The Daycare Assignment: A Dynamic Matching Problem*†

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Abstract

The problem of allocating children to public daycares differs from the school choice problem in two fundamental ways: there is entry and exit of agents over time, and the priorities of schools over children are history dependent. We illustrate with the Danish case and show that there is no mechanism that is strategy-proof and yields a stable matching. We propose an algorithm in which parents sequentially choose menus of schools, ordered by the child's birth date. This mechanism is strategy-proof, Pareto efficient, eliminates ex-post uncertainty, and may be considered fair: parents face similar choice sets, which increase over time.

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1 Introduction

The decision of which daycare to enroll a child is an important and difficult one. This caution is justified by mounting evidence that early childhood care facilities are heterogeneous and crucial to the development of important non-cognitive skills that have a significant impact on the child's future career and social opportunities (Chetty et al., 2010; Heckman, 2008). In addition, important risks are associated with opting out of a daycare facility in favor of home care. For example, Goldin (1994) argues that home care is a major barrier to the advancement of female careers because it undermines mothers' time at work during those years when the possibilities for career advancement are at their fullest.

Many daycare systems are publicly funded and centrally administered, particularly in European countries. Our lead example is the case of Denmark. Copenhagen has more than 400 independently managed daycare facilities and parents can, in principle, choose to have their children assigned to any one of these facilities. However, each facility has strict capacity constraints due to the mandated number of children per pedagogue as well as inherent space restrictions. As such, parents may sign up for a particular daycare, but they are not guaranteed to have their children accepted into it. In practice, some daycares are very popular and difficult to enter while other daycares are less popular and easier to enter.[1]

A centralized daycare system attempts to balance parental choice with the priorities of the public daycares regarding the various children. A general sense exists that daycares are a useful tool for social integration via the non-cognitive skill development of less fortunate children (see Warren (2010)). Therefore, the priorities of public daycares are often set to accommodate disadvantaged groups. Other criteria for priorities might exist, for example the assignment system currently in place in Denmark is such that the oldest unassigned child is given high priority in a daycare where presently no capacity restriction exists– a concept called "child care guarantee."

In the current paper, we study the problem of the centralized assignment of

[1]For example, there is much heterogeneity in the wait list lengths of each daycare in Copenhagen, as is evidenced by current statistics published (in Danish) at http://www.kk.dk/Redirections/daginstitutioner.aspx

children to daycares. Parents report their preferences concerning institutional and home daycare. Given these reports and the priorities of each daycare, the central authorities decide on the assignment. Our goal is three-fold: illustrate the problem by presenting what is currently done in Denmark; extend well-known concepts from the static problem to a dynamic one; and, finally, propose an algorithm that has several advantages over the current one and other commonly used mechanisms. Most importantly, in the process of doing this, we believe that some of the concepts developed in this paper will be useful to other applications and to the theory of dynamic matching in general.

In the static problem known as *school choice problem*, children of a specific age are assigned to different schools.[2] The problem proposed in this paper extends the school choice problem in two fundamental ways.[3] First, the daycare assignment problem has a dynamic structure: each child may attend daycare for several periods, but not necessarily the same facility. Moreover, in any given period, children of different ages may be allocated to the same daycare. For example, in Denmark, children attending the same daycare range in age from 6 months to 3 years. Every month, a group of young children start daycare while those children who turn 3 years leave for the next level of pre-school. The second defining feature of the daycare assignment problem is that the schools' priorities are history dependent: a school gives priority to children previously allocated to it. In Denmark, it is also the case that children that were not assigned to any school in a given period are given high priority in all schools in the subsequent period.

One of the main objectives in the *school choice* literature has been to identify mechanisms that satisfy one or more well-defined positive properties, such as stability, Pareto efficiency, and strategy-proofness. Stability has been interpreted as eliminating "justified envy": a mechanism that leads to an allocation in which no child would prefer a different school to her current one and, at the same time, find a student in that preferred school with a lower priority than her. Pareto efficiency, on the other hand, refers to the preferences of the students, and, thus, ignores the

[2]See Abdulkadiroğlu and Sönmez (2003) for an important paper in the area, and also Pathak (2011) for a recent survey.

[3]Henceforth, we will refer to our problem as the daycare assignment problem mainly due to what we see as its prototypical application.

schools' priorities. Finally, an algorithm is said to be strategy-proof if reporting the true preference profile is a weakly dominant strategy. Abdulkadiroğlu and Sönmez (2003) discuss two important mechanisms to be used in this allocation problem: the Gale-Shapley Deferred Acceptance algorithm, which is shown to be strategy-proof and to yield a stable algorithm; and the Top-Trading Cycles, which is strategy-proof and yields an efficient matching. Here, we extend the above mentioned concepts to the daycare assignment problem and study whether these concepts are compatible with one another in this new– dynamic– environment.

In our setting, the concept of stability must be strengthened to be meaningful. The main intuition here is that justified envy becomes harder to define when the priorities of each school depend on the allocation of the previous year. For example, a child who stays home in period t might have a higher priority in her preferred daycare in period $t+1$ (in particular, this is true under the assignment mechanism currently in place in Denmark). Thus, in the discussion of the concept of justified envy for period $t+1$, it is not clear whether the allocation to which it should be analyzed is the one in t or the one in $t+1$.

To account for issues such as the ones raised in the previous paragraph, we propose a refinement for the concept of stability, which we denote *strong stability*. A strongly stable matching is one in which there is no profile of schools such that an agent that prefers this profile over her assigned profile has higher priorities in these schools *even* if she were to move to these referred schools. To find a strongly stable matching we show that one can treat the daycare assignment problem as separate school choice problems in different periods and find stable matchings in each period, sequentially starting from period 0. The well known Gale-Shapley deferred acceptance algorithm satisfies strong stability. We also show that it is not Pareto dominated by any other mechanism that satisfies strong stability, and, if there exists an efficient and strongly stable matching, it must be the Gale-Shapley one.

Importantly, though, we show that the mechanism described above is not strategy proof: parents might have incentives to misreport their true preferences. This negative result holds even if we restrict attention to a restricted domain of preferences and priorities. Specifically, for the most part we assume that priorities of schools are history dependent in only a rather weak sense: the priority ranking of each school will only change for children previously allocated to it, while for all other children, the priorities will remain the same. We denote this condition by *independence of previous assignment*. Moreover, we also consider a restriction on preferences, which we call *separability*. This restriction implies that preferences over schools are somehow stable and consistent; in particular, there are no complementarities. Even with only this weak link between periods, the problem becomes substantially different from the static case and the Gale-Shapley is not strategy-proof.

The result above raises the question of whether there is *any* mechanism that would be strategy-proof and stable (even if only weakly). This question is a very important one in the context of the school choice literature, where much attention has been given to stability and the Gale-Shapley mechanism (which has since been adopted in New York and Boston). However, the search for a stable and strategy-proof mechanism is not straightforward, since the class of all possible mechanisms is, of course, very large. Here we prove an impossibility result: there does not exist a mechanism that is both strategy-proof and stable.

To prove the result above (Theorem 3), we construct an example in which for different preference profiles there is a unique stable outcome. Thus, if there is a strategy-proof and stable mechanism, it must yield the unique stable allocation for each reported preference profile. We then proceed to show that a player may benefit from a unilateral deviation in her reported preference profile–which must yield the unique stable matching for that reported profile. This impossibility result does not rely on the concept of strong-stability, but it holds even with the weaker concept of static stability. Moreover, the result is true even in the restricted domain of preferences and priorities referred above.

We turn the focus to finding a strategy-proof and Pareto efficient mechanism. Unlike the case of stability, extending the concept of *efficiency* in the daycare assignment problem is straightforward– at least conceptually. However, although in static settings it is impossible to find a child who would agree to trade her placement for a worse one, in a dynamic setting this may be possible as long as the child obtains a better placement in the other period. Hence, as long as there are two or more "willing" participants of such a trade, there is room for Pareto improvement even if none exists by changing one period matchings only. We show that due to this motive, the Top Trading Cycles is not efficient. We also show that it is not strategy-proof and that even a variation of this algorithm, which we call Top Trading Cycles by cohort, is not strategy-proof.

