Notes on the Revenue Equivalence Theorem
Jonathan Levin, Econ 136

The revenue equivalence theorem states that for certain economic environments, the expected revenue and bidder profits for a broad class of auctions will be the same provided that bidders use equilibrium strategies. The purpose of these notes is provide a statement of the result and explain it.

The model

- Seller has a single item to sell.
- Two potential bidders, both risk-neutral.
- Each bidder has a value drawn from a uniform distribution on $[0, 1]$.

Standard auctions

A "standard auction" is an auction in which:

- bidders are asked to submit bids;
- the bidder who submits the highest bid wins the object;
- bidder i is asked to pay $\tau(b_i, b_j)$.

and furthermore there is a Nash equilibrium of the auction game in which (a) the bidders use a strategy $b(v)$ that is increasing in v, so that (b) the bidder with higher value wins; and also (c) a bidder who has value $v = 0$ ends up paying nothing in equilibrium.

The Revenue Equivalence Theorem

Theorem 1 *If there are two bidders with values drawn from $U[0, 1]$, then any standard auction has an expected revenue $1/3$ and gives a bidder with value v and expected profit of $v^2/2$, the same as the second-price auction.*

The more general version of the theorem, which we won't prove, asserts that if there are N bidders with values drawn from the same continuous value distribution, then any standard auction will lead to the same revenue and expected bidder profit as the second-price auction (the expected revenue may not be $1/3$ however). In the N bidder case, the payment rule for bidder i can depend on i's bid b_i and all of the competing bids (i.e. pay your bid if your bid is higher than all the rest of the bids would be the first price auction payment rule).

Theorem 2 *If there are N bidders with values drawn from a continuous distribution (e.g. uniform on $[a, b]$), then any standard auction leads to the same expected revenue, and same expected bidder profit, as a second-price auction.*

Example: Second Price Auction

The second price auction is a standard auction. The payment rule has $\tau(b_i, b_j)$ equal to zero if $b_i < b_j$, and equal to b_j if $b_i > b_j$. In equilibrium, each bidder bids his value, so the equilibrium strategy is $b(v) = v$. In equilibrium the bidder with the higher value will win, and pay the bid (or equivalently value) of the lower-valued bidder. The expected revenue is 1/3. Why 1/3? If we take two draws from a uniform distribution on $[0, 1]$, the higher draw will be on average 2/3 and the lower draw will be on average 1/3. Notice also that if a bidder has value v, he expects to win whenever the other bidder has a value less than v, which happens with probability equal to v. If he does win, he expects to pay $v/2$. So the expected profit of a bidder with value v is $U(v) = v \cdot (v - v/2) = v^2/2$.

Example: First Price Auction

The first price auction is also a standard auction. The payment rule has $\tau(b_i, b_j)$ equal to zero if $b_i < b_j$, and equal to b_i if $b_i > b_j$. In equilibrium each bidder submits a bid equal to half his value, so the equilibrium bid strategy is $b(v) = v/2$. Because in equilibrium the high-valued bidder wins and pays his bid.The expected revenue is again 1/3. Why? On average the higher of the two values is 2/3, and the higher of the two bids is 1/3. A bidder with value v again expects a profit $v^2/2$. Again, the reason is that if a bidder has value v, he expects to win whenever the other bidder has a value below v (because the first bidder will bid $v/2$, and the second bidder will bid less than this whenever its value is below v). So the bidder with value v expects to win with probability v, and if he does win expects to pay $v/2$, his equilibrium bid. Therefore $U(v) = v \cdot (v - v/2) = v^2/2$.

Example: Other Standard Auctions

There are many other standard auctions. Here are a few examples:

- All-pay auction: bidders submit bids, the high bid wins and *all* bidders pay their bids, even losing bidders. The payment rule is $\tau(b_i, b_j) = b_i$.
- Mixed first-price/second-price: High bid wins and winner pays the average of the first and second highest bids. So $\tau(b_i, b_j)$ equals zero if $b_i < b_j$, and $(b_i + b_j)/2$ if $b_i > b_j$.
- Ascending auction: Can be viewed as a standard auction if we think about each bidder giving instructions ("bid up to some amount b") to a proxy bidder that executes the strategy.

