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The geometry of revealed preference

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ABSTRACT

In this paper, we examine how the geometry underlying revealed preference determines the set of preferences that can be revealed by choices. Specifically, given an arbitrary binary relation defined on a finite set, we ask if and when there exists a data set which can generate the given relation through revealed preference. We show that the dimension of the consumption space affects the set of revealed preference relations. If the consumption space has more goods than observations, any revealed preference relation can arise. Conversely, if the consumption space has low dimension relative to the number of observations, then there exist both rational and irrational preference relations that can never be revealed by choices.

1. Introduction

Classical revealed preference theory provides a simple, intuitive and nonparametric way of testing the most basic assumption of economics—that agents are rational. In this approach, observed choices by individuals "reveal" a (potentially incomplete) preference relation over the set of consumption bundles. It is well known that the necessary and sufficient condition for the observed choices of an individual to be consistent with utility maximization is that the preference relation revealed by the choices should be acyclic or equivalently should satisfy the Generalized Axiom of Revealed Preference or GARP (Varian, 1982). There is a large body of empirical work that checks this condition in a variety of different settings both in the field and in the lab (references can be found in the survey by Varian, 2007).

Revealed preference in the standard consumption setting is a geometric property—a chosen bundle is revealed preferred to all the bundles that were affordable but not chosen. Put differently, revealed preference is determined by where points corresponding to choices lie in the consumption space relative to the planes determined by the budgets. A bundle is revealed preferred to another if the latter lies underneath the budget plane on which the former lies. In this paper, we examine how this geometry underlying revealed preference determines the set of possible preference relations that can be revealed by choices. This provides ex-ante information about the preference relations that are possible given the number of goods and the number of observations in the data and this can be useful for experimental design. Formally, suppose we are given a set $\{1, \ldots, N\}$ with a relation $\succ$ defined on it. We ask whether there exists a price consumption data set $\{p_i, x_i\}_{i=1}^N$ consisting of K goods such that for all $i \neq j$, $p_i x_i > p_i x_j$ whenever $i \succ j$ and $p_i x_i < p_i x_j$ whenever $i \not\succ j$. The analysis of the above question involves examining whether budget sets can be chosen to intersect in appropriate ways to allow for choices of consumption bundles which will generate relation $\succ$ through revealed preference.

All possible relations may not arise from revealed preference as it may not be possible to separate the consumption space into the required number of regions using "downward sloping" budget planes. Hence, the set of possible revealed preference relations may depend on the dimension K of the consumption space. A higher dimensional consumption space may allow for the budget planes to separate more regions in the space potentially leading to a larger set of revealed preference relations. The main aim of this paper is to examine the relationship between the number of goods in a data set and the set of relations that can be generated through revealed preference. It is well known that when there are only two goods ($K = 2$), the Weak Axiom of Revealed Preference (WARP) is equivalent to GARP (Rose, 1958). This implies that when $K = 2$, there cannot be choices that satisfy WARP but violate GARP which, of course, is possible for $K \geq 3$.[1] However, little is known about the relationship between the set of possible revealed preference relations and the dimension of the consumption space when $K \geq 3$.

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[1] For example, a relation which is cycle consisting of three elements can never be generated by revealed preference when $K = 2$.

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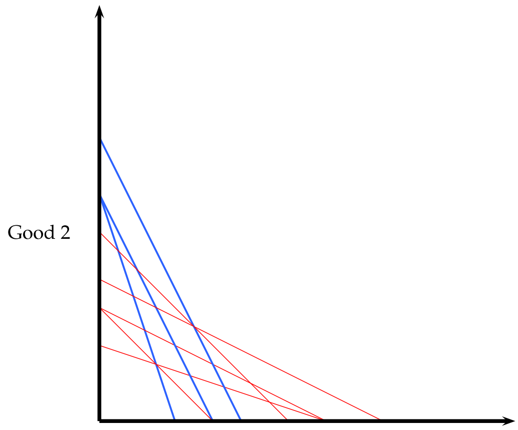

Fig. 1. Budget sets from Andreoni and Miller (2002).

