Econometrica Supplementary Material

SUPPLEMENT TO "IMPLEMENTATION WITH CONTINGENT CONTRACTS"

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IN THIS SUPPLEMENT, WE FIRST PRESENT AN EXAMPLE that shows that Theorem 1 breaks down when the type space is countably infinite. We then present a few extensions to the results in the paper.

Throughout this supplement, we conduct the analysis for an arbitrary agent i, fix $v_{-i} \in V_{-i}$, and, for notational convenience, we suppress the dependence on v_{-i}. Recall that we can do so because the incentive compatibility requirement is for each agent i and all possible reports v_{-i} of the other agents.

NONEQUIVALENCE WITH COUNTABLY INFINITE TYPES

We construct an example of a type space with countably infinite types and a social choice function (SCF) where the latter can be implemented by a contingent contract but not by a linear contract. We begin by showing that acyclicity is sufficient for implementability even when types are countably infinite.

LEMMA A: *Suppose the type space is countable. If an SCF is acyclic, it can be implemented* (*using a contingent contract*).

PROOF: Consider an SCF f. We need to show that there exists a contingent contract s_i such that for every $v_i, v_i' \in V_i$, we have

$$s\big(v_i\big(f(v_i)\big), v_i\big) \geq s\big(v_i\big(f(v_i')\big), v_i'\big). \tag{S1}$$

We will define an incomplete binary relation $\succ_s, \sim_s$ over tuples $\{v_i(f(v_i')), v_i'\}$ for all $v_i, v_i' \in V_i$. These tuples correspond to a type v_i making a report of v_i'. We first define the relation $\succ_{s_0}$ and $\sim_s$:

$$\begin{aligned}
&\{v_i(f(v_i')), v_i'\} \succ_{s_0} \{v_i'(f(v_i')), v_i'\}, \quad \text{if} \quad v_i(f(v_i')) > v_i'(f(v_i')),\\
&\{v_i'(f(v_i')), v_i'\} \succ_{s_0} \{v_i(f(v_i')), v_i'\}, \quad \text{if} \quad v_i(f(v_i')) < v_i'(f(v_i')),\\
&\{v_i(f(v_i')), v_i'\} \sim_s \{v_i'(f(v_i')), v_i'\}, \quad \text{if} \quad v_i(f(v_i')) = v_i'(f(v_i')),\\
&\{v_i'(f(v_i')), v_i'\} \sim_s \{v_i(f(v_i')), v_i'\}, \quad \text{if} \quad v_i(f(v_i')) = v_i'(f(v_i')),\\
&\{v_i(f(v_i)), v_i\} \succ_{s_0} \{v_i(f(v_i')), v_i'\}, \quad \text{for all } v_i' \neq v_i.
\end{aligned}$$

We define $\succ_s$ as the transitive closure of $\succ_{s_0}$. Formally, we say $\{v_i(f(v_i')), v_i'\} \succ_s \{\hat{v}_i(f(\hat{v}_i')), \hat{v}_i'\}$ if there exists a finite sequence $\{\{v_i^1(f(v_i'^1)), v_i'^1\}, \ldots,$

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$\{v_i^K(f(v_i'^K)), v_i'^K\}\}$ such that

$$\{v_i(f(v_i')), v_i'\}R_1\{v_i^1(f(v_i'^1)), v_i'^1\}R_2 \cdots \\ R_K\{v_i^K(f(v_i'^K)), v_i'^K\}R_{K+1}\{\hat{v}_i(f(\hat{v}_i')), \hat{v}_i'\},$$

where $R_k \in \{\succ_{s_0}, \sim_s\}$ and at least one $R_k \equiv \succ_{s_0}$. It is easy to argue that acyclicity of f implies that the relation $\succ_s$ is irreflexive.

Since $\succ_s$ is irreflexive and transitive and V_i is countable, we can then use a standard representation theorem (Fishburn (1970)) that guarantees the existence of a function s_i that respects $\succ_s$. *Q.E.D.*

Note that scaled cycle monotonicity (s.c.m.) is sufficient for implementation by a linear contract even for uncountably infinite type spaces (Proposition 2). In the example below, acyclicity is satisfied but s.c.m. is not.

EXAMPLE A: Consider a single agent with the countably infinite type space

$$V_1 = \{v_1^2, v_1^3, \dots\} \cup \{v_1^\infty\}.$$

Suppose the set of alternatives A has equal cardinality and consider an SCF f that satisfies

$$f(v_1^k) \neq f(v_1^{k'}) \quad \text{for all } k \neq k'.$$

Define the type space such that

$$v_1^k(f(v_1^{k'})) = \begin{cases} \dfrac{2}{k'}, & \text{if } k' < k, \\ \dfrac{1}{k}, & \text{if } k' = k, \\ \dfrac{1}{2k'}, & \text{if } k' > k, \\ 0, & \text{if } k' = \infty. \end{cases}$$

Finally, we define payoff for type v_1^∞ as

$$v_1^\infty(f(v_1^{k'})) = \begin{cases} \dfrac{2}{k'}, & \text{if } k' < \infty, \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to see that f is acyclic. This is because $v_1^k \succ v_1^{k'}$ for all $k' < k \leq \infty$ as

$$v_1^k(f(v_1^{k'})) = \frac{2}{k'} > \frac{1}{k'} = v_1^{k'}(f(v_1^{k'})).$$

Moreover, when $k < k' < \infty$, then $v_1^k \not\succeq v_1^{k'}$ as

$$v_1^k\big(f\big(v_1^{k'}\big)\big) = \frac{1}{2k'} < \frac{1}{k'} = v_1^{k'}\big(f\big(v_1^{k'}\big)\big).$$

Finally, $v_1^k \not\succeq v_1^\infty$,

$$v_1^k\big(f\big(v_1^\infty\big)\big) = 0 < 1 = v_1^\infty\big(f\big(v_1^\infty\big)\big).$$

Lemma A shows that acyclicity remains a sufficient condition for implementability if the type space is countable and, hence, f can be implemented using a contingent contract.

