INTERTEMPORAL PRICE DISCRIMINATION WITH STOCHASTIC VALUES

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ABSTRACT. We study the infinite-horizon pricing problem of a seller facing a buyer with single-unit demand, whose private valuation changes over time. Specifically, a stochastic shock to the buyer's initial private value arrives at a random time that is unanticipated by both the buyer and the seller and unobserved by the latter. We show that, under certain conditions, the seller's optimal pricing strategy with commitment is simple and consists of two prices: a low introductory price, and a constant higher price thereafter. This provides a novel explanation for introductory pricing as a means of intertemporal price discrimination.

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1. INTRODUCTION

There is a large body of research in economic theory devoted to understanding intertemporal price discrimination. A significant proportion of this literature studies incentives which cause the seller to lower his price over time, a phenomenon which is common in practice. However, in many instances, sellers choose to raise their prices instead. Perhaps the most common example of such a pricing strategy is introductory pricing where new products are offered at a discount when they are launched after which the price is raised. Additionally, the seller often commits to the price increase by advertising both the expiry date of the introductory price and the subsequent regular price that will be charged. If a buyer has rational expectations, then there is no reason to expect a buyer with a constant valuation to wait and buy the product at a higher price. One explanation for introductory pricing is that the seller may have capacity constraints (see for instance Dana 1998, Gale & Holmes 1993). However, in certain settings, scarce capacity may not play a role. For instance, capacity constraints are absent for digital goods such as movie downloads, MP3s, software etc. where introductory pricing is often observed (by Amazon, iTunes, Google play etc.). In this paper, we provide a different explanation. We develop a model in which the buyer has a stochastic valuation and we derive conditions under which introductory pricing is the optimal pricing strategy for the seller.

A consumer's valuation can be influenced by a variety of different media such as product reviews, advertising, word of mouth etc. In practice, the seller can observe neither when the buyer's valuation changes as a result of new information nor what the revised valuation is. No seller can plausibly hope to know when his customers have read or will read a product review or for that matter which review they read and as a result what their revised valuation will be. Additionally, a buyer's valuation could simply be affected by her mood, by an impulse to make a purchase or by other such behavioral factors.

The canonical intertemporal pricing model considers a buyer with a private valuation that does not change across time. In this model, the seller offers a price path and the game ends whenever the buyer chooses to purchase the good. The fact that the buyer chooses not to purchase the good at a certain time provides incentives for the seller to lower his price in the future to serve the buyer, who has revealed that she has a low valuation. However, when prices fall over time, it induces some buyers with rational expectations to delay their purchases. When the monopolist cannot commit to a sequence of prices, this intertemporal competition can be severe. The Coase conjecture (formally shown by Stokey (1981), Gul et al. (1986) for stationary strategies) states that, without commitment, it may not be possible for the monopolist to exercise any market power whatsoever. When the seller can make offers frequently, the competitive market outcome occurs despite the fact that the durable good is being supplied by a monopolist. There has been a large body of work examining market conditions under which the Coase conjecture does and does not apply.

A complementary line of research studies optimal pricing when the seller has commitment power. Here, the seminal work is Stokey (1979), who argued that when the monopolist can commit to a sequence of prices at the beginning of time, he chooses to offer the constant monopoly price at each time period. Hence, he makes sales only in the first instant, forgoing all future sales.

This is a surprising result as the monopolist could potentially commit to dropping prices only in the distant future thereby making future sales while minimizing intertemporal competition. Hence, her result shows that the driving concern for the monopolist is to restrict the buyer's option value (from postponing her purchasing decision), to the extent that he makes no future sales, in order to dissuade the buyer from waiting. Since every equilibrium of the pricing game without commitment can be implemented by the seller when he has commitment power, the seller cannot expect to get more than the single period monopoly profit in equilibrium (with or without commitment). As a result, the solution with commitment constitutes an upper bound for the revenue the seller can receive in equilibrium. This upper bound can be useful as Ausubel & Deneckere (1989) show that a sufficiently patient seller can get arbitrarily close to the single period monopoly profit even without being able to commit.

The above results depend critically on the assumption that the buyer's valuation is constant across time. In this paper, we study the intertemporal pricing problem of a seller trying to sell a durable good over an infinite horizon to a buyer with a stochastic value. The buyer's initial private value changes due to the arrival of an unanticipated stochastic shock. Conditional on receiving the shock, she draws a new independent private valuation. Both the time of arrival of the shock and the resulting valuation are not known ex-ante by either the buyer or the seller. The shock arrives via an exponential process and the seller does not observe when the buyer's valuation changes. This shock models behavioral factors or the arrival of information that may result in the buyer reassessing her value of the good.

In such a model, the seller's optimal pricing policy can potentially take a very complex form. Surprisingly, we show that there are conditions under which introductory pricing becomes the optimal way to price discriminate when the seller has commitment power. Formally, the seller charges a low "introductory" price at the first instant and then charges a fixed higher "regular" price at all points of time thereafter. If a buyer does not purchase at the first instant, she will only buy the good in the future if she receives the shock and her resulting valuation is higher than the regular price. Since the buyer can receive the shock at any time, the seller makes a sale at all points of time with strictly positive probability. As we discussed above, Stokey (1979) shows that when the buyer's valuation is constant, the seller's optimal contract is equivalent to that of a static single period monopoly problem. By contrast, we show that the solution to the single shock model is essentially the solution to a two period model where the buyer has a new independent private valuation in each period.

There are considerable technical difficulties associated with deriving the profit maximizing price path. Optimal behavior by the buyer involves solving a complex optimal stopping problem that depends on the prices set by the seller and the seller must take the strategic behavior of the buyer into account when deriving optimal prices. Moreover, the expected distribution of buyer types at any time also depends on the entire price path and on the time that the buyer received the shock - an event that is unobserved by the seller. We argue that it is difficult for the seller to solve this optimal control problem and define instead an appropriate "relaxed problem." In this relaxed problem, we assume that the seller can observe the shock and can condition his prices on this information (these conditional price functions are committed to at the beginning of time). This removes one level of asymmetric information, namely, the arrival time of the shock. The seller can always do weakly better in the relaxed problem as he can choose to ignore this extra information. We show that when the solution to the relaxed problem features increasing prices, the same revenue can be achieved by the seller without observing the arrival of the shock.

1.1. *Related Literature*

As we mentioned above, this paper is related to the literature on durable goods pricing. The original papers in this literature made two critical assumptions: That no new buyers enter the market and that the valuation of the buyer is constant across time. Conlisk et al. (1984) and Sobel (1991) relax the former assumption by allowing the entry of an identical cohort of new buyers in each period. Board (2008) introduces a model in which new heterogeneous consumers can enter in each period and he derives the optimal seller contract under commitment. By contrast, there are considerably fewer papers that relax the latter assumption of constant valuations. Conlisk (1984) and Biehl (2001) analyze a two period, two type model. Conlisk derives the optimal contract with and without commitment, whereas Biehl compares sales to leasing and shows that under certain parameter values sales may dominate leasing.

Nocke et al. (2011) examine a two period model with stochastic values and derive conditions under which advance purchase discounts are optimal when the seller can commit. Of course, in their two period setting, the only possible pricing strategy for the seller is to post two prices, one for each period. Hence, their analysis focuses on isolating the conditions under which a lower first period price is optimal. By contrast, one of the main contributions of this paper is to show that despite being able to use complicated pricing paths over an infinite horizon, the seller may find it optimal to use a simple pricing strategy consisting of only two prices.