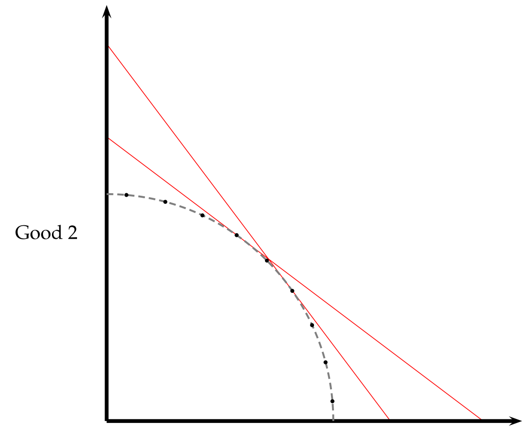

Fig. 2. Maximum WARP violations.

We show that when the dimension of the consumption space is large relative to the number of observations, revealed preference can generate any relation and conversely, certain rational and irrational preferences cannot be revealed by choices if the consumption space has low dimension relative to the number of observations.

As we will elaborate below, we feel that the results in this paper are useful for the design of experiments for models, the tests of which depend on revealed preference. A different but related problem which has received substantial recent attention is how to interpret the results of GARP tests. Suppose a researcher is interested in testing rationality on a given data set by checking for GARP violations. If the choices satisfy GARP, is it because the consumer is rational or is it because the budget sets provided little opportunity for rejecting rationality? Conversely, if the choices reject GARP, is there a simple way to interpret the degree of irrationality? These questions are addressed in a number of papers which have suggested both power (for example, Andreoni et al., 2013) and goodness of fit measures (for example, Beatty and Crawford, 2011) for a given collection of budget sets faced by a consumer. These measures are ideal to interpret results of GARP tests on a given price consumption data set.

By contrast, in experimental settings, the researcher is free to choose the budget sets, the dimension of the consumption space and the number of observations. Here, the experimental setup cannot be informed by the subject choices which are yet to be made at the design stage. As an example, consider the influential altruism experiment of Andreoni and Miller (2002). In this experiment, they varied relative prices and budgets and made individuals choose between keeping money for themselves and giving it to another subject. They then tested if charitable giving is rational. Fig. 1 shows the eight budget sets that they presented to their subjects. To provide a measure of the degree of irrationality of the subjects' choices, one of the statistics they reported was the number of WARP violations. They find that the choices of most irrational subjects contain only a single WARP violation (see Table 2 in their paper).

Suppose a researcher is interested in designing an experiment to study the irrationality of subjects measured by the number of WARP violations. What budget sets should be provided to the subjects? Presumably, the design should allow for choices which result in a large number of WARP violations. Clearly, the theoretical upper bound for the number of WARP violations in a subject's choices is achieved when every pair of her choices violate WARP. Fig. 2 shows that it is possible to provide subjects with budget sets on which such choices are possible. Here budget sets are chosen as different tangents to a given arc. Notice that if a subject's choices were the tangent point (or close to it) then every consumption bundle would be strictly revealed preferred to every other. In other words, every pair of such choices would violate WARP. Hence, by providing these budget sets to subjects, it is at least theoretically possible to observe choices with the maximal amount of irrationality. By contrast, consider the budget sets in Fig. 1 which were chosen by Andreoni and Miller. Note that the bold blue budget sets are such that any choices made on these budget sets will satisfy WARP. Hence, the most possible WARP violations that can be observed on these budgets sets are far fewer than the theoretical maximum.

Of course, testing GARP is just one instance of a test involving the revealed preference relation. Knowledge of the set of relations that can arise is important in experimental design for more general models as well. As an example, consider the design of an experiment to test the multiple rationale model of Kalai et al. (2002). In this model, an agent's preference depends on states. For instance, an agent may have different preferences depending on whether she is in a happy or a sad state. If these different states are unobserved by the researcher, then the observed choices from such an individual may be construed as irrational. Formally, in this setting, an individual has M rationales if there are M different states and the agent has a distinct utility function corresponding to each state. A data set is said to be "rationalized by M rationales" if each observed choice is the utility maximizing bundle corresponding to one of these preferences. The revealed preference test for M rationales simply involves partitioning the data into M subsets, such that GARP is satisfied on each of these subsets separately. A good experimental design for such general models should allow for the possibility that the hypotheses we want to test for can possibly arise from choices on the given budget sets. In particular, an important design choice is the number of goods in the subjects' choice sets.

