ECO 426 (Market Design) - Lecture 8

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Revenue equivalence

- **Model:**
  - $N$ bidders
  - Bidder $i$ has valuation $v_i$
  - Each $v_i$ is drawn independently from the same distribution $F$ (e.g. $U[0, 1]$)

- **Theorem** In any auction such that in equilibrium:
  - the winner with the highest valuation wins, and
  - the bidder with the lowest possible valuation pays nothing,
  - the average revenue are the same, and the average bidder profits are the same.

- DP, SP, FP and AP share the properties that
  - equilibrium outcome is efficient (i.e. the highest value bidder wins the auction)
  - a bidder with a 0 valuation pays nothing.
Consider a maximization problem

$$\max_b u(b, v)$$

where $u()$ is differentiable in $b$ and $v$

The solution $b^*(v)$ is a function of $v$ and satisfies the FOC

$$u_b(b^*(v), v) = 0$$

The value of the maximization problem is $U(v) \equiv u(b^*(v), v)$ and by the chain rule of differentiation

$$U'(v) = u_b(b^*(v), v)b^*(v) + u_v(b^*(v), v)$$

The envelope theorem says that

$$U'(v) = u_v(b^*(v), v)$$
A bidder with valuation $v$ choosing to submit a bid $b$ solves

$$\max_b v \Pr(\text{win}|b) - \mathbb{E}[\text{Payment}|b]$$

In an auction the objective function is

$$u(b, v) = v \Pr(\text{win}|b) - \mathbb{E}[\text{Payment}|b]$$

$$u_v(b, v) = \Pr(\text{win}|b)$$

If $b^*(v)$ is the equilibrium bidding strategy, the envelope theorem says that the bidder expected profit $U(v)$ satisfies

$$U'(v) = \Pr(\text{win}|b^*(v)) = \text{Eq. Prob. value-}v\text{ bidder wins}$$

Integrating

$$U(v) = U(0) + \int_0^v \Pr(\text{win}|\tilde{v}) d\tilde{v}$$
Revenue equivalence theorem

\[ U(v) = U(0) + \int_0^v Pr(win|\tilde{v}) d\tilde{v} \]

- A bidder expected profit only depends on
  - his probability of winning as function of his valuation (i.e. \( Pr(win|\tilde{v}) \))
  - his expected profit when he has the lowest possible valuation (i.e. \( U(0) \))
  - Both are identical across the four auction formats we considered

- Critical assumptions
  - bidder know their own values (their values do not depend on others private information)
  - values are statistically independent
  - bidders only care about their profit (i.e. payoff equals valuation minus price paid)
The revenue equivalence theorem implies that: in any auction where, in equilibrium, the highest valuation bidder wins the object
- the expected revenue to the seller is constant
- the expected surplus to each bidder is constant

In a second price auction:
- the highest value bidder wins the object
- equilibrium strategies are easily characterized (dominant strategy)
- bidders expected surplus and sellers revenue are easily characterized

Can use the bidder expected revenue characterization in a second price auction to derive the (less obvious) equilibrium strategies of other auctions
First price auction

- Two bidders - valuations are independent draws from $U[0, 1]$
- Second price auction
  - Each bidder bids his valuation
  - A bidder with valuation $v$
    - wins with probability $v$ (i.e. the probability his opponent value is less than $v$)
    - the expected payment upon winning is $v/2$ (i.e. the expected valuation of his opponent, provided his opponent has a valuation smaller than $v$)

First price auction

- Suppose, in equilibrium, the highest valuation bidder wins
- A bidder with valuation $v$
  - wins with probability $v$ (i.e. the probability his opponent value is less than $v$)
  - by the revenue equivalence theorem, his expected payment upon winning must be the same as in a SP auction (i.e. $v/2$)
  - since the payment of the winner in a FP auction equals his own bid, the equilibrium bid of a bidder with valuation $v$ must be $v/2$ (i.e. the equilibrium bidding strategy is $b(v) = v/2$.)
All pay auction

- Each bidder submits a sealed bid
- Bids are open
  - Bidder who submitted the highest bid wins the object
  - **Each bidder** pays a price to the seller equal to his own bid

What should bidders do?
- Suppose there is an equilibrium where the highest valuation bidder wins
  - use the revenue equivalence theorem to solve for the candidate equilibrium bidding strategies
  - ex-post verify that the strategies constitute an equilibrium (i.e. no bidder has any incentive to deviate)
Two bidders - valuations are independent draws from $U[0, 1]$

Suppose, in equilibrium, the highest valuation bidder wins

A bidder with valuation $v$

- wins with probability $v$ (i.e. the probability his opponent value is less than $v$)
- pays his own bid, $b(v)$, regardless of whether he wins or not
- his expected profit is then

$$\text{Prob}(\text{win}|v) \times v - b(v) = v^2 - b(v)$$

- in a second price auction has an expected profit of

$$v(v - v/2) = v^2/2$$

- from the revenue equivalence theorem

$$v^2 - b(v) = v^2/2$$

- the equilibrium bidding strategy in an all pay auction must be

$$b(v) = v^2/2$$
A reserve price is a price below which the seller is not willing to give up the object.

