A market approach to fractional matching*

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First draft: Oct 25, 2010
This version: February 20, 2012

Abstract

We define a general fractional matching model. Each person has a set of potential partners and consumes a bundle of partnerships. Each person consumes the same quantity of a particular partnership as his partner does. Each person's preferences are defined over partnership bundles.

This model has several natural applications: the assignment of probability distributions over deterministic matchings for marriage problems, school choice, scheduling workers at various work sites, organizing paired activities among a group, and so on.

For this novel model, we define a *price* based solution. We show that the *core* of each problem is non-empty. We show that our solution selects a subset of the *core*. We also show that if the number of people involved increases—in a way that there is a fixed number of "kinds" of people—the gains from misreporting preferences diminish.

JEL classification: C71, C78, D51, D61

Keywords: *Randomization, matching, marriage problems, school choice, general equilibrium, trading favors, networks*

*I thank William Thomson for the many invaluable discussions that we have had. I also thank Paulo Barelli, Eric Buddish, John Duggan, Lars Ehlers, Michael Insler, Bariş Kaymak, Bettina Klaus, Karol Szwagrzak, Romans Pancs, Gabor Virag, workshop participants at the University of Rochester, participants of the 2011 Conference on Economic Design, and seminar audiences at Université de Montréal, Yeshiva University, Université de Lausanne, Aalto University, the University of Western Ontario, Universidad de Los Andes, and Université du Québec à Montréal for helpful comments and suggestions.

1 Introduction

Professor X has several graduate students in his charge. Each of his students has a fixed amount of time to devote to research (in a given semester, let us say). Research interests dictate which students are compatible as co-authors. Professor X needs a "good" way to decide which of his students should collaborate, and for how long.

Our goal is to solve problems like the one faced by Professor X.

The abstract model that we study is one where each person has a set of potential partners. His consumption space is the set of all combinations of potential partners (i.e. the simplex whose dimension is the number of his potential partners). His preferences are convex, continuous, and locally non-satiated (except at the maxima), over this consumption space. A feasible allocation is one where, for each pair i and j, the amount of partnership with j that i consumes is exactly the same as the amount of partnership with i that j consumes, and no person is partnered for more than his availability.

Our model is very general. Among others, it includes the following models as special cases:

1. Fractional (heterosexual) marriage (Rothblum 1992, Roth, Rothblum and Vande Vate 1993, Aldershof, Carducci and Lorenc 1999, Baïou and Balinski 2000, Klaus and Klijn 2006, Bogomolnaia and Moulin 2004, Sethuraman, Teo and Qian 2006, Manjunath 2011): Since the potential partners are determined by a bi-partition, any feasible allocation is a bistochastic matrix and thus a probability distribution over deterministic matchings. In fact, our model covers a fractional version of the *roommate problem* (Gale and Shapley 1962).[1]

2. Trade under bilateral constraints (Bochet, İlkılıç, Moulin and Sethuraman Forthcoming, Szwagrzak 2012c, Szwagrzak 2012a, Szwagrzak 2012b): This is the case where each person is indifferent between trading partners and has single peaked preferences only over the volume of his trade.

3. School choice (Abdulkadiroğlu and Sönmez 2003, Abdulkadiroğlu, Pathak, Roth and Sönmez 2005, Erdil and Ergin 2008, Abdulkadiroğlu, Che and Yasuda 2010): While a school does not have preferences over the children that are admitted to it, each school is associated with a priority order over children. These priorities dictate which children are to be favored at each school. Just as in the fractional marriage model, feasible allocations are bistochastic matrices.

[1] However, fractional matchings for the roommate problem cannot necessarily be expressed as probability distributions over deterministic matchings (Budish, Che, Kojima and Milgrom 2010).

4. Matching differently skilled workers to various employers who desire particular combinations of skills. We interpret a feasible allocation as a schedule that determines the time each worker spends at each job. To the extent of our knowledge, though related to the "stable schedule problem" (Baïou and Balinski 2002, Alkan and Gale 2003), the model presented in Section 5.4 is novel.

5. Trading favors over a network: Each person is connected to a set of people with whom he can engage in pairwise activities. Since each member may value such activities differently, each person may consider activities with some partners to be be "costly" while those with others to be "beneficial." In this way, we can inerpret our model of one where favors are traded. At a feasible allocation, each person takes part in various quantities of paired activities with different partners. We can interpret the allocation as one where a person does "favors" for some people (a *costly* activity for him that is *beneficial* for the other) and receives favors from others (a *beneficial* activity for him that is *costly* for the other).

Just as with more familiar matching models without transfers, such as the marriage problem (Gale and Shapley 1962), we can think of fractional matching problems as resource allocation problems where the resources are people. That is, the resources that need to be allocated have preferences over whom they are consumed by. The only difference is, that in our model, these resources are divisible. In terms of applications, there are scenarios where considering fractional matchings can have benefits in terms of both fairness and efficiency. The idea that randomization permits the fair allocation of discrete goods dates back the the Bible (Hofstee 1990). The same goes for time-sharing as a means to fairness. The following example demonstrates the gains in efficiency.

Example 1. *Ex-ante efficiency gains from randomization.*

Consider the following unhappy situation involving two men, m_1 and m_2, and two women, w_1 and w_2. Suppose that m_1 prefers w_1 to being single, and being single to w_2. Similarly, w_1 prefers m_2 to being single to m_1, m_2 prefers w_2 to being single to w_1, and finally w_2 prefers m_1 to being single to m_2.

The requirement of "individual rationality" says that each person has the right to remain single. That is, no person should find his partner to be worse than being single. If we restrict our attention to discrete allocations, the only *individually rational* allocation is for each person to remain single. To consider fractional allocations, we need more information about preferences. Suppose that each person's preferences over bundles of being with their most preferred mate, least preferred mate, and alone are as depicted in Figure 1. As shown in the figure, assigning a

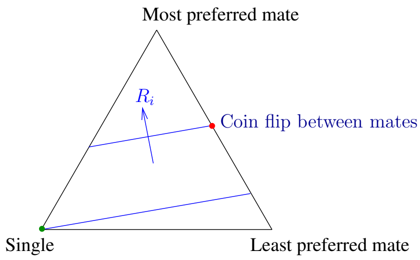

Figure 1: Preferences of m_1, m_2, w_1 and w_2: Since preferences are over lotteries, we represent this by the simplex with three corners. The least and most preferred mates are each at a corner and being single is at the other. We have depicted an indifference curves through the deterministic outcome of being single as well as a coin flip between the two possible mates. The arrow shows the direction of increasing preferences.

lottery between both possible mates is an ex-ante Pareto improvement over the only individually rational discrete allocation.

Our main contributions are:

1. We set up a new model of fractional matching. We show that it can be interpreted as a production model. A key insight is that a partnership is a public good in some senses and a private good in others. It is like a public good in that if a person i consumes a certain amount of partnership with j, then j necessarily consumes the same amount of partnership with i. It is a private good in that i excludes all others from that amount of partnership with j. Understanding this helps us define a price-based solution for such economies. Since "resources have preferences," the price system that we consider will have to reflect them. We achieve this through "double-indexed" prices. A natural interpretation for double-indexed prices is that since each i has preferences over whom he partners with, it is natural that he would charge different partners different prices. Specifically, if he prefers j to k, then he would charge j less than he would charge k.

2. We define an appropriate notion of the *core* and show that our solution is a sub-correspondence of it. This establishes non-emptiness of the *core*.

3. We add some structure to our model to incorporate the concept of a "kind" of person. In this more general version of our model, each person has preferences over the kinds of his potential partners. Think of each person as having a set of "external characteristics" or a *kind*. For the examples listed above, these could be a man or woman's income, and education, a trader's

connections, a school child's neighborhood and number of siblings at different schools, and a worker's qualifications. Not only are these characteristics observable by others, but they are what preferences are based on.

We consider a generalization of our model where a person is identified by a *kind* along with his preferences over bundles of kinds. We extend our solution by indexing prices by kind rather than identity. This is important for two reasons. First, from a fairness point of view, two people who are exactly the same ought to be given exactly the same opportunities. Second, if the market is "thick" (there are many people of each kind), for given utility representations, the gains from misreporting preferences is small for each person of each kind.

While Shapley and Shubik (1969) have defined *competitive equilibria* for matching problems, we remind the reader that the model that they consider involves monetary transfers. Subsequently, Kelso and Crawford (1982) have shown the nonemptiness of the *core* and its equivalence with the set of *competitive allocations* for matching markets involving money. Cole and Prescott (1997), Bikhchandani and Ostroy (2002), and Sun and Yang (2006) have also used non-anonymous prices. Our model differs from the ones studied in these papers in an important way: monetary transfers are not possible in our model. Further, unlike Bikhchandani and Ostroy (2002) and Sun and Yang (2006) the goods in our model are divisible.