- Descending auction: Also can be viewed as a standard auction if we think about each bidder giving an instruction ("stop the auction if the price falls to b") to a proxy bidder.

There are lots of other auctions that are standard auctions or can be interpreted as such. For instance, an auction where the winner pays half his bid, or his bid plus ten dollars. Or an auction that is ascending until the price reaches \$0.50, at which point there is a final round of sealed bids.

"Proof" of the Theorem

You can find a mathematical proof in the Milgrom book, or in the book *Auction Theory* by Vijay Krishna. I'm going to provide an intuitive proof that's longer but helps you think about why this result is true.

The main step, which we'll get to below, is to show that bidder profits are the same in any standard auction. In particular, we're going to show that if $U(v)$ is the expected profit a bidder with value v expects in some standard auction, then whatever the exact rules of the standard auction, $U'(v) = v$ (more generally, $U'(v)$ equals the probability that a bidder with value v is the equilibrium winner, or has the highest value; with two bidders and $U[0,1]$, this probability is v). Let's assume we can show this.

Now, by the definition of a standard auction, $U(0) = 0$, i.e. a bidder with the lowest possible value makes zero expected profit. By the Fundamental Theorem of Calculus, and what we've already argued, we have:

$$U(v) = U(0) + \int_0^v U'(x)dx = \int_0^v xdx = \frac{1}{2}v^2$$

This tells us the profit of a bidder with value v.

We can also get the average profit of a given bidder by taking the expectation of $U(v)$. That is,

$$\mathbb{E}_v[U(v)] = \int_0^1 U(v)dv = \int_0^1 \frac{1}{2}v^2 dv = \frac{1}{6}$$

So on average bidder one can expect a profit of 1/6 and bidder 2 can expect a profit of 1/6, and therefore the expected total bidder profit is 1/3.

$$\mathbb{E}[\text{Total Bidder Profit}] = 2 \cdot \mathbb{E}[U(v)] = \frac{1}{3}.$$

Now, observe that in any standard auction, the bidder with the high value wins. So the expected *surplus* (the value created by transferring the object to the winner) is $\max\{v_1, v_2\}$ (let's assume the seller has zero value, so there isn't an opportunity cost of transferring the object). Therefore:

$$\mathbb{E}[\text{Surplus}] = \mathbb{E}\left[\max\{v_1, v_2\}\right] = \frac{2}{3}.$$

Of course, if the auction creates a an expected total surplus $S = 2/3$, and the bidders each expect a profit of $1/6$, there is $1/3$ left over. This must go to the seller as revenue because

$$\text{Surplus} = \text{Revenue} + \text{Total Bidder Profit}.$$

So we have

$$\mathbb{E}[\text{Revenue}] = \mathbb{E}[\text{Surplus}] - \mathbb{E}[\text{Total Bidder Profit}] = \frac{1}{3}.$$

How do we pin down bidder profits?

The tricky step of the proof is to argue that whatever standard auction we are studying, we must have $U'(v) = v$. This argument relies on a mathematical result called the envelope theorem, which is explained below. The way I'm going to explain it here is to explain first why it's true in a second price auction, then why it's true in a first price auction, then why it's true in any standard auction.

An important equation here is:

$$U(v) = \Pr(\text{Win}) \cdot v - \mathbb{E}[\text{Payment}].$$

Note that if only the winner pays (as in a first price auction or second price), then $\mathbb{E}[\text{Payment}]$ is equal to $\Pr(\text{Win})$ times $\mathbb{E}[\text{Payment}|\text{Win}]$, but in general we might have losers paying so we'll use the more general equation for $U(v)$.

Second Price Auction

Let's start with a mechanical argument. In the second price auction, we know that $b(v) = v$. The probability of winning with value v is $\Pr(v_j < v) = v$. And the expected payment is equal to $v/2$ if you do win, or $v/2 \cdot v = v^2/2$ in total.

$$U(v) = \Pr(\text{Win}) \cdot v - \mathbb{E}[\text{Payment}] = v^2 - v^2/2 = v^2/2.$$

Therefore by calculus, $U'(v) = v$.

