise any contractual options not to purchase (and claim any associated refunds) before recontracting. Thus, $\tilde{p}_{12}(\lambda)$ reflects the buyer's gains, if any, from her optimal exit decision.

Finally, the overall expected payoff of a cohort-one buyer is

$$V_{11}(\lambda) := x_1(\lambda)U_{11}(\lambda) + (1 - x_1(\lambda))\widetilde{U}_{11}(\lambda).$$

Of course, the outcomes (and corresponding payoffs) must be consistent with equilibrium in the underlying contracting game and its various subgames. We again work backwards and start from the second period. Note that in equilibrium, it must be the case that each type of buyer must prefer behaving in accordance with their type's strategy rather than with the strategy of any other type. Therefore, for cohort-one buyers continuing in a period-one contract, we must have

$$U_{12}(v, \lambda) \geq q_1(v', \lambda)v - p_{12}(v', \lambda) \text{ for all } v, v' \in \mathbf{V} \text{ and } \lambda \in \Lambda. \tag{IC$_{12}$}$$

A similar requirement holds for buyers entering into a period-two contract: we must have

$$U_{22}(v) \geq q_2(v')v - p_{22}(v') \text{ for all } v, v' \in \mathbf{V}. \tag{IC$_{22}$}$$

In period one, we require that (conditional on entering into a contract) a type-λ buyer must prefer her own contract to that chosen by any other type $\lambda' \in \Lambda$.[12] This implies that we must have

$$U_{11}(\lambda) \geq -p_{11}(\lambda') + \int_{\mathbf{V}} V_{12}(v, \lambda')dG(v|\lambda) \text{ for all } \lambda, \lambda' \in \Lambda. \tag{IC$_{11}$}$$

In addition, since each buyer is free to postpone contracting, her overall expected payoff must be bounded below by the option value of delay. Moreover, individual delay decisions (as recommended by the seller) must be optimal: a buyer's period-one participation decision must maximize her expected payoff, and a buyer should be willing to randomize only if she is indifferent between delaying and contracting immediately. This is summarized as

$$V_{11}(\lambda) \geq \widetilde{U}_{11}(\lambda) \text{ for all } \lambda \in \Lambda, \text{ with equality if } x_1(\lambda) < 1. \tag{SD}$$

A similar constraint is necessary when buyers are free to anonymously recontract in period two. In that setting, a cohort-one buyer's continuation utility (within a period-one contract) is bounded below by the option value of recontracting; thus, we must have

$$V_{12}(v, \lambda) \geq \widetilde{U}_{12}(v, \lambda) \text{ for all } v \in \mathbf{V} \text{ and } \lambda \in \Lambda, \text{ with equality if } x_2(v, \lambda) < 1. \tag{$\widetilde{\text{RC}}$}$$

(When recontracting is not permitted, we simply impose $x_2(v, \lambda) = 1$ for all $v \in \mathbf{V}$ and $\lambda \in \Lambda$.)

Finally, buyer participation must be voluntary. This requires that, for all $\lambda \in \Lambda$ and $v \in \mathbf{V}$,

$$U_{22}(v) \geq 0, \; U_{12}(v, \lambda) \geq 0, \text{ and } V_{11}(\lambda) \geq 0. \tag{IR}$$

Notice, however, that the option value of strategic delay is sufficient to make the period-one participation constraint redundant whenever the period-two contract is individually rational.[13]

[12] Since initial types are payoff-irrelevant in period two, constraints (IC$_{12}$) and (IC$_{22}$) preclude compound "misreports."

[13] Unlike the participation constraints in Krähmer and Strausz (2015), we place no restrictions on ex post payoffs.

2.5. The seller's problem

The seller's expected profits, from the perspective of period one, may be expressed as

$$\Pi_1 := \iint\limits_{\Lambda\times\mathbf{V}} x_1(\lambda)x_2(v,\lambda)[p_{11}(\lambda)+p_{12}(v,\lambda)]dG(v|\lambda)dF(\lambda)$$
$$+\iint\limits_{\Lambda\times\mathbf{V}} x_1(\lambda)(1-x_2(v,\lambda))[p_{11}(\lambda)-\tilde{p}_{12}(\lambda)+p_{22}(v)]dG(v|\lambda)dF(\lambda)$$
$$+\iint\limits_{\Lambda\times\mathbf{V}} (1-x_1(\lambda))p_{22}(v)dG(v|\lambda)dF(\lambda)+\gamma\int\limits_{\mathbf{V}} p_{22}(v)dH(v).$$

In the absence of full commitment, the seller's second-period contract must be sequentially rational; in particular, the seller's period-two mechanism must maximize her continuation profits

$$\Pi_2 := \Pi_1 - \iint\limits_{\Lambda} x_1(\lambda)p_{11}(\lambda)dF(\lambda).$$

Therefore, when buyers cannot recontract in the second period, the sequential rationality constraint can be written as

$$\{q_2, p_{22}\} \in \arg\max\{\Pi_2\} \text{ subject to (IC}_{22}\text{) and (IR).} \tag{SR}$$

If, instead, cohort-one buyers *are* free to recontract in period two, then this constraint becomes

$$\{q_2, p_{22}\} \in \arg\max\{\Pi_2\} \text{ subject to (IC}_{22}\text{), (IR), and (}\widetilde{\text{RC}}\text{).} \tag{$\widetilde{\text{SR}}$}$$

Note that the seller's continuation profits Π_2 (and therefore problems (SR) and ($\widetilde{\text{SR}}$)) are unaffected by measure-zero changes to the set of buyers that choose to delay or to recontract, as such changes do not affect the resulting distributions of values.[14]

Thus, when cohort-one buyers cannot recontract in period two, the seller's problem is to

$$\max_{x_1,p_{11},p_{12},q_1,p_{22},q_2}\{\Pi_1\} \text{ subject to (IC}_{11}\text{), (IC}_{12}\text{), (IR), (SD), and (SR).} \tag{$\mathcal{P}$}$$

Similarly, when recontracting is permitted in period two, the seller solves

$$\max_{x_1,p_{11},p_{12},q_1,x_2,p_{22},q_2}\{\Pi_1\} \text{ subject to (IC}_{11}\text{), (IC}_{12}\text{), (IR), (SD), and (}\widetilde{\text{SR}}\text{).} \tag{$\widetilde{\mathcal{P}}$}$$

[14] Recall that our model features a continuum of buyers instead of finitely many agents. This implies that a strategic deviation by any (infinitesimal) individual buyer in our setting does not affect the distribution of agents contracting in each period, leaving the expected path of play unchanged. We thank an anonymous referee for pointing out an alternative approach that delivers identical results: suppose a single buyer privately arrives with positive probability at each date. In this case, strategic delay by the period-one buyer remains on path, so the period-two contract does not depend on whether the buyer actually arrived in period one or two.

3. Preliminary observations

It is relatively straightforward to simplify the seller's problem and its various constraints.[15] Incorporating the incentive compatibility and strategic delay constraints into the seller's objective function yields a more useable (and familiar) virtual surplus form expressing that the seller's payoff as a function of the "effective" allocation rule $\bar{q}_1(v,\lambda)$ alone, where we define

$$\bar{q}_1(v,\lambda) := x_1(\lambda)x_2(v,\lambda)q_1(v,\lambda) + (1 - x_1(\lambda)x_2(v,\lambda))q_2(v).$$

In particular, we can write the seller's ex ante payoff as

$$\begin{aligned}\Pi_1 = &\iint_{\Lambda\times\mathbf{V}} \bar{q}_1(v,\lambda)\varphi_1(v,\lambda)dG(v|\lambda)dF(\lambda) - V_{11}(\underline{\lambda})\\ &+ \gamma\int_{\mathbf{V}} q_2(v)\psi_2(v)dH(v) - \gamma U_{22}(\underline{v}),\end{aligned}$$

where the cohort-one and cohort-two virtual surplus functions are given by

$$\varphi_1(v,\lambda) := v + \frac{G_\lambda(v|\lambda)}{g(v|\lambda)}\frac{1-F(\lambda)}{f(\lambda)} \text{ and } \psi_2(v) := v - \frac{1-H(v)}{h(v)},$$

respectively. Similarly, the seller's continuation payoff is

$$\begin{aligned}\Pi_2 = &\iint_{\Lambda\times\mathbf{V}} \bar{q}_1(v,\lambda)\psi_1(v,\lambda)dG(v|\lambda)dF(\lambda)\\ &- \int_{\Lambda} [x_1(\lambda)V_{12}(\underline{v},\lambda) + (1 - x_1(\lambda))U_{22}(\underline{v})]dF(\lambda)\\ &+ \gamma\int_{\mathbf{V}} q_2(v)\psi_2(v)dH(v) - \gamma U_{22}(\underline{v}),\end{aligned}$$

where the virtual surplus for a cohort-one buyer with *known* initial type $\lambda \in \Lambda$ is given by

$$\psi_1(v,\lambda) := v - \frac{1-G(v|\lambda)}{g(v|\lambda)}.$$

Finally, the assumption of deterministic mechanisms implies that the allocation rules correspond to cutoff policies: there exists a function $k_1 : \Lambda \to \mathbf{V}$ and a constant $\alpha \in \mathbf{V}$ such that

$$q_1(v,\lambda) = \begin{cases} 0 & \text{if } v < k_1(\lambda), \\ 1 & \text{if } v \geq k_1(\lambda); \end{cases} \text{ and } q_2(v) = \begin{cases} 0 & \text{if } v < \alpha, \\ 1 & \text{if } v \geq \alpha. \end{cases}$$

This cutoff behavior is inherited by the effective allocation $\bar{q}_1(v,\lambda)$, which can be represented as a cutoff policy with threshold $\bar{k}_1(\lambda)$ for all $\lambda \in \Lambda$.[16] Of course, the stochastic order on

[15] The observations in this section, Lemma 1 aside, follow relatively standard arguments; see Appendix B for details.

[16] If $\bar{q}_1$ is not a cutoff rule, then there exists an interval (λ_1,λ_2) with $x_1(\lambda) \in (0,1)$ but $k_1(\lambda) \neq \alpha$ for all $\lambda \in (\lambda_1,\lambda_2)$. But if $k_1(\lambda) < \alpha$ and $q_1(\cdot,\lambda)$ is more generous than $q_2(\cdot)$, first-order stochastic dominance implies that all types $\lambda' > \lambda$ strictly prefer $q_1(\cdot,\lambda)$ to $q_2(\cdot)$. This, of course, contradicts the optimality of delay for buyers in the interval (λ_1,λ_2). A symmetric argument rules out the possibility that $k_1(\lambda) > \alpha$ for $\lambda \in (\lambda_1,\lambda_2)$. Thus, any agent that mixes between immediate contracting and delay must be receiving identical cutoffs in both.

$\{G(\cdot|\lambda)\}_{\lambda\in\Lambda}$ implies that higher initial types are more likely to realize values above any given cutoff; due to this single-crossing property, any implementable $\bar{k}_1(\lambda)$ must be (weakly) decreasing.

Recall that when buyers cannot recontract in period two, the seller's period-two problem (SR) requires maximizing continuation profits from the combined population of cohort-two buyers and cohort-one buyers that delayed contracting. As is well known, the optimal allocation with linear payoffs and indivisible goods is implemented by a posted price; the optimal such price solves

$$\max_{\alpha}\left\{\int_{\Lambda}(1-x_1(\lambda))\pi_\lambda(\alpha)dF(\lambda)+\gamma\pi_H(\alpha)\right\}. \tag{SR$'$}$$

The period-two problem $(\widetilde{\text{SR}})$ when cohort-one buyers are free to recontract in the second period is somewhat more subtle, as the seller's choice of period-two mechanism influences—through constraint $(\widetilde{\text{RC}})$ and its impact on $x_2(v,\lambda)$—the set of buyers that choose to recontract; that is, the set of participating buyers in period two is endogenously determined by the seller's choice of contract. Despite this additional complication, we show that a simple price is still optimal for the seller in period two (see Lemma B.4 in Appendix B, which shows that the seller does not use subsidies to encourage recontracting). Moreover, when faced with a price, cohort-one buyers will choose to recontract whenever that price is more generous (that is, lower) than their already-contracted cutoff. Therefore, the seller's period-two problem $(\widetilde{\text{SR}})$ becomes

$$\max_{\alpha}\left\{\int_{\Lambda}\left(x_1(\lambda)\left[\pi_\lambda(\min\{k_1(\lambda),\alpha\})-U_{12}(\underline{v},\lambda)\right]+(1-x_1(\lambda))\pi_\lambda(\alpha)\right)dF(\lambda)\right.$$
$$\left.+\gamma\pi_H(\alpha)\right\}. \tag{$\widetilde{\text{SR}}'$}$$

Thus, regardless of whether buyers can recontract in period two or not, the period-two contract takes the form of a price $\alpha\in\mathbf{V}$. However, this second-period price also serves as a type-dependent outside option for each cohort-one buyer. The following result helps characterize when the resulting endogenous "participation" constraint (SD) binds and how it impacts the seller's problem.

Lemma 1. *Suppose constraint (SD) binds at some $\hat{\lambda}\in\Lambda$ in the solution to either $(\mathcal{P})$ or $(\widetilde{\mathcal{P}})$. Then (SD) binds at all $\lambda<\hat{\lambda}$; $\bar{k}_1(\lambda)=\alpha$ for all $\lambda\in(\underline{\lambda},\hat{\lambda})$; and $\bar{k}_1(\lambda)\le\alpha$ for all $\lambda>\hat{\lambda}$.*[17]

The intuition behind this result is relatively straightforward. Suppose that, in an optimal contract, the strategic delay constraint (SD) binds for some buyer with initial type λ_1 but is slack for some $\lambda_2<\lambda_1$. Incentive compatibility implies that type λ_1 prefers strategic delay to λ_2's contract, which type λ_2 in turn strictly prefers to strategic delay. Since Assumption 1 implies first-order stochastic dominance (so higher types have a stronger preference for the "quantity" allocated), this is only possible if λ_2's contract is less generous than the optimal period-two contract. Thus, we must have $\bar{k}_1(\lambda_2)>\alpha$ for any $\lambda_2<\lambda_1$ for whom constraint (SD) is slack. If

[17] Note that Lemma 1 holds under both full and limited commitment, whether or not recontracting is possible.

the seller instead offers type λ_2 the period-two contract in advance (by setting $k_1(\lambda_2) = \alpha$), she can raise her profits (the surplus from trade increases while type λ_2's payoff decreases) without affecting any incentive or strategic delay constraints (as all types already have the option of strategic delay).

