Trading Networks and Equilibrium Intermediation

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20 April 2012

Abstract

We consider a network of intermediaries facilitating exchange between a buyer and a seller. Intermediary traders face a private trading cost, a network characterizes the set of feasible transactions, and an auction mechanism sets prices. We investigate stable and equilibrium network configurations. Bottlenecks in the trading network arise naturally in a free-entry competitive equilibrium and equilibrium markets organize into an asymmetric structure with many traders near the buyer and fewer traders near the seller. Such asymmetries can render equilibrium markets fragile by amplifying the shocks experienced by key intermediaries in the economy.

Keywords: Networks, Intermediation, Trade, Network Formation
JEL: D85, L14, D44

This study is still a work in progress. Please contact the authors for the latest draft before citing. All comments and suggestions are greatly appreciated. First draft: 29 February 2012.

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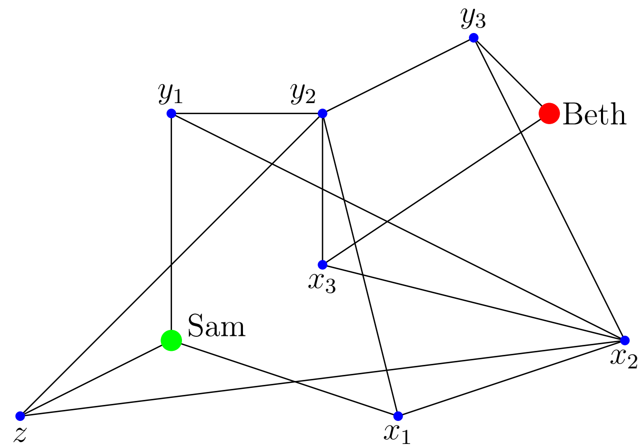

Figure 1: A trading network.

Intermediation in markets is commonplace. Consider the situation depicted in Figure 1. Sam is a farmer growing watermelons in California while Beth is a consumer of watermelons in New England. There are gains from trade between Sam and Beth; however, rarely will Sam and Beth trade directly—they may not even know each other. Instead, trade between them is mediated by a network of intermediary agents $\{x_1, x_2, x_3, y_1, y_2, y_3, z\}$. These intermediaries—such as wholesalers, transporters, distributors, or retailers—have invested in market-specific technologies and have developed a web of trading relationships through which Sam and Beth are linked. As Figure 1 emphasizes, there are many paths in the economy through which Sam's produce can arrive on Beth's picnic table.

We study two cross-cutting dimensions of intermediary networks. First, networks encourage *competition* among intermediaries, especially those with similar links and relationships. For example, noting their positions, x_3 and y_3 can perform similar tasks in the market of Figure 1. Intuitively, markets with a high degree of competition among intermediaries are robust—a shock experienced by a particular intermediary is unlikely to lead to market breakdown since competitors can step in. However, are equilibrium intermediary networks destined to be robust or are fragilities and bottlenecks, which exaggerate the shocks experienced by certain agents, likely to emerge? How severe are any associated welfare consequences?

Second, the persistence of a network of intermediaries as a facilitator of exchange implies that there must be some definable distance between the seller and the buyer—a *degree of intermediation*. In the market of Figure 1, at least three intermediaries are necessary to deliver Sam's watermelons to Beth. But why three? For example, $\{x_1, x_2, x_3\}$ could merge together, change the network of relationships in the economy, and thus shorten the economic distance between Sam and Beth. In this regard, intermediary networks encourage cooperation among dissimilar or complementary traders. Which economic forces counteract such integrative impulses and add stability to the network of intermediary relationships? Are all networked markets stable in the long term or are some network configurations unlikely to persist?

To explore both dimensions of intermediation, we develop a model of endogenous network formation. Intermediary traders have private information concerning their trading cost, an auction mechanism sets prices, and a free-entry/zero-profit condition drives the equilibrium network formation process. Our study speaks to the fragility and complexity of intermediary markets. We argue that intermediary networks have a natural tendency to organize into a structure that exaggerates the negative shocks experienced by some traders. Bottlenecks between the buyer and the seller can arise in equilibrium with negative consequences for market robustness as a whole. Moreover, network externalities imply that many network structures—with significant differences in efficiency—are often possible in equilibrium. We also argue that traders' incentives to shorten distances in the economy are closely tied to the underlying trading technology. Indeed, a market may exhibit a very large distance between buyers and sellers and be robust to traders integrating or forming collusive relationships with neighbors. Despite not always reflecting an efficient market organization, very extensive and complex intermediary markets can be stable.

We develop our argument progressively. Section 1 formally introduces our model. While our lighthearted introductory vignette with Sam and Beth focused on the produce market, the applicability of our analysis to many real and financial markets is immediate. Indeed, Spulber (1996) argues that intermediation, broadly interpreted, accounts for a quarter of U.S. economic output. Beyond the noted application to the exchange of physical goods, in section 1 we offer alternative interpretations of our model with applications to production and to financial markets. Indeed, many economic activities can be considered as involving a high degree of intermediation services. Our analysis stresses unpacking the black box between buyers and sellers so that the network of relationships underlying these transactions becomes a key contributor behind successful markets.

As a prelude to our main analysis, Section 2 briefly discusses price-formation and exchange taking the intermediary network as fixed and exogenous. Section 3 tackles the first dimension of intermediation by endogenizing the level of competition among intermediaries while holding the degree of intermediation fixed. Here we show that competitive intermediary markets have a natural tendency to assume an equilibrium form which does not tend to promote market robustness. Roughly, intermediaries will congregate near the buyer with fewer intermediaries linked to the seller. The asymmetric structure begets fragility in the market as a whole. Section 4 considers the second dimension of intermediation and introduces our notion of stable markets, which are immune to collusive arrangements among neighboring intermediaries. Finally, noting a tension between stable markets and those with equilibrium levels of trader competition, Section 5 reconciles the two ideas. We place our analysis in the context of the wider literature on networked markets before concluding. An appendix collects longer proofs not in the main text.

1 Model

An economy is characterized by three elements. First, agents are organized in a network defining the set of trading possibilities. Second, each agent has a private trading cost determining the prudence of exchange. Finally, a trading protocol sets prices. After introducing our model, we comment on our assumptions and we offer interpretations in relation to the exchange of physical goods, to intermediate goods production, and to financial intermediation.

Trading Possibilities Trading possibilities are summarized by a directed graph $\langle \mathcal{N}, \mathcal{E} \rangle$. $\mathcal{N}$ is the set of agents (nodes) and $ij \in \mathcal{E}$ is an edge indicating that agent i can sell an item to agent j. Suppose that the network is given exogenously and that it is common knowledge. In section 3 we discuss network formation.

Our network topology generalizes the incomplete trading networks analyzed by Gale & Kariv (2009) and Figure 2 presents a typical example. Agents are arranged in rows $\{0, 1, \ldots, R+1\}$.[1] Let $\mathcal{N}_r$ be the set of agents in row r. Row $R+1$ is inhabited only by the seller (S). The seller is the originator of an asset. We assume that the seller is always willing to sell her creation at a price normalized to zero. Row 0 is inhabited only by the buyer (B) who is offering to pay $v > 0$ for the seller's asset. Since we seek to study intermediary behavior, for simplicity we assume that the buyer and the seller are passive agents. The seller and buyer can be interpreted as metaphors for larger upstream and downstream markets that the intermediaries take as given. Extensions allowing for multiple buyers and sellers or

[1] Our row numbering is opposite to the convention followed by Gale & Kariv (2009).

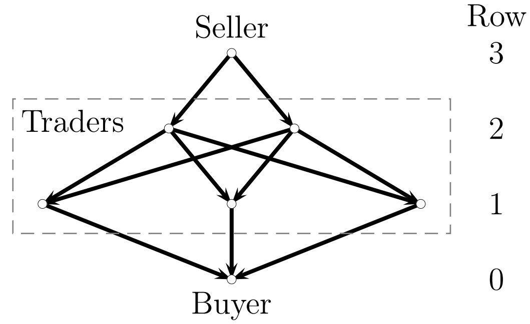

Figure 2: A trading network with configuration $\mathbf{n} = (3, 2)$.

uncertain buyer valuations are easy to accommodate.

There exist gains from transferring the asset from the seller to the buyer; however, the seller and the buyer cannot trade directly: $SB \notin \mathcal{E}$. Instead, there is a set of intermediary traders who may buy and (re)sell the asset. Intermediaries inhabit rows $r \in \{1, \ldots, R\}$. Row r has n_r traders and we call the vector $\mathbf{n} = (n_1, \ldots, n_R)$ the *configuration* of intermediary traders.[2] Traders do not value the asset per se. Rather, they seek to earn trading profits by facilitating network-conforming trades which move the asset from the seller to the buyer. We assume that the trading network is not complete and instead constrains trade as follows:

> An agent in row r can purchase the asset from any agent in row $r + 1$ and can sell the asset to any agent in row $r - 1$. Other trades are not feasible.[3]

The example in Figure 2 is a trading network with $R = 2$ and a trader configuration $\mathbf{n} = (3, 2)$. Traders in row 2 can buy directly from the seller and can resell the asset to traders in row 1. We interpret R as measuring the *degree of intermediation* in an economy. The number of traders in each row, $\mathbf{n} = (n_1, \ldots, n_R)$, measures the competition among intermediaries. Our analysis focuses on these two dimensions of trading markets.

Trading Costs We assume that each trader i in row r has a private trading cost $\theta_{ri} \in \{0, \bar{c}\}$ where $0 < v < \bar{c}$. At the time of trading, each agent knows her own private trading cost but is unaware of others' realized trading costs. In this sense, the network structure captures the set of (ex ante) potential trading relationships while agents hold residual uncertainty

[2] We use standard shorthand: $\mathbf{n}_{-r} = (n_1, \ldots, n_{r-1}, n_{r+1}, \ldots, n_R)$ and $\mathbf{n} = (n_r, \mathbf{n}_{-r})$.

[3] If $i \in \mathcal{N}_r$ and $j \in \mathcal{N}_{r-1}$, then $ij \in \mathcal{E}$. Similarly, if $i \in \mathcal{N}_r$, $j \in \mathcal{N}_{r'}$ and $r' \neq r - 1$, then $ij \notin \mathcal{E}$.

concerning the set of economically worthwhile trading opportunities in a given trading period. We take the distribution of trading costs to be common knowledge. Costs are distributed independently and identically. The probability that a trader has a low trading cost is $\Pr[\theta_{ri} = 0] = p \in (0, 1)$. We consider private trading costs to behave like an inventory cost. A trader incurs the private cost θ_{ri} upon acquiring the asset even if who only intends to resell it. Traders that do not encounter the asset do not incur a trading cost.

We interpret p as describing the trading technology. If p is very low, traders find it difficult to trade since they are exposed to negative cost shocks with high probability. Alternatively, $1-p$ reflects the expected opportunity cost that a specific trader faces in participating in this particular market versus other (not modeled) markets. If $1 - p$ is very high, despite being a member of our trading network, agents have on average more promising opportunities elsewhere.

Trading Protocol We assume that all trade occurs via second-price, sealed-bid auctions and proceeds along the following timeline.

1. An agent in row $r + 1$ has the asset. She organizes an auction to sell it to someone in row r.

2. Each agent in row r submits a bid from the set $\mathcal{B} = \{\ell\} \cup \mathbb{R}_+$.

   (a) The bid $\ell < 0$ is a non-competitive bid equivalent to not participating in the auction. An agent bidding ℓ cannot win the auction nor will she ever incur the trading cost θ_{ri}. If all agents in row r bid ℓ, the asset is not sold and it expires. In this case, trade is said to break down.