The model in this paper most closely resembles that of Fuchs & Skrzypacz (2010). In their model, there is a single exogenous event that arrives from an exponential process and which terminates the game. Upon termination, the seller and the buyer receive payoffs given by an exogenous function that depends on the buyer's private valuation. They derive the stationary equilibrium when the seller cannot commit to the price path and show that the revenue of the seller is driven down to his outside option as the time between successive offers goes to 0. Apart from the fact that we study the optimal contract when the seller has commitment power, the single shock model in this paper differs in two additional fundamental ways. Firstly, in this paper, upon arrival of the event, the continuation values of the buyer and the seller are determined endogenously by the prices set by the seller. Secondly, the seller does not observe the arrival of the event which, in particular, implies the game does not stop at this point.

Finally, this paper is related to the literature on dynamic mechanism design stemming from the works of Baron & Besanko (1984) and Courty & Li (2000) (see Bergemann & Said (2011) for a survey of recent work). This literature considers a mechanism design environment where a principal is trying to dynamically contract with an agent who has changing private information. The literature differs from this paper in that the agent must make an irreversible decision at the beginning of time as to whether she will contract with the principal or not. If she chooses not to, the game ends. In other words, she is not allowed to wait for her value to change and then enter the contract which, of course, is the only decision that the agent makes in the durable goods context. However, it should be pointed out that following this paper, there has been work examining environments where agents with changing private information can delay their contracting decision. A notable such work is Garrett (2013) who tries to explain price cycles by developing a stationary two type Markov environment where buyers exogenously arrive and leave the market.

1.2. *Organization*

This paper has been organized into the following sections. Section 2 describes the model. In Section 3, we describe some of the difficulties inherent in solving the seller's problem and present a relaxed approach which circumvents these issues. Section 4 presents the optimal contract and discusses its properties. Finally, Section 5 provides some concluding remarks. The proofs not included in the body of the paper are in the appendix.

2. The Model

We consider a continuous time, infinite horizon model consisting of a single representative buyer[1] facing a monopolist seller where time is indexed by $t \in [0, \infty)$. The seller wants to sell a single unit of a perfectly durable good, the cost of which is assumed to be constant over time and is normalized to 0. Due to this cost normalization, we use the terms revenue and profit interchangeably. The buyer stays in the market until she makes a purchase (if ever) and the game ends when she does. We assume that both the seller and the buyer discount the future exponentially with a common discount rate $r \in (0, \infty)$.

The buyer's type or valuation is denoted by θ. She draws an initial private valuation from a distribution F_0 at the beginning of time 0. There is an exogenous shock which arrives in the market from an exponential distribution with parameter λ. If the shock arrives at the beginning of time t, the buyer draws a new valuation independently from a distribution F_1 and the valuation is persistent thereafter. The seller can observe neither the arrival of the shock nor the realized valuation. For $i \in \{0, 1\}$, the distributions F_i are assumed to have continuous densities f_i which are positive on the supports $[\underline{\theta}_i, \overline{\theta}_i]$, where $\overline{\theta}_i > 0$. Note that this value process is such that the buyer's value is correlated across time (since it may not change) but the change itself is independent. A similar stochastic process is considered by Skrzypacz & Toikka (2013) who refer to it as a renewal process.[2]

The seller commits to a measurable price function $p : [0, \infty) \to [0, 1]$ at the beginning of time 0 with the aim of maximizing profits. The buyer observes this entire price path at time 0 and her strategy consists of the time at which she will make a purchase conditional on whether the shock has arrived or not. Note that we do not restrict the seller to offering continuous or differentiable price paths which are commonly made assumptions in such models. As we mentioned in the introduction, the optimal contract can feature introductory pricing (which is discontinuous) and hence allowing for this generality reaps dividends.

[1] It is equivalent to think of the distribution of initial values representing a measure of infinitesimal buyers each of whom receives the shock independently.

[2] The assumption of independent value changes has also been used in papers on sequential auctions such as Engelbrecht-Wiggans (1994) and Said (2012).

If the buyer purchases the good at time t and her value at that time is θ, she receives discounted payoff $e^{-rt}[\theta - p(t)]$ and the seller gets $e^{-rt}p(t)$. This formulation allows us to capture goods that are storable for the seller but are single use for the buyers such as books or movies. Additionally, this can also capture goods that are durable for the buyer. Here, θ represents the expected discounted value (which, of course, incorporates the fact that the future value might change).[3]

We denote the maximal profit maximizing price corresponding to the distributions F_i by p_{F_i}. Formally,

$$p_{F_i} = \sup\left\{\underset{p}{\operatorname{argmax}}\, p[1 - F_i(p)]\right\}.$$

The supremum above reflects the fact that, without additional assumptions, there may be multiple prices which maximize monopoly profits.

The distribution F_i is said to satisfy the monotone hazard rate condition if $\frac{f_i(\cdot)}{1-F_i(\cdot)}$ is increasing. This is a standard condition in mechanism design and pricing problems since Myerson (1981) and is satisfied by most commonly used distributions. This assumption implies that the first order condition of the monopolist's profit function has a unique solution and, therefore, that there is a unique profit maximizing price.

3. Solving the Seller's Problem

In this section, we identify the conditions under which introductory pricing is optimal. In order to do so, we do not directly solve for the seller's optimal price path in the original problem. Instead, we first define a relaxed problem in which the seller has more information and then identify conditions under which the solution to the relaxed problem can be achieved without giving the seller this additional information. The difficulty in solving the seller's problem directly is that we have to derive the profit maximizing price path taking into account the utility maximizing behavior of the buyer. For a given price path $p(\cdot)$, the buyer decides when to purchase the good (optimally stop) taking into account the fact that her value might, or already has, changed. This optimal stopping problem corresponding to the buyer's purchasing decision is itself a difficult problem to solve in closed form. This is due to the stochasticity of the buyer's value and because the seller is not restricted to offering monotone or continuous price paths.[4] In the interest of brevity, we have deliberately chosen to not set up the original problem of the seller directly. This requires additional notation and offers no further insight.

In the relaxed problem, we assume that the seller observes the arrival of the shock to the buyer's valuation. Moreover, we allow the seller to condition the prices he offers on this information. Any price path in the original problem can be implemented in the relaxed problem because the seller can simply choose to ignore the information about the arrival of the shock. Hence, the highest

[3] If we interpret θ to be flow values, then the lifetime value corresponding to a given θ at t before the shock arrives is just a linear transformation $\frac{\theta}{r+\lambda} + \frac{\lambda}{r(r+\lambda)}\int_{\underline{\theta}_1}^{\overline{\theta}_1} \theta' dF_1(\theta')$. Similarly, if the shock has already arrived, then flow value θ at t corresponds to a lifetime valuation of $\frac{\theta}{r}$.

[4] Zuckerman (1986) and Stadje (1991) study a job search model with random wage draws in continuous time where offers arrive stochastically. While they allow for multiple draws, they assume that the cost of job search is nondecreasing in time which corresponds to nondecreasing prices $p(\cdot)$ in our context. To the best of our knowledge, the problem with non-monotone costs has not been solved.

revenue in the relaxed problem must be weakly greater than in the original. We will identify conditions under which the solution to this relaxed problem features increasing prices where the seller does not profit from the additional information. Therefore, under these conditions, the solution to the relaxed problem coincides with the original problem.

