Existence of Monotone Pure Strategy Equilibria in First-Price Auctions with Resale

PRELIMINARY. NOT FOR CIRCULATION.

Charles Z. Zheng*

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Abstract

A perfect Bayesian equilibrium with monotone pure strategies exists in an independent private value first-price auction followed by the winner's choosing a mechanism to offer resale. We allow for any number of ex ante different bidders and impose no restriction on the choice of resale mechanisms. Since a bidder's ex post valuation of the auction outcome is derived from endogenous resale, the major conditions for the fixed-point method on Bayesian games cannot be obtained through assumptions of the valuation function. Based on a comparative static analysis of the mechanism-design decision at resale, we obtain single-crossing and continuity properties at non-tying bids. Tying bids are disentangled according to the direction of resale conditional on the tie.

*Department of Economics, University of Western Ontario, London, Ontario, charles.zheng@uwo.ca, http://economics.uwo.ca/faculty/zheng/.

1 Introduction

Analyses of economic institutions are based on existence of equilibria of the underlying games. Although Athey [1], McAdams [8], Reny and Zamir [13], Reny [12], and Jackson, Simon, Swinkels and Zame [5] have established an ingenious fixed-point approach to equilibrium existence for Bayesian games such as static auctions, the approach has not been applied to dynamic games such as auctions with resale. The main difficulty is that, whereas a bidder's ex post valuation of the auction outcome is exogenous in the aforementioned works, in an auction followed by possible resale, the valuation is derived from the associated continuation equilibrium at resale, which in turn may be manipulated by actions taken during the auction.

In the auction-resale literature, the state of the knowledge regarding equilibrium existence in first-price auctions is that equilibrium exists in an independent private value model where there are only two kinds of bidders ex ante, those whose use values are drawn from a distribution say s, and the others whose use values drawn from a distribution say w. Garratt and Tröger [2] handled the case where the w-distribution is degenerate and there is only one w-bidder. Hafalir and Krishna [4] handled the 2-bidder case with one w-bidder and one s-bidder and they proved an elegant symmetrization result. Lebrun [6, 7] handled similar 2-bidder models with more general bid-disclosure policies and resale-bargaining power allocations. Virág [14] proved existence in the multiple-bidder 2-distribution case and showed that the symmetrization result of Hafalir and Krishna does not extend beyond 2-bidder models.

The methods of these works are based on explicit characterizations of the associated continuation equilibrium at resale, which are plugged into a bidder's objective function viewed at the start of the auction. The differential equation obtained thereof pins down a bid function required by the first-order necessary condition for an equilibrium. The second-order condition is then verified by considerations of all on- and off-path cases at resale. Such a procedure is feasible in these works due to their 2-distribution assumption. The assumption implies that on the equilibrium path a reseller has only one kind of potential buyers to deal with and, furthermore, conditional on almost every winning bid, only one kind of bidders wants to offer resale upon winning. Neither properties would hold in a more general model with more than two distributions of bidders. An alternative method would help.

The aforementioned fixed-point approach is an alternative though challenged by new issues due to the possibility of resale. Take the issue of tying bids for example. In the fixed-point approach, ties cause troubles only at the limit of the sequence of equilibria in the approximating finite-action games. With resale, ties may cause a bidder's valuation of the auction outcome, assumed to be a continuous function of one's bid in the fixed-point approach, to be discontinuous in his bid. That is because the valuation depends on the bidder's expected revenue at resale in case of winning the auction. If there is a mass of his rivals' types who bid x in the auction, then in outbidding them through switching his bid from $x - \epsilon$ to $x + \epsilon$, the bidder adds a mass of bidder-types to his clientele for resale, thereby generating a non-infinitesimal change in his expected revenue at resale.

The issue of ties is further complicated by the fact that a bidder who has lost in an auction may buy the good from the reseller. Furthermore, if a bidder plays the role of a speculator, some other bidders may prefer losing to winning in the auction. Thus, it is not obvious that a bidder wants to win at all conditional on tying with his rivals.

As another example, the fixed-point approach would require in an auction-resale game that a bidder's valuation of the auction outcome should satisfy the single crossing condition in his bid and use value. That in turn requires knowledge about the associated continuation equilibrium at resale, since one's bid has a learning effect on resale: the winning bid signals upper bounds of the losing bidders' use values, and these upper bounds affect the choice of the resale mechanism.

This paper welds together the fixed-point approach for bidding equilibrium with a mechanism-design approach for resale equilibria. Based on Myerson's [11] characterization of optimal auctions, we obtain fundamental comparative static properties of resale regarding the choice of resale mechanisms and bidding behaviors in these mechanisms. These properties imply a single crossing property, one of the two building blocks for the fixed-point approach.

The other building block, that a bidder prefers to win conditional on his bid's tied or (in the converging sequence of equilibria) clustered with his rivals' bids, is more difficult to establish. In the literature, this building block is established mainly by the assumption that a bidder's ex post valuation of the auction outcome is weakly increasing in his rivals' types. This assumption does not hold in my model, because a bidder's payoff from losing need not be decreasing in rivals' types. To establish the building block, we disentangle the clustering bidders according to who could possibly resell to whom conditional on the cluster, thereby rendering immaterial the nonmononicity of the loser's payoff (Proposition 8).

Another important assumption in the fixed-point literature is that a bidder's ex post valuation is continuous in his bid. As the valuation is endogenous in our model, the continuity has to be proved rather than assumed. We establish the property based on examinations of a reseller's problem of choosing an optimal resale mechanism (Proposition 5).

The result is that monotone pure-strategy equilibrium exists in an independent private value first-price auction followed by the winner's offering resale through a mechanism chosen by the winner (Theorem 1).

This paper has significantly developed the mechanism-design approach to resale previously used by Zheng [15] and Garratt, Tröger and Zheng [3]. Reconciling a conflict between the traditional optimal auction theory that assumes away resale and its typical implication that players could gain from resale, Zheng proposed sufficient conditions for continuation equilibria at resale based on which the initial seller can design an optimal auction that incorporates resale possibilities. One of those sufficient conditions, "resale monotonicity," partially anticipated the learning effect proved in this paper (Proposition 1). Garratt, Tröger and Zheng characterized the continuation equilibrium at resale, after an English or second-price auction, through Myerson's characterization of optimal auctions and the Miglrom-Segal [9] envelope condition. They have found comparative statics results that anticipated the monopoly effect obtained in this paper (Proposition 2 here).

In both papers, tying bids are nonissue as the initial auction does not have to be a first-price auction, whereas this paper needs to handle the issue of ties because the occurrence of ties is an unavoidable possibility of first-price auctions and the posterior beliefs and the choice of resale mechanisms all depend crucially on the information about ties. The other comparative static properties obtained in this paper, including the strict single crossing (Proposition 3), the upper and lower bounds of the marginal expected payoff from winning (Proposition 4) and continuity at nonatomic bids (Proposition 5), are new in the study of auctions with resale.

2 The Model and the Theorem

Let us consider a two-period dynamic game of an auction with resale. In period 1, a good is sold via first-price auction to one of finitely many bidders, constituting a set I. Each bidder i's use value of the good is independently drawn from a commonly known distribution. Privately informed of his own use value, bidder i submits a sealed bid from $B_i := \{l\} \cup (r_i, \infty) \subseteq \mathbb{R}$, with reserve price $r_i \geq 0$ for bidder i and $l < 0$ being the losing bid that amounts to nonparticipation in the period-1 auction. If bidder i's bid does not exceed r_i then it amounts to l. Ties are broken randomly and uniformly with equal probabilities. At the end of the auction, the highest bid and the identity of the the winner (after tie-breaking if a tie occurs) are announced publicly and nothing else is disclosed. In period 2, the bidder who won in period 1 offers resale to the other bidders (in the set I) by committing to a selling mechanism. Every bidder is assumed risk-neutral in his net payoff, which is defined to be his use value, if he is the final owner of the good, plus the net monetary transfer he receives from others. Discounting is assumed away for notational simplicity.

Assume that every bidder i's prior distribution of his use value is F_i, with positive continuous density f_i on its support $T_i := [\underline{t}_i, \bar{t}_i] \subseteq \mathbb{R}_+$, such that the hazard rate $f_i(t_i)/(1 - F(t_i))$ is weakly increasing in t_i on T_i. Denote $T_{-i} := \Pi_{k \in I \setminus \{i\}} T_k$ and $T := \Pi_{k \in I} T_k$.

Our solution concept is the standard notion of perfect Bayesian equilibrium with the only modification that sequential rationality is not required at any node where the reseller has just chosen a resale mechanism that precludes existence of equilibrium in the continuation game. This notion has been formalized by Zheng [15].

Theorem 1 *The auction-resale game defined thereof with the above-stated assumptions admits a perfect Bayesian equilibrium where each bidder's period-one bid is a weakly increasing function of the bidder's use value.*

The assumption that a bidder i's non-l bids (called serious bids) constitute an open set (r_i, ∞) is important. By contrast, the sets of serious bids in Athey [1] and Reny and Zamir [13] are closed such as $[r_i, \infty)$. This difference is driven by the fact that, due to resale possibility in our model, a bidder's ex post valuation of the auction outcome is not necessarily an increasing function of his rivals' types. To approximate a arbitrarily chosen bid as an alternative to the equilibrium bid, the reasoning of Athey and Reny and Zamir is that it does not hurt a bidder to bid slightly above a tying bid. Due to the problem of nonmonotonicity, I need to give the bidder the option to drop slightly below the typing bid, which requires the openness of the set of serious bids.

We can dispense with the assumption that the set of highest bidders is not announced, but doing so would need more cumbersome notations. The hazard rate assumption is not uncommon in the literature. It facilitates tractability of the endogenous resale mechanisms.

3 Preliminaries

Denote bidder i's use value, or *type*, by $\mathbf{t}_i$ as the random variable and t_i as the realized value. Denote $\mathbf{t}_{-i} := (\mathbf{t}_k)_{k \in I \setminus \{i\}}$ and $t_{-i} := (t_k)_{k \in I \setminus \{i\}}$ as the random vector and the realization for the type profile across rivals of i. Analogously, denote $\mathbf{t} := (\mathbf{t}_i, \mathbf{t}_{-i}) := (\mathbf{t}_k)_{k \in I}$, $t := (t_i, t_{-i}) := (t_k)_{k \in I}$, $\mathbf{t}_{\neg\{i,j\}} := (\mathbf{t}_k)_{k \in I \setminus \{i,j\}}$ and $t_{\neg\{i,j\}} := (t_k)_{k \in I \setminus \{i,j\}}$. Denote $\mathbb{E}[g(\mathbf{x})]$ for the expected value of any function g of the random variable or random vector $\mathbf{x}$, with the random variable/vector boldfaced, and $\mathbb{E}[g(\mathbf{x}) \mid E]$ for the expected value conditional on event E. For any set S, $|S|$ denotes the cardinality of S. If a subset S of T_i is empty, we let $\sup S := \inf S := \underline{t}_i$. Denote $\mathbf{1}_E$ for the indicator function for the event or statement denoted by E.

3.1 Monotone Bid Functions

For any bidder i, let $\beta_i : T_i \to B_i$ denote i's period-1 bid function. Throughout this paper we consider only weakly increasing bid functions. Let

$$\begin{aligned} \beta_i^{-1}(x) &:= \{t_i \in T_i : \beta_i(t_i) = x\}, \\ \beta_i^{-1}(\leq x) &:= \{t_i \in T_i : \beta_i(t_i) \leq x\}. \end{aligned}$$

By monotonicity of β_k, $\beta_k^{-1}(x)$ and $\beta_i^{-1}(\leq x)$ are intervals.

For any $b \in \mathbb{R}_+$ and any bidder i, we say that b is an *atom of* β_i if and only if $\beta_i^{-1}(b)$ is a nondegenerate interval, i.e., the cardinality of b's inverse image $\left|\beta_i^{-1}(b)\right| > 1$. A bid b is said to be *atom of* β_{-i} if and only if b is an atom of β_j for some $j \in I \setminus \{i\}$.

3.2 Posterior Beliefs and Virtual Utilities

The *public history* at the end of period one, commonly known, consists of the identity of the winner w and the winning bid b_w. In addition to the public history (w, b_w), each bidder i also privately knows, at the end of period one, his bid b_i in period one. The list $(w, b_w; b_i)$ is called i's *personal history*.

If bidder i is believed to have played according to bid function β_i and lost in the auction, then the posterior distribution of $\mathbf{t}_i$ conditional on any public history (w, b_w), denoted by $F_i(\cdot \mid w, b_w)$, is derived from Bayes's rule based on the observation that i has been defeated either because $\beta_i(\mathbf{t}_i) < b_w$ or because $\beta_i(\mathbf{t}_i) = b_w$ and i did not win the tie-breaking lottery. Note that the support of $F_i(\cdot \mid w, b_w)$ is $\beta_i^{-1}(\leq b_w)]$, called *posterior support*.

Lemma 1 *For any distinct bidders i and w and any $b_w \in B_w$, $\frac{1-F_i(t_i|w,b_w)}{f_i(t_i|w,b_w)}$ is weakly decreasing in t_i on its posterior support and is strictly increasing in* $\sup \beta_i^{-1}(\leq b_w)$.

