

Ranking Asymmetric Auctions: Filling the Gap between a Distributional Shift and Stretch*

René Kirkegaard

Department of Economics

University of Guelph

rkirkega@uoguelph.ca

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Abstract

I consider first-price auctions (FPA) and second-price auctions (SPA) with two asymmetric bidders. The FPA is known to be more profitable than the SPA if the strong bidder's distribution function is convex and the weak bidder's distribution is obtained by truncating or horizontally shifting the former. In this paper, I employ a new mechanism design result to show that the FPA remains optimal if the weak bidder's distribution falls between the two benchmarks in a natural way. The same conclusion holds if the strong bidder's distribution is concave, but with a vertical shift replacing the horizontal shift. A result with a similar flavor holds if the strong bidder's distribution is neither convex nor concave. The dispersive order and the star order prove useful in comparing the weak bidder's distribution to the benchmarks. A key step establishes a relationship between the dispersive and star orders, truncations, and reverse hazard rate dominance.

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1 Introduction

In the by now standard independent private values model, the celebrated *Revenue Equivalence Theorem* implies that the auction format is irrelevant for a risk-neutral seller whenever bidders are homogeneous ex ante. While Vickrey (1961) discovered an early version of the Revenue Equivalence Theorem, he also proved that it does not extend to a more realistic setting with heterogeneous bidders. Since then, a literature has emerged comparing the standard auctions, specifically the first-price auction (FPA) and the second-price auction (SPA). This literature has identified a number of isolated environments or examples, each of which allows a revenue ranking to be obtained. In this paper, I identify robust classes of environments where the FPA can be shown to be superior to the SPA.¹

In a seminal paper, Maskin and Riley (2000) study three particular environments. In the first model, the strong bidder's distribution, F_s , is obtained by shifting the weak bidder's distribution, F_w , horizontally to the right. In the second model, F_s is obtained by "stretching" F_w . In both models, the FPA dominates the SPA under certain curvature assumptions. However, the SPA dominates in their third model, in which F_s and F_w share the same support. A central result in the current paper takes Maskin and Riley's (2000) first two models as "benchmarks" and then proves that the FPA is superior whenever the auction environment "lies between" the two benchmarks in a natural way. Several results of this type are presented.

More concretely, consider for the moment the most stringent assumptions in Maskin and Riley (2000), namely that the distributions are convex and log-concave. An alternative way, used from now on, of thinking about their "stretch" model is that F_w is a truncation of F_s , which I denote F_s^t .² In their "shift" model, F_w is a horizontal, left-ward shift of F_s , denoted F_s^h . These cases are depicted in Figure 1.

¹Vickrey (1961), Lebrun (1996), and Cheng (2006) analytically compare revenue in models where bidders draw types from restricted classes of power distributions. Cheng (2010) contains additional examples. Gaviols and Minchuk (2011) study auctions with "small" asymmetries. Maskin and Riley (1985) and Doni and Menicucci (2011) assume distributions are discrete. Maskin and Riley's (2000) paper is discussed momentarily. The numerical literature dates back to Marshall et al (1994) and also includes Fibich and Gaviols (2003), Li and Riley (2007), and Gayle and Richard (2008).

²Defining F_w as a truncation of F_s is arguably a more parsimonious way of describing the same setting. In either case, F_s is a multiple of F_w on the shared support, but when F_s is thought of as a stretch of F_w one has to also formulate an extension of F_s on the rest of its support. As noted in Kirkegaard (2011b), this forces Maskin and Riley (2000) to impose some unnecessarily strong assumptions.

Of course, holding F_s fixed, F_w can take many other forms. Here, the FPA is shown to dominate if F_w is between F_s^t and F_s^h and satisfies certain regularity conditions. These conditions are satisfied if F_w is more disperse than F_s^t , but less disperse F_s^h . Similar results obtain if F_s is concave, but with a vertical shift of F_s taking the place of the horizontal shift. A related result for non-monotonic densities is also derived.

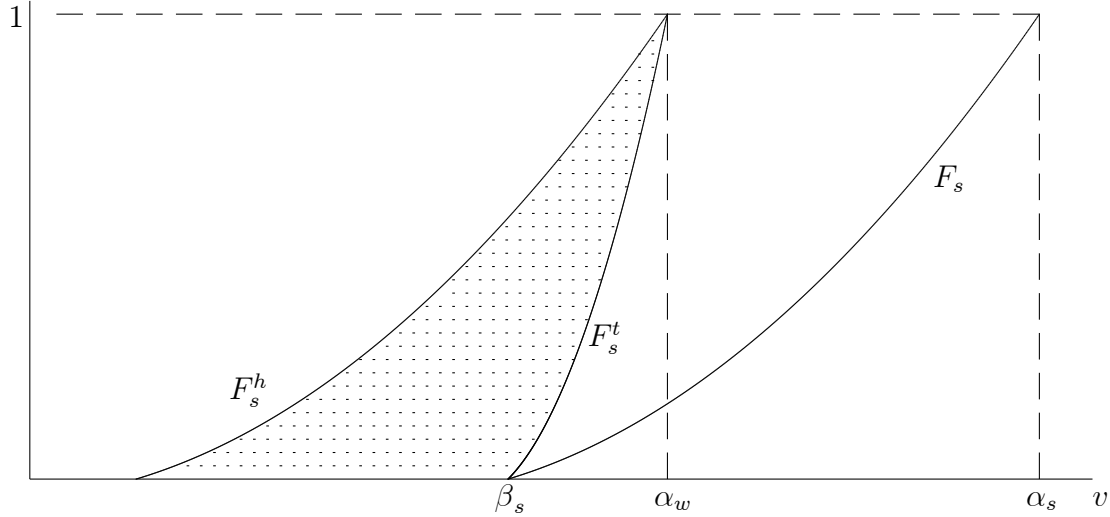


Figure 1: Truncations and horizontal shifts.

Note: F_s has support $[\beta_s, \alpha_s]$. Consider some F_w whose support ends at $a_w \in (\beta_s, \alpha_s)$. Generate the appropriate truncation, F_s^t , and horizontal shift, F_s^h , of F_s , such that these end at α_w . The FPA dominates if F_w is more disperse than F_s^t but less disperse than F_s^h .

The analysis takes as its starting point a new revenue ranking result, due to Kirkegaard (2011b). Kirkegaard's (2011b) objective is to demonstrate that mechanism design methods can be used to simplify the problem of ranking asymmetric auctions. As a result, he proves a theorem establishing the superiority of the FPA under two conditions. First, F_s must dominate F_w in terms of the reverse hazard rate. This assumption allows some inferences concerning bidding behavior in the FPA. The second assumption is, roughly speaking, that F_s is flatter and more disperse than F_w . Indeed, Kirkegaard (2011b) shows that Maskin and Riley's (2000) first two models satisfy the conditions.³ The idea in this paper is to invoke Kirkegaard's (2011b) theorem. However, the challenge remains to describe environments where both, possibly contradictory, conditions are satisfied simultaneously.

³Given the two conditions, Kirkegaard (2011b) also shows that the revenue ranking is unaffected by a reserve price. Moreover, it also holds if there are more than one weak bidder.

It is easy to show that reverse hazard rate dominance does not apply if F_w lies below F_s^t in Figure 1. Thus, a key step is to more precisely establish when reverse hazard rate dominance applies. Under mild conditions, I show that if F_w is more disperse than F_s^t (which implies $F_w \geq F_s^t$) then it is also the case that F_s dominates F_w in terms of the reverse hazard rate (Lemma 3). That is, there is an intimate relationship between the dispersive order, truncations, and reverse hazard rate dominance. Thus, F_s^t is a useful benchmark. However, if F_w is too far above F_s^t , then the second condition in Kirkegaard’s (2011b) theorem is violated. In fact, when F_s is convex, the condition is violated if F_w is ever above F_s^h in Figure 1. Thus, Maskin and Riley’s (2000) examples are essentially on opposite boundaries of Kirkegaard’s (2011b) theorem. Here, I establish the regularity assumptions needed to conclude that the FPA is revenue superior to the SPA if F_w falls between F_s^t and F_s^h .