Allocation models where resources are not associated with preferences can be encoded as instances of the model that we study. However, the analysis presented here is not as interesting as it is for the case where resources *are* associated with preferences. In particular, we need not resort to double-indexed prices since single-indexed *competitive equilibria from equal income* typically do exist for these problems (Hylland and Zeckhauser 1979, Budish forthcoming).

Though there are papers on two-sided "probabilistic" (or fractional) matching, such as those mentioned above, their focus has been on the *ex-post core* (Rothblum 1992, Roth et al. 1993, Aldershof et al. 1999, Baïou and Balinski 2000, Klaus and Klijn 2006, Bogomolnaia and Moulin 2004, Sethuraman et al. 2006, Manjunath 2011). Exceptions are Bogomolnaia and Moulin (2004) and Manjunath (2011). However, Bogomolnaia and Moulin (2004) study problems where preferences are "dichotomous." For this very restricted class of problems, they propose a rule that fulfills certain efficiency and fairness criteria. Manjunath (2011) studies various *ex-ante core* notions and their logical relations.

The remainder of the paper is organized as follows. In Section 2 we formally introduce the model and define key concepts. In Section 4 we prove that our solution is well defined and that it is a selection from the core. We particularize our model for specific applications in Section 4. In Section 6 we generalize our model to accommodate *kinds* of people.

2 The Model

Let N be a set of people. Let each $i \in N$ be associated with a set of **potential partners** $\boldsymbol{S_i} \subseteq N$ such that $i \in S_i$.[2] For each $i \in N$, let i's **availability** be $a_i \in \mathbb{R}$ such that $a_i > 0$. Let i's **consumption set** be $\Delta(a_i, S_i) \equiv \{x \in \mathbb{R}_+^{S_i} : \sum_{j \in S_i} x_j = a_i\}$. Let $\boldsymbol{R_i}$ be i's preference relation over $\Delta(a_i, S_i)$. We require that R_i be continuous and convex. Further we require that it be locally non-satiated, except at its maxima on $\Delta(a_i, S_i)$. Let $\mathcal{R}_i$ be the set of all such preferences. For each pair $x, y \in \Delta(a_i, S_i)$, if i finds x to be at least as desirable as y under preference relation R_i, we write $x \; R_i \; y$. Similarly, if i prefers x to y, we write $x \; P_i \; y$. If he is indifferent between them, we write $x \; I_i \; y$.

An **economy** is fully described by a profile of preferences $R \in \mathcal{R} \equiv \times_{i \in N} \mathcal{R}_i$.

A **feasible allocation** specifies for each $i \in N$ a consumption bundle $\pi_i \in \Delta(a_i, S_i)$ in a way that for each $i \in N$ and each $j \in S_i$, $\pi_{ij} = \pi_{ji}$. Let $\boldsymbol{\Pi}$ be the set of feasible allocations. We represent a feasible allocation by a symmetric $N \times N$ matrix, of which, the columns sum to one and for each $i \in N$, the i^{th} row sums to a_i.

A **solution**, $\phi : \mathcal{R} \rightrightarrows \Pi$, associates each economy with a set of feasible allocations.

The *availability* of each $i \in N$ plays no significant role in the remainder of the definitions and the results that we present. In order to simplify notation, we normalize so that for each $i \in N$, $a_i = 1$ and denote i's consumption space by $\Delta(S_i)$. Nonetheless, the results carry through for arbitrary profiles of *availabilities*.

3 Solutions

We start with some normative concepts. The first one reflects a very familiar notion of *efficiency*. For each $R \in \mathcal{R}$ and $\pi \in \Pi$, we say that π is **Pareto-efficient at $\boldsymbol{R}$** if there is no $\pi' \in \Pi$ such that for each $i \in N, \pi'_i \; P_i \; \pi_i$. Let $\boldsymbol{P(R)}$ be the set of *Pareto-efficient allocations at R.*

The Next is an expressesion of the principle that each person has the right to "consume" himself. Let $\boldsymbol{\delta} \in \Pi$ be such that for each $i \in N$, $\delta_{ii} = 1$. For each $R \in \mathcal{R}$ and $\pi \in \Pi$, we say that π is **individually rational at $\boldsymbol{R}$** if for each $i \in N, \pi_i \; R_i \; \delta_i$. Let $\boldsymbol{I(R)}$ be the set of allocations that are *individually rational at R.*

[2]While this specification of *potential partners* might remind the reader of that in Sönmez (1996), it ought to be noted that Sönmez's analysis is not restricted to bilateral situations such as ours.

At one end, we have defined the *Pareto* solution that picks allocations that society as a whole cannot improve upon. At the other end, the *individually rational* solution respects the rights of individuals. The following solution extends these principles to groups of all sizes. For each $R \in \mathcal{R}$, $\pi \in \Pi$, and $S \subseteq N$, $\boldsymbol{S}$ **blocks** $\boldsymbol{\pi}$ **at** $\boldsymbol{R}$ if there is $\pi^S \in \Pi$ such that for each $i \in S$,

$$i) \quad \sum_{j \in S} \pi^S_{ij} = 1 \text{ and}$$

$$ii) \quad \pi^S_i \; P_i \; \pi_i.$$

The **core at** $\boldsymbol{R}$**,** $\boldsymbol{C(R)}$**,** is the set of allocations that are not *blocked* by any coalition at R. We will see, in the sequel, that $C(R)$ is never empty.

As expressed when we defined *individual rationality*, it is natural to think of each person as "owning" himself. The next obvious step is to wonder whether each person can trade what they own for something they prefer. That is, can we find prices that dictate such trades in a meaningful way?

We begin our search for a price-based solution with a naïve first attempt. Since each person "owns" himself, we assign a price to each person and allow people to trade parts of themselves for parts of others. An allocation $\pi \in \Pi$ is a **Walrasian allocation at** R if there is a vector $p \in \mathbb{R}^N$ such that for each $i \in N$,

$$\pi_i \in \underset{\pi'_i \in \Delta(S_i)}{\text{argmax}} \; R_i$$
$$\text{subject to}$$
$$\underbrace{\sum_{j \in S_i} \pi_{ij} p_j}_{\text{Price of } \pi'_i} \leq \underbrace{1 \cdot p_i}_{i\text{'s income}} .$$

We refer to (π, p) as a **Walrasian equilibrium**. Let $\boldsymbol{W(R)}$ be the set of all *Walrasian allocations at* R. As demonstrated by Example 2, $W(R)$ may be empty.

Example 2. *An economy with no* Walrasian allocation.

Let $N \equiv \{m_1, m_2, w_1, w_2\}$ and

$$\begin{aligned} S_{m_1} &\equiv \{m_1, w_1, w_2\}, \\ S_{m_2} &\equiv \{m_2, w_1, w_2\}, \\ S_{w_1} &\equiv \{w_1, m_1, m_2\}, \text{ and} \\ S_{w_1} &\equiv \{w_2, m_1, m_2\}. \end{aligned}$$

Let $R \in \mathcal{R}$ be such that the following are numerical representations:

$$\begin{aligned}
&\text{For each } \pi_{m_1} \in \Delta(S_{m_1}), u_{m_1}(\pi_{m_1}) = 2\pi_{m_1 w_1} + \pi_{m_1 w_2},\\
&\text{for each } \pi_{m_2} \in \Delta(S_{m_2}), u_{m_2}(\pi_{m_2}) = 2\pi_{m_2 w_2} + \pi_{m_2 w_1},\\
&\text{for each } \pi_{w_1} \in \Delta(S_{w_1}), u_{w_1}(\pi_{w_1}) = 2\pi_{w_1 m_2} + \pi_{w_1 m_1}, \text{ and}\\
&\text{for each } \pi_{w_2} \in \Delta(S_{w_2}), u_{w_2}(\pi_{w_2}) = 2\pi_{w_2 m_1} + \pi_{w_2 m_2}.
\end{aligned}$$

Let $p \in \mathbb{R}^N_+$. Suppose that (π, p) is a *Walrasian equilibrium*. Suppose $m_1 \in \underset{i \in N}{\text{argmax}}\, p_i$. Then, $\pi_{m_1 w_1} = 1$. By feasibility, $\pi_{m_2 w_1} = 0$. So, $p_{w_1} > p_{m_2}$. This implies that $\pi_{w_1 m_2} = 1$ and contradicts $\pi_{m_1 w_1} = 1$. Since the problem is symmetric (each person has the same preferences over bundles of being single, with the most preferred mate and with the least preferred mate), we reach a similar contradiction if $m_1 \notin \underset{i \in N}{\text{argmax}}\, p_i$. ◦

The reason that a *Walrasian allocation* may not exist is that some of the consumption goods in these economies are *not* private goods: a partnership involves both members.