   (b) Bids $b_{ri} \neq \ell$ are competitive bids. The agent submitting the highest competitive bid wins the auction. A lottery among high bidders resolves any ties.

3. The agent winning the auction makes a payment equal to the second-highest competitive bid (or zero if all others bid ℓ) and incurs her private trading cost. Other agents do not incur any costs.

4. The process repeats until the asset reaches the buyer or trade breaks down. By assumption, the buyer pays v to purchase the asset from a trader in row 1.

We assume that all traders are risk neutral and wish to maximize trading profits net of trading costs. If a trader never acquires the asset her payoff is zero. If a trader purchases the asset her payoff is given by

$$\text{Resale Price} - \text{Price Paid} - \theta_{ri}.$$

Discussion and Interpretations Our first key assumption concerns the economy's network structure. Our network is not complete but it has a regularized form making the direction of trade discernible. The topology captures the notion that many traders can perform a similar function in the chain of intermediation. Furthermore, we believe that the structure captures many of the qualitative features of trade in complex economic networks. For example, we would not expect trade to exhibit cycles (Gale & Kariv, 2007). Similarly, some branches of a more general network may be pruned if they are ancillary to trade.[4] We introduce additional asymmetries into our network structure when we investigate market stability in Section 4.

Throughout we assume that all traders are ex ante symmetric, except for their position in the network. Trader symmetry allows us to isolate the effects of the network structure alone. Extending the analysis to ex ante asymmetric traders opens many new questions, such as the selection of traders into systematically different intermediary roles, which we hope to explore in future work.

Like Kranton & Minehart (2001) or Patil (2011), we rely on a second-price (equivalently, an ascending auction) to structure exchange. Beyond capturing the flavor behind a competitive bidding process, this format allows us to bracket trader bidding behavior and to move quickly into a discussion of equilibrium network structures. Bidding one's value is still an equilibrium strategy and we exploit this simplification fully. That said, much of our analysis is robust to a change in the auction format. In an earlier working paper (Kotowski & Leister, 2012), we developed our arguments around first-price sealed-bid auctions. This alternative mechanism, where traders follow mixed strategies in equilibrium, offers a rich set of predictions concerning price formation as trade unfolds. Our conclusions about equilibrium networks carry over to alternative auction formats provided revenue equivalence obtains. We leave to future research the introduction of other trading schemes or of more exotic auction mechanisms. For example, Rubinstein & Wolinsky (1987) study a consignment mechanism with middlemen while Zheng (2002) considers optimal auctions with resale. Alternative treatments of network-mediated exchange focus on bargaining (Corominas-Bosch, 2004; Manea, 2011; Elliott, 2011; Siedlarek, 2011).

[4]For example, if the network had traders who are not on any path between the buyer and the seller, they would not wish to participate in this market anyway.

To aid intuition, above we described the trading procedure as proceeding sequentially. In many markets, this assumption is natural. However, our trading game also has a static counterpart. In this version, all bids are submitted simultaneously and the asset immediately traverses the network, greedily following the path of the highest bids. The static variant allows for a simple adaptation of our model as a (normal-form) experiment. It is a reasonable approximation of a market where trade happen very rapidly, as in finance, with minimal strategic revisions by traders during the course of particular transaction. The experimental study of Gale & Kariv (2009) is an implementation of such a design.

In the introduction, we sketched our model's interpretation concerning the exchange of tangible goods. As in our produce example, a network of intermediaries acts as a geographic and temporal bridge between producers and consumers. Another interpretation of our set-up is as a model of production with intermediate inputs. A consumer wishes to purchase one unit of good q_1 at a price of v. Only firms $i \in \mathcal{N}_1$ have the technology to produce good q_1. The production function of good q_1 combines one unit of labor (for example), at cost θ_{1i}, with one unit of intermediate good q_2; and so on up to intermediate good q_{R+1}, which is a natural resource available at a rate of one unit per trading period. Interpreted in this light, our model emphasizes the importance of complementaries in production—a theme explored extensively in the literature on economic development (Kremer, 1993). Nagurney & Qiang (2009) present a network model, with a topology reminiscent of our networks, to describe production inside a firm.[5]

Our model also has an interpretation when viewed as a financial market.[6] Suppose an investor (the seller) has one unit of capital available. A safe asset offers a return normalized to zero. Some firm seeking financing (the buyer) offers an expected rate of return of $v > 0$ for the funds. Intermediary financial institutions—banks, brokers, insurance companies, mutual funds, etc.—link the investor and the firm. The investor initially allocates her funds with the intermediary promising the highest return. The intermediary does the same, and so on until the resources reach the firm. Intermediaries skim small fractions of the expected return promised by the firm as a payment for their intermediation services. On average intermediaries earn non-negative profits but particular intermediaries may experience losses if they are unlucky in allocating the funds among downstream interests.

[5]Noting the production interpretation, our model therefore covers cases in between pure intermediation and pure production. For example, intermediaries may "add value" to an item by introducing packaging or a specific branding. Often which activities are labeled as "intermediation" is quite flexible and our model accommodates many such instances.

[6]This interpretation may be easier to appreciate if prices are set with a first-price auction.

We analyze our model in three steps. Section 2 briefly examines intermediary behavior holding the trading network fixed. Section 3 endogenizes the degree of competition among traders ($\mathbf{n}$) while Section 4 examines the stability of the market's other dimension, the degree of intermediation (R).

2 Exogenous Trading Networks

We begin by recording some facts about trader behavior in a fixed trading network. Fix an economy with R rows of intermediaries and a market configuration $\mathbf{n} = (n_1, \ldots, n_R)$. It is well-known that the second-price auction admits multiple equilibria, and indeed this observation carries over to our model. Following tradition, however, we restrict our analysis to the equilibrium where all traders truthfully bid their value for the item that they can buy.

Theorem 1. *Let $\mathbf{n} = (n_1, \ldots, n_R)$ be a fixed network configuration and define*

$$\delta(n) = 1 - (1-p)^{n-1}(1 + p(n-1)).$$

Let $\nu_r = \prod_{k=1}^{r-1} \delta(n_k)v$. The strategy profile where each bidder $i \in \mathcal{N}_r$ bids ν_r if $\theta_{ri} = 0$ and ℓ otherwise is a Bayesian-Nash equilibrium of the trading game.

Proof. In a second-price auction, truthfully bidding one's value is a dominant strategy. Thus, it suffices to show that ν_r is the value of the asset to a trader in row r given the bids of others.[7] The argument is by induction.

All low-cost traders in row 1 will bid v since this is the price they can secure by selling the asset to the buyer. High cost traders bid ℓ since $v < \bar{c}$. Now consider a trader in row 2. If she has the asset, one of three events may happen when she tries to resell it to an agent in row 1:

1. There are at least two low-cost agents in row 1. Since each such trader will bid v, the item will transact at that price.

2. There is exactly one low-cost agent in row 1. In this case the asset sells at a price of zero since there is only one competitive bid submitted.

3. There are no low-cost agents in row 1. In this case the asset does not sell at all.

[7] Generally, the strategy in Theorem 1 is not a dominant strategy since ν_r depends on the strategies of other bidders.

The probability of case 1 is $1-(1-p)^{n_1}-n_1p(1-p)^{n_1-1}=\delta(n_1)$. Therefore the expected resale value is $\nu_2=\delta(n_1)v$. By a similar argument, if each low-cost agent in row $r-1$ bids ν_{r-1}, the expected value of the asset to a trader in row r is $\nu_r=\delta(n_{r-1})\nu_{r-1}$. The conclusion follows. □

While the equilibrium bidding strategy outlined in Theorem 1 is simple, it is nevertheless reflective of many qualitative features of exchange. First, since $\delta(n)<1$ expected prices and bids by low-cost traders are nondecreasing as the asset approaches the final buyer. Second, since $\delta(n)$ is increasing in both p and n (Lemma A.1 in the appendix), transaction prices increase as prevalence of low-cost traders increases ($p\uparrow$) and as the degree of competition in the economy increases ($n_r\uparrow$). Changes in n_r, have an asymmetric effect for different parts of the trading network. Changes in this value only impact the bids and transaction prices seen in rows $r'>r$.[8]

Despite its simplicity and intuitive appeal, in practical terms the equilibrium outlined in Theorem 1 actually demands a high degree of sophistication from traders. In particular, traders must be able to correctly forecast downstream trader's strategies and associated prices to bid effectively. Strategic uncertainty or a failure of the inductive reasoning behind equilibrium bidding can in principle lead to very different non-equilibrium outcomes. Recent laboratory experiments studying a similar trading environment by Gale *et al.* (2012) suggest that human subjects (i.e. Berkeley undergraduate students) form reasonably accurate estimates of downstream prices and equilibrium predictions accord well with observed market outcomes. We view this as supportive evidence for the equilibrium analysis we are undertaking.

3 Endogenous Competition

In this section we study network formation when R is fixed. Many economic interactions are defined by a fixed degree of intermediation due to geographic, legal, or transaction-cost considerations. In the subsequent section we pose the complementary question and ask whether R is a stable degree of intermediation given the trading technology and the prevailing trader configuration.

Fix R and suppose there is a large group of potential traders who may enter the market at any of the R rows. To enter the market, a trader must incur an entry cost of $\kappa>0$. We interpret κ as an irreversible investment in market-specific skills or technology. For

[8] The invariance of bids in row r to n_r is an artifact of the second-price auction.

example, it may be the cost of forming the relevant relationships to be a part of the trading community or network. Once all traders have made their entry and location decisions, the network configuration $\mathbf{n}$ becomes known. Each trader then learns her private cost θ_{ri} and exchange unfolds as before. Agents not entering the market receive a payoff of zero. We wish to define an equilibrium concept in our model that consistent with the following basic principle.

Assumption A1. *There is free entry into the trading market. Traders enter until no further profitable entry is possible into any row.*

To operationalize the essence behind Assumption A1 requires some new notation. Let $\pi_r(\mathbf{n})$ denote the expected profits (before the realization of private trading costs) of a trader in row r given the trading network $\mathbf{n} = (n_1, \ldots, n_R)$.

Lemma 1. *Suppose* $\mathbf{n} = (n_1, \ldots, n_R)$ *is a configuration of traders. If*

$$\delta(n_k) = 1 - (1-p)^{n_k - 1}(1 + p(n_k - 1))$$
$$\mu(n_k) = 1 - (1-p)^{n_k}$$

then for each $r \in \{1, \ldots, R\}$,

$$\pi_r(\mathbf{n}) = \left[\prod_{k=1}^{r-1} \delta(n_k)\right] \left[p(1-p)^{n_r - 1}\right] \left[\prod_{k=r+1}^{R} \mu(n_k)\right] v. \tag{1}$$

Moreover, $\pi_r(n_r, \mathbf{n}_{-r})$ *is increasing in* $\mathbf{n}_{-r}$ *and decreasing in* n_r.

Proof. Consider bidder i in row r. Suppose this bidder is bidding to purchase the asset. If this bidder has a low trading cost, this bidder will bid ν_r. With probability $(1-p)^{n_r - 1}$, all other bidders in row r have a high trading cost; thus, bidder i receives a surplus of ν_r. (She expects to resell the asset at price ν_r but pays zero to buy it.) With probability $1-(1-p)^{n_r-1}$, this bidder receives a surplus of zero. Either she does not win the auction or if she wins, she must pay the second highest bid, which is also ν_r. Thus, conditional on bidding for the asset, a low-cost trader in row r has an expected payoff of $\left[\prod_{k=1}^{r-1} \delta(n_k) v\right] (1-p)^{n_r - 1}$. The probability that this bidder draws a low trading cost is p. Finally, the probability that the asset reaches row $r+1$ (and thus, a bidder in row r can bid to purchase the asset) is the probability that each row $k \in \{r+1, r+2, \ldots, R\}$ has at least one low-cost trader. This probability is $\prod_{k=r+1}^{R} \mu(n_k)$. Combining the above observations gives (1).