Strategy-proofness is more difficult to achieve in the dynamic environment that we consider since there is an additional potential benefit for a player from misreporting her true preferences: to affect the priority rankings of schools in the subsequent period. This motive is indeed very strong and is the driving force of some of our negative results. In addition, note that if an assignment algorithm in place is not strategy-proof, then computing the optimal strategy for the parents is substantially more complicated in a dynamic problem than it is in a static one.[4]

We propose a mechanism and denote it the Sequential Choice Mechanism. In this mechanism, which is a version of the well-known Serial Dictatorship, children are exogenously ordered by the planner and they choose a menu of schools over time according to their position in the queue. The Serial Dictatorship is not strategy-proof nor efficient in our problem if applied period-by-period, but this extended version satisfies both properties. Moreover, as Pathak (2011) argued, in the school choice problem there is no natural way of ordering the agents, so the Serial Dictatorship mechanism may seem "unfair." In contrast, our dynamic problem has a natural way of ordering the agents: each child's date of birth. If the Sequential Choice Mechanism is used in practice, each child has the right to choose at some point in time, over all menus available at that moment.[5] This proposed mechanism has the advantage of being simple and transparent. In addition, once parents make their choices, they know the daycares that their child will attend at each different period. In contrast, in the mechanism currently in place in Denmark, parents sign up

[4]Our problem is not part of the literature on multi-unit allocation. Papai (2001), and Ehlers and Klaus (2003), for example, have obtained negative results concerning strategy-proofness and efficiency, however, the problem here is substantially different and their results do not apply to our setting. Many of the results in that literature depend on the feature that each agent might consume several goods. In contrast, here, each agent consumes only a fixed number of goods (one per period). In addition, due to the overlapping generations, some goods are only available in future periods.

[5]Our model is discrete and each child attends daycares for two periods only. This is not important for the results, though: a more general model with each child living for n periods would still give us the same results.

to multiple waiting lists, but they have little idea of how these choices will translate into dates of acceptance.

We should highlight that although our problem is motivated by the assignment of children to daycares, it has many other applications. A firm with offices in different cities must also solve a dynamic allocation problem; workers must be allocated according to their preferences and the priorities of each office.[6]

Other interesting applications are the assignment of teachers to public schools, diplomats to different embassies, or high-level bureaucrats to different regions (see Bloch and Cantala (2008)). A problem related to this one is the market for new physicians in the United Kingdom, where each doctor is allocated to two six-month positions, a medical post and a surgical post (see Roth (1991) and I. (1998)).

Since the work of Abdulkadiroğlu and Sönmez (2003), mechanism design has been used by many researchers to design new algorithms for the assignment of children to schools. This literature has shown that some of the systems currently in place have many shortcomings, and new systems that overcome some of these problems have been proposed. In particular, as we have mentioned before, special attention has been given to the Gale-Shapley Deferred Acceptance algorithm and the Top-Trading Cycles. These new mechanisms were recently adopted in Boston and New York school systems, and the early evidence suggests that these mechanisms are an improvement over the previous systems. See Abdulkadiroğlu et al. (2009) and Abdulkadiroğlu et al. (2005) for a discussion of the practical considerations in the student assignment mechanisms in these two cities.

The theory of market design and dynamic allocation is very recent. Ünver (2010) extends the literature on centralized matching for kidney exchanges to a dynamic environment in which the pool of agents evolves over time. Kurino (2009) studied the housing allocation problem with an overlapping generations structure. There, stability is not discussed, since there is no concept of priorities in the housing allocation model. Bloch and Cantala (2008) consider a dynamic matching problem, but their focus is on the long-run properties of different assignment rules, which

[6]This is exemplified by the following excerpt from a McKinsey quarterly report:
*"It is thus no surprise that a systematic and continuous approach to fitting the right person to the right job at the right time has long been the Holy Grail of workforce organization." (*Agrawal et al. (2003)).

makes their analysis substantially different from ours.

The structure of this paper is as follows. In Section 2, we present a short description of the daycare system currently in place in Denmark. In Section 3, we describe the model in detail. In Section 4, we study stable matchings and their properties. In Section 5, we prove an impossibility result relating strong stability and strategy-proofness. In Section 6, we study efficiency and propose an algorithm that yields efficiency and strategy-proofness. In Section 7, we provide a brief conclusion. Longer proofs are left in the appendix.

2 The Danish Daycare System

The local municipalities in Denmark use broadly similar mechanisms to assign children to daycares. For specificity, below we highlight the essential features of the Aarhus mechanism, which are also common to most municipalities in Denmark, including Copenhagen.

Children can start a daycare at the age of 6 months and when she turns 3 years she must exit, moving to the next level of pre-schooling. The assignment algorithm runs once a month and each parent reports the preference for her top 3 choices among all daycares. They also report whether they want the option for what is called as a "guaranteed spot," in case the child is currently unassigned. The parents can enroll their child any time after birth. Even if a child has a spot in some daycare she can participate in the assignment algorithm without having to give up her spot, i.e. she may sign up for two different daycares and will be placed in a waiting list for these two daycares. It is important to highlight that children currently allocated to a daycare, will not be displaced from that daycare involuntarily.

When a spot opens in a daycare, a child will be allocated according to a general priority ordering. Below is brief description of the priority orderings of the daycares from the assignment algorithm currently in place in the Aarhus Municipality.[7] Once a spot opens, it is offered to a child according to the following order:

1. Children with special needs, e.g., children with disabilities.

[7] For the original document see:
https://www.borger.dk/selvbetjening/sider/fakta.aspx?sbid=8632

2. Children with siblings in the same daycare.

3. Immigrant children who after expert evaluations are considered in need of special assistance in daycare.

4. The oldest child who is listed for a *guaranteed place* in his or her own district i.e., not at a particular daycare.[8]

5. The oldest child who is listed for a guaranteed place in the local warranty district. Aarhus Municipality is divided into 8 major warranty districts. A warranty district consists of one to several districts.

6. The oldest child listed for a guaranteed place from a different warranty district.

7. The oldest child from the waiting list of a particular daycare. This offer is also made to a child already in a daycare (unless the child was assigned to a guaranteed place under rules 4-6).

With this mechanism, it is not clear what is the best way to report the preferences from the point of view of the parents. A very popular daycare will take longer to open a position, so it may be desirable to choose a less popular daycare. Moreover, the guaranteed place may be used strategically as illustrated in the following example. Suppose that a parent knows (by talking to a principal, for example) that a spot is likely to open in her preferred daycare 5 months from now. In this case, she may be inclined to wait for 2 months before asking for the guaranteed spot. This way, when the spot opens in 5 months, chances are that she will be in need of a place according to the guaranteed spot concept. Thus, this mechanism is not

[8]"You can choose a guaranteed place and also a desired place with one or more specific institutions. These requests will be taken into account when we find a place for you. However, we cannot guarantee your desired institution. If your desired institutions does not have an opening, you will be offered a "guaranteed place. A guaranteed place is a place within the district you live in, or at a distance from your home which involves no more than half an hour of extra transport each way to and from work. The municipal placement guarantee is satisfied when you have been offered a place. To be assigned a guaranteed seat at a desired time, the application must be received by the placement guarantee office no later than 3 months before the place is desired." (Translated from https://www.borger.dk)

strategy-proof. Also the Aarhus mechanism fails efficiency and stability. However, we present the formal proof in Section 3.4 because we need to adapt the concepts of efficiency and stability to our setting.

The Aarhus mechanism is also not fully transparent: parents sign up to daycare wait lists, but have very little information about the waiting time. Moreover, given that it is not clear where and when a new spot will be offered to a child, the mechanism generates uncertainty from the parents' point of view.

3 Model

In Section 3.1 we define the concept of matching in our setting. Moreover, we define the preference relation of the children over the different profiles of daycares and the priority orderings of the daycares over the set of children. In Section 3.2 we define the concepts of a Pareto efficient matching and (weak) stability. Further, we extend the well known concept of stability, and denote it strong stability. In Section 3.3 we define a mechanism and its properties. In particular, we define strategy-proofness. Finally in Section 3.4, we revisit the Aarhus mechanism and show its weaknesses.

3.1 Setup

Time is discrete and $t = -1, 0, \cdots, \infty$. There are a finite number of infinitely lived schools/daycares. Let $S = \{s_1, \cdots, s_m\}$ be the set of schools. Each school $s \in S$ has a maximal capacity r_s which we assume is constant. Children can attend school when they are 1 and 2 years old. School attendance is not mandatory. Let h stand for the option of staying home. Let $\bar{S} = S \cup \{h\}$. For technical convenience, we treat h as a school with unbounded capacity. In each period t, a new set of 1-year old children $I_t = \{1, \cdots, n_t\}$ arrives. Consequently, at any period t the set of school-age children is $I_{t-1} \cup I_t$. As time passes the set of school-age children evolves in the "overlapping generations" (OLG) fashion. The set of all children is $I = \cup_t I_t$.

First, we extend the definition of matching to a dynamic context. For the static problem, matching maps the set of children to the set of schools. Here, a matching is a collection of functions indicating which school-age child is assigned to which school at every period.

Definition 1 (Matching)**.** *A period t matching μ^t is a function $\mu^t : I_t \cup I_{t-1} \times \bar{S} \to \{0,1\}$ such that*

1. *For all $i \in I_{t-1} \cup I_t$, $\sum_{s\in\bar{S}} \mu^t(i,s) = 1$,*
2. *For all $s \in S$, $\sum_{i\in I_{t-1}\cup I_t} \mu^t(i,s) \leq r_s$.*

A matching μ is a collection of period matchings $\mu = (\mu^{-1}, \mu^0, \cdots, \mu^t, \cdots)$.