The dispersive order plays a role both in Kirkegaard’s (2011b) theorem as well as in the aforementioned Lemma 3 of this paper. A related order, the star order, also proves useful. In Section 5 I show that these orders, or comparisons between distributions, are equivalent to comparisons of various notions of price sensitivity in different markets. Recall that Bulow and Roberts (1989) have argued that there are parallels between the auction design problem and the monopoly pricing problem.

The dispersive order has recently attracted some attention in the theoretical auction literature. Jia et al (2010), Katzman et al (2010), and Szech (2011) examine various comparative statics in symmetric auctions when bidders’ distributions become more disperse. Ganuza and Penalva (2010) consider symmetric auctions in which the seller can influence the precision of bidders’ information by making their signals more or less disperse. Johnson and Myatt (2006) examine a related question in the context of a monopoly. Their “rotation order” is also used to compare how spread out two distributions are. In asymmetric auctions, the dispersive order plays a role in determining the qualitative features of revenue-enhancing interventions into particular auction formats, as demonstrated by Kirkegaard (2011c) and Mares and Swinkels (2011a, 2011b). The results in these papers are particularly strong when densities are monotonic. Hopkins (2007) describe qualitative features of bidding behavior in auctions where distribution functions cross and one is smaller than the other in the dispersive order. However, apart from Kirkegaard (2011b), the current paper is the first to explicitly use the dispersive order to rank revenue across standard auctions with asymmetric bidders.

2 Model and preliminaries

Two risk neutral bidders compete in an auction. Bidder s is perceived as strong and bidder w as weak. Roughly speaking, the former is more likely to have a high willingness to pay; a more precise definition is postponed. Independently of the other bidder, bidder i draws a type or valuation from a distribution function, F_i , which is continuously differentiable on its support, $S_i = [\beta_i, \alpha_i]$, $i = s, w$. Mass points are ruled out and the density, f_i , is strictly positive on $(\beta_i, \alpha_i]$, with $\alpha_i > \beta_i \geq 0$, $i = s, w$.⁴ Finally, $\beta_w \leq \beta_s$ and $\alpha_w < \alpha_s$.

The common support is denoted by $C = S_s \cap S_w$, with $C = [\beta_s, \alpha_w]$ if the supports overlap. It is useful to define $F_i(v) = f_i(v) = 0$ for all $v < \beta_i$, such that $f_w(v) \geq f_s(v)$ for all $v < \beta_s$.

Two distinct kinds of stochastic orders are useful for comparing F_s and F_w . Thus, stochastic orders of strength and stochastic orders of dispersion and spread are reviewed next.

2.1 Stochastic orders of strength

There are several ways to formalize the idea that one bidder is stronger than another, depending on how F_s and F_w are related on C :

1. F_s dominates F_w i.t.o. the likelihood ratio, $F_w \leq_{lr} F_s$: $\frac{f_s(v)}{f_w(v)}$ is increasing on C .⁵
2. F_s dominates F_w i.t.o. the reverse hazard rate, $F_w \leq_{rh} F_s$: $\frac{f_s(v)}{F_s(v)} \geq \frac{f_w(v)}{F_w(v)}$, $\forall v \in C$.
3. F_s dominates F_w i.t.o. the hazard rate, $F_w \leq_{hr} F_s$: $\frac{f_s(v)}{1-F_s(v)} \leq \frac{f_w(v)}{1-F_w(v)}$, $\forall v \in C$.
4. F_s first order stochastically dominates F_w , $F_w \leq_{st} F_s$: $F_s(v) \leq F_w(v)$, $\forall v \in C$.

See Krishna (2002) for an introduction to these stochastic orders and their use in auction theory. See Shaked and Shanthikumar (2007) for an in-depth treatment. The first order implies the other orders. The second and third both imply the fourth.

⁴Maskin and Riley (2000) allow a mass point at β_i in two of their models.

⁵In this paper, increasing is taken to mean non-decreasing; decreasing means non-increasing. The abbreviation i.t.o. stands for “in terms of”.

Following Lebrun (1999) and Maskin and Riley (2000), qualitative features of bidder interaction in the FPA are known under the assumption that $F_w \leq_{rh} F_s$.⁶ To be more specific, let $r(v) = F_s^{-1}(F_w(v))$, $v \in S_w$. In Hopkins' (2007) terminology, bidder s with type $r(v)$ has the same *rank* as bidder w with type v since $F_s(r(v)) = F_w(v)$. Note that $F_w \leq_{st} F_s$ is equivalent to $r(v) \geq v$ for all $v \in S_w$. Given $F_w \leq_{rh} F_s$, Maskin and Riley (2000) show that in a FPA, bidder w with type v either submits a non-serious bid (one that is so low that it never wins) or he submits a bid of the same magnitude as a bid submitted by the strong bidder of some type, $k_1(v)$, somewhere in the interval $[v, r(v)]$.⁷ In other words, the weak bidder is more aggressive than the strong bidder, but not aggressive enough to make up for the difference in strength. The bid is strictly increasing in type for those that submit serious bids. Moreover, bidder w with type α_w submits the same bid as bidder s with type α_s . Hence, $k_1(\alpha_w) = \alpha_s = r(\alpha_w)$.

In a SPA, it is a weakly dominant strategy to submit a bid equal to the bidder's type. Since the auction is efficient, bidder w with type v wins if and only if bidder s has a type below $k_2(v) = \max\{\beta_s, v\}$.

2.2 Stochastic orders of dispersion and spread

Given $r(v)$ plays an important role bounding bidder w 's winning probability, r 's characteristics are of some interest. To this end, the following orders of dispersion and spread are relevant for the current paper, in descending order of importance:

1. F_w is smaller than F_s in the dispersive order, $F_w \leq_{disp} F_s$: $r(v) - v$ is increasing on S_w .
2. F_w is smaller than F_s in the star order, $F_w \leq_* F_s$: $\frac{r(v)}{v}$ is increasing on S_w .
3. F_w is smaller than F_s in the convex transform order, $F_w \leq_c F_s$: $r(v)$ is convex on S_w .

⁶See Milgrom (2004), Hopkins (2007), or Kirkegaard (2011b) for alternative proofs. Kirkegaard (2009) analyses environments where reverse hazard rate dominance (or even first order stochastic dominance) does not hold.

⁷A non-serious bid is made only if bidder w 's type, v , is sufficiently far below β_s . A non-serious bid wins with probability 0 = $F_s(\beta_s)$. Since $\beta_s \in [v, r(v)]$ when $v \leq \beta_s$, letting $k_1(v) = \beta_s$ for all v that submit non-serious bids implies that $k_1(v) \in [v, r(v)]$ for all $v \in S_w$.

Shaked and Shanthikumar (2007) review these stochastic orders.⁸ In words, $F_w \leq_{disp} F_s$ if the distance between the types that are at the same percentile is increasing. Thus, geometrically, the dispersive order implies that the horizontal difference between F_s and F_w is increasing. If F_s is more disperse than F_w then it has larger variance and wider support, $\alpha_s - \beta_s \geq \alpha_w - \beta_w$. It is useful to note that if $F_w \leq_{disp} F_s$ then $r'(v) \geq 1$ or $f_w(v) \geq f_s(r(v))$ for all $v \in S_w$. I write $F_w =_{disp} F_s$ if $r(v) - v$ is constant.