Now let's try an intuitive argument. Suppose you have value v and submit your equilibrium bid $b(v) = v$. You expect to win with probability v, and you expect on average to end up paying $v^2/2$ (you only pay if $b(v) = v$ is a winning bid, and given that you expect to pay $v/2$). After sending in your bid you realize that your value is actually $v + \varepsilon$. This doesn't change your expected payment, which depends only on your bid which is already in the mail. But if your bid is a winner, it means you get an extra ε in profit. So the increase in expected profit is ε times the probability that you win, which is v, or εv. Now, when we calculate $U(v + \varepsilon) - U(v)$, we should also factor in that if your value increases from v to $v + \varepsilon$, you'd want to increase your bid, but the envelope theorem tells us not to worry about this last part — it is a negligible effect if ε is small.[1] Therefore $U(v+\varepsilon) - U(v) \approx \varepsilon v$, or $U'(v) = v$.

First Price Auction

Now let's try the argument for the first price auction. The equilibrium bid function is $b(v) = v/2$. The probability of winning with value v is v. The expected payment is equal to $b(v) = v/2$ if you do win, or $v \cdot v/2 = v^2/2$ in total. So again:

$$U(v) = \Pr(\text{Win}) \cdot v - \mathbb{E}[\text{Payment}] = v^2 - v^2/2 = v^2/2,$$

and by calculus $U'(v) = v$.

The intuitive argument works the same as above. Suppose you have value v and mail in your bid $b(v) = v/2$. Now you realize you have value $v+\varepsilon$. This doesn't change your expected payment, which depends only on your bid which has been mailed already. But it gives you an extra ε in the probability v event that your win, or an extra $v\varepsilon$. Again to calculate $U(v+\varepsilon) - U(v)$ we should factor in that having the higher value makes you want to slightly increase your bid (from $v/2$ to $(v+\varepsilon)/2$) but the envelope theorem say the second effect is not important. So we can ignore it and write $U(v+\varepsilon) - U(v) \approx \varepsilon v$, or $U'(v) = v$.

Other Standard Auctions

Now we'll make the case for some arbitrary standard auction. Because it's a standard auction, it has an equilibrium with strategy $b(v)$, and in equilibrium the high-valued bidder wins (because $b(v)$ is increasing in v). We can't use the mechanical argument, however because even though:

$$U(v) = \Pr(\text{Win}) \cdot v - \mathbb{E}[\text{Payment}] = v^2 - \mathbb{E}[\text{Payment}],$$

we don't know yet what expected payment to plug in!

So let's try the intuitive argument. Suppose you have value v and mail in your equilibrium bid $b(v)$ (whatever that happens to be). You expect this bid to be highest with probability v, because it will be highest if the opponent submits a bid less than $b(v)$ and this will happen if his value is less than v.

Now you realize you have value $v+\varepsilon$. Again, this doesn't change your expected payment, which depends only on the bid that's already in the mail. (Think about why this .... it's because the seller has to calculate the payments from the bids — she doesn't know the values.) But having the higher value gives you an extra ε if you win. You expect $b(v)$ to be the winning bid with probability v, so you expect an extra ε with probability v. Again to calculate

[1] In the second price auction, the reason it is negligible is that if you raise your bid from v to $v+\varepsilon$, you increase your chance of winning by ε, but the additional victories all come in cases when the opponent bid is between v and $v+\varepsilon$, so if your value is $v+\varepsilon$, the additional victories yield profits of at most ε. So the expected gain from moving from the sub-optimal v bid to the optimal $v+\varepsilon$ bid is of the order ε^2, or negligible compared to the direct ε effect of having the higher value.

$U(v+\varepsilon) - U(v)$ we should factor in that having the higher value makes you want to slightly increase your bid (from $b(v)$ to $b(v+\varepsilon)$) but again the envelope theorem says we can ignore the gain that comes from the slight change in bid. So again we can ignore it and write $U(v+\varepsilon) - U(v) \approx \varepsilon v$, or $U'(v) = v$.