For our next attempt, we draw inspiration from the literature on public goods economies and introduce "double-indexed" prices, as follows.

Let $M \subseteq N \times N$ be such that $(i, j) \in M$ if and only if $j \in S_i$ and $i \in S_j$. We say that $\pi \in \Pi$ is a **double-indexed price (DIP) allocation at** R if there is a vector $p \in \mathbb{R}^M_+$ such that for each $i \in N$,

$$\begin{gathered}
\pi_i \in \underset{\pi'_i \in \Delta(S_i)}{\text{argmax}}\, R_i\\
\text{subject to}\\
\underbrace{\sum_{j \in S_i} \pi'_{ij} p_{ij}}_{\text{Price of } \pi'_i} \leq \underbrace{\sum_{j \in S_i} \pi_{ji} p_{ji}}_{i\text{'s income at } \pi},
\end{gathered}$$

and $\pi \in \Pi$ (this ensures that the "market clears"). We interpret the price vector as follows: for each $(i, j) \in M$, p_{ij} *is the price that* i *pays for* j.

We refer to (π, p) as a **double-indexed price equilibrium**. Let $\boldsymbol{D(R)}$ be the set of all *DIP allocations at* R.

It is easy to see that a *Walrasian equilibrium*, if it exists, is also a *DIP equilibrium*: Let (π, q) be a *Walrasian equilibrium* and define $p \in \mathbb{R}^M_+$ by setting, for each $i \in N$ and each $j \in S_i, p_{ji} = q_i$. It follows directly from the two definitions that (π, p) is a *DIP equilibrium*.

Remark 1. Our definition of a *DIP equilibrium* has the flavor of a "Lindahl equilibrium" (Lindahl 1958). The reason is that a positive amount of a partnership between i and $j \in N$ is not a private good (nor is it purely a public good). If i consumes a certain amount of this partnership, say π_{ij}, then he excludes all others from consuming it. Yet, j is not excluded. Note that partnerships are not "common goods" or "club goods" either.[3] $\triangle$

Unfortunately, even *DIP allocations* may not exist as demonstrated by Example 3.

Example 3. *An economy with no* DIP allocation.[4]

Let $N \equiv \{1, 2\}$ and for each $i \in N$, $S_i \equiv N$. Let $R \in \mathcal{R}$ be such that for each $i \in N, R_i$ is represented by $u_i : \Delta(S_i) \to \mathbb{R}$ defined as follows:

$$\begin{array}{l} \text{For each } \pi_1 \in \Delta(S_1), u_1(\pi_1) = \pi_{12} \text{ and} \\ \text{for each } \pi_2 \in \Delta(S_2), u_2(\pi_2) = -(\frac{1}{4} - \pi_{21})^2. \end{array}$$

Suppose that $(\pi, p) \in \Pi \times \mathbb{R}^M_+$ is a *DIP equilibrium*. Only the relationship between the prices p_{12} and p_{21} is relevant. For each possibility, we show that $\pi \notin \Pi$: If $p_{12} > p_{21}$, then $\pi_{12} = 0$ and $\pi_{21} = 1$. If $p_{12} < p_{21}$, then $\pi_{12} = \frac{1}{4}$ and $\pi_{21} = 0$. Finally, if $p_{12} = p_{21}$, then $\pi_{12} = \frac{1}{4}$ and $\pi_{21} = 1$. $\circ$

The difficulty here arises from the fact that "endowments" of each person are on the boundaries of their consumption spaces. As with exchange economies, this sometimes precludes the existence of equilibria. We propose the following way of dealing with this:

1. For each $i \in N$, we extend each i's preference relation from $\Delta(S_i)$ to $\Lambda(S_i) \equiv \{\lambda \in \mathbb{R}^{S_i}_+ : \sum_{j \in S} \lambda_j \leq 1\}$. We take care to make sure that the extended preference relation satisfies certain properties (that we will describe below).

2. We redistribute an abitrarily small amount of each person's endowment among others.

3. We define a price-equilibrium for this modified economy.

We first explain how to extend preferences. Clearly, for each $i \in N, \Delta(S_i) \subset \Lambda(S_i)$. For each $R \in \mathcal{R}$, and each $i \in N$, let $\hat{R}_i$ be an extension of R_i from $\Delta(S_i)$ to $\Lambda(S_i)$ such that $\hat{R}_i$ is:

- *strictly monotonic* over $\Lambda(S_i) \setminus \Delta(S_i)$,

[3]A *common good* is one that is not excludable but has congestion effects. A *club good* is one that can be consumed by any number of people simultaneously but is excludable.

[4]Person 2 in Example 3 does not have linear preferences. However, we can construct examples in the linear domain, involving more people, where a *DIP allocations* do not exist.

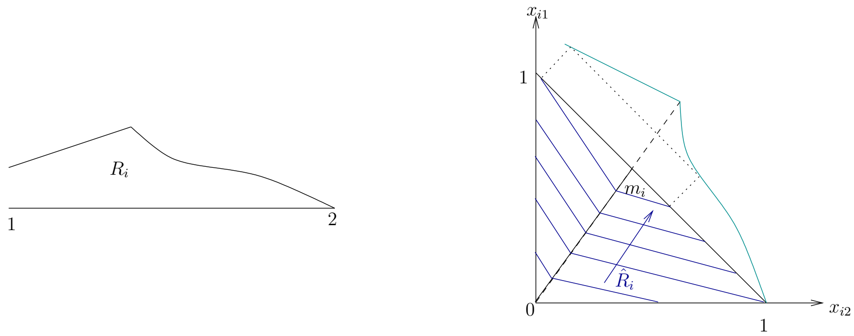

Figure 2: The preference relation $\hat{R}_i$ which is an extension of R_i from $\Delta(S_i)$ to $\Lambda(S_i)$.

- *continuous*, and
- *convex*.

Claim 1. *Such $\hat{R}$ exists.*

Proof: We describe the construction of one such profile (see Figure 2). First we extend R_i to $\{x_i \in \mathbb{R}^{S_i} : \sum_{j \in S_i} x_{ij} = 1\}$ and then define $\hat{R}_i$ over $\Lambda(S_i)$.

Let $\tilde{R}_i$ be a continuous and convex extension of R_i from $\Delta(S_i)$ to $\{x_i \in \mathbb{R}^{S_i} : \sum_{j \in S_i} x_{ij} = 1\}$ that is locally non-satiated except at the maxima of R_i over $\Delta(S_i)$.

Let $M_i \equiv \operatorname*{argmax}_{\Delta(S_i)} R_i$. For each $x \in \Delta(S_i)$, let $I_i(x) \equiv \{y \in \Delta_i : x_i \; I_i \; y_i\}$. For each $x \in \Delta(S_i)$, let

$$d(x) \equiv \min_{\substack{y \in M_i \\ z \in I_i(x)}} ||z - y||.$$

That is, $d(x)$ is the shortest distance between the indifference class of x and M_i. Note that for all x, $d(x) \leq \sqrt{2}$. Define $\hat{R}_i$ as follows: for each $x_i \in \Delta(S_i)$, let the upper contour set of $\hat{R}_i$ at x_i be

$$U(\hat{R}_i, x) \equiv \text{convex hull} \left(\left(1 - \frac{d(x)}{2} \right) M_i \cup U(\tilde{R}_i, x_i) \right) \cap \mathbb{R}^{S_i}_{+}.$$

Let $w_i \in \{x_i \in \Delta(S_i) : \text{for each } y_i \in \Delta(S_i), y_i \; R_i \; x_i\}$. That is, w_i is one of i's least preferred points in $\Delta(S_i)$ at preference relation R_i. The preference map over $\Lambda(S_i) \setminus U(\hat{R}_i, w_i)$ is completed by translating the level set of w_i.

It is easy to verify that $\hat{R}_i$ is convex, continuous, and strictly monotonic over $\Lambda(S_i) \setminus \Delta(S_i)$.

We are now ready to define our next solution.