We now show that f cannot be implemented by a linear contract. Let us assume to the contrary that it is implementable by (r_1, t_1). Then adding the two incentive compatibility conditions for types v_1^k and v_1^∞ misreporting as each other, we get

$$\begin{aligned} & r_1\big(v_1^k\big)\big[v_1^k\big(f\big(v_1^k\big)\big) - v_1^\infty\big(f\big(v_1^k\big)\big)\big] \\ & \quad + r_1\big(v_1^\infty\big)\big[v_1^\infty\big(f\big(v_1^\infty\big)\big) - v_1^k\big(f\big(v_1^\infty\big)\big)\big] \geq 0 \\ & \quad \Rightarrow \quad \frac{r_1(v_1^\infty)}{r_1(v_1^k)} \geq -\frac{\dfrac{1}{k} - \dfrac{2}{k}}{1-0} = \frac{1}{k}. \end{aligned}$$

Similarly, incentive compatibility for types v_1^k and v_1^{k+1} implies

$$\begin{aligned} & r_1\big(v_1^k\big)\big[v_1^k\big(f\big(v_1^k\big)\big) - v_1^{k+1}\big(f\big(v_1^k\big)\big)\big] \\ & \quad + r_1\big(v_1^{k+1}\big)\big[v_1^{k+1}\big(f\big(v_1^{k+1}\big)\big) - v_1^k\big(f\big(v_1^{k+1}\big)\big)\big] \geq 0 \\ & \quad \Rightarrow \quad \frac{r_1(v_1^{k+1})}{r_1(v_1^k)} \geq -\frac{\dfrac{1}{k} - \dfrac{2}{k}}{\dfrac{1}{k+1} - \dfrac{1}{2(k+1)}} = \frac{2(k+1)}{k}. \end{aligned}$$

Multiplying inequalities for succeeding $k = 2, \dots, K-1$, we get

$$\frac{r_1(v_1^K)}{r_1(v_1^2)} \geq 2^{K-3}K.$$

Combining inequalities, we get

$$r_1\big(v_1^\infty\big) \geq \frac{r_1(v_1^K)}{K} \geq 2^{K-3} r_1\big(v_1^2\big).$$

Taking the limit $K \to \infty$, we observe that the right side diverges, which implies that $r_1(v_1^\infty)$ must be ∞, which is a contradiction. Hence, f cannot be implemented using a linear contract.

PARTIALLY CONTRACTIBLE PAYOFFS

We note that on many occasions the entire payoff may not be contractible. However, our results will continue to hold in some such situations. Suppose the payoff of an agent has a contractible and a noncontractible component. We assume that the noncontractible component of the payoff is a monotone function of the contractible component and the alternative chosen. Formally, type v_i now reflects the *contractible component* of agent i's payoff over various alternatives.

There is a map

$$g_i : \mathbb{R} \times A \to \mathbb{R}$$

that gives the *noncontractible payoff* of agent i. We assume that g_i is nondecreasing in the first argument.

Consider an SCF f. Given a contingent contract s_i, the net payoff of agent i by reporting v_i' with true type v_i is given by

$$s_i\big(v_i\big(f\big(v_i'\big)\big), v_i'\big) + g_i\big(v_i\big(f\big(v_i'\big)\big), f\big(v_i'\big)\big).$$

Here, we consider linear contracts where the royalty term is not bounded from above (by 1) or $r_i : V_i \to (0, \infty)$. Given a linear contract (r_i, t_i), the net payoff of agent i by reporting v_i' with true type v_i is given by

$$r_i\big(v_i'\big)v_i\big(f\big(v_i'\big)\big) + g_i\big(v_i\big(f\big(v_i'\big)\big), f\big(v_i'\big)\big) - t_i\big(v_i'\big).$$

We will show that Theorem 1 continues to hold even under this setting. Since Theorem 1 continues to hold, with an appropriate redefinition of aggregate payoff maximizers, we can also show that Theorem 2 holds.

As before, for any SCF f, we define the binary relation $\succ^f$ as follows. For any $v_i, v_i' \in V_i$, we say $v_i \succ^f v_i'$ if $v_i(f(v_i')) > v_i'(f(v_i'))$. We also define the binary relation $\succeq^f$ as follows. For any $v_i, v_i' \in V_i$, we say $v_i \succeq^f v_i'$ if $v_i(f(v_i')) \geq v_i'(f(v_i'))$.

DEFINITION A: An SCF f is acyclic if for any sequence of types $v_i^1, \ldots, v_i^k$ with $v_i^1 \succeq^f v_i^2 \succeq^f \cdots \succeq^f v_i^k$, we have $v_i^k \not\succ^f v_i^1$.

As before, we can show the necessity of acyclicity.