4. Optimal contracts

We now characterize the solution to the seller's optimal contracting problem. We contrast results across four environments that are ordered with respect to the seller's commitment power: full commitment when buyers cannot delay contracting; full commitment with strategic delay; limited commitment without recontracting; and limited commitment with recontracting. Although our primary focus is on settings with partial commitment, the full-commitment settings provide useful benchmarks for understanding the impact of limitations on the seller's ability to commit to future contractual terms.[18]

4.1. Full commitment

Suppose first that cohort-one buyers are unable to delay contracting or recontract in period two. In this case, the seller is able to treat the two cohorts of buyers separately, with no regard to the potential impact of the second-period contract on buyers in the first cohort. Thus, the seller simply charges the monopoly price p_H to cohort-two buyers, while the optimal cohort-one contract (depicted in Fig. 1a) is essentially that of Courty and Li (2000). In particular, the linearity of the seller's profits with respect to $\bar{q}_1$ implies that the optimal cohort-one contract is a set of (call) options with strike prices $k^{ND}(\lambda)$ defined by

$$\varphi_1(k^{ND}(\lambda), \lambda) = 0.$$

(To ensure that this function is well-defined, we set $k^{ND}(\lambda) := \underline{v}$ if $\varphi_1(\underline{v}, \lambda) > 0$.) Assumption 4 implies that these strike prices are decreasing in λ. Buyers also pay an (increasing) upfront premium $p_{11}^{ND}(\lambda)$ that is determined using the initial-period sorting constraint (IC$_{11}$).

Theorem 1. *(See Courty and Li, 2000.) Suppose that cohort-one buyers cannot delay contracting until the second period, so $x_1^{ND}(\lambda) := 1$ for all $\lambda \in \Lambda$. Then the seller maximizes profits by offering a period-one contract $\{q_1^{ND}, p_{11}^{ND}, p_{12}^{ND}\}$, where*

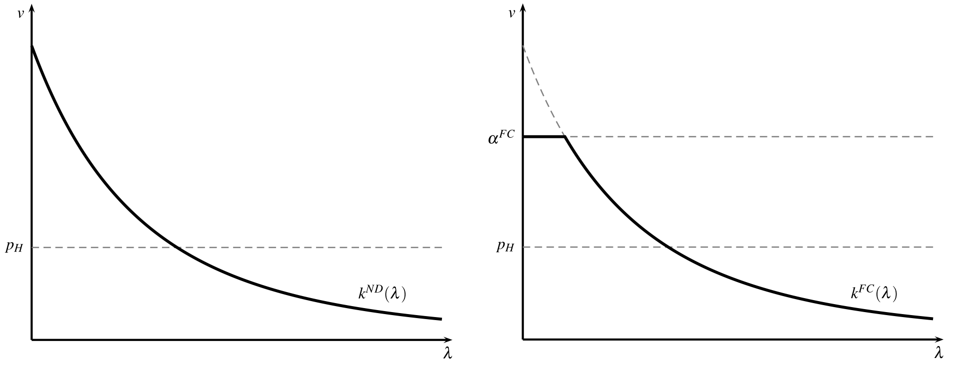
$$q_1^{ND}(v,\lambda) := \begin{cases} 0 & \text{if } v < k^{ND}(\lambda), \\ 1 & \text{if } v \geq k^{ND}(\lambda), \end{cases} \quad p_{12}^{ND}(v,\lambda) := q_1^{ND}(v,\lambda) k^{ND}(\lambda), \text{ and}$$

$$p_{11}^{ND}(\lambda) := \int_{k^{ND}(\lambda)}^{\bar{v}} (1 - G(v|\lambda))dv + \int_{\underline{\lambda}}^{\lambda} \int_{k^{ND}(\mu)}^{\bar{v}} G_\lambda(v|\mu) dv d\mu,$$

and a period-two contract $\{q_2^{ND}, p_{22}^{ND}\}$ corresponding to the fixed price p_H.

When buyers *can* strategically delay contracting, the seller's period-two price α now serves as a type-dependent outside option for cohort-one buyers and constraint (SD) must be considered. The argument of Lemma 1 applies immediately in this setting to show that whenever this

[18] For the sake of brevity, we omit proofs of the full-commitment results in Section 4.1, but they are available on request.



(A) Without strategic delay (B) With strategic delay

Fig. 1. Optimal contracts with full commitment.

constraint binds for some type $\hat{\lambda} \in \Lambda$, it must also bind for all $\lambda < \hat{\lambda}$. This observation implies that, when the seller can fully commit to a second-period price α, the optimal cohort-one contract (depicted in Fig. 1b) remains relatively simple: cohort-one buyers are offered a set of call options with strike prices $k^{FC}(\lambda)$ defined by

$$k^{FC}(\lambda) := \min\{\alpha, k^{ND}(\lambda)\};$$

the upfront premium for $k^{FC}(\lambda) = \alpha$ is zero; the premiums for the remaining options are discounted from the no-delay premiums $p_{11}^{ND}(\lambda)$ by the lowest type's option value of waiting; and no buyers delay contracting (since a "free" call option with strike price α is equivalent to waiting).

Theorem 2. *Suppose the seller fully commits to the second-period contract $\{q_2^{FC}, p_{22}^{FC}\}$ corresponding to a fixed price α, and also that cohort-one buyers can strategically delay contracting. Then the period-one contract $\{q_1^{FC}, p_{11}^{FC}, p_{12}^{FC}\}$ maximizes the seller's cohort-one profits, where*

$$q_1^{FC}(v, \lambda) := \begin{cases} 0 & \text{if } v < k^{FC}(\lambda), \\ 1 & \text{if } v \geq k^{FC}(\lambda); \end{cases} \quad p_{12}^{FC}(v, \lambda) := q_1^{FC}(v, \lambda) k^{FC}(\lambda);$$

$$p_{11}^{FC}(\lambda) := \int_{k^{FC}(\lambda)}^{\bar{v}} (1 - G(v|\lambda)) dv + \int_{\underline{\lambda}}^{\lambda} \int_{k^{FC}(\mu)}^{\bar{v}} G_\lambda(v|\mu) dv d\mu - \int_{\alpha}^{\bar{v}} (1 - G(v|\underline{\lambda})) dv;$$

and all cohort-one buyers optimally choose to contract immediately, so $x_1^{FC}(\lambda) := 1$ for all $\lambda \in \Lambda$. Moreover, the seller's optimal period-two contract corresponds to the fixed price α^{FC} that maximizes

$$\Pi^{FC}(\alpha) := \int_{\Lambda} \int_{k^{FC}(\lambda)}^{\bar{v}} \varphi_1(v, \lambda) dG(v|\lambda) dF(\lambda) - \int_{\alpha}^{\bar{v}} (1 - G(v|\underline{\lambda})) dv + \gamma \pi_H(\alpha),$$

the seller's total profits (from both cohorts of buyers) under full commitment. Finally, $\alpha^{FC} > p_H$.

Thus, a seller with full commitment power optimally increases the second-period price α^{FC} above the price charged in the absence of strategic delay. Doing so decreases profits derived from second-cohort buyers—in the absence of potential buyers from cohort one, the seller would simply charge p_H in the second period. This decrease is, of course, offset by two effects. First, the outside option of *all* first-cohort buyers is reduced, as delayed contracting involves a higher price; this implies that the induced participation constraint (SD) is relaxed somewhat, reducing the rents left to cohort-one buyers. Second, raising the second-period contract price allows the seller to mimic the distortions of the no-delay optimal contract for a larger subset of cohort-one buyers.

In the limit as γ shrinks to zero, the impact of cohort-two entrants on the seller's profits vanishes, and the tradeoff between maximizing profits from cohort-two buyers and reducing the outside option of cohort-one buyers disappears. Thus, in the limit, the seller commits to $\alpha^{FC} \geq \bar{v}$ and no sales in the final period, thereby reducing the value of strategic delay to zero. Of course, it is in precisely this situation that the seller's ability to commit is most valuable: in the absence of commitment to future contractual terms, cohort-one buyers will anticipate the seller's incentive to decrease her second-period price should they delay contracting. It is with this in mind that we now turn to the seller's problem in the face of limited commitment.

4.2. *Limited commitment without recontracting*

Consider the seller's problem ($\mathcal{P}$) when cohort-one buyers cannot recontract in period two. Using the observations of Section 3, this problem can be simplified and rewritten as

$$\max_{x_1,\hat{\lambda},p_{11},p_{12},k_1,\alpha} \left\{ \int_{\underline{\lambda}}^{\hat{\lambda}} \int_{\alpha}^{\bar{v}} \varphi_1(v,\lambda) dG(v|\lambda) dF(\lambda) + \int_{\hat{\lambda}}^{\bar{\lambda}} \int_{k_1(\lambda)}^{\bar{v}} \varphi_1(v,\lambda) dG(v|\lambda) dF(\lambda) + \gamma \pi_H(\alpha) - V_{11}(\underline{\lambda}) \right\}$$

$$\text{subject to } x_1(\lambda) = 1 \text{ for all } \lambda > \hat{\lambda},\ V_{11}(\underline{\lambda}) \geq \widetilde{U}_{11}(\underline{\lambda}),\ \bar{k}_1(\lambda) \text{ decreasing, and (SR}'). \qquad (\mathcal{P}')$$

(Note that we denote by $\hat{\lambda}$ the upper bound of the interval, possibly degenerate, of types λ for whom (SD) binds; its existence follows from Lemma 1.)

It is quite easy to see that the constraint $V_{11}(\underline{\lambda}) \geq \widetilde{U}_{11}(\underline{\lambda})$ (which substitutes the tighter delay constraint (SD) for the participation constraint in (IR)) must bind. Therefore, we must have

$$V_{11}(\underline{\lambda}) = \int_{\mathbf{V}} \max\{v-\alpha, 0\} dG(v|\underline{\lambda}) = \int_{\alpha}^{\bar{v}} (v-\alpha) dG(v|\underline{\lambda}) = \int_{\alpha}^{\bar{v}} (1 - G(v|\underline{\lambda})) dv.$$

Moreover, once we fix $\hat{\lambda}$ and $x_1(\cdot)$, we can isolate the question of optimally choosing the cutoffs $k_1(\lambda)$.[19] Recall from Lemma 1 that we must have $k_1(\lambda) \leq \alpha$ for all $\lambda > \hat{\lambda}$. It is easy to see (by pointwise maximization for each $\lambda > \hat{\lambda}$) that the optimal allocation and cutoff must then be

$$q_1^{LC}(v,\lambda) := \begin{cases} 0 & \text{if } v < k^{LC}(\lambda), \\ 1 & \text{if } v \geq k^{LC}(\lambda); \end{cases} \quad \text{where } k^{LC}(\lambda) := \begin{cases} \alpha & \text{if } \lambda \leq \hat{\lambda}, \\ \min\{\alpha, k^{ND}(\lambda)\} & \text{if } \lambda > \hat{\lambda}. \end{cases}$$

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19 Assumption 2 implies that, for any delay decisions $x_1(\cdot)$, the objective function in (SR′) is strictly concave and thus admits a unique maximizer. Therefore, by fixing $\hat{\lambda}$ and $x_1(\cdot)$, we also implicitly pin down α.

Note, as before, that Assumption 4 implies that $k^{LC}(\cdot)$ is decreasing, and so the induced allocation rule is increasing in λ and the initial-period incentive compatibility constraint (IC_{11}) is satisfied.

We can also determine the payment rules for the optimal period-one contract. We let

$$p_{12}^{LC}(v,\lambda) := q_1^{LC}(v,\lambda)k^{LC}(\lambda), \tag{1}$$

which corresponds to simply charging each buyer a strike price in period two equal to her (type-dependent) cutoff. Meanwhile, $p_{11}^{LC}(\lambda)$ is pinned down via the envelope condition corresponding to (IC_{11}), where the constant of integration is simply the payoff $V_{11}(\underline{\lambda})$ received by the lowest type:

$$p_{11}^{LC}(\lambda) := \int_{k^{LC}(\lambda)}^{\bar{v}} (1 - G(v|\lambda))dv + \int_{\underline{\lambda}}^{\lambda} \int_{k^{LC}(\mu)}^{\bar{v}} G_\lambda(v|\mu)dvd\mu - \int_{\alpha}^{\bar{v}} (1 - G(v|\underline{\lambda})). \tag{2}$$

Notice that $p_{11}^{LC}(\lambda)$ is the immediate (limited commitment) analogue of $p_{11}^{FC}(\lambda)$ from Theorem 2; it is identical in "form" to the full commitment case, but the cutoffs involved now differ.

Finally, we turn to characterizing the set of buyers that delay contracting. It is without loss of generality to consider contracts that induce *all* buyers within an *interval* to delay contracting (with probability one), leaving all other buyers to contract immediately. To see why this should be the case, consider a contract in which the seller recommends delay (with probability 1) for all buyers with initial types in the set $\Lambda_1 := [\lambda_1, \lambda_2] \cup [\lambda_3, \lambda_4]$ (where $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$), and assume (for the sake of illustration only; the remaining possibilities are covered in Appendix A) that the resulting period-two price is some $\alpha_1 < p_{\lambda_2}$, where p_{λ_2} is the monopoly price corresponding to $G(\cdot|\lambda_2)$. Suppose that the seller instead recommends the delay of all buyers with initial types in the set $\Lambda_2 := \Lambda_1 \cup (\lambda_2, \lambda_2 + \epsilon]$ for sufficiently small $\epsilon > 0$. Since Assumption 1 implies that type-specific monopoly prices are increasing in λ, the resulting period-two price will be some $\alpha_2 > \alpha_1$. Of course, Assumption 2 implies that the second-period price varies continuously as the measure of delayed buyers changes; therefore, there exists some $\delta > 0$ such that when the set of delayed buyers is $\Lambda_3 := \Lambda_2 \setminus (\lambda_4 - \delta, \lambda_4]$, the period-two price is $\alpha_3 = \alpha_1$. Therefore, the seller is able to maintain the same period-two price (leaving the tradeoff between cohort-one outside options and cohort-two profits unaffected) while decreasing the upper bound of the set of delayed buyers.

