INTERDEPENDENT PREFERENCES, POTENTIAL GAMES AND HOUSEHOLD CONSUMPTION

RAHUL DEB†

ABSTRACT. This paper presents a nonparametric model of interdependent preferences, where an individual's consumption may be an externality on the preferences of other consumers. We assume that individual price consumption data is observed for all consumers and prove that the general model imposes few restrictions on the observed data, where the consistency requirement is Nash rationalizability. We motivate potential games as an important sub class of games where the family of concave potential games is refutable and imposes stronger restrictions on observed data. As an application of this model, we discuss inter-household consumption data. Finally, we use this framework to extend the analysis of Brown and Matzkin (1996) on refutable pure exchange economies to pure exchange economies with externalities.

1. INTRODUCTION

In general, economic theory assumes individual demand is the result of consumers maximizing independent utility functions subject to budget constraints. Some economists however, have questioned this assumption of convenience. As early as 1899, Veblen observed that social status was an important consideration for the nouveau riche of 19^{th} century capitalist societies and used the term 'conspicuous consumption' to describe the practice of lavish spending to display wealth. Duesenberry in his classic work (Duesenberry (1949)) concerning the consumption function problem, attempted to explain the statistical discrepancy between Kuznet's data on aggregate savings and income in the period 1869-1929, and budget study data for 1935-1936 and 1941-1942, by challenging the conventional assumption of independent preferences. Work on interdependent preferences can also be found in Hopkins and Kornienko (2004), Pollak (1976), Postlewaite (1998), Schall (1972) and Sobel (2005).

The empirical research on interdependent preferences assume parametric specifications. Our analysis is nonparametric and extends Afriat's seminal nonparametric analysis of independent preferences (Afriat (1967)) to interdependent preferences. Afriat provides nonparametric necessary and sufficient conditions for a finite set of observations on prices and individual demands to be consistent with independent utility maximization. The consistency requirement for interdependent

†DEPARTMENT OF ECONOMICS, YALE UNIVERSITY, BOX 208268, NEW HAVEN, CT 06520-8268
E-mail address: rahul.deb@yale.edu.
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preferences is pure-strategy Nash equilibrium. We begin, by analyzing a general model of interdependent agent preferences. This general framework imposes very weak restrictions on observed data.

Bracha (2005) introduced potential games as a model for interdependent preferences in the context of individual decision making under uncertainty and risk. We characterize differentiable potential games and show that for two data points, potential games impose the same weak restrictions on the data as the general model. Our main result is that the class of utility functions which generate concave potentials impose refutable restrictions on observed data while still encompassing a large set of preferences. That is, we present necessary and sufficient conditions for constructing a concave potential function which Nash rationalizes the data.

Our model of interdependent preferences can be extended to study aggregate data. Carvajal (2007) also studies interdependent preferences (using aggregate data) where he fixes a single good as the externality good. His results are largely negative unless separability in the externality is imposed. Our general model imposes no restriction if we consider aggregate data. The concave potential model however, does impose refutable restrictions on aggregate data. We show that this is a strengthening of the results of Brown and Matzkin (1996) on refutable pure exchange economies to pure exchange economies with interdependent preferences.

The initial motivation for this paper came from the problem of studying household consumption data. There has been a substantial amount of research on household consumption. The introduction of the collective household model of Chiappori (1988,1992) was the first notable divergence from the unitary approach, which modeled the household as a single unit, where members of the household were assumed to have common goals. This divergence was necessary as there was mounting empirical evidence that the unitary model was rejected on household data (for example Blundell, Pashardes and Weber (1993), Browning and Meghir (1991)). The collective model allows the members of the household to have different preferences and allows a household member's private unobserved consumption to act as an externality on another household member. In contrast, our framework assumes household member's have common goals (as in the unitary model) but allows a household's consumption to act as an externality on another household. An example of this, is the well known phenomenon of "keeping up with the Joneses". We discuss the application of our model to household data in section 4.3.

The paper is organized as follows. In section 2, we describe and characterize a general model of interdependent preferences. In section 3, we describe the class of potential games and characterize differentiable potential games. In section 4, we characterize observed data for consistency with potential games and concave potential games, as well as provide an example of data which refutes the latter model. We also discuss the application to household consumption data in this section.

2. THE GENERAL MODEL

The economy consists of N individuals and L goods. Person i's consumption bundle is denoted by $x^i \in \mathbb{R}^L_+$. Person i has a utility function $u^i : \mathbb{R}^{NL}_+ \to \mathbb{R}$. Person i's utility level is $u^i(x^1, \ldots, x^i, \ldots, x^N)$ (henceforth represented as $u^i(x^i, x^{-i})$) which depends not only on her consumption x^i but also on the consumption of the other individuals x^{-i} in the economy. Prices are denoted by p. When we consider aggregate data we will denote person i's income by I^i and aggregate consumption by w.

We first consider the case where we observe market prices and individual level consumption data. The data consists of repeated observations of prices and consumption bundles in the economy. Hence, the observed data is of the form $D = \{(p_t, x^1_t, \ldots, x^i_t, \ldots, x^N_t)\}^T_{t=1}$, where the subscripts denote the time period of the observation. We pose the following question- when is the observed data consistent with utility maximization, where utility functions are interdependent? The relevant notion of individual optimization is Nash equilibrium. We say this data is *Nash Rationalizable*, if there exist utility functions u^i such that for all i and t, we have

$$x^i_t = \underset{p_t x \leq p_t x^i_t}{\operatorname{argmax}}\, u^i(x, x^{-i}_t)$$

That is, each player chooses a consumption bundle in her budget set which is a best response to every other player's actions in each observation. Following Varian (1982), we define an intuitive necessary condition, Conditional GARP (CGARP). For player i, it can be thought that for the subset of the data where every other players' actions stay the same, there are no externalities. Hence, it follows from Afriat's theorem that player i's actions must satisfy GARP on this subset of data. It is surprising that this condition is also sufficient for rationalizing the data. More formally:

$$D^i_t = \{(p_{t'}, x^i_{t'}) \quad : \quad x^{-i}_{t'} = x^{-i}_t, \quad 1 \leq t' \leq T\}$$

In words D^i_t is the set of all price consumption bundles of person i when everyone else is consuming the same as they did in period t. We now define the standard Generalized Axiom of Revealed Preference as in Varian (1982). The following definition assumes that only the consumption of a single individual is observed.