While our primary focus is not on the relaxed problem, it should be pointed out that it is itself of independent interest. This problem can model commonly occurring situations where a firm knows that a certain observable event is imminent but does not know the exact time that this event will occur. For instance, a firm may be aware that a rival is planning to release a product and that the consumers' willingness to pay might change due to this release (because of competition or other factors). The rival's product launch is observable and the firm can condition prices on this event.

Formally, in this relaxed problem, the seller commits at time 0 to conditional price functions $p_0(\cdot)$, $p_1(\cdot,\cdot)$ where

$$p_0(t) := \text{Price offered at } t \text{ if the buyer has not received the shock yet,}$$
$$p_1(t,t') := \text{Price offered at } t' \text{ if the buyer received the shock at } t \leq t'.$$

The seller commits to these functions at the beginning of the game with the intention of maximizing profits. The price at time 0 is $p_0(0)$. If the buyer has not received the shock until time t, the price she faces is $p_0(t)$. If the buyer receives the shock at time t, she faces price $p_1(t,t)$ at t and prices $p_1(t,t')$ at all times $t' > t$ in the future. We assume that $p_0(\cdot)$ is measurable with respect to the Lebesgue measure on $[0,\infty)$ and $p_1(\cdot,\cdot)$ is measurable with respect to the Lebesgue measure on $[0,\infty) \times [0,\infty)$. The buyer knows these price functions at the beginning of time and behaves optimally in response. We denote the seller's revenue in the relaxed problem by R.

We first observe that the seller's maximum profit is higher in this relaxed problem than in the original. Let $p^*(\cdot)$ be the optimal price path of the original problem. The seller can ignore information of the arrival of the shock by setting prices $p_0(t) = p^*(t)$ and $p_1(s,t) = p^*(t)$ for all $s \leq t$. Since this yields the same revenue as p^*, the maximal profit in the relaxed problem must be weakly higher than the original.

The price functions $p_0(\cdot), p_1(\cdot,\cdot)$ induce continuation payoffs for each type at each point of time conditional on the information regarding the arrival (or not) of the shock. The continuation payoff of a type θ who has not received the shock till time t is denoted by

$$V_0(t,\theta) := \text{Continuation payoff of type } \theta \text{ at time } t \text{ when the shock has not arrived.}$$

In other words, $V_0(t,\theta)$ is the payoff to a type θ from not purchasing the good at time t and behaving optimally in the future where she will face different prices depending on when the shock arrives. If the buyer receives the shock at time t, the continuation payoff of a type θ (which is drawn as a realization of the shock) at time $t' \geq t$ is denoted by

$$V_1(t,t',\theta) := \text{Continuation payoff of type } \theta \text{ at time } t' \text{ when the shock arrived at } t.$$

We define $\Theta_0(t)$ as the set of types remaining in the market at the beginning of time t conditional on the buyer not having received the shock till t. Similarly, we define $\Theta_1(t,t')$ to be the set of types remaining in the market at time t' conditional on the buyer having received the shock at t.

As is standard in mechanism design problems, it helps to eliminate prices and and maximize over the space of allocations instead. The optimal buyer response to given price paths p_0, p_1 in our model follows cutoff type behavior: If a type θ that hasn't (has) received the shock finds it optimal to purchase at a given time, so do all higher types $\theta' > \theta$ that are currently still in the market and haven't (have) received the shock. The fact that optimal behavior can be captured by cutoff types is a standard result from the literature (see, for example, Lemma 1 in Board 2008). This result extends to our setting as the buyer's payoff is linear (in her valuation and the price) and because, conditional on receiving the shock, the buyer draws a new value independently.

Hence, the allocations to the buyer can be captured by cutoff types which we denote by

$$c_0(t) := \sup\{\theta : \theta \in \Theta_0(t) \text{ and } \theta - V_0(t,\theta) \leq p_0(t)\},$$
$$c_1(t,t') := \sup\{\theta : \theta \in \Theta_1(t,t') \text{ and } \theta - V_1(t,t',\theta) \leq p_1(t,t')\}.$$

Cutoff type $c_0(t)$ represents the highest type left in the market that hasn't received the shock and is unwilling to purchase the good at $p_0(t)$. Notice that this does not imply that type $c_0(t)$ is indifferent between purchasing at $p_0(t)$ or waiting. It is possible that $p_0(t)$ is high enough such that all the remaining types in the market who haven't made a purchase yet strictly prefer to wait as opposed to buying at $p_0(t)$. $c_1(t,t')$ is the analogous cutoff type at t' when the shock arrived at t. By definition,

$$c_0(t) \text{ is non-increasing in } t \text{ and } c_1(t,t') \text{ is non-increasing in } t' \text{ for all } t.$$

Note that, since $c_0(\cdot)$ and $c_1(t,\cdot)$ are monotone, they are differentiable almost everywhere (henceforth abbreviated to a.e.).

It is possible derive an expression for the seller's revenue solely in terms of the cutoffs $c_0(\cdot)$ and $c_1(\cdot,\cdot)$. As mentioned earlier, this simplifies the problem as it allows us to ignore prices and maximize over the cutoff space which already incorporates optimal buyer behavior. The cutoffs can then be used to back out the optimal prices (which need not be unique), for instance, by simply setting $p_0(t) = c_0(t) - V_0(t, c_0(t))$ and $p_1(t,t') = c_1(t,t') - V_1(t,t',c_1(t,t'))$. When these particular prices are chosen, the cutoff type is always indifferent between purchasing at t or delaying and behaving optimally. In what follows, the prices are always assumed (without loss of generality) to be chosen in this way.

As a first step in eliminating prices from the seller's profit function, we derive an expression for the continuation payoff of the cutoff type. For simplicity, the continuation payoff $V_0(t,c_0(t))$ of the cutoff type $c_0(t)$ is denoted by shortened notation

$$V_0(t) := V_0(t, c_0(t)).$$

Note that $V_0(t)$ must be continuous in t as otherwise the cutoff type $c(t)$ could do strictly better by infinitesimally delaying or preponing her purchase (depending on the direction of the discontinuity). Similarly, we use

$$V_1(t) := \int_{\underline{\theta}_1}^{\overline{\theta}_1} V_1(t,t,\theta) dF_1(\theta),$$

to denote the expected continuation payoff if the buyer receives the shock at t.