If child i is placed at school s in period t, then $\mu^t(i,s) = 1$. Requirement (1) above says that each child is placed at one school, while requirement (2) says that each school cannot house more children than its capacity. We assume that at time $t = -1$ the matching is exogenously given (for example, it may be that these initial children stay at home in their first year). In other words, each matching we consider has a common period -1 matching.

With slight abuse of notation, $\mu^t(i)$ denotes the school at which child i is placed under μ^t, i.e., $\mu^t(i) = s$ whenever $\mu^t(i,s) = 1$, for each $i \in I_{t-1} \cup I_t$. Similarly, $\mu^t(s)$ denotes the set of children who are placed at school s under μ^t, i.e., $\mu^t(s) = \{i \in I_{t-1} \cup I_t : \mu^t(i,s) = 1\}$.

Children's Preferences

Each child is characterized by a strict preference relation $\succ_i$ over $\bar{S}^2$. The notation (s,s') denotes the allocation in which a child is placed at school s at age 1 and at school s' at age 2. We write $(s,s') \succeq_i (\bar{s},\bar{s}')$ if either $(s,s') \succ_i (\bar{s},\bar{s}')$ or $(\bar{s},\bar{s}') = (s,s')$. Throughout the paper, we maintain the following assumptions on preferences:

Assumption 1 (Preferences)**.** *Each child i's preferences satisfy:*

1. *(No complementarities) If $(s,s) \succ_i (s',s')$ for some $s,s' \in \bar{S}$, then $(s,s) \succ_i (s,s')$ and $(s,s) \succ_i (s',s)$.*

2. *(Weak separability) If* $(s,s) \succ_i (s',s')$ *for some* $s,s' \in \bar{S}$*, then* $(s,s'') \succ_i (s',s'')$ *and* $(s'',s) \succ_i (s'',s')$ *for any* $s'' \neq s'$*.*

No complementarities and the strictness of preferences yield that for any $s,s' \in \bar{S}$ and i, at least one of the following conditions is satisfied

$$(i)\ (s,s) \succ_i (s,s') \text{ and } (s,s) \succ_i (s',s); \text{ or}$$
$$(ii)\ (s',s') \succ_i (s,s') \text{ and } (s',s') \succ_i (s',s).$$

Moreover, the two conditions above may be satisfied at the same time. This would be the case, for example, if a child incurs a large enough cost (not necessarily monetary) from changing schools.

In this paper, we often consider a stronger version of the weak separability assumption which we call *separability*. Recall that if child's preferences satisfy weak separability, then whenever attending school s in both periods is preferred to attending school s' in both periods, attending s and a third school s'' must be better than attending s' and s''. However, weak separability does not rule out the possibility that the child prefers attending school s' in both periods to attending s in one period and s' in the other. Separability, however, rules out this possibility.

Definition 2 (Separability). *Child i's preferences are* separable *if, for any* $s,s' \in \bar{S}$

$$(s,s) \succ_i (s',s') \Longleftrightarrow (s,s'') \succ_i (s',s'') \text{ and } (s'',s) \succ_i (s'',s') \text{ for all } s'' \in \bar{S}.$$

Schools' Priorities

At any time $t \geq 0$, each school ranks all the school-age children by priority. Priorities do not represent school preferences but rather, they are imposed by local municipality. For example, in the existing assignment mechanism in Denmark, all schools give priority to their currently enrolled children. Similarly, the children with special needs are given higher priority by the schools tailored to meet those needs.

Henceforth, we assume that each institution gives the highest priority to its currently enrolled children, which is a feature of the assignment mechanism currently in place in Denmark. A rationale behind this priority is that no school forces its current enrollee out in order to free a spot for some other child. Because of this assumption, the priority ranking of each school is history dependent, i.e., a school's priority ranking depends on its attendees of the previous period.

One could argue that even in the school choice problem, the schools' priorities are history dependent because a typical school gives priority to children whose siblings are in it. In other words, the matchings of the previous periods affect how the schools rank the new applicants. However, in the school choice literature, this history dependence of the schools' priorities is not modelled explicitly. This omission is justified if the older siblings make decision without caring about the younger ones, i.e., one sibling's well-being is not dependent on another's. However, in our model, the children participate in the assignment mechanism twice and of course, any child's well being depends on the schools she attends in different periods. Therefore, in our model, we have to take the history dependence of the schools' priorities seriously.

We will denote the strict, binary relation which generates the priority ranking of school s at period t by $\rhd_s^t(\mu^{t-1})$. That is, if at period t child i has a higher priority than child j at school s given the period $t-1$ matching μ^{t-1}, then we denote $i \rhd_s^t \left(\mu^{t-1}\right) j$. We write $i \unrhd_s^t \left(\mu^{t-1}\right) j$ if either $i \rhd_s^t \left(\mu^{t-1}\right) j$ or $i = j$.

We impose the following assumptions on the schools' priorities.

Assumption 2 (Priorities)**.** *Each school's priorities satisfy:*

1. *(Priority for currently enrolled children) If $i \in I_{t-1}$ and $i \in \mu^{t-1}(s)$ for some $s \in S$, then $i \rhd_s^t (\mu^{t-1}) j$ for all $j \notin \mu^{t-1}(s)$.*

2. *(Weak consistency of different period rankings) If $i \rhd_s^{t-1} (\mu^{t-2}) j$ for some $i, j \in I_{t-1}$, $s \in S$ and μ, then $i \rhd_s^t \left(\mu^{t-1}\right) j$ in any of the following cases:*

   - $\mu^{t-1}(i) = \mu^{t-1}(j) = s$
   - $\mu^{t-1}(i) = s, h$ *and* $\mu^{t-1}(j) = h$
   - $\mu^{t-1}(j) \neq s, h$

3. *(Weak irrelevance of previous assignment) If $i \rhd_s^t (\mu^{t-1})\, j$ for some $i, j \in I_{t-1}$, $s \in S$, and μ with $\mu^{t-1}(i) \neq s, h$ and $\mu^{t-1}(j) \neq s, h$, then $i \rhd_s^t \left(\bar{\mu}^{t-1}\right) j$ for any $\bar{\mu}$ satisfying one of the following conditions.*

   - $\bar{\mu}^{t-1}(i) = \bar{\mu}^{t-1}(j) = s$
   - $\bar{\mu}^{t-1}(i) = s, h$ *and* $\bar{\mu}^{t-1}(j) = h$
   - $\bar{\mu}^{t-1}(j) \neq s, h$

4. *(Weak irrelevance of difference in age) If $i \rhd_s^t (\mu^{t-1})\, j$ for some $i \in I_{t-1}$, $j \in I_t$, $s \in S$, and μ with $\mu^{t-1}(i) \neq s, h$, then $i \rhd_s^t \left(\bar{\mu}^{t-1}\right) j$ for all $\bar{\mu}$. In addition, if $j \rhd_s^t (\mu^{t-1}) i$ for some $i \in I_{t-1}$, $j \in I_t$, $s \in S$, and μ with $\mu^{t-1}(i) \neq s, h$, then $j \rhd_s^t \left(\bar{\mu}^{t-1}\right) i$ for all $\bar{\mu}$ with $\bar{\mu}^{t-1}(i) \neq s, h$.*

Loosely speaking, the last three assumptions mean that the priorities of any school do not depend on the attendees of other schools (excluding staying home). Specifically, the second one says that if child i has higher priority than child j at school s in period $t-1$, then child i keeps her advantage over child j in the following period unless child j attends school s (h) while child i does not attend s (s or h). The third one says that at any period, school s's relative ranking of any two children is not affected by the fact that one child has attended school $s' \neq s$ and the other $s'' \neq s$. The fourth assumption says that at any period school s's relative ranking of any two children is not affected by the fact that one child has attended school $s' \neq s$ at period $t-1$ while the other is one year old at period t. Here we remark that Assumption 2 does not rule out the possibility that a school s gives priorities to the children who have not attended any school over the ones who have attended some school other than s in the previous period. This possibility is ruled out if the schools' priorities satisfy the *Independence of Past Attendance* property which we define below.

Definition 3 (*Independence of Past Attendance*)**.** *School s's priorities satisfy the* Independence of Past Attendance *(IPA) property if the conditions below are satisfied:*

2a. *(Consistency of different period rankings) If $i \rhd_s^{t-1} (\mu^{t-2})\, j$ for some $i, j \in I_{t-1}$, $s \in S$ and μ, then $i \rhd_s^t \left(\mu^{t-1}\right) j$ in any of the following cases:*

– $\mu^{t-1}(i) = \mu^{t-1}(j) = s$

– $\mu^{t-1}(j) \neq s$

3a. *(Irrelevance of previous assignment) If* $i \rhd_s^t (\mu^{t-1})\, j$ *for some* $i, j \in I_{t-1}$, $s \in S$, *and* μ *with* $\mu^{t-1}(i) \neq s$ *and* $\mu^{t-1}(j) \neq s$, *then* $i \rhd_s^t \left(\bar{\mu}^{t-1}\right) j$ *for any* $\bar{\mu}$ *satisfying one of the following conditions.*

– $\bar{\mu}^{t-1}(i) = \bar{\mu}^{t-1}(j) = s$

– $\bar{\mu}^{t-1}(j) \neq s$

4a. *(Irrelevance of difference in age) If* $i \rhd_s^t (\mu^{t-1})\, j$ *for some* $i \in I_{t-1}$, $j \in I_t$, $s \in S$, *and* μ *with* $\mu^{t-1}(i) \neq s$, *then* $i \rhd_s^t \left(\bar{\mu}^{t-1}\right) j$ *for all* $\bar{\mu}$. *In addition, if* $j \rhd_s^t (\mu^{t-1}) i$ *for some* $i \in I_{t-1}$, $j \in I_t$, $s \in S$, *and* μ *with* $\mu^{t-1}(i) \neq s$, *then* $j \rhd_s^t \left(\bar{\mu}^{t-1}\right) i$ *for all* $\bar{\mu}$ *with* $\bar{\mu}^{t-1}(i) \neq s$.