Definition 2.1 (GARP)**.** Given arbitrary data set $D = \{(p_t, x_t)\}^T_{t=1}$. For any two consumption bundles x_t and $x_{t'}$ we say $x_t \succ_{R^0} x_{t'}$ if $p_t x_t \geq p_t x_{t'}$. We say $x_t \succ_P x_{t'}$ if $p_t x_t > p_t x_{t'}$. Finally we say $x_t \succ_R x_{t'}$ if for some sequence of observations $(x_{t_1}, x_{t_2}, \ldots, x_{t_m})$ we have $x_t \succ_{R^0} x_{t_1}, x_{t_1} \succ_{R^0} x_{t_2}, \ldots, x_{t_m} \succ_{R^0} x_{t'}$. In other words relation $\succ_R$ is the transitive closure of $\succ_{R_0}$. The data D satisfies ***GARP*** if

$$x_t \succ_R x_{t'} \implies x_{t'} \not\succ_P x_t \qquad \forall x_t, x_{t'}$$

This leads to the following definition of CGARP.

Definition 2.2 (CGARP)**.** The data set $D = \{(p_t, x^1_t, \ldots, x^i_t, \ldots, x^N_t)\}^T_{t=1}$ satisfies ***CGARP*** if for all $i \in N$ and all $t \in T$, D^i_t satisfies GARP.

We can now characterize data that is Nash rationalizable.

Theorem 2.1. *For an observed data set $D = \{(p_t, x_t^1, \dots, x_t^i, \dots, x_t^N)\}_{t=1}^T$ the following are equivalent*

(1) *The data set is Nash rationalized by utility functions which are non-satiated in the individual consumption.*
(2) *The data set satisfies CGARP.*
(3) *The following system of inequalities*

$$U_{t'}^i \leq U_t^i + \lambda_t^i p_t (x_{t'}^i - x_t^i) + \sum_{j \neq i} \mu_t^{ij} (x_{t'}^j - x_t^j)$$
$$\text{for } i, j = 1, 2, \dots, N \qquad \text{and } t, t' = 1, 2, \dots, T$$

has positive solutions for utility values U and strictly positive solutions for marginal utilities λ, μ where μ is a vector.
(4) *The data set is Nash rationalized by utility functions which are strictly monotone, continuous and concave in all arguments.*

The following is an example of data which violates CGARP.

Example 2.1. Consider a two person economy with two goods and the following data set consisting of two observations

$$p_1 = (1,2) \qquad x_1^1 = (0,1) \qquad x_1^2 = (0,1)$$
$$p_2 = (2,1) \qquad x_2^1 = (1,0) \qquad x_2^2 = (0,1)$$

Clearly as person 2's consumption does not change even with the change in prices, person 1's choices cannot violate GARP, and it is straightforward to check that they do. Thus the above data will violate the inequalities of Theorem 1 and is not Nash rationalizable.

The above example shows that the general model is refutable although it imposes weak restrictions on data (due to the unrestrictive nature of CGARP). To show that the general model of interdependent preferences is refutable, we need to show that the multivariate polynomial inequalities of theorem 2.1 are refutable. To show that the system of inequalities is consistent we can construct an example of data which satisfies them (consider two observations where both people consume different bundles). To show that the inequalities are not always satisfied, it suffices to construct an example of data which violates CGARP which is example 2.1. Hence it follows from the Tarski-Seidenberg theorem (Tarski (1951)) that the system of inequalities are refutable as the inequalities are solvable for some but not all consumption data sets (see Brown and Kubler (2008) for discussion). Carvajal (2007) finds the same result for an economy where only one good acts as an externality.

3. Potential Games

In this section we study the important class of potential games and do not restrict the discussion to interdependent preferences and consumption. Potential games were introduced in Monderer and Shapley (1996). A classical example of a potential game is a Cournot oligopoly game. Monderer and Shapley (1996) show that every congestion game (Rosenthal (1973)) is a potential game.

Other studies which use potential games are Mäler et al (2003) in their study of the economics of shallow lakes, Konishi et al (1998) study a poll tax scheme for the provision for a public good, Garcia and Arbeláez (2002) evaluate impacts of mergers in Colombian wholesale market for electricity. Bracha and Brown (2007) propose a behavioral theory of choice, where the individual is a composite agent consisting of a rational and emotional process which is represented as a potential game.

Let $\Gamma(u^1, \ldots, u^N)$ be a game of N players where player i has strategy set $Y^i \subseteq \mathbb{R}^{n_i}$ (n_i being a positive integer). Player i's payoff function is $u^i : Y \to \mathbb{R}$ where $Y = Y^1 \times \cdots \times Y^N$. A function $P : Y \to \mathbb{R}$ is an *ordinal potential* if for every $i \in N$ and for every $y^{-i} \in Y^{-i}$

$$u^i(x, y^{-i}) - u^i(z, y^{-i}) > 0 \qquad \text{iff} \qquad P(x, y^{-i}) - P(z, y^{-i}) > 0 \qquad \text{for all } x, z \in Y^i$$

An *exact potential* or simply a *potential* is a function $P : Y \to \mathbb{R}$ such that for every $i \in N$ and for every $y^{-i} \in Y^{-i}$

$$u^i(x, y^{-i}) - u^i(z, y^{-i}) = P(x, y^{-i}) - P(z, y^{-i}) \qquad \text{for all } x, z \in Y^i$$

Γ is called an ordinal potential game if it admits an ordinal potential and a potential game if it admits an exact potential. Monderer and Shapley (1996) characterize potential games and the associated characterization problem for ordinal potential games was solved by Voorneveld and Norde (1997). Here we characterize differentiable potential games using Poincaré's lemma for differential forms (see Weintraub (1996)). The characterization below has a nice intuitive interpretation.

Theorem 3.1. *Given a game Γ. Assume the utility of each player is defined on an open convex set [1] Z such that $Y_1 \times \cdots \times Y_N \subseteq Z$, or in other words $u^i : Z \to \mathbb{R}$. Moreover assume the utility functions are twice continuously differentiable on Z. Then Γ is a potential game if and only if*

$$\frac{\partial^2 u^i}{\partial y^i_{m_i} \partial y^j_{m_j}} = \frac{\partial^2 u^j}{\partial y^j_{m_j} \partial y^i_{m_i}} \qquad \text{for every } i, j \in N \text{ and for all } 1 \le m_i \le n_i, 1 \le m_j \le n_j$$

This result implies that existence of a potential is merely a restriction on the mixed partial derivatives of the utility functions. This restriction requires symmetry on how a player's action can affect the marginal utility of another player. The special case where strategies are intervals of real numbers appears in Monderer and Shapley (1996).

In most games, strategy sets are usually closed and convex, but it is straightforward to smoothly extend the utility function to an open convex set containing the strategy set.

[1] More generally, utilities need only be defined on an open contractible set.