Now, suppose cutoff type $c_0(t)$ decides to delay her purchase to $t+\Delta t$. Her payoff from this would be

$$\int_t^{t+\Delta t} \lambda e^{(s-t)(r+\lambda)} V_1(s)ds + e^{-(r+\lambda)\Delta t}[c_0(t) - p_0(t+\Delta t)]$$
$$= \int_t^{t+\Delta t} \lambda e^{(s-t)(r+\lambda)} V_1(s)ds + e^{-(r+\lambda)\Delta t}[c_0(t) - (c_0(t+\Delta t) - V_0(t+\Delta t))]$$

The first term is the expected utility of the buyer if the shock arrives between t and $t+\Delta t$, and the second term is the utility if it doesn't and a purchase is made at $t+\Delta t$. Since $c(t)$ is indifferent between purchasing at t and optimally delaying, this implies that $V_0(t)$ must be greater than the above expression (which is the utility from a particular delayed purchase strategy) or

$$V_0(t) \geq \int_t^{t+\Delta t} \lambda e^{(s-t)(r+\lambda)} V_1(s)ds + e^{-(r+\lambda)\Delta t}[c_0(t) - (c_0(t+\Delta t) - V_0(t+\Delta t))].$$

This can be rearranged to

$$V_0(t+\Delta t) - V_0(t) \leq e^{-(r+\lambda)\Delta t}[c_0(t+\Delta t) - c_0(t)] - \int_t^{t+\Delta t} \lambda e^{(s-t)(r+\lambda)} V_1(s)ds + (1 - e^{-(r+\lambda)\Delta t})V_0(t+\Delta t). \tag{1}$$

Similarly, by definition, type $c_0(t+\Delta t)$ is in the market at $t+\Delta T$. This must imply that this type weakly prefers to wait until at least $t+\Delta t$ to purchase as opposed to purchasing early at t. Formally, this can be written as

$$c_0(t+\Delta t) - (c_0(t) - V_0(t)) \leq \int_t^{t+\Delta t} \lambda e^{(s-t)(r+\lambda)} V_1(s)ds + e^{-(r+\lambda)\Delta t} V_0(t+\Delta t).$$

The left side of the inequality is the utility from purchasing at t at price $p_0(t) = c_0(t) - V_0(t)$ and the right side is the expected utility from delaying for an additional Δt. Rearranging, we get

$$V_0(t+\Delta t) - V_0(t) \geq [c_0(t+\Delta t) - c_0(t)] + (1 - e^{-(r+\lambda)\Delta t})V_0(t+\Delta t) - \int_t^{t+\Delta t} \lambda e^{(s-t)(r+\lambda)} V_1(s)ds. \tag{2}$$

Dividing (1) and (2) on both sides by Δt and taking the limit $\Delta t \to 0$, we conclude that $V(t)$ is differentiable a.e. (since c_0 is differentiable a.e. and the right side of both inequalities converge to the same limit). The derivative $V'(t)$ satisfies the differential equation

$$V_0'(t) = c_0'(t) + (r+\lambda)V_0(t) - \lambda V_1(t).$$

This equation solves to

$$V_0(t) = \int_t^\infty \lambda e^{-(r+\lambda)(s-t)} V_1(s)ds - \int_t^\infty e^{-(r+\lambda)(s-t)} c_0'(s)ds,$$

where we have used the fact that $\lim_{t\to\infty} e^{-(r+\lambda)t} V_0(t) = 0$ (this follows from the fact that $V_0(\cdot)$ is bounded). Finally, using integration by parts on the second term, we get

$$V_0(t) = \int_t^\infty \lambda e^{-(r+\lambda)(s-t)} V_1(s)ds - e^{-(r+\lambda)(s-t)} c_0(s)|_t^\infty - (r+\lambda)\int_t^\infty e^{-(r+\lambda)(s-t)} c_0(s)ds,$$

$$= \int_t^\infty \lambda e^{-(r+\lambda)(s-t)} V_1(s) ds + c_0(t) - (r+\lambda) \int_t^\infty e^{-(r+\lambda)(s-t)} c_0(s) ds. \tag{3}$$

The continuation payoff of the cutoff type has two components. The first term is the expected payoff over time from receiving the shock. The remaining terms constitute the payoff from waiting and when the shock does not arrive.

We denote the expected surplus and expected revenue conditional on receiving a shock at time t (but before values are realized) by $S_1(t)$ and $R_1(t)$. Note that these satisfy

$$R_1(t) := S_1(t) - V_1(t).$$

The probability that the buyer is still in the market at the end of time t when the shock hasn't arrived is simply $F_0(c_0(t))$. Since, $c_0(\cdot)$ is differentiable a.e. and F_0 is differentiable, $F_0(c_0(t))$ is differentiable a.e. with respect to t. Then, $-\frac{dF_0(c_0(t))}{dt}$ is the flow mass of types who purchase the good at time t.

We are now in a position to write the seller's revenue R in the relaxed problem in terms of $c_0(\cdot)$, $V_0(\cdot)$ and $R_1(\cdot)$ as

$$R := \left\{ \underbrace{\int_0^\infty \lambda e^{-(r+\lambda)t} R_1(t) F_0(c_0(t)) dt}_{\text{Expected revenue from sales after shock arrives}} + \underbrace{[1 - F_0(c_0(0))][c_0(0) - V_0(0)] - \int_0^\infty e^{-(r+\lambda)t} [c_0(t) - V_0(t)] \frac{dF_0(c_0(t))}{dt} dt}_{\text{Expected revenue from sales before shock arrives}} \right\}.$$

The first term is the expected profit from sales after the shock arrives. This term is weighted by the probability $F_0(c_0(t))$ that the buyer hasn't made a purchase prior to t. The second term is the expected profit from sales prior to the arrival of the shock. There is a mass of sales at 0 following which the price at t is $p_0(t) = c_0(t) - V_0(t)$ and the flow mass of sales is $-\frac{dF_0(c_0(t))}{dt}$.

The seller maximizes R by choosing cutoff functions $c_0(\cdot)$ which is nonincreasing and $c_1(\cdot, \cdot)$ which is nonincreasing in the second argument. Note that this is still a difficult problem to solve as c_1 is a function of two variables. We show that the choice space of the seller can be reduced to two single variable functions which considerably simplifies the problem.

Plugging in $c_0(t) - V_0(t)$ from (3), using integration by parts and finally plugging in $c_0(0) - V_0(0)$ also from (3), we get

$$\begin{aligned}
R &= \int_0^\infty \lambda e^{-(r+\lambda)t} R_1(t) F_0(c_0(t)) dt + [1 - F_0(c_0(0))][c_0(0) - V_0(0)] \\
&\quad - \int_0^\infty e^{-(r+\lambda)t} \left[\int_t^\infty (r+\lambda) e^{-(r+\lambda)(s-t)} c_0(s) ds - \lambda \int_t^\infty e^{-(r+\lambda)(s-t)} V_1(s) ds \right] \frac{dF_0(c_0(t))}{dt} dt \\
&= \int_0^\infty \lambda e^{-(r+\lambda)t} R_1(t) F_0(c_0(t)) dt + [1 - F_0(c_0(0))][c_0(0) - V_0(0)] \\
&\quad - \int_0^\infty \left[\int_t^\infty (r+\lambda) e^{-(r+\lambda)s} c_0(s) ds - \lambda \int_t^\infty e^{-(r+\lambda)s} V_1(s) ds \right] \frac{dF_0(c_0(t))}{dt} dt \\
&= \int_0^\infty \lambda e^{-(r+\lambda)t} R_1(t) F_0(c_0(t)) dt - \left[\int \frac{dF_0(c_0(t))}{dt} dt \right] \left[\int_t^\infty (r+\lambda) e^{-(r+\lambda)s} c_0(s) ds - \lambda \int_t^\infty e^{-(r+\lambda)s} V_1(s) ds \right] \Bigg|_0^\infty \\
&\quad - \int_0^\infty \left[\int \frac{dF_0(c_0(t))}{dt} dt \right] \left[(r+\lambda) e^{-(r+\lambda)t} c_0(t) - \lambda e^{-(r+\lambda)t} V_1(t) \right] dt + [1 - F_0(c_0(0))][c_0(0) - V_0(0)]
\end{aligned}$$

$$= \int_0^\infty \lambda e^{-(r+\lambda)t} R_1(t) F_0(c_0(t)) dt + [c_0(0) - V_0(0)]$$
$$- \int_0^\infty F_0(c_0(t)) \left[(r+\lambda) e^{-(r+\lambda)t} c_0(t) - \lambda e^{-(r+\lambda)t} V_1(t) \right] dt$$
$$= \int_0^\infty \lambda e^{-(r+\lambda)t} \left\{ R_1(t) F_0(c_0(t)) - [1 - F_0(c_0(t))] V_1(t) \right\} dt + (r+\lambda) \int_0^\infty e^{-(r+\lambda)t} c_0(t) [1 - F_0(c_0(t))] dt$$
$$= \int_0^\infty \lambda e^{-(r+\lambda)t} \left\{ S_1(t) F_0(c_0(t)) - V_1(t) \right\} dt + (r+\lambda) \int_0^\infty e^{-(r+\lambda)t} c_0(t) [1 - F_0(c_0(t))] dt. \tag{4}$$