As briefly mentioned above, our contribution is in the form of two theorems. In the first, we show that as long as the data set contains as many goods as observations ($K \geq N$), every possible binary relation can be generated using revealed preference. The proof is constructive and demonstrates how to generate a data set corresponding to a given relation. This can be viewed as a positive result for designing experiments which test properties of the revealed preference relation.

While the above result is positive, it suggests that there may be a connection between the number of goods and the nature of preferences that can be revealed by choices. Our second result shows that this is indeed the case. We prove that for every K, there exists a binary relation $\succ$ defined on a set with $N = O(2^K)$ elements such that there is no price consumption data set consisting of N

observations with K goods which generates $\succ$ through revealed preference. This result suggests that the number of goods in a data set may be an important choice variable for experimental design when testing general choice models.

Interestingly, we can construct relations on sets of size $N = O(2^K)$ which are acyclic but nevertheless cannot be generated if the consumption space has dimension K. This shows that a data set with a low number of goods relative to observations rules out not only certain preferences which violate GARP but also rules out certain preferences for which GARP is satisfied.

2. Preliminaries

The number of goods is denoted by K. We denote price vectors by p and consumption bundles by x. An observed *data set* is a finite set of price, consumption vectors $D = \{(p_i, x_i)\}_{i=1}^N$, where $(p_i, x_i) \in \mathbb{R}_{++}^K \times (\mathbb{R}_{+}^K \setminus \{0\})$ and $1 \leq N < \infty$. p_i^k and x_i^k denote the price and quantity consumed of the kth good in the ith observation, respectively. Utility functions are given by $U : \mathbb{R}_{+}^K \setminus \{0\} \to \mathbb{R}_+$.

Given a data set D, a consumption bundle x_i is said to be revealed preferred to another bundle x_j if the latter was affordable under prices p_i but was not chosen. Formally, x_i is *revealed preferred* to x_j if $p_i x_i \geq p_i x_j$. If the inequality is strict, then revealed preference is said to be strict. The Generalized Axiom of Revealed Preference (GARP) of Varian (1982) requires the revealed preference relation to be acyclic in the following sense.

Definition 2.1 (*GARP*)**.** Suppose we are given an arbitrary data set $D = \{(p_i, x_i)\}_{i=1}^N$. For any two consumption bundles x_i, x_j we say that $x_i R_0 x_j$ if x_i is revealed preferred to x_j. We say $x_i R x_j$ if for some sequence of observations $(x_1, x_2, \ldots, x_m)$, we have $x_i R^0 x_1$, $x_1 R^0 x_2, \ldots, x_m R^0 x_j$ (R is the transitive closure of R_0). The data set D satisfies *GARP* if

$$x_i R x_j \implies p_j x_j \leq p_j x_i \quad \forall i, \in \{1, \ldots, N\} \text{ where } i \neq j.$$

We use $\succ$ to denote an arbitrary binary relation defined on a set $\{1, \ldots, N\}$. We now define formally what is meant by a relation being generated by revealed preference.

Definition 2.2 (*Generating a Relation*)**.** A data set $\{(p_i, x_i)\}_{i=1}^N$ is said to *generate relation* $\succ$ defined on a set $\{1, \ldots, N\}$ if for all $i \neq j$

$$p_i x_i > p_i x_j \quad \text{if } i \succ j \quad \text{and} \quad p_i x_i < p_i x_j \quad \text{if } i \not\succ j.$$

Notice that in the above definition we require x_i to be strictly revealed preferred to x_j whenever $i \succ j$. None of our results are affected by replacing the strict by a weak inequality. Our choice of the strict inequality reflects the fact that in real data it is almost never the case that two different choice bundles cost exactly the same amount. Also note that the definition is in terms of the direct revealed preference relation R_0 and not in terms of the indirect revealed preference relation R. This reflects the fact that we are agnostic about whether the underlying consumer is a standard utility maximizer or whether, for instance, she has multiple rationales. Hence, the transitivity of preferences (implicit in the relation R) is not a natural assumption for our setting.