In practice, *IPA* is often not satisfied: many schools give priority to two year old children who have not attended any school in the previous period over one year old children and the two year old children who have attended school in the previous period. In particular, given a concept called "guaranteed spots," *IPA* is not satisfied in the current Danish daycare assignment mechanism, but Assumption 2 is satisfied.

The school choice problem is a special case of the daycare assignment problem. To see this, suppose that each child is one at period -1 when they stay home. The schools' priorities are well defined at period 0. In addition, the children rank the schools at period 0 fixing that their period -1 matches are h. Now one can see that this special case of our daycare assignment problem is a school choice problem.

The OLG structure of the daycare assignment problem is one of its distinguishing features from the school choice problem. To be specific, due to the OLG structure, schools could have different number of open slots in different periods. Hence, a child may face a situation in which her preferred school does not have any open slot when she is one but does have one when she is two. This type of possibility must affect the child's decision. To illustrate why the OLG structure is crucial, let us consider the following dynamic model. Let all the children in the model be born at the same time and attend school for two periods. Given Assumption 1, the children can rank schools by their preferences under the assumption that they will attend the same school in both periods. We can treat the problem as a static problem in which each child is assigned to the same school in both periods. Consequently, all the results from the school choice problem will extend to this special case.

We also remark that the history dependence of the schools' priorities plays a crucial role in our analysis. However, let us postpone this discussion until we study strategy-proofness.

3.2 Properties of a Matching: Efficiency and Stability

The matching literature has identified Pareto efficiency and stability as the two main desirable properties. The main goal of this subsection is to adapt these concepts to our daycare assignment problem.

The definition of Pareto efficiency in our setting coincides with the one in the school assignment problem: a matching μ is *Pareto efficient* if no other matching strictly improves at least one child without hurting the others.

Definition 4 (Pareto Efficiency)**.** *A matching $\bar{\mu}$ Pareto dominates μ if for some $t \geq 0$ and some $i \in I$,*

$$\left(\bar{\mu}^t(i), \bar{\mu}^{t+1}(i)\right) \succ_i \left(\mu^t(i), \mu^{t+1}(i)\right)$$

and for $\forall j \in I$,

$$\left(\bar{\mu}^t(j), \bar{\mu}^{t+1}(j)\right) \succeq_j \left(\mu^t(j), \mu^{t+1}(j)\right).$$

A matching μ is Pareto efficient if no matching $\bar{\mu}$ Pareto dominates μ.

Adapting the definition of stable matching in our setting is much less straightforward as the dynamic nature of our setting presents some challenges, absent in the school choice problem. We propose two stability concepts based on the idea of *justified envy* freeness[9]: weak stability and the strong stability. A matching is weakly stable if no child can *justify* her envy of another child at some period, i.e., at any period t, if child i improves by moving to school s from her currently matched

[9] In static settings in which one side of the market has priorities but not preferences, stable matchings are defined as the ones free of justified envy. See Balinski and Sonmez (1999) and Abdulkadiroğlu and Sönmez (2003) for examples.

school only at t while keeping her past/future then s must not assign a seat to any child who has lower priority than i. In a way, for weak stability, we are analyzing the problem at fixed period t, assuming that the matching of every other period $t' \neq t$ is fixed. In this sense, the weak stability concept is analogous to the stability concept in the school choice problem.

Definition 5 (Weak Stability)**.** *A matching μ is* weakly stable *if at any period $t \geq 0$, there does not exist a school-child pair (s,i) such that (1) and (2) below hold at the same time*

1. *(a)* $(s,\mu^{t+1}(i)) \succ_i (\mu^t(i),\mu^{t+1}(i))$, *or*

   (b) $(\mu^{t-1}(i),s) \succ_i (\mu^{t-1}(i),\mu^t(i))$,

2. $|\mu^t(s)| < r_s$ *or/and* $i \rhd_s^t (\mu^{t-1})\, j$ *for some* $j \in \mu^t(s)$.

Condition (1) above refers to the fact that child i would be strictly better off by switching to some school s rather than the school specified by the matching μ. On top of that, condition (2) implies that either there are unfilled spots at the preferred school s of child i, or the school is in full capacity but some child j placed at this school under the matching μ has lower priority than child i.

In the definition of weak stability, one considers only the one period deviations which has two shortcomings: (1) because the children can attend school for two periods, a child can imagine situations in which she changes her match in both periods and (2) the schools' priorities, which have to be considered for stability, evolve depending on the past matchings.

To account for the issues raised above, we define a stronger concept of stability. Mainly, under strong stability a child takes into consideration that priorities are history-dependent, so that justified envy is not simply based on the current period's matching. Before formally defining the concept, we need to define the following notation.

For any $i, j \in I_t$, $s \in \bar{S}$ and μ such that $\mu(i) \neq \mu(j)$ and $\mu(j) \in S$, let

$$\bar{M}^t(i,j,\mu) \equiv \left\{\bar{\mu}^t : \bar{\mu}^t(i) = \mu^t(j), \bar{\mu}^t(j) \neq \mu^t(j) \,\&\, \bar{\mu}^t(i') = \mu^t(i') \,\forall i' \neq i, j \in I_{t-1} \cup I_t\right\}.$$

That is, the set $\bar{M}^t(i,j,\mu)$ is a set of matchings at period t such that j is replaced by i in the allocation specified by the matching μ^t, j is placed at a different school and all other children's placements remain unchanged. One may think of this as the set of all hypothetical matchings at time t such that i replaces j who then finds a school somewhere else — perhaps home, or some other school — and all other children remain in the same school. Implicit in the solution concept of strong stability and the construction of the set $\bar{M}^t(i,j,\mu)$ is the assumption that children are not "farsighted." Under this view, an allocation of a particular period is considered "unfair" (or subject to justified envy) if the child takes the matching of all other children at all other periods as given. In particular, when the child "feels" that she has justified envy over some child in a particular school, for the following period, she imagines that this child over whom she had priority will either stay at home, or be placed in some other school that will not affect the next period's matching and all other children remain matched as originally. When evaluating that the matching μ is subject to justified envy, the child does not evaluate the entire general equilibrium effect of a new allocation that would take into consideration her justified envy and possibly everyone else's.

Definition 6 (Strong Stability)**.** *Matching μ is* strongly stable *if it is weakly stable and at any period $t \geq 0$, there does not exist a triplet (s,s',i) such that*

$$(s,s') \succ_i (\mu^t(i),\mu^{t+1}(i)),$$

for $s \neq \mu^t(i)$, $s' \neq \mu^{t+1}(i)$ and one of the following conditions holds:

1. *$|\mu^t(s)| < r_s$ and $\left|\mu^{t+1}(s')\right| < r_{s'}$,*

2. *$|\mu^t(s)| < r_s$, $\left|\mu^{t+1}(s')\right| = r_{s'}$, and, for some $j' \in \mu^{t+1}(s')$, $i \rhd_{s'}^{t+1} (\bar{\mu}^t) j'$ where $\bar{\mu}^t$ is the period t matching with $\bar{\mu}^t(i) = s$ and $\bar{\mu}^t(i') = \mu^t(i')$ for all $i' \neq i \in I_{t-1} \cup I_t$,*

3. *$|\mu^t(s)| = r_s$, $\left|\mu^{t+1}(s')\right| < r_{s'}$, and, for some $j \in \mu^t(s)$, $i \rhd_s^t (\mu^{t-1}) j$,*

4. *$|\mu^t(s)| = r_s$, $\left|\mu^{t+1}(s')\right| = r_{s'}$, for some $j \in \mu^t(s)$, $j' \in \mu^{t+1}(s')$ and for any*

$\bar{\mu}^t \in \bar{M}(i, j, \mu)$, $i \rhd_s^t (\mu^{t-1}) j$ *and* $i \rhd_{s'}^{t+1} (\bar{\mu}^t) j'$.[10]

We interpret justified envy in the dynamic context as the existence of a pair of schools for which a child prefers to its current match and such that in some "reasonable" way it would be "fair" for her to go to the preferred schools. Specifically, a reasonable way may mean one the four cases: (1) both of these schools have unassigned spots; (2) in the first period a preferred school has an unassigned spot and in the second, the child has a higher priority over another child allocated at a preferred school; (3) a preferred school in the second period is operating with less than full capacity and in the first period the child is placed on a higher priority than some other child already allocated there, and finally (4) in the first year the child has a higher priority than some other child in a particular school and in the second year, the child has a higher priority than some other child even if there had been a reallocation in the first period, in which she replaced some child in year 1, as long as in this new allocation, all other children remained in the same school.