Now notice that the maximum value of R can be derived by first maximizing with respect to $c_1(t, \cdot)$ for a fixed $c_0(t)$ and then maximizing over $c_0(\cdot)$. Additionally, note that c_1 only appears in the first term $\int_0^\infty \lambda e^{-(r+\lambda)t} [S_1(t) F_0(c_0(t)) - V_1(t)] dt$ from (4). The maximum for this term is achieved by maximizing $S_1(t) F_0(c_0(t)) - V_1(t)$ pointwise for each time t.

We now observe that the term $S_1(t) F_0(c_0(t)) - V_1(t)$ is simply the profit of the seller from the standard durable goods problem of Stokey (1979) where types are drawn from $[\underline{\theta}_1, \overline{\theta}_1]$, the valuation corresponding to a type θ is $F_0(c_0(t))\theta$. She shows that a constant price path maximizes $S_1(t) F_0(c_0(t)) - V_1(t)$. We can use her result to conclude that $p_1(t,t) = p_1(t,t')$ which in turn implies $c_1(t,t) = c_1(t,t')$ for all $t' > t$.

It should be pointed out that this observation is not obvious from the outset. At first glance, it might seem that after the buyer receives a shock at t, we are effectively in the setting of Stokey (1979) as the buyer's value no longer changes. However, the critical difference is that the prices set by the seller after the shock arrives at t determine the continuation payoffs and hence the behavior of the buyer *before* t as well. The key insight of rewriting the seller's revenue in the form of expression (4) is that this additional effect can be captured by simply scaling $S_1(t)$ to $S_1(t) F_0(c_0(t))$. The economic intuition for this expression is provided in Section 4.1.

This observation implies that we can now reduce the dimension of the seller's problem. The seller now simply needs to choose cutoffs $c_0(t)$ and cutoffs $c_1(t,t)$ as it is not optimal for him to make sales to any persistent types at a time $t' > t$ conditional on having observed a shock at t. Notice that this also implies that in the optimal price path satisfies $p_1(t,t) = c_1(t,t)$ or, in words, that the cutoff type is equal to the price. For notational convenience, we can now use this observation to drop the extra argument and denote $c_1(t,t)$ as simply $c_1(t)$.

This allows us to substitute the simplified expressions for the surplus and continuation payoff conditional on the arrival of the shock,

$$S_1(t) = \int_{c_1(t)}^1 \theta dF_1(\theta) \text{ and } V_1(t) = \int_{c_1(t)}^1 \left[\theta - c_1(t) \right] dF_1(\theta) = \int_{c_1(t)}^1 \left[\frac{1 - F_1(\theta)}{f_1(\theta)} \right] dF_1(\theta),$$

into (4) and write the seller's maximization problem as

$$\max_{c_0(\cdot), c_1(\cdot)} \left\{ \int_0^\infty \lambda e^{-(r+\lambda)t} \int_{c_1(t)}^1 \left[F_0(c_0(t))\theta - \frac{1 - F_1(\theta)}{f_1(\theta)} \right] dF_1(\theta) dt + (r+\lambda) \int_0^\infty e^{-(r+\lambda)t} c_0(t) [1 - F_0(c_0(t))] dt \right\}, \tag{5}$$

such that $c_0(\cdot)$ is nonincreasing.

This is now a well defined calculus of variations problem. Notice now that the above problem can be solved by the pointwise maximization of

$$\lambda \int_{c_1(t)}^{\overline{\theta}_1} \left[F_0(c_0(t))\theta - \frac{1 - F_1(\theta)}{f_1(\theta)} \right] dF_1(\theta) + (r+\lambda)c_0(t)[1 - F_0(c_0(t))] \tag{6}$$

at each point of time t by choosing $c_0(t)$ and $c_1(t)$. Since these cutoffs are chosen from compact sets and the maximand is continuous, the pointwise maximum must exist and, moreover, the set of maximizers of (6) must be the same for all t. Let c_0^* and c_1^* be one such pair of maximizers. Then, by setting $c_0(t) = c_0^*$ and $c_1(t) = c_1^*$ for all t, the monotonicity restriction on $c_0(\cdot)$ is satisfied and this would be a solution to (6). It is important to point out that the existence of a solution in such problems can be hard to prove and often has to be assumed. Our approach is constructive in that we show the existence of a solution by explicitly deriving it.

Since there are only two cutoffs in the relaxed problem, what we have effectively shown is that the solution to the relaxed problem consists of two prices such that

$$p_0(t) = p_0^* \qquad \text{for all } t \geq 0$$

$$p_1(t,t') = p_1^* \qquad \text{for all } 0 \leq t \leq t'.$$

The seller chooses to make some sales in the first instant. If the buyer does not purchase in the first instant, she waits until she receives the shock at some time t. If her value θ drawn from F_1 is greater than the price p_1^* she makes a purchase else she never buys the good. This is because the seller never chooses to drop the price and serve any of the persistent types.

When there is an interior solution, the maximizer of (6) must satisfy the first order conditions. These are obtained by differentiating (6) with respect to $c_0(t)$ and $c_1(t)$ to get

$$c_0^* - \frac{1 - F_0(c_0^*)}{f_0(c_0^*)} = \frac{\lambda}{r+\lambda} \int_{c_1^*}^{\overline{\theta}_1} \theta dF_1(\theta),$$

$$F_0(c_0^*)c_1^* - \frac{1 - F_1(c_1^*)}{f_1(c_1^*)} = 0.$$

When F_0, F_1 both satisfy the monotone hazard rate condition, it is easy to observe that the solution to the above first order conditions satisfies $c_0^* \in (p_{F_0}, \overline{\theta}_0)$ and $c_1^* \in (p_{F_1}, \overline{\theta}_1)$. This is intuitive. There is an opportunity cost to making sales at the first instant, namely, that sales cannot be made to those types in the future. Additionally, due to positive continuation values, the price must be lower than the cutoff ($p_0^* < c_0^*$). Both these forces compel the seller to make fewer sales at $t = 0$ than he would have as a static monopolist facing F_0. Similarly, making more sales at any t is at the expense of lowering profits from sales at $t' < t$ due to the increase in the continuation value. Therefore, once again, the seller makes fewer sales after the arrival of the shock than he would have in a static monopoly pricing problem with distribution F_1.

4. Optimality of Introductory Pricing

Whenever one of the solutions to the relaxed problem has increasing prices, the solution to the original problem is introductory pricing. Recall, that we have argued that the seller is always weakly better in the relaxed problem as he has additional information that he can choose to ignore.