LEMMA B: *If an SCF is implementable by a contingent contract, then it is acyclic.*

PROOF: Suppose SCF f is implementable by a contingent contract s_i. Consider any sequence of types $v_i^1, \dots, v_i^k$ with $v_i^1 \succeq^f v_i^2 \succeq^f \cdots \succeq^f v_i^k$. Choose $j \in \{1, \dots, k-1\}$. Since f is implementable by s_i, we get that

$$\begin{aligned} & s_i\big(v_i^j\big(f\big(v_i^j\big)\big), v_i^j\big) + g_i\big(v_i^j\big(f\big(v_i^j\big)\big), f\big(v_i^j\big)\big) \\ & \quad \geq s_i\big(v_i^j\big(f\big(v_i^{j+1}\big)\big), v_i^{j+1}\big) + g_i\big(v_i^j\big(f\big(v_i^{j+1}\big)\big), f\big(v_i^{j+1}\big)\big) \\ & \quad \geq s_i\big(v_i^{j+1}\big(f\big(v_i^{j+1}\big)\big), v_i^{j+1}\big) + g_i\big(v_i^{j+1}\big(f\big(v_i^{j+1}\big)\big), f\big(v_i^{j+1}\big)\big), \end{aligned}$$

where the second inequality used the fact that $v_i^j \succeq^f v^{j+1}$, s_i is increasing in the first argument, and g_i is nondecreasing in the first argument. Hence, we get that for any $j \in \{1, \dots, k-1\}$, we have

$$\begin{aligned} & s_i\big(v_i^j\big(f\big(v_i^j\big)\big), v_i^j\big) + g_i\big(v_i^j\big(f\big(v_i^j\big)\big), f\big(v_i^j\big)\big) \\ & \quad \geq s_i\big(v_i^{j+1}\big(f\big(v_i^{j+1}\big)\big), v_i^{j+1}\big) + g_i\big(v_i^{j+1}\big(f\big(v_i^{j+1}\big)\big), f\big(v_i^{j+1}\big)\big). \end{aligned} \tag{S2}$$

Adding inequality (S2) for all $j \in \{1, \dots, k-1\}$ and telescoping, we get

$$\begin{aligned} & s_i\big(v_i^1\big(f\big(v_i^1\big)\big), v_i^1\big) + g_i\big(v_i^1\big(f\big(v_i^1\big)\big), f\big(v_i^1\big)\big) \\ & \quad \geq s_i\big(v_i^k\big(f\big(v_i^k\big)\big), v_i^k\big) + g_i\big(v_i^k\big(f\big(v_i^k\big)\big), f\big(v_i^k\big)\big). \end{aligned} \tag{S3}$$

Since f is implementable, we have $s_i(v_i^k(f(v_i^k)), v_i^k) + g_i(v_i^k(f(v_i^k)), f(v_i^k)) \geq s_i(v_i^k(f(v_i^1)), v_i^1) + g_i(v_i^k(f(v_i^1)), f(v_i^1))$. This along with inequality (S3) gives us

$$\begin{aligned} & s_i\big(v_i^1\big(f\big(v_i^1\big)\big), v_i^1\big) + g_i\big(v_i^1\big(f\big(v_i^1\big)\big), f\big(v_i^1\big)\big) \\ & \quad \geq s_i\big(v_i^k\big(f\big(v_i^1\big)\big), v_i^1\big) + g_i\big(v_i^k\big(f\big(v_i^1\big)\big), f\big(v_i^1\big)\big). \end{aligned} \tag{S4}$$

Now assume, for contradiction, $v_i^k \succ^f v_i^1$. Then $v_i^k(f(v_i^1)) > v_i^1(f(v_i^1))$. Since s_i is strictly increasing in the first argument and g_i is nondecreasing in the first argument, we get that

$$\begin{aligned} & s_i\big(v_i^k\big(f\big(v_i^1\big)\big), v_i^1\big) + g_i\big(v_i^k\big(f\big(v_i^1\big)\big), f\big(v_i^1\big)\big) \\ & \quad > s_i\big(v_i^1\big(f\big(v_i^1\big)\big), v_i^1\big) + g_i\big(v_i^1\big(f\big(v_i^1\big)\big), f\big(v_i^1\big)\big). \end{aligned} \tag{S5}$$

This is a contradiction to inequality (S4). *Q.E.D.*

We now proceed to show that the remainder of the proof of Theorem 1 can be adapted straightforwardly. First, we define some terminology. For any $v_i, v_i' \in V_i$, let

$$d\big(v_i, v_i'\big) := v_i\big(f(v_i)\big) - v_i'\big(f(v_i)\big)$$

and

$$d'(v_i, v_i') := g_i(v_i(f(v_i)), f(v_i)) - g_i(v_i'(f(v_i)), f(v_i)).$$

DEFINITION B: An SCF f is *generalized scaled cycle monotone* if there exists $\lambda_i : V_i \to (0, \infty)$ such that for every sequence of types $(v_i^1, \dots, v_i^k, v_i^{k+1} \equiv v_i^1)$, we have

$$\sum_{j=1}^{k} \left[\lambda_i(v_i^j) d(v_i^j, v_i^{j+1}) + d'(v_i^j, v_i^{j+1})\right] \geq 0.$$

PROPOSITION A: *An SCF f is implementable by a linear contract if and only if it is generalized scaled cycle monotone.*