Proof For any bidder $k \neq w$, if $\beta_k^{-1}(b_w) = \varnothing$ then let $c_k := d_k := \sup \beta_k^{-1}(\leq b_w)$; else let

$$c_k := \inf \beta_k^{-1}(b_w), \tag{1}$$

$$d_k := \sup \beta_k^{-1}(b_w) \; (= \sup \beta_k^{-1}(\leq b_w)). \tag{2}$$

Denote $\psi_i(w, b_w)$ for the probability of the event that bidder w, with bid b_w, wins the tie-breaking lottery conditional on tying with bidder i (and possibly with some others). Then

$$\psi_i(w, b_w) = \frac{1}{2} + \sum_{\varnothing \neq S \subseteq I \setminus \{i,w\}} \frac{1}{|S|+2} \left(\Pi_{k \in S}(F_k(d_k) - F_i(c_k))\right) \left(\Pi_{j \in I \setminus (S \cup \{i,w\})} F_j(c_j)\right). \tag{3}$$

Note that $\psi_i(w, b_w)$ is independent of (c_i, d_i, t_i). By Bayes's rule,

$$F_i(t_i \mid w, b_w) = \begin{cases} \frac{F_i(t_i)}{F_i(c_i)+(F_i(d_i)-F_i(c_i))\psi_i(w,b_w)} & \text{If } t_i \leq c_i \\ \frac{F_i(c_i)+(F_i(t_i)-F_i(c_i))\psi_i(w,b_w)}{F_i(c_i)+(F_i(d_i)-F_i(c_i))\psi_i(w,b_w)} & \text{If } c_i \leq t_i \leq d_i. \end{cases} \tag{4}$$

Then

$$\frac{1-F_i(t_i \mid w, b_w)}{f_i(t_i \mid w, b_w)} = \begin{cases} \frac{\psi_i(w,b_w)F_i(d_i)+(1-\psi_i(w,b_w))F_i(c_i)-F_i(t_i)}{f_i(t_i)} & \text{If } t_i \leq c_i \\ \frac{F_i(d_i)-F_i(t_i)}{f_i(t_i)} & \text{If } c_i \leq t_i \leq d_i. \end{cases} \tag{5}$$

Since F_i is assumed strictly increasing on its support and $\psi_i(w, b_w) > 0$, it follows that $\frac{1-F_i(t_i|w,b_w)}{f_i(t_i|w,b_w)}$ is strictly increasing in d_i, i.e., $\sup \beta_i^{-1}(\leq b_w)$. To prove that $\frac{1-F_i(t_i|w,b_w)}{f_i(t_i|w,b_w)}$ is decreasing in t_i, note that $\psi_i(w, b_w)F_i(d_i) + (1 - \psi_i(w, b_w))F_i(c_i)$ is less than or equal to one and if $t_i < c_i$ then $\psi_i(w, b_w)F_i(d_i) + (1 - \psi_i(w, b_w))F_i(c_i) > F_i(t_i)$. Thus, for any $t_i \neq c_i$ there is a constant $e \in [F_i(t_i), 1]$ such that

$$\frac{d}{dt_i}\left(\frac{1-F_i(t_i \mid w, b_w)}{f_i(t_i \mid w, b_w)}\right) = \frac{d}{dt_i}\left(\frac{e-F_i(t_i)}{f_i(t_i)}\right) = -1 - \frac{f_i'(t_i)}{f_i(t_i)^2}\left(e - F_i(t_i)\right).$$

If $f_i'(t_i) \geq 0$ then this derivative is negative, as desired. Suppose $f_i'(t_i) < 0$. Because $F_i(t_i) \leq e \leq 1$, $-f_i'(t_i)(e - F_i(t_i)) \leq -f_i'(t_i)(1 - F_i(t_i))$ and hence

$$\frac{d}{dt_i}\left(\frac{1-F_i(t_i \mid w, b_w)}{f_i(t_i \mid w, b_w)}\right) \leq \frac{d}{dt_i}\left(\frac{1-F_i(t_i)}{f_i(t_i)}\right),$$

which is nonpositive due to the hazard rate assumption of the priors. ■

Corollary 1 *For any distinct bidders i and w and any $b_w \in B_w$, the* posterior virtual utility *defined by*

$$V_{i,w,b_w}(t_i) := V_i(t_i \mid w, b_w) := t_i - \frac{1 - F_i(t_i \mid w, b_w)}{f_i(t_i \mid w, b_w)} \quad \left(\forall t_i \in \beta_i^{-1}(\leq b_w)\right) \tag{6}$$

is continuous in t_i and $\sup \beta_i^{-1}(\leq b_w)$, strictly increasing in t_i, and weakly decreasing in b_w.

Proof It follows directly from Eq. (5) that $V_{i,w,b_w}(t_i)$ is continuous in t_i and $\sup \beta_i^{-1}(\leq b_w)$; coupled with Lemma 1, Eq. (5) impies that $V_{i,w,b_w}(t_i)$ is strictly increasing in t_i. By Eq. (3), $\psi_i(w, b_w)$ is weakly increasing in b_w (through the d_k); by Eqs. (5)–(6), $V_{i,w,b_w}(t_i)$ is weakly decreasing in $\psi_i(w, b_w)$ because $F_i(d_i) \geq F_i(c_i)$. These combined with the fact that $V_{i,w,b_w}(t_i)$ is decreasing in $\sup \beta_i^{-1}(\leq b_w)$ and $\sup \beta_i^{-1}(\leq b_w)$ is weakly decreasing in b_w, imply that $V_{i,w,b_w}(t_i)$ is weakly decreasing in b_w. ■

3.3 Continuation Equilibria at Resale

Given any public history (w, b_w), winner w's optimal action in the continuation game is to offer resale through a mechanism that maximizes w's expected payoff based on the posterior distributions given by Eq. (4). By the strict monotonicity of the posterior virtual utilities (Corollary 1), Myerson's [11] complete characterization of optimal auctions implies

Lemma 2 *Given any history (w, b_w), if the winner w believes that every bidder $k \neq w$ followed a weakly increasing bid function β_k in period one, then a resale allocation can be supported as a continuation equilibrium conditional on (w, b_w) if and only if it satisfies—*

a. for any $t_w \in T_w$ and for almost every $t_{-w} \in T_{-w}$, if

$$\exists i \neq w : V_i(t_i \mid w, b_w) \geq \max\left\{t_w, \max_{k \notin \{w,i\}} V_k(t_k \mid w, b_w)\right\} \tag{7}$$

then bidder i is the final owner of the good; else player w is the final owner; and—

b. for any $i \neq w$, the type-$\underline{t}_i$ bidder i gets zero expected payoff.

Provisions (a) and (b) of Lemma 2 determine via the envelope formula a class of payment schemes, all payoff-equivalent, that implement the optimal allocation. One of such payment rules (constructed by Myerson) is that bidder i's payment given $t \in T$ is equal to

$$p_{i,w,b_w}(t) = \inf\left\{t_i \in \beta_i^{-1}(\leq b_w) : V_{i,w,b_w}(t_i) \geq \max\left\{t_w, \max_{k \in I \setminus \{i,w\}} V_{k,w,b_w}(t_k)\right\}\right\} \tag{8}$$

if i wins at resale and is equal to zero if otherwise.

Lemma 2 provides a sufficient and necessary condition for a continuation equilibrium at resale. Moreover, it implies that a continuation equilibrium at resale exists and is effectively unique (unique allocation almost surely and unique payment scheme in expected payoffs), hence we shall refer to *the* continuation equilibrium in the sequel. Thus,

Lemma 3 *If a profile $(\beta_i)_{i\in I}$ of weakly increasing period-1 bid functions constitutes a Bayesian Nash equilibrium (BNE) provided that the continuation play conditional on any public history (w, b_w) satisfies conditions (a) and (b) in Lemma 2, then $(\beta_i)_{i\in I}$ coupled with the continuation play constitutes a perfect Bayesian equilibrium (PBE) of the auction-resale game.*

Therefore, existence of PBE of the dynamic game becomes existence of BNE in a static game where bidders' expected payoffs are *endogenously derived* from the continuation equilibrium characterized in Lemma 2. To analyze these endogenous expected payoffs, we establish in the next section properties of any continuation equilibrium at the resale stage.

4 The Strategy of the Proof

The building blocks for the fixed approach; problems due to resale possibilities; and solutions. Next version.

5 Comparative Statics

5.1 Initial Bids and Resale Acquisition

Fix a profile $(\beta_i)_{i\in I}$ of weakly increasing bid functions. For any $i \in I$ and any $b_i \in B_i$, define

$$\tau_{-i}(b_i) := \{t_{-i} \in T_{-i} : \forall j \in I \setminus \{i\} \, [b_i \geq \beta_j(t_j)]\} \, . \tag{9}$$

For any bidder i, any $b_i \in B_i$ and any $t := (t_i, t_{-i}) \in T$ such that $t_{-i} \in \tau_{-i}(b_i)$, define $Q_i(b_i, t)$ to be the probability with which i is the final owner of the good in the continuation equilibrium (à la Lemma 2) conditional on the history (i, b_i)—that i won with bid b_i—provided that everyone else abides by the period-1 bid functions β_{-i}; if $t_{-i} \notin \tau_{-i}(b_i)$ then define $Q_i(b_i, t) := 0$. By Lemma 2.a, for any b_i, any $t_i \in T_i$ and for almost every $t_{-i} \in \tau_{-i}(b_i)$,

$$Q_i(b_i, t) = \mathbf{1}_{t_i \geq \max_{k\in I\setminus\{i\}} V_k(t_k | i, b_i)}. \tag{10}$$

For any $t := (t_i, t_{-i}) \in T$ and any $j \in I \setminus \{i\}$, define $q_{ij}(b_i, t)$ to be the probability with which i is the final owner of the good in the continuation equilibrium conditional on the personal history $(j, \beta_j(t_j); b_i)$,[1] provided that everyone else abided by β_{-i}. Lemma 2.a implies that, for any b_i, any $t_i \in T_{-i}$ and almost every t_{-i},

$$q_{ij}(b_i, t) = \mathbf{1}_{V_i(t_i|j,\beta_j(t_j)) \geq \max\left\{t_j, \max_{k \in I \setminus \{i,j\}} V_k(t_k|j,\beta_j(t_j))\right\}} =: q_{ij}(t), \tag{11}$$

where the second equality follows from the first equality, which implies that $q_{ij}(b_i, t)$ is independent of bidder i's (privately known) period-1 bid b_i.

Proposition 1 (learning effect) *For any bidder i, any $b_i \in B_i$, and any $t \in T$, if every bidder $k \neq i$ abides by a weakly increasing bid function β_k and if $b_i'' > b_i'$, then for almost every $t_{-i} \in T_{-i}$, $Q_i(b_i'', t) \geq Q_i(b_i', t)$.*

Proof Let $b_i'' > b_j'$. If $Q_i(b_i', t) = 0$ then $Q_i(b_i'', t) \geq Q_i(b_i', t)$. Suppose $Q_i(b_i', t) > 0$. Then, by definition of Q_i, $t_{-i} \in \tau_{-i}(b_i')$; by Eq. (9), for each $k \neq i$, $\beta_k(t_k) \leq b_i' < b_i''$, so $t_{-i} \in \tau_{-i}(b_i'')$. Thus, for each $b_i \in \{b_i', b_i''\}$ Eq. (10) holds for almost every $t_{-i} \in \tau_{-i}(b_i)$. Hence there is no loss of generality to assume Eq. (10) for this $t_{-i} \in \tau_{-i}(b_i)$. Then $Q_i(b_i', t) > 0$ implies

$$k \in I \setminus \{i\} : t_i \geq V_k(t_k \mid i, b_i').$$

Since $V_k(t_k \mid i, b_i)$ is weakly decreasing in b_i (Corollary 1),

$$k \in I \setminus \{i\} : V_k(t_k \mid i, b_i') \geq V_k(t_k \mid i, b_i'').$$

These two inequalities combined, Eq. (10) implies $Q_i(b_i'', t) = 1 \geq Q_i(b_i', t)$. ■

Proposition 2 (monopoly effect) *For any bidder i, if every bidder $k \neq i$ abides by a weakly increasing bid function β_k, then for any bidder $j \neq i$, any $b_i \in B_i$ such that for any $k \in I \setminus \{i\}$, b_i is not an atom of β_k, and for almost every $t_{-i} \in \tau_{-i}(b_i)$, $Q_i(b_i, t) \geq q_{ij}(t)$.*

Proof Pick any $t_{-i} \in \tau_{-i}(b_i)$. If $q_{ij}(t) = 0$ then the conclusion is vacuously true. Hence suppose that $q_{ij} > 0$ and so Eq. (11) implies

$$V_i(t_i \mid j, \beta_j(t_j)) \geq \max\left\{t_j, \max_{k \in I \setminus \{i,j\}} V_k(t_k \mid j, \beta_j(t_j))\right\}.$$

[1] I.e., bidder i has bid b_i and lost (possibly due to the tie-breaking randomization) in period one while bidder j has won with winning bid $\beta_j(t_j)$.

By Eqs. (10), it suffices to prove

$$t_i \geq \max_{k\in I\setminus\{i\}} V_k(t_k \mid i, b_i).$$

By Eq. (6), $t_i \geq V_i(t_i \mid j, \beta_j(t_j))$ and $t_j \geq V_j(t_j \mid i, b_i)$. Thus, we need only to prove

$$\forall k \in I \setminus \{i, j\} : V_k(t_k \mid j, \beta_j(t_j)) \geq V_k(t_k \mid i, b_i). \tag{12}$$

To this end, pick any $k \notin \{i, j\}$. By the hypothesis of the lemma, b_i is not an atom of β_k, hence Eqs. (5) and (6) imply that

$$V_k(t_k \mid i, b_i) = t_k - \frac{F_k(\sup \beta_k^{-1}(\leq b_i)) - F_k(t_k)}{f_k(t_k)}.$$

Eqs. (5) and (6) also imply

$$V_k(t_k \mid j, \beta_j(t_j)) = \begin{cases} t_k - \frac{\psi_k(j,\beta_j(t_j))F_k(d_k)+(1-\psi_k(j,\beta_j(t_j)))F_k(c_k)-F_k(t_k)}{f_k(t_k)} & \text{If } t_k \leq c_k \\ t_k - \frac{F_k(d_k)-F_k(t_k)}{f_k(t_k)} & \text{If } c_k \leq t_k \leq d_k, \end{cases} \tag{13}$$

where $c_k = \inf \beta_k^{-1}(\beta_j(t_j))$ and $d_k = \sup \beta_k^{-1}(\beta_j(t_j))$ if $\beta_k^{-1}(\beta_j(t_j)) \neq \varnothing$, else $c_k = d_k = \inf \beta_k^{-1}(\leq \beta_j(t_j))$. Since $t_{-i} \in \tau_{-i}(b_i)$, $\beta_j(t_j) \leq b_i$, hence the monotonicity of β_k implies that $\sup \beta_k^{-1}(\leq b_i)) \geq d_k$. Thus, the two equations displayed above imply (12), as desired. ■

Remark: While not needed in the sequel, generalization of the conclusion of Proposition 2 can be done except for the case where $b_i = \beta_j(t_j)$ is an atom of both β_i and β_k for some $k \in I \setminus \{i, j\}$. In that case, I have not ruled out the possibility that $\psi_k(i, b_i) < \psi_k(j, \beta_j(t_j))$ for some $t_k < c_k$ and hence possibly $V_k(t_k \mid i, b_i) > V_k(t_k \mid j, \beta_j(t_j))$ for such t_k.