Lemma A.1 shows that $\delta(n)$ is increasing in n. That $\mu(n)$ is increasing in n is obvious; thus, $\pi_r(n_r, \mathbf{n}_{-r})$ is increasing in $\mathbf{n}_{-r}$. By inspection, $\pi_r(\mathbf{n})$ is decreasing in n_r. $\square$

Remark 1. Since $\pi_r(\mathbf{n}) \propto v$ it is without loss of generality that we henceforth normalize the value of the asset to the buyer at $v = 1$.

The μ- and δ-terms in (1) capture the positive externality experienced by a trader in row r from an increase in the number of traders at upstream (μ) and downstream (δ) positions in the network. A trader benefits from more downstream traders since the more intense competition leads to higher (re)sale prices. The benefit a trader receives from upstream traders is more subtle. Recall that a trader can only earn profits if the asset actually reaches her row and she is fortunate enough to sell it. With increased upstream competition, this event becomes more likely and the probability of a premature market breakdown declines.

Noting the presences of externalities, we propose two equilibrium notions compatible with our free entry assumption.

Definition 1 (Local Equilibrium)**.** $\mathbf{n}^* = (n_1^*, \ldots, n_R^*)$ is a *local equilibrium configuration* of traders if for all $r \in \{1, \ldots, R\}$,

$$\pi_r(n_1^*, \ldots, n_r^*, \ldots, n_R^*) - \kappa \geq 0$$

and

$$\pi_r(n_1^*, \ldots, n_{r-1}^*, n_r^* + 1, n_{r+1}^*, \ldots, n_R^*) - \kappa < 0.$$

In a local equilibrium all traders earn nonnegative profits and no single additional trader can enter profitably. We consider the equilibrium to be "local" since the configuration $\mathbf{n}^*$ is robust only to one-agent deviations. Our second equilibrium notion looks at large changes in the network structure. A global equilibrium is a local equilibrium with the added requirement that no group of potential traders, coordinating their entry decision, can all enter profitably.

Definition 2 (Global Equilibrium)**.** $\mathbf{n}^* = (n_1^*, \ldots, n_R^*)$ is a *global equilibrium configuration* of traders if for all $r \in \{1, \ldots, R\}$,

$$\pi_r(n_1^*, \ldots, n_r^*, \ldots, n_R^*) - \kappa \geq 0$$

and for all $\mathbf{a} = (a_1, \ldots, a_R) \in \mathbb{Z}_+^R$, $\mathbf{a} \neq 0$, there exists r such that $a_r \geq 1$ and

$$\pi_r(n_1^* + a_1, \ldots, n_r^* + a_r, \ldots, n_R^* + a_R) - \kappa < 0.$$

Several features of a global equilibrium are worth noting. First, our definition allows for coordinated entry but we do not allow coordinated side-payments between new entrants. In future work, we wish to extend our model by allowing more sophisticated economic relationships among entering parties but for the moment we suppress such embellishments. Second, the intuition behind a global equilibrium highlights the backward and forward linkages in our market. A new trader entering the market will generate a positive externality on both upstream and downstream market participants. The existence of externalities suggests there is a scope for mis-coordination and equilibrium multiplicity. Finally, we show below that a global equilibrium is unique. Hence, the stronger equilibrium concept offers a natural selection criterion.

We call a network *trivial* if $\mathbf{n} = (0, \dots, 0)$ and nontrivial otherwise. When $R \geq 2$ there always exists a trivial local equilibrium. No agent wishes to enter row 2 since there are no traders in row 1 to buy the asset. No agent wishes to enter row 1 since the asset is never made available to row 1 traders. In both cases, incurring the fixed entry costs is not worthwhile. Similar reasoning applies to all rows. Although an important case—arguably many (unobserved) markets depending on complementary relationships could exist but do not because of (unobserved) "coordination" on the no trade equilibrium—we will focus only on nontrivial equilibria in the following discussion.

Using (1) we can derive several conclusions concerning equilibrium market configurations. First, local and global equilibria exist under fairly weak conditions. Second, equilibria form a directed set with a unique global equilibrium dominating all local equilibria. Finally, equilibrium configurations will assume an asymmetric structure similar to a pyramid. Hence, our market does not naturally organize itself into a robust network structure and it may be very fragile despite our free-entry assumptions. Theorems 2–4 formalize these points.

Theorem 2. *Let* $\bar{n} \equiv \left\lceil 1 + \frac{\log(\kappa) - \log(p)}{\log(1-p)} \right\rceil$ *and define* $\mathscr{X} = \{\mathbf{n} \in \mathbb{Z}^R \colon 0 \leq n_r \leq \bar{n}\}$. *There exists a nontrivial local equilibrium configuration if and only if there exists* $\mathbf{n} \in \mathscr{X}$ *such that for all* r, $\pi_r(\mathbf{n}) - \kappa \geq 0$.

The proof of Theorem 2 relies on a tâtonnement procedure. When it is started at any configuration $\mathbf{n}$ satisfying $\pi_r(\mathbf{n}) - \kappa \geq 0$, it converges to a local equilibrium. An immediate corollary and shown en route in the proof of Theorem 2 is the following.

Corollary 1. *If* $\mathbf{n}^*$ *is a local equilibrium configuration, then* $\mathbf{n}^* \in \mathscr{X}$.

Since $\mathscr{X}$ is a finite set, Corollary 1 says that an exhaustive search in $\mathscr{X}$ will identify all equilibria in our model. A naive search is acceptable when R is small (as in our examples below) but is impractical for economies with a very large degree of intermediation. A related concern is that if R is large, verifying the necessary and sufficient condition for an equilibrium to exist may be prohibitively difficult. A very conservative sufficient condition for a nontrivial equilibrium is that $\min\{(1-p)p^{2R-2}, p^{2R-1}\} > \kappa$ (Lemma A.4 in the appendix).

Building on Theorem 2 we can show that the set of equilibrium networks has a natural directed structure and that there exists a unique global equilibrium configuration.

Definition 3. Let $\mathbf{n} = (n_1, \ldots, n_R)$ and $\mathbf{m} = (m_1, \ldots, m_R)$ be network configurations. We say that $\mathbf{n} \geq \mathbf{m} \iff n_r \geq m_r$ for each r.

Theorem 3. *Let $\mathbf{n}^*$ and $\mathbf{m}^*$ be local equilibrium configurations.*

1. *There exists a local equilibrium configuration $\mathbf{x}^*$ such that $\mathbf{x}^* \geq \mathbf{n}^*$ and $\mathbf{x}^* \geq \mathbf{m}^*$.*

2. *There exists a unique global equilibrium configuration $\mathbf{q}^*$ and $\mathbf{q}^* \geq \mathbf{n}^*$ for all local equilibrium configurations $\mathbf{n}^*$.*

Example 1 illustrates the coexistence and ordering of distinct equilibria. We discuss its implications in detail below.

Example 1. Suppose $R = 6$, $p = 0.5$, $\kappa = 0.01$. There exist two equilibrium networks:

$$\mathbf{n}^* = (4, 4, 3, 3, 2, 1)$$
$$\mathbf{m}^* = (6, 6, 6, 6, 5, 5)$$

Figures 3 and 4 illustrate these networks. $\mathbf{m}^*$ is a global equilibrium configuration while $\mathbf{n}^*$ is only a local equilibrium.

The two equilibrium markets in Example 1 provoke several observations. First, both networks share a distinctive "pyramid" structure with more traders locating near the buyer than near the seller in equilibrium. This is a characteristic feature of all equilibrium markets.

Theorem 4. *If $\mathbf{n}^* = (n_1^*, \ldots, n_R^*)$ is a local equilibrium configuration then for all r, $n_r^* \geq n_{r+1}^*$.*

The intuition behind Theorem 4 is best illustrated with the following thought experiment. Suppose that a market has an equal number of traders in each row. Although the network structure is balanced, there is (typically) an imbalance in the (ex ante) expected profits of traders in different rows. Specifically, noting (1) and because $\mu(n) > \delta(n)$ for all $p \in (0, 1)$,

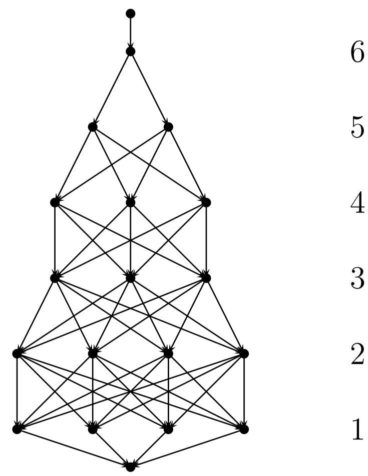

Figure 3: A local equilibrium configuration ($\mathbf{n}^*$) in Example 1.

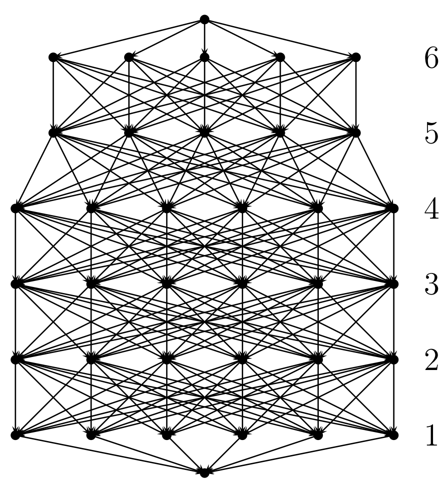

Figure 4: The global equilibrium configuration ($\mathbf{m}^*$) in Example 1.

the expected profits of a trader in row 1 are greater than the expected profits of a trader in row R. Provided $\pi_1(\mathbf{n})$ is sufficiently large, if additional traders were to enter the market now, they would be naturally drawn to rows closer to the buyer since the expected returns are greatest there. As entry occurs, asymmetric competition equalizes profits across rows at zero.

The simple observation that $\mu(n) > \delta(n)$ reflects the distinct roles played by upstream and downstream trading relationships. From the perspective of a trader in row r, the only relevant characteristic of upstream trades is that they occur. The prices at which traders in rows $r+1, \ldots, R$ trade is immaterial. From a downstream relationship, however, a trader in row r cares not only that exchange can happen but she also has an interest in the prices at which it happens. The distinction between μ and δ thus reflects this added dependence on downstream trading relationships.

A second observation we can draw from equilibrium markets concerns welfare. Up to integer constraints, traders in an equilibrium market earn zero profits. Therefore, we instead adopt the following metric to summarize the welfare generated by a market.

Definition 4. The *capability* of a market, denoted χ, is the probability that the asset reaches the buyer taking the configuration of traders as given.

In our model χ is a direct function of $\mathbf{n}$, p, and R. Specifically,

$$\chi(\mathbf{n}, p, R) = \prod_{k=1}^{R} \mu(n_k).$$

In equilibrium, however, market capability is also an indirect function of κ since the equilibrium configuration $\mathbf{n}^*$ is itself a function κ as well. Changes in the environment which boost competition enhance capability.

Under this criterion, a global equilibrium is seen to be far more capable of ensuring consistent delivery of the asset from the seller to the buyer. In example 1 the probability that the asset reaches the buyer when the market configuration is $\mathbf{n}^*$ is approximately 0.25. In the global equilibrium, the market's capability jumps to about 0.88. In this regard, local and global equilibria need not be nearby and substantial gains are possible if coordination problems surrounding trading network formation are overcome.