The dispersive order and the star order are obviously related. In particular,

$$F_w(v) \leq_* F_s(v) \iff F_w(e^v) \leq_{disp} F_s(e^v),^9 \quad (1)$$

which helps explain why both $F_i(e^v)$ and $F_i(v)$, $i = s, w$, will play a role in this paper.

The dispersive order, star order, and convex transform order have natural economic interpretations, all related to various notions of price sensitivity. A discussion of these interpretations are postponed until Section 5, however.

Note it is possible that $F_w \leq_{disp} F_s$ and yet $F_s \leq_* F_w$. The assumption that $F_s \leq_* F_w$ plays a role in Kirkegaard's (2011c) analysis of favoritism in asymmetric all-pay auctions. Mares and Swinkels (2011b) consider favoritism in procurement auctions in which the buyer has a preference for a specific bidder (seller). Translating their procurement setting into a standard auction, one of their assumptions is that $F_s \leq_c F_w$.

3 Ranking asymmetric auctions

Using mechanism design techniques, Kirkegaard (2011b) proves the following result.^{10,11}

⁸The literature on the star order and convex transform order should be read with some care. In this literature, it is often assumed that $\beta_s = \beta_w = 0$, and a number of results rely on this assumption (for example, if $\beta_s = \beta_w = 0$ then $\leq_c \implies \leq_*$). In the current paper, β_s and β_w are allowed to be strictly positive.

⁹Consider two random variables, X and Y . By Theorem 4.B.1 in Shaked and Shanthikumar (2007), $X \leq_* Y \iff \log X \leq_{disp} \log Y$. The relationship in (1) comes from the fact that if X is distributed according to $F(x)$ then $\log X$ is distributed according to $F(e^x)$.

¹⁰The following version of Kirkegaard's (2011b) theorem incorporates his discussion of (3); see the discussion around equation (8) in Kirkegaard (2011b).

¹¹See Kirkegaard (2011a) for a review of examples in the literature that do not satisfy the conditions of Theorem 1, but where a revenue ranking is nevertheless possible.

Theorem 1 Assume $F_w \leq_{rh} F_s$. Then, the FPA generates strictly higher expected revenue than the SPA if

$$f_w(v) \geq f_s(x) \text{ for all } x \in [v, r(v)] \text{ and all } v \in S_w \quad (2)$$

or, more generally, if

$$\int_v^{k(v)} (f_w(v) - f_s(x)) dx \geq 0 \text{ for all } k(v) \in [v, r(v)] \text{ and all } v \in S_w. \quad (3)$$

Condition (2) implies condition (3) and is typically easier to check than condition (3). If (2) is satisfied then F_s is flatter and more disperse than F_w . Together, $F_w \leq_{rh} F_s$ and condition (2) imply that $F_w \leq_{hr} F_s$.

As demonstrated in Kirkegaard (2011b), Maskin and Riley's (2000) shift and stretch models satisfy $F_w \leq_{rh} F_s$ and condition (2). Thus, their results are corollaries of Theorem 1. The shift and stretch models are discussed further in Section 4.

To invoke Theorem 1, the challenge is to describe environments where both $F_w \leq_{rh} F_s$ and either (2) or (3) hold simultaneously. For example, (2) requires that $f_w(v) \geq f_s(v)$ for any $v \in C$, while, by definition, $F_w \leq_{rh} F_s$ can hold only if $f_w(v)$ is not too much larger than $f_s(v)$ for any $v \in C$. Thus, there is a tension between the two types of conditions.

Note also that $F_w \leq_{rh} F_s$ is equivalent to assuming that F_w/F_s is a decreasing function on C . However, for any $v \in C$,

$$\frac{d}{dv} \left(\frac{F_w(v)}{F_s(v)} \right) = \frac{d}{dv} \left(\frac{F_s(r(v))}{F_s(v)} \right) \propto \frac{f_s(r(v))r'(v)}{F_s(r(v))} - \frac{f_s(v)}{F_s(v)}. \quad (4)$$

Condition (2) implies $F_w \leq_{disp} F_s$, or $r'(v) \geq 1$. Thus, if F_s is locally log-convex ($\frac{f_s(v)}{F_s(v)}$ is locally increasing) the right hand side may easily be positive, thereby violating $F_w \leq_{rh} F_s$. Thus, the dual assumption of $F_w \leq_{disp} F_s$ and $F_w \leq_{rh} F_s$ is more likely to be satisfied when F_s is log-concave, when $C \neq \emptyset$. Incidentally, Lebrun (2006) has shown that equilibrium in the FPA is essentially unique if $\beta_s > \beta_w$, or if $\beta_s = \beta_w$ and F_i is strictly log-concave close to β_s , $i = s, w$.¹² Maskin and Riley (2000) assume F_s

¹²More precisely, equilibrium is essentially unique when bidders never bid above their valuations. However, Kaplan and Zamir (2011) prove there are other equilibria in which the non-serious bids (that never win) exceed the weak bidder's valuation. Note bidder w is using a weakly dominated strategy in this case. Moreover, Kirkegaard's (2011b) theorem applies to all equilibria.

is log-concave in their examples in which the FPA dominates. This assumption will typically also be imposed here. Indeed, at times the stronger assumption that $F_s(e^v)$ is log-concave will be imposed. Since $F_s(e^v)$ is log-concave if and only if the function $v \frac{f_s(v)}{F_s(v)}$ is decreasing, log-concavity of $F_s(e^v)$ requires that the reverse hazard rate falls sufficiently rapidly.

The following example shows that log-concavity of F_s is not necessary to invoke Theorem 1, however.

EXAMPLE 0 (CONCAVE VS. CONVEX): Assume F_w is concave and F_s is convex, with $f_w(\alpha_w) \geq f_s(\alpha_s)$. F_s need not be log-concave. Note that $r(v)$ must be concave, or $F_s \leq_c F_w$. The curvature assumptions imply $F_w \leq_{lr} F_s$ and therefore $F_w \leq_{rh} F_s$. Condition (2) is satisfied since densities are monotonic and $f_w(\alpha_w) \geq f_s(\alpha_s)$. Thus, Theorem 1 applies. $\blacktriangle$

The two types of conditions in Theorem 1 are examined in turn. Section 4 completes the analysis by describing environments where Theorem 1 can be invoked.

3.1 Conditions (2) and (3)

At this point, it is desirable to establish more direct conditions on the primitives, F_s and F_w , which would imply (2) or (3). The following lemmata are new, i.e. not included in Kirkegaard (2011b).

First, note that condition (2) is particularly simple to check if f_s is monotonic. The proof of the following Lemma is trivial and is therefore omitted.

Lemma 1 *Condition (2) is satisfied if:*

1. F_s is convex on S_s and $f_w(v) \geq f_s(r(v))$ for all $v \in S_w$ (i.e. $F_w \leq_{disp} F_s$), or
2. F_s is concave on S_s , $f_w(v) \geq f_s(v) \forall v \in C$, and f_w is decreasing on $[\beta_w, \beta_s]$.

Turning to condition (3), a counterpart to the first part of Lemma 1 can be developed. Lemma 2 should be seen in light of the relationship described in (1).