Let $\varepsilon \in (0,1)$. An allocation $\pi \in \Pi$ is an **ε-double-indexed price (εDIP) allocation** if there is $p \in \mathbb{R}_+^M$ such that:

1. For each pair $(i,j) \in M$,

$$p_{ii} + p_{jj} \geq p_{ij} + p_{ji}.$$

2. For each pair $(i,j) \in M$ such that $\pi_{ij} > 0$.

$$p_{ii} + p_{jj} = p_{ij} + p_{ji}.$$

3. For each $i \in N$,

$$\begin{gathered} \pi_i \in \underset{x_i \in \Lambda(S_i)}{\text{argmax}}\ R_i \\ \text{subject to} \\ \underbrace{\sum_{j \in S_i} x_{ij} p_{ij}}_{\text{Price of } x_i'} \leq \underbrace{(1-\varepsilon) p_{ii} + \left(\frac{\varepsilon}{|n|-1}\right) \left(\sum_{j \in N \setminus \{i\}} p_{jj}\right)}_{i\text{'s income}}. \end{gathered}$$

4. $\pi \in \Pi$.

We refer to (π, p) as an **εDIP equilibrium**.

The first two conditions, on the price vector, relate the price of individuals as inputs and partnerships as outputs. Let $\boldsymbol{D^\varepsilon(R)}$ be the set of all *εDIP allocations at R*. As we will show, for each $\varepsilon \in (0,1)$ and each $R \in \mathcal{R}$, there is an *εDIP* allocation. Note that if we set $\varepsilon = 0$, the above definition coincides with that of a *DIP* allocation.

We now define our last solution. An allocation $\pi \in \Pi$ is a **limit DIP (lim-DIP) allocation** if there is a sequence $\{\pi^\varepsilon\}_{\varepsilon \in (0,1)} \in \Pi$ such that

$$\begin{aligned} i) &\quad \text{for each } \varepsilon \in (0,1), \pi^\varepsilon \in D^\varepsilon(R) \text{ and} \\ ii) &\quad \lim_{\varepsilon \to 0} \pi^\varepsilon = \pi. \end{aligned}$$

Let $\boldsymbol{D^l(R)}$ be the set of all *limit DIP allocations.* In Section 4 we will show that $D^l(R) \neq \varnothing$ and that $D^l(R) \subseteq C(R)$, thereby establishing that $C(R) \neq \varnothing$ as well.

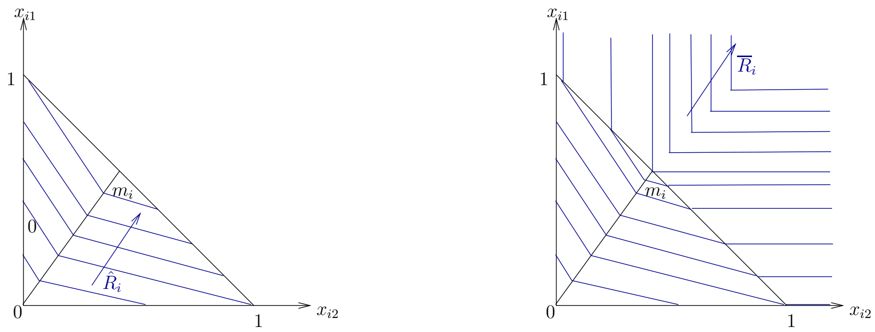

Figure 3: The preference relation $\hat{R}_i$ which is an extension of R_i from $\Delta(S_i)$ to $\mathbb{R}_+^{S_i}$.

4 Nonemptiness of *lim-DIP*

We begin this section by proving that an ε*DIP* always exists. Since Π is compact, this means that a *lim-DIP* exists (Theorem 1). We conclude by showing that every *lim-DIP allocation* is in the *core*, thereby establishing the existence of a *core* allocation.

Proposition 1. *For each $\varepsilon \in (0,1)$ and $R \in \mathcal{R}$, $D^\varepsilon(R) \neq \varnothing$.*

Proof: We proceed by embedding R in an Arrow-Debreu economy. We then show the existence of a *competitive equilibrium* of this augmented economy (McKenzie 1959, Arrow and Hahn 1971). We conclude by showing that this *competitive equilibrium* corresponds to an ε*DIP* of R.

Step 1: *Embed R in an Arrow-Debreu economy, E.*

For each $i \in N$, let i's consumption space be $X_i \subseteq \mathbb{R}_+^M$ defined by

$$x \in X_i \Leftrightarrow \ (x_{ij})_{j \in S_i} \in \mathbb{R}_+^{S_i} \text{ and for each pair } (j,k) \in M \text{ such that } j \neq i, x_{jk} = 0.$$

That is, $X_i \equiv \mathbb{R}_+^{S_i} \times \{(0, \ldots, 0)\}$. By definition, X_i is closed and convex.

A notable feature of these consumption spaces is that for each pair $i, j \in N$ $X_i \cap X_j = \{0\}$.

Note that $\Lambda(S_i) \times \{0\} \subseteq X_i$. For each $i \in N$, we now extend $\hat{R}_i$ (defined in Section 3) from $\Lambda(S_i)$ to $\overline{R}_i$ over X_i (see Figure 3).

For each $x_i \in \Lambda(S_i)$, let the upper contour set of $\overline{R}_i$ at x_i be

$$U(\overline{R}_i, x_i) \equiv \text{comp } U(\hat{R}_i, x_i).^5$$

[5]Denote the "upper comprehensive hull" of $X \subseteq \mathbb{R}^l$ by comp $(X) \equiv X + \mathbb{R}_+^l$.

Since for each $m_i \in M_i$, $U(\overline{R}_i, m_i)$ is convex, the preference map in this region is completed by translating the level set of M_i.

Clearly, $\overline{R}_i$ is *continuous, convex, monotone*, and *strictly monotone over* $\Lambda(S_i) \setminus \Delta(S_i)$

Let F be a set of $\frac{|M|-|N|}{2}$ firms. Label these firms by unordered (distinct) pairs from N, a generic member being $\{i,j\}$. The production set of $\{i,j\} \in F$ is

$$Y_{\{i,j\}} \equiv \left\{y \in \mathbb{R}^{\{ij,ji,ii,jj\}} : y_{ij} = y_{ji} = -y_{ii} = -y_{jj}\right\} \times \{0\} \subset \mathbb{R}^M.$$

Note that $Y_{\{i,j\}}$ is closed and convex.

For each $\{j,k\} \in F$ and each $i \in N$, let $\sigma_i(\{j,k\})$ be i's share of $\{j,k\}$. Thus, for each $\{j,k\} \in F$, $\sum_{i\in N} \sigma_i(\{j,k\}) = 1$.

Finally, for each $i \in N$, let $\omega^i \in \mathbb{R}^M_+$ be such that for each pair $k,j \in N$,

$$\omega^i_{kj} = \begin{cases} 1-\varepsilon & \text{if } i = j = k, \\ \frac{\varepsilon}{|N|-1} & \text{if } i \neq j = k, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\omega \equiv (\omega^i)_{i\in N}$.

We have now specified an Arrow-Debreu economy $E \equiv (X, Y, \overline{R}, \omega, \sigma)$.

Step 2: *Check that E has a competitive allocation.*

Since the set of goods that each person is endowed with is the same, E is "irreducible" (McKenzie 1959) (alternatively, we could have shown that it satisfies "resource relatedness" (Arrow and Hahn 1971)). Let $Y \equiv \sum_{\{i,j\}\in M} Y_{\{i,j\}} + \omega$ and $X \equiv \sum_{i\in N} X_i$. We have the following:

1. For each $i \in N$, X_i is convex, closed, and bounded from below.

2. For each $i \in N$, $\overline{R}_i$ is continuous, convex, and weakly monotonic.

3. For each $i \in N, X_i \cap Y \neq \varnothing$.

4. For each $\{i,j\} \in M$, $Y_{\{i,j\}}$ is closed and convex.

5. $Y \cap \mathbb{R}^M_+ = \{0\}$.

6. ω is in the relative interiors of Y and X.

7. Irreducibility (McKenzie 1959): For each bi-partition N_1, N_2 of N, if $x_{N_1} \in Y - \sum_{i\in N_2} X_i$, then there is $w \in Y - \sum_{i\in N_2} X_i$ and $x' \in X$ such that $w = \sum_{i\in N_1} x'_i - \sum_{i\in N_2} x_i$ and for each $i \in N_1, x'_i \ R_i \ x_i$ with $x'_i \ P_i \ x_i$ for at least one $i \in N_1$.

By Theorem 2 of McKenzie (1959), E has a *competitive allocation* $(x, y, p) \in X \times Y \times \mathbb{R}^M_+$.

Step 3: *Show that* $(x, p) \in D^\varepsilon(R)$.