When viewed from the perspective of capability, however, Theorem 4 can assume a somewhat negative interpretation. Roughly the asymmetric market structure that emerges in equilibrium suggests that intermediary markets have a natural tendency to organize in a manner which creates market fragility. Asymmetries imply that the economy's equilibrium network functions both as an absorber and as an amplifier of idiosyncratic risks.[9] This characteristic is easiest to appreciate in the equilibrium of Figure 3. Idiosyncratic cost shocks experienced by traders in rows one and two are effectively absorbed by the network. Even if a specific trader is unlucky and is unable to trade, there are many others to facilitate trade. In contrast, a negative cost shock experienced by the trader in rows five or six has a much more severe implication for the functioning of the market as a whole. The equilibrium market has a bottleneck in this region of the economy. Indeed, if the trader in row six experiences a cost shock, trade breaks down entirely.

4 Stable Intermediation

Our analysis thus far has focused on the horizontal dimension of trading network formation. In this section we extend our discussion by identifying a stable degree of intermediation—R. Given a market configuration $\mathbf{n}$, a natural and compelling intuition is that neighboring traders have an incentive to merge, to collude, or to partner. By behaving as a single collective, they can together span multiple links in the chain leading from the seller to the buyer. A stable value for R, as a measure of the distance between seller and buyer, ensures that such integrative impulses are kept at bay. To make the above intuition precise, we begin by introducing the collusive actions traders may take to compromise the stability of a market.

Definition 5. A *(vertical) partnership spanning rows r to s* is a set of traders $\mathcal{P}$ such that for all $k \in \{r, r+1, \ldots, s-1, s\}$ there exists a unique trader from row k in $\mathcal{P}$. We may write $\mathcal{P}_r^s$ to emphasize a partnership's span.

Intuitively, a partnership will resemble a vertical merger. We assume that agents can form a partnership *after* all agents have entered the market and the resulting configuration $\mathbf{n}$ is known but *before* traders learn their private costs for the particular trading period. Our definition therefore abstracts from the selection of traders into partnerships given their observed costs. Partnership formation is costless.

We model a partnership as a unitary actor maximizing its trading profit. Since a partnership is composed of individual traders we assume that it inherits its constituents' characteristics. First, partnerships are connected with other traders. A partnership $\mathcal{P}_r^s$ has precisely the following links to the wider economy:

[9] We thank Richard Zeckhauser for suggesting to us the amplifier/absorber metaphor.

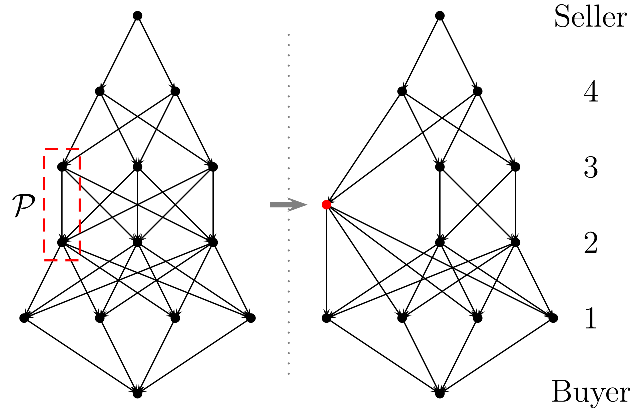

Figure 5: The formation of a partnership spanning rows 2 and 3.

1. A partnership $\mathcal{P}_r^s$ can purchase the asset from any agent in row $s+1$ by participating in the established trading protocol.

2. A partnership $\mathcal{P}_r^s$ can sell the asset to any agent in row $r-1$ using the established trading protocol.

When agents form a partnership they collectively deviate to an alternative network structure. Figure 5 illustrates this deviation in a typical market. The two agents forming a partnership maintain links to traders in rows 1 and 4. However, links with rows 2 and 3 have been severed. In this regard, partnerships are stronger than a loose coalition among individual traders. Instead, they involve a deep vertical integration with commitment and the severing of intermediate inbound and outbound links.

Second, a partnership will incur a trading cost if it engages in exchange. We assume that the partnership's cost type $\theta_{\mathcal{P}}$ is a function of its members' types. For simplicity, suppose

$$\theta_{\mathcal{P}} = \sum_{j \in \mathcal{P}} \theta_j. \tag{2}$$

The distribution of $\theta_{\mathcal{P}}$ is induced from the distribution of its members' types. Our motivation for (2) is simple. A partnership moves the asset multiple steps in the market and it accomplishes the function of $s-r+1$ traders simultaneously. We can interpret each member of $\mathcal{P}$ as being specialized in moving the asset through their step of the trading chain. Our specification of a partnership's cost abstracts from any so-called cost synergies that may accompany a merger. This abstraction is plausible in the short-run when post-merger firms often continue operating as separate, but coordinated, divisions for some time. Our results will extend naturally (by continuity) to the case of cost synergies, provided cost savings are not too strong.

Finally, the actions available to a partnership mirror those available to a trader. $\mathcal{P}_r^s$ places a bid in the auction organized by an agent in row $s+1$ and it organizes an auction to sell to the agent(s) in row $r-1$ (or it sells directly to the buyer if $r=1$). If the asset bypasses the partnership entirely, it incurs zero costs. The following lemma summarizes the consequences of the preceding assumptions.

Lemma 2. *Let $\mathcal{P}_r^s$ be a partnership and suppose all $i \notin \mathcal{P}_r^s$ are truthfully bidding their expected value for the asset. Then it is a best response for the partnership to bid $\nu_r = \prod_{k=1}^{r-1} \delta(n_k)$ if $\theta_{\mathcal{P}_r^s} = 0$ and ℓ otherwise. The partnership's ex ante expected payoff is*

$$\pi_{\mathcal{P}_r^s}(\mathbf{n}) = \left[(1-p)^{n_s-1} + (1-(1-p)^{n_s-1})\left(1 - \prod_{j=r}^{s-1} \delta(n_j - 1)\right)\right] \nu_r p^{s-r+1} \prod_{k=s+1}^{R} \mu(n_k). \quad (3)$$

Proof. The partnership sells the asset to an agent in row $r-1$ at an expected price of ν_r; thus, it is optimal for the partnership to submit such a bid when it has a chance to buy the item and it has low costs. (See the proof of Theorem 1.) If $\theta_{\mathcal{P}} \neq 0$, then a bid of ℓ is a best response.

To derive the ex ante expected profits of the partnership we argue analogously to Lemma 1. Conditional on $\theta_{\mathcal{P}_r^s} = 0$, the partnership bids against $n_s - 1$ other agents to purchase the asset. With probability $(1-p)^{n_s-1}$, all of these competitors have a high cost and bid ℓ. In this case the partnership enjoys a surplus of ν_r. If instead at least one other trader in row s has a low trading cost, which happens with probability $1-(1-p)^{n_s-1}$, they will submit a bid equal to their value for the item: $\hat{\nu}_s = \prod_{j=r}^{s-1} \delta(n_j - 1)\nu_r$, which accounts for the reduced number of traders in rows $r, \ldots, s-1$ owing to the partnership's existence and the associated severing of links. Since $\nu_r > \hat{\nu}_s$, the partnership will win this auction and will benefit from a surplus of $\nu_r - \hat{\nu}_s = \left(1 - \prod_{j=r}^{s-1} \delta(n_j - 1)\right) \nu_r$. With probability p^{s-r+1} the partnership has a low trading cost and with probability $\prod_{k=s+1}^{R} \mu(n_k)$ the asset traverses rows R to $s+1$ to be available for purchase. Combining the above expressions gives $\pi_{\mathcal{P}_r^s}(\mathbf{n})$ as above. □

What incentive do agents have to form partnerships? Given that there are no cost advantages, the motivation to merge in our model must be driven by more indirect forces. There are at least two factors behind partnership formation. The first centers on an information advantage that a partnership brings. A partnership is aware of the costs of its members and thus can bid appropriately knowing whether or not it can transport the asset multiple steps efficiently. Non-partnership agents at intermediate levels face residual uncertainty concerning the costs and behavior of neighbors.

The second advantage rests on a strategic externality resulting from the presence of a partnership and the resulting change in network structure. A partnership $\mathcal{P}_r^s$ severs many links in the economy and this negatively impacts traders exterior to the partnership. Traders in rows $r+1, r+2, \ldots, s$ must anticipate reselling the item at a lower price than otherwise since they have fewer prospective buyers. By reducing the expected resale prices of these agents, the partnership reduces the competing bids it faces when it bids for the item. Since it pays a price equal to the second highest bid, the lower bids by competitors directly benefit the partnership's payoffs. As we show below, these slight and subtle asymmetries in market organization are often sufficient to encourage partnership formation and to render some market structures vertically unstable.

We consider a market to be vertically stable if no partnership can form profitably taking as given the configuration of traders, $\mathbf{n}$. Recall that $\pi_r(\mathbf{n})$ is the expected profit of a trader in row r, as defined in (1), while $\pi_{\mathcal{P}}(\mathbf{n})$ is the expected profit of a partnership, as defined in (3).

Definition 6. A market with configuration $\mathbf{n}$ is *vertically stable*[10] if for all partnerships $\mathcal{P}$,

$$\sum_{k \in \mathcal{P}} \pi_k(\mathbf{n}) \geq \pi_{\mathcal{P}}(\mathbf{n}).$$

In a vertically stable market, the sum of traders' individual expected trading profits exceeds the profits of any possible partnership they may form. The spirit of the core underlies our definition of vertical stability. Example 2 illustrates the mechanics of vertical stability in a simple market.

Example 2. Suppose $\mathbf{n} = (3, 2)$ is the configuration of traders. Absent a partnership, the expected trading profits of a trader in rows one and two are

$$\begin{aligned} \pi_1(\mathbf{n}) &= -p^5 + 4p^4 - 5p^3 + 2p^2 \\ \pi_2(\mathbf{n}) &= 2p^5 - 5p^4 + 3p^3 \end{aligned}$$

The only nontrivial partnership that can form is between a trader in row one and a trader

[10] This definition of vertical stability is subtlety distinct from a definition we presented in the working paper Kotowski & Leister (2012) since $\pi_{\mathcal{P}_r^s}$ is computed in a different manner; otherwise the concept is the same.

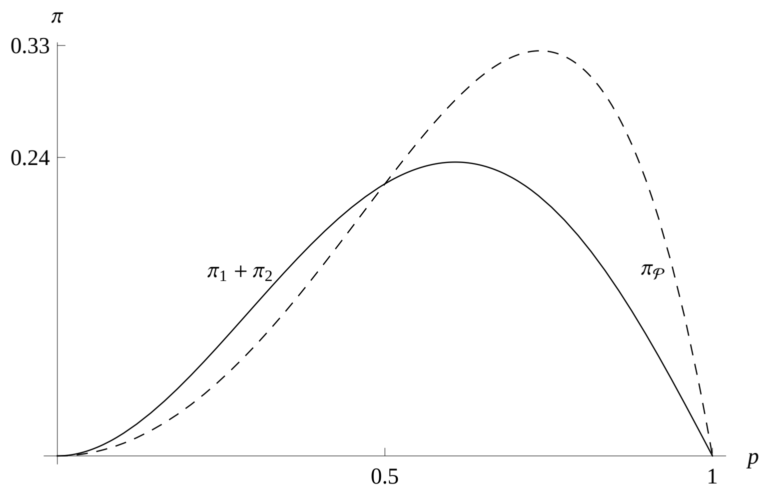

Figure 6: Trading profits in Example 2.

in row two. The expected trading profits accruing to this coalition are

$$\pi_{\mathcal{P}}(\mathbf{n}) = p^2 - p^5.$$

The configuration $\mathbf{n} = (3, 2)$ is vertically stable when $\pi_1 + \pi_2 \geq \pi_{\mathcal{P}}$. Figure 6 presents both sides of this inequality as a function of p. When p is low the network configuration is vertically stable. When p is high the converse is true.