Of course, Lemma 1 implies that (SD) binds and $\bar{k}_1(\lambda) = \alpha_1$ for all $\lambda \leq \lambda_4$ when the delayed set is Λ_1, while $\bar{k}_1(\lambda) = \alpha_1$ for all $\lambda \leq \lambda_4 - \delta$ when the delayed set is Λ_3. But since $\delta > 0$, delaying buyers in Λ_3 provides the seller with greater freedom in choosing optimal cutoffs for types $\lambda \in (\lambda_4 - \delta, \lambda_4]$; that is, by decreasing the upper bound of the set of delayed buyers from λ_4 to $\lambda_4 - \delta$, the seller relaxes an implicit constraint in her problem and (weakly) increases profits.

Thus, any contract in which there are "gaps" in the set of delayed buyers—that is, where the set of delayed buyers is not connected—can be improved upon by concentrating the mass of delayed buyers and "closing" any gaps. The same argument also implies that there is no benefit to delaying only a fraction $x_1(\lambda) \in (0, 1)$ of type-λ buyers. Since the strategic delay constraint (SD) binds at λ if $x_1(\lambda) < 1$, delaying only a fraction of type-λ buyers has the same impact on the seller's ability to price discriminate in period one as delaying *all* type-λ buyers. However, delaying only a fraction of type-λ buyers attenuates the impact of delay on the second-period price (and ultimately requires a larger set of types to delay with positive probability, causing (SD) to bind over a larger interval). Thus, the seller optimally recommends deterministic delay to an interval of buyers.

Lemma 2. *Fix any period-one contract* $\{x_1, q_1, p_{11}, p_{12}\}$ *with* $\int_\Lambda (1 - x_1(\lambda))dF(\lambda) > 0$, *period-two price* α, *and total profits* Π. *Then there exists some* $\mu_1, \mu_2 \in \Lambda$ *and a period-one contract* $\{\hat{x}_1, \hat{q}_1, \hat{p}_{11}, \hat{p}_{12}\}$ *with*

$$\hat{x}_1(\lambda) := \begin{cases} 0 & \text{if } \lambda \in [\mu_1, \mu_2], \\ 1 & \text{otherwise,} \end{cases}$$

such that the induced second-period price is $\hat{\alpha} = \alpha$ *and total profits are* $\widehat{\Pi} \geq \Pi$.

With this characterization in hand, we can therefore rewrite the seller's problem as simply a choice of the interval of delayed types; that is, the seller must solve

$$\max_{\mu_1,\mu_2,\alpha} \left\{ \int_{\underline{\lambda}}^{\mu_2} \int_{\alpha}^{\bar{v}} \varphi_1(v,\lambda)dG(v|\lambda)dF(\lambda) + \int_{\mu_2}^{\bar{\lambda}} \int_{k^{LC}(\lambda)}^{\bar{v}} \varphi_1(v,\lambda)dG(v|\lambda)dF(\lambda) + \gamma\pi_H(\alpha) - \int_{\alpha}^{\bar{v}} (1 - G(v|\underline{\lambda}))dv \right\}$$

$$\text{subject to } \mu_1 \leq \mu_2 \text{ and (SR}'\text{).} \qquad (\mathcal{R})$$

The solution to this problem yields the seller's optimal contract under limited commitment.

Theorem 3. *Let* μ_1^{LC} *and* μ_2^{LC} *solve problem* $(\mathcal{R})$, *define*

$$x_1^{LC}(\lambda) := \begin{cases} 0 & \text{if } \lambda \in [\mu_1^{LC}, \mu_2^{LC}], \\ 1 & \text{otherwise,} \end{cases}$$

and let α^{LC} *denote the solution to* (SR′) *given* x_1^{LC}. *Then*

1. *the contract* $\{x_1^{LC}, q_1^{LC}, p_{11}^{LC}, p_{12}^{LC}\}$ *solves the seller's problem* $(\mathcal{P})$;
2. $\alpha^{LC} \in [p_H, \alpha^{FC}]$, *and if* $\alpha^{LC} \in (p_H, \alpha^{FC})$, *then* $\mu_1^{LC} = \bar{\mu} < \mu_2^{LC}$, *where* $\bar{\mu} > \hat{\mu}$ *is defined by* $p_{\bar{\mu}} = \alpha^{LC}$; *and*
3. *the seller's profits are given by*

$$\Pi^{LC} := \Pi^{FC}(\alpha^{LC}) - \int_{(k^{ND})^{-1}(\alpha^{LC})}^{\max\{(k^{ND})^{-1}(\alpha^{LC}),\mu_2\}} \int_{k^{ND}(\lambda)}^{\alpha^{LC}} \varphi_1(v,\lambda)dG(v|\lambda)dF(\lambda).$$

There are several key features of the characterization in Theorem 3 (illustrated in Fig. 2) to note. First, the seller delays an interval of cohort-one buyers; by doing so, she is able to increase the second-period price α^{LC} above the cohort-two monopoly price p_H. This has two effects: it relaxes the participation constraint (SD) induced by the possibility of strategic delay, thereby increasing the seller's profits from cohort-one buyers; but it also moves prices away from the cohort-two optimum, thereby decreasing the seller's profits from cohort-two buyers. Notice, however, that the induced second-period price α^{LC} is lower than the second-period price under full commitment α^{FC}. Thus, in the absence of commitment concerns, the seller would continue trading off cohort-two profits in order to decrease the implicit outside option available to cohort-one buyers. This tradeoff is not feasible when the seller has limited commitment, however; in

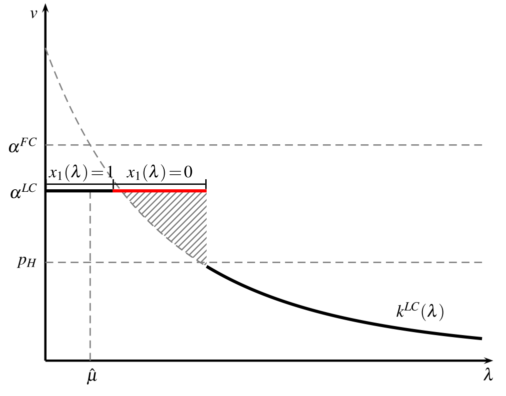

Fig. 2. Optimal contract with limited commitment and no recontracting.

order to continue increasing the period-two price, the seller needs to induce buyers with even higher types to delay contracting. While doing so succeeds in reducing the endogenous outside option in period one, it also reduces the seller's ability to screen buyers in the first period. In particular, each additional buyer that delays contracting until the second period is a buyer whom the seller cannot separate in the first period and sequentially screen. This reduced screening ability reflects the deadweight loss of limited commitment: in order to reduce the option value of strategic delay, the seller must inefficiently exclude buyers with initial types $\lambda \in ((k^{ND})^{-1}(\alpha^{LC}), \mu_2^{LC})$ and realized values $v \in (k^{ND}(\lambda), \alpha^{LC})$. If, on the other hand, the seller were able to fully commit in advance to a period-two price equal to α^{LC}, these buyers (indicated by the shaded region in Fig. 2) would not be excluded. This deadweight loss (quantified by the integral in part (3) of Theorem 3) is, of course, in addition to the fact that α^{LC} is suboptimally low for a seller with full commitment.

Inducing delay by "intermediate" buyers remains optimal even if we drop Assumption 2 and weaken Assumption 1 to simple first-order stochastic dominance. In the absence of these regularity assumptions, the set of buyers who choose to delay contracting may consist of disjoint intervals instead of a single connected set; that is, Lemma 2 (which showed that any contract can be improved by "coalescing" all delayed buyers into an interval) may fail in the absence of concavity and continuity. Weaker assumptions (such as monotone hazard rates or appropriately ordered virtual values) are not sufficient, as they generally imply only that the monopoly profit functions $\pi_\lambda(\cdot)$ are quasi-concave without imposing additional structure on the behavior of these functions away from their (single) peaks. This in turn implies that it is possible to achieve a higher period-two price by delaying the disjoint interval of types $[\underline{\mu}_1, \underline{\mu}_2] \cup [\overline{\mu}_1, \overline{\mu}_2]$ (with $\underline{\mu}_2 < \overline{\mu}_1$) instead of delaying any single interval $[\mu_1', \mu_2']$ with $\underline{\mu}_1 \leq \mu_1' \leq \mu_2' \leq \overline{\mu}_2$. Even when this is the case, however, these disjoint intervals remain "intermediate" in the sense of Theorem 3. This follows from Lemma 1's implication that delay by a relatively "high" buyer type forces the pooling of all lower types. This pooling prevents the seller from offering more efficient contracts to the more valuable high-λ buyers and then extracting that increased surplus via screening and price discrimination.

Note further that there is, in general, a gap between the type $\hat{\mu}$ corresponding to the cohort-two monopoly price and the lower bound of the interval of delayed types. To see why this is the case, notice that as we increase the mass of cohort-one buyers who delay contracting, we decrease the responsiveness of the sequentially rational second-period price to the composition of that set. Thus, increasing the second-period price is most efficiently achieved by inducing the delay of a relatively small set of buyers with moderately high monopoly prices (as opposed to a large set of buyers with low-to-intermediate monopoly prices). This implies that the optimal contract in the case of limited commitment will generally display an interesting non-monotonicity: cohort-one buyers with relatively low and relatively high initial-period types will contract in the initial period, while intermediate initial-period types will delay contracting to the second period.

Finally, note that we have thus far analyzed this dynamic contracting problem using a "reduced form" approach that focuses on the allocations and payments generated by equilibrium behavior. We have not yet verified, however, that there indeed exists a perfect Bayesian equilibrium in the underlying game that implements the optimal contract described above. Suppose, however, that the seller offers the menu of call options $\mathcal{M} := \{(0, \alpha^{LC})\} \cup \{(p_{11}^{LC}(\lambda), k^{LC}(\lambda)) | \lambda > \mu_2^{LC}\}$ in the first period, where each $(\tau_1, \tau_2) \in \mathcal{M}$ denotes an upfront premium τ_1 that guarantees the period-two strike price τ_2. If each cohort-one buyer expects that all buyers with initial-period types $\lambda \in [\mu_1^{LC}, \mu_2^{LC}]$ will delay contracting, then they will expect the seller to (rationally) set a price α^{LC} in the second period. Moreover, as each buyer is infinitesimal, a unilateral deviation in the timing of contracting will not affect the seller's second-period pricing problem (SR). Since constraints (IC_{11}) and (SD) are satisfied in the contract described in Theorem 3, this implies that it is, in fact, optimal for buyers with $\lambda < \mu_1^{LC}$ to choose the $(0, \alpha^{LC})$ option; for intermediate buyers with $\lambda \in [\mu_1^{LC}, \mu_2^{LC}]$ to reject the menu and delay contracting; and for buyers with $\lambda > \mu_2^{LC}$ to choose the $(p_{11}^{LC}(\lambda), k^{LC}(\lambda))$ option. This, of course, justifies the seller setting a period-two price α^{LC}. Finally, the strike prices have been chosen to implement the optimal cohort-one allocation rule q_1^{LC}. Thus, the contract described in Theorem 3 is indeed implementable in a perfect Bayesian equilibrium.

4.3. *Limited commitment with recontracting*

We now turn to the seller's problem ($\widetilde{\mathcal{P}}$) when cohort-one buyers are free to recontract in the second period. The solution is complicated by the fact that, unlike the no-recontracting case, the seller's period-two problem ($\widetilde{\text{SR}}'$) depends not only on the distribution of cohort-one buyers who delayed contracting, but also on the allocations promised to buyers who did not delay. This is because recontracting is advantageous for all buyers whose allocation from a contract signed in period one is less generous (has a higher cutoff) than the period-two price. Note that this will be the case irrespective of how the seller chooses to implement (via option contracts, refunds etc.) the period-one contract.

Again taking advantage of the observations in Section 3, the seller's problem ($\widetilde{\mathcal{P}}$) becomes

$$\max_{x_1, \hat{\lambda}, p_{11}, p_{12}, k_1, \alpha} \left\{ \int_{\underline{\lambda}}^{\hat{\lambda}} \int_{\alpha}^{\bar{v}} \varphi_1(v, \lambda) dG(v|\lambda) dF(\lambda) + \int_{\hat{\lambda}}^{\bar{\lambda}} \int_{k_1(\lambda)}^{\bar{v}} \varphi_1(v, \lambda) dG(v|\lambda) dF(\lambda) \right.$$
$$\left. + \gamma \pi_H(\alpha) - V_{11}(\underline{\lambda}) \right\}$$

$$\text{subject to } x_1(\lambda) = 1 \text{ for all } \lambda > \hat{\lambda},\ V_{11}(\underline{\lambda}) \geq \widetilde{U}_{11}(\underline{\lambda}),\ \bar{k}_1(\lambda) \text{ decreasing, and } (\widetilde{\text{SR}}'). \qquad (\widetilde{\mathcal{P}}')$$

(Note that $\hat{\lambda}$ again denotes the upper bound of the interval, possibly degenerate, on which (SD) binds; its existence follows from Lemma 1.)

As in the case where buyers cannot recontract in period two, the effective participation constraint $V_{11}(\underline{\lambda}) \geq \widetilde{U}_{11}(\underline{\lambda})$ must bind; therefore, we must have $V_{11}(\underline{\lambda}) = \int_{\alpha}^{\bar{v}}(1 - G(v|\underline{\lambda}))dv$. We can also pin down the optimal payment rules for buyers contracting in period one. We again make use of option contracts in which buyers are charged a period-two strike price equal to their contracted cutoffs, and the upfront premiums are determined via the envelope formulation of (IC$_{11}$). Therefore, the payment rules are as in (1) and (2), but using the appropriate cutoffs $k_1^{\widetilde{LC}}(\lambda)$. As before, these payments implement the optimal allocations if the effective cutoffs $\bar{k}_1^{\widetilde{LC}}(\lambda)$ are decreasing.

We first observe that, as in the case without recontracting, the optimal period-two price cannot be less than p_H. Recall that when recontracting was prohibited, the seller could always ensure that the period-two price was at least p_H (for instance, by contracting with all cohort-one buyers in the first period). The intuition for this similar result with recontracting is more subtle. Suppose the optimal period-two price were $\alpha' < p_H$. This would only be possible if seller was contracting in period two with some cohort-one buyers (either via delay or recontracting) in addition to the mass of newly arrived cohort-two buyers. Instead, the seller could "lock in" buyers who take advantage of that price by offering a free (zero premium) option with strike price α' in period one. Doing so will induce the same cohort-one allocation and profits (without affecting any incentives). Note however, that by doing so, it is no longer optimal for the seller to lower her price to α' in period two. Instead, the sequentially rational period-two price would be p_H. Consequently, this alternate contract would result in a gain of profits from cohort-two buyers and a reduced value for the delay option in period one (which implies the seller can charge more in the first period) contradicting the optimality of $\alpha' < p_H$.