We check that for each $i \in N, x_i \in \Delta(S_i)$. From this, we conclude that $x \in D^\varepsilon(R)$. Suppose that there is $i \in N$ such that $x_i \notin \Delta(S_i)$.

Case 1: $\sum_{j\in S_i} x_{ij} > 1$. Then, $\sum_{j\in S_i\setminus\{i\}} x_{ij} + x_{ii} > 1 = \sum_{j\in N} \omega^j_{ii}$.

For each $j \in S_i \setminus \{i\}$, $x_{ij} \leq y^{\{i,j\}}_{ij} = -y^{\{i,j\}}_{ii}$.

Thus, $\sum_{j\in S_i\setminus\{i\}} x_{ij} \leq -\sum_{j\in S_i\setminus\{i\}} y^{\{i,j\}}_{ii}$,

Finally, we establish that $x_{ii} \geq \omega_{ii} + \sum_{j\in S_i\setminus\{i\}} y^{\{i,j\}}_{ii}$. This violates the feasibility of (x, y) for E.

Case 2: $\sum_{j\in S_i} x_{ij} < 1$. Let $\alpha = 1 + \sum_{j\in S_i\setminus\{i\}} y^{\{i,j\}}_{ii}$. By feasibility, $\alpha \geq x_{ii}$. If $\alpha > x_{ii}$ then, let $x' \in X$ be such that $x'_{ii} = \alpha$ and for each $j \in N \setminus \{i\}, x_j{}' = x_j$ and $x'_{ij} = x_{ij}$. Since $\overline{R}_i$ is strictly monotone at x, we know that $x'_i \ \overline{P}_i \ x_i$. This violates the *Pareto-efficiency* of (x, y) at $\overline{R}$ (which is a *competitive allocation* for E).

Thus, $1 + \sum_{j\in S_i\setminus\{i\}} y^{\{i,j\}}_{ii} = x_{ii}$. So, $x_{ii} - \sum_{j\in S_i\setminus\{i\}} y^{\{i,j\}}_{ii} = 1 > \sum_{j\in S_i} x_{ij}$. From this, we conclude that there is $j \in S_i \setminus \{i\}$ such that $x_{ij} < -y^{\{i,j\}}_{ii} = y^{\{i,j\}}_{ij}$. Let $x' \in X$ be such that for each $k \in N \setminus \{i\}, x_k{}' = x_k$, for each $k \in S_i \setminus \{j\}, x'_{ik} = x_{ik}$, and $x'_{ij} = y^{\{i,j\}}_{ij}$. Since $\overline{R}_i$ is strictly monotone at x, we know that $x'_i \ \overline{P}_i \ x_i$. This violates the *Pareto-efficiency* of (x, y) at $\overline{R}$ (which is a *competitive allocation* for E).

Since $\sum_{j\in S_i} x_{ij} = 1$ and $x_{ii} = 1 - \sum_{j\in S_i} y^{\{i,j\}}_{ij}$, we have $\sum_{j\in S_i} x_{ij} = \sum_{j\in S_i} y^{\{i,j\}}_{ij}$. Since for each $j \in S_i \setminus \{i\}, x_{ij} \leq y^{\{i,j\}}_{ij}$, we have $x_{ij} = y^{\{i,j\}}_{ij}$. Since for each $\{i, j\} \in F, y^{\{i,j\}}_{ij} = y^{\{i,j\}}_{ji}$, we deduce that $x_{ij} = x_{ji}$.

Since (x, y, p) is an equilibrium, for each pair $i, j \in N$, if $y^{\{i,j\}}_{ij} = y^{\{i,j\}}_{ij} > 0$ then $p_{ij} + p_{ji} = p_{ii} + p_{jj}$. Otherwise, $p_{ij} + p_{ji} \geq p_{ii} + p_{jj}$.

As we have established, for each $i \in N, x_i \in \Delta(S_i)$. From the definition of $Y_{\{i,j\}}$ for each $\{i, j\} \in M$, we have $x_{ij} = x_{ji}$. Thus, $x \in \Pi$. It is clear that that for each

$i \in N$,

$$\left\{\pi_i \in \Delta(S_i) : \sum_{j \in S_i} \pi_{ij} p_{ij} \leq (1-\varepsilon) p_{ii} + \left(\frac{\varepsilon}{|n|-1}\right)\left(\sum_{j \in N \setminus \{i\}} p_{jj}\right)\right\}$$

$$\cap\!\!\!|$$

$$\left\{x_i \in X_i : \sum_{j \in S_i} x_{ij} p_{ij} \leq p \cdot \omega^i\right\}$$

Thus, (x, p) is a *εDIP equilibrium at R* and $x \in D^{\varepsilon}(R)$. □

We now establish that a *lim-DIP* allocation always exists.

Theorem 1. *For each* $R \in \mathcal{R}$, $D^l(R) \neq \varnothing$.

Proof: For each $\varepsilon \in (0,1)$, let $\pi^{\varepsilon} \in D^{\varepsilon}(R)$ (this is possible since $D^{\varepsilon}(R) \neq \varnothing$). Since Π is compact, $\pi \equiv \lim_{\varepsilon \to 0} \pi^{\varepsilon}$ is well defined and $\pi \in D^l(R)$. □

An appealing property of D^l is that it is a sub-correspondence of the *core*.

To prove this, we will use the following definitions. Recall the definition, for each $R \in \mathcal{R}$, of $\hat{R}$ in Section 3.

Let $\varepsilon \in (0,1)$. Let $S \subseteq N$. We say that S ***ε*-blocks** $\pi \in \Pi$ if there is $x^S \in \times_{i \in S} \Lambda(S_i)$ such that:

1. For each $i \in S$ and each $j \in S \cap S_i$, $x^S_{ij} = x^S_{ji}$, and
2. For each $i \in S$,

$$i) \quad \sum_{j \in S_i \cap S} x^S_{ij} = 1 - \left(\frac{|N| - |S|}{|N| - 1}\right)\varepsilon,$$

$$ii) \quad \sum_{j \in S_i \setminus S} x^S_{ij} = 0, \text{ and}$$

$$ii) \quad x^S_i \; \hat{P}_i \; \pi_i.$$

The ***ε*-core, $C^{\varepsilon}(R)$**, is the set of allocations are not ε-blocked by any coalition.

Lemma 1. *For each* $R \in \mathcal{R}$, $D^{\varepsilon}(R) \subseteq C^{\varepsilon}(R)$.

Proof: Let $\pi \in D^{\varepsilon}(R)$ and (π, p) is an *εDIP equilibrium*. Suppose that $S \subseteq N$ *ε-blocks* x^S. Then, for each $i \in S$,

$$\sum_{j \in S_i \cap S} p_{ij} x_{ij}^S > p_{ii}(1-\varepsilon) + \left(\sum_{j \in N \setminus \{i\}} p_{jj}\right) \frac{\varepsilon}{|N|-1}.$$

Summing over all members of S,

$$\sum_{i,j \in S} p_{ij} x_{ij}^S > \left(\sum_{i \in S} p_{ii}\right) \left(1 - \frac{|N|-|S|}{|N|-1}\varepsilon\right) + \left(\sum_{i \in N \setminus S} p_{ii}\right) \frac{|S|\varepsilon}{|N|-1}.$$

However, for each $(i,j) \in M$, $p_{ii} + p_{jj} \geq p_{ij} + p_{ji}$ and so,

$$\sum_{i,j \in S} p_{ij} x_{ij}^S \leq \left(\sum_{i \in S} p_{ii}\right) \left(1 - \frac{|N|-|S|}{|N|-1}\varepsilon\right).$$

From this contradiction we conclude that $\pi \in C^{\varepsilon}(R)$. □

Next, we show that the limit of a sequence of *ε-core* allocations, as ε goes to zero, is a *core* allocation.

Lemma 2. *For each $R \in \mathcal{R}$, and each sequence $\{\pi^{\varepsilon}\}_{\varepsilon \in (0,1)}$ such that for each $\varepsilon \in (0,1)$, $\pi^{\varepsilon} \in C^{\varepsilon}(R)$, we have $\lim_{\varepsilon \to 0} \pi^{\varepsilon} \in C(R)$.*

Proof: Let $\pi \in \lim_{\varepsilon \to 0} \pi^{\varepsilon}$. Suppose that $\pi \notin C(R)$. Then there is $S \subseteq N$ and π^S such that for each $i \in S$,

$$\begin{array}{rl} i) & \sum_{j \in S} \pi_{ij}^S = 1 \text{ and} \\ ii) & \pi_i^S \; P_i \; \pi_i. \end{array}$$

Let V be a neighborhood of π and V^S be a neighborhood of π^S such that for each $v \in V$, each $v^S \in V^S$, and each $i \in S$,

$$v_i^S \; \hat{R}_i \; v_i.$$

Since $\hat{R}_i$ is continuous, such V and V^S exist. For ε small enough, $\pi^{\varepsilon} \in V$ and $(1 - \left(\frac{|N|-|S|}{|N|-1}\right) \varepsilon)\pi^S \in V^S$. This contradicts $\pi^{\varepsilon} \in C^{\varepsilon}(R)$. □

We finally establish that the set of *lim-DIP* allocations is a subset of the *core*.