Consider the following contract using the optimal prices from the relaxed problem. The seller charges the price p_0^* at the first instant $t = 0$ and p_1^* at all $t > 0$ thereafter. When $p_1^* > p_0^*$, the types that do not purchase at the first instant will not do so until they receive a new shock as the prices have gone up but the continuation value is still the same. When they do receive a shock at t, they will only make a purchase if the type θ they draw from F_1 is greater than p_1^*. This behavior is identical to the behavior of buyer in the relaxed problem. Hence, this price function yields the same revenue for the seller as that in optimal solution to the relaxed problem. This argument yields the main result of the paper.

Proposition 1. *Suppose one of the solutions to the relaxed problem features increasing prices. Then, the optimal price path in the original problem consists of two prices - an introductory price p_0^* at time 0 and price p_1^* at all times $t > 0$. Additionally, when there is an interior solution, these prices induce cutoff types c_0^*, c_1^* which are solutions to the following two equations*

$$c_0^* - \frac{1 - F_0(c_0^*)}{f_0(c_0^*)} = \frac{\lambda}{r + \lambda} \int_{c_1^*}^{\overline{\theta}_1} \theta dF_1(\theta), \tag{7}$$

$$F_0(c_0^*)c_1^* - \frac{1 - F_1(c_1^*)}{f_1(c_1^*)} = 0. \tag{8}$$

These cutoff types lie in the open intervals $(p_{F_0}, \overline{\theta}_0)$ and $(p_{F_1}, \overline{\theta}_1)$ respectively. Prices are given by

$$p_0^* = \frac{1 - F_0(c_0^*)}{f_0(c_0^*)} + \frac{\lambda}{r + \lambda} c_1^* [1 - F_1(c_1^*)],$$

$$p_1^* = c_1^*.$$

That the optimal contract can consist of increasing prices is intuitive. The increasing price implies that if the buyer does not have a high enough valuation in the first instant to make a purchase, she will not do so in the future unless she receives new information about the product that makes her revise her valuation upwards. From the seller's perspective, the standard intuition for the durable goods monopoly problem applies. Intertemporal competition is too costly. The gain the seller gets from serving a type who chose to not purchase at time 0 and who has not received a shock is offset by the loss due to the additional rent he has to give the types who make a purchase in the first instant. This is because the seller can only make sales to types that haven't received the shock by dropping the price and this increases the continuation payoff at time 0.

Once the seller infers that it is not optimal for him to make a sale to any type who did not purchase at time 0 and who has not received a shock, the stationarity of the price after time 0 follows from the arrival process of the shock. The seller only wants to serve the buyer when she receives a shock and her valuation as a result is sufficiently high. By an identical argument to that given above, if the buyer receives a shock at time t and her type is too low to be served at time t, the seller has no incentive to serve that type in the future. But since the shock arrives from an exponential process, the expected duration of arrival of the shock is the same at every point of time conditional on not having received it. Hence, in essence, the problem after period 0 is stationary.

It is clear from Proposition 1 that it is easy to construct numerous examples of F_0 and F_1 for which the result applies. That said, it is important to point out that the above result does not require that the distribution F_1 dominates F_0 in a first order or other sense. Put differently, it is not the case the price is higher after $t = 0$ simply because the the demand has gone up. It is possible to have introductory pricing even when the distribution F_1 corresponds to a lower demand than F_0 and has a lower monopoly price ($p_{F_1} < p_{F_0}$). The following lemma characterizes one such case.

Lemma 1. *Let $F_0(\theta) = (\theta - \varepsilon)^\alpha$ on support $[\varepsilon, 1 + \varepsilon]$ and $F_1(\theta) = \theta^\alpha$ on support $[0, 1]$ for $\alpha > 0$. Then there is a $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in [0, \bar{\varepsilon}]$, introductory pricing is optimal.*

In the above lemma, whenever $\varepsilon > 0$, the distribution F_0 first order stochastically dominates F_1. Despite this, the seller may choose to set a low introductory price (when $0 \leq \varepsilon \leq \bar{\varepsilon}$). Note that this family of distributions covers both the case where densities are increasing ($\alpha > 1$) and decreasing ($\alpha < 1$). Finally, note that for this family of distributions, introductory prices are optimal irrespective of the arrival or discount rate which, of course, is not true in general.

4.1. *Equivalence to the Two Period Problem*

As we have mentioned earlier, Stokey (1979) showed that seller's maximum profit in the standard durable goods monopoly problem with constant buyer valuations is the same as what he can achieve if he has only a single period in which to sell the good. We now argue that when there is a single stochastic shock to the buyer's value, the solution to the seller's problem similarly coincides with that of a two period problem. This analogy also makes the intuition for the optimality of introductory pricing more transparent.

Consider a discrete time two period problem where the buyer draws an initial private valuation in period 0 from F_0 and a new independent private valuation from F_1 in period 1. The common discount rate is given by δ. The seller sets prices p_0 in period 0 and p_1 in period 1. These prices induce cutoffs c_0 and c_1 in periods 0 and 1 respectively. Given that the game ends after period 1, the buyer's continuation value in this period is zero or $c_1 = p_1$.

Let the continuation payoff of the buyer in period 0 be given by V_0. Since the buyer draws a new valuation in period 1, this continuation payoff is type independent and satisfies

$$V_0 = \delta \int_{c_1}^{\bar{\theta}_1} [\theta - c_1] dF_1(\theta) = \delta \int_{c_1}^{\bar{\theta}_1} \frac{1 - F_1(\theta)}{f_1(\theta)} dF_1(\theta).$$

The seller solves the following problem:

$$\max_{c_0, c_1} \left\{ [c_0 - V_0][1 - F_0(c_0)] + \delta F_0(c_0) c_1 [1 - F_1(c_1)] \right\}. \tag{9}$$

In the above expression, the expected revenue in period 0 is $p_0[1 - F_0(c_0)]$ where $p_0 = c_0 - V_0$. The probability that the buyer reaches period 1 is $F_0(c_0)$ and conditional on reaching period 1, the expected revenue is given by $p_1[1 - F_1(p_1)] = c_1[1 - F_1(c_1)]$. Assuming an interior solution, the solution satisfies the first order conditions with respect to c_0 and c_1, which are

$$c_0^* - \frac{1 - F_0(c_0^*)}{f(c_0^*)} = \delta \int_{c_1^*}^{\bar{\theta}_1} \theta dF_1(\theta), \tag{10}$$

$$F_0(c_0^*) c_1^* - \frac{1 - F_1(c_1^*)}{f_1(c_1^*)} = 0 \tag{11}$$

respectively. It is immediate from the above first order conditions that the solution to such a two period problem satisfies the same equations as those of the continuous time, single shock model (7 and 8) where the discount rate is $\delta = \lambda/(r+\lambda)$.