Lemma 2 *Assume $F_s(e^v)$ is convex and $F_w \leq_* F_s$. Then, condition (3) is satisfied.*

Proof. Since $f_w(v) = f_s(r(v))r'(v)$, the left hand side of (3) equals

$$f_s(r(v))r(v) \left[\frac{r'(v)}{r(v)} (k(v) - v) - \int_v^{k(v)} \frac{f_s(x)x}{f_s(r(v))r(v)} \frac{v}{x} \frac{1}{v} dx \right],$$

where the first two terms under the integration is less than one because (i) $vf_s(v)$ is increasing for all $v \leq \alpha_s$ (since $F_s(e^v)$ is convex with $f_s(x) = 0$ for $x < \beta_s$) and $x \leq k(v) \leq r(v)$ and (ii) $v \leq x$, respectively. The sign of the above expression is determined by the terms inside the square brackets, which is then no smaller than

$$\frac{r'(v)}{r(v)} (k_1(v) - v) - \int_v^{k_1(v)} \frac{1}{v} dx = (k_1(v) - v) \left[\frac{r'(v)}{r(v)} - \frac{1}{v} \right] \geq 0,$$

where the inequality is due to $F_w \leq_* F_s$. Hence, condition (3) is satisfied. ■

The important difference between Lemma 1 and Lemma 2 is that the density need not be monotonic in the latter. Thus, in the analysis Lemma 1 and condition (2) will be invoked when f_s is monotonic and Lemma 2 and condition (3) when it is not.

3.2 Reverse hazard rate dominance

The requirement of reverse hazard rate dominance in Theorem 1 has bite only if $C \neq \emptyset$. For any $\alpha \in (\beta_s, \alpha_s)$, define

$$F_s^t(v|\alpha) = \frac{F_s(v)}{F_s(\alpha)}, \quad v \in [\beta_s, \alpha] \quad (5)$$

as the right-truncation of F_s with truncation point α . Since $F_s^t(\cdot|\alpha)$ has the *same* reverse hazard rate as F_s , $F_s^t(\cdot|\alpha) \leq_{rh} F_s$. This suggests that whenever $\alpha_w \in (\beta_s, \alpha_s)$, it may be fruitful to compare F_w to the benchmark $F_s^t(\cdot|\alpha_w)$, as highlighted by the next proposition.

Proposition 1 (Necessary condition) *Assume the upper end-point of F_w 's support is $\alpha_w \in (\beta_s, \alpha_s)$. Then $F_w \leq_{rh} F_s \implies F_w \leq_{st} F_s^t(\cdot|\alpha_w)$; if $F_s^t(\cdot|\alpha_w)$ does not first order stochastically dominate F_w then F_s does not reverse hazard rate dominate F_w .*

Proof. Assume $F_w(x) < F_s^t(x|\alpha_w)$ for some $x \in [\beta_s, \alpha_w]$. Then, $F_w(\alpha_w) = 1 = F_s^t(\alpha_w|\alpha_w)$ necessitates $f_w(v) > f_s^t(v|\alpha_w)$ for some $v \in [x, \alpha_w]$ where $F_w(v) < F_s^t(v|\alpha_w)$

and thus $F_w \not\leq_{rh} F_s^t$. Since $F_s^t(\cdot|\alpha_w)$ has the same reverse hazard rate as F_s , $F_w \not\leq_{rh} F_s$ as well. ■

Thus, if $C \neq \emptyset$, $F_w \leq_{st} F_s^t(\cdot|\alpha_w)$ is necessary to invoke Theorem 1. Note the implication that $F_w(v) \geq F_s^t(v|\alpha_w) \geq F_s(v)$ for all $v \in C$, meaning that F_w cannot be “too close” to F_s . Otherwise, a fundamental problem arises; if $F_w \not\leq_{rh} F_s$ then it is no longer necessarily the case that bidder w is more aggressive than bidder s . See Maskin and Riley (2000) and Kirkegaard (2009). The method of proof in both Maskin and Riley (2000) and Kirkegaard (2011b) relies crucially on the property that bidder w is more aggressive.

As explained previously, the dispersive and star orders are sometimes sufficient for condition (2) or (3) to hold. Thus, it would be desirable to also link reverse hazard rate dominance to these stochastic orders. The following lemma establishes such a link and thus constitutes a crucial building block. The key to linking the stochastic orders is to recognize the pivotal role of the benchmark distribution F_s^t .

Lemma 3 (Sufficient conditions) *Assume the upper end-point of F_w 's support is $\alpha_w \in (\beta_s, \alpha_s)$. Then:*

1. *If F_s is log-concave then $F_s^t(\cdot|\alpha_w) \leq_{disp} F_s$; if $F_s^t(\cdot|\alpha_w) \leq_{disp} F_w$ then $F_w \leq_{rh} F_s$.*
2. *If $F_s(e^v)$ is log-concave then $F_s^t(\cdot|\alpha_w) \leq_* F_s$; if $F_s^t(\cdot|\alpha_w) \leq_* F_w$ then $F_w \leq_{rh} F_s$.*

Proof. Assume first that F_s is log-concave. F_s^t can be written in one of two ways, $F_s^t(v|\alpha_w) = \frac{F_s(v)}{F_s(\alpha_w)}$ or $F_s^t(v|\alpha_w) = F_s(r^t(v))$. Thus, $F_s(r^t(v)) = \frac{F_s(v)}{F_s(\alpha_w)}$ and so

$$r^{tt}(v) = \frac{1}{F_s(\alpha_w)} \frac{f_s(v)}{f_s(r^t(v))} = \frac{f_s(v)}{F_s(v)} \frac{F_s(r^t(v))}{f_s(r^t(v))} \geq 1$$

by log-concavity, as $r^t(v) \geq v$. Thus, $F_s^t(\cdot|\alpha_w) \leq_{disp} F_s$. Next, $F_s^t(\cdot|\alpha_w) \leq_{disp} F_w \implies F_w \leq_{st} F_s^t(\cdot|\alpha_w)$ since the upper bound of the supports, α_w , are the same. Thus, $S_w \supset C$. Since $F_s^t(\cdot|\alpha_w) \leq_{disp} F_w$, $f_w(v) \leq f_s^t(x|\alpha_w)$ must hold for any $v \in S_w$, where x satisfies $F_w(v) = F_s^t(x|\alpha_w)$. Since $F_w \leq_{st} F_s^t(\cdot|\alpha_w)$, $x \geq v$. Thus, for any $v \in C$,

$$\frac{f_w(v)}{F_w(v)} = \frac{f_w(v)}{F_s^t(x|\alpha_w)} \leq \frac{f_s^t(x|\alpha_w)}{F_s^t(x|\alpha_w)} = \frac{f_s(x)}{F_s(x)} \leq \frac{f_s(v)}{F_s(v)},$$

where the second inequality comes from the log-concavity of F_s . This proves the first part of the Lemma. By (1) and the assumed log-concavity of the function $F_s(e^v)$, the

proof of the first part can be applied to prove the second part. ■

In words, if F_w is more disperse than $F_s^t(\cdot|\alpha_w)$ then it is possible to conclude that $F_w \leq_{rh} F_s$. Note, as explained in the proof, that $F_s^t(\cdot|\alpha_w) \leq_{disp} F_w$ implies $F_w \leq_{st} F_s^t(\cdot|\alpha_w)$. Together, Proposition 1 and Lemma 3 signify that $F_w \leq_{st} F_s^t(\cdot|\alpha_w)$ is necessary for $F_w \leq_{rh} F_s$ and, in a sense, “almost sufficient” as well. Note that if F_w coincides with F_s^t then $F_s^t(\cdot|\alpha_w) \leq_{disp} F_w$ and $F_w \leq_{rh} F_s$ are trivially satisfied.

4 Between a shift and a stretch

This section begins by considering four benchmark examples. The idea is then to use these benchmarks to establish robust classes of environments where the FPA can be shown to dominate the SPA.