To further illustrate the need for the concept of strong stability, we consider the following two examples.

Example 1 (Justified Envy under Failure of Separability). *Consider a matching that places child i at school s' when she is both 1 and 2 years old. However, there is another school s such that child i improves only if she switches to school s in both periods. Observe that child i's preferences are not* separable. *Moreover, suppose that when child i is 1 year old, at school s she has priority over another child i' who is placed at school s at that time. With this information, we cannot rule out the possibility that the matching is* weakly stable. *This is because child i prefers attending s' for 2 periods to attending school s when she is 1 and s' when she is 2.*

However, one can argue that child i's envy of i' is justified: she has a right to attend school s ahead of i' at age 1. Then, in the following period, she will be in the highest priority group at school s. This gives her a right to attend school s when she is 2. This argument is captured in requirement 4 in the strong stability definition. $\diamond$

Example 2 (Justified Envy under Failure of IPA). *Suppose there are 2 schools, s and s', with respective capacities of 1 and 2 children. Children i and i' are born*

[10] Observe that $\mu^t(j) = s \neq h$ as h has an unlimited capacity. Hence, $M^t(i, j, \mu)$ is well defined.

at the same period and their preferences satisfy the following property: $(s,s) \succ (s',s) \succ (h,s) \succ (s',s')$. *Suppose that school s gives higher priority to child i than i' at period t when the children are 1 year old. However, i' is given higher priority over child i by school s at period $t+1$ if at period t, i' does not attend any school while i attends s'. Observe that school s's priorities do not satisfy* IPA.

Consider a matching which places both children at school s' in period t but places child i at school s and child i' at school s' in period $t+1$. Implicitly, the period t spot of school s is assigned to some other child who has higher priority at school s over both children. With this information only, we cannot prove that the matching is not weakly stable.

However, one can argue that child i' envies i in a justified manner: if she is stays home at period t and attends school s at period $t+1$, then she would definitely improve. In addition, she would have had priority over i at school s in period $t+1$. This argument is captured in requirement 2 of the strong stability definition. $\diamond$

Strong stability is a refinement of weak stability and we believe that it is a natural concept that captures the meaning of justified envy in our setting. Yet we must remark that the definition of strong stability is stronger than what Examples 1 and 2 call for. In other words, one can slightly weaken definition 6 so that a matching is strongly stable if it is weakly stable and free of justified envy discussed in Examples 1 and 2. However, this does not change any of the results in the next section. Given this, weakening the definition of strong stability is not beneficial from a technical perspective.

Examples 1 and 2 show that the strong and weak stability concepts are not equivalent if one of separability or IPA is not satisfied. But what if both of them are satisfied? Indeed in this case, it turns out that two concepts of stability are equivalent. As this result is somewhat technical, we refer the interested readers to Appendix A.

3.3 Mechanism and Its Properties

Let P_i denote the reported preference ordering of child $i \in I$ and P be the product of the reported preferences of every child i. A mechanism φ is an algorithm that constructs, sequentially, a matching for the daycare assignment problem, given the reported preferences and the priorities. That is, mechanism φ maps the reported preferences P and the function $\rhd^t(\cdot)$ to a matching μ. Let $\varphi_i(P, \rhd^t(\cdot))$ denote the pair of schools in which child i is placed. Strategy-proofness is defined as an incentive for reporting the true preferences. Formally, reporting the true preferences is a weakly dominant strategy for the children.

Definition 7 (Strategy-Proofness)**.** *A mechanism* φ *is* strategy-proof *if for all* $i \in I$*, all* $\rhd^t(\cdot)$*, all* P_i*, all* $t \geq 0$*, all* $\hat{P}_i$*, and all* $\hat{P}_{-i}$*,*

$$\varphi_i\left(P_i, \hat{P}_{-i}, \rhd^t(\cdot)\right) \succeq_i \varphi_i\left(\hat{P}_i, \hat{P}_{-i}, \rhd^t(\cdot)\right)$$

where P_i *is* i*'s true preferences while* $\hat{P}_i$ *and* $\hat{P}_{-i}$ *are the reported preferences of* i *and the others.*

Definition 8 (Stability and Efficiency)**.** *A mechanism* φ *is* efficient (strongly/weakly stable)*, if for all* P *and* $\rhd^t(\cdot)$*, it yields an efficient (strongly/weakly stable) matching.*

3.4 Danish Mechanism Revisited

In this subsection, we revisit the Danish mechanism. For specificity, we focus on the Aarhus mechanism presented in section 2 and show that the mechanism does not satisfy any of the desirable properties discussed in the previous subsection.

Example 3 (Aarhus Mechanism)**.** *Suppose there are 2 schools,* $\{s_1, s_2\}$ *and each school has a capacity of one child. In each period, 1 child is born, but children are identical in all other aspects. Their preferences satisfy the following property:* $(s_1, s_1) \succ (s_2, s_1) \succ (h, s_1) \succ (s_2, s_2)$*.*

Consider the following strategy: each child participates in the Aarhus mechanism when she is 2. Each child also participates in the Aarhus mechanism when she is one if and only if the child from the previous generation attended school s_2 *in the previous period. Whenever a child participates her reported preferences rank the schools as follows:* s_1, h, s_2*.*

The resulting matching from the strategy described above is that (h, s_1) *for each child. It is easy to see that this strategy profile is an (subgame perfect) equilibrium:*

no child wants to deviate because she cannot attend school s_1 when she is 1. If she attends school s_2 when she is 1, then she cannot attend s_1 when she is 2 because she will lose her priority over the younger child in that period.

Clearly, the Aarhus mechanism is not efficient as each child matching with (s_2, s_1) Pareto dominates (h, s_1). Furthermore, in each period, the younger child can attend school s_2 as it has an unfilled spot. Consequently, the Aarhus allocation mechanism is not weakly stable. Finally, in the Aarhus mechanism, each child reports that h is preferred to s_2. Thus, the mechanism fails strategy-proofness too.

4 Stable Matchings

Now we turn our attention to the question of whether strongly stable matchings exist. We first show that if the schools' priority rankings do not satisfy *IPA*, then a strongly stable matching might not exist. Later, we show that *IPA* is a sufficient condition for existence of *strongly* stable matchings.

Theorem 1. *If the schools' priorities do not satisfy* IPA*, then the existence of strongly stable matchings is not guaranteed.*

Proof. Consider the following example in which *IPA* is violated. There are 2 schools, s and s' with respective capacities of 1 and 3. In each period, there are two identical one-year old children. Their preferences are separable and satisfy the following property: $(s,s) \succ (h,s) \succ (s',s') \succ (h,h)$.

Each period, the schools rank the children in which the highest priority groups are: (1) the previous period's attendees (2) two year old children who have not attended any school in the previous period. (Note that condition (2) violates *IPA*). Now we show that strongly stable matchings do not exist in this example. On contrary, suppose μ be a strongly stable matching.

1. Suppose there exist i and t such that $\mu^t(i) = h$. Then because there are 4 school age children and 4 school spots at period t, at least one unassigned spot must exist. Child i improves if she claims this spot at t, which is a contradiction.

2. Suppose for some i and t, $(\mu^t(i), \mu^{t+1}(i)) = (s, s')$. Clearly, i has the highest priority at school s in period $t+1$. In addition, as $(s,s) \succ_i (s,s')$ by separability, child i can be improved in a justified manner. This is a contradiction.

3. Suppose for $i \in I_t$, $\mu^{t+1}(i) = s$. Then one of the following happens: (1) $\mu^{t+2}(s) = j$ for some $j \in I_{t+1}$ or (2) $\mu^{t+2}(s) \neq j$ for all $j \in I_{t+1}$. In the former case, the matching of j is (s', s); otherwise, we are back to case 1. Consequently, the matching of $\bar{j} \neq j \in I_{t+1}$ is (s', s'). If $\bar{j}$ stays home at $t+1$, at $t+2$ she has priority over any one-year old or j (who attended s' at $t+1$). In addition, $\bar{j}$ prefers (h, s) to (s', s'). Hence, $\bar{j}$ can be improved in a justified manner. In case (2), either we are back to case 1 or both children born at I_{t+1} match with (s', s'). At $t+2$ both of these children have priority over any one year old at school s. In addition, (s', s) is preferred to (s', s'). Hence, both children child can be improved in a justified manner.