5.2 Envelope Conditions of the Payoffs for Winners and Losers

For any bidder i, suppose that everyone else abides by a profile β_{-i} of weakly increasing period-1 bid functions. For any $b_i \in B_i$ and any type profile $t_{-i} \in T_{-i}$, let $j = h_i(b_i, t_{-i})$ denote the event that bidder j turns out to be the winner in the period-one auction, either because he is the only highest bidder or because he ties with some rival(s) and wins the tie-breaking lottery; also denote $j = h_i(t_{-i})$ for the event that j is the winner conditional on i's not being the winner (possibly through tie-breaking lotteries). Formally, denote

$$j = h_i(b_i, t_{-i}) \quad \Leftrightarrow \quad i\text{'s personal history is } \begin{cases} (j, \beta_j(t_j); b_i) & \text{if } j \neq i \\ (i, b_i; b_i) & \text{if } j = i; \end{cases} \tag{14}$$

$$j = h_i(t_{-i}) \quad \Leftrightarrow \quad \begin{bmatrix} j \in \arg\max_{j\neq i} \beta_j(t_j) \\ |\arg\max_{j\neq i} \beta_j(t_j)| > 1 \Rightarrow j \text{ wins the tie-breaking lottery.} \end{bmatrix} \tag{15}$$

Since we assume the uniform tie-breaking rule, the realized outcome of tie-breaking lotteries is stochastically independent of bidder-types and hence is omitted in the notation $h_i(b_i, t_{-i})$.

For any $t_i \in T_i$, any $t_{-i} \in T_{-i}$ with $t := (t_i, t_{-i})$, and any bidder $j \neq i$, denote:

- $W_i(b_i, t) :=$ type-t_i bidder i's expected payoff in the continuation equilibrium conditional on the personal history $(i, b_i; b_i)$ (which implies $i = h_i(b_i, t_{-i})$), provided that everyone else abides by β_{-i};

- $L_{ij}(b_i, t) :=$ type-t_i bidder i's expected payoff in the continuation equilibrium given the personal history $(j, \beta_j(t_j); b_i)$ (which implies $j = h_i(b_i, t_{-i})$), provided that everyone else abides by β_{-i}.

Lemma 4 *For any distinct $i, j \in I$ and any $t \in T$, $L_{ij}(b_i, t)$ is constant to all b_i and hence is denoted $L_{ij}(t)$ in the sequel.*

Proof Conditional on any public history $(j, \beta_j(t_j))$, the resale mechanism at the continuation equilibrium, characterized in Lemma 2, is determined by the reseller w's identity and its winning bid $\beta_j(t_j)$ and is independent of bidder i's privately known bid b_i. Given realized type profile t, i's expected payoff in the resale mechanism is uniquely determined. ■

Define:

$$\begin{aligned}
\overline{W}_i(b_i, t_i) &:= \mathbb{E}\left[W_i(b_i, t_i, \mathbf{t}_{-i}) \mid i = h_i(b_i, \mathbf{t}_{-i})\right], && (16)\\
\overline{L}_{ij}(b_i, t_i) &:= \mathbb{E}\left[L_{ij}(t_i, \mathbf{t}_{-i}) \mid j = h_i(b_i, \mathbf{t}_{-i})\right], && (17)\\
\overline{Q}_i(b_i, t) &:= \mathbb{E}\left[Q_i(b_i, t_i, \mathbf{t}_{-i}) \mid i = h_i(b_i, \mathbf{t}_{-i})\right],\\
\overline{q}_{ij}(b_i, t_i) &:= \mathbb{E}\left[q_{ij}(t_i, \mathbf{t}_{-i}) \mid j = h_i(b_i, \mathbf{t}_{-i})\right].
\end{aligned}$$

Lemma 5 *For any bidder i, any $b_i \in B_i$ and any $j \neq i$, $\overline{W}_i(b_i, \cdot)$ and $\overline{L}_{ij}(b_i, \cdot)$ are absolutely continuous and for any $t_i \in T_i$,*

$$\overline{W}_i(b_i, t_i) = \overline{W}_i(b_i, \underline{t}_i) + \int_{\underline{t}_i}^{t_i} \overline{Q}_i(b_i, \tau_i) d\tau_i, \tag{18}$$

$$\overline{L}_{ij}(b_i, t_i) = \int_{\underline{t}_i}^{t_i} \overline{q}_{ij}(b_i, \tau_i) d\tau_i. \tag{19}$$

Proof Eq. (18) follows directly from the incentive compatibility of player i in the continuation equilibrium conditional on the event that i is the period-1 winner.[2] Analogously, in the event where i loses and j wins, upon which i knows j's period-1 bid $\beta_j(t_j)$ and hence j's type t_j, we have

$$\overline{L}_{ij}(b_i, t_i \mid t_j) = \overline{L}_{ij}(b_i, \underline{t}_i \mid t_j) + \int_{\underline{t}_i}^{t_i} \overline{q}_{ij}(b_i, \tau_i \mid t_j) d\tau_i,$$

where

$$\begin{aligned}
\overline{L}_{ij}(b_i, t_i \mid t_j) &:= \mathbb{E}\left[L_{ij}(t_i, t_j, \mathbf{t}_{\neg\{i,j\}}) \mid j = h_i(b_i, \mathbf{t}_{-i})\right], \\
\overline{q}_{ij}(b_i, t_i \mid t_j) &:= \mathbb{E}\left[q_{ij}(t_i, , \mathbf{t}_{\neg\{i,j\}}) \mid j = h_i(b_i, \mathbf{t}_{-i})\right].
\end{aligned}$$

Since the resale mechanism in the continuation game is chosen by the reseller, the winner j, for the mechanism to be optimal for the reseller, we have $\overline{L}_{ij}(b_i, \underline{t}_i \mid t_j) = 0$. Hence

$$\overline{L}_{ij}(b_i, t_i \mid t_j) = \int_{\underline{t}_i}^{t_i} \overline{q}_{ij}(b_i, \tau_i \mid t_j) d\tau_i.$$

Integrating this equation across $t_j \in T_j$ gives Eq. (19). ■

Denote

$$L_i(t) := \sum_{j \neq i} \mathbf{1}_{j=h_i(t_{-i})} L_{ij}(t), \tag{20}$$

$$q_i(t) := \sum_{j \neq i} \mathbf{1}_{j=h_i(t_{-i})} q_{ij}(t). \tag{21}$$

5.3 The Single Crossing Property

Denote $U_i(b_i, t_i; \beta_{-i})$ for type-t_i bidder i's expected payoff from bidding b_i in period one, provided that everyone else abides by a profile β_{-i} of weakly increasing period-1 bid functions and the continuation play constitutes a continuation equilibrium. With the notations W_i

[2] In choosing a resale mechanism, the winner-turned reseller player w effectively inputs an alleged type $\hat{t}_w$ into the formula in Lemma 2 that outputs a mechanism optimal for $\hat{t}_w$. Then w's expected probability of being the final owner (no resale) is $\overline{Q}_w(b_w, \hat{t}_w)$, and the expected revenue w receives is $R_w(b_w, \hat{t}_w)$. Then w's expected payoff in period 2 is $t_w \overline{Q}_w(b_w, \hat{t}_w) + R_w(b_w, \hat{t}_w)$. Picking the optimal resale mechanism means setting $\hat{t}_i = t_i$. Then (18) follows from the envelope theorem of Milgrom and Segal [9].

and L_i defined in §5.2, we have

$$U_i(b_i, t_i; \beta_{-i}) = \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i,\mathbf{t}_{-i})}\left(W_i(b_i, t_i, \mathbf{t}_{-i}) - b_i\right) + \sum_{j\neq i} \mathbf{1}_{j=h_i(b_i,\mathbf{t}_{-i})} L_{ij}(t_i, \mathbf{t}_{-i})\right] \quad (22)$$

$$= \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i,\mathbf{t}_{-i})}\right]\left(\overline{W}_i(b_i, t_i) - b_i\right) + \sum_{j\neq i} \mathbb{E}\left[\mathbf{1}_{j=h_i(b_i,\mathbf{t}_{-i})}\right] \overline{L}_{ij}(b_i, t_i), \quad (23)$$

where the second line follows from Eqs. (16) and (17).

The strict single crossing condition in the literature is that if $b''_i \geq b'_i$ and $t''_i \geq t'_i$ then

$$U_i(b''_i, t'_i; \beta_{-i}) \geq (\text{resp.} >)\, U_i(b'_i, t'_i; \beta_{-i}) \Rightarrow U_i(b''_i, t''_i; \beta_{-i}) \geq (\text{resp.} >)\, U_i(b'_i, t''_i; \beta_{-i}).$$

The next proposition claims a stronger conclusion, increasing difference, provided that the higher bid does not tie with rivals' bids with strictly positive probabilities.

Proposition 3 (single crossing) *For any bidder i, any profile β_{-i} of weakly increasing bid functions, and any $b'_i, b''_i \in B_i$ such that $b''_i \geq b'_i$, if for any $k \neq i$, b''_i is not an atom of β_k, then for any $t'_i, t''_i \in T_i$ such that $t''_i \geq t'_i$,*

$$U_i(b''_i, t''_i; \beta_{-i}) - U_i(b'_i, t''_i; \beta_{-i}) \geq U_i(b''_i, t'_i; \beta_{-i}) - U_i(b'_i, t'_i; \beta_{-i}). \quad (24)$$

Proof Applying the envelope equations (18) and (19) to Eq. (23), we have

$$\begin{aligned}
\frac{\partial}{\partial t_i} U_i(b_i, t_i; \beta_{-i}) &= \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i,\mathbf{t}_{-i})}\right] \overline{Q}_i(b_i, t_i) + \sum_{j\neq i} \mathbb{E}\left[\mathbf{1}_{j=h_i(b_i,\mathbf{t}_{-i})}\right] \overline{q}_{ij}(b_i, t_i) \\
&= \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i,\mathbf{t}_{-i})}\right] \mathbb{E}\left[Q_i(b_i, t_i, \mathbf{t}_{-i}) \mid i = h_i(b_i, \mathbf{t}_{-i})\right] \\
&\quad + \sum_{j\neq i} \mathbb{E}\left[\mathbf{1}_{j=h_i(b_i,\mathbf{t}_{-i})}\right] \mathbb{E}\left[q_{ij}(t_i, \mathbf{t}_{-i}) \mid j = h_i(b_i, \mathbf{t}_{-i})\right] \\
&= \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i,\mathbf{t}_{-i})} Q_i(b_i, t_i, \mathbf{t}_{-i}) + \sum_{j\neq i} \mathbf{1}_{j=h_i(b_i,\mathbf{t}_{-i})} q_{ij}(t_i, \mathbf{t}_{-i})\right].
\end{aligned}$$

If $j \neq i$ then

$$\mathbf{1}_{j=h_i(b_i,\mathbf{t}_{-i})} = \mathbf{1}_{i\neq h_i(b_i,\mathbf{t}_{-i})} \mathbf{1}_{j=h_i(\mathbf{t}_{-i})}, \quad (25)$$

with the notation $h_i(\mathbf{t}_{-i})$ defined in (15). Thus, the above calculation gives us

$$\begin{aligned}
\frac{\partial}{\partial t_i} U_i(b_i, t_i; \beta_{-i}) &= \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i,\mathbf{t}_{-i})} Q_i(b_i, t_i, \mathbf{t}_{-i}) + \sum_{j\neq i} \mathbf{1}_{i\neq h_i(b_i,\mathbf{t}_{-i})} \mathbf{1}_{j=h_i(\mathbf{t}_{-i})} q_{ij}(t_i, \mathbf{t}_{-i})\right]. \\
&\overset{(21)}{=} \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i,\mathbf{t}_{-i})} Q_i(b_i, t_i, \mathbf{t}_{-i}) + \mathbf{1}_{i\neq h_i(b_i,\mathbf{t}_{-i})} q_i(t_i, \mathbf{t}_{-i})\right].
\end{aligned}$$

For any $b_i'' > b_i'$, the previous equation implies

$$\begin{aligned}\frac{\partial}{\partial t_i}U_i(b_i'', t_i; \beta_{-i}) - \frac{\partial}{\partial t_i}U_i(b_i', t_i; \beta_{-i}) &= \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i'',\mathbf{t}_{-i})}Q_i(b_i'', t_i, \mathbf{t}_{-i}) - \mathbf{1}_{i=h_i(b_i',\mathbf{t}_{-i})}Q_i(b_i', t_i, \mathbf{t}_{-i})\right] \\ &\quad + \mathbb{E}\left[\mathbf{1}_{i\neq h_i(b_i'',\mathbf{t}_{-i})}q_i(t_i, \mathbf{t}_{-i}) - \mathbf{1}_{i\neq h_i(b_i',\mathbf{t}_{-i})}q_i(t_i, \mathbf{t}_{-i})\right].\end{aligned}$$

Since $b_i'' > b_i'$, $i = h_i(b_i', t_{-i}) \Rightarrow \forall k \neq i\,[\beta_k(t_k) \leq b_i' < b_i''] \Rightarrow i = h_i(b_i'', t_{-i})$. Thus,

$$\begin{aligned}\frac{\partial}{\partial t_i}U_i(b_i'', t_i; \beta_{-i}) - \frac{\partial}{\partial t_i}U_i(b_i', t_i; \beta_{-i}) &= \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i',\mathbf{t}_{-i})}\left(Q_i(b_i'', t_i, \mathbf{t}_{-i}) - Q_i(b_i', t_i, \mathbf{t}_{-i})\right)\right] \\ &\quad + \mathbb{E}\left[\mathbf{1}_{h_i(b_i',\mathbf{t}_{-i})\neq i=h_i(b_i'',\mathbf{t}_{-i})}\left(Q_i(b_i'', t_i, \mathbf{t}_{-i}) - q_i(t_i, \mathbf{t}_{-i})\right)\right] \\ &\geq \mathbb{E}\left[\mathbf{1}_{h_i(b_i',\mathbf{t}_{-i})\neq i=h_i(b_i'',\mathbf{t}_{-i})}\left(Q_i(b_i'', t_i, \mathbf{t}_{-i}) - q_i(t_i, \mathbf{t}_{-i})\right)\right],\end{aligned}$$

where the inequality follows from Proposition 1. By Eq. (21),

$$\begin{aligned}Q_i(b_i'', t_i, \mathbf{t}_{-i}) - q_i(t_i, \mathbf{t}_{-i}) &= Q_i(b_i'', t_i, \mathbf{t}_{-i}) - \sum_{j\neq i}\mathbf{1}_{j=h_i(t_{-i})}q_{ij}(t_i, \mathbf{t}_{-i}) \\ &= \sum_{j\neq i}\mathbf{1}_{j=h_i(t_{-i})}\left(Q_i(b_i'', t_i, \mathbf{t}_{-i}) - q_{ij}(t_i, \mathbf{t}_{-i})\right),\end{aligned}$$

where the second line is due to the fact that the events $j = h_i(\mathbf{t}_{-i})$ and $j' = h_i(\mathbf{t}_{-i})$ are mutually exclusive for any $j \neq j'$. By Proposition 2 and the hypothesis of this proposition that b_i'' is not an atom of β_{-i}, $Q_i(b_i'', t_i, \mathbf{t}_{-i}) - q_{ij}(t_i, \mathbf{t}_{-i}) \geq 0$ for each $j \neq i$; thus,

$$\frac{\partial}{\partial t_i}U_i(b_i'', t_i; \beta_{-i}) - \frac{\partial}{\partial t_i}U_i(b_i', t_i; \beta_{-i}) \geq 0. \tag{26}$$