Lemma 3. *In an optimal contract, the solution α^* to problem ($\widetilde{\text{SR}}'$) is such that $\alpha^* \geq p_H$.*

We focus on the case where the seller optimally induces a period-two price strictly greater than p_H. This case is particularly interesting as the optimal contract is qualitatively similar to the no-recontracting optimum described in Theorem 3, thereby isolating the influence of recontracting possibilities on the timing of contracting and the resulting impact on the optimal contract. As before, the seller pools all buyers with initial types below a cutoff $\hat{\lambda}$, providing them with effective allocations equivalent to the period-two contract; all types above the cutoff receive the full-commitment-optimal allocations. By contrast, the period-two price is the seller's optimal price when selling to cohort two combined with the *entire* interval $[\underline{\lambda}, \hat{\lambda}]$: it is no longer sequentially rational for the seller to charge a greater price that would only be optimal when delaying a strict subset of that interval consisting of relatively higher types. This is because the seller would have an incentive (that is not present when recontracting is prohibited) to drop her price and capture the additional period-two gains from recontracting with relatively lower types. Therefore, while the pattern of (additional) distortions is the same, their magnitude is greater when buyers are free to recontract in period two.

Theorem 4. *Suppose that the optimal contract induces a period-two price $\alpha^{\widetilde{LC}} > p_H$. Then there exists some $\hat{\lambda} \geq (k^{ND})^{-1}(\alpha^{\widetilde{LC}})$ such that*

$$\alpha^{\widetilde{LC}} \in \arg\max_{\alpha} \left\{ \int_{\underline{\lambda}}^{\hat{\lambda}} \pi_\mu(\alpha) dF(\mu) + \gamma \pi_H(\alpha) \right\} \text{ and } \bar{k}_1^{\widetilde{LC}}(\lambda) = \begin{cases} \alpha^{\widetilde{LC}} & \text{if } \lambda \leq \hat{\lambda}, \\ k^{ND}(\lambda) & \text{if } \lambda > \hat{\lambda}. \end{cases}$$

Moreover, the seller's profits are $\Pi^{\widetilde{LC}} \leq \Pi^{LC}$.

While Theorem 4 characterizes the allocations to period-one buyers, it is relatively unrestrictive along one important dimension. While the equilibrium requires $x_1^{\widetilde{LC}}(\lambda) = 1$ for all $\lambda > \hat{\lambda}$, as these initial types receive strictly more generous contracts than what might be achieved via delay, our result is completely agnostic, however, about the delay decisions for initial types $\lambda \leq \hat{\lambda}$. The outcomes would be identical if all such buyers delayed contracting—the conditions in Theorem 4 guarantee that the seller's optimal price given delay by all such buyers is equal to $\alpha^{\widetilde{LC}}$. Likewise, the seller could simply offer all buyers in this interval a contract in period one that "locks in" the ultimate period-two outcome in advance (as was optimal in the case where contracts could not be broken); the resulting optimal period-two price that solves ($\widetilde{\text{SR}}$) would again be exactly $\alpha^{\widetilde{LC}}$. Indeed, many other types of contracts, as well as different combinations thereof across buyer types, yield the same outcome. And while this multiplicity in possible delay decisions stands in contrast to the sharp characterization in Theorem 3, it should not be too surprising: just as renegotiation-proof contracts can be used to preclude the possibility of contract renegotiation on the equilibrium path, it is possible to write "recontracting-proof" contracts in the present setting that preempt the possibility of recontracting on the equilibrium path.

Of course, the irrelevance of the recontracting constraint for the timing of contracting on the equilibrium path does *not* imply payoff irrelevance; rather, the recontracting constraint that differentiates problem (SR) from ($\widetilde{\text{SR}}$) has a negative impact on the seller's rent extraction ability. This is most simply demonstrated by a revealed-preference argument. Theorem 4 demonstrates that the optimal contract when recontracting is possible can be achieved by delaying contracting with *all* buyers in the interval $[\underline{\lambda}, \hat{\lambda}]$ for some $\hat{\lambda} \in \Lambda$, while offering all buyers with types $\lambda > \hat{\lambda}$ the full-commitment allocations $k^{ND}(\lambda)$ described in Theorem 1. This is, of course, a feasible contract under the limited commitment *without* recontracting regime. In this no-recontracting regime, however, the interval of delayed types (characterized in part (2) of Theorem 3) is strictly interior; therefore, the ability of buyers to recontract in the second period decreases the seller's payoff. Intuitively, the seller can always sell to all cohort-one types in period two by offering a low enough price (irrespective of whether they contracted in the first period or not) whereas, without recontracting, low cohort-one types can effectively be removed from the period-two market. As a consequence, a seller who can choose whether or not to bar recontracting and exclude buyers who contracted in the past would like to "renegotiate" with some cohort-one buyers and capture a portion of the efficiency gains arising from a lower price. Precisely in order to avoid this temptation, the seller in period one benefits from banning recontracting entirely (perhaps by keeping records on buyer identities and imposing a "one-transaction-per-customer" rule).

The following proposition identifies an intuitive sufficient condition under which the period-two price in the optimal contract satisfies $\alpha^{\widetilde{LC}} > p_H$ (the required assumption for Theorem 4).

Proposition 5. *Define* $\lambda_H := (k^{ND})^{-1}(p_H)$ *and suppose that* $\lambda_H > \underline{\lambda}$. *In addition, suppose that* $p_H < \arg\max_\alpha \left\{ \int_{\underline{\lambda}}^{\lambda_H} \pi_\mu(\alpha) dF(\mu) \right\}$. *Then the optimal contract induces a period-two price* $\alpha^{\widetilde{LC}} > p_H$.

The condition in Proposition 5 holds in settings where both (a) the period-two monopoly price p_H is low relative to the prices attainable by delaying cohort-one buyers; and (b) the optimal full-commitment contract from Theorem 1 entails substantially larger distortions away from efficiency for low initial-period types than for cohort-two buyers.[20] Intuitively, maintaining a period-two price greater than p_H in such settings allows the seller to simultaneously reduce the incentives for delay while ensuring that allocations to cohort-one buyers are closer to the no-delay optimal schedule $k^{ND}(\cdot)$.

It is also possible for the optimal contract to induce a period-two price equal to the cohort-two monopoly price (that is, $\alpha^{\widetilde{LC}} = p_H$).[21] The key difficulty in characterizing the optimal contract in such cases is that the strategic delay constraint (SD) may only bind at the lowest type $\underline{\lambda}$. When this is the case, Lemma 1's implication that constraint (SD) leads to pooling no longer has any bite. Therefore, the seller's contracting problem involves optimally choosing a decreasing schedule of cutoffs that trade off proximity to the no-delay optimal schedule $k^{ND}(\cdot)$ with resilience to the incentive to cut prices. Although this appears to be a natural candidate for an optimal control approach, the sequential rationality constraint ($\widetilde{\text{SR}}'$) is not easily incorporated into such a problem.

This is not to say, however, that the optimal contract is entirely indeterminate when the seller optimally implements a period-two price $\alpha^{\widetilde{LC}} = p_H$. It is straightforward to argue that the optimal cutoffs in such a case must "eventually" coincide with $k^{ND}(\cdot)$ (that is, $\bar{k}_1^{\widetilde{LC}}(\lambda) = k^{ND}(\lambda)$ for sufficiently large $\lambda \in \Lambda$). This follows from the usual "no-distortion-at-the-top" logic in which the surplus obtainable from the highest cohort-one types is large enough that—absent sequential rationality considerations—the seller wishes to offer them approximately efficient contracts. Such types are therefore promised allocations with cutoffs so low as to not be tempted by any prices that the seller might rationally charge in the second period. Thus, the sequential rationality constraint is generally satisfied "for free" at the top end of the type distribution.

5. Concluding remarks

A critical assumption in our analysis is our assumption, as is common in the dynamic mechanism design literature, that the distribution of values $G(\cdot|\lambda)$ for any type λ first-order stochastically dominates that of any lower type $\lambda' < \lambda$. A natural alternative ordering is one in which higher types face greater uncertainty about their realized values (in the sense of second-order stochastic dominance). Indeed, Courty and Li (2000) also examined a special case of second-order stochastic dominance in which the family of distributions $\{G(\cdot|\lambda)\}_{\lambda\in\Lambda}$ are rotation ordered.[22] They showed that, under certain additional conditions on the allocation rule, monotonicity of the optimal cutoffs $k^{ND}(\cdot)$ continues to be sufficient for incentive compatibility and hence optimality. The additional assumptions ensure that the cutoffs $k^{ND}(\cdot)$ are in regions where the cumulative distributions are ordered in a well-behaved manner similar to first-order stochastic dominance.

Unfortunately, our analysis in the present work does not in general extend to such type spaces. (In the special case where both $k^{ND}(\cdot)$ and p_H are larger than the distributional rotation point, our results extend immediately. In this case, all the relevant allocations occur in a region where types

[20] Consider a simple power family of distributions as described in Section 2.2: $\Lambda = \mathbf{V} = [0, 1]$, $F(\lambda) = \lambda$, $G(v|\lambda) = v^{\lambda}$, and $H(v) = G(v|\hat{\mu})$. Then the condition in Proposition 5 is satisfied whenever $\hat{\mu}$ is less than approximately 0.3.

[21] This occurs, for example, when $\Lambda = [1, 2]$, $\mathbf{V} = [0, 1]$, $F(\lambda) = \lambda$, $G(v|\lambda) = 1 - (1 - v)^{1/\lambda}$, $H(v) = G(v|1.5)$, and we impose a positive marginal cost $c = 0.2$. Even if the mass of cohort-two buyers is very small ($\gamma = 0.001$, for instance), the optimal contract with recontracting is characterized by $\alpha^{\widetilde{LC}} := p_H > k^{ND}(\lambda) =: \bar{k}_1^{\widetilde{LC}}(\lambda)$ for all $\lambda \in \Lambda$.

[22] In the rotation order, all distributions $G(v|\lambda)$ pass through a single point $z \in \mathbf{V}$. Moreover, $G(v|\lambda)$ is increasing in λ for all $v < z$ and decreasing in λ for all $v > z$. For more on this order, see Johnson and Myatt (2006).

are ordered "as though" by first-order stochastic dominance.) One difficulty arises in identifying the set of types for which the strategic delay constraint (SD) binds in an optimal contract—even for a fixed period-two price. In particular, finding an analogue to Lemma 1 is difficult because there is no general relationship governing the relative rankings of the cutoffs $k^{ND}(\cdot)$ and the period-two price α; the utility from either may be increasing, decreasing, or simply non-monotone in λ. An additional complication arises in fully characterizing initial-period incentive compatibility: without the structure imposed by first-order stochastic dominance, general necessary and sufficient conditions for incentive compatibility are not known. This precludes the "first-order approach" to solving for optimal contracts, thereby necessitating a consideration of "global" incentive compatibility constraints as in Battaglini and Lamba (2015).

There are a number of other natural generalizations of our model. One such extension is to allow for uncertain market conditions in the second period; for example, suppose that the mass γ of cohort-two entrants is randomly drawn from some distribution. When the seller can observe the realization of this draw before choosing her period-two mechanism, the price charged in the second period becomes (from the perspective of a cohort-one buyer) a random variable. Clearly, there exists some deterministic "certainty equivalent" contract for each initial-period type λ such that the buyer receives exactly the same utility from contracting immediately instead of delaying contracting to period two. However, first order stochastic dominance is not sufficient to guarantee that this certainty equivalent price is well-behaved and monotonically decreasing in λ. Since incentive compatibility requires monotonicity in the effective allocation rule, this implies that there can no longer be an interval of types contracting in the initial period for whom the "induced participation constraint" binds. This, in turn, complicates the seller's revenue management problem as she now must provide rents (beyond this endogenous outside option) to low initial types to prevent them from delaying contracting until the second period.

A second natural extension of our framework provides a tractable model to study a seller without commitment who has a limited supply of the good.[23] Since we have a continuum of buyers in each period, there is no aggregate uncertainty regarding the distribution of values of cohort-one buyers in the second period. Thus, for a given contract, the seller knows exactly the quantity promised to cohort-one buyers who contract in the first period. This precludes over-sale situations in which the seller promises a greater quantity than her available supply. Moreover, it is easy to see that the optimal second-period price is simply determined by market-clearing. Despite this apparent simplicity, however, the case of limited supply introduces some additional complexity: in addition to determining which cohort-one buyer purchases should be postponed to the second period, the seller must also determine the constraint on period-two supply remaining after period-one contracting. In particular, offering generous contracts in the first period ensures that there is greater scarcity and increased competition in the second period, thereby inducing higher prices.[24] Understanding the tradeoffs involved when the seller has an additional avenue for endogenously generating commitment power is certainly an interesting question, but is also one that is beyond the scope of the present work.

[23] In a full-commitment model very similar to our own, Ely et al. (2015) show that "overbooking" and then repurchasing pre-committed capacity can improve profits over a simple auction for limited capacity. In related work, Hinnosaar (2015) shows that the sale of conditional promises of allocation can be revenue-maximizing when overlapping generations of buyers compete for limited supply.

[24] This is similar to the intuition (in a model with fully persistent private information) of Dilme and Li (2014), where the seller holds occasional fire-sales in order to endogenously commit to future supply restrictions.

Finally, we emphasize that the nature of the seller-optimal equilibrium depends delicately on the class of contracts available to her. As discussed in Section 2.3, if the seller can offer period-one contracts that condition (either explicitly or implicitly) on the period-two price, she can generate additional commitment power and profits. Thus, our characterization of the full- and limited-commitment outcomes provides bounds on the best- and worst-case payoffs the seller can achieve. The mapping between the richness of a contractual language and the payoff it allows (within these bounds) is a natural question that we leave for future study.