4. Potential Games And Interdependent Preferences

An example to motivate potential games is the following. Assume that the Smith's and the Jones' utility functions are composed of two separate additive components- utility from consumption and utility from 'status'. Then a potential game implies that the Smiths and the Joneses care equally about status (that is, they have the same status term), however they may get different utilities from consumption. For example, although both the Smiths and the Joneses want to have a better car than their neighbor's, the Smiths settle for the minivan as they get more utility from going out to expensive dinners, even though they both have the same status terms in their utility functions. This can be modeled by making the consumption term of the Smiths, large with respect to the status term in their utility function. Moreover, by making the consumption term of the Smiths arbitrarily large, we can approximately model the case where the Joneses try to compete with the Smiths but not vice versa. Thus, although potential games do restrict the possible set of preferences, they impose a reasonable restriction which still encompasses a large set of interdependent preferences. Concavity is motivated as an assumption in section 4.2.

4.1. **Potential Games.** We first consider the general class of potential games. As was shown in section 3, the restriction to utility functions which generate a potential is simply a restriction on the mixed partial derivatives, much like concavity is a restriction on the Hessian. This class of games is large and it allows goods to be either positive or negative externalities, and this could differ across players. However, the potential function itself must be strictly monotone in all arguments. This reflects the fact that each player's utility function is strictly monotone in individual consumption.

In order to characterize data for consistency with utility functions that generate a potential, we employ an argument different from the standard proofs inspired by Afriat (1967). Our approach uses the extension theorem of Herden (2008) that does not depend on the properties of concave functions and can be used to test the refutability of models with externalities. The following is the result for two data points

Theorem 4.1. *For an observed data set* $D = \{(p_t, x_t^1, \ldots, x_t^i, \ldots, x_t^N)\}_{t \in \{1,2\}}$ *the following are equivalent*

(1) *The data set is Nash rationalized by utility functions which are non-satiated in the individual consumption.*
(2) *The data set satisfies CGARP.*
(3) *The data set is Nash rationalized by utility functions which admit a continuous, strictly monotonic, potential function.*

The above result says that if the data is Nash rationalizable, then it is Nash rationalizable by a potential game. Hence, for two data points, potential games have the same explanatory power as the general model. However, this implies that the class of potential games also imposes only weak restrictions on observed data. The same conclusions will also hold for the more general class of pseudo-potential games defined in Dubey et al. (2006).

4.2. **Concave Potential Games.** Concavity is a desirable property of potential functions as it is for utility functions. In a smooth potential game with a concave potential function, a strategy profile is a pure-strategy Nash equilibrium, if and only if, it is a potential maximizer. In addition, if the potential function is strictly concave then there is a unique equilibrium. Neyman (1997) shows that smooth strictly concave potential games have a unique Nash equilibrium and a unique correlated equilibrium. Since our characterization of concave potential games is constructive, this uniqueness allows us to predict future consumer behavior, by utilizing the tools developed in Bracha and Brown (2007), who use the potential function to predict affective demand in insurance markets.

Hence, we now restrict our attention to utility functions which generate a potential which is concave in all arguments. This does not require utility functions themselves to be concave in all arguments. As with non-concave potentials, this allows for both positive and negative externalities. The following is an example of a concave potential game where individual utilities are not concave in all arguments and the externalities are negative. Consumers get utility from the consumption of their bundle as well as from their relative consumption. This relative consumption paradigm was suggested by Duesenberry (1949). Consider the case of two investment bankers. Investment banker A owns a Porsche 911 Carrera which by virtue of allowing her to drive to work, provides her with a certain level of utility. However, her utility decreases when her coworker B, purchases a Lamborghini Murcielago. The decrease in utility can be attributed to relative consumption. Below is an example of such a model.

Example 4.1. Consider the case of two investment bankers who consume two goods (suits and cars) where each individual has the following utility function

$$u^i(x^i, x^{-i}) = v^i(x^i_1, x^i_2) + \ln\left(\frac{x^i_1}{x^1_1 + x^2_1}\right) + \ln\left(\frac{x^i_2}{x^1_2 + x^2_2}\right)$$

where $i \in \{1, 2\}$ and v^i is concave. The interpretation of the above utility function is as follows. Each person gets utility v^i from the consumption of the goods she purchases. The second term is the relative utility she gets with regard to what her coworker is consuming.

Clearly u^i is concave in x^i fixing x^{-i}. However u^i is not concave in x^{-i}. This model corresponds to the following concave potential function.

$$P(x^1, x^2) = v^1(x^1_1, x^1_2) + v^2(x^2_1, x^2_2) + \ln\left(\frac{x^1_1 x^2_1}{x^1_1 + x^2_1}\right) + \ln\left(\frac{x^1_2 x^2_2}{x^1_2 + x^2_2}\right)$$

This example shows that utility functions which generate a concave potential need not be concave themselves. Ui (2007) shows that requiring the potential to be concave imposes an additional constraint on the gradients of the utility functions. He shows that a smooth potential game has a concave potential, if and only if, the gradients of the utility functions are monotone. This additional restriction makes concave potential games refutable as can be seen by the following characterization of concave potential games.

Theorem 4.2. *For an observed data set $D = \{(p_t, x_t^1, \ldots, x_t^i, \ldots, x_t^N)\}_{t=1}^T$ the following are equivalent*

(1) *The data set is Nash rationalized by utility functions which admit a strictly monotonic, concave, ordinal potential function.*

(2) *The following system of inequalities*

$$V_{t'} \leq V_t + \sum_{i=1}^{N} \lambda_t^i p_t (x_{t'}^i - x_t^i)$$

have positive solutions for potential values V and strictly positive solutions for marginal utilities λ for all i, t, t'.

(3) *The data set is Nash rationalized by utility functions which admit a continuous, strictly monotonic, concave potential function.*

We can also consider the implications of this model on aggregate data in an exchange economy, where consumers have interdependent preferences. The relevant notion of equilibrium is Nash-Walras equilibrium (see Ghoshal and Polemarchakis (1997)). Since our aim is to rationalize the data, we can follow Brown and Matzkin (1996) and ignore the fact that we are no longer simply dealing with a consumption game but instead are in the setting of an abstract economy (see chapter 19 of Border (1989)). This is clearly illustrated by the following.

An observed aggregate data set $D = \{(p_t, \{I_t^i\}_{i=1}^N, w_t)\}_{t=1}^T$ is consistent with Nash-Walras equilibrium, if we can find feasible consumptions x_t^i for each individual i at each observation t, such that $p_t x_t^i \leq I_t^i$ and $\sum_{i=1}^N x_t^i = w_t$, and each consumption bundle x_t^i is a best response to x_t^{-i} for all t, i. Since rationalization merely involves finding feasible consumption bundles such that p_t is an equilibrium price vector, we can avoid the complications of redefining our setting as an abstract economy.