4.1 Four benchmark examples

To provide a first illustration of Theorem 1, consider the following four examples. The first two examples essentially coincide with Maskin and Riley’s (2000) stretch and shift models, respectively. These were also examined and more thoroughly discussed in Kirkegaard (2011b).

EXAMPLE 1 (TRUNCATIONS AND STRETCHES): Assume F_s is log-concave and that F_w is a truncation of F_s , i.e. $F_w = F_s^t$ as defined earlier. Alternatively, F_s can be viewed as a “stretched” version of F_w , which is the interpretation given in Maskin and Riley (2000). It has already been established that $F_s^t \leq_{rh} F_s$ (Lemma 3). By log-concavity,

$$\frac{f_w(v)}{F_w(v)} = \frac{f_s(v)}{F_s(v)} \geq \frac{f_s(x)}{F_s(x)}$$

for any $x \in [v, r(v)]$. Since $F_s(x) \leq F_w(v)$ for any $x \in [v, r(v)]$, the inequality necessitates $f_s(x) \leq f_w(v)$ for all $x \in [v, r(v)]$, implying (2). Theorem 1 now applies.

▲

EXAMPLE 2 (HORIZONTAL SHIFTS): Assume that F_s is convex and log-concave. Assume F_w is obtained by shifting F_s to the left, or $F_w(v) = F_s(v+a)$, for $v \in [\beta_w, \alpha_w]$, where $a = \beta_s - \beta_w = \alpha_s - \alpha_w > 0$ and $\beta_w \geq 0$. Since $r(v) = v + a$, $F_w =_{disp} F_s$. By

Lemma 1, (2) is satisfied. By logconcavity,

$$\frac{f_w(v)}{F_w(v)} = \frac{f_s(v+a)}{F_s(v+a)} \leq \frac{f_s(v)}{F_s(v)}, \text{ for all } v \in C,$$

or $F_w \leq_{rh} F_s$. Theorem 1 can now be invoked. $\blacktriangle$

EXAMPLE 3 (VERTICAL SHIFTS): Assume F_s and F_w are concave and that F_w is a vertical shift of F_s on $C \neq \emptyset$. That is, $F_w(v) = F_s(v) + 1 - F_s(\alpha_w)$ for $v \in [\beta_s, \alpha_w]$, where $\alpha_s > \alpha_w > \beta_s > 0$. On $[\beta_w, \beta_s)$, F_w is some (unspecified) concave function, with $\beta_w \geq 0$. For $v \in C$,

$$\frac{F_w(v)}{F_s(v)} = \frac{1 - F_s(\alpha_w)}{F_s(v)} + 1,$$

which is decreasing. Hence, $F_w \leq_{rh} F_s$. By concavity, $f_w(v) \geq f_s(x)$ for all $x \in [v, \alpha_s]$, implying that (2) is satisfied as well. Theorem 1 applies once again. $\blacktriangle$

REMARK A: Comparing Examples 2 and 3, the former satisfies $f_w(v) = f_s(r(v))$ and the latter $f_w(v) = f_s(v)$ on C . Hence, (2) is satisfied “with equality” at one of the endpoints of the interval $[v, r(v)]$. Indeed, if F_w lies anywhere above the horizontal or vertical shift of F_s whose support has the same end-point, α_w , then (2) cannot be satisfied.¹³ Thus, Examples 2 and 3 identify the boundaries of condition (2) and thus complement Proposition 1. $\triangle$

EXAMPLE 4 (RESCALING): Assume $F_s(e^v)$ is convex but log-concave. Assume $S_w = \left[\frac{\beta_s}{\gamma}, \frac{\alpha_s}{\gamma}\right]$, where $\gamma > 1$. Thus, either $\beta_s = \beta_w = 0$ or $\beta_s > \beta_w > 0$. Finally, assume $r(v) = \gamma v$, which implies $F_w =_* F_s$. If $C \neq \emptyset$, then by Lemma 3, $F_s^t \leq_* F_s =_* F_w$ and therefore $F_w \leq_{rh} F_s$. Lemma 2 implies condition (3) is satisfied. Thus, the FPA dominates the SPA. $\blacktriangle$

REMARK B: The difference between Examples 1 and 4 is significant. In the former, the truncation changes the shape of the density. For example, f_w may be monotonic even if f_s is not. In contrast, the rescaling in Example 4 preserves the shape of the density. The examples coincide only if F_s is a power distribution with $\beta_s = 0$. There are two ways of transforming F_s to get F_w ; F_w can be written $F_w(v) = G(F_s(v))$ or

¹³If $F_w(x) > F_s(v + \alpha_s - \alpha_w)$ for some $x \in (\beta_w, \alpha_w)$ then F_w cannot be less disperse than $F_s(v + \alpha_s - \alpha_w)$ and still satisfy $F_w(\alpha_w) = 1 = F_s(\alpha_w + \alpha_s - \alpha_w)$. Since $F_s^h =_{disp} F_s$, condition (2) is then violated. For similar reasons, condition (2) is violated if $F_w(x) > F_s^v(x|\alpha_w)$ for some $x \in C$.

$F_w(v) = F_s(r(v))$. G is a linear transformation in Example 1. On the other hand, it is r that is linear in Example 4.¹⁴

However, Examples 2 and 4 are intimately related. The difference between bidders' valuations is an additive term in Example 1, $r(v) = v + a$. Thus, $F_w =_{disp} F_s$; $r(v) - v$ is constant because the dispersive order is location free. In Example 4, the difference is a multiplicative term, $r(v) = \gamma v$. Therefore, $F_w =_* F_s$; $\frac{r(v)}{v}$ is constant because the star order is scale free.¹⁵ For completeness, note that the convex transform order is scale and location free; if $r(v) = \gamma v + a$ then $F_w =_c F_s$. $\triangle$

The more interesting and challenging case is when $C \neq \emptyset$ since both conditions in Theorem 1 then comes into play. Therefore, consider F_s fixed and assume $\alpha_w > \beta_s$.

For any $\alpha \in (\beta_s, \alpha_s)$, let $F_s^h(v|\alpha) = F_s(v + \alpha_s - \alpha)$ denote a horizontal and left-ward shift of F_s , such that the new distribution's support ends at α . The weak bidder's distribution in Example 2 takes this form. Let β^h denote the lowest end-point of this distribution's support. Technically, it is possible that $\beta^h < 0$ (depending on α), in which case the distribution does not satisfy the assumptions made in Section 2. Nevertheless, it remains a useful benchmark. Let $r^h(v|\alpha) = F_s^{-1}(F_s^h(v|\alpha))$. Similarly, let $F_s^v(v|\alpha)$ denote a vertical shift of F_s (as in Example 3), with the added requirement that $f_s^v(v|\alpha) = f_s(\beta_s)$ for all $v \in [\beta^v, \beta_s]$. Here, it is also possible that $\beta^v < 0$. Note that if f_s is increasing then $F_s^v(v|\alpha_w) \leq_{st} F_s^h(v|\alpha_w)$. The opposite holds if f_s is decreasing. Finally, let $F_s^r(v|\alpha_w)$ denote a rescaling of F_s such that $\frac{\alpha_s}{\gamma} = \alpha_w$ (as in Example 4). Let $r^m(v|\alpha) = F_s^{-1}(F_s^m(v|\alpha))$, where $m \in \{t, h, v, r\}$, denote the r function in the four benchmarks.

Given assumptions of log-concavity of F_s or $F_s(e^v)$, Examples 2 – 4 establishes that $F_s^h \leq_{rh} F_s$, $F_s^v \leq_{rh} F_s$, and $F_s^r \leq_{rh} F_s$. Thus, by Proposition 1, F_s^t first order stochastically dominates the other three benchmarks. In other words, there is a gap between F_s^t and F_s^h , say. In the following, I will derive conditions that ensure the FPA is more profitable than the SPA whenever F_w lies between F_s^t and one of the other benchmarks.