Before proceeding to our results, we state the classic result of Varian (1982) which states that a data set is consistent with utility maximization if and only if it satisfies GARP.

Theorem 1 (*Varian, 1982*)**.** *Let $D = \{(p_i, x_i)\}_{i=1}^N$ be a price consumption data set. The following are equivalent:*

(1) *Data set D is consistent with utility maximization. In other words, there exists a nonsatiated utility function U such that for each observation i*

$$U(x') \leq U(x_i), \quad \text{for all } x' \text{ satisfying } p_i x' \leq p_i x_i.$$

(2) *Data set D satisfies GARP.*

3. Results

Our first result states that revealed preference can generate any binary relation as long as the number of goods is at least as many as the number of observations.

Theorem 2. *Given an arbitrary relation $\succ$ defined on a set $\{1, \ldots, N\}$. There exists a data set $\{(p_i, x_i)\}_{i=1}^N$ consisting of N goods ($K = N$) which generates $\succ$.*

Proof. We label each good in a bundle by using a superscript. Thus, the jth good in the ith observation is represented by x_i^j, and the price of the jth good in the ith observation by p_i^j, where $1 \leq i, j \leq N$.

We now construct the data set $D = \{(p_i, x_i)\}_{i=1}^N$ as follows

$$\begin{aligned} &p_i^i = 1 \quad x_i^i = 1 \\ &p_i^j = 0 \quad x_i^j = 0 \text{ if } j \neq i \text{ and } j \succ i \\ &p_i^j = 0 \quad x_i^j = 2 \text{ if } j \neq i \text{ and } j \not\succ i. \end{aligned} \tag{1}$$

We now check to see if this data set indeed generates $\succ$. For an arbitrary $i \neq j$, if we have $i \succ j$ then

$$\begin{aligned} &p_i x_i = p_i^i x_i^i = 1 \\ &p_i x_j = p_i^i x_j^i = 0 \\ &\implies p_i x_i > p_i x_j. \end{aligned}$$

Similarly if we have $i \not\succ j$ then

$$\begin{aligned} &p_i x_i = p_i^i x_i^i = 1 \\ &p_i x_j = p_i^i x_j^i = 2 \\ &\implies p_i x_i < p_i x_j. \end{aligned}$$

Clearly the above data set generates the relation $\succ$. However, the proof is not complete because we do not allow the observed data to contain zero prices. But of course, it is easy to replace every instance of a 0 price by a small enough positive $\varepsilon > 0$ in Eq. (1). Since the above inequalities are strict, for a small enough ε, they will not be violated and this completes the proof. □

Theorem 2 shows that any relation can be generated by revealed preference; however, it requires the number of goods in the data set to be increasing in the size of the panel. Note that, it does not claim that N is the minimum number of goods required to generate any relation defined on a set of size N. This immediately leads to two questions. What is the minimum number of goods required to generate any relation and does this minimum number depend on N? While we do not have an answer to the first question we can provide an answer to the latter. Our second result shows that for every K, there are relations $\succ$ defined on a set of size $N > K$ which cannot be generated when the observed data has K goods.

Theorem 3. *For any $K \geq 2$, there is a relation $\succ$ defined on a set of size $N = O(2^K)$ such that no data set consisting of N observations and K goods can generate relation $\succ$. Moreover, relation $\succ$ can be chosen to be acyclic.*

The above result implies that there does not exist a dimension of the consumption space K such that all relations can be generated irrespective of the number of observations N. This in turn implies that experiments in which subjects are presented with many choice problems should perhaps have a larger number of goods. As we mentioned in the Introduction, the fact that there are acyclic relations that cannot be generated shows that a consumption space of low dimension rules out certain rational preferences along with certain irrational preferences.

4. Concluding remarks

In this paper, we studied the extent to which the geometric basis of revealed preference affects the set of underlying relations it can generate. We showed that every possible relation can be generated if there are enough goods in the data relative to the number of observations. Conversely, we showed that when this is not the case, there are situations where certain relations cannot be generated by any data set.