The analogue of theorem 4.2 for aggregate consumption data can be written as follows and the proof is omitted as it is a straightforward extension of theorem 4.2.

Theorem 4.3. *For an observed data set $D = \{(p_t, \{I_t^i\}_{i=1}^N, w_t)\}_{t=1}^T$ the following are equivalent*

(1) *There exist utility functions which admit a strictly monotonic, concave ordinal potential function such that at each observation t, p_t is a Nash-Walras equilibrium price vector for the exchange economy.*

(2) *The following system of inequalities*

$$V_{t'} \leq V_t + \sum_{i=1}^{N} \lambda_t^i p_t (x_{t'}^i - x_t^i)$$

$$p_t x_t^i = I_t^i$$

$$\sum_{i=1}^{N} x_t^i = w_t$$

have positive solutions for potential values V, feasible consumptions x and strictly positive solutions for marginal utilities λ for all i, t, t'.

(3) *There exist utility functions which admit a continuous, strictly monotonic, concave potential function such that at each observation t, p_t is a Nash-Walras equilibrium price vector for the exchange economy.*

We will concentrate on the implications of the latter result. Any refutable restrictions on aggregate data will carry over to individual data. Theorems 2.1 and 4.1 imply that the general model and the potential game impose no restrictions on aggregate data, as it is easy to find feasible bundles which will not violate CGARP (by virtue of having each person consuming a different bundle in each time period). This is not the case with the concave potential game.

The inequalities of theorem 4.3 are simply the sum of the equilibrium inequalities of theorem 2 in Brown and Matzkin (1996). This means that any aggregate data set that satisfies their inequalities will also satisfy ours. Hence, their model is a special case of our concave potential model. A more intuitive explanation is the following.

Brown and Matzkin find necessary and sufficient conditions for the existence of independent utility functions and feasible consumption bundles, so that the observed aggregate data corresponds to an equilibrium of the exchange economy. These conditions are also necessary and sufficient for there to exist *concave* independent utility functions and feasible consumption bundles, so that the observed aggregate data corresponds to an equilibrium of the exchange economy. Concave independent utility functions constitute a concave potential game where the potential function is simply the sum of the utility functions. Hence, the exchange economy with independent preferences is a special case of a concave potential game. We now construct an example of aggregate data which satisfies the inequalities of theorem 4.3 but does not satisfy the equilibrium inequalities of Brown and Matzkin. Hence, theorem 4.3 is a true generalization of their results.

Example 4.2. Consider the following aggregate consumption data

$$\begin{array}{llll} p_1 = (1,2) & I_1^1 = 14 & I_1^2 = 1 & w_1 = (3,6) \\ p_2 = (2,1) & I_2^1 = 14 & I_2^2 = 1 & w_1 = (6,3) \end{array}$$

For player 1, every feasible consumption bundle in observation 1 is affordable under observation 2 and vice versa. However the same is not true for player 2. In particular the bundle $(1, 0)$ is affordable for player 2 under period 1 prices and income but not under those of period 2. Thus although these observations would not satisfy the inequalities of Brown and Matzkin (1996), we can find potential levels V and Lagrangian multipliers λ such that they satisfy the inequalities of theorem 4.3.

It is surprising to find that a general equilibrium model, where we allow interdependent preferences, has refutable restrictions on aggregate data. But this is in fact the case for the concave potential model as is shown by the following example.

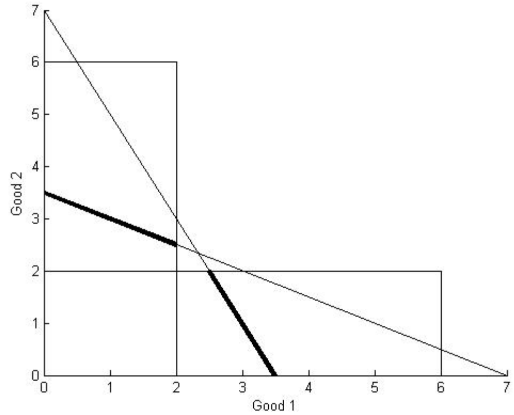

FIGURE 1

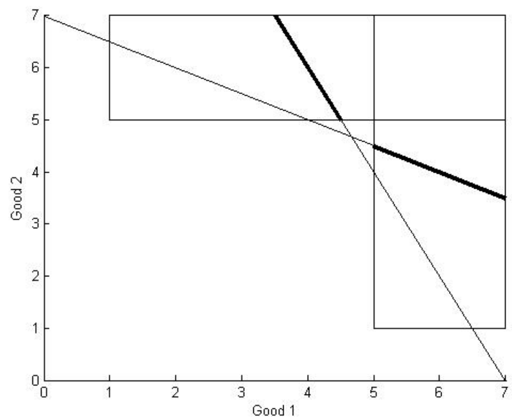

FIGURE 2

Example 4.3. Consider the following aggregate consumption data.

$$p_1 = (1,2) \qquad I_1^1 = 7 \qquad I_1^2 = 7 \qquad w_1 = (2,6)$$
$$p_2 = (2,1) \qquad I_2^1 = 7 \qquad I_2^2 = 7 \qquad w_1 = (6,2)$$

The observed aggregate consumption only allows feasible individual consumptions that lie in the Edgeworth boxes shown in figures 1 and 2. Figure 1 is similar to figure 1 in Brown and Matzkin (1996) and it shows that for player 1, every feasible consumption bundle in observation 1 is affordable under observation 2 income and prices and vice versa. The feasible bundles for player 1 are shown by the dark shaded line in each box. Figure 2 provides an equivalent analysis for player 2. This is the key difference between this example and that of Brown and Matzkin. The situation described by figure 1 was sufficient to violate their equilibrium inequalities. We now show that these data points cannot satisfy the inequalities of theorem 4.3.

Let us individually consider the two inequalities which need to be satisfied.

$$V_1 \leq V_2 + \lambda_2^1 p_2(x_1^1 - x_2^1) + \lambda_2^2 p_2(x_1^2 - x_2^2) \tag{1}$$

$$V_2 \leq V_1 + \lambda_1^1 p_1(x_2^1 - x_1^1) + \lambda_1^2 p_1(x_2^2 - x_1^2) \tag{2}$$

For any feasible consumption bundles for player 1, we know from Fig 1 that $p_2(x_1^1 - x_2^1) < 0$ and $p_1(x_2^1 - x_1^1) < 0$. Similarly from Fig 2 we know that the same holds for player 2, that is, $p_2(x_1^2 - x_2^2) < 0$ and $p_1(x_2^2 - x_1^2) < 0$. Hence, inequality (1) will imply $V_2 > V_1$ as the λ's must be strictly positive. Similarly, (2) will imply $V_1 > V_2$ which is the contradiction we seek.