Note that $U_i(b_i'', \cdot) - U_i(b_i', \cdot)$ is an absolutely continuous function, as Eq. (23) says that, given any b_i, $U_i(b_i, \cdot)$ is equal to the sum of functions $\overline{W}_i(b_i, \cdot)$ and $\overline{L}_{ij}(b_i, \cdot)$, each absolutely continuous by Lemma 5. Thus, apply Ineq. (26) to $U_i(b_i'', \cdot) - U_i(b_i', \cdot)$ and we obtain (24). ∎

5.4 Estimating the Marginal Payoff from Winning

For any bidder i, any profile β_{-i} of bid functions, and any $x, y \in B_i$ with $x \leq y$, denote

$$E(x, y) := \{t_{-i} \in T_{-i} : \exists j \neq i\,[x \leq \beta_j(t_j) \leq y; \forall k \notin \{i, j\}\,[\beta_k(t_k) \leq y]]\}, \tag{27}$$

i.e., the set of type profiles across i's rivals such that bidder i's status switches from losing to winning in the auction when i's bid switches from x to y. The next lemma says that $E(x, y)$ is essentially the shorthand for the event $h_i(x, t_{-i}) \neq i = h_i(y, t_{-i})$.

Lemma 6 *For any bidder i, $x, y \in B_i$, $t_{-i} \in E(x, y)$ if and only if there exist realized outcomes ω, ω' for tie-breaking lotteries such that $h_i(x, t_{-i}) \neq i$ at ω and $h_i(y, t_{-i}) = i$ at ω'.*

Proof Suppose that $t_{-i} \in E(x, y)$. By Eq. (27), in bidding x, bidder i is not the unique highest bidder and hence i is either outbid by some $j \neq i$ or, if i ties with some rivals, there exists a tie-breaking realized outcome such that i loses in the lottery. By contrast, in bidding$y$, bidder i is one of the highest bidders by Eq. (27), hence either i outbids the rivals or there exists a tie-breaking realized outcome at which i wins in the lottery. Thus, the "only if" assertion holds. To prove the converse, suppose that there exist realized outcomes ω, ω' for tie-breaking lotteries such that $h_i(x, t_{-i}) \neq i$ at ω and $h_i(y, t_{-i}) = i$ at ω'. Then for some $j \neq i$ $\beta_j(t_j)$ is equal to the highest bid when i bids x, and y is the highest bid when i bids y. Thus, $\beta_j(t_j) \geq x$ and $\beta_k(t_k) \leq y$ for all $k \neq i$. Hence $t_{-i} \in E(x, y)$ by (27). ■

Denote

$$\overline{W}_i(b_i, t_i \mid x, y) := \mathbb{E}\left[W_i(b_i, t_i, \mathbf{t}_{-i}) \mid \mathbf{t}_{-i} \in E(x, y)\right], \tag{28}$$

$$\overline{L}_{ij}(t_i \mid x, y) := \mathbb{E}\left[L_{ij}(t_i, \mathbf{t}_{-i}) \mid \mathbf{t}_{-i} \in E(x, y)\right], \tag{29}$$

$$\Delta_i(b_i, t_i \mid x, y) := \mathbb{E}\left[W_i(b_i, t_i, \mathbf{t}_{-i}) - \sum_{j \neq i} \mathbf{1}_{\mathbf{1}_{j=h_i(\mathbf{t}_{-i})}} L_{ij}(t_i, \mathbf{t}_{-i}) \middle| E(x, y)\right] - b_i. \tag{30}$$

Hence $\Delta_i(b_i, t_i \mid x, y)$ is the expected value of the net gain from winning in the auction with bid b_i conditional on the rivals' types being in $E(x, y)$ (though in picking a resale mechanism upon winning, i does not know that event $E(x, y)$ occurs).

Proposition 4 *For any bidder i, any type $t_i \in T_i$, any profile β_{-i} of weakly increasing bid functions, any $b_i, b_i', b_i'' \in B_i$, if $b_i' < b_i < b_i''$ then (with the notation $b_i - l := b_i$)*

$$U_i(b_i'', t_i; \beta_{-i}) - U_i(b_i, t_i; \beta_{-i}) \geq \mathbb{E}\left[\mathbf{1}_{E(b_i, b_i'')}\right] \Delta_i(b_i, t_i \mid b_i, b_i'') - (b_i'' - b_i)\mathbb{E}\left[\mathbf{1}_{i=h_i(b_i'', \mathbf{t}_{-i})}\right] \tag{31}$$

$$U_i(b_i', t_i; \beta_{-i}) - U_i(b_i, t_i; \beta_{-i}) \geq -\mathbb{E}\left[\mathbf{1}_{E(b_i', b_i)}\right] \Delta_i(b_i, t_i \mid b_i', b_i) + (b_i - b_i')\mathbb{E}\left[\mathbf{1}_{i=h_i(b_i', \mathbf{t}_{-i})}\right] \tag{32}$$

Proof By Eq. (25) and with the notation defined in Eq. (20), we have

$$\sum_{j \neq i} \mathbf{1}_{j=h_i(x, \mathbf{t}_{-i})} L_{ij}(t_i, \mathbf{t}_{-i}) = \mathbf{1}_{i \neq h_i(x, \mathbf{t}_{-i})} L_i(t_i, \mathbf{t}_{-i}).$$

Hence Eq. (22) becomes

$$U_i(b_i, t_i; \beta_{-i}) = \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i, \mathbf{t}_{-i})}\left(W_i(b_i, t_i, \mathbf{t}_{-i}) - b_i\right) + \mathbf{1}_{i \neq h_i(b_i, \mathbf{t}_{-i})} L_i(t_i, \mathbf{t}_{-i})\right].$$

Thus,

$$\begin{aligned}
U_i(b_i'', t_i; \beta_{-i}) - U_i(b_i, t_i; \beta_{-i}) &= \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i'',\mathbf{t}_{-i})}\left(W_i(b_i'', t_i, \mathbf{t}_{-i}) - b_i''\right) + \mathbf{1}_{i\neq h_i(b_i'',\mathbf{t}_{-i})}L_i(t_i, \mathbf{t}_{-i})\right] \\
&- \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i,\mathbf{t}_{-i})}\left(W_i(b_i, t_i, \mathbf{t}_{-i}) - b_i\right) + \mathbf{1}_{i\neq h_i(b_i,\mathbf{t}_{-i})}L_i(t_i, \mathbf{t}_{-i})\right].
\end{aligned}$$

Noting the fact $i = h_i(b_i, t_{-i}) \Rightarrow i = h_i(b_i'', t_{-i})$, we recombine the the previous terms.

$$\begin{aligned}
& U_i(b_i'', t_i; \beta_{-i}) - U_i(b_i, t_i; \beta_{-i}) \\
= \; & \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i'',\mathbf{t}_{-i})}W_i(b_i'', t_i, \mathbf{t}_{-i})\right] - \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i,\mathbf{t}_{-i})}W_i(b_i, t_i, \mathbf{t}_{-i})\right] \\
& -\mathbb{E}\left[\mathbf{1}_{h_i(b_i,\mathbf{t}_{-i})\neq i=h_i(b_i'',\mathbf{t}_{-i})}L_i(t_i, \mathbf{t}_{-i}) + b_i\right] - (b_i'' - b_i)\,\mathbb{E}\left[\mathbf{1}_{i=h_i(b_i'',\mathbf{t}_{-i})}\right].
\end{aligned} \tag{33}$$

To prove (31), note that

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{i=h_i(b_i'',\mathbf{t}_{-i})}W_i(b_i'', t_i, \mathbf{t}_{-i})\right] &= \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i'',\mathbf{t}_{-i})}\right]\mathbb{E}\left[W_i(b_i'', t_i, \mathbf{t}_{-i}) \mid i = h_i(b_i'', \mathbf{t}_{-i})\right] \\
&\geq \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i'',\mathbf{t}_{-i})}\right]\mathbb{E}\left[W_i(b_i, t_i, \mathbf{t}_{-i}) \mid i = h_i(b_i'', \mathbf{t}_{-i})\right] \qquad (34) \\
&= \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i'',\mathbf{t}_{-i})}W_i(b_i, t_i, \mathbf{t}_{-i})\right],
\end{aligned}$$

where the inequality is due to the fact that $\mathbb{E}\left[W_i(b_i'', t_i, \mathbf{t}_{-i}) \mid i = h_i(b_i'', \mathbf{t}_{-i})\right]$ is equal to i's expected payoff in the resale mechanism that maximizes his expected payoff conditional on the event $i = h_i(b_i'', t_{-i})$. It therefore follows from Eq. (33) that

$$\begin{aligned}
& U_i(b_i'', t_i; \beta_{-i}) - U_i(b_i, t_i; \beta_{-i}) \\
\geq \; & \mathbb{E}\left[\mathbf{1}_{h_i(b_i,\mathbf{t}_{-i})\neq i=h_i(b_i'',\mathbf{t}_{-i})}\left(W_i(b_i, t_i, \mathbf{t}_{-i}) - L_i(t_i, \mathbf{t}_{-i}) - b_i\right)\right] - (b_i'' - b_i)\,\mathbb{E}\left[\mathbf{1}_{i=h_i(b_i'',\mathbf{t}_{-i})}\right] \\
= \; & \mathbb{E}\left[\mathbf{1}_{h_i(b_i,\mathbf{t}_{-i})\neq i=h_i(b_i'',\mathbf{t}_{-i})}\right]\Delta_i(b_i, t_i \mid b_i, b_i'') - (b_i'' - b_i)\,\mathbb{E}\left[\mathbf{1}_{i=h_i(b_i'',\mathbf{t}_{-i})}\right],
\end{aligned}$$

where the equality follows from Eq. (30) and Lemma 6. Hence (31) is proved.

To prove (32), substitute (b_i, b_i') for the (b_i'', b_i) in Eq. (33) and we have

$$\begin{aligned}
& U_i(b_i', t_i; \beta_{-i}) - U_i(b_i, t_i; \beta_{-i}) \\
= \; & -\mathbb{E}\left[\mathbf{1}_{i=h_i(b_i,\mathbf{t}_{-i})}W_i(b_i, t_i, \mathbf{t}_{-i})\right] + \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i',\mathbf{t}_{-i})}W_i(b_i', t_i, \mathbf{t}_{-i})\right] \\
& +\mathbb{E}\left[\mathbf{1}_{h_i(b_i',\mathbf{t}_{-i})\neq i=h_i(b_i,\mathbf{t}_{-i})}\left(L_i(t_i, \mathbf{t}_{-i}) + b_i'\right)\right] + (b_i - b_i')\,\mathbb{E}\left[\mathbf{1}_{i=h_i(b_i,\mathbf{t}_{-i})}\right].
\end{aligned}$$

Analogous to (34), a revealed-preference argument from a reseller's standpoint,

$$\mathbb{E}\left[\mathbf{1}_{i=h_i(b_i',\mathbf{t}_{-i})}W_i(b_i', t_i, \mathbf{t}_{-i})\right] \geq \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i',\mathbf{t}_{-i})}W_i(b_i, t_i, \mathbf{t}_{-i})\right].$$

Thus, the previous equation implies

$$\begin{aligned} & U_i(b_i', t_i; \beta_{-i}) - U_i(b_i, t_i; \beta_{-i}) \\ \geq\ & -\mathbb{E}\left[\mathbf{1}_{i=h_i(b_i,\mathbf{t}_{-i})} W_i(b_i, t_i, \mathbf{t}_{-i})\right] + \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i',\mathbf{t}_{-i})} W_i(b_i, t_i, \mathbf{t}_{-i})\right] \\ & + \mathbb{E}\left[\mathbf{1}_{h_i(b_i',\mathbf{t}_{-i}) \neq i = h_i(b_i,\mathbf{t}_{-i})} \left(L_i(t_i, \mathbf{t}_{-i}) + b_i'\right)\right] + (b_i - b_i')\, \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i,\mathbf{t}_{-i})}\right]. \end{aligned}$$

Hence (32) follows from Eq. (30) and Lemma 6. ■

5.5 Continuity at Nonatomic Bids

For any bidder i and any $x \in \mathbb{R}_+$ such that $\left|\beta_i^{-1}(x)\right| \leq 1$, define:

$$\varphi_i(x) := \sup \beta_i^{-1}(\leq x). \tag{35}$$

Lemma 7 *If $\left|\beta_i^{-1}(x)\right| \leq 1$, then for any neighborhood M of $\varphi_i(x)$ in T_i there exists a neighborhood N of x in B_i such that for any $b_i \in N$,*

a. if $\beta_i^{-1}(b_i) \neq \varnothing$ then closure $\beta_i^{-1}(b_i) \subseteq M$*,*

b. if $\beta_i^{-1}(b_i) = \varnothing$ then $\varphi_i(b_i) \in M$.