¹⁴Bagnoli and Bergstrom (2005, Section 5.3) write that a truncation is a “linear transformation” of the original distribution function. Unfortunately, they appeal to a result that relies on r , not G , being linear in order to “prove” their Theorem 9, which is erroneous. It is easy to construct examples where their claims concerning log-convex functions are untrue.

¹⁵Kirkegaard (2011b) shows that the FPA is also superior in the two models if the terms a and γ , respectively, are private information.

4.2 F_s is convex

To begin, consider the strongest curvature assumptions in Maskin and Riley's (2000) two models, namely that F_s is convex but log-concave. In this case, Corollary 1, below, unifies and extends Maskin and Riley's (2000) results.

Corollary 1 (Intermediate dispersion) *Fix F_s and $\alpha_w \in (\beta_s, \alpha_s)$. Assume F_s is convex but log-concave. Then, the FPA yields strictly higher expected revenue than the SPA if $F_s^t(\cdot|\alpha_w) \leq_{disp} F_w \leq_{disp} F_s^h(\cdot|\alpha_w) =_{disp} F_s$.¹⁶*

Proof. Lemma 1 implies (2) is satisfied. Lemma 3 establishes $F_w \leq_{rh} F_s$. ■

Figure 1 in the introduction illustrates Corollary 1. It applies if, for instance,

$$F_w(v) = \frac{F_s(v + \beta_s - \beta_w)}{F_s(\alpha_w + \beta_s - \beta_w)}, \quad v \in [\beta_w, \alpha_w], \quad (6)$$

such that F_w is obtained by first shifting F_s leftward, and then truncating it.¹⁷ Maskin and Riley (2000, p. 423) allude to this possibility, but do not provide any details or proof.

Using Proposition 1, $F_w \leq_{rh} F_s$ is violated if $F_w \leq_{disp} F_s^t(\cdot|\alpha_w)$, since $F_s^t(\cdot|\alpha_w) \leq_{st} F_w$ in that case. Likewise, by Remark A, if $F_s^h(\cdot|\alpha_w) \leq_{disp} F_w$ then condition (2) is violated. Thus, a problem arises if F_w is either too disperse or not disperse enough. In this light, Corollary 1 signifies that intermediate dispersion of F_w , compared to the benchmarks, are “almost” necessary and sufficient for the conditions of Theorem 1 to hold when F_s is convex and log-concave. The qualifier is due to the fact that the dispersive order is not a complete order.

Corollary 1 relates F_w to the benchmark distributions $F_s^h(v|\alpha_w)$ and $F_s^t(v|\alpha_w)$. It is also of interest to compare F_w directly to F_s . In the following, the assumption that F_s is log-concave is strengthened. It is then possible to describe qualitative relationships

¹⁶The assumptions in the proposition imply that $\beta_w \in [\beta_w^h, \beta_s]$. It should be understood from the description of the model in Section 2 that it is also required that $\beta_w \geq 0$. From Lemma 3, $F_s^t(\cdot|\alpha_w) \leq_{disp} F_s =_{disp} F_s^h$, so the set of F_w functions satisfying the condition in Corollary 1 is non-empty.

¹⁷Write $F_w(v) = F_s^t(\hat{r}(v)|\alpha_w)$. If F_w is described by (6) then $\hat{r}(v) \leq v + \beta_s - \beta_w$ and

$$\hat{r}'(v) = \frac{f_s(v + \beta_s - \beta_w)}{F_s(v + \beta_s - \beta_w)} \frac{F_s(\hat{r}(v))}{f_s(\hat{r}(v))} \leq 1,$$

by log-concavity. Thus, $F_s^t(\cdot|\alpha_w) \leq_{disp} F_w$. The proof that $F_w(v) \leq_{disp} F_s^h(\cdot|\alpha_w)$ is similar.

between F_w and F_s that are sufficient for the FPA to dominate the SPA. Recall that if $F_w \leq_{disp} F_s$, then the absolute distance between $r(v)$ and v is increasing. The implication of the next result is that the FPA is superior if the asymmetry between bidders do not increase too fast with type, or $\frac{r(v)}{v}$ is decreasing.

Corollary 2 *Assume $F_s(v)$ is convex and $F_s(e^v)$ is log-concave. Then, the FPA yields strictly higher expected revenue than the SPA if $F_w \leq_{disp} F_s \leq_* F_w$.¹⁸*

Proof. Assume $\alpha_w \geq \beta_s$. By Lemma 3, $F_s \leq_* F_w$ implies $F_s^t(\cdot|\alpha_w) \leq_* F_s \leq_* F_w$ and therefore $F_w \leq_{rh} F_s$. Since $F_w \leq_{disp} F_s$ and F_s is convex, condition (2) is satisfied as well. The result also holds if $\alpha_w < \beta_s$, since only condition (2) is required in that case. ■

In the spirit of Corollary 1, it is also possible to show that the FPA dominates when $r(v)$ takes intermediate values and satisfies certain regularity conditions. Note, by Lemma 3, that $r^{t'}(v|\alpha_w) \geq 1 = r^{h'}(v|\alpha_w)$ for all $v \in C$ when F_s is log-concave.

Proposition 2 (Rank-mixtures) *Fix F_s and $\alpha_w \in (\beta_s, \alpha_s)$. Assume F_s is convex and log-concave. Then, the FPA yields strictly higher expected revenue than the SPA if $r(v|\alpha_w)$ is steeper than $r^h(v|\alpha_w)$ but flatter than $r^t(v|\alpha_w)$; $r'(v|\alpha_w) \geq r^{h'}(v|\alpha_w)$ for all $v \in S_w$ and $r'(v|\alpha_w) \leq r^{t'}(v|\alpha_w)$ for all $v \in C$.*

Proof. The assumptions that $r'(v|\alpha_w) \geq r^{h'}(v|\alpha_w) = 1$ and F_s is convex mean that condition (2) is satisfied, by the first part of Lemma 1. The assumptions in the proposition also imply that $r^t(v|\alpha_w) \leq r(v|\alpha_w) \leq r^h(v|\alpha_w)$ on C . Thus, for $v \in C$,

$$\frac{f_w(v)}{F_w(v)} = \frac{f_s(r(v|\alpha_w))}{F_s(r(v|\alpha_w))} r'(v|\alpha_w) \leq \frac{f_s(r^t(v|\alpha_w))}{F_s(r^t(v|\alpha_w))} r'(v|\alpha_w) \leq \frac{f_s(r^t(v|\alpha_w))}{F_s(r^t(v|\alpha_w))} r^{t'}(v|\alpha_w) = \frac{f_s(v)}{F_s(v)},$$

where the first inequality comes from $r(v|\alpha_w) \geq r^t(v|\alpha_w)$ and the log-concavity of F_s . The second inequality comes from $r'(v|\alpha_w) \leq r^{t'}(v|\alpha_w)$. ■

The proposition applies if F_w is a “rank-mixture” of F_s^h and F_s^t . Clearly, this has a similar flavor as Corollary 1. However, neither implies the other.

¹⁸If F_s is a convex power distribution with $\beta_s = 0$ then $\ln F_s(e^v)$ is linear and the conditions in Theorem 1 are satisfied *if and only if* $r(0) = 0$, $r'(v) \geq 1$, and $r(v)/v$ is decreasing. It is possible to construct examples where f_s is increasing but f_w has a peak. Assume $f_s(v) = 2v$, $v \in [0, 1]$ and $r(v) = 4ve^{-v}$, $v \in [0, 0.357]$. Then, $f_w(v) = 2r(v)r'(v)$ is non-monotonic. This result should be contrasted with Maskin and Riley’s (2000) two models, in which f_w never has more peaks than f_s (see Remark B).