Theorem 2. *For each $R \in \mathcal{R}$, $D^l(R) \subseteq C(R)$.*

Proof: This follows directly from Lemmas 1 and 2. □

As a corollary of Theorems 1 and 2, we show that the core is never empty.[6]

Corollary 1. *For each* $R \in \mathcal{R}, C(R) \neq \varnothing$.

5 Applications

In this section, we consider each of the applications mentioned in the introduction in more detail.

5.1 Probabilistic (heterosexual) marriage problems

Let M be a set of men and W be a set of women. A *deterministic matching* either associates each person with a mate of the opposite sex or leaves them single.

As with other problems involving indivisibilities, randomization is one way to bring a sense of justice to a matching process (Aldershof et al. 1999, Klaus and Klijn 2006, Sethuraman et al. 2006). A common approach is to randomize only over the *ex post core* (or the *stable set*) (Sethuraman et al. 2006). However, if groups are able to commit to probabilistic allocations among themselves, a notion of *ex ante stability* is called for. That is, we should look for probabilistic matchings that are in the *core* with respect to their preferences over lotteries.[7]

To encode these problems in our model, let $N \equiv M \cup W$. For each $m \in M$, let $S_m \equiv \{m\} \cup W$ and for each $w \in W$, let $S_w \equiv \{w\} \cup M$. For each $i \in N$, let R_i be i's linear (von Neumann-Morgenstern) preferences over $\Delta(S_i)$.

The following is an implication of Theorem 1:

Corollary 2. *Every probabilistic marriage problem has a* lim-DIP *allocation.*

By considering preferences over lotteries, rather than just preferences over individual partners, we are able to account for intensities of preferences and achieve *ex ante* efficiency gains. The example described in the introduction demonstrates this.

5.2 Trade under bilateral constraints

Suppose there is a group V of vendors of some good, and a group of buyers B. However, not every vendor can sell to every buyer. Instead, a graph $G \subseteq V \times B$

[6] This can also be shown by proving that, for each problem, the non-transferable-utility (NTU) game associated with each economy is "balanced" (Scarf 1967).

[7] See Manjunath (2011) for more on the core of probabilistic marriage problems.

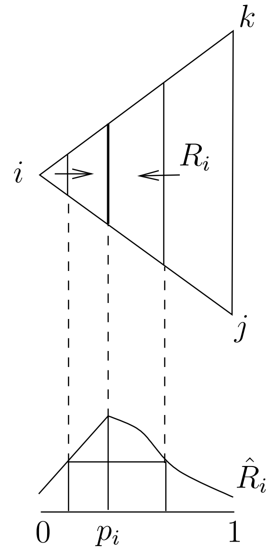

Figure 4: Let $i \in N$ be such that $S_i \equiv \{i, j, k\}$. We construct R_i from $\hat{R}_i$.

dictates which vendor-buyer pair can trade (Bochet et al. Forthcoming): the pair $v \in V, b \in B$ can trade only if $(v, b) \in G$. Each $v \in V$ has single peaked preferences, $\hat{R}_v$, over the amount that he sells. Each $b \in B$ has single peaked preferences, $\hat{R}_b$, over the amount that he purchases. Since preferences are defined over the real line, we pick suitable bounds and normalize so that the maximum any buyer can purchase or seller can sell is one unit.[8] The goal is then to specify an amount for each vendor to sell and for each buyer to purchase.

This model can be embedded in ours as follows: Let $N \equiv V \cup B$. For each $v \in V$, let $S_v \equiv \{v\} \cup B$ and for each $b \in B$, let $S_b \equiv \{b\} \cup V$. For each $i \in N$, let R_i be such that for each $\pi_i, \pi'_i \in \Delta(S_i)$, (see Figure 4)

$$\pi_i \mathcal{R}_i \pi'_i \Leftrightarrow \left(\sum_{j \in S_i} \pi_{ij} \right) \hat{R}_i \left(\sum_{j \in S_i} \pi'_{ij} \right).$$

The following is an implication of Theorem 1:

Corollary 3. *Every problem of trade under bilateral constraints has a* lim-DIP *allocation.*

Our model can accommodate two natural generalizations of these problems:

[8]While Bochet et al. (Forthcoming) do not assume that such a bound exists, if we apply their "voluntary participation" axiom, such a normalization is possible.

1. *Diverse vendors and buyers.* Corollary 3 holds for more general preferences on the part of both buyers and vendors. For instance, the vendors need not sell identical goods. The only restrictions on preferences are that, as listed earlier, they be continuous, convex, and locally non-satiated except at the maxima.

2. *More general graphs.* Rather than work with a bipartite graph such as G, Theorem 1 applies to a larger set of trading constraints. For instance, we can consider a situation where each person i owns an input that he can either sell to a set of buyers B_i or can combine with other inputs that he buys from the vendors V_i. Then, for each $i \in N$, and each $v \in V_i$, $i \in B_v$ and for each $b \in B_i, i \in V_b$. Thus, $S_i \equiv \{i\} \cup V_i \cup B_i$ and R_i is defined over triples of the form (b_i, o_i, v_i) where,

$$\begin{array}{lcll} b_i & = & \sum_{b \in B_i} \pi_{ib} & \text{is the amount of goods that } i \text{ sells to his available buyers,} \\ o_i & = & \pi_{ii} & \text{is the amount of } i\text{'s good that he keeps, and} \\ v_i & = & \sum_{v \in V_i} \pi_{iv} & \text{is the amount of goods that } i \text{ buys from his available sellers.} \end{array}$$

5.3 School Choice

Let S be a set of schools and C be a set of children. For each $c \in C$, let R_c be c's (more likely, his parents') *von Neumann-Morgenstern* preferences over $\Delta(S)$. For each $s \in S$, let $\prec_s$ be a priority ordering of children at school s which involves large indifference classes. Let R_s be a *von Neumann-Morgenstern* index over $\Delta(C)$ that is consistent with $\prec_s$. Call (R_C, R_S) an "augmented school choice problem."

We can now select an *lim-DIP* allocation.

Corollary 4. *Every augmented school choice problem has a* lim-DIP *allocation.*

In real-world school choice problem, ties are broken randomly (Erdil and Ergin 2008, Abdulkadiroğlu et al. 2010, Pathak and Sethuraman forthcoming) and used as inputs for deterministic algorithms like the *Boston* and *deferred acceptance* algorithms. Since these algorithms only consider ordinal information in students' preferences, there are *ex ante* efficiency losses (Abdulkadiroğlu et al. 2010). These losses can be avoided by modeling these problems as fractional matching problems.

Example 4. *A school choice problem.*[9]

Let $S \equiv \{s_1, s_2, s_3\}$ and $C \equiv \{c_1, c_2, c_3\}$. For each $s \in S$, let $\prec_s$ be degenerate so that each child has the same priority. Let R_C be defined by the following *von*

[9] This example is from Abdulkadiroğlu et al. (2010).

Neumann-Morgenstern indices:

	u_{c_1}	u_{c_2}	u_{c_3}
s_1	0.8	0.8	0.6
s_2	0.2	0.2	0.4
s_3	0.0	0.0	0.0

Since each child has the same preferences over individual schools, both the *Boston* and *deferred acceptance* algorithms single out the same recommendation: equal probability for each child at each school. This, however, is inefficient. Consider the allocation $\pi \in \Pi$ such that $\pi_{c_1 s_1} = \pi_{c_1 s_3} = 0.5$, $\pi_{c_2} = \pi_{c_1}$ and $\pi_{c_3 s_2} = 1$. Clearly π *Pareto-dominates* equal division (see Figure 5). Further, since for each $s \in S$, R_s is complete indifference, $\pi \in D^l(R_C, R_S)$.