Proof Pick any $\epsilon > 0$. Since $|\beta_i^{-1}(x)| \leq 1$ and β_i weakly increasing, by (35) we have

$$\beta_i\left(\varphi_i(x) - \epsilon/2\right) < x < \beta\left(\varphi_i(x) + \epsilon/2\right). \tag{36}$$

Then

$$N(x; \epsilon/2) := \left(\frac{x + \beta_i(\varphi_i(x) - \epsilon/2)}{2}, \frac{x + \beta_i(\varphi_i(x) + \epsilon/2)}{2}\right)$$

is a neighborhood of x in B_i. With β_i weakly increasing and Eq. (36),

$$\begin{aligned} \forall t_i' \in \beta_i^{-1}\left(\frac{x + \beta_i(\varphi_i(x) - \epsilon/2)}{2}\right) &: \quad t_i' \geq \varphi_i(x) - \epsilon/2, \\ \forall t_i' \in \beta_i^{-1}\left(\frac{x + \beta_i(\varphi_i(x) + \epsilon/2)}{2}\right) &: \quad t_i' \leq \varphi_i(x) + \epsilon/2. \end{aligned}$$

Thus, by monotonicity of β_i,

$$\begin{aligned} x' \in N(x; \epsilon/2) \quad &\Rightarrow \quad \beta_i^{-1}(x') \subseteq [\varphi_i(x) - \epsilon/2, \varphi_i(x) + \epsilon/2] \\ &\Rightarrow \quad \text{closure}\, \beta_i^{-1}(x') \subseteq (\varphi_i(x) - \epsilon, \varphi_i(x) + \epsilon). \end{aligned}$$

Hence claim (a) is proved.

To prove claim (b), we may assume without loss of generality that there is an infinite sequence $(x^m)_{m=1}^{\infty}$ converging to x such that $\beta_i^{-1}(x^m) = \varnothing$ for each m (otherwise claim (b) is vacuously true because there is a sufficiently small neighborhood of x that contains none of such x^m). If $\varphi_i(x^m) \to \varphi_i(x)$ then we are done. Suppose $(\varphi_i(x^m))_{m=1}^{\infty}$ is bounded away from $\varphi_i(x)$, hence $\liminf_{m\to\infty}\varphi_i(x^m) > \varphi_i(x)$ or $\limsup_{m\to\infty}\varphi_i(x^m) < \varphi_i(x)$.

Consider the case

$$\liminf_{m\to\infty}\varphi_i(x^m) > \varphi_i(x). \tag{37}$$

Let

$$t_i := \frac{1}{2}\left(\varphi_i(x) + \liminf_{m\to\infty}\varphi_i(x^m)\right).$$

Since $t_i < \liminf_{m\to\infty}\varphi_i(x^m)$, monotonicity of β_i implies that

$$\beta_i(t_i) \le \beta_i\left(\liminf_{m'\to\infty}\varphi_i(x^{m'})\right) \le x^m$$

for all sufficiently large m. Since $t_i > \varphi_i(x)$, we have $x \le \beta_i(t_i)$, otherwise $x > \beta_i(t_i)$ leads to a contradiction $\sup \beta_i^{-1}(< x) \ge t_i > \varphi_i(x) = \sup \beta_i^{-1}(< x)$. Thus, for all sufficiently large m,

$$x \le \beta_i(t_i) \le \beta_i\left(\liminf_{m'\to\infty}\varphi_i(x^{m'})\right) \le x^m.$$

Since $x^m \to x$, the above inequality implies that $\beta(t_i)$ is arbitrarily close to x. Then claim (a) of this lemma implies that the entire closure of $\beta_i^{-1}(\beta_i(t_i))$, including t_i, is contained by an arbitrarily small neighborhood of $\varphi_i(x)$. But by its definition, t_i is bounded away from $\varphi_i(x)$. This contradiction implies that (37) cannot be true. By the same token, it cannot be true that $\limsup_{m\to\infty}\varphi_i(x^m) < \varphi_i(x)$. Thus, claim (b) is proved. ■

Proposition 5 *For any bidder i, if everyone else abides by a profile β_{-i} of weakly increasing bid functions, then for any $t_i \in T_i$ and any $b_i^* \in B_i$ such that $\left|\beta_k^{-1}(b_i^*)\right| \le 1$ for all $k \ne i$, $\overline{W}_i(\cdot, t_i)$, $\overline{L}_{ij}(\cdot, t_i)$ ($\forall j \ne i$) and $U_i(\cdot, t_i; \beta_{-i})$ are each continuous at b_i^*.*

Proof In the continuation equilibrium conditional on the the event that bidder i has bid b_i and won in period one, i's best response, characterized by Myerson [11], is to maximize

$$\mathbb{E}\left[\sum_{k\ne i}\pi_k(\mathbf{t}_{-i})\left(V_k(\mathbf{t}_k \mid i, b_i) - t_i\right)\right] \tag{38}$$

among all allocations $(\pi_k)_{k\neq i} : T_{-i} \to [0,1]^{|I|-1}$ subject to the resource feasibility constraint

$$\sum_{k\neq i} \pi_k(t_{-i}) \leq 1 \quad \forall t_{-i} \in T_{-i} \tag{39}$$

and the incentive constraint that $\mathbb{E}\pi_k(t_k, \mathbf{t}_{\neg\{i,k\}})$ is weakly increasing in t_k for all $k \neq i$. By Corollary 1, the posterior virtual utilities in the integral (38) is strictly increasing in t_k for every bidder $k \neq i$. Therefore, as in Myerson's regular case, the incentive constraint is automatically satisfied by any solution that maximizes (38) subject to (39). By Eqs. (5) and (6), the objective (38) is equal to $\mathbb{E}\left[\sum_{k\neq i} g_k(\pi, b_i, t_i, \mathbf{t}_{-i})\right]$, where

$$g_k(\pi, b_i, t_i, t_{-i}) := \begin{cases} \pi_k(t_{-i})\left(t_k - \frac{\psi_k(i,b_i)F_k(d_k)+(1-\psi_k(i,b_i))F_k(c_k)-F_k(t_k)}{f_k(t_k)} - t_i\right) & \text{If } t_k \leq c_k \\ \pi_k(t_{-i})\left(t_k - \frac{F_k(d_k)-F_k(t_k)}{f_k(t_k)} - t_i\right) & \text{If } c_k \leq t_k \leq d_k, \end{cases}$$

where c_k and d_k, defined by Eqs. (1)–(2), are the lower and upper bounds of closure $\beta_k^{-1}(b_i)$ if $\beta_k^{-1}(b_i) \neq \varnothing$, else $c_k = d_k = \varphi_k(b_i)$, with $\varphi_k(b_i)$ defined in (35). Since $\psi_k(i, b_i)F_k(d_k) + (1 - \psi_k(i, b_i))F_k(c_k)$ is a convex combination between $F_k(d_k)$ and $F_k(c_k)$ and F_k is continuous,

$$g_k(\pi, b_i, t_i, t_{-i}) \begin{cases} \in \left\{\pi_k(t_{-i})\left(t_k - \frac{F_k(s_k)-F_k(t_i)}{f_k(t_k)} - t_i\right) : s_k \in \text{closure}\, \beta_k^{-1}(b_i)\right\} & \text{if } \beta_k^{-1}(b_i) \neq \varnothing \\ = \pi_k(t_{-i})\left(t_k - \frac{F_k(\sup \beta_k^{-1}(\leq b_i))-F_k(t_i)}{f_k(t_k)} - t_i\right) & \text{if } \beta_k^{-1}(b_i) = \varnothing. \end{cases} \tag{40}$$

Thus, for any $b_i \in B_i$ and any $t_i \in T_i$,

$$\overline{W}_i(b_i, t_i) = \max_{(\pi_k)_{k\neq i}:T_{-i}\to[0,1]^{|I|-1}} \mathbb{E}\left[\sum_{k\neq i} g_k(\pi, b_i, t_i, \mathbf{t}_{-i})\right] \quad \text{subject to (39).} \tag{41}$$

Claim 1: For any π, any t_i, any t_{-i} and any $k \neq i$, $g_k(\pi, \cdot, t_i, t_{-i})$ is continuous at b_i^*. Recall from the hypothesis that $\left|\beta_k^{-1}(b_i^*)\right| \leq 1$. Then Eq. (40) implies that

$$g_k(\pi, b_i^*, t_i, t_{-i}) = \pi_k(t_{-i})\left(t_k - \frac{F_k\left(\varphi_k(b_i^*)\right) - F_k(t_i)}{f_k(t_k)} - t_i\right),$$

with φ_k defined in Eq. (35), and Lemma 7 applies. Thus, if $b_i^m \to_m b_i^*$ then the closure of $\beta_k^{-1}(b_i^m)$ (when $\varphi_k(b_i^m) \neq \varnothing$) and $\varphi_k(b_i^m)$ (when $\varphi_k(b_i^m) = \varnothing$) both converge to $\varphi_k(b_i^*)$, hence by Eq. (40) $g_k(\pi, b_i^m, t_i, t_{-i}) \to_m g_k(\pi, b_i^*, t_i, t_{-i})$.

Claim 2: For any π, any t_i, and any t_{-i}, $\mathbb{E}\left[\sum_{k\neq i} g_k(\pi, \cdot, t_i, \mathbf{t}_{-i})\right]$ is continuous at b_i^*. Claim 1 and the bounded convergence theorem implies that for each $k \neq i$, $\mathbb{E}g_k(\pi, \cdot, t_i, \mathbf{t}_{-i})$ is continuous at b_i^*. Summing across $k \neq i$ and switching positions between $\mathbb{E}$ and $\sum$, we get Claim 2.

Claim 3: $\overline{W}_i(\cdot,t_i)$ is continuous at b_i^*. Let $(b_i^m)_{m=1}^\infty$ converge to b_i^*. For each m, let π^m be an optimal solution for the problem in Eq. (41). Since (39) defines a compact set, the infinite sequence $(\pi^m)_{m=1}^\infty$ has an infinite subsequence converging to some π^* that satisfies (39). Relabeling if necessary, let $\pi^m \to \pi^*$. Pick any π that satisfies (39). For each m, the fact that π^m is an optimal solution given b_i^m implies that

$$\mathbb{E}\left[\sum_{k\neq i} g_k(\pi^m, b_i^m, t_i, \mathbf{t}_{-i})\right] \geq \mathbb{E}\left[\sum_{k\neq i} g_k(\pi, b_i^m, t_i, \mathbf{t}_{-i})\right].$$

Taking the limit when $\pi^m \to \pi^*$ and $b_i^m \to b_i^*$ and using Claim 2, we have

$$\mathbb{E}\left[\sum_{k\neq i} g_k(\pi^*, b_i^*, t_i, \mathbf{t}_{-i})\right] \geq \mathbb{E}\left[\sum_{k\neq i} g_k(\pi, b_i^*, t_i, \mathbf{t}_{-i})\right].$$

With π arbitrarily chosen, it follows that

$$\overline{W}_i(b_i^*, t_i) = \mathbb{E}\left[\sum_{k\neq i} g_k(\pi^*, b_i^*, t_i, \mathbf{t}_{-i})\right] = \lim_{m\to\infty} \mathbb{E}\left[\sum_{k\neq i} g_k(\pi^m, b_i^m, t_i, \mathbf{t}_{-i})\right] = \lim_{m\to\infty} \overline{W}_i(b_i^m, t_i).$$

Thus, $\overline{W}_i(\cdot,t_i)$ is continuous at b_i^*.

Claim 4: $U_i(\cdot, t_i; \beta_{-i})$ is continuous at b_i^*. To prove the claim, note from Eq. (22) that

$$U_i(b_i, t_i; \beta_{-i}) = \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i,\mathbf{t}_{-i})}\right]\left(\overline{W}_i(b_i,t_i) - b_i\right) + \mathbb{E}\left[\sum_{j\neq i} \mathbf{1}_{j=h_i(b_i,\mathbf{t}_{-i})} L_{ij}(t_i, \mathbf{t}_{-i})\right].$$

Let $(b_i^m)_{m=1}^\infty$ be an infinite sequence in B_i that converges to b_i^*. By hypothesis, $|\beta_k^{-1}(b_i^*)| \leq 1$ for all $k \neq i$. Thus, Lemma 7.a implies that

$$\lim_{m\to\infty} \text{Prob}\left\{\exists k \neq i\, [\beta_k(\mathbf{t}_k) = b_i^m]\right\} \leq \lim_{m\to\infty} \sum_{k\neq i} \text{Prob}\left(\text{closure}\, \beta_k^{-1}(b_i^m)\right) = \text{Prob}\left(\beta_j^{-1}(b_i^*)\right) = 0.$$

Thus,

$$\begin{aligned}\lim_{m\to\infty} U_i(b_i^m, t_i; \beta_{-i}) \quad &= \quad \lim_{m\to\infty} \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i^m,\mathbf{t}_{-i})} \mid \forall k \neq i\, [b_i^m \neq \beta_k(\mathbf{t}_k)]\right]\left(\overline{W}_i(b_i^m,t_i) - b_i^m\right) \\ &+ \quad \lim_{m\to\infty} \mathbb{E}\left[\sum_{j\neq i} \mathbf{1}_{j=h_i(b_i^m,\mathbf{t}_{-i})} L_{ij}(t_i,\mathbf{t}_{-i}) \middle| \forall k \neq i\, [b_i^m \neq \beta_k(\mathbf{t}_k)]\right].\end{aligned}$$

By Claim 3 and the hypothesis that b_i^* is not an atom of β_{-i}, $\lim_{m\to\infty} \overline{W}_i(b_i^m, t_i) = \overline{W}_i(b_i^*, t_i)$. Hence

$$\begin{aligned}\lim_{m\to\infty} U_i(b_i^m, t_i; \beta_{-i}) \quad &= \quad \lim_{m\to\infty} \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i^m,\mathbf{t}_{-i})} \mid \forall k \neq i\, [b_i^m \neq \beta_k(\mathbf{t}_k)]\right]\left(\overline{W}_i(b_i^*,t_i) - b_i^*\right) \\ &+ \quad \lim_{m\to\infty} \mathbb{E}\left[\sum_{j\neq i} \mathbf{1}_{j=h_i(b_i^m,\mathbf{t}_{-i})} L_{ij}(t_i,\mathbf{t}_{-i}) \middle| \forall k \neq i\, [b_i^m \neq \beta_k(\mathbf{t}_k)]\right].\end{aligned}$$

Note that, for any $b_i \in B_i$,

$$\begin{aligned} \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i,\mathbf{t}_{-i})} \mid \forall k \neq i\,[b_i \neq \beta_k(\mathbf{t}_k)]\right] &= \mathbb{E}\left[\mathbf{1}_{b_i>\max_{k\neq i}\beta_k(\mathbf{t}_k)}\right], \\ \mathbb{E}\left[\sum_{j\neq i}\mathbf{1}_{j=h_i(b_i,\mathbf{t}_{-i})}L_{ij}(t_i,\mathbf{t}_{-i})\middle|\forall k \neq i\,[b_i \neq \beta_k(\mathbf{t}_k)]\right] &= \mathbb{E}\left[\sum_{j\neq i}\mathbf{1}_{\beta_j(\mathbf{t}_j)\geq\max_{k\neq i}\beta_k(\mathbf{t}_k)>b_i}L_{ij}(t_i,\mathbf{t}_{-i})\right]. \end{aligned}$$