PROOF: The necessity of generalized scaled cycle monotonicity follows by adding any cycle of incentive constraints. For sufficiency, suppose f satisfies generalized scaled cycle monotonicity. Let $\lambda_i : V_i \to (0, \infty)$ be the corresponding multiplier. Then, by the Rochet–Rockafellar theorem, there exists a map $W : V_i \to \mathbb{R}$ such that for every $v_i, v_i' \in V_i$, we have

$$W(v_i) - W(v_i') \leq \left[\lambda_i(v_i) d(v_i, v_i') + d'(v_i, v_i')\right]. \tag{S6}$$

Now, for any $v_i \in V_i$, let

$$t_i(v_i) := \lambda_i(v_i) v_i(f(v_i)) + g_i(v_i(f(v_i)), f(v_i)) - W(v_i).$$

Now, substituting in inequality (S6), we get for every $v_i, v_i' \in V_i$,

$$\begin{aligned} W(v_i) - W(v_i') &= \lambda_i(v_i) v_i(f(v_i)) + g_i(v_i(f(v_i)), f(v_i)) - t_i(v_i) \\ &\quad - \lambda_i(v_i') v_i'(f(v_i')) - g_i(v_i'(f(v_i')), f(v_i')) + t_i(v_i') \\ &\leq \lambda_i(v_i) v_i(f(v_i)) - \lambda_i(v_i) v_i'(f(v_i)) \\ &\quad + g_i(v_i(f(v_i)), f(v_i)) - g_i(v_i'(f(v_i)), f(v_i)). \end{aligned}$$

Canceling terms, we get

$$\begin{aligned} &\lambda_i(v_i') v_i'(f(v_i')) + g_i(v_i'(f(v_i')), f(v_i')) - t_i(v_i') \\ &\quad \geq \lambda_i(v_i) v_i'(f(v_i)) + g_i(v_i'(f(v_i)), f(v_i)) - t_i(v_i). \end{aligned}$$

This gives us the desired incentive constraints. *Q.E.D.*

We will now show that if f is acyclic, then it is generalized scaled cycle monotone. To do so, we first observe that if f is acyclic, then we can apply Lemma 2 to claim that the type space can be f-order-partitioned. Now, we can use this to construct a $\lambda_i : V_i \to (0, \infty)$ map recursively. Let $(V_i^1, \dots, V_i^K)$ be an f-ordered partition of V_i. First, we set for all $v_i \in V_i^K$,

$$\lambda_i(v_i) := 1.$$

Having defined $\lambda_i(v_i)$ for all $v_i \in V_i^{k+1} \cup \dots \cup V_i^K$, we define $\lambda_i(v_i)$ for all $v_i \in V_i^k$. Let C be any cycle of types $(v_i^1, \dots, v_i^q, v_i^1)$ involving types in $V_i^k \cup \dots \cup V_i^K$ with at least one type in V_i^k and at least one type in $V_i^{k+1} \cup \dots \cup V_i^K$. Let $\mathcal{C}$ be the set of all such cycles. Now define for each cycle $C \equiv (v_i^1, \dots, v_i^q, v_i^{q+1} \equiv v_i^1) \in \mathcal{C}$,

$$L(C) := \sum_{v_i^j \in C \cap (V_i^{k+1} \cup \dots \cup V_i^K)} \lambda_i(v_i^j) d(v_i^j, v_i^{j+1}) + \sum_{j=1}^{q} d'(v_i^j, v_i^{j+1})$$

and

$$\ell(C) := \sum_{v_i^j \in V_i^k \cap C} d(v_i^j, v_i^{j+1}).$$

Now, consider two possible cases.

- If $L(C) \geq 0$ for all $C \in \mathcal{C}$, then set $\lambda_i(v_i) = 1$ for all $v_i \in V_i^k$.
- If $L(C) < 0$ for some $C \in \mathcal{C}$, we proceed as follows. Since V_i is f-order-partitioned, for every $v_i \in V_i^k$ and $v_i' \in (V_i^{k+1} \cup \dots \cup V_i^K)$, we have $d(v_i, v_i') > 0$ (Property P2). Similarly, for every $v_i, v_i' \in V_i^k$, we have $d(v_i, v_i') \geq 0$ (Property P1). Then, for every $C \in \mathcal{C}$, we must have $\ell(C) > 0$, since it involves at least one type from V_i^k and at least one type from $(V_i^{k+1} \cup \dots \cup V_i^K)$. Now, for every $v_i \in V_i^k$, define

$$\lambda_i(v_i) := \max_{C \in \mathcal{C}} \frac{-L(C)}{\ell(C)}.$$

We thus recursively define the λ_i map.

PROPOSITION B: *If f is acyclic, then λ_i makes f generalized scaled cycle monotone.*

PROOF: Consider any cycle $C \equiv (v_i^1, \dots, v_i^q, v_i^{q+1} \equiv v_i^1)$. We will show that

$$\sum_{j=1}^{q} \lambda_i(v_i^j) d(v_i^j, v_i^{j+1}) + d'(v_i^j, v_i^{j+1}) \geq 0. \tag{S7}$$

If $C \subseteq V_i^K$, then $d(v_i^j, v_i^{j+1}) \geq 0$, $d'(v_i^j, v_i^{j+1}) \geq 0$, and $\lambda_i(v_i^j) = \lambda_i(v_i^{j+1})$ for all $v_i^j, v_i^{j+1} \in C$. Hence, inequality (S7) holds. Now, suppose inequality (S7) is true for all cycles $C \subseteq (V_i^{k+1} \cup \dots \cup V_i^K)$. Consider a cycle $C \equiv (v_i^1, \dots, v_i^q,$