□

In the proof of Theorem 1, we use a counter example with separable preferences of the children. However, *separability* has no role in Theorem 1, i.e., one can construct an example needed for Theorem 1 in which the children's preferences *are not separable*. Hence, we conclude that the existence of strongly stable matchings is not guaranteed without *IPA* regardless of whether *separability* is satisfied or not. But with *IPA*, is the existence guaranteed? The answer to this question is positive, but first let us introduce the algorithm used for the existence result.

The Gale-Shapley Deferred Acceptance Algorithm and Its Properties

The Gale and Shapley deferred acceptance algorithm (GS algorithm) was originally designed to deal with static two-sided matching problems. To run this algorithm at certain period t, one needs to know the schools' priorities over all school-age children as well as the children's preferences over schools. In our setting, the schools' priorities are well defined given the previous period's matching. However, the children's preferences are defined over the pairs of schools. Hence, to run the original GS mechanism, one needs to derive one period preferences for each child at a given period, based on the past matchings and the original preferences of the children over the pairs of schools; we do not want to derive one period preferences based on the future matchings as the current matchings affect next period's priority rankings of the schools.

For now, let us assume that at period t, we have derived the one period preference relation $\mathcal{P}_i(\mu^{t-1})$ for each $i \in I_{t-1} \cup I_t$ depending on μ^{t-1} matchings. Let $\mathcal{P}(\mu^{t-1}) = \left\{\mathcal{P}_i(\mu^{t-1})\right\}_{i \in I_{t-1} \cup I_t}$. Thus, $s\mathcal{P}_i(\mu^{t-1})s'$ means that at time t, player i prefers school s to s' given the period $t-1$ matching μ^{t-1}. Now we define stability in a static context that will be used in some of our proofs.

Definition 9 (Static Stability)**.** *Period t matching μ^t is statically stable under preferences $\mathcal{P}(\mu^{t-1})$ and μ^{t-1}, if there exists no school-child pair (s,i) such that*

1. $s\mathcal{P}_i(\mu^{t-1})\mu^t(i)$,
2. $|\mu^t(s)| < r_s$ *or/and* $i \rhd_s^t (\mu^{t-1}) j$ *for some* $j \in \mu^t(s)$

Now we will define the one-period preferences that we will use for the GS algorithm.

Definition 10 (Isolated Preference Relation)**.** *For given μ^{t-1},*

1. *the* isolated preference relation *for $i \in I_t$ is the preference relation $\succ_i^1$ such that $s' \succ_i^1 s''$ if and only if $(s',s') \succ_i (s'',s'')$ for any $s' \neq s'' \in \bar{S}$,*
2. *the* isolated preference relation *for $i \in I_{t-1}$ is the preference relation $\succ_i^2 (\mu^{t-1})$ depending on previous period's matching and such that $s' \succ_i^2 (\mu^{t-1})s''$ if and only if $(\mu^{t-1}(i),s') \succ_i (\mu^{t-1}(i),s'')$ for any $s' \neq s'' \in \bar{S}$.*

Now we will state the formal definition of the Gale and Shapley deferred acceptance algorithm. The algorithm is the same in each period, and it only uses the matching of the preceding period. In period $t \geq 1$, assume that the previous period's matching is obtained by using the GS algorithm. At period t, the schools assign their spots to the all school-age children in finite rounds as follows:

Round 1: Each child proposes to her first choice according to her isolated preferences. Each school tentatively assigns its spots to the proposers according to its priority ranking. If the number of proposers to school s is greater than the number of available spots r_s, then the remaining proposers are rejected.

In general, at:

Round k: Each child who was rejected in the previous round proposes to her next choice according to her isolated preferences. Each school considers the pool of children who it had been holding plus the current proposers. Then it tentatively assigns its spots to this pool of children according to its priority ranking. The remaining proposers are rejected.

The algorithm terminates when no proposal is rejected and each child is assigned her final tentative assignment.

Given that the children's preferences as well as schools' priority rankings are strict, it is easy to see that the GS algorithm yields a unique matching. We refer to this matching as the GS matching and use the notation μ_{GS} for it.

With the next result we show that when assuming *IPA*, strong stability is equivalent to static stability under isolated preferences.

Lemma 1. *If μ is strongly stable then for all $t \geq 0$, μ^t is statically stable under isolated preferences and μ^{t-1}. If for all $t \geq 0$, μ^t is statically stable under isolated preferences and μ^{t-1}, then $\mu = (\mu^{-1}, \cdots, \mu^t, \cdots)$ is*

1. *weakly stable.*

2. *strongly stable if each school's preferences satisfy* IPA.

Proof. See Appendix C. □

Lemma 1 implies that to find a strongly stable matching, it suffices to find a stable matching under isolated preferences in each period, sequentially starting from period 0. In other words, for the purpose of finding a stable matching, one can treat the daycare assignment problem as separate school choice problems in different periods. Consequently, the GS matching is strongly stable as Gale and Shapley (1962) shows that the GS algorithm yields a stable matching in a static setting. We state the result below.

Theorem 2 (Existence of Strongly Stable Matching)**.** *The GS matching is weakly stable. Furthermore, if the priority ranking of each school satisfies* IPA*, then the GS matching is strongly stable.*

As we already mentioned, examples 1 and 2 illustrate the need of strengthening the weak stability concept into the strong stability one if *separability* or *IPA* is not satisfied. However, Theorem 2 demonstrates that *IPA* is a sufficient condition for the existence of strongly stable matchings even if *separability* is not satisfied. In addition, Theorem 1 shows that with or without *separability*, the existence of strongly stable matchings is not guaranteed without *IPA*. In this sense, *IPA* might be considered a more critical condition than *separability* for the existence of strongly stable matchings.

One of the most important results in the matching literature is that the GS matching Pareto dominates all other stable matching. We study how GS matching compares to the other stable matchings in more detail in Appendix B as these result are somewhat technical. We summarize our findings in the following proposition.

Proposition 1. *The GS is matching does not necessarily Pareto dominate all other stable matchings. However, it is not Pareto dominated by any stable matching. Moreover, if any matching is stable and efficient, then it must be the GS matching.*

Henceforth, we will always assume that the children's preferences satisfy *separability* and the schools' priorities satisfy *IPA* because these assumptions do not play any role in the results we will present next. In other words, we are concentrating on the cases with a minimal history dependence.

5 Strategy-Proofness and Stability

It is well known that in static settings, when the GS mechanism is strategy-proof. We show below, that in a our setting this no longer holds. In fact, the result below is much stronger: there is no mechanism that is strategy-proof and strongly stable.

Theorem 3 (Impossibility Result)**.** *Weakly stable and strategy proof mechanism may not exist.*

Proof. Consider the following example: there are 4 schools $\{s, \bar{s}, s_1, s_2\}$ and each school have a capacity of one child. There is no school-age child until period $t-1$. Suppose $I_{t-1} = \{i, \bar{\imath}\}$, $I_t = \{i_1, i_2\}$, $I_{t+1} = \{i'\}$ and $I_\tau = \emptyset$ for all $\tau \geq t+2$. In addition, school $s' = s, \bar{s}, s_1, s_2$ prioritizes the children as follows under the assumption that no child attended s' in the previous period:

$$
\begin{array}{ccccccccc}
i & \rhd_s & i' & \rhd_s & i_1 & \rhd_s & i_2 & \rhd_s & \bar{\imath} \\
i & \rhd_{s_1} & i_1 & \rhd_{s_1} & i_2 & \rhd_{s_1} & i' & \rhd_{s_1} & \bar{\imath} \\
i & \rhd_{s_2} & i_1 & \rhd_{s_2} & i' & \rhd_{s_2} & i_2 & \rhd_{s_2} & \bar{\imath} \\
\bar{\imath} & \rhd_{\bar{s}} & i_1 & \rhd_{\bar{s}} & i' & \rhd_{\bar{s}} & i_2 & \rhd_{\bar{s}} & i
\end{array}
$$

We consider two preference profiles which differ from each other in child i_1's preferences. Child i's top choice is (s,s) while child $\bar{\imath}$'s is $(\bar{s},\bar{s})$. The preferences of children i_2 and i' satisfy the following conditions:

$$
\begin{array}{ccccccc}
(s_2,s_2) & \succ_{i_2} & (s_1,s_1) & \succ_{i_2} & (s,s) & \succ_{i_2} & (\bar{s},\bar{s}) \\
(s_2,s_2) & \succ_{i'} & (s,s) & \succ_{i'} & (s_1,s_1) & \succ_{i_2} & (\bar{s},\bar{s})
\end{array}
$$

Child i_1's preference ordering is $\succ_{i_1}^1$ under preference profile 1 and is $\succ_{i_1}^2$ under profile 2. These preferences are given as follows:

$$
\begin{array}{ccccccc}
(s,s) & \succ_{i_1}^1 & (s_1,s_1) & \succ_{i_1}^1 & (s_2,s_2) & \succ_{i_1}^1 & (\bar{s},\bar{s}) \\
(s,s) & \succ_{i_1}^2 & (\bar{s},\bar{s}) & \succ_{i_1}^2 & (s_2,s_2) & \succ_{i_1}^2 & (s_1,s_1)
\end{array}
$$

In addition, suppose $(s_2,s) \succ_{i_1}^1 (s_1,s_1)$.