We can now interpret these equations by comparing them to the first order condition of a standard static monopoly pricing problem where the seller has constant marginal costs. We first rewrite the seller's revenue maximization problem (9) in integral form (using integration by parts) as

$$\max_{c_0,c_1}\left\{\int_{c_0}^{\overline{\theta}_0}\left(\underbrace{\theta}_{\text{Period 1 Surplus}} - \underbrace{\frac{1-F_0(\theta)}{f(\theta)}}_{\text{Period 1 Buyer's Rent}}\right)dF_0(\theta) + \delta\int_{c_1}^{\overline{\theta}_1}\left(\underbrace{F_0(c_0)\theta}_{\text{Period 2 Surplus}} - \underbrace{\frac{1-F_1(\theta)}{f_1(\theta)}}_{\text{Period 2 Buyer's Rent}}\right)dF_1(\theta)\right\}.$$

In this two period problem, the cost of serving the buyer in period 0 is essentially the opportunity cost of not serving her in period 1 instead. By decreasing the period 0 cutoff type c_0, the seller profits more from sales in period 0, however, this reduces the chance of making a sale in period 1. Similarly, the cost of serving the buyer at price $p_1 = c_1$ in period 1 is the opportunity cost of the continuation value it provides to the buyer in period 0. By decreasing c_1, the seller increases his revenue from sales in period 1 but he loses revenue in period 0 as now the continuation payoff to the buyer has gone up. Notice that the period 1 surplus term is scaled by the probability of reaching period 1. This reflects the fact that a sale can only be made in period 1 if the buyer chose not to purchase in the first. However, the rents to the buyer in period 1 are not scaled. This is because whether the buyer makes a purchase in period 0 or not, she gets at least her type independent continuation payoff V_0. In other words, these rents have to be paid to all types.

Equations (10) and (11) capture the above intuition. Consider the first order condition (10) with respect to c_0. The term on the left side of the equation is the marginal gain in revenue from serving more types in period 0. The marginal cost of serving additional types in period 0 is the marginal loss in expected surplus from period 1 as the probability of reaching period 1 goes down. All types in period 0 (not just the types who purchase the object) receive the continuation payoff from the price p_1 in period 1 and hence the period 1 rent does not enter the equation.

Rewriting the first order condition (11) with respect to c_1 as

$$F_0(c_0)\left[c_1 - \frac{1-F_1(c_1)}{f_1(c_1)}\right] = [1-F_0(c_0)]\left[\frac{1-F_1(c_1)}{f_1(c_1)}\right],$$

provides a similar interpretation. The term on the left side is once again the (ex-ante) marginal gain in revenue at period 1 from serving additional types. Since the seller is maximizing revenue at the beginning of period 0, this term includes the probability that the buyer is still in the market at period 1. Moreover, this term also accounts for the information rent to the buyer in the second period. The term on the right accounts for the marginal impact a change in period 1 price has on the rents (in the form of continuation values) to types at period 0.

4.2. *Comparative Statics*

In this section, we argue that introductory pricing is more likely to arise when the seller's effective discount factor $\lambda/(r+\lambda)$ decreases. When the effective discount factor decreases, sales in the future become less lucrative and so the seller prefers to make upfront sales at $t=0$. The seller can achieve this by reducing the incentive for the buyer to delay. This can be done by raising the price at $t>0$, however, this may not be necessary as the buyer now values the future less. The following proposition states that if introductory pricing is optimal at a given effective discount factor, it will remain optimal whenever the effective discount factor is lower.

Proposition 2. *Suppose F_0 and F_1 satisfy the monotone hazard rate condition and that introductory pricing is optimal for a given $\overline{r}, \underline{\lambda} > 0$. Moreover, suppose that there is an interior solution and that the first order conditions have a unique solution for all $r, \lambda > 0$ such that $\frac{\lambda}{r+\lambda} \leq \frac{\underline{\lambda}}{\overline{r}+\underline{\lambda}}$. Then introductory pricing remains optimal for all $r, \lambda > 0$ such that $\frac{\lambda}{r+\lambda} \leq \frac{\underline{\lambda}}{\overline{r}+\underline{\lambda}}$.*

The proof of the above proposition shows that as the effective discount factor decreases, the difference between the introductory and continuation price $p_0^* - p_1^*$ increases (which implies that our relaxed approach will continue to work). As an intermediate step, we show that the seller makes more sales up front (c_0^* decreases) at the expense of later sales (c_1^* increases). A consequence of this is that while p_1^* goes up, p_0^* can either increase or decrease in response to a change in the effective discount factor. The ambiguous effect on p_0^* is driven by the fact that while the cutoff c_0^* goes does, the continuation payoff to the buyer goes down as well and recall that the price is given by the difference between these two.

A few comments are in order on the additional conditions in the above proposition. In order to get the comparative static, we need to work with the first order conditions. These characterize the solution if, and only if, it is in the interior. However, the first order conditions may have multiple solutions which prevents us from implicitly differentiating as, in this case, the solution may be discontinuous in r, λ. That said, it appears that these conditions aren't very stringent, in particular, they are satisfied by the family of distributions in Lemma 1.[5] Simulations seem to suggest that these conditions hold in general for most commonly used distributions (with appropriately chosen supports).

5. Concluding Remarks

In this paper, we developed a model of a monopolist seller of a durable good who faces a buyer with a stochastic valuation. In the model, the buyer has an initial private value and then subsequently receives new private information which arrives in the form of a single randomly arriving shock. We show that, with commitment, introductory pricing can be optimal for the seller. The main insight is that, despite having very complicated price paths at his disposal, the seller can find it optimal to use a very simple pricing strategy. The result is driven by the fact that, under certain conditions, it is optimal to make upfront sales by reducing the continuation payoff of the buyer even at the expense of making fewer sales in the future. Surprisingly, the

[5]Note that the distribution $F_0(\theta) = \theta^\alpha$ does not satisfy the monotone hazard rate condition globally when $0 < \alpha < 1$. But it does satisfy the monotone hazard rate condition to the right of the monopoly price p_{F_0} which is sufficient.

solution to the problem turns out to be equivalent to a two period pricing problem (in which the buyer draws an independent private value in each period) in the same way that the solution to the standard durable goods monopoly problem (with persistent values) is equivalent to a static monopoly problem (Stokey 1979).

A few comments about the robustness of the results are in order. The analysis in the paper was done in continuous time primarily because it makes the intuition transparent and it simplifies algebra. Additionally, it allows us to contrast our results to Stokey (1979) who also uses a continuous time framework. The identical arguments in the paper will work for a discrete time environment as well.

Unfortunately, it is very hard to characterize the seller's optimal price path when introductory pricing is not optimal. In this case, the relaxed approach in this paper will not work. When the relaxed problem does not feature increasing prices, it is even hard to show that the seller's optimal price path is decreasing as we cannot rule out the optimality of nonmonotone price paths.[6] This is because the buyer's optimal stopping decision for a nonmonotone price path at each t can depend on the entire future price path as opposed to being summarized in time t cutoffs that depend on the the price and the slope of the price path at t alone.

Finally, we end with a discussion of a few possible generalizations of the model. There are a number of ways in which the model of the paper can be generalized. Perhaps the most obvious way is to introduce some correlation in the buyer's value after she receives the shock. Recall that we assume that, conditional on receiving a shock, the new valuation is drawn independently. It is possible to generalize the results to allow for 'a small amount' of correlation. For instance, the buyer's value after the shock can be taken to be a weighted sum of the original and new value where the weight on the original value is small. Allowing for arbitrary amounts of correlation, conditional on a shock, significantly complicates the analysis. The primary difficulty is that optimal behavior may no longer be a cutoff strategy. A higher type may find it optimal to wait when a lower type finds it optimal to purchase. As a result, we can no longer to work in the cutoff space and optimal buyer behavior must be derived by solving complicated optimal stopping problems, the solutions to which are not known in general.