As revealed preference theory develops for more sophisticated choice models, we feel that an important result would be a complete characterization of the set of relations that can potentially be generated for a given N, K (where $N > K$). We leave this ambitious problem for future research.

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Appendix. Proof of Theorem 3

We begin the proof by defining an auxiliary problem. Suppose we are given a relation $\succ$ and N consumption bundles $\{x_i\}_{i=1}^N$ each with the K goods. When are there prices $\{p_i\}_{i=1}^N, p_i \in \mathbb{R}^K_{++}$, such that $\{(p_i, x_i)\}_{i=1}^N$ generates exactly the relation $\succ$? This is essentially the problem we are studying in the paper, but it assumes that consumption bundles are observed.

Mathematically, prices that generate $\succ$ will exist if the following system of linear inequalities has a solution for all $i, j \in \{1, \dots, N\}$ where $i \neq j$.

$$p_i \cdot (x_i - x_j) > 0 \quad \text{when } i \succ j, \tag{2a}$$

$$p_i \cdot (x_j - x_i) > 0 \quad \text{when } i \not\succ j, \tag{2b}$$

$$p_i \gg 0. \tag{2c}$$

We will now use a version of Farkas' lemma which will allows us to examine the Farkas alternative of the above system. This lemma is stated below.

Lemma 1. *For any matrix $A \in \mathbb{R}^{m\times n}$, either there exists a $y \in \mathbb{R}^n$ such that:*

$$Ay = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad [1 \ \cdots \ 1]y = 1,\ y \geq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},$$

or there exists a $x \in \mathbb{R}^m$ such that:

$$x^T A \gg [0 \ \cdots \ 0].$$

Proof. Let $\mathbf{i} \in \mathbb{R}^n, \mathbf{i} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \mathbf{0} \in \mathbb{R}^m, \mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$. The original system can be written as:

$$\begin{bmatrix} A \\ \mathbf{i}^T \end{bmatrix} y = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}, \quad y \geq \mathbf{0}.$$

By the Farkas lemma (see, e.g. Theorem 2.4 of Vohra, 2005), either the system above is feasible or there exists a solution $\tilde{x} \in \mathbb{R}^{m+1}$ to the system:

$$\tilde{x}^T \begin{bmatrix} A \\ \mathbf{i}^T \end{bmatrix} \geq [0 \ \cdots \ 0], \tag{3}$$

$$\tilde{x}^T \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} < 0. \tag{4}$$

Rewriting $\tilde{x}^T = [x, \hat{x}]$ where $x \in \mathbb{R}^m, \hat{x} \in \mathbb{R}$, we see that (4) is equivalent to $\hat{x} < 0$. Substituting this into (3), we get

$$x^T A \geq [-\hat{x} - \hat{x} \dots - \hat{x}] \gg [0 \ \cdots \ 0],$$

which concludes the proof. □

We now use Lemma 1 to take the Farkas alternative for the above system (2a)–(2c). We denote the dual variable for the inequality corresponding to the directed pair $i \succ j$ as y_{ij}, and the dual variable corresponding to the price of the kth good in the ith observation p_i^k as η_i^k. The Farkas alternative is:

$$\sum_{i=1}^N \sum_{j\neq i} y_{ij} + \sum_{i=1}^N \sum_{k=1}^K \eta_i^k = 1,$$

$$\sum_{\{j|i\succ j\}} y_{ij}(x_i^k - x_j^k) + \sum_{\{j|i\not\succ j\}} y_{ij}(x_j^k - x_i^k) + \eta_i^k = 0$$

$$\text{for all } 1 \leq i \leq N,\ 1 \leq k \leq K,$$

$$y_{ij}, \eta_i^k \geq 0 \quad \text{for all } 1 \leq i \neq j \leq N, 1 \leq k \leq K.$$

We eliminate the η variables to get an equivalent system which consists of fewer unknowns:

$$\sum_{i=1}^N \sum_{j\neq i} y_{ij} > 0,$$

$$\sum_{\{j|i\succ j\}} y_{ij}(x_i^k - x_j^k) + \sum_{\{j|i\not\succ j\}} y_{ij}(x_j^k - x_i^k) \leq 0$$

$$\text{for all } 1 \leq i \leq N,\ 1 \leq k \leq K,$$

$$y_{ij} \geq 0 \quad \text{for all } 1 \leq i \neq j \leq N.$$

The intuition of this elimination is straightforward. There can be no solution to the original system where $\sum_{i=1}^N \sum_{j\neq i} y_{ij} = 0$. This is because if any η_i^k is positive then at least one y_{ij} must be positive to satisfy the second equation.