Example 2 offers several lessons. First, partnerships are not universally dominant propositions for traders. Second, whether a specific market configuration is vertically stable depends critically on the underlying trading technology. When p is very high, traders are likely to have low costs and therefore they are often eager to trade in this particular market. As a consequence, the collusive partnership which brings together complementary traders is also likely to have low cost sufficiently often to be worth forming. When p is low and low-cost traders are infrequent, it is very unlikely that the partnership will have low costs. Instead traders are better-off acting independently and the shock-absorber role of a networked market becomes especially valuable. Multiple independent trading paths insulate downstream traders from cost shocks experienced by upstream traders. This benefit of independent trade is sufficient to discourage the formation of partnerships and indeed applies generally.

Theorem 5. *Fix a configuration* $\mathbf{n} = (n_1, \ldots, n_R)$. *There exists* $\hat{p} > 0$ *such that for all* $p < \hat{p}$ *the configuration* $\mathbf{n}$ *is vertically stable.*

Theorem 5 complements the analysis of Kranton & Minehart (2000) who also conclude, in a model of production, that networks increase in importance when firm-specific shocks are high and flexibility is a key consideration determining overall welfare. This is precisely the situation when p is low in our model.

5 Stable and Competitive Intermediary Markets

Having a description of competition between traders and a notion of stability in the degree of intermediation, we wish to combine the ideas. Up to this point we have analyzed the free entry of traders into a market taking R as given. Similarly, we have confirmed that $\mathbf{n}^*$ can be a vertically stable configuration if p is chosen correctly; however, would $\mathbf{n}^*$ ever arise as an equilibrium configuration at that specific p (for some κ) if traders could enter the market freely? We will call a configuration of traders *stable* if it is both vertically stable and a local equilibrium configuration.

We begin reconciling the two dimensions of market organization by considering a simple example. The example highlights the tensions which render the coexistence of the two concepts for any R a nontrivial matter.

Example 3. Suppose $R = 2$. We will identify the set of (p, κ) such that $\mathbf{n}^* = (3, 2)$ is a vertically-stable, local equilibrium configuration. (p, κ) must simultaneously satisfy the following five non-linear (in p) inequalities:

$$\begin{aligned}
\pi_1(3,2) - \kappa &\geq 0 \\
0 &> \pi_1(4,2) - \kappa \\
\pi_2(3,2) - \kappa &\geq 0 \\
0 &> \pi_2(3,3) - \kappa \\
\pi_1(3,2) + \pi_2(3,2) &\geq \pi_{\mathcal{P}_1^2}(3,2)
\end{aligned}$$

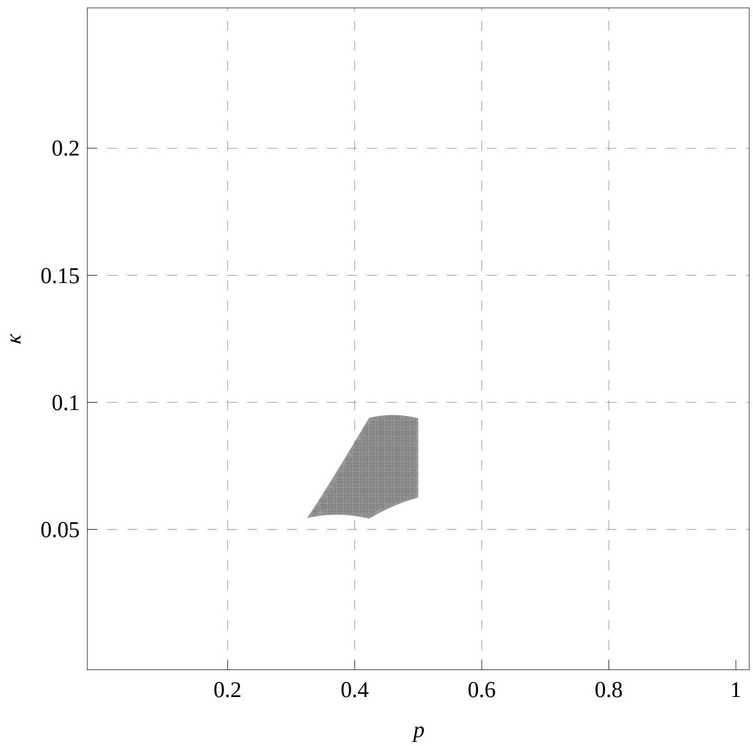

Figure 7: When (p, κ) are in the shaded region, $\mathbf{n}^* = (3, 2)$ is both vertically-stable and a (local) equilibrium configuration.

The first four inequalities are the definition of a local equilibrium. The final inequality concerns vertical stability. Figure 7 presents the irregularly-shaped set of pairs (p, κ) which satisfy the above conditions. By Theorem 4, $p \approx 0$ guarantees that this market is vertically stable; however, $\mathbf{n}^* = (3, 2)$ is not compatible with a free entry equilibrium at such a low value of p. Likewise, when κ is reduced, additional entry also compromises the equilibrium.

Suppose instead $\mathbf{n}^* = (3, 3)$. This configuration is *not* vertically stable when it is a local equilibrium. Thus, it is not stable for any (p, κ). This configuration is vertical stable only if $p < 0.445$ (approximately). The smallest value of p for which it is a local equilibrium for any κ is $p \approx 0.468$. Therefore, not all configurations are assured to be stable.

In moving beyond the case of $R = 2$, the general existence of nontrivial stable markets is not apparent. First, the number of constraints defining our stability and equilibrium notions grows with R. To prevent all partnerships while maintaining free-entry requires simultaneously satisfying $\frac{R^2+3R}{2}$ non-linear inequality constraints. If we demand additionally that a stable market is also a global equilibrium configuration, the number of constraints is much larger.

Although the above discussion suggests that there could exist a limit to the degree of intermediation in a stable market—an R for which there does not exist a stable configuration of intermediaries—this is actually not the case. Indeed, for any R we can find a trading technology (p, κ) which admits a stable nontrivial configuration of traders.

Theorem 6. *Fix $R \in \mathbb{N}$. There exists (p, κ) such that the economy has a nontrivial, global (and therefore local) equilibrium configuration that is vertically stable.*

The proof of Theorem 6 depends on the careful joint adjustment of both p and κ. We identify a sequence of equilibrium markets as $p \to 0$ and $\kappa \to 0$ together. By bounding the rate at which p tends to zero, we are able to use a limiting argument to confirm vertical stablity.[11]

Theorem 6 confirms that, in principle, arbitrarily large degrees of intermediation are possible in a stable market. However, taken together, this section's discussion paints a nuanced picture. Stable markets generally exist, but which markets are stable is closely tied with the economy's underlying technology. Taking the technology as given, the notions of vertical stability and free-entry prevent many network structures from persisting as equilibrium market organizations. The configuration $\mathbf{n} = (3, 3)$ in Example 3 is just one instance. Likewise, if we observe a stable market, its stability is consistent with only a relatively small set of trading technologies. A technological disruption, such as a change in p or κ, can therefore compromise an existing network of intermediary relationships along both of its dimensions. Similarly, noting the multiplicity of equilibria, small changes in the economy's primitives can lead to drastic reorganizations of the intermediary market and of the network of stable trading relationships.

6 Context, Extensions, and Conclusions

We have developed a model of network formation to study the competition among intermediary traders and the degree of intermediation in the economy. Our model shows that intermediary markets may not naturally assume the most capable market organization. Bottlenecks, which amplify market fragility, can arise in a free-entry equilibrium. Similarly we have argued that the degree of intermediation in the economy is closely tied with the underlying trading technology. In principle, the degree of intermediation can be quite extensive

[11] A related inquiry concerning stability may ask whether for each (p, κ) there exists an R such that the market is stable. The answer is essentially trivial since all local equilibria are vertically stable when $R = 1$.

and may go far beyond the bipartite buyer-seller networks traditionally explored in the literature. Our analysis has stressed both substitutive and complementary factors that shape equilibrium trading networks.

In studying intermediation, our study builds on earlier analyses in several literatures. Networks provide a natural forum for studying exchange and the relationships among economic agents. In particular, our equilibrium stresses the complementaries among agents in the presence of network externalities (Economides, 1996).[12] Intuitively, traders who perform similar tasks in the intermediation process (i.e. those who have the same "friends") function as substitutes. In contrast, traders who are in distant regions of the economy complement each other. Downstream traders enhance competition and thus bid up resale prices. Upstream traders enhance the frequency of exchange; idiosyncratic shocks are less likely to compromise the market's operation.

Like Bala & Goyal (2000), Kranton & Minehart (2001), or more recently Condorelli & Galeotti (2012), we study network formation. Our network-formation process builds around free entry and contrasts with their focus on strategic link formation. Additionally, our analysis moves away from bipartite buyer-seller networks by incorporating layers of intermediaries or middlemen. In this regard, our study follows most closely recent work by Gale & Kariv (2007, 2009) who also study intermediation with a network of intermediaries. Unlike these papers we endow traders in our model with private information. Recognizing the importance of market "middlemen," Rubinstein & Wolinsky (1987) offer a lucid analysis based on the random matching of buyers and sellers with intermediaries. They do not explicitly model a network but their model accommodates alternative institutional arrangements, such as consignment sales, which we do not consider.

Our analysis stresses the competitive and complementary pressures seen by markets with intermediaries. The free entry assumption is ubiquitous when analyzing market organization and, like here, has been noted to imply cross-cutting implications for social efficiency (Mankiw & Whinston, 1986). At a more abstract level, our concept of a local equilibrium is closest to the "equilibrium configurations" analyzed by Gary-Bobo (1990) in a general class of asymmetric entry models. Our model falls outside that paper's purview since traders' payoffs in our model do not satisfy his monotonicity condition.

Similarly, the role of vertical relationships and integration has also received considerable attention. Like Kranton & Minehart (2000), we consider our network-based model well suited for investigating this market dimension. Although many motives can drive firms to integrate

[12]Jackson (2008) offers a comprehensive survey of the literature on economic networks.

vertically, our model stresses information asymmetries and the advantages that they may bring. Arrow (1975), for example, describes such incentives in the context of a production line. We additionally highlight the strategic externality resulting from the change in the economy's network of trading relationships, which serve to benefit the integrating agents.

Our model can be extended along many dimensions and incorporated into broader studies of trade where intermediaries play an important role. A particularly promising direction concerns developing a more comprehensive understanding of the stability and robustness of networked markets. This is especially salient if traders can form more elaborate network configurations than what we have considered. Similarly, we have focused on a specific market institution, an auction, as mediating exchange. Allowing for alternative or endogenous institutional arrangements—such as consignment contracts, bargaining, or optimal trading mechanisms—among buyers, sellers, and intermediaries, is but one exciting avenue for further analysis.

A Proofs

In the appendix we may write $\delta_p(n)$ and $\mu_p(n)$ to emphasize the dependence of these expressions on p. Lemmas A.1 and A.2 record facts about $\delta_p(n)$ and $\mu_p(n)$ which are used throughout the analysis.