Another way in which the analysis can be generalized is to consider the seller optimal equilibrium when he cannot commit to the entire price path. As we have mentioned earlier, such a model would be a generalization of Fuchs & Skrzypacz (2010) to an environment where the arrival of shocks is unobserved by the seller and where the continuation payoffs are endogenous (as opposed to be determined by a given exogenous function). This is an interesting but extremely challenging problem which we leave for future research.

[6]It is possible to show that increasing price paths are no longer optimal.

APPENDIX

Proof of Lemma 1. We will argue that when $\varepsilon = 0$, then $p_1^* > p_0^*$. The result then follows from continuity as p_1^* must also be greater than p_0^* in a neighborhood $\varepsilon \in [0, \bar{\varepsilon}]$ of 0. Since $F_0(\theta) = F_1(\theta) = \theta^\alpha$ and $p_{F_0} = p_{F_1}$ for $\varepsilon = 0$, we represent them as F and p_F respectively for brevity.

We first make a few observations about the distribution $F(\theta) = \theta^\alpha$. The virtual value

$$\theta - \frac{1 - F(\theta)}{f(\theta)} = \frac{\theta^\alpha(\alpha+1) - 1}{\alpha\theta^{\alpha-1}},$$

for this class of distributions is negative when $\theta < p_F$ and positive when $\theta > p_F$ where $p_F = \left(\frac{1}{1+\alpha}\right)^{\frac{1}{\alpha}}$. An implication of this is that both c_0^*, c_1^* are greater than p_F since neither equation (7) nor (8) can hold when $c_0^*, c_1^* < p_F$ as their left hand sides would be negative. Additionally, the expression

$$\frac{F(\theta)(1 - F(\theta))}{f(\theta)} = \frac{1}{\alpha}(\theta - \theta^{\alpha+1}),$$

is decreasing whenever $\theta \geq p_F$.

Due to the fact that $F_1 = F_2$ and that the virtual values are strictly negative at $\theta = 0$, the solution lies in the interior. Then recall from Proposition 1 that prices p_0^* and p_1^* satisfy

$$p_0^* = \frac{1 - F(c_0^*)}{f(c_0^*)} + \frac{\lambda}{r+\lambda} c_1^*[1 - F(c_1^*)]$$

and

$$p_1^* = c_1^*.$$

Subtracting, we get

$$\begin{aligned} p_0^* - p_1^* &= \frac{1 - F(c_0^*)}{f(c_0^*)} + \frac{\lambda}{r+\lambda} c_1^*[1 - F(c_1^*)] - c_1^* \\ &< \frac{1 - F(c_0^*)}{f(c_0^*)} + c_1^*[1 - F(c_1^*)] - c_1^* = \frac{1 - F(c_0^*)}{f(c_0^*)} - c_1^* F(c_1^*). \end{aligned}$$

Now suppose the converse holds or that $p_0^* - p_1^* > 0$. This in turn implies that

$$\frac{1 - F(c_0^*)}{f(c_0^*)} - c_1^* F(c_1^*) > 0$$

and that $c_0^* > c_1^*$ since $c_0^* \geq p_0^*$. Using the first order condition (8), we can substitute $c_1^* = \frac{1 - F(c_1^*)}{F(c_0^*) f(c_1^*)}$ in the above inequality to get

$$\frac{1 - F(c_0^*)}{f(c_0^*)} > c_1^* F(c_1^*) \iff \frac{F(c_0^*)[1 - F(c_0^*)]}{f(c_0^*)} > \frac{F(c_1^*)[1 - F(c_1^*)]}{f(c_1^*)}.$$

But since $c_0^*, c_1^* > p_F$, the above inequality implies that $c_0^* < c_1^*$ which is a contradiction. Hence, we have shown that $p_0^* < p_1^*$ which completes the proof. □

Proof of Proposition 2. For this proof, we denote the implicit functions (c_0 in terms of c_1) defined in both the first order conditions (7) and (8) as $c_0^a(c_1)$, $c_0^b(c_1)$ respectively. We first argue that when the monotone hazard rate condition holds, these implicit functions are downward sloping $\left(\frac{dc_0^a(c_1)}{dc_1} < 0, \frac{dc_0^b(c_1)}{dc_1} < 0\right)$. In equation (7), an increase in c_1 decreases the right side and therefore, the monotone hazard rate condition implies that c_0 must go down in order for the equation to hold. Similarly, in equation (8), the monotone hazard rate condition implies that an increase in c_1 raises the left side and hence, c_0 must go down in order for the equation to hold.

Now observe that $c_0^a(\overline{\theta}_1) \geq c_0^b(\overline{\theta}_1)$. This follows from $c_0^a(\overline{\theta}_1) = p_{F_0}$ and $c_0^b(\overline{\theta}_1) = \underline{\theta}_0$. By assumption, an interior solution exists for all $\lambda < \underline{\lambda}$ and, in particular for λ close to 0, which then implies $p_{F_0} \geq \underline{\theta}_0$ (from the solution to (7)). Additionally, since we have assumed that these equations have a unique solution, the function c_0^a must cross c_0^b only once from below.

Now, when either λ increases or r decreases, the effective discount factor $\delta = \frac{\lambda}{r+\lambda}$ increases. This implies that for all c_1, $c_0^a(c_1)$ increases while $c_0^b(c_1)$ remains unaffected. Since, c_0^a and c_0^b are both downward sloping and cross only once, this implies that an increase δ leads to an increase in $c_0^*(\delta)$ and a decrease in $c_1^*(\delta)$. We use $c_0^*(\delta)$, $c_1^*(\delta)$, $p_0^*(\delta)$ and $p_1^*(\delta)$ to denote the optimal cutoffs and prices corresponding to the effective discount factor δ.

Now suppose that introductory pricing is optimal for a given $\overline{r}$, $\underline{\lambda}$. Then for any $\delta \leq \frac{\underline{\lambda}}{\underline{r}+\underline{\lambda}}$,

$$p_1^*(\delta) - p_0^*(\delta) = c_1^*(\delta) + \delta \int_{c_1^*(\delta)}^{\overline{\theta}_1} (\theta - c_1^*(\delta)) dF_1(\theta) - c_0^*(\delta),$$

which in turn implies that

$$\frac{\partial p_1^*(\delta)}{\partial \delta} - \frac{\partial p_0^*(\delta)}{\partial \delta} = \underbrace{\frac{\partial c_1^*(\delta)}{\partial \delta}}_{\text{-ve}} \underbrace{[1 - \delta(1 - F_1(c_1^*(\delta)))]}_{\text{+ve}} - \underbrace{\frac{\partial c_0^*(\delta)}{\partial \delta}}_{\text{+ve}} < 0.$$

In other words, if $p_1^*\left(\frac{\underline{\lambda}}{\underline{r}+\underline{\lambda}}\right) \geq p_0^*\left(\frac{\underline{\lambda}}{\underline{r}+\underline{\lambda}}\right)$ for a given $\overline{r}$, $\underline{\lambda}$, then it follows that $p_1^*\left(\frac{\lambda}{r+\lambda}\right) \geq p_0^*\left(\frac{\lambda}{r+\lambda}\right)$ for all r, λ such that $\frac{\lambda}{r+\lambda} \leq \frac{\underline{\lambda}}{\underline{r}+\underline{\lambda}}$. □

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