$v_i^{q+1} \equiv v_i^1$) involving types in $(V_i^k \cup \cdots \cup V_i^K)$. If each type in C is in V_i^k, then again $d(v_i^j, v_i^{j+1}) \geq 0$, $d'(v_i^j, v_i^{j+1}) \geq 0$, and $\lambda_i(v_i^j) = \lambda_i(v_i^{j+1})$ for all $v_i^j, v_i^{j+1} \in C$. Hence, inequality (S7) holds. By our hypothesis, if all types in C belong to $(V_i^{k+1} \cup \cdots \cup V_i^K)$, then again inequality (S7) holds. So assume that C is a cycle that involves at least one type from V_i^k and at least one type from $(V_i^{k+1} \cup \cdots \cup V_i^K)$. Let $\lambda_i(v_i) = \mu$ for all $v_i \in V_i^k$. By definition,

$$\sum_{v_i^j \in C} \big[\lambda_i\big(v_i^j\big) d\big(v_i^j, v_i^{j+1}\big) + d'\big(v_i^j, v_i^{j+1}\big)\big] = L(C) + \mu \ell(C) \geq 0,$$

where the last inequality follows from the definition of μ. Hence, inequality (S7) again holds. Proceeding like this inductively, we complete the proof. *Q.E.D.*

To summarize, we have shown the following result.

THEOREM A: *Consider the partially contractible environment and suppose the type space is finite. Then, for any SCF f, the following statements are equivalent*:

(i) *f is implementable by a contingent contract.*
(ii) *f is acyclic.*
(iii) *f is generalized scaled cycle monotone.*
(iv) *f is implementable by a linear contract.*

AN INFINITE TYPE SPACE WHERE THE EQUIVALENCE HOLDS

We now show that the equivalence established in Theorem 1 extends to a model with uncountably infinite type spaces under additional conditions. We make the following assumptions.

B1. The set of alternatives A is a metric space.

B2. The set of types V_i is a compact metric space and each $v_i \in V_i$ is continuous in a.

B3. The SCF $f(\cdot)$ is continuous in v_i.

THEOREM B: *Suppose assumptions* B1–B3 *hold and, additionally, suppose an SCF f can be implemented by a contingent contract s_i that is twice continuously (partially) differentiable in the first argument. Then f can also be implemented by a linear contract.*

PROOF: We are given that f can be implemented by a contingent contract $s_i : \mathbb{R} \times V_i \to \mathbb{R}$ that is twice continuously differentiable in the first argument. We first show that this implies that f can also be implemented by a contingent contract $\tilde{s}_i$ that is convex in the first argument. Consider the following transformation of s:

$$\tilde{s}_i = e^{\gamma s_i}, \quad \text{where} \quad \gamma > 0.$$

Clearly, since $\tilde{s}_i$ is a monotone transformation of s_i, it is both strictly increasing in the first argument and incentive compatible. Therefore, it also implements f. We denote partial derivatives of $\tilde{s}_i$ with respect to the first argument by $\frac{\partial \tilde{s}_i}{\partial u_i}$.

Since s_i is twice differentiable in the first argument, so is $\tilde{s}_i$ and its second partial derivative is given by

$$\frac{\partial^2 \tilde{s}_i}{\partial u_i^2} = \gamma e^{\gamma s_i} \left(\frac{\partial \tilde{s}_i}{\partial u_i}\right)^2 \left(\frac{\partial^2 \tilde{s}_i / \partial u_i^2}{(\partial \tilde{s}_i / \partial u_i)^2} + \gamma\right).$$

Now, since V_i is compact, f is continuous, and s_i is twice continuously differentiable in the first argument, this implies that

$$\frac{\partial^2 \tilde{s}_i(v_i(f(v_i')), v_i')/\partial u_i^2}{(\partial \tilde{s}(v_i(f(v_i')), v_i')/\partial u_i)^2} \quad \text{is bounded from below for all } v_i, v_i' \in V_i.$$

This is because the above function is continuous on the compact set $V_i \times V_i$ and, hence, must attain a minimum.

This in turn implies that there exists a large and finite $\gamma > 0$ such that $\tilde{s}_i$ is convex in the first argument. Incentive compatibility and convexity of $\tilde{s}_i$ applied in turn then yield the inequality, for all $v_i, v_i' \in V_i$,

$$\begin{aligned}
&\tilde{s}_i\big(v_i\big(f(v_i)\big), v_i\big) \\
&\quad \geq \tilde{s}_i\big(v_i\big(f\big(v_i'\big)\big), v_i'\big) \\
&\quad \geq \tilde{s}_i\big(v_i'\big(f\big(v_i'\big)\big), v_i'\big) + \frac{\partial \tilde{s}(v_i'(f(v_i')), v_i')}{\partial u_i}\big[v_i\big(f\big(v_i'\big)\big) - v_i'\big(f\big(v_i'\big)\big)\big].
\end{aligned}$$

Now set multipliers $\lambda_i(v_i') = \frac{\partial \tilde{s}_i(v_i'(f(v_i')), v_i')}{\partial u_i} > 0$ for all $v_i' \in V_i$ and notice that f will satisfy scaled cycle monotonicity with these multipliers. Of course, this implies that f can be implemented by a linear contract (Proposition 2), which completes the proof. *Q.E.D.*

IMPLEMENTATION IN A LINEAR ONE DIMENSIONAL UNCOUNTABLE MODEL

In this section, we describe a simple model of a one dimensional type space with uncountable types, where 2-acyclicity is sufficient.