4.3 F_s is concave

A counterpart to Corollary 2 exists when F_s is concave.

Corollary 3 *Assume F_s is concave and that $F_w(v) = G(F_s(v))$, $v \in [\beta_s, \alpha_w]$, with $\alpha_w > \beta_s$ and $G'(\cdot) \geq 1$. If $G(x)/x$ is decreasing then the FPA yields strictly higher expected revenue than the SPA. $G(x)/x$ is decreasing if G is concave.¹⁹*

Proof. $\frac{F_w(v)}{F_s(v)} = \frac{G(F_s(v))}{F_s(v)}$ is decreasing by assumption, implying $F_w \leq_{rh} F_s$. Since $f_w(v) = G'(F_s(v))f_s(v) \geq f_s(v)$, Lemma 1 ensures condition (2) is satisfied. ■

However, the main result of this subsection is a counterpart to Corollary 1 and Proposition 2. Recall that $f_s^t(v|\alpha_w) \geq f_s^v(v|\alpha_w)$ for all $v \in C$.

Proposition 3 (Mixtures) *Fix F_s and $\alpha_w \in (\beta_s, \alpha_s)$. Assume F_s is concave. Then, the FPA yields strictly higher expected revenue than the SPA if F_w is steeper than F_s^v but flatter than F_s^t ; $f_w(v) \geq f_s^v(v|\alpha_w)$ for all $v \in S_w$ and $f_w(v) \leq f_s^t(v|\alpha_w)$ for all $v \in C$.*

Proof. By Lemma 1, the concavity of F_s and $f_w(v) \geq f_s^v(v|\alpha_w) \geq f_s(\max\{v, \beta_s\})$ for all $v \in S_w$ ensures condition (2) is satisfied. Since $f_w(v) \leq f_s^t(v|\alpha_w)$ for all $v \in C$, $F_w \leq_{st} F_s^t(\cdot|\alpha_w)$ and therefore

$$\frac{f_w(v)}{F_w(v)} \leq \frac{f_s^t(v|\alpha_w)}{F_s^t(\cdot|\alpha_w)} = \frac{f_s(v)}{F_s(v)},$$

thereby proving $F_w \leq_{rh} F_s$. ■

Proposition 3 applies if F_w is a convex combination (a mixture) of F_s^v and F_s^t on C , (with an appropriate differentiable extension on S_w/C).

4.4 $F_s(e^v)$ is convex

Example 4 does not require f_s to be monotonic. Assume from now on that $F_s(e^v)$ is convex but log-concave.²⁰ Hence, $vf_s(v)$ is increasing but $\frac{vf_s(v)}{F_s(v)}$ is decreasing. The density need not be monotonic.

¹⁹The transformation in Example 1 is linear and thus a special case of the one in Corollary 3. However, with the linear transformation in Example 1 it is possible to weaken the assumption that F_s is concave and instead assume only that it is log-concave.

²⁰If F is the uniform distribution then $F(e^v)$ is both strictly convex and strictly log-concave whenever $\beta_s > 0$. Write $F_s(v) = (1 - \varepsilon)F(v) + \varepsilon H(v)$, $\varepsilon \in (0, 1)$, where $F(v)$ is the uniform

Once again, a counterpart to Corollary 1 can be derived. Recall from Example 4 that $F_s^t \leq_* F_s^r =_* F_s$.

Proposition 4 *Fix F_s and $\alpha_w \in (\beta_s, \alpha_s)$. Assume $F_s(e^v)$ is convex but log-concave. Then, the FPA yields strictly higher expected revenue than the SPA if $F_s^t(\cdot|\alpha_w) \leq_* F_w \leq_* F_s^r(\cdot|\alpha_w)$.*

Proof. Since $F_s^t(\cdot|\alpha_w) \leq_* F_w$, Lemma 2 ensures $F_w \leq_{rh} F_s$. Lemma 3 guarantees condition (3) is satisfied. ■

Proposition 4 applies if F_w is obtained by first scaling down F_s , and then truncating it, $F_w(v) = \frac{F_s(\gamma v)}{F_s(\gamma \alpha_w)}$, where $\beta_w = \frac{\beta_s}{\gamma}$, $\alpha_w \in \left(\frac{\beta_s}{\gamma}, \frac{\alpha_s}{\gamma}\right]$, and $\gamma \geq 1$.

5 Interpretation & application of $\leq_{disp}$, $\leq_*$, and $\leq_c$

Bulow and Roberts (1989) argue that the auction design problem is analogous to the monopoly pricing problem. Intuitively, comparing different distributions is equivalent to comparing different demand functions. Thus, it is well known that the common stochastic orders of strength can be used to compare demand functions in intuitive ways. Here, I show that the stochastic orders of dispersion and spread also have natural interpretations.

Compare two distribution functions, F_1 and F_2 , with support S_1 and S_2 respectively. Define $r(v) = F_1^{-1}(F_2(v))$. Thinking of v as a price, the survival function $q_i(v) \equiv 1 - F_i(v)$ has the properties of a demand curve in a market with a continuum of consumers of mass one, distributed on S_i , $i = 1, 2$. For the most part, no stochastic order of strength need to be imposed in this section, but to fit with the model in Section 2 bidder 1 could be thought of as strong and bidder 2 as weak. For example, if $F_2 \leq_{st} F_1$, then $q_1(v) \geq q_2(v)$. It is well known, and easy to show, that hazard rate dominance would allow the elasticities in the two markets to be ordered (see (7), below).

distribution and $H(v)$ is a distribution on S_s . Assume H has finite density. If $\beta_s > 0$, $F_s(e^v)$ is then convex and log-concave when ε is sufficiently small. Note that f_s may have arbitrarily many peaks. As another example, assume F_s is obtained by truncating a Normal distribution with mean $.5(\beta_s + \alpha_s)$. If $\beta_s > 0$ and the variance is sufficiently large, then $F_s(e^v)$ is convex and log-concave (the truncated Normal distribution converges to the uniform distribution as the variance increases).

The relative change in demand following a marginal price increase can be measured by

$$\left| \frac{q'_i(v)}{q_i(v)} \right| = \frac{f_i(v)}{1 - F_i(v)} \text{ and } \varepsilon_i(v) = \left| \frac{v q'_i(v)}{q_i(v)} \right| = \frac{v f_i(v)}{1 - F_i(v)}. \quad (7)$$

For future reference, define marginal revenue evaluated at price v as

$$J_i(v) = v \left[1 - \frac{1}{\varepsilon_i(v)} \right] = v - \frac{1 - F_i(v)}{f_i(v)}.$$

The interpretation of J_i as marginal revenue is due to Bulow and Roberts (1989). Myerson (1981) refers to J_i as bidder i 's virtual valuation.

Stochastic orders of strength can be used to compare demand functions at any given price. The orders of dispersion and spread instead allow comparisons to be made at any given quantity (or probability of sale), as illustrated next. The third part assumes that densities are differentiable.