Here we prove a weaker version of the theorem: that the GS mechanism is not strategy-proof in the above example. We leave the formal proof in Appendix C. Under profile 1, the GS matching is as follows: $\mu^{t-1}(i) = \mu^t(i) = s$, $\mu^{t-1}(\bar{\imath}) = \mu^t(\bar{\imath}) = \bar{s}$, $\mu^t(i_1) = \mu^{t+1}(i_1) = s_1$, $\mu^t(i_2) = \mu^{t+1}(i_2) = s_2$, $\mu^{t+1}(i') = s$ and $\mu^{t+2}(i') = s_2$. (In fact, it is the unique weakly stable matching here).

Under profile 2, the GS matching $\bar{\mu}$ is as follows: $\bar{\mu}^{t-1}(i) = \bar{\mu}^t(i) = s$, $\bar{\mu}^{t-1}(\bar{\imath}) = \bar{\mu}^t(\bar{\imath}) = \bar{s}$, $\bar{\mu}^t(i_1) = s_2$, $\bar{\mu}^t(i_2) = s_1$, $\bar{\mu}^{t+1}(i_1) = s$, $\bar{\mu}^{t+1}(i_2) = s_1$, $\bar{\mu}^{t+1}(i') = s_2$ and $\bar{\mu}^{t+2}(i') = s_2$.

Under profile 1, child i_1's matching is (s_1,s_1) if she reports her preference truthfully but it is (s_2, s) if she misreports her preference as if under profile 2. But $(s_2, s) \succ_{i_1}^1 (s_1, s_1)$. Consequently, child i_1 misreports her preferences under profile 1. Hence, the GS mechanism is not strategy-proof. $\square$

Even when *separability* and *IPA* are satisfied, strategy-proofness is more difficult to achieve in the daycare assignment problem. In static problems, a child has a motive to misreport her preferences only if she can obtain a better placement. This motive is also present in the daycare assignment problem. To be specific, a child will misreport her preferences if she can improve her present placement without hurting her placement in the other period. This motive, as known from the school choice literature, is eliminated if the mechanism is the GS or Top Trading Cycles mechanism. However, in our setting, there is an extra motive absent in the school choice problem: one might misrepresent her preferences to affect the schools' priorities in the subsequent period. This way, she could obtain a better future placement by sacrificing her current one.

In the example used for the proof of Theorem 3, type 1 child i_1 likes school s better than any other school, but attending s in period t is impossible for her. Again, in period $t+1$, she cannot attend s because child i' attends s. But observe that child i' wants to attend school s_2 but cannot do so because child i_2 attends s_2. The most important aspect is that child i_2 has higher priority over child i' at school s_2 in period $t+1$ only because she attends school s_2 in period t. Child i_1 can eliminate child i_2's advantage over i' if she attends school s_2 in period t. By doing this, i_1 enables i' to attend s_2 at $t+1$. Ultimately, she frees a spot at school s for herself at $t+1$. This is the reason why type 1 child i_1 has an incentive to misreport her preferences.

Remark 1. *For Theorem 3, both the OLG structure and the history dependence of the schools' priorities play indispensable roles. We have already mentioned that without OLG structure, all the existing results in the school choice problem will be valid. Now let us discuss why the history dependence of the schools' priorities is critical for Theorem 3 even with the OLG structure. To see this, suppose that the children's preferences are* separable *and somehow the schools' priorities at any period are independent of the previous period's matching–in particular, at some school, a child who did not attend the school in the previous period can have* *priority over some other child who did attend the school. In this case, the GS algorithm must be strategy-proof. Let us discuss why this is the case. For the GS algorithm, one has to report her preferences over the pairs of schools. But this, in fact, is equivalent to the case in which the school-age children report their isolated preferences in each period and the algorithm is run sequentially because the GS algorithm uses the isolated preference. As the preferences satisfy* separability *and the schools' preferences are independent of history, any child's reported isolated preferences in one period do not affect her placement in the other period. Now recall that the GS algorithm is strategy-proof in the static settings. Hence, by misreporting one's isolated preferences in some period, she is worse off in that period without affecting her placement in the other period. Accordingly, no one misreports her isolated preferences. Thus, the GS mechanism is strategy-proof.*[11]

Remark 2. *In the previous remark, we argued that the history dependence of the schools' priorities is crucial for Theorem 3. However, if schools' priorities are history independent, then in some cases, some children will be forced out of the schools they attended in the previous period. For example, in the example used in the proof of Theorem 3, child i_2 is forced out of school s_2 at period $t+1$. We firmly believe that this should be avoided. Therefore, under the restriction that no 2-year old child can be forced out of the school she attended in the previous period, Theorem 3 is valid even when the schools' priorities are independent of the previous period's matching.*

Theorem 3 has two important, direct consequences which we present next.

Corollary 1.

1. *Strategy proof and strongly stable mechanism may not exist.*
2. *The GS mechanism is not necessarily strategy-proof.*

Proof. Recall that each strongly stable matching is weakly stable. This and Theorem 3 prove item 1 of the corollary. □

[11] For this argument, the assumption that children's preferences are *separable* plays an important role. In fact, if the children's preferences do not satisfy separability, then an impossibility result similar to Theorem 3 arises even if the schools' priorities are independent of history. This result can be obtained from the authors.

6 Efficiency and Strategy-Proofness

In this section, we first define the concept of efficiency in a static sense, which we denote Autarkic Efficiency. We show that a matching that satisfies autarkic efficiency may exhibit opportunities for Pareto improving trades across members of the same cohort or across members of different cohorts. Then, in Section 6.2 we study the Top-Trading Cycles in detail. We show that it is neither Pareto efficient nor strategy-proof. Finally, in Section 6.3 we propose a new mechanism, which is both strategy-proof and efficient.

6.1 Efficient Matchings

We have shown that stability and strategy-proofness may be incompatible for the daycare assignment problem. In the remaining sections of this paper, we investigate whether strategy-proofness is compatible with efficiency. However, before doing so, let us consider some properties of efficient matchings.

From the school choice literature, we know that the Top Trading Cycles (TTC) or the Serial Dictatorship mechanisms yield stable matchings. Hence, one might expect that these algorithms using the isolated preferences of the children yield efficient matchings. In other words, one may expect that a result analogous to the result of Lemma 2 will hold for efficiency as well. We will demonstrate that this is not necessarily the case. But first, let us define the Autarkic efficiency concept.

Definition 11 (Autarkic Efficiency)**.** *Matching μ is Autarkic Efficient if for any $t \geq 0$, there does not exist period t matching $\bar{\mu}^t$ such that $(\mu^{-1}, \cdots, \mu^{t-1}, \bar{\mu}^t, \mu^{t+1}, \cdots)$ Pareto dominates μ.*

For Autarkic efficiency, one considers only one period deviations. Hence, it is clear that all efficient matchings satisfy Autarkic efficiency. We present two examples to show that Autarkic efficiency is not equivalent to efficiency. The first example shows that a matching might satisfy Autarkic efficiency but fail to be efficient because of the intergenerational trades. We demonstrate this point below. The second example, more standard, shows that a matching may fail to be efficient due to intra-generational trades, and we present it in Appendix C.

Example 4 (Pareto Improving Trade Across Cohorts)**.** *In each period, there are two 1-year old children in each period $\{i^t, j^t\}$ and there are four schools $\{s_1, s_2, s_3, s_4\}$ with each having a capacity of 1 child. For this example, we will only specify the schools' top ranked school-age child if all of them stayed home in the previous period. School s_1 and s_2 give their respective highest priorities to children i and j who are 1 in odd periods. On the other hand, school s_3 and s_4 give their respective highest priorities to children i and j who are 1 in even periods. In the following table we summarize each school's top ranked child:*

s_1	s_2	s_3	s_4
i^k	j^k	i^{k+1}	j^{k+1}

where k is odd. The children's preferences are as follows.

- *Child i^{-1}'s top choice is s_1 while for child j^{-1}, $s_3 \succ^1_{j^{-1}} s_2 \succ^1_{j^{-1}} s_4$.*
- *For child i^0,$s_1 \succ^1_{i^0} s_3 \succ^1_{i^0} s_2$,while child j^0's top choice is s_4.*
- *For child i^1, $s_4 \succ^1_{i^1} s_1 \succ^1_{i^1} s_3$, while child j^1's top choice is s_2.*
- *Child i^2's top choice is s_3 while for child j^2, $s_2 \succ^1_{j^2} s_4 \succ^1_{j^2} s_1$.*
- *For $t \geq 3$, child i^t (j^t) has the same preferences as child i^{t-4} (j^{t-4}).*

In addition, each child prefers being placed at the school of her third choice when she is 1 and at her most preferred school when she is two to being placed at the school of her second choice for 2 periods.