Lemma A.1. *For $p \in (0,1)$ and $n \in \mathbb{N}$, let $\delta_p(n) = 1 - (1-p)^{n-1}(1 + p(n-1))$. Then,*

1. *$\delta_p(0) = \delta_p(1) = 0$; for all $n \geq 1$, $\delta_p(n) < \delta_p(n+1)$; and, $\lim_{n\to\infty} \delta_p(n) = 1$.*

2. *If $n \geq 2$, then $\frac{d}{dp}\delta_p(n) > 0$; $\lim_{p\to 0} \delta_p(n) = 0$; $\lim_{p\to 1} \delta_p(n) = 1$*

Proof.

1. $\delta_p(0) = \delta_p(1) = 0$ is trivial. To prove the second statement, let $q(n) = (1-p)^{n-1}(1 + p(n-1))$. It is sufficient to show that $q(n+1) < q(n)$. $q(n+1) = (1-p)^n(1+pn) = (1-p)^{n-1}(1+p(n-1))\frac{(1-p)(1+pn)}{(1+p(n-1))} = q(n)\frac{1+p(n-1)-np^2}{1+p(n-1)} < q(n)$. To confirm the third point, $\lim_{n\to\infty} \frac{q(n+1)}{q(n)} = \lim_{n\to\infty} \frac{(1-p)(np+1)}{(n-1)p+1} = 1 - p < 1$. Thus, $\lim_{n\to\infty} q(n) = 0$, which implies $\lim_{n\to\infty} \delta_p(n) = 1$.

2. It is sufficient to show that $(1-p)^m(1+pm)$, $m \geq 1$, is decreasing in p:

$$\frac{d}{dp}\left(m \log(1-p) + \log(1+pm)\right) = m\frac{-p - pm}{(1+pm)(1-p)} < 0.$$

   The limit results are immediate from inspection.

□

Lemma A.2. *Let $n \geq 1$, $p \in (0,1)$, and $\mu_p(n) = 1 - (1-p)^n$. (1) $\mu_p(n)/p \geq 1$. (2) $\lim_{p\to 0} \frac{\mu_p(n)}{p} = n$. (3) $\lim_{p\to 1} \frac{\mu_p(n)}{p} = 1$. (4) If $n \geq 2$, $\frac{d}{dp}\left(\frac{\mu_p(n)}{p}\right) < 0$.*

Proof. For (1) note that $1-(1-p)^n \geq 1-(1-p)^1 = p \geq p$. To show (2) apply l'Hôpital's Rule: $\lim_{p\to 0} \frac{1-(1-p)^n}{p} = \lim_{p\to 0} \frac{n(1-p)^{n-1}}{1} = n$. (3) follows from inspection. To see (4) we can compute the derivative:

$$\frac{d}{dp}\left(\frac{\mu_p(n)}{p}\right) = \frac{np(1-p)^{n-1} + (1-p)^n - 1}{p^2}.$$

Since $\frac{d}{dp}\left(\frac{\mu_p(n)}{p}\right)\Big|_{n=1} = 0$, it is sufficient to show that $np(1-p)^{n-1} + (1-p)^n - 1$ is decreasing in n:

$$\frac{d}{dn}\left(np(1-p)^{n-1} + (1-p)^n - 1\right) = (1-p)^{n-1}(((n-1)p+1)\log(1-p) + p) < 0,$$

since $\log(1-p)+p<0$ when $p \in (0,1)$. $\square$

Lemma A.3. *Fix $p \in (0,1)$ and $\kappa > 0$. Suppose $\mathbf{n}^*$ is a local equilibrium configuration.*

1. *For each r, $n_r^* \leq \bar{n} = \left\lceil 1 + \frac{\log(\kappa)-\log(p)}{\log(1-p)} \right\rceil$. (This is Corollary 1.)*
2. *For each $r \in \{1,\dots,R-1\}$, $n_r^* \neq 1$.*

Proof. To prove part 1, it is sufficient to derive a bound the number of traders any given row can support. From (1), $\pi_r(\mathbf{n}) \leq p(1-p)^{n_r-1}$. In an equilibrium $\pi_r(\mathbf{n}^*) - \kappa \geq 0$, hence for each r

$$p(1-p)^{n_r^*-1} - \kappa \geq 0 \implies n_r^* \leq \bar{n} = \left\lceil 1 + \frac{\log(\kappa)-\log(p)}{\log(1-p)} \right\rceil.$$

To prove part 2 suppose that for some $r \in \{1,\dots R-1\}$, $n_r^* = 1$. Then $\delta(n_r^*) = 0$. Thus, for all $r' > r$, $\pi_{r'}(\mathbf{n}^*) = 0$. But this implies $\pi_{r'}(\mathbf{n}^*) - \kappa < 0$, which is a contradiction if $n_{r'} \geq 1$. If instead $n_{r'} = 0$, then $\mu(n_{r'}) = 0$ and $\pi_r(\mathbf{n}^*) = 0$. Hence $\pi_r(\mathbf{n}^*) - \kappa < 0$, which again is a contradiction. $\square$

Proof of Theorem 2. ($\Rightarrow$) Follows from the definition of equilibrium. ($\Leftarrow$) We define a tâtonnement-style mapping that converges to a local equilibrium. First, let $\mathbf{n}^0 \in \mathscr{X}$ be such that $\pi_r(\mathbf{n}^0) - \kappa \geq 0$ for all r. Let $\mathscr{Q}_r(\mathbf{n}) = \{\tilde{n}_r \geq n_r \colon \pi_r(\tilde{n}_r, \mathbf{n}_{-r}) - \kappa \geq 0, \tilde{n}_r \in \mathbb{Z}_+, \tilde{n}_{r'} \leq \bar{n}\}$. Let $\hat{n}_r = \max \mathscr{Q}_r(\mathbf{n})$. Next, define $A_r \colon \mathscr{X} \to \mathscr{X}$ to be

$$A_r(\mathbf{n}) = \begin{cases} (\hat{n}_r, \mathbf{n}_{-r}) & \text{if } \mathscr{Q}_r(\mathbf{n}) \neq \emptyset \\ \mathbf{n}^0 & \text{if } \mathscr{Q}_r(\mathbf{n}) = \emptyset \end{cases}$$

Composing these mapping together, define $A \colon \mathscr{X} \to \mathscr{X}$ as

$$A(\mathbf{n}) = (A_1 \circ \cdots \circ A_R)(\mathbf{n}) \tag{4}$$

We first argue that A has a fixed point, $A(\mathbf{n}^*) = \mathbf{n}^*$. Afterward, we show that $\mathbf{n}^*$ is an equilibrium configuration.

To show that A has a fixed point we first establish that if $\pi_r(\mathbf{n}) - \kappa \geq 0$ for all r, then $A(\mathbf{n}) \geq \mathbf{n}$. Suppose $\pi_r(\mathbf{n}) - \kappa \geq 0$. Then $\mathscr{Q}_R(\mathbf{n}) \neq \emptyset$. So, $A_R(\mathbf{n}) \geq \mathbf{n}$ since n_R may have increased (but certainly did not decrease). Now consider any r and let $\tilde{\mathbf{n}} = (n_1, \dots, n_r, \tilde{n}_{r+1}, \dots, \tilde{n}_R)$ where the first r terms are unchanged relative to $\mathbf{n}$ and $(\tilde{n}_{r+1}, \dots, \tilde{n}_R) \geq (n_{r+1}, \dots, n_R)$. Then $\pi_r(n_r, \tilde{\mathbf{n}}_{-r}) - \kappa \geq \pi_r(n_r, \mathbf{n}_{-r}) - \kappa \geq 0$. Therefore,

$A_r(\tilde{\mathbf{n}}) \geq \tilde{\mathbf{n}}$. It follows immediately that $A(\mathbf{n}) \geq \mathbf{n}$. Note also that for all r, $\pi_r(A(\mathbf{n}))-\kappa \geq 0$. Indeed, if we let $\tilde{\mathbf{n}} = A(\mathbf{n})$, we see that

$$\pi_r(\tilde{n}_1, \ldots, \tilde{n}_{r-1}, \tilde{n}_r, \ldots, \tilde{n}_R) - \kappa \geq \pi_r(n_1, \ldots, n_{r-1}, \tilde{n}_r, \ldots, \tilde{n}_R) - \kappa \geq 0.$$

Finally, consider the sequence $\mathbf{n}^{t+1} = A(\mathbf{n}^t)$ starting at $\mathbf{n}^0$. Since $\pi_r(\mathbf{n}^0)-\kappa \geq 0$ for every r, $\mathbf{n}^t$ is an increasing sequence and for each t, $\pi_r(\mathbf{n}^t) - \kappa \geq 0$ for every r. Since $\mathbf{n}^t \in \mathscr{X}$ and $\mathscr{X}$ is a finite set, the sequence $\{\mathbf{n}^t\}$ converges to a limit $\mathbf{n}^* \in \mathscr{X}$. Since the space is discrete, convergence implies $\mathbf{n}^t = A(\mathbf{n}^t)$ for all $t \geq T$. Thus, there exists a configuration such that $\mathbf{n}^* = A(\mathbf{n}^*)$

Take a fixed point $\mathbf{n}^* = A(\mathbf{n}^*) \geq \mathbf{n}^0$ and suppose that $\mathbf{n}^*$ is not an equilibrium configuration. Therefore, there exists some row $\hat{r}$ such that either (1) $\pi_{\hat{r}}(\mathbf{n}^*) - \kappa < 0$ or (2) $\pi_{\hat{r}}(n^*_{\hat{r}} + 1, \mathbf{n}^*_{-\hat{r}}) - \kappa \geq 0$. We address both cases.

Suppose that $\pi_{\hat{r}}(\mathbf{n}^*) - \kappa < 0$. Then, $A_{\hat{r}}(\mathbf{n}^*) = \mathbf{n}^0$ since $\mathscr{Q}_{\hat{r}}(\mathbf{n}^*) = \emptyset$. Therefore $\mathbf{n}^* = (n^*_1, \ldots, n^*_{\hat{r}-1}, n^0_{\hat{r}}, \ldots, n^0_R)$. Thus, recalling that $\pi_r(n_r, \mathbf{n}_{-r})$ is increasing in $\mathbf{n}_{-r}$ and $\mathbf{n}^* \geq \mathbf{n}^0$,

$$\pi_{\hat{r}}(\mathbf{n}^*) - \kappa = \pi_{\hat{r}}(n^*_1, \ldots, n^*_{\hat{r}-1}, n^0_{\hat{r}}, \ldots, n^0_R) - \kappa \geq \pi_{\hat{r}}(\mathbf{n}^0) - \kappa \geq 0,$$

which is a contradiction.

Suppose instead that $\pi_{\hat{r}}(n^*_{\hat{r}} + 1, \mathbf{n}^*_{-\hat{r}}) - \kappa \geq 0$. But then, from the definition of $\mathscr{Q}_{\hat{r}}$, $n^*_{\hat{r}} + 1 \in \mathscr{Q}_r(\mathbf{n}^*)$. This implies $n^*_{\hat{r}} \geq n^*_{\hat{r}} + 1$, which is a contradiction with the definition of A. Therefore a fixed point of A is an equilibrium configuration. $\square$

Lemma A.4. *Let* $\mathbf{n} = (2, \ldots, 2, 1)$. $\pi_r(\mathbf{n}) > \kappa$ *for all* r *if and only if* $\min\{(1-p)p^{2R-2}, p^{2R-1}\} > \kappa$.