Finally to end this section, we would like to provide an example which reiterates the importance of concavity in potential games. This example shows that the weaker assumption of a biconcave potential game imposes substantially weaker restrictions on observed data than the concave potential game. Biconcave potentials are generated by utility functions which are concave only in individual consumption. Biconcave potentials are hence concave in consumption x^i of arbitrary player $i \in N$ for fixed x^{-i} of the other players, but need not be concave in all arguments. Strict biconcavity ensures unique best responses.

Example 4.4. Consider the following two person, two good example where each person has the same following utility function

$$u^1(x^1, x^2) = u^2(x^1, x^2) = P(x^1, x^2) = x_1^1 x_1^2 + x_2^1 x_2^2$$

We observe that this potential function is biconcave, as it linear in x^1 when we fix x^2 and vice versa. The following observations are equilibria of the above game.

$$p_t = (2, 1) \qquad x_t^1 = x_t^2 = (1, 0)$$
$$p_{t'} = (1, 2) \qquad x_{t'}^1 = x_{t'}^2 = (0, 1)$$

Using an identical argument to that of example 4.3, it can be shown that these observations will violate the equilibrium inequalities of theorem 4.2.

4.3. **Potential Games and the Household.** A natural application of concave potential games is to analyze household consumption data. By assuming members of the household have common goals, the preferences of the household can be modeled by a single utility function and we can use theorem 4.2 on household consumption data. In practice, it is easier to get consumption data at the level of the household, than it is to get consumption data for individuals within the household. Unobserved private consumption of household members is the reason that makes nonparametric tests of the collective model difficult to implement.

Browning and Chiappori (1998) provide a rigorous theoretical framework for testing consistency of data with the collective model. They assume members of the household may have different, possibly divergent preferences, where each a household member's private consumption might act as an externality for another household member. The only assumption is that regardless of how decisions are made, the outcome must be efficient. Thus, if the household consists of a husband and a wife whose utilities are given by u^h and u^w, efficiency implies that the household's utility is given by $u^h + \lambda u^w$, where λ is a measure of the "distribution of power". The model's explanatory power lies in the fact that although the utility functions stay the same, λ can be different across different observations. Lastly, they assume individual consumption is not observed. They provide a parametric characterization of this model.

The assumption of changing "distributions of power" across observations seems somewhat ad hoc. Also, empirical studies such as Udry (1996) show that even the assumption of efficiency is not as innocuous as it may seem. Finally, nonparametric tests of the this model (Cherchye et al. (2007)) are only testable for a special case of the collective model, moreover, Deb (2007) shows them to be computationally inefficient even for this special case.

By contrast, the nonparametric test for the concave potential model is easily and efficiently implemented, since, if we observe household consumption, then the inequalities (of theorem 4.2) are linear in the unknowns. Moreover, the assumption that households violate the unitary model because their preferences are influenced by the consumption of others is intuitive and the simplicity of the nonparametric test makes testing such a model an interesting empirical exercise. The intrahousehold collective model derives its explanatory power by allowing private consumptions of household members which in practice are unobserved. In contrast, our interhousehold model utilizes the observed consumptions of the other households in the economy in order to rationalize the consumption of a particular household. Lastly, our approach is also constructive. We construct a concave potential function which Nash rationalizes the data, and this allows us to use the tools in Bracha and Brown (2007) to predict future household behavior.

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Appendix A. Proofs Of Theorem 2.1

Proof of Theorem 2.1. $(1) \implies (2)$
Let us consider arbitrary D^i_t. If it is a singleton set then it trivially satisfies GARP. For non singleton sets the proof is identical to the first step of the proof of Afriat's theorem in the appendix of Varian (1982).

$(2) \implies (3)$
This is the main step in the proof. Consider arbitrary person i. First consider each D^i_t. From Afriat's theorem, we know for all $x^i_{t_1}, x^i_{t_2} \in D^i_t$ we can find positive numbers $V^i_{t_1}, V^i_{t_2}$ and strictly

positive $\zeta^i_{t_1}$ such that $V^i_{t_2} \leq V^i_{t_1} + \zeta^i_{t_1} p_{t_1}(x^i_{t_2} - x^i_{t_1})$ because D^i_t satisfies GARP. Having found such numbers for each D^i_t let us define η as follows.

$$\eta = \max_{i \in N} \max_{t,t' \in T} V^i_{t'} - V^i_t - \zeta^i_t p_t(x^i_{t'} - x^i_t)$$

If $\eta < 0$ then, for all j, t we can set $U^i_t = V^i_t$, $\lambda^i_t = \zeta^i_t$ and we can set μ^{ij}_t as strictly positive and small in order to satisfy the inequalities and we are done. If $\eta > 0$ then we take any smooth, strictly monotone and strictly concave function $f : \mathbb{R}^{(N-1)L}_+ \rightarrow \mathbb{R}_+$ and define $W^i_t = f(x^{-i}_t)$ and $\mu^{ij}_t = \partial f(x^{-i}_t)/\partial x^j$. Finally we define γ as follows

$$\gamma = \max_{x^{-i}_t \neq x^{-i}_{t'}} W^i_{t'} - W^i_t - \sum_{j \neq i} \mu^{ij}_t (x^j_{t'} - x^j_t)$$

Since f is strictly concave we will get $\gamma < 0$ ($\eta > 0$ implies there exists t, t' such that $x^{-i}_{t'} \neq x^{-i}_t$). Hence, we can set $U^i_t = W^i_t - \frac{\gamma}{\eta} V^i_t$ and $\lambda^i_t = -\frac{\gamma \zeta^i_t}{\eta}$ and hence we have found positive solutions for

$$U^i_{t'} - U^i_t - \lambda^i_t p_t(x^i_{t'} - x^i_t) - \sum_{j \neq i} \mu^{ij}_t (x^j_{t'} - x^j_t) \leq 0$$

for $t, t' = 1, 2, \ldots, T$

For $\eta = 0$ we can do the same as when $\eta > 0$. In this case, $\gamma \leq 0$ and hence we set $U^i_t = W^i_t + V^i_t$ and $\lambda^i_t = \zeta^i_t$ and we are done. We repeat this for all i to get solutions to the desired inequalities.