Appendix A. Omitted proofs from Section 4

Proof of Lemma 1.[25] Fix any $\hat{\lambda}$ at which constraint (SD) binds, and fix any $\lambda' \in \Lambda$. Of course, if $x_1(\lambda') = 0$, $\bar{k}_1(\lambda') = \alpha$ by definition; therefore, assume that $x_1(\lambda') > 0$. Recall that

$$\begin{aligned} V_{11}(\hat{\lambda}) &:= x_1(\hat{\lambda})U_{11}(\hat{\lambda}) + (1 - x_1(\hat{\lambda}))\widetilde{U}_{11}(\hat{\lambda}) \\ &\geq x_1(\lambda')\left(\int_{\mathbf{V}} V_{12}(v,\lambda')dG(v|\hat{\lambda}) - p_{11}(\lambda')\right) + (1 - x(\lambda'))\widetilde{U}_{11}(\hat{\lambda}) \\ &= V_{11}(\lambda') + x_1(\lambda')\int_{\mathbf{V}} [V_{12}(v,\lambda') - U_{22}(v)]d[G(v|\hat{\lambda}) - G(v|\lambda')] \\ &\quad + (\widetilde{U}_{11}(\hat{\lambda}) - \widetilde{U}_{11}(\lambda')) \\ &\geq x_1(\lambda')\int_{\mathbf{V}} [V_{12}(v,\lambda') - U_{22}(v)]d[G(v|\hat{\lambda}) - G(v|\lambda')] + \widetilde{U}_{11}(\hat{\lambda}), \end{aligned}$$

where the final inequality is strict if (SD) is slack at λ'. (The first equality is the definition of $V_{11}(\hat{\lambda})$; the next line follows from (IC_{11}); the third from the definition of $V_{11}(\lambda')$; and the final line from (SD).) Note, however, that (SD) binds at $\hat{\lambda}$, implying that $V_{11}(\hat{\lambda}) = \widetilde{U}_{11}(\hat{\lambda})$. Therefore,

$$\begin{aligned} 0 &\geq \int_{\mathbf{V}} [V_{12}(v,\lambda') - U_{22}(v)]d[G(v|\hat{\lambda}) - G(v|\lambda')] \\ &= \int_{\mathbf{V}} [q_2(v) - \bar{q}_{12}(v,\lambda')][G(v|\hat{\lambda}) - G(v|\lambda')]dv = \int_{\alpha}^{\bar{k}_{12}(\lambda')} [G(v|\hat{\lambda}) - G(v|\lambda')]dv, \end{aligned}$$

where the first equality follows from integration by parts, and the second from the cutoff property of the allocations. Since $\{G(\cdot|\lambda)\}_{\lambda\in\Lambda}$ is ordered by first-order stochastic dominance, however, this inequality holds only if $\bar{k}_1(\lambda') \leq \alpha$ for all $\lambda' > \hat{\lambda}$, and $\bar{k}_1(\lambda') \geq \alpha$ for all $\lambda' < \hat{\lambda}$ (with strict inequalities if (SD) is slack at λ').

With this observation in hand, fix an optimal contract (denoted with a * superscript) and let

$$\hat{\lambda} := \sup\{\lambda | V_{11}(\lambda) = \widetilde{U}_{11}(\lambda)\};$$

[25] Note that the various steps of this proof apply, except where explicitly noted, to *both* problems ($\mathcal{P}$) and ($\widetilde{\mathcal{P}}$); the argument also extends immediately to the full-commitment setting described in Section 4.1.

that is, $\hat{\lambda}$ is the largest type for which constraint (SD) binds. Now suppose that there exists some $\tilde{\mu} < \hat{\lambda}$ such that (SD) is slack. (Note that this requires $x_1^*(\tilde{\mu}) = 1$. Moreover, both of these two properties must hold in a neighborhood of $\tilde{\mu}$ due to the absolute continuity of both V_{11} and $\widetilde{U}_{11}$.)

We now define a new contract, which we denote with a ** superscript, by

$$q_1^{**}(v,\lambda) := \begin{cases} q_2^*(v) & \text{if } \lambda \leq \hat{\lambda}, \\ q_1^*(v,\lambda) & \text{if } \lambda > \hat{\lambda}; \end{cases} \quad p_{11}^{**}(\lambda) := \begin{cases} 0 & \text{if } \lambda \leq \hat{\lambda}, \\ p_{11}^*(\lambda) & \text{if } \lambda > \hat{\lambda}; \end{cases} \quad \text{and}$$

$$p_{12}^{**}(v,\lambda) := \begin{cases} p_{22}^*(v) & \text{if } \lambda \leq \hat{\lambda}, \\ p_{12}^*(v,\lambda) & \text{if } \lambda > \hat{\lambda}. \end{cases}$$

We will show that this contract, when combined with an appropriate delay recommendation x_1^{**}, has three important properties: (a) it induces a period-two price $\alpha^{**} = \alpha^*$; (b) it satisfies the various incentive compatibility constraints; and (c) increases the seller's profits.

We begin by defining the x_1^{**} separately for problems $(\mathcal{P})$ and $(\widetilde{\mathcal{P}})$, and show that $\alpha^* = \alpha^{**}$:

- *Problem* $(\mathcal{P})$: for all $\lambda \in \Lambda$, let $x_1^{**}(\lambda) = x_1^*(\lambda)$. Since the composition of the set of delayed buyers is identical in the two contracts, the period-two problem in (SR′) is unchanged. Therefore, $\alpha^{**} = \alpha^*$.
- *Problem* $(\widetilde{\mathcal{P}})$: for all $\lambda \in \Lambda$, let $x_1^{**}(\lambda) = 1$. Note that we can write the seller's objective function from $(\widetilde{\text{SR}}')$ as

$$\Pi_2(\alpha|x_1,k_1) = \gamma\pi_H(\alpha) + \int_\Lambda (1 - x_1(\lambda))\pi_\lambda(\alpha)dF(\lambda) + \int_\Lambda x_1(\lambda)\pi_\lambda(\min\{\alpha, k_1(\lambda)\})dF(\lambda),$$

ignoring the $\int_\Lambda x_1(\lambda)U_{12}(\underline{v},\lambda)dF(\lambda)$ term that is independent of α. For all $\alpha \leq \alpha^*$, we have

$$\begin{aligned}
\Pi_2(\alpha|x_1^*,k_1^*) &= \gamma\pi_H(\alpha) + \int_{\underline{\lambda}}^{\hat{\lambda}} (1 - x_1^*(\lambda))\pi_\lambda(\alpha)dF(\lambda) + \int_{\hat{\lambda}}^{\bar{\lambda}} (1 - x_1^*(\lambda))\pi_\lambda(\alpha)dF(\lambda) \\
&\quad + \int_{\underline{\lambda}}^{\hat{\lambda}} x_1^*(\lambda)\pi_\lambda(\min\{\alpha, k_1^*(\lambda)\})dF(\lambda) \\
&\quad + \int_{\hat{\lambda}}^{\bar{\lambda}} x_1^*(\lambda)\pi_\lambda(\min\{\alpha, k_1^*(\lambda)\})dF(\lambda) \\
&= \gamma\pi_H(\alpha) + \int_{\underline{\lambda}}^{\hat{\lambda}} \pi_\lambda(\alpha)dF(\lambda) + \int_{\hat{\lambda}}^{\bar{\lambda}} \pi_\lambda(\min\{\alpha, k_1^*(\lambda)\})dF(\lambda) \\
&= \gamma\pi_H(\alpha) + \int_{\underline{\lambda}}^{\hat{\lambda}} \pi_\lambda(\min\{\alpha, \alpha^*\})dF(\lambda) + \int_{\hat{\lambda}}^{\bar{\lambda}} \pi_\lambda(\min\{\alpha, k_1^*(\lambda)\})dF(\lambda) \\
&= \Pi_2(\alpha|x_1^{**},k_1^{**}),
\end{aligned}$$

where the first equality is the definition of Π_2; the second follows from $k_1^*(\lambda) \geq \alpha^*$ for all $\lambda < \hat{\lambda}$ and $x_1^*(\lambda) = 1$ for all $\lambda > \hat{\lambda}$ (as constraint (SD) must be slack there); and the last line from the definitions of x_1^{**} and k_1^{**}. Thus, since α^* is optimal under the * contract, we must have $\Pi_2(\alpha^*|x_1^{**}, k_1^{**}) \geq \Pi_2(\alpha|x_1^{**}, k_1^{**})$ for all $\alpha \leq \alpha^*$. Moreover, for any $\alpha' > \alpha^*$,

$$\begin{aligned}\Pi_2(\alpha|x_1^{**}, k_1^{**}) &= \gamma\pi_H(\alpha) + \int_\Lambda \pi_\lambda(\min\{\alpha, k_1^{**}(\lambda)\})dF(\lambda)\\ &= \gamma\pi_H(\alpha) + \int_\Lambda \pi_\lambda(k_1^{**}(\lambda))dF(\lambda),\end{aligned}$$

since $k_1^{**}(\lambda) = \alpha^*$ for all $\lambda \leq \hat{\lambda}$ and $k_1^{**}(\lambda) = k_1^*(\lambda) < \alpha^*$ for all $\lambda > \hat{\lambda}$ (where the strict inequality follows from the observation above). Therefore,

$$\Pi_2(\alpha|x_1^{**}, k_1^{**}) - \Pi_2(\alpha^*|x_1^{**}, k_1^{**}) = \gamma(\pi_H(\alpha) - \pi_H(\alpha^*)) < 0,$$

where the inequality follows from the strict concavity of π_H Lemma 3. Thus, $\alpha^{**} = \alpha^*$.

Since $\alpha^{**} = \alpha^*$, the value of delay is unchanged from the * contract, so (SD) remains satisfied. In addition, the ** contract satisfies (IC_{12}) and (IC_{22}) because it yields cutoff allocations. Constraint (IC_{11}) is trivially satisfied for types $\lambda > \hat{\lambda}$, as they see no change in their "designated" contract and have fewer possible deviations. As for types $\lambda \leq \hat{\lambda}$, note that for all $\lambda' > \hat{\lambda}$,

$$\begin{aligned}&V_{11}^{**}(\lambda) - \left(-p_{11}^{**}(\lambda') + \int_{\mathbf{V}} V_{12}^{**}(v,\lambda')dG(v|\lambda)\right)\\ &= \int_{\mathbf{V}} \left[U_{22}^{**}(v) - V_{12}^{**}(v,\lambda')\right]dG(v|\lambda) + p_{11}^{**}(\lambda')\\ &\geq \int_{\mathbf{V}} \left[U_{22}^{**}(v) - V_{12}^{**}(v,\lambda')\right]dG(v|\hat{\lambda}) + p_{11}^{**}(\lambda')\\ &= V_{11}^*(\hat{\lambda}) - \left(-p_{11}^*(\lambda') + \int_{\mathbf{V}} V_{12}^*(v,\lambda')dG(v|\hat{\lambda})\right) \geq 0.\end{aligned}$$

The first equality above follows from (SD) binding at λ; the first inequality from the facts that $G(\cdot|\hat{\lambda})$ stochastically dominates $G(\cdot|\lambda)$ and that $\frac{\partial}{\partial v}[U_{22}^{**}(v) - V_{12}^{**}(v,\lambda')] = q_2^*(v) - q_1^*(v,\lambda') \leq 0$; the next equality from the definition of $\hat{\lambda}$ and the fact that our construction leaves the contracts for all $\lambda' > \hat{\lambda}$ unchanged; and the final inequality follows from the incentive compatibility of the original * contract. Thus, the new contract ** satisfies (IC_{11}).

We now consider the change in the seller's period-one profits, which can be written as

$$\Pi_1^{**} - \Pi_1^* = \int_{\underline{\lambda}}^{\hat{\lambda}} \int_{\mathbf{V}} v[\bar{q}_1^{**}(v,\lambda) - \bar{q}_1^*(v,\lambda)]dG(v|\lambda)dF(\lambda) + \int_{\underline{\lambda}}^{\hat{\lambda}} [V_{11}^*(\lambda) - V_{11}^{**}(\lambda)]dF(\lambda).$$

However, $\bar{q}_1^{**}(v,\lambda) \geq \bar{q}_1^*(v,\lambda)$ for all $v \in \mathbf{V}$ and all $\lambda \in \Lambda$, while $V_{11}^*(\lambda) \geq V_{11}^{**}(\lambda)$ for all $\lambda \in \Lambda$ (with strict inequality in a neighborhood of $\tilde{\mu}$). Thus, $\Pi_1^{**} - \Pi_1^* > 0$, contradicting the optimality of the original * contract. Therefore, we may conclude that (SD) must bind for all $\lambda < \hat{\lambda}$, as desired.

Finally, note that this implies that (SD) binds at $\underline{\lambda}$. Then for any $\mu \in (\underline{\lambda}, \hat{\lambda})$, the first observation of this proof implies $\alpha \geq \bar{k}_1(\mu) \geq \alpha$, so $\bar{k}_1(\mu) = \alpha$. □

Proof of Lemma 2. Consider the original contract, and let $\bar{\mu} := \sup\{\lambda | x_1(\lambda) < 1\}$ denote the largest initial-period type that delays contracting with positive probability. Note that Assumption 2 implies that the objective function in (SR′) is strictly concave, so its associated first-order condition is necessary and sufficient to characterize the induced period-two price; that is, α solves

$$\int_{\Lambda} (1 - x_1(\lambda))\pi'_\lambda(\alpha) dF(\lambda) + \gamma \pi'_H(\alpha) = 0.$$

Recalling that Assumption 3 guarantees the existence of some initial-period type $\hat{\mu} \in (\underline{\lambda}, \bar{\lambda})$ such that $p_H = p_{\hat{\mu}}$, Assumptions 1 and 2 jointly imply that there also exists some $\mu^* \in \Lambda$ such that $\alpha = p_{\mu^*}$. Moreover, we must have $\pi'_\lambda(\alpha) > 0$ for all $\lambda > \mu^*$, and that $\pi'_\lambda(\alpha) < 0$ for all $\lambda < \mu^*$.

Thus, the first-order condition induced by x_1 can be written as $X_1 + X_2 + \pi'_H(\alpha) = 0$, where

$$X_1 := \int_{\underline{\lambda}}^{\min\{\mu^*, \bar{\mu}\}} (1 - x_1(\lambda))\pi'_\lambda(\alpha) dF(\lambda) \text{ and } X_2 := \int_{\min\{\mu^*, \bar{\mu}\}}^{\bar{\mu}} (1 - x_1(\lambda))\pi'_\lambda(\alpha) dF(\lambda),$$

and note that $X_1 \leq 0 \leq X_2$. Finally, we define

$$Y_1(z) := \int_{z}^{\min\{\mu^*, \bar{\mu}\}} \pi'_\lambda(\alpha) dF(\lambda) \text{ and } Y_2(z) := \int_{\min\{\mu^*, \bar{\mu}\}}^{z} \pi'_\lambda(\alpha) dF(\lambda).$$

Notice that $Y_1(\min\{\mu^*, \bar{\mu}\}) = 0$ and $Y_1(\underline{\lambda}) \leq X_1$, and that $Y_1(z)$ is strictly increasing, so there exists a unique $\mu_1 \in [\underline{\lambda}, \min\{\mu^*, \bar{\mu}\}]$ with $Y_1(\mu_1) = X_1$. Similarly, $Y_2(\min\{\mu^*, \bar{\mu}\}) = 0$, and $Y_2(\bar{\mu}) \geq X_2$, and $Y_2(z)$ is strictly increasing, so there exists a unique $\mu_2 \in [\min\{\mu^*, \bar{\mu}\}, \bar{\mu}]$ with $Y_2(\mu_2) = X_2$.