Proof. Let $\mathbf{n} = (2, \ldots, 2, 1)$ and fix $\hat{r}$. If $\hat{r} \neq R$,

$$\pi_{\hat{r}}(\mathbf{n}) = p \left(1 - (1-p)^2\right)^{R-\hat{r}-1} p(1-p) \left(1 - (1-p)(1+p)\right)^{\hat{r}-1}.$$

Taking logarithms and collecting terms gives:

$$\log(\pi_{\hat{r}}(\mathbf{n})) = \hat{r} \left(\log(p^2) - \log(1 - (1-p)^2)\right) + K(R, p)$$

where $K(R,p)$ is a term that is independent of $\hat{r}$. Since $p^2 < 1 - (1-p)^2$ when $p \in (0,1)$, the above expression is decreasing in $\hat{r}$. Therefore for all $\hat{r} \leq R-1$, $\pi_{\hat{r}}(\mathbf{n}) \geq \pi_{R-1}(\mathbf{n}) =$

$(1-p)p^{2R-2}$. If instead $\hat{r} = R$, then $\pi_R(\mathbf{n}) = p^{2R-1}$. Therefore, we have the conclusion that $\pi_r(\mathbf{n}) - \kappa > 0 \iff \min\{(1-p)p^{2R-2}, p^{2R-1}\} > \kappa$. $\square$

Proof of Theorem 3. If the only local equilibrium is trivial, then it will also be the unique global equilibrium. Suppose instead that $\mathbf{n}^* \neq (0, \ldots, 0)$ is a local equilibrium and $\mathbf{m}^*$ is some other local equilibrium. Therefore, $\pi_r(\mathbf{n}^*) - \kappa \geq 0$ for every r.

Following standard notation, let $\mathbf{n}^* \vee \mathbf{m}^* \equiv (\max(n_1^*, m_1^*), \ldots, \max(n_R^*, m_R^*))$ and consider the alternative configuration $\mathbf{x} = \mathbf{n}^* \vee \mathbf{m}^*$. Choose some row r, and without loss of generality suppose $n_r^* \geq m_r^*$. Then,

$$\pi_r(\mathbf{x}) - \kappa = \pi_r(n_r^*, \mathbf{n}_{-r}^* \vee \mathbf{m}_{-r}^*) - \kappa \geq \pi_r(n_r^*, \mathbf{n}_{-r}^*) - \kappa \geq 0.$$

Therefore, the configuration $\mathbf{x}$ satisfies the conditions of Theorem 2. Using the mapping $A(\cdot)$ defined in (4), we can construct an increasing sequence of configurations $\mathbf{x}^{\mathbf{t+1}} = A(\mathbf{x}^t)$, $\mathbf{x}^0 = \mathbf{x}$, which converges to some configuration $\mathbf{x}^* \geq \mathbf{x}$. $\mathbf{x}^*$ is another nontrivial local equilibrium configuration. Since $\mathbf{n}^*$ and $\mathbf{m}^*$ were arbitrary local equilibrium configurations and there is a finite number of such configurations ($\mathscr{X}$ is a finite set), there exists a local equilibrium configuration $\mathbf{q}^* \geq \mathbf{n}^*$ for all local equilibria $\mathbf{n}^*$. Thus, if a global equilibrium exists, it must be $\mathbf{q}^*$.

We claim that $\mathbf{q}^*$ is global equilibrium configuration. To verify this claim, suppose not. Then there exists an $\mathbf{a} = (a_1, \ldots, a_R) \in \mathbb{Z}_+^R$, with at least one $a_k \geq 1$, such that $\pi_r(\mathbf{q}^* + \mathbf{a}) - \kappa \geq 0$ for all r. But, following the same argument as above and applying the $A(\cdot)$ mapping, this implies there exists a nontrivial local equilibrium configuration $\hat{\mathbf{q}}$ such that $\hat{\mathbf{q}} \geq \mathbf{q}^* + \mathbf{a} \geq \mathbf{q}^*$ and $\hat{\mathbf{q}} \neq \mathbf{q}^*$, which is a contradiction. $\square$

Proof of Theorem 4. The proof is by contradiction. Suppose that $\mathbf{n}^* = (n_1^*, \ldots, n_R^*)$ is an equilibrium configuration such that for some r, $n_r^* < n_{r+1}^*$. Since $\mathbf{n}^*$ is an equilibrium configuration, it satisfies the following inequalities:

$$\prod_{k=1}^{r-1} \delta(n_k^*) \left[p(1-p)^{n_r^*-1}\mu(n_{r+1}^*)\right] \prod_{k=r+2}^{R} \mu(n_k^*) \geq \kappa > \prod_{k=1}^{r-1} \delta(n_k^*) \left[p(1-p)^{n_r^*}\mu(n_{r+1}^*)\right] \prod_{m=r+2}^{R} \mu(n_m^*)$$

$$\prod_{k=1}^{r-1} \delta(n_k^*) \left[\delta(n_r^*)p(1-p)^{n_{r+1}^*-1}\right] \prod_{k=r+2}^{R} \mu(n_k^*) \geq \kappa > \prod_{k=1}^{r-1} \delta(n_k^*) \left[\delta(n_r^*)p(1-p)^{n_{r+1}^*}\right] \prod_{m=r+2}^{R} \mu(n_m^*)$$

To simplify, let

$$\tilde{\kappa} = \frac{\kappa}{\prod_{k=1}^{r-1} \delta(n_k^*) \prod_{m=r+2}^{R} \mu(n_m^*)}.$$

The above inequalities are equivalent to

$$\begin{aligned} p(1-p)^{n_r^*-1}\mu(n_{r+1}^*) \geq \tilde{\kappa} > p(1-p)^{n_r^*}\mu(n_{r+1}^*) \\ \delta(n_r^*)p(1-p)^{n_{r+1}^*-1} \geq \tilde{\kappa} > \delta(n_r^*)p(1-p)^{n_{r+1}^*} \end{aligned}$$

From these inequalities, we see that

$$\delta(n_r^*)(1-p)^{n_{r+1}^*-1} > (1-p)^{n_r^*}\mu(n_{r+1}^*).$$

However, since $n_{r+1}^* \geq n_r^* + 1$, $(1-p)^{n_{r+1}^*-1} \leq (1-p)^{n_r^*}$. Similarly, by Lemma A.1, $\delta(n_r^*) \leq \delta(n_r^* + 1) \leq \delta(n_{r+1}^*) < \mu(n_{r+1}^*)$. Hence, $\delta(n_r^*)(1-p)^{n_{r+1}^*-1} < (1-p)^{n_r^*}\mu(n_{r+1}^*)$, which is a contradiction. □

Proof of Theorem 5. There are several cases depending on the network configuration $\mathbf{n} = (n_1, \ldots, n_R)$. Let $\mathcal{P}_r^s$ be a partnership spanning rows r to s. If $n_k = 0$ for any k, then all traders earn zero expected profits independently and as members of a vertical partnership; thus, suppose $n_k \geq 1$ for all k.

1. Suppose $n_k = 1$ for some $k \leq r-1$. Then, $\delta(n_k) = 0$ and consequently, $\nu_r = \prod_{j=1}^{r-1} \delta(n_j) = 0$. Thus, $\pi_{\mathcal{P}_r^s}(\mathbf{n}) = 0$ and $\pi_k(\mathbf{n}) = 0$ for all $r \leq k \leq s$. Hence, the partnership does not strictly benefit its members.

2. Suppose $n_k \geq 2$ for all $k \leq r-1$, but $n_r = 1$. In this case, $\pi_{\mathcal{P}_r^s}(\mathbf{n}) = \nu_r p^{s-r+1} \prod_{k=s+1}^{R} \mu(n_k)$. Therefore,

$$\frac{\pi_r(\mathbf{n})}{\pi_{\mathcal{P}_r^s}(\mathbf{n})} = \frac{\nu_r p \left(\prod_{k=r+1}^{s} \mu(n_k)\right) \left(\prod_{k=s+1}^{R} \mu(n_k)\right)}{\nu_r p^{s-r+1} \left(\prod_{k=s+1}^{R} \mu(n_k)\right)} = \prod_{k=r+1}^{s} \frac{\mu(n_k)}{p} \geq 1.$$

   And so, $\sum_{k=r}^{s} \pi_k(\mathbf{n}) \geq \pi_r(\mathbf{n}) \geq \pi_{\mathcal{P}_r^s}(\mathbf{n})$. Hence, this partnership does not strictly benefit its members.

3. Suppose $n_k \geq 2$ for all $k \leq r$, but $n_{r+1} = 1$. For notation, let $Q(p, \mathbf{n}) = (1-p)^{n_s-1} + (1-(1-p)^{n_s-1})(1-\prod_{j=r}^{s-1} \delta(n_j - 1))$. We note that $0 < Q(p, \mathbf{n}) \leq 1$ and $\lim_{p \to 0} Q(p, \mathbf{n}) = 1$.

Then, we can write[13]

$$\frac{\pi_r(\mathbf{n})+\pi_{r+1}(\mathbf{n})}{\pi_{\mathcal{P}_r^s}(\mathbf{n})} = \left[\frac{\mu(n_{r+1})}{p}(1-p)^{n_r-1}+\frac{\delta(n_r)}{p}\right]\underbrace{\frac{1}{Q(p,\mathbf{n})}\prod_{k=r+2}^{s}\frac{\mu(n_k)}{p}}_{\geq 1}$$

Therefore, to establish that $\pi_r(\mathbf{n})+\pi_{r+1}(\mathbf{n}) \geq \pi_{\mathcal{P}_r^s}(\mathbf{n})$, it is sufficient to show that

$$\left[\frac{\mu(n_{r+1})}{p}(1-p)^{n_r-1}+\frac{\delta(n_r)}{p}\right] = \frac{-1+p+(1-p)^{n_r}(1+(n_r-2)p)}{p(p-1)} \tag{5}$$

is greater than 1. When $n_r = 2$, (5) equals 1 for all $p \in (0,1)$. Thus, it is sufficient to show that (5) is nondecreasing in n_r when $p \in (0,1)$. Differentiating with respect to n_r,

$$\frac{d}{dn_r}\left[\frac{-1+p+(1-p)^{n_r}(1+(n_r-2)p)}{p(p-1)}\right] = -\frac{(1-p)^{n_r-1}(((n_r-2)p+1)\log(1-p)+p)}{p} > 0,$$

since $(n_r-2)p\log(1-p)+\log(1-p)+p < 0$.

4. Suppose $n_k \geq 2$ for all $k \leq r+1$. Then,

$$\frac{\pi_r(\mathbf{n})}{\pi_{\mathcal{P}_r^s}(\mathbf{n})} = \left(\prod_{k=r+1}^{s}\frac{\mu(n_k)}{p}\right)\left[\frac{(1-p)^{n_r-1}}{(1-p)^{n_s-1}+(1-(1-p)^{n_s-1})(1-\prod_{j=r}^{s-1}\delta(n_j-1))}\right].$$

Since the term in square brackets tends to 1 as $p \to 0$, while by Lemma A.2 $\lim_{p\to 0}\prod_{k=r+1}^{s}\frac{\mu(n_k)}{p} = \prod_{k=r+1}^{s} n_k \geq 2$, for p sufficiently small, $\pi_r(\mathbf{n}) > \pi_{\mathcal{P}_r^s}(\mathbf{n})$. Hence, this partnership does not strictly benefit its members for p sufficiently small.

□

Lemmas A.5—A.9 are used in the proof of Theorem 6.

Lemma A.5. *Let $R \geq 2$ and $n \geq 2$. There exists a unique $p_n \in (0,1)$ such that $1-p_n = \delta_{p_n}(n)^{R-1}$.*

Proof. By Lemma A.1, $\delta_{p_n}(n)^{R-1}$ is strictly increasing and $\delta_0(n)^{R-1} = 0$, while $\delta_1(n)^{R-1} = 1$. Thus, the conclusion follows. □

[13] If $s < r+2$, then it is understood that $\prod_{k=r+2}^{s}\frac{\mu(n_k)}{p} = 1$.