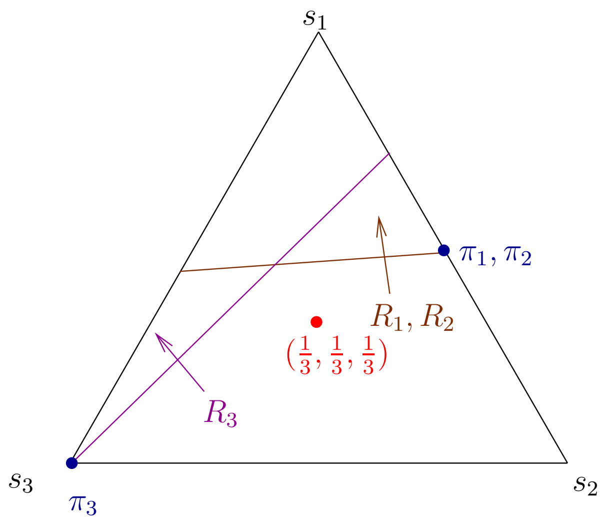

Figure 5: Clearly, equal division is dominated by π at (R_C, R_S).

By Theorem 2, we know that a *lim-DIP* allocation is in the *core*. It is in this sense that the *augmented* priorities of the schools are respected. While it is true that the allocation realized *ex-post* may violate priorities, the priorities are not entirely ignored. It is by weakening the role of priorities that we achieve gains in efficiency.

5.4 Workers and employers

Let E be a set of employers and W be a set of workers. For each $e \in E$, let $\overline{R}_e$ be e's preferences over R^W_+. For each $w \in W$, let $\overline{R}_w$ be w's preferences over R^E_+.

Each $w \in W$ has a unit supply of labor and each $e \in E$ can hire at most one unit of labor. The goal is to assign a work schedule to each worker. An allocation in the core of such a problem ensures participation of all groups. It is easy to see that, as in the applications above, this problem is a special case of our model.

The following is an implication of Theorem 1:

Corollary 5. *Every problem of workers and employers has a* lim-DIP allocation.

6 Kinds of people

In this section, we describe a more general model than the one analyzed so far. In particular, we introduce the notion of a *kind* for each person and show that the prices of an *lim-DIP* allocation can be indexed by *kinds* rather than identities.

Recall that the *lim-DIP* equilibria are somewhat like *Lindahl equilibria* as explained in Remark 1. A common indictment of *Lindahl* allocations, however, is that as the number of people involved increases, the number of prices must also increase. While prices are "personalized" in the definition of an *lim-DIP* equilibrium, we show here that they only need to be indexed by the *kind* of person and not his identity. The role of double-indexing is to reflect the "preferences of the resource." Suppose that two people are identical to the rest of the world. Since they are identical, anyone matched to them is indifferent between the two. The two should then face the same prices. Here, we generalize our earlier definitions and results to reflect this.

Let $\boldsymbol{K}$ be a set of *kinds*. For each $\tau \in K$, let $S_\tau \subseteq K$ be such that $\tau \in S_\tau$. The set of **potential partner kinds** of τ are $S_\tau \setminus \{\tau\}$. As before, let N be the set of people involved. Let $\kappa \in K^N$ be such that for each $i \in N$, i's kind is κ_i. For each pair $\upsilon, \tau \in K$ if $\upsilon \in S_\tau$ then $\tau \in S_\upsilon$. For each $i \in N$, i's **consumption set** is $\Delta(S_{\kappa_i})$. Let $\boldsymbol{R_i}$, i's preference relation over $\Delta(S_{\kappa_i})$, be continuous and convex. We also require R_i to be locally non-satiated, except at its maxima on $\Delta(S_{\kappa_i})$. Let $\mathcal{R}_i$ be the set of all such preferences. Let $(N_\tau)_{\tau \in K}$ be a partition of N such that for each $\tau \in K, N_\tau \equiv \{i \in N : \kappa_i = \tau\}$. An **economy** is described by a profile of preferences $R \in \mathcal{R} \equiv \times_{i \in N} \mathcal{R}_i$ and a profile of kinds $\kappa \in K^N$.

A **feasible allocation** specifies for each $i \in N$ a consumption bundle $\pi_i \in \Delta(S_{\kappa_i})$ in a way that for each $\tau \in K$ and each $\upsilon \in S_\tau$,

$$\sum_{i \in N_\tau} \pi_{i\upsilon} = \sum_{i \in N_\upsilon} \pi_{i\tau}.$$

Let $\boldsymbol{\Pi}$ be the set of feasible allocations.

Let $M \subseteq K \times K$ be such that $(\upsilon, \tau) \in M$ if and only if $\upsilon \in S_\tau$ and $\tau \in S_\upsilon$. An allocation $\pi \in \Pi$ is a **double-indexed price (DIP) allocation at** R if there is a vector $p \in \mathbb{R}^M_+$ such that for each $i \in N$,

$$\pi_i \in \underset{\pi'_i \in \Delta(S_{\kappa_i})}{\text{argmax}}\ R_i$$
$$\text{subject to}$$
$$\underbrace{\sum_{\tau \in S_{\kappa_i}} \pi'_{i\tau} p_{\kappa_i \tau}}_{\text{Price of } \pi'_i} \leq \underbrace{\sum_{\tau \in S_{\kappa_i}} \sum_{j \in N_\tau} \pi'_{j\kappa_i} p_{\tau\kappa_i}}_{i\text{'s income at } \pi'},$$

and $\pi \in \Pi$. We interpret the price vector as follows: for each $(\upsilon, \tau) \in M$, $p_{\upsilon\tau}$ is the price that a person of kind υ pays for a partnership with someone of kind τ.

We refer to (π, p) as a **DIP equilibrium**. Let $\boldsymbol{D(R)}$ be the set of all *DIP allocations at R.*

Of course, for the same reasons as before, $D(R)$ may be empty. So we define *εDIP* and *lim-DIP* allocations here as well. While the definition of *εDIP equilibrium* is nearly the same as before, there are a few minor differences that we will highlight.

For each $i \in N$, $\Lambda(S_{\kappa_i}) \equiv \{\lambda \in \mathbb{R}^{S_{\kappa_i}}_+ : \sum_{\tau \in S_{\kappa_i}} \lambda_\tau \leq 1 + \varepsilon |N_{\kappa_i}|\}$. Clearly, for each $i \in N, \Delta(S_{\kappa_i}) \subset \Lambda(S_{\kappa_i})$. For each $R \in \mathcal{R}$ and each $\varepsilon \in (0,1)$, let R^ε_i be an extension of R_i from $\Delta(S_{\kappa_i})$ to $\Lambda(S_{\kappa_i})$ such that R^ε_i is:

- *monotonic*,
- *strictly monotonic* over $\{\lambda \in \Lambda(S_{\kappa_i}) : \sum_{\tau \in S_{\kappa_i}} \lambda_\tau < 1\}$,
- *continuous*,
- *convex*, and
- For each pair $x, y \in \Lambda(S_{\kappa_i})$ if $\sum_{\tau \in S_{\kappa_i}} x_\tau < \min\{1-\varepsilon, \sum_{\tau \in S_{\kappa_i}} y_\tau\}$, then $y \ P^\varepsilon_i \ x$. That is, if the sum of x's coordinates are less than $1-\varepsilon$, then any point whose coordinates have a greater sum is preferred to x.

Claim 2. *Such R^ε exists.*

Proof: The proof is identical to that of Claim 1 with only a few changes. For each $x \in \Delta(S_{\kappa_i})$,

$$U(\hat{R}_i, x) \equiv \text{convex hull}\left(\left(1 - \frac{d(x)\varepsilon}{2}\right) M_i \cup U(\tilde{R}_i, x)\right) \cap \mathbb{R}^{S_{\kappa_i}}_+$$

Then, we complete the preference map over $\Lambda(S_{\kappa_i})$ in a way that indifference curves through any point x such that $\sum_{\tau \in S_{\kappa_i}} x_\tau \leq 1-\varepsilon$ are parallel to the simplex $\Delta(S_{\kappa_i})$.