$(3) \implies (4)$
For any (x^i, x^{-i}) define

$$u^i(x^i, x^{-i}) = \min_t \left[U^i_t + \lambda^i_t p_t(x^i - x^i_t) + \sum_{j \neq i} \mu^{ij}_t (x^j - x^j_t) \right]$$

We first show $u^i(x^i_t, x^{-i}_t) = U^i_t$. This can be shown as follows

$$\begin{aligned} u^i(x^i_t, x^{-i}_t) &= U^i_{t'} + \lambda^i_{t'} p_{t'}(x^i_t - x^i_{t'}) + \sum_{j \neq i} \mu^{ij}_{t'} (x^j_t - x^j_{t'}) \qquad \text{for some } t' \\ &\leq U^i_t + \lambda^i_t p_t(x^i_t - x^i_t) + \sum_{j \neq i} \mu^{ij}_t (x^j_t - x^j_t) \\ &= U^i_t \end{aligned}$$

This inequality cannot be strict as it would violate condition (3) and hence $u^i(x^i_t, x^{-i}_t) = U^i_t$. Finally, for each t and i if $p_t x^i \leq p_t x^i_t$ then

$$\begin{aligned} u^i(x^i, x^{-i}_t) &\leq U^i_t + \lambda^i_t p_t(x^i - x^i_t) + \sum_{j \neq i} \mu^{ij}_t (x^j_t - x^j_t) \\ &= U^i_t + \lambda^i_t p_t(x^i - x^i_t) \\ &\leq U^i_t \end{aligned}$$

Hence, each person is best responding. Since the minimum of concave functions is concave, u^i is a concave function. Since λ and μ are strictly positive, u^i is strictly monotone.

$(4) \implies (1)$

This is obvious and it completes the proof. □

Appendix B. Proof of Theorem 3.1

Proof of Theorem 3.1. We first show the necessity. Since Γ is a potential game, for every $i \in N$ and for every $y^{-i} \in Y^{-i}$ we have

$$u^i(x, y^{-i}) - u^i(z, y^{-i}) = P(x, y^{-i}) - P(z, y^{-i}) \qquad \text{for all } x, z \in Y^i$$

Since u^i is differentiable it is straightforward to observe that P is a potential function if and only if P is differentiable and

$$\frac{\partial u^i}{\partial y^i_{m_i}} = \frac{\partial P}{\partial y^i_{m_i}}$$
$$\frac{\partial u^j}{\partial y^j_{m_j}} = \frac{\partial P}{\partial y^j_{m_j}}$$
$$\text{for every } i, j \in N \text{ and for all } 1 \leq m_i \leq n_i, 1 \leq m_j \leq n_j$$

Differentiating the first equation with respect to $y^j_{m_j}$ and the second with respect to $y^i_{m_i}$ we get

$$\frac{\partial^2 u^i}{\partial y^i_{m_i} \partial y^j_{m_j}} = \frac{\partial^2 P}{\partial y^i_{m_i} \partial y^j_{m_j}} = \frac{\partial^2 P}{\partial y^j_{m_j} \partial y^i_{m_i}} = \frac{\partial^2 u^j}{\partial y^j_{m_j} \partial y^i_{m_i}}$$
$$\text{for every } i, j \in N \text{ and for all } 1 \leq m_i \leq n_i, 1 \leq m_j \leq n_j$$

which proves the necessity.

We now prove the sufficiency using Poincaré's lemma. This result states that if a differentiable p form defined on a contractible open subset of $\mathbb{R}^n$ is closed then it is also exact. Since every convex set is contractible, hence each u^i is defined on an open contractible set and we can use Poincaré's lemma. Since each u^i is differentiable we can define the following 1-form

$$\alpha = \sum_{i=1}^{N} \sum_{m_i=1}^{n_i} \frac{\partial u^i}{\partial y^i_{m_i}} dy^i_{m_i}$$

Since u^i is twice differentiable we can take the derivative of the above 1-form to get

$$d\alpha = \sum_{i=1}^{N} \Bigg[\sum_{1 \leq l_i, m_i \leq n_i} \frac{\partial^2 u^i}{\partial y^i_{m_i} \partial y^i_{l_i}} dy^i_{m_i} dy^i_{l_i} + \sum_{m_i=1}^{n_i} \sum_{j \neq i} \sum_{m_j=1}^{n_j} \frac{\partial^2 u^i}{\partial y^i_{m_i} \partial y^j_{m_j}} dy^i_{m_i} dy^j_{m_j} \Bigg]$$

From elementary properties of differential forms we know that $\frac{\partial^2 u^i}{\partial y^i_{m_i} \partial y^i_{m_i}} dy^i_{m_i} dy^i_{m_i} = 0$ and, $\frac{\partial^2 u^i}{\partial y^i_{m_i} \partial y^i_{l_i}} dy^i_{m_i} dy^i_{l_i} = -\frac{\partial^2 u^i}{\partial y^i_{l_i} \partial y^i_{m_i}} dy^i_{l_i} dy^i_{m_i}$ for all i, m_i and l_i hence $d\alpha$ is simply

$$d\alpha = \sum_{i=1}^{N} \Big[\sum_{m_i=1}^{n_i} \sum_{j \neq i} \sum_{m_j=1}^{n_j} \frac{\partial^2 u^i}{\partial y^i_{m_i} \partial y^j_{m_j}} dy^i_{m_i} dy^j_{m_j} \Big]$$

But by assumption, we know that $\frac{\partial^2 u^i}{\partial y^i_{m_i} \partial y^j_{m_j}} = \frac{\partial^2 u^j}{\partial y^j_{m_j} \partial y^i_{m_i}}$, for all m_i, m_j when $i \neq j$. Also $\frac{\partial^2 u^j}{\partial y^i_{m_i} \partial y^j_{m_j}} dy^i_{m_i} dy^j_{m_j} = -\frac{\partial^2 u^j}{\partial y^j_{m_j} \partial y^i_{m_i}} dy^j_{m_j} dy^i_{m_i}$ and therefore $d\alpha = 0$. Hence α is closed. But by Poincaré's lemma we know that there must exist a twice differentiable 0-form P such that $dP = \alpha$. But then, this function then has the property that

$$\frac{\partial u^i}{\partial y^i_{m_i}} = \frac{\partial P}{\partial y^i_{m_i}}$$
$$\text{for every } i \in N \text{ and for all } 1 \leq m_i \leq n_i$$

and hence P is a potential function for game Γ and this completes the proof. □

APPENDIX C. PROOF OF THEOREMS 4.1 AND 4.2

Before we can prove theorem 4.1 we need the following notation. Let (X, t) be a topological space, C a subset of X, $\precsim$ a preorder and f a function on X. Then $t_{|C}$ is the relativized topology on C, that is the topology induced by t, $\precsim_{|C}$ is the restriction of $\precsim$ to C and $f_{|C}$ is the restriction of f to C. $\precsim$ is said to be closed if $\precsim$ is a closed subset of $X \times X$ endowed with the product topology $t \times t$. Finally, t_{nat} is the natural topology on $\mathbb{R}^n$. We are now in a position to define the lifting theorem of Herden (2008).