We assume that the set of alternatives A is finite. Additionally, we assume that types are one dimensional and linear. Formally, for every alternative $a \in A$, there exists $\kappa_a \geq 0$ and γ_a such that for all i,

$$v_i(a) = \kappa_a v_i + \gamma_a, \quad \text{where} \quad v_i \in V_i \subseteq \mathbb{R}.$$

The following theorem is the characterization result.

THEOREM C: *Suppose A is finite and the types are one dimensional and linear. Then the following conditions on an SCF f are equivalent*:

(i) *f satisfies 2-acyclicity.*
(ii) *f is scaled 2-cycle monotone.*
(iii) *f is implementable by a linear contract.*
(iv) *f is implementable by a contingent contract.*

PROOF: (i) ⇒ (ii). Define the map $\nu : A \to \mathbb{R}_+$ as follows. For every $a \in A$,

$$\nu(a) = \begin{cases} \dfrac{1}{\kappa_a}, & \text{if } \kappa_a \neq 0, \\ 0, & \text{if } \kappa_a = 0. \end{cases}$$

Further, define $\nu^* := \max_{a \in A} \nu(a)$ and $V_i^0 := \{v_i \in V_i : \kappa_{f(v_i)} = 0\}$. Now define $r_i : V_i \to (0, 1]$ as follows. Fix an $\varepsilon \in (0, 1]$. For every $v_1 \in V_i$,

$$r_i(v_i) = \begin{cases} \varepsilon & \forall v_i \in V_i^0, \\ \dfrac{\nu(f(v_i))}{\nu^*} & \forall v_i \in V \setminus V_i^0. \end{cases}$$

Now note that if $v_i \in V_i^0$, then $r_i(v_i)\kappa_{f(v_i)} = 0$, and if $v_i \in V \setminus V_i^0$, then $r_i(v_i)\kappa_{f(v_i)} = \frac{1}{\nu^*}$. Hence, for every $v \in V_i^0$ and $v' \in V \setminus V_i^0$, we have

$$r_i(v_i')\kappa_{f(v_i')} > r_i(v)\kappa_{f(v_i)}. \tag{S8}$$

Now consider any $v_i, v_i' \in V$. Since f is 2-acyclic, it means $v_i' \succeq^f v_i$ implies $v_i \not\succ^f v_i'$. Equivalently, $(v_i' - v_i)\kappa_{f(v_i)} \geq 0$ implies $(v_i' - v_i)\kappa_{f(v_i')} \geq 0$. Equivalently, if $v_i > v_i'$ and $\kappa_{f(v_i)} = 0$, then $\kappa_{f(v_i')} = 0$. This further means that if $v_i \in V_i^0$ and $v_i' < v_i$, then $v_i' \in V_i^0$. Hence, using inequality (S8), we get that if $v_i' > v_i$, then

$$r_i(v_i')\kappa_{f(v_i')} \geq r_i(v_i)\kappa_{f(v_i)}. \tag{S9}$$

Now, for any $v_i, v_i' \in V$ with $v_i' > v_i$, scaled 2-cycle monotonicity requires that

$$(v_i' - v_i)\big(r_i(v_i')\kappa_{f(v_i')} - r_i(v_i)\kappa_{f(v_i)}\big) \geq 0. \tag{S10}$$

This is true because of inequality (S9).

(ii) ⇒ (iii). Using Proposition 2, it is enough to show that if f is scaled 2-cycle monotone, then it is scaled cycle monotone. Because f satisfies scaled 2-cycle monotonicity, for any $v_i' > v_i$, inequality (S10) is satisfied. But this implies that inequality (S9) is satisfied.

Assume for contradiction that f fails scaled cycle monotonicity. Let k be the smallest integer such that f fails scaled k-cycle monotonicity. Since f satisfies scaled 2-cycle monotonicity, $k \geq 3$. This means for every $r_i : V_i \to (0, 1]$ and for some finite sequence of types $(v_i^1, \dots, v_i^k)$, we have

$$\sum_{j=1}^{k} \ell^{f,r_i}\big(v_i^j, v_i^{j+1}\big) < 0,$$

where $v_i^{k+1} \equiv v_i^1$ and $\ell^{f,r_i}(v_i^j, v_i^{j+1}) := r_i(v_i^j)[v_i^j(f(v_i^j)) - v_i^{j+1}(f(v_i^j))]$. Consider a $r_i : V_i \to (0, 1]$. Let $v_i^j > v_i^p$ for all $p \in \{1, \dots, k\} \setminus \{j\}$. We will show that $\ell^{f,r_i}(v_i^{j-1}, v_i^j) + \ell^{f,r_i}(v_i^j, v_i^{j+1}) - \ell^{f,r_i}(v_i^{j-1}, v_i^{j+1}) \geq 0$. To see this,

$$\begin{aligned}
&\ell^{f,r_i}\big(v_i^{j-1}, v_i^j\big) + \ell^{f,r_i}\big(v_i^j, v_i^{j+1}\big) - \ell^{f,r_i}\big(v_i^{j-1}, v_i^{j+1}\big) \\
&\quad = v_i^j\big[r_i\big(v_i^j\big)\kappa_{f(v_i^j)} - r_i\big(v_i^{j-1}\big)\kappa_{f(v_i^{j-1})}\big] \\
&\qquad + v_i^{j+1}\big[r_i\big(v_i^{j+1}\big)\kappa_{f(v_i^{j+1})} - r_i\big(v_i^j\big)\kappa_{f(v_i^j)}\big] \\
&\qquad - v_i^{j+1}\big[r_i\big(v_i^{j+1}\big)\kappa_{f(v_i^{j+1})} - r_i\big(v_i^{j-1}\big)\kappa_{f(v_i^{j-1})}\big] \\
&\quad = \big(v_i^j - v_i^{j+1}\big)\big[r_i\big(v_i^j\big)\kappa_{f(v_i^j)} - r_i\big(v_i^{j-1}\big)\kappa_{f(v_i^{j-1})}\big] \\
&\quad \geq 0,
\end{aligned}$$