Expected Payment in a Standard Auction

One reason the RET is useful is because of the calculation we just did showing that in any standard auction $U'(v) = v$, or — integrating up as explained above — $U(v) = v^2/2$. Notice that we figured this out without first figuring out the equilibrium strategy $b(v)$ and also without figuring out the expected payment a value v bidder makes in equilibrium!

What is really useful is that this lets us come back and figure out what the expectd payment must be! In a standard auction

$$U(v) = \Pr(\text{Win}) \cdot v - \mathbb{E}[\text{Payment}].$$

To be a little more precise, by Pr(Win) we mean the probabiltiy of winning with your equilibrium bid $b(v)$, which is equal to v. By $\mathbb{E}$[Payment], we mean the expected amount a bidder with value v expects to make if they send in their equilibrium bid $b(v)$, i.e. $\mathbb{E}$[Payment given eqm bid $b(v)$].

Combining this with the RET result that $U(v) = v^2/2$, we have

$$U(v) = \Pr(\text{Win}) \cdot v - \mathbb{E}[\text{Payment}] = v^2 - \mathbb{E}[\text{Payment}] = v^2/2$$

Or by re-arranging

$$\mathbb{E}[\text{Payment - given eqm bid } b(v)] = v^2/2.$$

Solving Auctions the RET Way

We just argued that in any standard auction, if $b(v)$ is the equilibrium bid of a bidder with value v, then:

$$\begin{aligned} \Pr[\text{Win} \mid \text{bid } b(v)] &= v \\ \mathbb{E}[\text{Payment} \mid \text{bid } b(v)] &= v^2/2 \end{aligned}$$

Let's see how to we can use this to "solve" for the equilibrium $b(v)$ in different standard auctions:

First Price Auction

In a first price auction, you pay your bid if you win, so

$$\mathbb{E}[\text{Payment} \mid \text{bid } b(v)] = \Pr[\text{Win} \mid \text{bid } b(v)] \cdot b(v)$$

So

$$b(v) = \frac{\mathbb{E}[\text{Payment} \mid \text{bid } b(v)]}{\Pr[\text{Win} \mid \text{bid } b(v)]} = \frac{v^2/2}{v} = \frac{1}{2}v$$

All Pay Auction

In an all-pay auction, you pay your bid whether you win or lose, so

$$\mathbb{E}[\text{Payment} \mid \text{bid } b(v)] = b(v) = \frac{v^2}{2}.$$

More Tricks

We'll do some more tricks with the RET in class...

Background: The Envelope Theorem

For completeness, this section explains the envelope theorem. The envelope theorem is a general theorem that applies when you make small changes in a function your are trying to maximize.

Suppose that $u(b, v)$ is a continuous function of a choice variable b and a parameter v, and that for each value v, $b^*(v)$ maximizes $u(b, v)$. Define $U(v) = \max_b u(b, v)$, or equivalently $U(v) = u(b^*(v), v)$. The envelope theorem says that: $U'(v) = u_v(b^*(v), v)$.

To see why this could be slightly surprising, observe that $U'(v)$ is the value of having a slightly higher v assuming that whatever v you have, you will choose b optimally. The theorem says that to compute the value of a slight increase in v you can ignore the effect that comes from re-optimizing the choice. One way to see this is that if $b^*(v)$ is a continuous function of v, then for a bidder with value v, there is very little difference between making the optimal choice $b^*(v)$ and a slightly different choice such as $b^*(v + \varepsilon)$ or $b^*(v - \varepsilon)$. So

$$u(b^*(v), v) \approx u(b^*(v + \varepsilon), v) \approx u(b^*(v - \varepsilon), v)$$

Consequently,

$$U(v + \varepsilon) = u(b^*(v + \varepsilon), v + \varepsilon) \approx u(b^*(v), v + \varepsilon)$$

and

$$U(v + \varepsilon) - U(v) \approx u(b^*(v), v + \varepsilon) - u(b^*(v), v)$$

which means that $U'(v) = u_v(b^*(v), v)$.