Proposition 5 *For any $v \in S_2$:*

1. $r(v) - v$ increasing $\iff \left| \frac{q'_2(v)}{q_2(v)} \right| \geq \left| \frac{q'_1(r(v))}{q_1(r(v))} \right|$.
2. $\frac{d}{dv} \left(\frac{r(v)}{v} \right) \geq 0 \iff \varepsilon_2(v) \geq \varepsilon_1(r(v))$.
3. $r''(v) \geq 0 \iff J'_2(v) \geq J'_1(r(v))$.²¹

Proof. The first part follows directly from (7) and $f_2(v) \geq f_1(r(v))$. The second part follows from

$$\frac{d}{dv} \left(\frac{r(v)}{v} \right) \propto r'(v) v - r(v) = \frac{f_2(v)}{f_1(r(v))} v - r(v) \propto f_2(v) v - f_1(r(v)) r(v) \propto \varepsilon_2(v) - \varepsilon_1(r(v)),$$

while the third part is due to

$$r''(v) \geq 0 \iff \frac{f'_2(v)}{(f_2(v))^2} \geq \frac{f'_1(r(v))}{(f_1(r(v)))^2} \iff J'_2(v) \geq J'_1(r(v)).$$

■

²¹ Obviously, if these properties hold for all $v \in S_2$ then $F_2 \leq_{disp} F_1$, $F_2 \leq_* F_1$, and $F_2 \leq_c F_1$, respectively.

In the monopoly interpretation, Proposition 5 implies that, starting at comparable quantities, a marginal price increase would have a greater impact on the less disperse market.

$F_2 \leq_{disp} F_1$ implies that the inverse demand curve $p_2(q) = F_2^{-1}(1 - q)$ is flatter than $p_1(q) = F_1^{-1}(1 - q)$. In contrast, $\frac{p_1(q)}{p_2(q)}$ is decreasing if $F_2 \leq_* F_1$.

Expressing marginal revenue as a function of quantity,

$$MR_i(q) \equiv J_i(F_i^{-1}(1 - q)) = \frac{f_i(F_i^{-1}(1 - q))F_i^{-1}(1 - q) - q}{f_i(F_i^{-1}(1 - q))}, \quad (8)$$

for $q \in [0, 1]$, it follows that

$$MR'_i(q) = J'_i(F_i^{-1}(1 - q)) \frac{-1}{f_i(F_i^{-1}(1 - q))}. \quad (9)$$

Assuming $F_2 \leq_{disp} F_1$, cases in which either $F_1 \leq_* F_2$ or $F_1 \leq_c F_2$ also hold have interesting interpretations.

Corollary 4 *If $F_2 \leq_{disp} F_1 \leq_c F_2$ then $|MR'_1(q)| \geq |MR'_2(q)|$. If $F_2 \leq_{disp} F_1 \leq_* F_2$ then $MR_1(q) \geq MR_2(q)$ whenever $MR_2(q) \geq 0$.²²*

Proof. For the first part, if $r(v) = F_1^{-1}(F_2(v))$ satisfies $r'(v) \geq 1$ and $r''(v) \leq 0$ then $f_1(F_1^{-1}(1 - q)) \leq f_2(F_2^{-1}(1 - q))$ and $J'_1(F_1^{-1}(1 - q)) \geq J'_2(F_2^{-1}(1 - q))$, respectively (see Proposition 5.3). The result then follows from (9). For the second part, $F_1 \leq_* F_2$ implies $f_1(F_1^{-1}(1 - q))F_1^{-1}(1 - q) \geq f_2(F_2^{-1}(1 - q))F_2^{-1}(1 - q)$. The result then follows from (8). ■

Wang (1993) compares auctions and posted-price selling in a model where otherwise symmetric bidders arrive sequentially. One of his comparative statics results is that if the marginal revenue curve is steeper for distribution F_1 than distribution F_2 , then auctions are more likely to dominate posted-price selling when all bidders draw types from F_1 rather than F_2 . Wang (1993) proves that for this to hold, F_1 must necessarily be more disperse than F_2 . Corollary 4 implies that $F_2 \leq_{disp} F_1$ combined with $F_1 \leq_c F_2$ is sufficient. First order stochastic dominance is not required for this result.

²²Mares and Swinkels (2011a) consider procurement auctions in which bidders' costs, c , are private information. Bidder i 's virtual cost is $\omega_i(c) = c + \frac{F_i(c)}{f_i(c)}$. Counterparts to Corollary 4 exist for "marginal costs" if (i) $F_2 \leq_{disp} F_1$ and $F_2 \leq_c F_1$ or (ii) $F_2 \leq_{disp} F_1$ and $F_2 \leq_* F_1$. Mares and Swinkels (2011a) assume (i).

Although Johnson and Myatt’s (2006) focus is very different from Wang’s (1993), many of the “ingredients” in their analysis are similar. Two distributions satisfy Johnson and Myatt’s (2006) rotation order if they cross precisely once.²³ They are also interested in distributions whose marginal revenue curves coincide at most once, which is obviously the case if one marginal revenue curve is steeper than the other. Both Wang (1993) and Johnson and Myatt (2006) explicitly mention variance ordered distributions, where F_i can be written $F_i(v) = F\left(\frac{v-\mu_i}{\sigma_i}\right)$. In this case, $r(v)$ is linear, with $r'(v) > 1$ whenever $\sigma_1 > \sigma_2$. Thus, $F_2 \leq_{disp} F_1$ but $F_1 =_c F_2$. The result in the previous paragraph therefore applies.

Moreover, assuming non-negative marginal costs, the important comparison of marginal revenues is at quantities where they are positive. It is irrelevant how many times marginal revenue curves cross below zero. Corollary 4 implies that marginal revenues are ordered in the positive quadrant if $F_2 \leq_{disp} F_1 \leq_* F_2$. Thus, this combination of the dispersive and star orders makes it possible to determine on which market the optimal quantity is highest. Note, however, that $F_2 \leq_{disp} F_1 \leq_* F_2$ implies $r(v) \geq r'(v)v \geq v$, or $F_2 \leq_{st} F_1$. With the stronger assumption that $F_2 \leq_{hr} F_1$, it is also possible to determine on which market the optimal price is the highest. Indeed, both the price and the quantity sold would be higher on market 1. In the auction setting, an optimal auction with n clones of bidder 1 would thus have a higher reserve price but nevertheless yield a higher probability of sale than an optimal auction with n clones of bidder 2.

6 Conclusion

Keeping the strong bidder’s distribution fixed, this paper establishes that the FPA dominates the SPA whenever the weak bidder’s distribution falls in the gap between a truncation and a shift of the strong bidder’s distribution, subject to some curvature and regularity assumptions. Thus, robust environments are described in which one auction dominates another. In contrast, the existing literature has focused mainly on ranking auctions in isolated examples.

The results of the paper rely critically on a new result that links the dispersive

²³The rotation order and the dispersive order are related. If two distributions cross and $r(v) - v$ is strictly increasing then they cross exactly once. However, as Ganuza and Penalva (2010) point out, dispersion only requires $r(v) - v$ to be weakly increasing, which permits the two distributions to coincide on an interval.

and star orders to reverse hazard rate dominance. Here, a pivotal role is played by the truncation of the strong bidder's distribution. In fact, a complementary result proves that reverse hazard rate dominance is violated if the weak bidder's distribution is between the strong bidder's distribution and a truncation hereof. In other words, reverse hazard rate dominance cannot apply unless the asymmetry is sufficiently large. A challenge for auction theory is thus identified; what are the properties of bidding strategies and revenue when the asymmetry is small?²⁴

The central role played by the dispersive order complements recent findings by Kirkegaard (2011c) and Mares and Swinkels (2011a, 2011b) in other asymmetric auction settings. Thus, the dispersive order may prove to be as useful for the analysis of auction design as the more commonly used stochastic orders of strength.

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²⁴Kirkegaard (2009) examine bidding behavior without any assumptions of reverse hazard rate dominance. Gaviols and Minchuk (2011) compare revenue in auctions with small asymmetries. However, both papers assume bidders share the same support, which is not the case in the current paper.

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