We then define $\hat{x}_1$ by $\hat{x}_1(\lambda) := 0$ for all $\lambda \in [\mu_1, \mu_2]$ and $\hat{x}_1(\lambda) := 1$ for all $\lambda \notin [\mu_1, \mu_2]$. Then

$$\int_{\Lambda} (1 - \hat{x}_1(\lambda))\pi'_\lambda(\alpha) dF(\lambda) + \gamma \pi'_H(\alpha) = Y_1(\mu_1) + Y_2(\mu_2) + \gamma \pi'_H(\alpha)$$

$$= X_1 + X_2 + \gamma \pi'_H(\alpha) = 0;$$

that is, the first-order condition induced by $\hat{x}_1$ also yields a period-two price $\hat{\alpha} = \alpha$. Thus, the option value of strategic delay is the same under both x_1 and the newly constructed $\hat{x}_1$.

Of course, Lemma 1 implies that (SD) binds and $\bar{k}_1(\lambda) = \alpha$ for all $\lambda < \bar{\mu}$ when delay is given by x_1, and also that $\bar{k}_1(\lambda) = \alpha$ for all $\lambda < \mu_2$ when delay is given by $\hat{x}_1$. But since $\mu_2 \leq \bar{\mu}$, delay given by $\hat{x}_1$ provides the seller with greater freedom in choosing optimal cutoffs for types $\lambda \in [\mu_2, \bar{\mu}]$. Thus, by decreasing the upper bound of the set of delayed buyers from $\bar{\mu}$ to μ_2, we have relaxed an "implicit" constraint on the seller's problem, implying that $\widehat{\Pi} \geq \Pi$. □

Proof of Theorem 3. Suppose first that μ_1^{LC} and μ_2^{LC} solve problem $(\mathcal{R})$. Note that constraints (IC_{12}) and (IC_{22}) are satisfied by construction; in addition, Assumption 4 implies that the effective allocation rule is monotone in λ. Invoking Lemmas B.1 and B.3, the contract is incentive compatible. Finally, it is trivial to verify that each buyer's expected utility from the cohort-one contract is greater than the expected value of delay, so constraint (SD) is also satisfied. Thus, the proposed contract does indeed solve the seller's problem $(\mathcal{P})$.

It is easy to see that we cannot have $\alpha^{LC} < p_H$. Delaying some cohort-one buyers in order to reduce the second-period price below p_H has two effects: it reduces the profits derived from cohort-two buyers (which are, by definition, maximized at p_H), and it also increases the utility that must be promised to *all* cohort-one buyers while decreasing the seller's ability to price discriminate. Each of these forces decreases overall profits, and so we must have $\alpha^{LC} \geq p_H$.

Denote by $\alpha(\mu_1, \mu_2)$ the period-two price induced when the set of delayed buyers is an interval $[\mu_1, \mu_2]$ (which is without loss by Lemma 2). Then the seller's profits from the objective in $(\mathcal{R})$ are

$$\begin{aligned}
\Pi^{LC}(\mu_1, \mu_2) :=& \int_{\underline{\lambda}}^{\mu_2} \int_{\alpha(\mu_1,\mu_2)}^{\bar{v}} \varphi_1(v, \lambda) dG(v|\lambda) dF(\lambda) \\
&+ \int_{\mu_2}^{\bar{\lambda}} \int_{\min\{\alpha(\mu_1,\mu_2), k^{ND}(\lambda)\}}^{\bar{v}} \varphi_1(v, \lambda) dG(v|\lambda) dF(\lambda) \\
&+ \gamma \pi_H(\alpha(\mu_1, \mu_2)) - \int_{\alpha(\mu_1,\mu_2)}^{\bar{v}} (1 - G(v|\underline{\lambda})) dv \\
=& \int_{\underline{\lambda}}^{\tilde{\lambda}(\alpha(\mu_1,\mu_2))} \int_{\alpha(\mu_1,\mu_2)}^{\bar{v}} \varphi_1(v, \lambda) dG(v|\lambda) dF(\lambda) \\
&+ \int_{\tilde{\lambda}(\alpha(\mu_1,\mu_2))}^{\max\{\tilde{\lambda}(\alpha(\mu_1,\mu_2)),\mu_2\}} \int_{\alpha(\mu_1,\mu_2)}^{\bar{v}} \varphi_1(v, \lambda) dG(v|\lambda) dF(\lambda) \\
&+ \int_{\max\{\tilde{\lambda}(\alpha(\mu_1,\mu_2)),\mu_2\}}^{\bar{\lambda}} \int_{k^{ND}(\lambda)}^{\bar{v}} \varphi_1(v, \lambda) dG(v|\lambda) dF(\lambda) \\
&+ \gamma \pi_H(\alpha(\mu_1, \mu_2)) - \int_{\alpha(\mu_1,\mu_2)}^{\bar{v}} (1 - G(v|\underline{\lambda})) dv,
\end{aligned}$$

where $\tilde{\lambda}(v) := (k^{ND})^{-1}(v)$ is the solution to $\varphi_1(v, \tilde{\lambda}(v)) = 0$. Rearranging this expression yields

$$\Pi^{LC}(\mu_1, \mu_2) = \Pi^{FC}(\alpha(\mu_1, \mu_2)) - \int_{\tilde{\lambda}(\alpha(\mu_1,\mu_2))}^{\max\{\tilde{\lambda}(\alpha(\mu_1,\mu_2)),\mu_2\}} \int_{k^{ND}(\lambda)}^{\alpha(\mu_1,\mu_2)} \varphi_1(v, \lambda) dG(v|\lambda) dF(\lambda),$$

where Π^{FC} is the full-commitment profit function as defined in Theorem 2. Notice that whenever $\mu_2 > \tilde{\lambda}(\alpha(\mu_1, \mu_2))$, the integrand in the last line above is always positive, as k^{ND} is decreasing in λ, and $\varphi_1(v,\lambda) \geq 0$ whenever $v \geq k^{ND}(\lambda)$. We may use this expression to evaluate the seller's profits when she induces a second-period price $\alpha^{LC} > \alpha^{FC}$.

Define $\Gamma := \{\mu \in \Lambda : \text{there exists } \mu' \leq \mu \text{ with } \alpha(\mu', \mu) = \alpha^{FC}\}$. If Γ is empty, then it is not possible to induce a period-two price α^{FC}: since $\alpha(\mu_1, \mu_2)$ is continuous and $\alpha(\underline{\lambda}, \underline{\lambda}) = p_H < \alpha^{FC}$, this implies that $\alpha^{LC} \leq \max_{\mu_1,\mu_2}\{\alpha(\mu_1, \mu_2)\} < \alpha^{FC}$. On the other hand, if Γ is nonempty, we may define $\mu_2^{FC} := \min\{\mu \in \Gamma\}$; that is, μ_2^{FC} is the smallest upper bound of the set of delayed buyers that is compatible with inducing α^{FC}. In addition, we implicitly define μ_1^{FC} by $\alpha(\mu_1^{FC}, \mu_2^{FC}) = \alpha^{FC}$.

Note that if $\mu_2^{FC} \leq \tilde{\lambda}(\alpha^{FC})$, the seller can replicate the effective allocation and payment rules from the full commitment case. This implies the payoff of the full-commitment problem is achievable even with limited commitment, and so $\alpha^{LC} = \alpha^{FC}$. So suppose instead that $\mu_2^{FC} > \tilde{\lambda}(\alpha^{FC})$, and note that Assumptions 1 and 2 imply that inducing $\alpha^{LC} > \alpha^{FC}$ requires $\mu_2^{LC} > \mu_2^{FC}$. Then

$$\Pi^{LC}(\mu_1^{FC}, \mu_2^{FC}) = \Pi^{FC}(\alpha^{FC}) - \int_{\tilde{\lambda}(\alpha^{FC})}^{\mu_2^{FC}} \int_{k^{ND}(\lambda)}^{\alpha^{FC}} \varphi_1(v,\lambda) g(v|\lambda) f(\lambda) dv d\lambda$$

$$= \Pi^{FC}(\alpha^{FC}) - \int_{k^{ND}(\mu_2^{FC})}^{\alpha^{FC}} \int_{\tilde{\lambda}(v)}^{\mu_2^{FC}} \varphi_1(v,\lambda) g(v|\lambda) f(\lambda) d\lambda dv, \text{ and}$$

$$\Pi^{LC}(\mu_1^{LC}, \mu_2^{LC}) = \Pi^{FC}(\alpha^{LC}) - \int_{\tilde{\lambda}(\alpha^{LC})}^{\mu_2^{LC}} \int_{k^{ND}(\lambda)}^{\alpha^{LC}} \varphi_1(v,\lambda) g(v|\lambda) f(\lambda) dv d\lambda$$

$$= \Pi^{FC}(\alpha^{LC}) - \int_{k^{ND}(\mu_2^{LC})}^{\alpha^{LC}} \int_{\tilde{\lambda}(v)}^{\mu_2^{LC}} \varphi_1(v,\lambda) g(v|\lambda) f(\lambda) d\lambda dv,$$

where the second equality in each expression follows from reversing the order of integration. Subtracting the two expressions above and then rearranging the integrals, we have

$$\Pi^{LC}(\mu_1^{FC}, \mu_2^{FC}) - \Pi^{LC}(\mu_1^{LC}, \mu_2^{LC})$$
$$= \Pi^{FC}(\alpha^{FC}) - \Pi^{FC}(\alpha^{LC}) + \int_{\alpha^{FC}}^{\alpha^{LC}} \int_{\tilde{\lambda}(v)}^{\mu_2^{FC}} \varphi_1(v,\lambda) g(v|\lambda) f(\lambda) d\lambda dv$$
$$+ \int_{k^{ND}(\mu_2^{LC})}^{\alpha^{LC}} \int_{\mu_2^{FC}}^{\mu_2^{LC}} \varphi_1(v,\lambda) g(v|\lambda) f(\lambda) d\lambda dv.$$

By definition, $\Pi^{FC}(\alpha^{FC}) \geq \Pi^{FC}(\alpha^{LC})$. Moreover, it is straightforward to verify that each of the integrands is positive, and therefore $\Pi^{LC}(\mu_1^{FC}, \mu_2^{FC}) > \Pi^{LC}(\mu_1^{LC}, \mu_2^{LC})$, contradicting the optimality of the proposed contract. Thus, the seller optimally induces a second-period price $\alpha^{LC} \leq \alpha^{FC}$.

Now suppose that the period-two price is $\alpha^{LC} \in (p_H, \alpha^{FC})$. Since $\alpha^{LC} > p_H$, Assumption 2 implies that $\pi'_H(\alpha^{LC}) < 0$. Similarly, Assumption 3 implies that $\pi'_{\hat{\mu}}(\alpha^{LC}) < 0 = \pi'_{\hat{\mu}}(p_H)$. Finally, notice that the first-order condition associated with (SR′) may be written as

$$\int_{\mu_1^{LC}}^{\mu_2^{LC}} \pi'_\mu(\alpha^{LC})dF(\lambda) + \gamma\pi'_H(\alpha^{LC}) = 0.$$

So suppose that $\mu_1^{LC} < \bar{\mu}$, where $\bar{\mu}$ is such that $p_{\bar{\mu}} = \alpha^{LC}$. Assumptions 1 and 2 then imply that $\pi'_{\mu_1^{LC}}(\alpha^{LC}) < 0$. Now define

$$X_1 := \int_{\mu_1^{LC}}^{\bar{\mu}} \pi'_\lambda(\alpha^{LC})dF(\lambda) \text{ and } Y_1(z) := \int_{\bar{\mu}}^{z} \pi'_\lambda(\alpha^{LC})dF(\lambda),$$

and note that $X_1 < 0$, $Y_1(\bar{\mu}) = 0$, $Y_1(\mu_2^{LC}) = -(\gamma\pi'_H(\alpha^{LC}) + X_1) > 0$, and $Y_1'(z) > 0$ for all $z > \bar{\mu}$. (In addition, note that we must have $\bar{\mu} < \mu_2^{LC}$; otherwise, the left-hand side of the FOC is strictly negative.) Therefore, there exists some $\mu' \in (\bar{\mu}, \mu_2^{LC})$ such that $Y_1(\mu') = -\gamma\pi'_H(\alpha^{LC})$. Thus, $\alpha(\bar{\mu}, \mu') = \alpha^{LC} = \alpha(\mu_1^{LC}, \mu_2^{LC})$, so the seller can achieve the same second-period price by delaying a (strictly) smaller subset of buyers. Since whenever constraint (SD) binds at a given type λ, it binds for all lower types (recall Lemma 1), the delay of a "lower" subset of buyers relaxes this constraint and increases profits. Therefore, we can choose $\mu_1^{LC} \geq \bar{\mu}$ without decreasing profits.