Therefore, given consumption bundles $\{x_i\}_{i=1}^N$, there exist prices which generate the relation $\succ$ if and only if the above system has no solution. Going one step further, therefore, given a relation $\succ$, if the above system has a solution for all choice of x, then there is no data set with K goods which can generate $\succ$. We collect this observation in the following lemma.

Lemma 2. *Suppose we are given a relation $\succ$. There does not exist a data set $\{(p_i, x_i)\}_{i=1}^N$, generating the relation $\succ$ if and only if the following system has a solution for every $\{x_i\}_{i=1}^N \in \mathbb{R}^{NK}_+$.*

$$\sum_{i=1}^N \sum_{j\neq i} y_{ij} > 0,$$

$$\sum_{\{j|i\succ j\}} y_{ij}(x_i^k - x_j^k) + \sum_{\{j|i\not\succ j\}} y_{ij}(x_j^k - x_i^k) \leq 0$$

$$\textit{for all } 1 \leq i \leq N,\ 1 \leq k \leq K,$$

$$y_{ij} \geq 0 \quad \textit{for all } 1 \leq i \neq j \leq N.$$

We now construct a relation $\succ$ which cannot be generated based on the following two observations.

Observation 1. The Farkas system has a solution if and only if the following subsystem has a solution *for some* i:

$$\sum_{j\neq i} y_{ij} > 0, \tag{5a}$$

$$\sum_{\{j|i\succ j\}} y_{ij}(x_i^k - x_j^k) + \sum_{\{j|i\not\succ j\}} y_{ij}(x_j^k - x_i^k) \leq 0$$

$$\text{for all } 1 \leq k \leq K, \tag{5b}$$

$$y_{ij} \geq 0 \quad \text{for all } 1 \leq j \neq i \leq N. \tag{5c}$$

Proof. When the above system has a solution, we can set the remaining $y_{i'j}$'s to 0 for all other $i' \neq i$ which would yield a solution to the original problem. □

Observation 2. Given K vectors $\{v_1, \ldots, v_K\}$ each in $\mathbb{R}^K$, the following system has a non-zero solution for λ_i's (equivalently $\{\lambda_i\}_{i=1}^K \in \mathbb{R}^K \setminus \{0\}$):

$$\sum_{i=1}^{K} \lambda_i v_i \leq 0.$$

Proof. If the given vectors are linearly independent, any vectors in the negative orthant can be generated by non-trivial linear combinations. If the given vectors are linearly dependent, there will be a non-trivial solution for the above expression holding with equality. □

We are now in a position to prove Theorem 3.

Proof of Theorem 3. We will construct a relation $\succ$ defined on a set of size $N = 2^{K+1} + K + 1$ and we will show that it cannot be generated by any data set consisting of N observations each with K goods.

In the proof, we will describe the essential pairs of the relation $\succ$. For all remaining pairs $i, j \in \{1, \ldots, N\}$, we can either assign $i \succ j$ or $i \not\succ j$ without affecting the proof. If we assign $i \not\succ j$ for all remaining i, j, then the relation $\succ$ will be acyclic.

The construction is as follows. We describe how the last 2^{K+1} elements $\{K+2, \ldots, N\}$ are related to the first $K+1$ elements $\{1, \ldots, K+1\}$. Consider each vector $e = (e_1, \ldots, e_K) \in \{0, 1\}^K$. This vector can be thought of as a binary representation of an integer and we denote the integer represented by the binary number e as E. We can now specify the crucial part of the relation $\succ$ corresponding to each e.

- $K + 2E + 2 \succ 1$.
- $K + 2E + 3 \not\succ 1$.
- For each $2 \leq i \leq K + 1$:
  - If $e_i = 0$: $K + 2E + 2 \succ i$ and $K + 2E + 3 \succ i$.
  - If $e_i = 1$: $K + 2E + 2 \not\succ i$ and $K + 2E + 3 \not\succ i$.