As $m \to \infty$, the indicator functions $\chi^m : t_{-i} \mapsto \mathbf{1}_{b_i^m>\max_{k\neq i}\beta_k(t_k)}$ converge to the function $t_{-i} \mapsto \mathbf{1}_{b_i^*>\max_{k\neq i}\beta_k(t_k)}$ pointwise, hence by the bounded convergence theorem,

$$\lim_{m\to\infty}\mathbb{E}\left[\mathbf{1}_{b_i^m>\max_{k\neq i}\beta_k(\mathbf{t}_k)}\right] = \mathbb{E}\left[\mathbf{1}_{b_i^*>\max_{k\neq i}\beta_k(\mathbf{t}_k)}\right].$$

Likewise,

$$\lim_{m\to\infty}\mathbb{E}\left[\sum_{j\neq i}\mathbf{1}_{\beta_j(\mathbf{t}_j)\geq\max_{k\neq i}\beta_k(\mathbf{t}_k)>b_i^m}L_{ij}(t_i,\mathbf{t}_{-i})\right] = \mathbb{E}\left[\sum_{j\neq i}\mathbf{1}_{\beta_j(\mathbf{t}_j)\geq\max_{k\neq i}\beta_k(\mathbf{t}_k)>b_i^*}L_{ij}(t_i,\mathbf{t}_{-i})\right].$$

Thus,

$$\begin{aligned} \lim_{m\to\infty}U_i(b_i^m,t_i;\beta_{-i}) &= \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i^*,\mathbf{t}_{-i})} \mid \forall k \neq i\,[b_i^* \neq \beta_k(\mathbf{t}_k)]\right]\left(\overline{W}_i(b_i^*,t_i)-b_i^*\right) \\ &\quad +\mathbb{E}\left[\sum_{j\neq i}\mathbf{1}_{j=h_i(b_i^*,\mathbf{t}_{-i})}L_{ij}(t_i,\mathbf{t}_{-i})\middle|\forall k \neq i\,[b_i^* \neq \beta_k(\mathbf{t}_k)]\right] \\ &= \mathbb{E}\left[\mathbf{1}_{i=h_i(b_i^*,\mathbf{t}_{-i})}\right]\left(\overline{W}_i(b_i^*,t_i)-b_i^*\right)+\mathbb{E}\left[\sum_{j\neq i}\mathbf{1}_{j=h_i(b_i^*,\mathbf{t}_{-i})}L_{ij}(t_i,\mathbf{t}_{-i})\right] \\ &= U_i(b_i^*,t_i;\beta_{-i}), \end{aligned}$$

where the second equality follows from the hypothesis that b_i^* is not an atom of β_{-i}. ■

6 Existence of Equilibrium

We shall now apply the fixed-point approach based on the comparative statics results. The general approach is to consider a sequence of approximation games and prove that a limit point of the sequence of equilibria of these approximation games turns out to be an equilibrium of the original game.

For any $m = 1, 2, \ldots$, define an *m-finite approximation game* by two conditions: first, replace for any bidder i the bid space B_i with a finite subset B_i^m of B_i such that

$$i \neq j \Longrightarrow B_i^m \cap B_j^m = \{l\}, \tag{42}$$

$$m < m' \Longrightarrow B_i^m \subseteq B_i^{m'}, \tag{43}$$

$$\min\{|b_i - b_i'| : b_i, b_i' \in B_i^m \setminus \{l\}; b_i \neq b_i'\} = 2^{-m}; \tag{44}$$

second, restrict each bidder i's period-1 bid function by a $1/m$*-perturbation* such that the function associates the losing bid l to every type $t_i \in [\underline{t}_i, \underline{t}_i + 1/m)$.

The first condition was devised by Reny and Zamir [13]; it ensures that the single crossing condition holds for each finite approximation, since our strict single results requires a no-tie condition. The $1/m$-perturbation was devised by Athey [1] and also used by Reny and Zamir; it guarantees that any serious bid (non-l bid) has a positive probability of winning in any approximation game.

Proposition 6 *For any $m, n \in \{1, 2, \ldots\}$, the m-finite approximation of the auction-resale game defined above admits a PBE, subject to the $1/m$-perturbation, such that every bidder's period-1 bid is a weakly increasing function of the bidder's use value.*

Proof By Lemma 3, it suffices to prove existence of a BNE of the auction game where each player i's interim expected payoff function $U_i(b_i, t_i; \beta_{-i})$ is given by Eq. (22). By (42), the no-tie condition of Proposition 3 is satisfied. Thus, the interim payoff function $U_i(b_i, t_i; \beta_{-i})$ has the strict single crossing property in (b_i, t_i) provided β_k is weakly increasing for each $k \neq i$. The rest of the proof is the same as Athey's proof [1, Theorem 1; Lemma 4] of existence of BNE in the finite-bid-space auction game. ■

Proposition 6 implies that for any $m = 1, 2, \ldots$, there exists an equilibrium $(\beta_i^m)_{i=1}^n$ given the profile of bid spaces $(B_i^m)_{i \in I}$ such that each β_i^m is weakly increasing and satisfies the $1/m$-perturbation. Then, taking a convergent subsequence if necessary, for any bidder i there exists a weakly increasing bid function $\beta_i^* : T_i \to B_i$ such that β_i^m converges to β_i^* pointwise on T_i.

By Lemma 3, it suffices to show that $(\beta_i^*)_{i \in I}$ constitutes a BNE given the interim payoff functions $(U_i)_{i \in I}$ calculated in Eq. (22).

For any $m = 1, 2, \ldots$, the fact that $(\beta_i^m)_{i=1}^n$ is a BNE in the $(B_i^m)_{i \in I}$-discrete-bid game implies that for any bidder i and any $t_i \in T_i$,

$$\forall b_i^m \in B_i^m : U_i(\beta_i^m(t_i), t_i; \beta_{-i}^m) \geq U_i(b_i^m, t_i; \beta_{-i}^m). \tag{45}$$

We are done if for almost every $t_i \in T_i$

$$\forall b_i \in B_i : U_i(\beta_i^*(t_i), t_i; \beta_{-i}^*) \geq U_i(b_i, t_i; \beta_{-i}^*). \tag{46}$$

6.1 Approachability of Tying Bids

In the sequel, we restrict attention to the continuation equilibrium where any reseller i follows, at any realized profile $t_{-i} \in T_{-i}$, the provisions in Lemma 2 and uses the payment scheme defined in Eq. (8). This continuation equilibrium exists by construction in Myerson. Based on this resale mechanism, the ex post payoffs from winning and losing exhibit certain monotonicity properties.

Lemma 8 *For any bidder i, any $t_i \in T_i$ and any $b_i \in B_i$,*

a. $W_i(b_i, t_i, t_{-i})$ is weakly increasing in t_{-i}, and

b. for any $j \in I \setminus \{i\}$ and any $t_j \in T_j$, $L_{ij}(t_i, t_j, t_{\neg(i,j)})$ is weakly decreasing in $t_{\neg(i,j)}$.

Proof For any $(t_i, t_{-i}) \in T$, as the provisions in Lemma 2 are followed,

$$W_i(b_i, t_i, t_{-i}) = \max_{(\pi_k)_{k\in I}\in[0,1]^I} \left(\pi_i t_i + \sum_{k\in I\setminus\{i\}} \pi_k V_k(t_k \mid i, b_i)\right) \quad \text{s.t.} \sum_{k\in I} \pi_k = 1. \tag{47}$$

Thus, it follows from the envelope theorem that $W_i(b_i, t_i, t_{-i})$ is increasing in the vector $(V_k(t_k \mid i, b_i))_{k\in I\setminus\{i\}}$. As $V_k(t_k \mid i, b_i)$ is strictly increasing in t_k for each k (Corollary 1), $W_i(b_i, t_i, t_{-i})$ is increasing in $(t_k)_{k\in I\setminus\{i\}}$. This proves claim (a).

For any $j \neq i$, by the provisions in Lemma 2 and the payment rule p_i given in Eq. (8),

$$L_{ij}(t_i, t_{-i}) = \left(t_i - p_{i,j,\beta_j(t_j)}(t_i, t_{-i})\right)^+,$$

where

$$p_{i,j,\beta_j(t_j)} = \max\left\{V^{-1}_{i,j,\beta_j(t_j)}(t_j), \max_{k\notin\{i,j\}} V^{-1}_{i,j,\beta_j(t_j)}(V_{k,j,\beta_j(t_j)}(t_k))\right\}$$

and $V_{k,j,\beta_j(t_j)}(t_k)$ obeys Eq. (13) for all $k \in I \setminus \{j\}$. Thus, $p_{i,j,\beta_j(t_j)}(t_i, t_{-i})$ is increasing in $(V_k(t_k \mid j, \beta_j(t_j)))_{k\in I\setminus\{i,j\}}$, which in turn is strictly increasing in $t_{\neg(i,j)}$ (Corollary 1). Hence $L_{ij}(t_i, t_{-i})$ is weakly decreasing in $t_{\neg(i,j)}$. Thus, claim (b) follows. ■

Lemma 9 *If $g : \mathbb{R} \to \mathbb{R}$ is a weakly increasing function and $g^{-1}(y)$ is bounded, then for any $\epsilon > 0$ there exists $\delta > 0$ such that the measure of $g^{-1}[(y - \delta, y + \delta)] \setminus g^{-1}(y)$ is less than ϵ.*

Proof We may assume without loss of generality that $g^{-1}(y) \neq \varnothing$; otherwise the claim is trivial. As g is weakly increasing and $g^{-1}(y)$ is bounded, $\sup g^{-1}(y)$ and $\inf g^{-1}(y)$ exist and

$$g\left(\inf g^{-1}(y) - \frac{\epsilon}{2}\right) < y < g\left(\sup g^{-1}(y) + \frac{\epsilon}{2}\right).$$

Then

$$\delta := \min\left\{\frac{g\left(\sup g^{-1}(y) + \frac{\epsilon}{2}\right) + y}{2} - y, y - \frac{g\left(\inf g^{-1}(y) - \frac{\epsilon}{2}\right) + y}{2}\right\}$$

works, again due to g's being weakly increasing. ■

Proposition 7 *For any bidder i, any $t_i \in T_i$ and any $b_i \in (r_i, \infty)$, if every other bidder $k \neq i$ abides by a profile β_{-i} of weakly increasing bid functions and*

$$\exists \underline{b} < b_i : \exists \pi > 0 : \text{Prob}\{\forall k \in I \setminus \{i, j\} : \beta_k(\mathbf{t}_k) \leq \underline{b}\} = \pi, \tag{48}$$

then there exists an infinite sequence $(b_i^n)_{n=1}^{\infty}$ in the bid space B_i such that, for each n, b_i^n is not an atom of β_{-i}, and

$$\lim_{n\to\infty} U_i(b_i^n, t; \beta_{-i}) \geq U_i(b_i, t; \beta_{-i}). \tag{49}$$

Proof If b_i is not an atom of β_{-i}, then the conclusion follows from the fact that a monotone function has at most countable many atoms, which ensures existence of a sequence $(b_i^n)_{n=1}^{\infty}$ of nonatomic bids converging to b_i, and Proposition 5, which ensures $\lim_{n\to\infty} U_i(b_i^n, t; \beta_{-i}) = U_i(b_i, t; \beta_{-i})$. Thus, suppose that b_i is an atom of β_{-i}, i.e., with the notation in Eq. (27),

$$\exists \mu > 0 : \text{Prob}(E(b_i, b_i)) = \mu. \tag{50}$$

Since $b_i \in (r_i, \infty)$ and since a monotone function has at most countable atoms, there exist infinite sequences $(c_i^n)_{i=1}^{\infty}$ and $(d_i^n)_{i=1}^{\infty}$ in the bid space B_i such that $c_i^n \uparrow b_i$, $d_i^n \downarrow b_i$, and, for each n, neither c_i^n nor d_i^n is an atom of β_{-i}. Note that

$$\begin{aligned} E(c_i^n, b_i) &= E(b_i, b_i) \sqcup \left\{t_{-i} \in T_{-i} : c_i^n \leq \max_{k\neq i} \beta_k(t_k) < b_i\right\}, \\ E(b_i, d_i^n) &= E(b_i, b_i) \sqcup \left\{t_{-i} \in T_{-i} : b_i < \max_{k\neq i} \beta_k(t_k) \leq d_i^n\right\}. \end{aligned}$$

Lemma 9 implies that

$$\lim_{n\to\infty} \text{Prob}\left\{t_{-i} \in T_{-i} : c_i^n \leq \max_{k\neq i} \beta_k(t_k) < b_i\right\} = 0, \tag{51}$$

$$\lim_{n\to\infty} \text{Prob}\left\{t_{-i} \in T_{-i} : b_i < \max_{k\neq i} \beta_k(t_k) \leq d_i^n\right\} = 0. \tag{52}$$

Case 1: $\Delta_i(b_i, t_i \mid b_i, b_i) = \rho > 0$ for some ρ. For each $n = 1, 2, \ldots$, Ineq. (31) implies

$$\begin{aligned}
& U_i(d_i^n, t_i; \beta_{-i}) - U_i(b_i, t_i; \beta_{-i}) \\
\geq \;& \text{Prob}\{E(b_i, d_i^n)\}\Delta_i(b_i, t_i \mid b_i, d_i^n) - (d_i^n - b_i)\text{Prob}\{\forall k \neq i : d_i^n > \beta_k(\mathbf{t}_k)\} \\
\geq \;& \text{Prob}\{E(b_i, b_i))\}\Delta_i(b_i, t_i \mid b_i, b_i) - \text{Prob}\left\{b_i < \max_{k \neq i} \beta_k(\mathbf{t}_k) \leq d_i^n\right\} \max_{k \in I} \bar{t}_k \\
& -(d_i^n - b_i)\text{Prob}\{\forall k \neq i : d_i^n > \beta_k(\mathbf{t}_k)\} \\
\overset{(50)}{\geq} \;& \mu\rho - \text{Prob}\left\{b_i < \max_{k \neq i} \beta_k(\mathbf{t}_k) \leq d_i^n\right\} \max_{k \in I} \bar{t}_k - (d_i^n - b_i)\text{Prob}\{\forall k \neq i : d_i^n > \beta_k(\mathbf{t}_k)\},
\end{aligned}$$

which is positive when n is sufficiently large, because when $n \to \infty$ the second and third terms in the last line converge to zero, with the secdon term does so due to Eq. (52).