The remainder of the proof remains the same as that of Claim 1 and the extension of $\hat{R}$ to the positive orthant in the proof of Proposition 1. $\square$

We say that $\pi \in \mathbb{R}_+^{\times_{i\in N} S_{\kappa_i}}$ is an **εDIP allocation** if there is $p \in \mathbb{R}_+^M$ such that for each pair $(\upsilon, \tau) \in M$,

$$p_{\upsilon\upsilon} + p_{\tau\tau} \geq p_{\upsilon\tau} + p_{\tau\upsilon},$$

for each (υ, t) such that $\sum_{i \in N_\tau} \pi_{i\upsilon} + \sum_{i \in N_\upsilon} \pi_{i\tau} > 0$,

$$p_{\upsilon\upsilon} + p_{\tau\tau} = p_{\upsilon\tau} + p_{\tau\upsilon},$$

for each $i \in N$,

$$\pi_i \in \underset{x_i \in \Lambda(S_{\kappa_i})}{\text{argmax}}\ R_i$$

$$\text{subject to}$$

$$\underbrace{\sum_{\tau \in S_{\kappa_i}} x_{ij} p_{\kappa_i \tau}}_{\text{Price of } x_i'} \leq \underbrace{(1-\varepsilon) p_{\kappa_i \kappa_i} + \sum_{\tau \in K \setminus \{\kappa_i\}} p_{\tau\tau} \frac{|S_\tau| \varepsilon}{|N| - |S_\tau|}}_{i\text{'s income}},$$

for each pair $(\upsilon, \tau) \in M$,

$$\sum_{i \in N_\tau} \pi_{i\upsilon} = \sum_{i \in N_\upsilon} \pi_{i\tau},$$

and for each $i \in N$,

$$1 - \varepsilon \leq \sum_{\tau \in S_{\kappa_i}} \pi_{i\tau} \leq 1 + \varepsilon |N_{\kappa_i}|.$$

The "clearing" condition here is less demanding than before. The allocation π need not a feasible allocation itself. That is, while it needs to clear in aggregate, at the individual level it only need be "within ε" of an element of a person's consumption set. [10]

[10] It is easy to find examples where the stronger clearing condition that requires, for each $i \in N, \pi_i \in \Delta(S_{\kappa_i})$ cannot be met. Consider a problem where $K = \{\upsilon, \tau\}$ and $N = \{1, 2, 3, 4\}$. Let $S_\upsilon = S_\tau = \{\upsilon, \tau\}$. Let $\kappa_1 = \kappa_2 = \upsilon$ and $\kappa_3 = \kappa_4 = \tau$. Let R_1 be represented by the utility function $u_0(\pi_0) = \pi_{0\upsilon}$ and let R_2, R_3, and R_4 be represented by the utility function $u_0'(\pi_0) = \pi_{0\tau}$. That is, 1 prefers to be matched to 3 or 4, 2 prefers to remain unmatched, and 3 and 4 prefer to be matched to 1 or 2. Suppose there is an equilibrium (p, π) where the stronger clearing condition holds. Since preferences are linear, each person chooses a corner solution. Further, by feasibility and the definition of R_1 and R_2 , neither $p_{\upsilon\tau}$ nor $p_{\upsilon\upsilon}$ can be zero. By the clearing condition, $\pi_{1\tau} = \pi_{2\upsilon} = 1$. By 2's budget constraint, $1p_{\upsilon\upsilon} = (1-\varepsilon)p_{\upsilon\upsilon} + \varepsilon p_{\tau\tau}$. So, $p_{\tau\tau} = p_{\upsilon\upsilon} = p^*$. By 1's budget constraint, $1p_{\upsilon\tau} = (1-\varepsilon)p_{\upsilon\upsilon} + \varepsilon p_{\tau\tau}$. This implies that $p_{\upsilon\tau} = p^*$. Finally, the equilibrium condition $p_{\tau\tau} + p_{\upsilon\upsilon} = p_{\tau\upsilon} + p_{\upsilon\tau}$ implies that $p_{\tau\upsilon} = p^*$ as well. Then, since 3 and 4 are maximizing in their budget sets, $\pi_{3\upsilon} = \pi_{4\upsilon} = 1$. This contradicts our assumption that the market clears in aggregate

Proposition 2. *For each $\varepsilon \in (0,1)$ and each $(R,\kappa) \in \mathcal{R} \times K^N, \varepsilon D(R,\kappa) \neq \varnothing$.*

Proof: This proof is very similar to that of Proposition 1. Preferences are extended to the positive orthant, and the problem is encoded as an Arrow-Debreu economy. For each $i \in N$, i's consumption space is

$$X_i \equiv \mathbb{R}_+^{S_{\kappa_i}} \times \{0\} \subset R_+^M.$$

Define the extension $\overline{R}_i^\varepsilon$ of R_i^ε to X_i exactly as in the proof of Proposition 1. For each $x \in \Lambda(S_{\kappa_i})$, set

$$U(\overline{R}_i^\varepsilon, x) \equiv \text{comp } U(R_i^\varepsilon, x).$$

Define M_i as before and and translate the preference map over $U(\overline{R}_i^\varepsilon, M_i)$.

Firms are defined the same way, except that they are indexed by pairs of kinds rather than pairs of people. The production set of firm $(\upsilon,\tau) \in F$ is

$$Y_{\{\upsilon,\tau\}} \equiv \left\{ y \in \mathbb{R}^{\{\upsilon\tau,\tau\upsilon,\upsilon\upsilon,\tau\tau\}} : y_{\upsilon\tau} = y_{\tau\upsilon} = -y_{\upsilon\upsilon} = -y_{\tau\tau} \right\} \times \{0\} \subset \mathbb{R}^M.$$

For each $i \in N$, i's endowment is $\omega^i \in \mathbb{R}_+^M$ such that for each pair $\upsilon, \tau \in K$,

$$\omega_{\upsilon\tau}^i = \begin{cases} 1-\varepsilon & \text{if } \upsilon = \tau = \kappa_i, \\ \frac{|S_\tau|\varepsilon}{|N|-|S_\tau|} & \text{if } \tau = \upsilon \neq \kappa_i, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

For each $\{\upsilon,\tau\} \in F$, and each $i \in N$, let $\sigma_i(\{\upsilon,\tau\})$ be i's share of $\{\upsilon,\tau\}$. Thus, for each $\{\upsilon,\tau\} \in F, \sum_{i \in N} \sigma_i(\{\upsilon,\tau\}) = 1$.

As before, the economy $E \equiv (X, Y, \overline{R}^\varepsilon, \omega, \sigma)$ has a *competitive allocation* $(x,y,p) \in X \times Y \times R_+^M$. We show that (x,p) is actually an *εDIP equilibrium*. Since (x,y,p) is a *competitive equilibrium*, for each $i \in N$, $x_i \; \overline{R}_i^\varepsilon \; (1-\varepsilon)\delta_{\kappa_i}$. Then, by definition of R^ε and therefore $\overline{R}^\varepsilon$, for each $i \in N$, $\sum_{\tau \in S_{\kappa_i}} x_{i\tau} \geq 1-\varepsilon$. Thus, by feasibility, for each $\tau \in K$ and $i \in N$, $\sum_{\upsilon \in S_{\kappa_i}} x_{i\upsilon} \leq 1 + \varepsilon|N_\tau|$. Finally, as argued in the proof of Proposition 1, by definition of Y and feasibility,

$$\sum_{i \in N_\tau} x_{i\upsilon} = \sum_{i \in N_\upsilon} x_{i\tau}.$$

□

As before, an allocation $\pi \in \Pi$ is a **limit DIP (lim-DIP) allocations** if there is a sequence $\{\pi^\varepsilon\}_{\varepsilon \in (0,1)} \in \Pi$ such that

$i)$ for each $\varepsilon \in (0,1), \pi^\varepsilon \in D^\varepsilon(R)$ and

$ii)$ $\lim_{\varepsilon \to 0} \pi^\varepsilon = \pi$.

Let $\boldsymbol{D^l(R)}$ be the set of all *limit DIP allocations.*.

From Proposition 2 we have the following.

Theorem 3. *For each $(R,\kappa) \in \mathcal{R}^N \times K^N$, $D^l(R,\kappa) \neq \varnothing$.*

6.1 Discussion regarding kinds

There are two distinct benefits to including kinds in our model. The first is with regards to fairness. Two people who are, for all intents and purposes, the same should be given the same opportunities. The *lim-DIP* solution does exactly that. Since prices are indexed by kind rather than identity, each person of a particular kind is faced with exactly the same "budget set." If prices are indexed by identities, then identical people may be treated differently.

The second is to apply our model to situations where no person is unique in the eyes of others. Take, for instance, a school district where each school has a large number of seats and large groups of students have identical priorities at each of the schools. Or think of a problem involving many workers and many tasks where there are many workers with identical skills and many tasks that are identical.

Unlike *Lindahl equilibria*, as the number of people involved increases, as long as there number of kinds remains fixed, the dimension of the price vector remains fixed. Since ε*DIP* equilibria are actually *competitive equilibria* of appropriately defined Arrow-Debreu models, for fixed utility representations, the gains from misreporting preferences diminish as the number of people of each kind increases.

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