Second price auction with a reserve price $r$

- the highest bidder wins the object if bid $> r$
- the winner pays a price equal to the largest between the second highest bid and the reserve price $r$

**Example 1**: Two bids, 0.3 and 0.6, and reserve price $r = 0.4$. The high bidder wins and pays 0.4.

**Example 2**: Two bids, 0.5 and 0.6, and reserve price $r = 0.4$. The high bidder wins and pays 0.5.

**Example 1**: Two bids, 0.3 and 0.36, and reserve price $r = 0.4$. Nobody wins, object remains with seller.
Two bidders - valuations are independent draws from $U[0, 1]$

It is a dominant strategy to:

- bid own valuation when $v > r$
- not bid when $v \leq r$ (or bid own valuation)

A bidder with valuation $v > r$

- wins with probability $v$
- when winning pays a price equal to:
  - opponent value, $\hat{v}$, if $\hat{v} > r$ (happens with probability $(v - r)/v$)
  - reserve price, $r$, if $\hat{v} \leq r$ (happens with probability $r/v$)
- expected payment when winning

$$\frac{r}{v} \times r + \left(\frac{v - r}{v}\right) \times \frac{v + r}{2} = r + \frac{(v - r)^2}{2v}$$
First price auction with reserve price

- Two bidders - valuations are independent draws from $U[0, 1]$
- Suppose, in equilibrium, the highest valuation bidder wins
- A bidder with valuation $v$
  - does not bid if $v \leq r$ (dominant strategy)
  - bids $b(v) = r + (v - r)^2/(2v)$ if $v > r$ (by the revenue equivalence theorem)
- note that $r + (v - r)^2/(2v)$ is strictly increasing in $v$, so in equilibrium the highest value bidder wins
Optimal reserve price

What reserve price maximizes the seller’s revenue?

Suppose there is just one bidder, with $U[0, 1]$ valuation

- reserve price is just a posted price
- sell at price equal $r$ if $v > r$
- do not sell otherwise

Expected revenue is

$$\text{Prob}(v > r) \times r = (1 - r) \times r$$

- monopolist’s revenue with demand function $Q(p) = 1 - p$
- revenue maximizing reserve price $r = 1/2$
- same as monopolist price
Optimal reserve price

- Two bidders - independent $U[0, 1]$ valuations
- Compare the revenue from marginally increasing the reserve price $r$ to $r + \epsilon$, across all possible pairs of valuations $v_l < v_h$

\[ \begin{align*}
0 & \quad v_l \quad v_h \quad r \quad v_h \quad v_l \quad v_h \quad v_h \quad 1 \\
\end{align*} \]

- $v_l < v_h < r$
  - no impact on revenue
- $r + \epsilon < v_l < v_h$
  - no impact on revenue
- $v_l < r < v_h$
  - $R$ increases by $\epsilon$ (probability $2r(1 - r)$)
- $v_l < r < v_h < r + \epsilon$
  - $R$ decreases by $r$ (probability $2\epsilon r$)

- Expected revenue change: $\Delta \text{Revenue} = \epsilon 2r(1 - r) - r 2\epsilon r$
- must be zero at the optimal reserve price $r^* = 1/2$
Optimal reserve price

- With $N > 2$ bidders same argument applies
  - The revenue only depends on the highest two bids
  - Similar calculation of impact on revenue
  - Optimal reserve price remains $r^* = 1/2$

- First price, second price, ascending price and descending price auctions all have the same optimal reserve price

- Optimal reserve price in an all pay auction?
  - Use the revenue equivalence theorem
  - Bidders with valuation below $r^*$ bid nothing
  - For $v \geq r^*$, solve for bidding strategy, $b(v)$, using the revenue equivalence theorem
  - Reserve price must be equal to $b(r^*)$
    - if it higher, the allocation rule is not the same as in the second price auction
    - if it is lower a bidder with value $r^*$ would have an incentive to lower its bid

- Homework: calculate the optimal reserve price with two bidders