where the last inequality follows from the fact that $v_i^j > v_i^{j+1}$ and applying inequality (S9). Since f satisfies scaled $(k-1)$-cycle monotonicity, we know that $\ell^{f,r_i}(v_i^1, v_i^2) + \cdots + \ell^{f,r_i}(v_i^{j-2}, v_i^{j-1}) + \ell^{f,r_i}(v_i^{j-1}, v_i^{j+1}) + \ell^{f,r_i}(v_i^{j+1}, v_i^{j+2}) + \cdots + \ell^{f,r_i}(v_i^k, v_i^1) \geq 0$. But because of the last inequality, we must have

$$\sum_{j=1}^{k} \ell^{f,r_i}\big(v_i^j, v_i^{j+1}\big) \geq 0,$$

which gives us a contradiction.

Of course, (iii) ⇒ (iv) and Lemma 1 establishes that (iv) ⇒ (i). This concludes the proof. *Q.E.D.*

REMARK: A closer look at the proof of Theorem C reveals that if $\kappa_a > 0$ for all $a \in A$, then for every SCF f, $V_i^0 = \emptyset$, and, hence, every SCF f satisfies 2-acyclicity vacuously. Thus, *every* SCF can be implemented using a linear contract.

SUFFICIENCY OF 2-ACYCLICITY IN A LINEAR TWO DIMENSIONAL COUNTABLE MODEL

In this section, we consider a linear two dimensional generalization of the model in the previous section. Formally, for every alternative $a \in A$, there exists $\kappa_{a_1} \geq 0$, $\kappa_{a_2} > 0$, and γ_a such that for all i,

$$v_i(a) = v_{i_1}\kappa_{a_1} + v_{i_2}\kappa_{a_2} + \gamma_a, \quad \text{where} \quad v_i \in V_i \subseteq \mathbb{R}.$$

The proof will use the normalized vector for an alternative $a \in A$, which we denote by

$$\kappa_a = \left(\frac{\kappa_{a_1}}{\kappa_{a_2}}, 1\right).$$

Note that since we have restricted $\kappa_{a_2} > 0$,[1] the above normalized vector is well defined.

The next result shows that 2-acyclicity is sufficient for implementability in the linear two dimensional environment. The proof shows that 2-acyclicity implies acyclicity. Thus, from Lemma A, we can conclude that in countable type spaces, 2-acyclicity is sufficient for implementability.

THEOREM D: *Suppose the type space is countable. Then an SCF f is implementable in the linear two dimensional environment if and only if it is 2-acyclic.*

PROOF: We need to show that 2-acyclicity implies k-acyclicity for all k. We will proceed by induction on k. The base case of $k = 2$ is trivially true. As the induction hypothesis, we assume the implication holds for some $k > 2$. We will now show the induction step that 2-acyclicity implies $k + 1$-acyclicity.

Suppose f is 2-acyclic. Consider a sequence $v_i^1, \dots, v_i^{k+1}$ with the following properties. For all $j \in \{1, \dots, k-1\}$, each element is weakly greater than the succeeding element and no element is strictly greater than any previous element in the sequence. Formally,

$$v_i^j \succeq v_i^{j+1} \quad \text{and} \quad v_i^{j+1} \not\succ v_i^{j'} \quad \text{for all } j' \in \{1, \dots, j\},$$

which is equivalent to

$$\kappa_{f(v_i^{j+1})}\left(v_i^j - v_i^{j+1}\right) \geq 0 \quad \text{and}$$

$$\kappa_{f(v_i^{j'})}\left(v_i^{j+1} - v_i^{j'}\right) \leq 0 \quad \text{for all } j' \in \{1, \dots, j\}.$$

Additionally, without loss of generality, we can take the inequality to be strict for v_i^1:

$$v_i^1 \succ v_i^2 \quad \text{or that} \quad \kappa_{f(v_i^2)}\left(v_i^1 - v_i^2\right) > 0.$$

[1]This assumption is required for the result. It is possible to construct a simple counterexample if we allow $\kappa_{a_2} = 0$.

The induction hypothesis (k-acyclicity) then implies that

$$v_i^{j'} \not\succeq v_i^1 \quad \text{or that} \quad \kappa_{f(v_i^1)}\big(v_i^{j'} - v_i^1\big) < 0 \quad \text{for all } j' \in \{2, \dots, k\}.$$

Finally, v_i^{k+1} is such that

$$v_i^k \succeq v_i^{k+1} \quad \text{or} \quad \kappa_{f(v_i^{k+1})}\big(v_i^k - v_i^{k+1}\big) \geq 0.$$

The induction hypothesis implies that

$$v_i^{k+1} \not\succ v_i^{j'} \quad \text{for all } j' \in \{2, \dots, k\} \quad \text{or}$$
$$\kappa_{f(v_i^{j'})}\big(v_i^{k+1} - v_i^{j'}\big) \leq 0 \quad \text{for all } j \in \{2, \dots, k\}.$$

It is sufficient to show that for such sequences it must be that

$$v_i^{k+1} \not\succeq v_i^1 \quad \text{or} \quad \kappa_{f(v_i^1)}\big(v_i^{k+1} - v_i^1\big) < 0.$$

We consider two cases, depending on how the first component of the normalized vector $\kappa_{f(v_i^1)}$ compares to the first components of the vectors $\kappa_{f(v_i^j)}$ for $j \in \{2, \dots, k+1\}$.