Theorem C.1. *Let (X, t) be a locally compact and second countable Hausdorff-space that is endowed with a preorder $\precsim$. Then the following assertions are equivalent:*

(1) *$\precsim$ is closed.*
(2) *For every compact subset C of X and every continuous and strictly increasing function $f : (C, \precsim_{|C}, t_{|C}) \to (\mathbb{R}, \leq, t_{nat})$, there exists a continuous and strictly increasing function $F : (X, \precsim, t) \to (\mathbb{R}, \leq, t_{nat})$ such that $F_{|C} = f$.*

For every natural number $n \geq 1$ Euclidean space $\mathbb{R}^n$ endowed with the natural topology is a locally compact and second countable Hausdorff-space. Moreover, the product ordering $\leq$ on $\mathbb{R}^n$ (or the natural partial order) that is induced by the natural linear ordering on $\mathbb{R}$ is closed with respect to $t_{nat} \times t_{nat}$. Hence on $\mathbb{R}^n$ we can always extend a strictly monotone and continuous function defined on a compact set to the entire space preserving the continuity and strict monotonicity. Now we are in a position to show our result.

Proof of Theorem 4.1. $(1) \implies (2)$ follows from the same step of theorem 2.1. We now show the critical step.

$(2) \implies (3)$
We will show this step for the case when there are two people in the economy or $N = 2$. For $N > 2$ the proof remains the same however the notation becomes cumbersome.

Consider an economy with 2 individuals where our data set consists of 2 data points or $D = \{(p_1, x_1, y_1), (p_2, x_2, y_2)\}$ where x_t is the consumption of player 1 at observation t and y_t is the consumption of player 2 at observation t. This is an abuse of notation and is used in order to make the proof more transparent.

The intuition of the proof is as follows. We will define a potential function V on a closed subset of $\mathbb{R}^{2L}_{+}$ such that the data set D is rationalized if and only if it is rationalized on this subset. We will then use the lifting theorem to extend the potential function to the rest of $\mathbb{R}^{2L}_{+}$.

We consider the case where $x_1 \neq x_2$ and $y_1 \neq y_2$. If either $x_1 = x_2$ or $y_1 = y_2$, then CGARP will imply that the inequalities of theorem 4.2 will be satisfied and our result is trivially true. Also we restrict our attention to the case where $p_1x_1 \geq p_1x_2$, $p_2x_2 \geq p_2x_1$ and $p_1y_1 \geq p_1y_2$, $p_2y_2 \geq p_2y_1$. This was true of the data in Example 4.3 and it is easy to show that this data cannot be rationalized by a concave potential. If this is not true then the data can be rationalized by a concave potential as we can satisfy the inequalities of theorem 4.2.

We will use the following notation

$$(\circ, y) = \{(x, y) : x \in \mathbb{R}^{L}_{+}\} \quad \subset \mathbb{R}^{2L}_{+}$$
$$(x, \circ) = \{(x, y) : y \in \mathbb{R}^{L}_{+}\} \quad \subset \mathbb{R}^{2L}_{+}$$

We define $C = (x_1, \circ) \cup (x_2, \circ) \cup (\circ, y_1) \cup (\circ, y_2)$. We will now define a potential function $V : C \to \mathbb{R}$ which will rationalize the data. We will do so by finding a positive solution to the following system of inequalities for scalar V's and for strictly positive vectors μ's, λ's

$$V_{11} \leq V_{12} + \mu_{12}(y_1 - y_2)$$
$$V_{12} \leq V_{11} + \mu_{11}(y_2 - y_1)$$
$$V_{22} \leq V_{12} + \lambda_{12}(x_2 - x_1)$$
$$V_{12} \leq V_{22} + \lambda_{22}(x_1 - x_2)$$
$$V_{22} \leq V_{21} + \mu_{21}(y_2 - y_1)$$
$$V_{21} \leq V_{22} + \mu_{22}(y_1 - y_2)$$
$$V_{11} \leq V_{21} + \lambda_{21}(x_1 - x_2)$$
$$V_{21} \leq V_{11} + \lambda_{11}(x_2 - x_1)$$

where $\lambda_{11} = \alpha_1 p_1$, $\lambda_{22} = \alpha_2 p_2$ for positive scalars α_1, α_2 and $\mu_{11} = \beta_1 p_1$, $\mu_{22} = \beta_2 p_2$ for positive scalars β_1, β_2.

Let us assume we have positive solutions to the above system. We then define V as follows

$$V : (\circ, y_1) \to \mathbb{R}, \qquad V(x, y_1) = \min_{t\in\{1,2\}} \{V_{t1} + \lambda_{t1}(x - x_t)\}$$
$$V : (\circ, y_2) \to \mathbb{R}, \qquad V(x, y_2) = \min_{t\in\{1,2\}} \{V_{t2} + \lambda_{t2}(x - x_t)\}$$
$$V : (x_1, \circ) \to \mathbb{R}, \qquad V(x_1, y) = \min_{t\in\{1,2\}} \{V_{1t} + \mu_{1t}(y - y_t)\}$$
$$V : (x_2, \circ) \to \mathbb{R}, \qquad V(x_2, y) = \min_{t\in\{1,2\}} \{V_{2t} + \mu_{2t}(y - y_t)\}$$

We need to show that this function is well defined on C. As $x_1 \neq x_2$ and $y_1 \neq y_2$, $(x_1, \circ) \cap (x_2, \circ) = \phi$ and $(y_1, \circ) \cap (y_2, \circ) = \phi$. Also $(x_1, \circ) \cap (\circ, y_1) = \{(x_1, y_1)\}$, $(x_1, \circ) \cap (\circ, y_2) = \{(x_1, y_2)\}$ etc. Using an identical argument to step 3 in the proof of theorem 2.1, we can show $V(x_1, y_1) = \min_{t\in\{1,2\}}\{V_{t1} + \lambda_{t1}(x_1 - x_t)\} = \min_{t\in\{1,2\}}\{V_{1t} + \mu_{1t}(y_1 - y_t)\} = V_{11}$ and similarly $V(x_1, y_2) = V_{12}$, $V(x_2, y_1) = V_{21}$ and $V(x_2, y_2) = V_{22}$. Hence, V is well defined. Moreover, V is trivially continuous on C. We now verify that V is strictly monotone on C with respect to the standard partial order on Euclidean spaces.