We now show that given the above the relation, the system (5a)–(5c) will have a solution for some $i \in \{K+2, \ldots, 2^K+K+1\}$ for every set of consumption bundles $\{x_i\}_{i=1}^N$. We first define for all $j = 1, \ldots, K$,

$$v_j = (x_1 - x_{j+1}).$$

By our construction of relation $\succ$, we have ensured that every possible combination of vectors $\{(-1)^{e_i} v_i\}_{i=1}^K$ where $e = (e_1, \ldots, e_K) \in \{0, 1\}^K$ is present on the left side of inequality (5b). We can then use Observation 2 which says that there is a non-trivial solution for λ to the following inequality:

$$\sum_{i=1}^{K} \lambda_i v_i \leq 0.$$

We now use the signs of the λ's to choose an e. For all $1 \leq i \leq K$

$$e_i = \begin{cases} 0 & \text{if } \lambda_i \geq 0, \\ 1 & \text{if } \lambda_i < 0. \end{cases}$$

Recall, E is the integer corresponding to binary number e defined above. If $\sum_{j=1}^K \lambda_j < 0$ we take $i = K + 2E + 2$ and sign variable $s = 0$; else we take $i = K + 2E + 3$ and sign variable $s = 1$. We now define $y_{ij'}$ for $1 \leq j' \leq K + 1$ as

$$y_{i1} = \left| \sum_{j=1}^{K} \lambda_j \right|,$$

$$y_{ij'} = \left| \lambda_{j'-1} \right| \quad \text{for all } 2 \leq j' \leq K + 1.$$

We set all remaining y's to 0. Formally,

$$y_{ij} = 0 \quad \text{for all } j > K + 1, \quad \text{and}$$

$$y_{ln} = 0 \quad \text{for all } l \neq i,\ 1 \leq n \leq N.$$

We now show that this choice of y leads to a solution of (5a)–(5c) for the above chosen i. Inequalities (5c) are satisfied as the chosen y's are nonnegative and inequality (5a) is satisfied due to Observation 2. It remains to be shown that inequality (5b) is satisfied.

We simplify the left side of inequality (5b) for our choice of i and an arbitrary $1 \leq k \leq K$ as follows:

$$\begin{aligned}
&\sum_{\{j|i \succ j\}} y_{ij}(x_i^k - x_j^k) + \sum_{\{j|i \not\succ j\}} y_{ij}(x_j^k - x_i^k) \\
&= (-1)^s y_{i1}(x_i^k - x_1^k) + \sum_{\{l|e_l=0\}} y_{il+1}(x_i^k - x_{l+1}^k) \\
&\quad + \sum_{\{l|e_l=1\}} y_{il+1}(x_{l+1}^k - x_1^k) \\
&= (-1)^s y_{i1}(x_i^k - x_1^k) + \sum_{\{l|e_l=0\}} y_{il+1}(x_i^k - x_1^k + x_1^k - x_{l+1}^k) \\
&\quad + \sum_{\{l|e_l=1\}} y_{il+1}(x_{l+1}^k - x_1^k + x_1^k - x_i^k) \\
&= (-1)^s y_{i1}(x_i^k - x_1^k) + \sum_{\{l|e_l=0\}} y_{il+1}(x_i^k - x_1^k + v_l^k) \\
&\quad + \sum_{\{l|e_l=1\}} y_{il+1}(-v_l^k + x_1^k - x_i^k) \\
&= (-1)^s \left| \sum_{l=1}^{K} \lambda_l \right| (x_i^k - x_1^k) + \sum_{l=1}^{K} \left[\lambda_l (x_i^k - x_1^k) + \lambda_l v_l^k \right] \\
&= \sum_{l=1}^{K} \lambda_l v_l^k \\
&\leq 0.
\end{aligned}$$

Thus for any choice of $\{x_i\}_{i=1}^N$, we can find an i such that (5a)–(5c) has a solution. Thus $\succ$ cannot be generated with K goods and this completes the proof. □

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