Lemma A.6. *Let $R \geq 2$ and consider the sequence $\{p_n\}_{n=2}^{\infty}$ defined implicitly by $1 - p_n = \delta_{p_n}(n)^{R-1}$. For each $n \in \{2, 3, \ldots\}$, there exists a nonempty set $[\underline{\kappa}_n, \bar{\kappa}_n]$ such that if $\kappa_n \in [\underline{\kappa}_n, \bar{\kappa}_n]$, then $\mathbf{n} = (\underbrace{n, n, \ldots, n}_{R})$ is a local equilibrium configuration when $p_n = \Pr[\theta_{ri} = 0]$ and the entry costs are κ_n.*

Proof. For each n, let

$$\begin{aligned}\bar{\kappa}_n &= p_n(1 - p_n)^{n-1}\delta_{p_n}(n)^{R-1} \\ \underline{\kappa}_n &= p_n(1 - p_n)^n \mu_{p_n}(n)^{R-1}\end{aligned}$$

Noting that $\underline{\kappa}_n = p_n(1 - p_n)^n \mu_{p_n}(n)^{R-1} < p_n(1 - p_n)^n = p_n(1 - p_n)^{n-1}\delta_{p_n}(n)^{R-1} = \bar{\kappa}_n$. Therefore $[\underline{\kappa}_n, \bar{\kappa}_n]$ is nonempty. Choose any $\kappa_n \in (\underline{\kappa}_n, \bar{\kappa}_n)$.

To simplify notation, if $\mathbf{n} = (n, \ldots, n)$, then let $\mathbf{n}_{r+} = (n \ldots, n + 1, \ldots, n)$ which is identical to $\mathbf{n}$ except there is an additional trader in row r. To show that $\mathbf{n}$ is a local equilibrium we verify the two inequalities for each r:

$$\begin{aligned}\pi_r(\mathbf{n}) - \kappa_n &= \delta_{p_n}(n)^{r-1} p_n(1 - p_n)^{n-1} \mu_{p_n}(n)^{R-r} - \kappa_n \\ &\geq \delta_{p_n}(n)^{R-1} p_n(1 - p_n)^{n-1} - \kappa_n \\ &> \delta_{p_n}(n)^{R-1} p_n(1 - p_n)^{n-1} - \bar{\kappa}_n = 0\end{aligned}$$

Similarly,

$$\begin{aligned}\pi_r(\mathbf{n}_{r+}) - \kappa_n &= \delta_{p_n}(n)^{r-1} p_n(1 - p_n)^n \mu_{p_n}(n)^{R-r} - \kappa_n \\ &\leq p_n(1 - p_n)^n \mu_{p_n}(n)^{R-1} - \kappa_n \\ &< p_n(1 - p_n)^n \mu_{p_n}(n)^{R-1} - \underline{\kappa}_n = 0\end{aligned}$$

Thus, $\mathbf{n} = (n, \ldots, n)$ is a local equilibrium configuration at (p_n, κ_n). $\square$

Lemma A.7. *Suppose $k \geq 0$ and $a \in [0, 1)$. Then, $\lim_{n\to\infty} n^k \left(1 - \frac{1}{n^a}\right)^n = 0$.*

Proof. Recalling that for all $x > 1$, $\left(1 - \frac{1}{x}\right)^x \leq \frac{1}{e}$, we can see that

$$0 \leq \lim_{n\to\infty} n^k \left(1 - \frac{1}{n^a}\right)^n = \lim_{n\to\infty} n^k \left[\left(1 - \frac{1}{n^a}\right)^{n^a}\right]^{n^{1-a}} \leq \lim_{n\to\infty} n^k \left[\frac{1}{e}\right]^{n^{1-a}} = 0.$$

$\square$

Lemma A.8. *Fix $R \geq 2$. And let $\{p_n\}_{n=2}^{\infty}$ be the sequence defined in Lemma A.5. For every $a \in (0,1)$ there exists $\bar{n}$ such that for all $n > \bar{n}$, $p_n < \frac{1}{n^a}$.*

Proof. The proof is by contradiction. Fix $a \in (0,1)$ and suppose that there exists a subsequence $\{n_k\}$ such that for all k, $p_{n_k} > \frac{1}{n_k^a}$. Since $p \mapsto \delta_p(n)$ is increasing, for all n_k:

$$\begin{aligned}
&1 - p_{n_k} = \delta_{p_{n_k}}(n_k)^{R-1} \\
&\implies 1 - \frac{1}{n_k^a} > \delta_{\frac{1}{n_k^a}}(n_k)^{R-1} \\
&\implies 1 - \frac{1}{n_k^a} > \left(1 - \left(1 - \frac{1}{n_k^a}\right)^{n_k - 1}\left(1 + \frac{1}{n_k^a}(n_k - 1)\right)\right)^{R-1} \\
&\implies 1 - \frac{1}{n_k^a} > 1 - (R-1)\left(1 - \frac{1}{n_k^a}\right)^{n_k - 1}\left(1 + \frac{1}{n_k^a}(n_k - 1)\right) \qquad (6) \\
&\implies (R-1)n_k^a\left(1 - \frac{1}{n_k^a}\right)^{n_k - 1}\left(1 + \frac{1}{n_k^a}(n_k - 1)\right) > 1 \qquad (7)
\end{aligned}$$

(6) follows from the fact that $(1-x)^R > 1 - Rx$ when $x \in (0,1)$. Using Lemma A.7, however, (7) is seen to fail for n_k sufficiently large—a contradiction. ☐

Lemma A.9. *Suppose $q \in (0,1)$, $a \in (0,1)$ and $k \geq e$. Then for all $z \geq 1$, $\frac{k^z}{1+a-q^z} \geq \frac{k}{1+a-q}$.*

Proof. Taking the derivative,

$$\frac{d}{dz}\left(\frac{k^z}{1+a-q^z}\right) = \frac{k^z}{(1+a-q^z)^2}\left[(1+a-q^z)\log(k) + q^z\log(q)\right].$$

Thus, it is sufficient to verify that $\phi(q) = (1+a-q^z)\log(k) + q^z\log(q)$ is nonnegative for all $q \in (0,1)$. Note that $\phi(1) = a\log(k) \geq 0$. Therefore it is sufficient to verify that $\phi(q)$ is non-increasing in q.

$$\phi'(q) = q^{z-1}(1 + z[\log(q) - \log(k)]) \leq q^{z-1}(1 + \log(q) - \log(k)) \leq 0.$$

The inequalities follow from $\log(q) \leq 0$ and $\log(k) \geq 1$. ☐

Proof of Theorem 6. Since the case of $R = 1$ is trivial, suppose $R \geq 2$. From Lemma A.6, take a sequence of economies—defined by $\{p_n\}_{n=2}^{\infty}$—such that for each n, $\mathbf{n} = (n, \ldots, n)$ is an equilibrium configuration of traders. We will show that for n sufficiently large, the configuration $\mathbf{n} = (n, \ldots, n)$ is vertically stable.

Suppose $\mathcal{P}_r^s$ is a partnership spanning rows r to s. We will show that for n sufficiently large $\pi_r(\mathbf{n}) > \pi_{\mathcal{P}_r^s}(\mathbf{n})$. Letting $z = s - r$ we can write

$$\begin{aligned}\frac{\pi_r(\mathbf{n})}{\pi_{\mathcal{P}_r^s}(\mathbf{n})} &= \left(\frac{\mu_{p_n}(n)}{p_n}\right)^z \cdot \frac{(1-p_n)^{n-1}}{(1-p_n)^{n-1} + (1-(1-p_n)^{n-1})(1-\delta_{p_n}(n-1)^z)} \\ &= \left(\frac{\mu_{p_n}(n)}{p_n}\right)^z \cdot \frac{(1-p_n)^{n-1}}{1-\delta_{p_n}(n-1)^z + (1-p_n)^{n-1}\delta_{p_n}(n-1)^z}\end{aligned}$$

Let $\tilde{p}_n = \frac{1}{n^{3/4}}$. By Lemma A.8 there exists an $\bar{n}_1$ such that for all $n > \bar{n}_1$, $\tilde{p}_n > p_n$. Using Lemma A.2, for $n > \bar{n}_1$,

$$\frac{\mu_{p_n}(n)}{p_n} > \frac{\mu_{\tilde{p}_n}(n)}{\tilde{p}_n} = \left(1 - \left(1 - \frac{1}{n^{3/4}}\right)^n\right) n^{3/4} \xrightarrow{n\to\infty} \infty.$$

Thus, there exists $\bar{n}_2$ such that for all $n > \bar{n}_3 = \max\{\bar{n}_1, \bar{n}_2, 4\}$, $\frac{\mu_{p_n}(n)}{p_n} > \frac{\mu_{\tilde{p}_n}(n)}{\tilde{p}_n} > e$.

Since $\delta_{p_n}(n) < 1$, we can use Lemma A.9 to note that for $n > \bar{n}_3$,

$$\begin{aligned}\frac{\pi_r(\mathbf{n})}{\pi_{\mathcal{P}_r^s}(\mathbf{n})} &= \left(\frac{\mu_{p_n}(n)}{p_n}\right)^z \cdot \frac{(1-p_n)^{n-1}}{1-\delta_{p_n}(n-1)^z + (1-p_n)^{n-1}\delta_{p_n}(n-1)^z} \\ &\geq \left(\frac{\mu_{p_n}(n)}{p_n}\right) \cdot \frac{(1-p_n)^{n-1}}{1-\delta_{p_n}(n-1) + (1-p_n)^{n-1}\delta_{p_n}(n-1)^z} \\ &\geq \left(\frac{\mu_{p_n}(n)}{p_n}\right) \cdot \frac{1-p_n}{2+(n-3)p_n} \\ &\geq \left(\frac{\mu_{\tilde{p}_n}(n)}{\tilde{p}_n}\right) \cdot \frac{1-\tilde{p}_n}{2+(n-3)\tilde{p}_n} \\ &= \frac{\left(1-\left(1-\frac{1}{n^{3/4}}\right)^n\right)\left(1-\frac{1}{n^{3/4}}\right)n^{3/2}}{2n^{3/4}+n-3} \xrightarrow{n\to\infty} \infty.\end{aligned}$$

Thus, there exists and n sufficiently large such that for all r and s, $\pi_r(\mathbf{n}) \geq \pi_{\mathcal{P}_r^s}(\mathbf{n})$. Therefore, the market is vertically stable.

To show that there exists a vertically stable market that is also a global equilibrium configuration, it suffices to identify an entry cost where $\mathbf{n} = (n, \ldots, n)$ is a global equilibrium. Recalling that $1 - p_n = \delta_{p_n}(n)^{R-1}$, set the cost of entry at $\kappa_n = p_n(1-p_n)^{n-1}\delta_{p_n}(n)^{R-1} = \bar{\kappa}_n$.

Choose r and let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}_+^R$, $a_r \geq 1$. Then,

$$\begin{aligned}
\pi_r(\mathbf{n}+\mathbf{a}) - \kappa_n &= \prod_{k=r+1}^{R} \mu_{p_n}(n_k + a_k) \left[p_n(1-p_n)^{n-1+a_r}\right] \prod_{k=1}^{r-1} \delta_{p_n}(n_k + a_k) - \kappa_n \\
&\leq p_n(1-p_n)^n - \kappa_n \\
&= p_n(1-p_n)^{n-1} \delta_{p_n}(n)^{R-1} - \kappa_n = 0.
\end{aligned}$$

Since r was arbitrary, $\mathbf{n} = (n, \ldots, n)$ is also a global equilibrium configuration. □

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