Since $x_1 \neq x_2$ and $p_1x_1 \geq p_1x_2$, $p_2x_2 \geq p_2x_1$, it is the case that $x_1 \ngeq x_2$ and $x_2 \ngeq x_1$. Hence for any $(x_1, y) \in (x_1, \circ)$ and $(x_2, y') \in (x_2, \circ)$ it must be the case that $(x_1, y) \ngeq (x_2, y')$ and $(x_2, y') \ngeq (x_1, y)$. We can make the same argument for sets $(\circ, y_1)$ and $(\circ, y_2)$. Consider arbitrary $(x_1, y) > (x, y_1)$ where $(x_1, y) \in (x_1, \circ)$ and $(x, y_1) \in (\circ, y_1)$. It must be the case that $(x_1, y) \geq (x_1, y_1) \geq (x, y_1)$ where one of the inequalities is strict. By construction we know that V is strictly monotone on $(x_1, \circ)$ and V is strictly monotone on $(\circ, y_1)$. Hence, $V(x_1, y) \geq V_{11} \geq V(x, y_1)$ where one of the inequalities is strict. We can extend this argument to all of C to conclude that V is strictly monotone on C.

We now need to verify that V rationalizes our data. For any $p_1x \leq p_1x_1$ we have

$$\begin{aligned} V(x, y_1) &= V_{t1} + \lambda_{t1}(x - x_t) \qquad \text{for some } t \in \{1, 2\} \\ &\leq V_{11} + \lambda_{11}(x - x_1) \\ &= V_{11} + \alpha_1 p_1(x - x_1) \\ &\leq V_{11} \end{aligned}$$

Therefore when player 2 consumes y_1, x_1 is a best response for player 1. We can do the same for player 2 at observation 1 and for both players at observation 2. Hence V rationalizes the data. Consider a large closed ball C'. We consider V restricted to $C \cap C'$. As C' is large is contains all points (x, y_1) such that $p_1x \leq p_1x_1$, (x_1, y) such that $p_1y \leq p_1y_1$ etc. Hence, V restricted to $C \cap C'$ is sufficient to rationalize the data. Now we can use the lifting theorem to extend $V_{|C\cap C'}$ to all of $\mathbb{R}^{2L}_+$ (as C is closed and hence $C \cap C'$ is compact) in a strictly monotonic and continuous way and hence we have a strictly monotonic and continuous potential function which rationalizes the data.

It remains to be shown that we can solve the above inequalities. We now construct a solution to the inequalities. We assign arbitrary strictly positive values to V_{11} and V_{22}. Assign $0 < V_{12}, V_{21} <$

$\min\{V_{11}, V_{22}\}$. Now we assign values to λ, μ to satisfy the first four inequalities and then we can do the same for the last four.

Since $y_1 \ngeq y_2$ and $y_2 \ngeq y_1$ we can always find strictly positive vector μ_{12} such that $\mu_{12}(y_1 - y_2) > 0$. We can make $\mu_{12}(y_1 - y_2) > 0$ arbitrarily large and hence we can satisfy $V_{11} \leq V_{12} + \mu_{12}(y_1 - y_2)$. By construction since $V_{11} > V_{12}$, we can take β_1 small so that $\beta_1 p_1(y_2 - y_1) < 0$ is small and inequality $V_{12} \leq V_{11} + \mu_{11}(y_2 - y_1)$ is satisfied.

Since $x_1 \ngeq x_2$ and $x_2 \ngeq x_1$ we can always find strictly positive vector λ_{12} such that $\lambda_{12}(x_2 - x_1) > 0$. We can make $\lambda_{12}(x_2 - x_1) > 0$ arbitrarily large and hence we can satisfy $V_{22} \leq V_{12} + \lambda_{12}(x_2 - x_1)$. By construction since $V_{22} > V_{12}$, we can take α_2 small so that $\alpha_2 p_2(x_1 - x_2) < 0$ is small and inequality $V_{12} \leq V_{22} + \lambda_{22}(x_1 - x_2)$ is satisfied.

We can do the same for the remaining four inequalities. Setting each person's utility function equal to the potential function V completes the step.

(3) $\implies$ (1) is obvious and this completes the proof. □

Proof of Theorem 4.2. (1) $\implies$ (2)
Since the potential function V is concave, it is continuous and subdifferentiable and hence a Nash equilibrium must satisfy

$$\frac{\partial V(x_t^i, x_t^{-i})}{\partial x^i} \leq \lambda_t^i p_t \qquad \text{for all } i \text{ and } t$$

where λ_t^i are the strictly positive Lagrangian multipliers. Also since V is concave we must have

$$V(x_{t'}^1, \ldots, x_{t'}^N) \leq V(x_t^1, \ldots, x_t^N) + \sum_{i=1}^{N} \frac{\partial V(x_t^1, \ldots, x_t^N)}{\partial x^i}(x_{t'}^i - x_t^i)$$
$$\implies \quad V(x_{t'}^1, \ldots, x_{t'}^N) \leq V(x_t^1, \ldots, x_t^N) + \sum_{i=1}^{N} \lambda_t^i p_t (x_{t'}^i - x_t^i)$$

Setting $V_t = V(x_t^1, \ldots, x_t^N)$ for all t, we have the required solutions for the inequalities.

(2) $\implies$ (3)
For arbitrary consumption bundles $x^1, \ldots, x^N$ we define the potential as follows

$$V(x^1, \ldots, x^N) = \min_{1 \leq t \leq T} \left[V_t + \sum_{i=1}^{N} \lambda_t^i p_t (x^i - x_t^i) \right]$$

Using an identical argument to theorem 2.1 we can show $V(x_t^1, \ldots, x_t^N) = V_t$ for $1 \leq t \leq T$. Finally, for arbitrary person i, if $p_t x^i \leq p_t x_t^i$ then

$$
\begin{aligned}
V(x^i, x_t^{-i}) &\leq V_t + \lambda_t^i p_t(x^i - x_t^i) + \sum_{j \neq i} \lambda_t^i p_t(x_t^j - x_t^j) \\
&= V_t + \lambda_t^i p_t(x^i - x_t^i) \\
&\leq V_t
\end{aligned}
$$

Hence, each person is best responding. Since the minimum of concave functions is concave, V is a concave function. Clearly it is also continuous and strictly monotone. Setting everyone's utility function equal to the potential function we have (3).

$(3) \implies (1)$
This is obvious and it completes the proof. □