CASE I: The first component of $\kappa_{f(v_i^1)}$ is the largest or smallest in the sequence.

This implies that either (i) the first component of $\kappa_{f(v_i^{k+1})}$ lies between the first components of $\kappa_{f(v_i^1)}$ and $\kappa_{f(v_i^2)}$ or that (ii) the first component $\kappa_{f(v_i^2)}$ lies between the first components of $\kappa_{f(v_i^1)}$ and $\kappa_{f(v_i^{k+1})}$.

Consider subcase (i) first. Here there must be an $\alpha \in [0, 1]$ such that $\kappa_{f(v_i^{k+1})} = \alpha\kappa_{f(v_i^1)} + (1-\alpha)\kappa_{f(v_i^2)}$. Then it must be that $v_i^1 \succ v_i^{k+1}$, which can be seen from the series of inequalities

$$\begin{aligned}
\kappa_{f(v_i^{k+1})}\big(v_i^1 - v_i^{k+1}\big) &= \kappa_{f(v_i^{k+1})}\big(v_i^1 - v_i^k\big) + \kappa_{f(v_i^{k+1})}\big(v_i^k - v_i^{k+1}\big) \\
&\geq \big(\alpha\kappa_{f(v_i^1)} + (1-\alpha)\kappa_{f(v_i^2)}\big)\big(v_i^1 - v_i^k\big) \\
&= \alpha\kappa_{f(v_i^1)}\big(v_i^1 - v_i^k\big) + (1-\alpha)\kappa_{f(v_i^2)}\big(v_i^1 - v_i^2\big) \\
&\quad + (1-\alpha)\kappa_{f(v_i^2)}\big(v_i^2 - v_i^k\big) \\
&> 0.
\end{aligned}$$

Note that the strictness follows from the fact that either or both of the first two terms in the above must be strictly positive depending on the value of α. But then applying 2-acyclicity to the sequence $\{v_i^1, v_i^{k+1}\}$ implies that $v_i^{k+1} \not\succeq v_i^1$.

Now consider subcase (ii). Here there must be an $\alpha \in [0, 1]$ such that $\kappa_{f(v_i^2)} = \alpha\kappa_{f(v_i^1)} + (1-\alpha)\kappa_{f(v_i^{k+1})}$. Observe that

$$\kappa_{f(v_i^2)}\big(v_i^1 - v_i^{k+1}\big) = \kappa_{f(v_i^2)}\big(v_i^1 - v_i^2\big) + \kappa_{f(v_i^2)}\big(v_i^2 - v_i^{k+1}\big) > 0,$$

which in turn implies that

$$\alpha\kappa_{f(v_i^1)}\big(v_i^1 - v_i^{k+1}\big) + (1-\alpha)\kappa_{f(v_i^{k+1})}\big(v_i^1 - v_i^{k+1}\big) > 0.$$

Hence, it must be that either $\kappa_{f(v_i^{k+1})}(v_i^1 - v_i^{k+1}) \leq 0$ and $\kappa_{f(v_i^1)}(v_i^1 - v_i^{k+1}) > 0$, in which case this subcase is completed, or that $\kappa_{f(v_i^{k+1})}(v_i^1 - v_i^{k+1}) > 0$. In the latter case, we can once again apply 2-acyclicity to the sequence $\{v_i^1, v_i^{k+1}\}$ and get the desired relation $v_i^{k+1} \not\succeq v_i^1$.

CASE II: The ratio of the components in $\kappa_{f(v_i^1)}$ lies between some $\kappa_{f(v_i^j)}$ and $\kappa_{f(v_i^{j+1})}$, where $j \in \{2, \ldots, k\}$. Then there must be an $\alpha \in [0, 1]$ such that $\kappa_{f(v_i^1)} = \alpha\kappa_{f(v_i^j)} + (1-\alpha)\kappa_{f(v_i^{j+1})}$. Then

$$\begin{aligned}
\kappa_{f(v_i^1)}\big(v_i^1 - v_i^{k+1}\big) &= \kappa_{f(v_i^1)}\big(v_i^1 - v_i^j\big) + \kappa_{f(v_i^1)}\big(v_i^j - v_i^{k+1}\big) \\
&> \big(\alpha\kappa_{f(v_i^j)} + (1-\alpha)\kappa_{f(v_i^{j+1})}\big)\big(v_i^j - v_i^{k+1}\big) \\
&\geq (1-\alpha)\kappa_{f(v_i^{j+1})}\big(v_i^j - v_i^{k+1}\big) \\
&= (1-\alpha)\kappa_{f(v_i^{j+1})}\big(v_i^j - v_i^{j+1}\big) \\
&\quad + (1-\alpha)\kappa_{f(v_i^{j+1})}\big(v_i^{j+1} - v_i^{k+1}\big) \\
&\geq 0,
\end{aligned}$$

which completes the proof. *Q.E.D.*

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