Similarly, suppose that $\alpha^{LC} > p_H$, but that $\mu_1^{LC} > \bar{\mu}$, where (again) $\bar{\mu}$ is defined by $p_{\bar{\mu}} = \alpha^{LC}$. Assumptions 1 and 2 then imply that $\pi'_{\mu_1^{LC}}(\alpha^{LC}) > 0$. So define

$$Y_2(z) := \int_{\bar{\mu}}^{z} \pi'_\lambda(\alpha^{LC})dF(\lambda),$$

and note that $Y_2(\bar{\mu}) = 0$, $Y_2(\mu_2^{LC}) > -\gamma\pi'_H(\alpha^{LC}) > 0$, and $Y_2'(z) > 0$ for all $z > \bar{\mu}$. Then there exists some $\mu' \in (\bar{\mu}, \mu_2^{LC})$ such that $Y_2(\mu') = -\gamma\pi'_H(\alpha^{LC})$. Thus, $\alpha(\bar{\mu}, \mu') = \alpha^{LC} = \alpha(\mu_1^{LC}, \mu_2^{LC})$, so the seller can achieve the same period-two price while decreasing the upper bound of the set on which (SD) must bind. Since whenever constraint (SD) binds at a given type λ, it binds for all lower types, relaxing this constraint increases profits. Thus, we can optimally set $\mu_1^{LC} = \bar{\mu}$. □

Proof of Lemma 3. Fix an optimal contract (denoted by *), and suppose that it induces a solution α^* to ($\widetilde{\text{SR}}'$) such that $\alpha^* < p_H$. Let $\Lambda^{**} := \Lambda \setminus \{\lambda | x_1^*(\lambda) = 1 \text{ and } k_1^*(\lambda) \leq \alpha^*\}$ be the set of types that delay contracting with positive probability or choose to recontract in period two because $k_1^*(\lambda) > \alpha^*$. Now define a new contract (denoted by **) with $x_1^{**}(\lambda) = 1$ for all $\lambda \in \Lambda$;

$$p_{11}^{**}(\lambda) := \begin{cases} 0 & \text{if } \lambda \in \Lambda^{**}, \\ p_{11}^*(\lambda) & \text{if } \lambda \notin \Lambda^{**}; \end{cases} \quad p_{12}^{**}(v,\lambda) := \begin{cases} p_{22}^*(v) & \text{if } \lambda \in \Lambda^{**}, \\ p_{12}^*(v,\lambda) & \text{if } \lambda \notin \Lambda^{**}; \end{cases} \text{ and}$$

$$q_1^{**}(v,\lambda) := \begin{cases} q_2^*(v) & \text{if } \lambda \in \Lambda^{**}, \\ q_1^*(v,\lambda) & \text{if } \lambda \notin \Lambda^{**}; \end{cases} \quad \text{for all } \lambda \in \Lambda \text{ and } v \in \mathbf{V}.$$

Note that we can write the objective function in $(\widetilde{\text{SR}}')$ as

$$\Pi_2(\alpha|x_1,k_1) := \int_\Lambda [x_1(\lambda)\pi_\lambda(\min\{\alpha,k_1(\lambda)\}) + (1-x_1(\lambda))\pi_\lambda(\alpha)]dF(\lambda) + \gamma\pi_H(\alpha),$$

where we ignore the $\int_\Lambda x_1(\lambda)U_{12}(\underline{v},\lambda)dF(\lambda)$ term as it does not depend on α. Then for all $\alpha \le \alpha^*$,

$$\begin{aligned}\Pi_2(\alpha|x_1^*,k_1^*) &= \int_\Lambda [x_1^*(\lambda)\pi_\lambda(\min\{\alpha,k_1^*(\lambda)\}) + (1-x_1^*(\lambda))\pi_\lambda(\alpha)]dF(\lambda) + \gamma\pi_H(\alpha) \\ &= \int_{\Lambda\setminus\Lambda^{**}} \pi_\lambda(\min\{\alpha,k_1^*(\lambda)\})dF(\lambda) + \int_{\Lambda^{**}} \pi_\lambda(\alpha)dF(\lambda) + \gamma\pi_H(\alpha) \\ &= \int_{\Lambda\setminus\Lambda^{**}} \pi_\lambda(\min\{\alpha,k_1^{**}(\lambda)\})dF(\lambda) \\ &\quad + \int_{\Lambda^{**}} \pi_\lambda(\min\{\alpha,k_1^{**}(\lambda)\})dF(\lambda) + \gamma\pi_H(\alpha) \\ &= \Pi_2(\alpha|x_1^{**},k_1^{**}),\end{aligned}$$

where the second equality follows from the fact that all buyers with $\lambda \in \Lambda^{**}$ ultimately consume the period-two contract (either by delaying or by recontracting), and the next equality from the fact that $k_1^{**}(\lambda) = \alpha^* \ge \alpha$ for all $\lambda \in \Lambda^{**}$. Therefore, since α^* maximizes $\Pi_2(\cdot|x_1^*,k_1^*)$, we must have $\Pi_2(\alpha^*|x_1^{**},k_1^{**}) \ge \Pi_2(\alpha|x_1^{**},k_1^{**})$ for all $\alpha \le \alpha^*$.

Now consider any price $\alpha' > \alpha^*$, and notice that $\Pi_2(\alpha'|x_1^{**},k_1^{**}) - \Pi_2(\alpha^*|x_1^{**},k_1^{**})$ equals

$$\begin{aligned}&\gamma[\pi_H(\alpha') - \pi_H(\alpha^*)] + \int_\Lambda [\pi_\lambda(\min\{\alpha',k_1^{**}(\lambda)\}) - \pi_\lambda(\min\{\alpha^*,k_1^{**}(\lambda)\})]dF(\lambda) \\ &= \gamma[\pi_H(\alpha') - \pi_H(\alpha^*)]\end{aligned}$$

since $k_1^{**}(\lambda) = \alpha^*$ for all $\lambda \in \Lambda^{**}$ and $k_1^{**}(\lambda) = k_1^*(\lambda) \le \alpha^*$ for all $\lambda \notin \Lambda^{**}$. Recall, however, that π_H is strictly concave and is maximized at p_H; this implies that the difference above is positive and is maximized at p_H. Therefore, the solution to $(\widetilde{\text{SR}}')$ given contract ** is $\alpha^{**} = p_H > \alpha^*$.

Of course, $\alpha^{**} > \alpha^*$ yields a decrease in the option value of delay, so constraint (SD) is now slack. This rent can be captured by charging *all* cohort-one buyers an additional lump sum of

$$\widetilde{U}_{11}^*(\underline{\lambda}) - \widetilde{U}_{11}^{**}(\underline{\lambda}) = \int_{\mathbf{V}} (\max\{v-\alpha^*,0\} - \max\{v-\alpha^{**},0\})dG(v|\underline{\lambda}) > 0.$$

Thus, augmenting the ** contract with the additional transfer above maintains the same allocation as * for period-one buyers while extracting additional surplus, and also extracts the *maximal* surplus from period-two buyers. This yields the seller a payoff strictly greater than the * contract, contradicting its presumed optimality. Therefore, we must have $\alpha^* \ge p_H$. □

Remark A. Define for all $\lambda \in \Lambda$ the function $\alpha(\lambda) := \arg\max_p \left\{\gamma\pi_H(p) + \int_{\underline{\lambda}}^{\lambda} \pi_\mu(p)dF(\mu)\right\}$ as the seller's optimal price when combining cohort two with the interval $[\underline{\lambda},\lambda]$. The concavity of π_H and π_μ imply that $\alpha(\cdot)$ is well-defined, and furthermore that $\alpha(\lambda)$ satisfies the first-order condition

$$\gamma\pi'_H(\alpha(\lambda)) + \int_{\underline{\lambda}}^{\lambda} \pi'_\mu(\alpha(\lambda))dF(\mu) = 0$$

for all $\lambda \in \Lambda$. Therefore, we can use implicit differentiation to show that

$$\alpha'(\lambda) = -\frac{\pi'_\lambda(\alpha(\lambda))f(\lambda)}{\gamma\pi''(\alpha(\lambda)) + \int_{\underline{\lambda}}^{\lambda} \pi''_\mu(\alpha(\lambda))dF(\mu)}.$$

Notice that $\alpha(\underline{\lambda}) = p_H > p_{\underline{\lambda}}$, implying that $\alpha'(\underline{\lambda}) < 0$. Moreover, $\alpha'(\mu) < 0$ for all μ with $p_\mu < \alpha(\mu)$, implying that $\alpha(\cdot)$ is initially decreasing (and strictly above p_μ). The function eventually reaches a minimum (at which $\alpha(\mu) = p_\mu$), and then it is strictly increasing with $\alpha(\mu) < p_\mu$. Furthermore, it is easy to show that $\alpha(\hat{\mu}) < p_H$, so the minimizer of $\alpha(\cdot)$ is smaller than $\hat{\mu}$, and $\alpha'(\mu) > 0$ for all $\mu \geq \hat{\mu}$.

Finally, fix any $\hat{\lambda} \in \Lambda$, and note that

$$\begin{aligned}
&\gamma[\pi_H(p_H) - \pi_H(\alpha(\hat{\lambda}))] \\
&\quad = \int_{\alpha(\hat{\lambda})}^{\alpha(\underline{\lambda})} \gamma\pi'_H(p)dp = \int_{\hat{\lambda}}^{\underline{\lambda}} \gamma\pi'_H(\alpha(\lambda))\alpha'(\lambda)d\lambda \\
&\quad = \int_{\hat{\lambda}}^{\underline{\lambda}} \left(-\int_{\underline{\lambda}}^{\lambda} \pi'_\mu(\alpha(\lambda))dF(\mu) \right) \alpha'(\lambda)d\lambda = \int_{\underline{\lambda}}^{\hat{\lambda}} \int_{\mu}^{\hat{\lambda}} \pi'_\mu(\alpha(\lambda))\alpha'(\lambda)d\lambda dF(\mu) \\
&\quad = \int_{\underline{\lambda}}^{\hat{\lambda}} \int_{\alpha(\mu)}^{\alpha(\hat{\lambda})} \pi'_\mu(p)dp dF(\mu) = \int_{\underline{\lambda}}^{\hat{\lambda}} [\pi_\mu(\alpha(\hat{\lambda})) - \pi_\mu(\alpha(\mu))]dF(\mu).
\end{aligned}$$

Therefore, for all $\hat{\lambda} \in \Lambda$, $\gamma\pi_H(p_H) + \int_{\underline{\lambda}}^{\hat{\lambda}} \pi_\mu(\alpha(\mu))dF(\mu) = \gamma\pi_H(\alpha(\hat{\lambda})) + \int_{\underline{\lambda}}^{\hat{\lambda}} \pi_\mu(\alpha(\hat{\lambda}))dF(\mu)$. ♢

Proof of Theorem 4. The proof proceeds in several steps. For ease of notation, we drop the $\widetilde{LC}$ superscripts throughout, and let α^* denote the optimal period-two price. We can also write the objective in $(\widetilde{\mathrm{SR}}')$—ignoring the $\int_\Lambda x_1(\lambda)U_{12}(\underline{v}, \lambda)dF(\lambda)$ term that does not depend on α—as

$$\begin{aligned}
\Pi_2(\alpha) &:= \gamma\pi_H(\alpha) + \int_\Lambda [(1 - x_1(\lambda))\pi_\lambda(\alpha) + x_1(\lambda)\pi_\lambda(\min\{\alpha, k_1(\lambda)\})]dF(\lambda) \\
&= \gamma\pi_H(\alpha) + \int_\Lambda \pi_\lambda(\bar{k}_1(\lambda))dF(\lambda).
\end{aligned}$$

The following series of claims derive properties of the optimal cutoffs $\bar{k}_1$ corresponding to α^*.

Claim. *There exists some $\lambda' > \underline{\lambda}$ such that $\bar{k}_1(\lambda') > p_H$.*

Proof of Claim. Suppose that $\bar{k}_1(\lambda) \leq p_H$ for all λ. Since $\alpha^* > p_H$, this implies that no buyers delay or recontract, so $x_1(\lambda) = 1$. But then this implies that the seller's period two payoff is

$$\Pi_2(\alpha^*) = \gamma\pi_H(\alpha^*) + \int_{\Lambda} \pi_\mu(k_1(\mu))dF(\mu)$$
$$< \gamma\pi_H(p_H) + \int_{\Lambda} \pi_\mu(k_1(\mu))dF(\mu) = \Pi_2(p_H),$$

where the inequality follows from the fact that $p_H = \arg\max_\alpha\{\pi_H(\alpha)\}$ and the strict concavity of π_H. This contradicts the sequential rationality of α^*. ♢

Claim. *Let λ^* be such that $p_{\lambda^*} = \alpha^*$. We must have $\bar{k}_1(\lambda) = \alpha^*$ for all $\lambda < \lambda^*$.*

Proof of Claim. Suppose not; that is, suppose that $\hat{\lambda} := \sup\{\lambda|\bar{k}_1(\lambda) = \alpha^*\}$ is such that $\hat{\lambda} < \lambda^*$. (Note that if $\bar{k}_1(\lambda) < \alpha^*$ for all $\lambda \in \Lambda$, we simply let $\hat{\lambda} = \underline{\lambda}$.) Also, define $\tilde{\lambda} := \sup\{\lambda|\bar{k}_1(\lambda) \geq \max\{p_H, p_\lambda\}\}$, and note that we must have $\tilde{\lambda} \in [\hat{\lambda}, \lambda^*)$.

With these definitions in mind, we let $\alpha' := \max\{p_H, p_{\tilde{\lambda}}\} < \alpha^*$. Note that $\bar{k}_1(\mu) \geq \alpha'$ for all $\mu \leq \tilde{\lambda}$, and $\bar{k}_1(\mu) < \alpha' := \max\{p_H, p_{\tilde{\lambda}}\}$ for all $\mu > \tilde{\lambda}$. Therefore,

$$\Pi_2(\alpha') = \gamma\pi_H(\alpha') + \int_{\underline{\lambda}}^{\tilde{\lambda}} \pi_\mu(\alpha')dF(\mu) + \int_{\tilde{\lambda}}^{\bar{\lambda}} \pi_\mu(\bar{k}_1(\mu))dF(\mu)$$
$$> \gamma\pi_H(\alpha^*) + \int_{\underline{\lambda}}^{\tilde{\lambda}} \pi_\mu(\bar{k}_1(\mu))dF(\mu) + \int_{\tilde{\lambda}}^{\bar{\lambda}} \pi_\mu(\bar{k}_1(\mu))dF(\mu) = \Pi_2(\alpha^*),$$

where the inequality follows from the strict concavity of the π functions and the fact that $p_H < \alpha' < \alpha^*$ and $p_\mu < \alpha' < \bar{k}_1(\mu)$ for all $\mu < \tilde{\lambda}$. This contradicts the sequential rationality of α^*. ♢

Claim. *Define μ^* by $\alpha(\mu^*) = \alpha^*$, where $\alpha(\cdot)$ is as defined in Remark A. Then $\bar{k}_1(\lambda) = \alpha^*$ for all $\lambda < \mu^*$.*

Proof of Claim. Suppose not; that is, suppose that $\hat{\lambda} := \sup\{\lambda|\bar{k}_1(\lambda) = \alpha^*\}$ is such that $\hat{\lambda} < \mu^*$. Note that the previous claim implies that $\hat{\lambda} \geq \lambda^*$. In addition, since $p_{\lambda^*} = \alpha^* > p_H$, the properties of $\alpha(\cdot)$ (see Remark A above) imply that $\alpha(\mu) < p_\mu$ for all $\mu \geq \tilde{\lambda}$.