Case 2: $\Delta_i(b_i, t_i \mid b_i, b_i) \leq 0$. For each $n = 1, 2, \ldots$, Ineq. (32) implies

$$\begin{aligned}
& U_i(c_i^n, t_i; \beta_{-i}) - U_i(b_i, t_i; \beta_{-i}) \\
\geq \;& -\text{Prob}\{E(c_i^n, b_i)\}\Delta_i(b_i, t_i \mid c_i^n, b_i) + (b_i - c_i^n)\text{Prob}\{\forall k \neq i : c_i^n > \beta_k(\mathbf{t}_k)\} \\
\geq \;& -\text{Prob}\{E(b_i, b_i))\}\Delta_i(b_i, t_i \mid b_i, b_i) - \text{Prob}\left\{c_i^n \leq \max_{k \neq i} \beta_k(\mathbf{t}_k) < b_i\right\} \max_{k \in I} \bar{t}_k \\
& +(b_i - c_i^n)\text{Prob}\{\forall k \neq i : c_i^n > \beta_k(\mathbf{t}_k)\} \\
\geq \;& -\text{Prob}\left\{c_i^n \leq \max_{k \neq i} \beta_k(\mathbf{t}_k) < b_i\right\} \max_{k \in I} \bar{t}_k + (b_i - c_i^n)\text{Prob}\{\forall k \neq i : c_i^n > \beta_k(\mathbf{t}_k)\},
\end{aligned}$$

which converges to zero when $n \to \infty$, with the first term in the last line converging to zero due to Eq. (51).

Both cases considered, the lemma is proved. ■

Corollary 2 *For any bidder i and any $t_i \in T_i \setminus \underline{t}_i$, if $\beta_i^*(t_i) = l$ or $\beta_i^*(t_i)$ is not an atom of β_{-i}^*, then (46) holds.*

Proof Pick any $t_i \in T_i$ and any $b_i \in B_i$. If $b_i = l$ then for any m large enough for $t_i > \underline{t}_i + 1/m$, $U_i(\beta_i^m(t_i), t_i; \beta_{-i}^m) \geq U_i(b_i, t_i; \beta_{-i}^m)$ by revealed preference, as $l \in B_i^m$. Since $\beta_i^*(t_i)$ is either the isolated point l or not an atom of β_{-i}^*, $U_i(\cdot, t_i; \beta_{-i}^*)$ is continuous at $\beta_i^*(t_i)$ by Proposition 5 and hence the left-hand side of the inequality converges to the left-hand side of (46). Hence Ineq. (46) holds.

Now consider the case where $b_i \neq l$, i.e., $b_i \in (r_i, \infty)$. Proposition 7 implies that there is a sequence $(b_i^n)_{n=1}^\infty$ in the bid space B_i such that, for each n, b_i^n is not an atom of β_{-i}^*, and (49) holds. Thus, for any $\epsilon > 0$ there exists N such that for any integer $n \geq N$,

$$U_i(b_i, t_i; \beta_{-i}^m) - \epsilon \leq U_i(b_i^n, t_i; \beta_{-i}^m).$$

For any such n, since $\cup_{m=1}^{\infty} B_i^m$ is dense in B_i and $(B_i^m)_{m=1}^{\infty}$ is nested, there exists an infinite sequence $(c_i^m)_{m=1}^{\infty}$ converging to b_i^n such that $c_i^m \in B_i^m$ for each m. For any m large enough for $t_i > \underline{t}_i + 1/m$,

$$U_i(\beta_i^m(t_i), t_i; \beta_{-i}^m) \geq U_i(c_i^m, t_i; \beta_{-i}^m)$$

by revealed preference. As in the previous case where $b_i = l$, the left-hand side of the inequality converges to $U_i(\beta_i^*(t_i), t_i; \beta_{-i}^*)$; the right-hand side converges to $U_i(b_i^n, t_i; \beta_{-i}^m)$ because b_i^n is not an atom of β_{-i}^* so Proposition 5 applies. Therefore,

$$U_i(\beta_i^*(t_i), t_i; \beta_{-i}^*) \geq U_i(b_i^n, t_i; \beta_{-i}^m) \geq U_i(b_i, t_i; \beta_{-i}^m) - \epsilon.$$

Since ϵ can be arbitrarily small, Ineq. (46) follows. ■

6.2 Passing to a Limit through Disentangling Ties

By Corollary 2, it suffices to handle the atom bids that are not the losing bid l. Since there are only countably many atoms of a monotone function, (46) holds for almost all $t_i \in T_i$, which means that we are done, if whenever a not-for-sure-losing bid $\beta_i^*(t_i)$ is an atom of β_{-i}^* then $\beta_i^*(t_i)$ is not an atom of β_i^*. Thus, the next proposition suffices.

Proposition 8 (no tie at the limit) *For any bid $x \neq l$, there is at most one bidder i such that x is atom of β_i^*.*

Proof Suppose by negation

$$K := \left\{ i \in I : |(\beta_i^*)^{-1}(x)| > 1 \right\}$$

is nonempty and nonsingleton. For each $i \in K$, with β_i^* monotone, $(\beta_i^*)^{-1}(x)$ is a nondegenerate interval in T_i with probability measure $\mu_i > 0$. For any $i \in I$, let

$$\begin{aligned} a_i &:= \inf(\beta_i^*)^{-1}(x), \\ z_i &:= \sup(\beta_i^*)^{-1}(x). \end{aligned}$$

Relabeling if necessary, list the bidders in K as $1, \ldots, |K|$ such that

$$a_1 \leq a_2 \leq \cdots \leq a_{|K|}. \tag{53}$$

For any $\delta > 0$, define the neighborhood $N_i(x; \delta)$ of the bid x for bidder i by

$$N_i(x; \delta) := (x - \delta, x + \delta) \cap (r_i, \infty).$$

Claim 1 For any $i \in I$, any $t_i \in T_i \setminus \{\underline{t}_i\}$, and any $m = 1, 2, \ldots$ such that $t_i > \underline{t}_i + 1/m$,

$$\Delta_i \left(\beta_i^m(t_i), t_i; \beta_{-i}^m \mid l, \beta_i^m(t_i)\right) \geq 0. \tag{54}$$

Since l is always an option and the $1/m$-restriction is not binding to type $t_i > \underline{t}_i + 1/m$, we have by revealed preference

$$\begin{aligned} 0 &\leq \mathbb{E}\left[\mathbf{1}_{i=h_i(\beta_i^m(t_i),\mathbf{t}_{-i})} W_i\left(\beta_i^m(t_i), t_i, \mathbf{t}_{-i}\right) - \beta_i^m(t_i) - L_i(t_i, \mathbf{t}_{-i})\right] \\ &= \mathbb{E}\left[\mathbf{1}_{i=h_i(\beta_i^m(t_i),\mathbf{t}_{-i})}\right] \underbrace{\mathbb{E}\left[W_i\left(\beta_i^m(t_i), t_i, \mathbf{t}_{-i}\right) - \beta_i^m(t_i) - L_i(t_i, \mathbf{t}_{-i}) \mid i = h_i(\beta_i^m(t_i), \mathbf{t}_{-i})\right]}_{=\Delta_i\left(\beta_i^m(t_i), t_i; \beta_{-i}^m \mid l, \beta_i^m(t_i)\right)}. \end{aligned}$$

Since β^m satisfies the $1/m$-restriction, $\mathbb{E}\left[\mathbf{1}_{i=h_i(\beta_i^m(t_i),\mathbf{t}_{-i})}\right] \geq \Pi_{j \in I \setminus \{i\}} F_j(\underline{t}_j + 1/m) > 0$, hence (54) follows. □

For any $m = 1, 2, \ldots$, any $\delta > 0$, any $b_i \in B_i^m \cap N_i(x; \delta)$, and any $t_{-i} := (t_j)_{j \in I \setminus \{i\}} \in E(b_i, x + \delta)$, define a projection

$$\text{proj}_{b_i}(t_{-i}) := (t'_j)_{j \in I \setminus \{i\}}$$

such that

$$b_i < \beta_j^m(t_j) < x + \delta \Longrightarrow t'_j := \inf\{t''_j \in T_j : \beta_j^m(t''_j) > b_i\}. \tag{55}$$

Note that $\text{proj}_{b_i}(t_{-i}) \in E(b_i, x + \delta)$ and $\text{proj}_{b_i}(t_{-i}) \leqq t_{-i}$ coordinatewise. Define

$$W_i^*\left(\beta_i^m(t_i), t_i, t_{-i}\right) := \begin{cases} W_i\left(\beta_i^m(t_i), t_i, \text{proj}_{\beta_i^m(t_i)}(t_{-i})\right) & \text{if } t_{-i} \in E(\beta_i^m(t_i), x + \delta) \\ W_i\left(\beta_i^m(t_i), t_i, t_{-i}\right) & \text{else.} \end{cases} \tag{56}$$

Since $W_i\left(\beta_i^m(t_i), t_i, \cdot\right)$ is an increasing function on T_{-i} by Lemma 8.a, so is $W_i^*\left(\beta_i^m(t_i), t_i, \cdot\right)$. Thus, by Theorem 5 of Milgrom and Weber [10],

$$\begin{aligned} \mathbb{E}\left[W_i^*\left(\beta_i^m(t_i), t_i, \mathbf{t}_{-i}\right) \mid E(\beta_i^m(t_i), x + \delta)\right] &\geq \mathbb{E}\left[W_i^*\left(\beta_i^m(t_i), t_i, \mathbf{t}_{-i}\right) \mid E(l, \beta_i^m(t_i))\right] \\ &= \mathbb{E}\left[W_i\left(\beta_i^m(t_i), t_i, \mathbf{t}_{-i}\right) \mid E(l, \beta_i^m(t_i))\right]. \end{aligned} \tag{57}$$

Denote $\text{proj}_{b_i}^j(t_{-i})$ for the component of $\text{proj}_{b_i}(t_{-i})$ that corresponds to bidder j's type. Applying the Milgrom-Segal envelope theorem to the maximization problem (47), we have for any $(t_j, t_{\neg(i,j)}) \in E(b_i, x + \delta)$ such that $b_i < \beta_j^m(t_j) < x + \delta$,

$$\begin{aligned} W_i\left(b_i, t_i, t_j, t_{\neg(i,j)}\right) - W_i\left(b_i, t_i, \text{proj}_{b_i}^j(t_{-i}), t_{\neg(i,j)}\right) &= \int_{\text{proj}_{b_i}^j(t_{-i})}^{t_j} \pi_j(t_i, t''_j, t_{\neg(i,j)}) \frac{\partial}{\partial t''_j} V_{j,i,b_i}(t''_j) dt''_j \\ &= \int_{\text{proj}_{b_i}^j(t_{-i})}^{t_j} \mathbf{1}_{t''_j > t_i} dt''_j \\ &= \left(t_j - \max\left\{t_i, \text{proj}_{b_i}^j(t_{-i})\right\}\right)^+; \end{aligned}$$

the second line follows from the fact that $\text{proj}^j_{b_i}(t_{-i})$ is above the posterior support of $\mathbf{t}_j$ conditional on the history (i, b_i); this fact implies

$$\forall t''_j \geq \text{proj}^j_{b_i}(t_{-i}) : V_{j,i,b_i}(t''_j) = t''_j$$

so $\frac{\partial}{\partial t''_j} V_{j,i,b_i}(t''_j) = 1$, and in the optimal resale mechanism conditional on the history (i, b_i), the good is resold to any j whose use value t''_j is above the reseller's t_i.

Thus, for any $t_{-i} \in E(b_i, x + \delta)$,

$$\begin{aligned} W_i(b_i, t_i, t_{-i}) - W^*_i(b_i, t_i, t_{-i}) &= W_i(b_i, t_i, t_{-i}) - W_i\left(b_i, t_i, \text{proj}_{b_i}(t_{-i})\right) \\ &= \sum_{j \in I \setminus \{i\}} \mathbf{1}_{b_i < \beta^m_j(t_j) < x+\delta} \left(t_j - \max\left\{t_i, \text{proj}^j_{b_i}(t_{-i})\right\}\right)^+ . \end{aligned} \tag{58}$$

Pick any $\eta > 0$. By Lemma 9, there exists a $\overline{\delta} > 0$ such that

$$\text{Prob}\left\{\exists j \in I \setminus \{i\} : \beta^*_j(\mathbf{t}_j) \in \left(x - \overline{\delta}, x\right) \cup \left(x, x + \overline{\delta}\right)\right\} < \eta / \max_{i \in I} \overline{t}_i.$$

As $\beta^m \to \beta^*$ pointwise, $\beta^m \to \beta^*$ uniformly except on a set of arbitrarily small measure (Littlewood's third principle or Egoroff's theorem). Thus, for any $\epsilon > 0$ such that

$$\epsilon < \min\left\{\min_{k \in K} \mu_k, \min_{k \in K}(z_k - a_k)\right\}, \tag{59}$$

there exist for each $i \in I$ an $E^*_i \subset T_i$ such that

$$\text{Prob}\left(E^*_i\right) < \epsilon \tag{60}$$

and β^m converges uniformly to β^* except on points in $\Pi_{i \in I} E^*_i$. For any $\delta > 0$ such that

$$\delta < \overline{\delta} \tag{61}$$

there exists $m_*(\eta, \epsilon, \delta)$ such that for every integer $m \geq m_*(\eta, \epsilon, \delta)$ and every $i \in K$, there exist

$$\begin{aligned} a^m_i &:= \inf\{t_i \in T_i : x - \delta < \beta^m_i(t_i)\}, \\ z^m_i &:= \sup\{t_i \in T_i : x + \delta < \beta^m_i(t_i)\} \end{aligned}$$

such that

$$\forall t_i \in (a^m_i, z^m_i) : \beta^m_i(t_i) \in (x - \delta, x + \delta), \tag{62}$$

$$\forall i \in K : |a_i - a^m_i| < \epsilon < \tfrac{1}{4}(z^m_i - a^m_i), \tag{63}$$

$$\textstyle\sum_{i \in I} \text{Prob}\left(([a_i, z_i] \setminus [a^m_i, z^m_i]) \cup ([a^m_i, z^m_i] \setminus [a_i, z_i])\right) < \epsilon / (2 \max_{i \in I} \overline{t}_i), \tag{64}$$

$$\forall k \in K \setminus \{1\} : \text{Prob}\left([a^m_k, a^m_k + |a_k - a^m_k| + |a_1 - a^m_1|\right) < \epsilon / (2 \max_{i \in I} \overline{t}_i), \tag{65}$$

$$\text{Prob}\left\{\exists j \notin K \left[x - \delta \leq \beta^m_j(\mathbf{t}_j) \leq x + \delta\right]\right\} < (\eta + \epsilon/2) / \max_{i \in I} \overline{t}_i. \tag{66}$$

For any $b \in \mathbb{R}$, any $m = 1, 2, \ldots$, and any $i \in I$, define

$$\lceil b \rceil_i \quad := \quad \min\{b' \in B_i^m : b' \geq b\}.$$

Now consider bidder 1. Pick any $t_1 \in (a_1^m, z_1^m)$ such that

$$0 < t_1 - a_1^m < \epsilon. \tag{67}$$

By definition of a_1^m and monotonicity of β_1^m, $\beta_1^m(t_1) \in N(x; \delta)$. For any

$$t_{-1} := (t_j)_{j \in I \setminus \{1\}} \in E(\beta_1^m(t_1), x + \delta)$$

with respect to β^m, if bidder $j \in K \setminus \{1\}$ wins, then

$$x + \delta > \beta_j^m(t_j) = \max_{k \in K \setminus \{1\}} \beta_k^m(t_k) \geq \beta_1^m(t_1) > x - \delta,$$

hence the definition of a_j^m implies $t_j \geq a_j^m$; conditional on $\mathbf{t}_j \geq a_j^m$, we know by Ineq. (65) that with probability at least $\epsilon/(2 \max_I \bar{t}_i)$,

$$t_j \geq a_j^m + |a_j^m - a_j| + |a_1^m - a_1^m|,$$

which implies

$$t_j \geq a_j + |a_1^m - a_1^m| \overset{(53)}{\geq} a_1 + |a_1^m - a_1^m| \geq a_1^m \overset{(67)}{>} t_1 - \epsilon,$$

hence bidder 1's payoff from acquiring the good from this type t_j of reseller j is at most ϵ. Other than buying from a bidder $j \in K \setminus \{1\}$ who is involved in the tie, the only other possibility for bidder 1 to buy at resale is to buy from a bidder $i \notin K$, which is an event of probability less than $(\eta + \epsilon/2)/\max_{i \in I} \bar{t}_i$ by Ineq. (66). Thus,

$$\mathbb{E}\left[\mathbf{1}_{\mathbf{t}_{-1} \in E(\beta_1^m(t_1), \lceil x + \delta \rceil_1)} L_1(t_1, \mathbf{t}_{-1})\right] \leq \frac{\eta + \epsilon/2}{\max_{i \in I} \bar{t}_i} \max_I \bar{t}_i + \frac{\epsilon}{2 \max_I \bar{t}_i} \epsilon < \eta + \epsilon, \tag{68}$$

i.e., the type-t_1 bidder 1's expected payoff from losing is negligible conditional on the event bidder 1 would have outbid the tying bidders had he raised the bid to $\lceil x + \delta \rceil_1$. Denote

$b_1 := \beta_1^m(t_1)$. We have

$$\begin{aligned}
& \mathbb{E}\left[\mathbf{1}_{\mathbf{t}_{-1}\in E(b_1,\lceil x+\delta\rceil_1)}\left(W_1\left(b_1,t_1,\mathbf{t}_{-1}\right)-b_1-L_1\left(t_1,\mathbf{t}_{-1}\right)\right)\right] \\
\overset{(68)}{>} \; & \mathbb{E}\left[\mathbf{1}_{\mathbf{t}_{-1}\in E(b_1,\lceil x+\delta\rceil_1)}\left(W_1\left(b_1,t_1,\mathbf{t}_{-1}\right)-b_1\right)\right]-\eta-\epsilon \\
= \; & \mathbb{E}\left[\mathbf{1}_{\mathbf{t}_{-1}\in E(b_1,\lceil x+\delta\rceil_1)}\left(W_1^*\left(b_1,t_1,\mathbf{t}_{-1}\right)-b_1\right)\right] \\
& +\mathbb{E}\left[\mathbf{1}_{\mathbf{t}_{-1}\in E(b_1,\lceil x+\delta\rceil_1)}\left(W_1\left(b_1,t_1,\mathbf{t}_{-1}\right)-W_1^*\left(b_1,t_1,\mathbf{t}_{-1}\right)\right)\right]-\eta-\epsilon \\
\overset{(57)}{\geq} \; & \mathbb{E}\left[\mathbf{1}_{\mathbf{t}_{-1}\in E(b_1,\lceil x+\delta\rceil_1)}\right]\mathbb{E}\left[W_1^*\left(b_1,t_1,\mathbf{t}_{-1}\right)-b_1 \mid E(l,b)\right] \\
& +\mathbb{E}\left[\mathbf{1}_{\mathbf{t}_{-1}\in E(b_1,\lceil x+\delta\rceil_1)}\left(W_1\left(b_1,t_1,\mathbf{t}_{-1}\right)-W_1^*\left(b_1,t_1,\mathbf{t}_{-1}\right)\right)\right]-\eta-\epsilon \\
\geq \; & \mathbb{E}\left[\mathbf{1}_{\mathbf{t}_{-1}\in E(b_1,\lceil x+\delta\rceil_1)}\right]\mathbb{E}\left[W_1^*\left(b_1,t_1,\mathbf{t}_{-1}\right)-b_1-L_1(t_1,\mathbf{t}_{-1}) \mid E(l,b)\right] \\
& +\mathbb{E}\left[\mathbf{1}_{\mathbf{t}_{-1}\in E(b_1,\lceil x+\delta\rceil_1)}\left(W_1\left(b_1,t_1,\mathbf{t}_{-1}\right)-W_1^*\left(b_1,t_1,\mathbf{t}_{-1}\right)\right)\right]-\eta-\epsilon \\
\overset{(54)}{\geq} \; & \mathbb{E}\left[\mathbf{1}_{\mathbf{t}_{-1}\in E(b_1,\lceil x+\delta\rceil_1)}\left(W_1\left(b_1,t_1,\mathbf{t}_{-1}\right)-W_1^*\left(b_1,t_1,\mathbf{t}_{-1}\right)\right)\right]-\eta-\epsilon, \qquad (69)
\end{aligned}$$

where the third, unlabeled, inequality is uses the fact that one's payoff from losing, $L_1(t)$, is always nonnegative, as well as the lower branch of Eq. (56).

Claim 2 For any $\eta > 0$, any $\epsilon > 0$ satisfying (59), any $\delta > 0$ satisfying (61), any integer $m \geq m_*(\eta, \epsilon, \delta)$, and any $b_1 \in B_1^m \cap N(x;\delta)$, if

$$\mathbb{E}\left[\mathbf{1}_{\mathbf{t}_{-1}\in E(b_1,\lceil x+\delta\rceil_1)}\left(W_1\left(b_1,a_1^m,\mathbf{t}_{-1}\right)-W_1^*\left(b_1,a_1^m,\mathbf{t}_{-1}\right)\right)\right] > \eta+\epsilon+2\delta \qquad (70)$$

then $\beta_1^m(t_1') > b_1$ for all $t_1' \in (a_1^m, z_1^m)$.

To prove the claim, it suffices to prove $\beta_1^m(t_1) > b_1$ for all t_1 in some neighborhood $(a_1^m, a_1^m+\epsilon')$. Monotonicity of β_1^m then extends the claim to the other t_1 in (a_1^m, z_1^m). Since $W_1(b_1, \cdot, t_{-1})$ is continuous, there is a neighborhood $(a_1^m, a_1^m+\epsilon')$ of a_1^m such that Ineq. (70) remains true when we replace a_1^m there by any point $t_1 \in (a_1^m, a_1^m+\epsilon')$. Thus, pick any t_1 specified by Ineq. (67). By (62), $\beta_1^m(t_1) \in N(x;\delta)$. Therefore, $\beta_1^m(t_1) > b_1$ is true if for any $b_1' \leq b_1$ with $b_1' \in N(x;\delta)$, $\beta_1^m(t_1) \neq b_1'$. Suppose by negation that $\beta_1^m(t_1) = b_1'$. We derive a contradiction by showing that the type-t_1 bidder 1 strictly prefers bidding $\lceil x+\delta\rceil_1$

to bidding b_1'. Applying Ineq. (31) to the case of $i = 1$ and $b_i'' = \lceil x + \delta \rceil_1$, we have

$$\begin{aligned}
U_1(\lceil x+\delta\rceil_1, t_1; \beta^m_{-i}) - U_1(b_1', t_1; \beta^m_{-i}) &\geq \mathbb{E}\left[\mathbf{1}_{\mathbf{t}_{-1}\in E(b_1,\lceil x+\delta\rceil_1)}\left(W_1\left(b_1,t_1,\mathbf{t}_{-1}\right) - b_1 - L_1\left(t_1,\mathbf{t}_{-1}\right)\right)\right] \\
&\quad -(\lceil x+\delta\rceil_1 - b_1')\mathbb{E}\left[\mathbf{1}_{1=h_1^*(\lceil x+\delta\rceil_1,\mathbf{t}_{-1})}\right] \\
&\overset{(69)}{>} \mathbb{E}\left[\mathbf{1}_{\mathbf{t}_{-1}\in E(b_1',\lceil x+\delta\rceil_1)}\left(W_1\left(b_1,t_1,\mathbf{t}_{-1}\right) - W_1^*\left(b_1,t_1,\mathbf{t}_{-1}\right)\right)\right] \\
&\quad -\eta - \epsilon - (\lceil x+\delta\rceil_1 - b_1')\mathbb{E}\left[\mathbf{1}_{1=h_1^*(\lceil x+\delta\rceil_1,\mathbf{t}_{-1})}\right] \\
&\overset{(70)}{\geq} \eta + \epsilon + 2\delta - \eta - \epsilon - 2\delta \\
&= 0,
\end{aligned}$$

as desired. □

Claim 3 If tie occurs with strictly positive probability at x at the limit β^* then

$$\text{Prob}\left\{1 = h_1^*(\beta_1^*(\mathbf{t}_1), \mathbf{t}_{-1}) \mid \forall k \in K\left[\beta_k^*(\mathbf{t}_k) = x\right]\right\} = 1. \tag{71}$$

To prove the claim, pick η, ϵ, δ and m as specified in Claim 2. Conditional on the event $\mathbf{t}_1 \in (a_1^m, z_1^m)$ and $\mathbf{t}_{-1} \in E(x-\delta, x+\delta)$ with respect to β^m, the complement of the event $1 = h_1^*(\beta_1^m(\mathbf{t}_1), \mathbf{t}_{-1})$ (i.e., bidder 1 wins) is contained in the event $\beta_1^m(\mathbf{t}_1) \leq \max_{I\setminus\{1\}} \beta_i^m(\mathbf{t}_i)$, which in turn is contained by the union of two events: (i) $\beta_1^m(\mathbf{t}_1) \leq \max_{K\setminus\{1\}} \beta_k^m(\mathbf{t}_i)$ and (ii) $\beta_1^m(\mathbf{t}_1) \leq \max_{I\setminus K} \beta_i^m(\mathbf{t}_i)$. By Ineq. (66), the probability of event (ii) is at most $(\eta + \epsilon/2)\max_I \bar{t}_i$. Let us consider event (i). For it to happen, there must be some $t_1 \in (a_1^m, z_1^m)$ such that $\beta_1^m(t_1) = b_1$ for some $b_1 \in N(x;\delta)$. Then the contrapositive of Claim 2 implies

$$\mathbb{E}\left[\mathbf{1}_{\mathbf{t}_{-1}\in E(b_1,\lceil x+\delta\rceil_1)}\left(W_1\left(b_1,a_1^m,\mathbf{t}_{-1}\right) - W_1^*\left(b_1,a_1^m,\mathbf{t}_{-1}\right)\right)\right] < \eta + \epsilon + 2\delta.$$

I.e., by Eq. (58),

$$\mathbb{E}\left[\mathbf{1}_{\mathbf{t}_{-1}\in E(b_1,\lceil x+\delta\rceil_1)} \sum_{j\in I\setminus\{1\}} \mathbf{1}_{b_1<\beta_j^m(t_j)<\lceil x+\delta\rceil_1}\left(t_j - \max\left\{a_1^m, \text{proj}_{b_1}^j(t_{-1})\right\}\right)^+\right] < \eta+\epsilon+2\delta. \tag{72}$$

By (55), $\text{proj}_{b_1}^j(t_{-1}) \leq t_{-1}$ and strictly so on the dimension j such that $b_i < \beta_j^m(t_j) < x+\delta$. Furthermore, by Ineq. (63),

$$a_1^m \leq a_1 + \epsilon \overset{(53)}{\leq} a_j + \epsilon \leq a_j^m + 2\epsilon \leq \frac{1}{2}(a_j^m + z_j^m) < z_j^m.$$

Thus, $a_1^m < \mathbf{t}_j$ for a positive measure of $\mathbf{t}_j$ for some j. Thus, every term inside the integral on the left-hand side of (72) is nonnegative and is positive on a positive measure. It follows that the probability of the event $E(\beta_1^m(t_1), x+\delta)$ is bounded from above by $O(\eta+\epsilon+2\delta)$. Coupled with the upper bound of the probability of event (ii), we have, for any $m \geq m_*(\eta, \epsilon, \delta)$,

$$\text{Prob}\left\{\beta_1^m(\mathbf{t}_1) \leq \max_{k\in K\setminus\{i\}} \beta_k^m(\mathbf{t}_k) \mid (a_1^m, z_1^m) \times E(x-\delta, x+\delta)\right\} < O(\epsilon) + O(\eta+\epsilon+2\delta).$$

No matter how small η, ϵ and δ are, in any m-approximation game with $m \geq m_*(\eta, \epsilon, \delta)$, Claims 1 and 2 are valid and hence the previous inequality is true. Thus, Eq. (71) follows. □

But then Eq. (71) contradicts the equal-probability uniform tie-breaking rule. Thus, the probability of tying at any bid $x > l$ according to the limit bid functions β^* is zero . ■

7 Conclusion

Next version.

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