Let $\tilde{\lambda} := \sup\{\lambda|\bar{k}_1(\lambda) \geq \alpha(\lambda)\}$, and note that we must have $\tilde{\lambda} \in [\hat{\lambda}, \mu^*)$. In addition, note that $\bar{k}_1(\lambda) \geq \alpha(\tilde{\lambda})$ for all $\lambda < \tilde{\lambda}$, while $\bar{k}_1(\lambda) < \alpha(\tilde{\lambda})$ for all $\lambda > \tilde{\lambda}$. Then

$$\Pi_2(\alpha^*) = \gamma\pi_H(\alpha^*) + \int_{\Lambda} \pi_\mu(\bar{k}_1(\mu))dF(\mu)$$
$$= \gamma\pi_H(\alpha^*) + \int_{\underline{\lambda}}^{\lambda^*} \pi_\mu(\alpha^*)dF(\mu) + \int_{\lambda^*}^{\tilde{\lambda}} \pi_\mu(\bar{k}_1(\mu))dF(\mu) + \int_{\tilde{\lambda}}^{\bar{\lambda}} \pi_\mu(\bar{k}_1(\mu))dF(\mu)$$

$$\leq \gamma\pi_H(\alpha^*) + \int_{\underline{\lambda}}^{\lambda^*} \pi_\mu(\alpha^*)dF(\mu) + \int_{\lambda^*}^{\tilde{\lambda}} \pi_\mu(\alpha^*)dF(\mu) + \int_{\tilde{\lambda}}^{\bar{\lambda}} \pi_\mu(\bar{k}_1(\mu))dF(\mu)$$

$$< \gamma\pi_H(\alpha(\tilde{\lambda})) + \int_{\underline{\lambda}}^{\lambda^*} \pi_\mu(\alpha(\tilde{\lambda}))dF(\mu) + \int_{\lambda^*}^{\tilde{\lambda}} \pi_\mu(\alpha(\tilde{\lambda}))dF(\mu) + \int_{\tilde{\lambda}}^{\bar{\lambda}} \pi_\mu(\bar{k}_1(\mu))dF(\mu)$$

$$= \Pi_2(\alpha(\tilde{\lambda})),$$

where the first equality rearranges the integral; the first inequality follows from observing that $\bar{k}_1(\mu) \leq \alpha^* < p_\mu$ for all $\mu > \lambda^*$ (where recall that $p_{\lambda^*} = \alpha^*$); the next inequality from the definition of $\alpha(\tilde{\lambda})$ as the maximizer of $\gamma\pi_H(p) + \int_{\underline{\lambda}}^{\tilde{\lambda}} \pi_\mu(p)dF(\mu)$; and the final equality from the definition of the seller's continuation payoffs. This, of course, contradicts the sequential rationality of α^*. ◊

With these claims in hand, it is easy to see that pointwise maximization of the objective function in $(\widetilde{\mathcal{P}}')$ (given the period-two price $\alpha^* > p_H$) yields cutoffs

$$\bar{k}_1^*(\lambda) = \begin{cases} \alpha^* & \text{if } \lambda < \mu^*, \\ \min\{\alpha^*, k^{ND}(\lambda)\} & \text{if } \lambda \geq \mu^*, \end{cases}$$

where the minimum operator arises from the observation that (IC_{11}) implies that $\bar{k}_1$ is decreasing.

Claim. *Define* $\tilde{\mu} := (k^{ND})^{-1}(\alpha^*)$. *We must have* $\tilde{\mu} \leq \mu^*$, *where* $\alpha(\mu^*) = \alpha^*$.

Proof of Claim. Define $\tilde{\lambda} := \max\{\mu^*, \tilde{\mu}\}$, and note that all $\lambda \leq \tilde{\lambda}$ are either delayed or receive contracts corresponding to the cutoff α^* (either via their initial contract or by recontracting), while for all $\lambda > \tilde{\lambda}$ we must have $x_1^*(\lambda) = 1$ and $k_1^*(\lambda) = k^{ND}(\lambda) < \alpha^*$. Thus, for any $\alpha \leq \alpha^*$, we have

$$\Pi_2^*(\alpha) = \gamma\pi_H(\alpha) + \int_\Lambda [(1 - x_1^*(\lambda))\pi_\lambda(\alpha) + x_1^*(\lambda)\pi_\lambda(\min\{\alpha, k_1^*(\lambda)\})]dF(\lambda)$$

$$= \gamma\pi_H(\alpha) + \int_{\underline{\lambda}}^{\tilde{\lambda}} \pi_\lambda(\alpha)dF(\lambda) + \int_{\tilde{\lambda}}^{\bar{\lambda}} \pi_\lambda(\min\{\alpha, k^{ND}(\lambda)\})dF(\lambda)$$

$$= \gamma\pi_H(\alpha) + \int_{\underline{\lambda}}^{\max\{(k^{ND})^{-1}(\alpha),\tilde{\lambda}\}} \pi_\lambda(\alpha)dF(\lambda) + \int_{\max\{(k^{ND})^{-1}(\alpha),\tilde{\lambda}\}}^{\bar{\lambda}} \pi_\lambda(k^{ND}(\lambda))dF(\lambda)$$

So, if $\tilde{\lambda} > (k^{ND})^{-1}(\alpha)$, it is easy to see that

$$\frac{\partial \Pi_2^*(\alpha)}{\partial \alpha} = \gamma\pi'_H(\alpha) + \int_{\underline{\lambda}}^{\tilde{\lambda}} \pi'_\lambda(\alpha)dF(\lambda).$$

On the other hand, if $\tilde{\lambda} \leq (k^{ND})^{-1}(\alpha)$, then

$$\frac{\partial \Pi_2^*(\alpha)}{\partial \alpha} = \gamma \pi_H'(\alpha) + \int_{\underline{\lambda}}^{(k^{ND})^{-1}(\alpha)} \pi_\lambda'(\alpha) dF(\lambda).$$

Thus, for all $\alpha < \alpha^*$,

$$\frac{\partial \Pi_2^*(\alpha)}{\partial \alpha} = \gamma \pi_H'(\alpha) + \int_{\underline{\lambda}}^{\max\{(k^{ND})^{-1}(\alpha), \tilde{\lambda}\}} \pi_\lambda'(\alpha) dF(\lambda),$$

implying that the left-hand derivative at α^* exists and is given by

$$\frac{\partial_- \Pi_2^*(\alpha^*)}{\partial \alpha} = \gamma \pi_H'(\alpha^*) + \int_{\underline{\lambda}}^{\tilde{\lambda}} \pi_\lambda'(\alpha^*) dF(\lambda).$$

Clearly, we must have $\frac{\partial_- \Pi_2^*(\alpha^*)}{\partial \alpha} \geq 0$; if not, then the period-two profits can be increased by infinitesimally decreasing the period-two price below α^*, contradicting its optimality. So suppose that $\frac{\partial_- \Pi_2^*(\alpha^*)}{\partial \alpha} > 0$, and define an alternative contract (denoted with a ** superscript) with

$$x_1^{**}(\lambda) = \begin{cases} 0 & \text{if } \lambda \leq \tilde{\lambda}, \\ 1 & \text{if } \lambda > \tilde{\lambda}; \end{cases} \text{ and}$$
$$\left\{p_{11}^{**}(\lambda), p_{12}^{**}(v,\lambda), q_1^{**}(v,\lambda)\right\} = \left\{p_{11}^{*}(\lambda), p_{12}^{*}(v,\lambda), q_1^{*}(v,\lambda)\right\}$$

for all $\lambda > \tilde{\lambda}$. Then in period two, the seller's problem is to $\max_\alpha\{\Pi_2^{**}(\alpha)\}$, where

$$\Pi_2^{**}(\alpha) = \gamma \pi_H(\alpha) + \int_{\underline{\lambda}}^{\tilde{\lambda}} \pi_\lambda(\alpha) dF(\lambda) + \int_{\tilde{\lambda}}^{\bar{\lambda}} \pi_\lambda(\min\{\alpha, k^{ND}(\lambda)\}) dF(\lambda).$$

It is easy to see that $\Pi_2^{**}(\alpha) = \Pi_2(\alpha)$ for all $\alpha \leq \alpha^*$; moreover, Π_2^{**} is continuous and differentiable at α^*. Thus, since $\frac{\partial_- \Pi_2^*(\alpha^*)}{\partial \alpha} > 0$, this implies that the optimal second period price under the ** contract is $\alpha^{**} > \alpha^*$. Thus, $\Pi_2^{**}(\alpha^{**}) > \Pi_2^{**}(\alpha^*) = \Pi_2(\alpha^*)$, so the new contract yields strictly greater period-two profits than the original * contract. Notice, however, that the contracts are identical for all types $\lambda > \tilde{\lambda}$, implying that first-period revenues are identical; thus, the ** contract yields a strictly greater overall payoff than the original contract, contradicting the latter's optimality.

Thus, we must have $\frac{\partial_- \Pi_2^*(\alpha^*)}{\partial \alpha} = 0$ at the optimal second-period price, and therefore

$$\gamma \pi_H'(\alpha^*) + \int_{\underline{\lambda}}^{\max\{\mu^*, \tilde{\mu}\}} \pi_\lambda'(\alpha) dF(\lambda) = 0.$$

For this condition to be consistent with the definition of μ^*, it must be the case that $\tilde{\mu} \leq \mu^*$. ◊

Thus, the optimal cutoff rule takes on the desired properties. Finally, note that it is feasible for the seller in $(\mathcal{R})$ (the program characterizing the optimal no-recontracting contract) to replicate this optimal cutoff we have characterized above by delaying all buyers in the interval $[\underline{\lambda}, \mu^*]$. Therefore, any allocation rule implementable with recontracting is also implementable in its absence, implying that the solution to $(\mathcal{R})$ must yield a greater value; that is, we must have $\Pi^{\widetilde{LC}} \leq \Pi^{LC}$. □

Proof of Proposition 5. Note that Assumption 2 implies that $\int_{\underline{\lambda}}^{\lambda_H} \pi_\mu(\alpha) dF(\mu)$ is strictly concave. Therefore, $p_H < \arg\max_\alpha \{\int_{\underline{\lambda}}^{\lambda_H} \pi_\mu(\alpha) dF(\mu)\}$ implies that $p_H < \alpha(\lambda_H)$ (where $\alpha(\cdot)$ is defined as in Remark A).

So define $\hat{\lambda}$ by $\alpha(\hat{\lambda}) = k^{ND}(\hat{\lambda})$, and note that this type is well-defined since $k^{ND}(\cdot)$ is strictly decreasing and $k^{ND}(\lambda_H) = p_H < \alpha(\lambda_H)$. In addition, let $\hat{\alpha} := \alpha(\hat{\lambda})$, and notice that $\hat{\alpha} > p_H$.

Fix any $\alpha' \in [p_H, \hat{\alpha}]$, and denote by λ' the type such that $\alpha(\lambda') = \alpha'$. For any $\alpha'' < \alpha'$, we have

$$
\begin{aligned}
&\gamma \pi_H(\alpha'') + \int_\Lambda \pi_\mu(\min\{\alpha'', k^{ND}(\mu)\}) dF(\mu) \\
&= \gamma \pi_H(\alpha'') + \int_{\underline{\lambda}}^{(k^{ND})^{-1}(\alpha'')} \pi_\mu(\alpha'') dF(\mu) + \int_{(k^{ND})^{-1}(\alpha'')}^{\bar{\lambda}} \pi_\mu(k^{ND}(\mu)) dF(\mu) \\
&< \gamma \pi_H(\alpha') + \int_{\underline{\lambda}}^{\lambda'} \pi_\mu(\alpha') dF(\mu) + \int_{\lambda'}^{(k^{ND})^{-1}(\alpha'')} \pi_\mu(\alpha'') dF(\mu) \\
&\quad + \int_{(k^{ND})^{-1}(\alpha'')}^{\bar{\lambda}} \pi_\mu(k^{ND}(\mu)) dF(\mu),
\end{aligned}
$$

since $\alpha' = \alpha(\lambda')$ is defined as the price maximizing the sum of the first two terms above. Now note that for all $\mu \in (\lambda', (k^{ND})^{-1}(\alpha''))$, $p_\mu > \alpha(\mu) > \alpha'$ (where the observation that $p_\mu > \alpha(\mu)$ is due to Remark A), and $p_\mu > \alpha(\mu) > k^{ND}(\mu) > \alpha''$. Therefore, $\pi_\mu(\min\{\alpha', k^{ND}(\mu)\}) > \pi_\mu(\alpha'')$ for all $\mu \in (\lambda', (k^{ND})^{-1}(\alpha''))$, implying that

$$
\begin{aligned}
&\gamma \pi_H(\alpha'') + \int_\Lambda \pi_\mu(\min\{\alpha'', k^{ND}(\mu)\}) dF(\mu) \\
&< \gamma \pi_H(\alpha') + \int_\Lambda \pi_\mu(\min\{\alpha', k^{ND}(\mu)\}) dF(\mu).
\end{aligned}
$$

Therefore, it is sequentially rational to set a period-two price α' when cohort-one buyers have been promised cutoffs $k_1'(\lambda) := \min\{\alpha', k^{ND}(\lambda)\}$. (It is never optimal to set a price greater than α' since it induces no recontracting while further reducing profits from cohort two.) Note, however, that these cutoffs correspond exactly to optimal contract from Theorem 2 when the seller has full commitment and sets (in advance) a period-two price equal to α', and so the seller's profit is equal to $\Pi^{FC}(\alpha')$, where $\Pi^{FC}(\cdot)$ is as defined in Theorem 2.

Since it is feasible for the seller (with limited commitment) to offer "full-commitment-equivalent" contracts for prices $\alpha \in [p_H, \hat{\alpha}]$, and it is easy to show that $\frac{\partial \Pi^{FC}}{\partial \alpha}(p_H) > 0$, it must therefore be the case that the optimal contract here induces a period-two price strictly greater than p_H. Therefore, the result of Theorem 4 applies immediately. □

Appendix B. Supplementary material

Supplementary material related to this article can be found online at http://dx.doi.org/10.1016/j.jet.2015.05.015.

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