Consider the following matching μ: in any period, school s_1 matches with the school-age child i who is 1 in an odd period, s_2 with j who is 1 in an odd period, s_3 with i who is 1 in an even period, and school s_4 with j who is 1 in an even period. Observe that in each period exactly 1 younger and 1 older children match with their second choice school. The others match with their top choice. It is easy to see that μ satisfies Autarkic efficiency. Now let us alter μ in the following way: in each period, the two children who are placed at her second choice school trades their schools. This way the older of the two children is placed at her first choice school while the younger one is placed at her third choice school. One can easily see that the new altered matching Pareto dominates μ.

Observe that in the above example, the infiniteness of time plays an important role. To see this, let us check why μ is not efficient. Matching μ places one younger and one older children at their second choice schools in each period. Each of these child prefers being placed in her third choice school when she is one and at her most preferred school in the following period to being placed at her second choice school in both periods. Hence, the younger child would agree to give her spot away and obtain a spot at a worse school as long as she obtains a spot at her most preferred school in the following period. Accordingly, μ is not efficient because one younger child can trade her spot with an older child in each period. If time stops at some point, then the younger child at that time would not agree to this trade. This is why the infiniteness of time is crucial in Example 4. This phenomenon is also observed in the standard overlapping generations models.

The examples 4, presented above, and 7, presented in Appendix C, have an important implication: not all mechanisms that deliver matchings satisfying Autarkic efficiency are necessarily efficient even if the children's preferences are *separable*. For example, the TTC mechanism using isolated preferences does not necessarily yield an efficient matching. As we will discuss whether the TTC mechanism is strategy-proof, let us consider the TTC mechanism in the next subsection.

6.2 The Top Trading Cycles Mechanism

The TTC mechanism was introduced in Abdulkadiroğlu and Sönmez (2003).[12] Next we will state the formal definition of the TTC mechanism.

In each period, we assume that the preceding period's matching is produced by the TTC mechanism according to the isolated preferences of children. In period t:

Round 1: Each child points to her preferred school. Each school points to its highest ranked child. The process goes on, until it reaches a cycle, which it eventually will. A cycle can be written as $\{i_1, s_1, i_2, s_2, \cdots, i_k, s_k\}$, where here, s_j is child $i_j's$ preferred school, whereas child i_l is the highest ranked child in school s_{l-1}, for

[12] The TTC mechanism is inspired by Shapley and Scarf (1974) and Roth and Postlewaite (1977).

$l = 2, \ldots, k$; and child i_1 is the highest ranked child at school s_k. All children in the cycle are allocated to their preferred school.

In general, at:

Round k: All children allocated in rounds 1,...,$k-1$ do not participate in step k. Each remaining child points to its preferred school, among the set of schools with remaining spots. Each pointed school points to the highest priority child among the remaining children. The process goes on until it reaches a cycle, which it eventually will. All children in the cycle are allocated to the schools that they have pointed to.

The process continues until all children are allocated.

We point out that the version of TTC we use is similar to the one Abdulkadiroğlu and Sönmez (1999) use in the housing allocation problem with existing tenants in the sense that in both versions, the object to be assigned points to its current owner unless s/he already obtained another object: in the case of Abdulkadiroğlu and Sönmez (1999), each house points to its current tenant unless she is already assigned a house while in our model, due to the fact that the schools give their highest priorities to its current enrollees, each school points to one of these children unless all of them are assigned to a school. However, the two versions of TTC are different in the sense that in Abdulkadiroğlu and Sönmez (1999), no house prioritizes the (non existing) tenants but in our model, each school prioritizes the children in different ways.

As we already hinted, the top trading cycle mechanism is not necessarily efficient. Given the importance of the TTC mechanism in the school choice problem, let us state this result in the following proposition.

Proposition 2 (TTC is not necessarily Pareto Efficient)**.** *The TTC mechanism is not necessarily Pareto efficient.*

Proof. Consider Example 4 and observe that μ is the matching from the TTC mechanism. As we mentioned μ is not efficient. $\square$

Note that in Example 4, not only the TTC mechanism is not necessarily efficient, but also a variation of it, done by cohorts. Precisely, consider the following mechanism. At any period t, the children born in period $t-1$ are allocated according to the TTC mechanism (see Abdulkadiroğlu and Sönmez (2003)). Once every children $i \in I_{t-1}$ is allocated, most schools will have less, if any, spots available. Consider only the schools with open spots and use the TTC mechanism for the generation born in period t, where from the initial number of spots for each school, we have subtracted the number of 2-year-old children already allocated. For this round, consider only the priority of schools over the children of generation t. i.e., a young child cannot replace an already allocated 2-year-old child. This variation of the TTC mechanism is also is not Pareto efficient.

In the example below, we show that the TTC mechanism (using isolated preferences) may not be strategy-proof.

Example 5 (TTC may not be Strategy-Proof)**.** *Assume that there are 4 schools $\{s, s_1, s_2, s_3\}$, and 4 children: $\{i, i_1, i_2, i_3\}$, with $i \in I_{-1}$ and $\{i_1, i_2, i_3\} \in I_0$. Assume also that $I_t = \varnothing$ for all $t \geq 1$. School $\bar{s} = s, s_1, s_2, s_3$ prioritizes the children as follows assuming that these children has not attended $\bar{s}$ in the previous period:*

$$\begin{array}{lllll} i & \rhd_s & i_2 & \rhd_s & i_1 \\ i_1 & \rhd_{s_1} & j, & \forall j \neq i_1 & \\ i_2 & \rhd_{s_2} & j, & \forall j \neq i_2 & \\ i_1 & \rhd_{s_3} & i_3 & \rhd_{s_3} & j, \quad \forall j \neq i_1, i_3 \end{array}$$

The children's preferences are:

$$\begin{array}{ccccccc} s & \succ_i & s_1 & \succ_i & s_2 & \succ_i & s_3 \\ s & \succ_{i_1} & s_1 & \succ_{i_1} & s_2 & \succ_{i_1} & s_3 \\ s_3 & \succ_{i_2} & s & \succ_{i_2} & s_2 & \succ_{i_2} & s_1 \\ s_3 & \succ_{i_3} & s_1 & \succ_{i_3} & s_2 & \succ_{i_3} & s \end{array}$$

In addition, child i_1 prefers (s', s) to (s_1, s_1).

The matching resulting from the TTC is: $\mu^0(i) = s$, $\mu^0(i_1) = s_1$, $\mu^0(i_2) = s_2$, $\mu^0(i_3) = s_3$, $\mu^1(i_1) = s_1$, $\mu^1(i_2) = s$ and $\mu^1(i_3) = s_3$. However, if i_1 misreports its preferences as $s \succ_{i_1} s_2 \succ_{i_1} s_1 \succ_{i_1} s_3$, while all others report truthfully. The resulting matching is: $\bar{\mu}^0(i) = s$, $\bar{\mu}^0(i_1) = s_2$, $\bar{\mu}^0(i_2) = s_3$, $\bar{\mu}^0(i_3) = s_1$, $\bar{\mu}^1(i_1) = s$, $\bar{\mu}^1(i_2) = s_3$

and $\bar{\mu}^1(i_3) = s_1$.

Note that under truth-telling, i_1's allocation was: (s_1, s_1), while after misreporting it is (s_2, s). Thus, i_1 has improve herself by misreporting. $\diamond$

Observe that the example above shows that a variation of the TTC which is done by cohorts is not strategy-proof.

6.3 Sequential Choice Mechanism

We propose an algorithm to be used in this problem of matching with entry and exit of agents, which we denote the Sequential Choice Mechanism. This is a version of the Serial Dictatorship algorithm, but uses an order of choices naturally provided the birth date of the children.

Formally, recall that at period t, there are n_t children who are 1-year-old and we exogenously label them from 1 through n_t. The algorithm runs as follows: at period 0, the 2-year-old children choose sequentially–according to their indices–one school from the set of schools that have available spots. Once all 2-year-old children have chosen, the 1-year old children choose sequentially a pair of schools following a restriction that a child can only choose a different school in period 2 if that school was not available to the child in period 1.[13] That is, the 1-year-old children choose a school s in the first period (from the set of schools with open slots) and a school for period 2. In period 2, the school can either be the same school s or a different school s' if s' was not available for this child in period 1 and not chosen by any child of the same age with a lower index.

At any given period there is a finite number of school-age children, therefore this is a well-defined mechanism that always converges to a unique matching. Moreover, this algorithm is strategy-proof and efficient. It is strategy-proof since each child can be allocated to the best available menu. Moreover, it is efficient since the

[13] Given our assumption of separability this restriction is not binding. Formally, if a child chooses the following menu: (s, s'), then it must be the case that s' was not available in period 1. To illustrate, consider a case in which s was chosen for the first period and the bundles (s, s) and (s', s') were available. If (s, s') was chosen, it implies that $(s, s') \succ (s, s)$ and $(s, s') \succ (s', s')$. This violates separability. We make this restriction on the menu of choices to avoid the possibility that assumption 2 is violated in some history "off-equilibrium path," i.e. some history inconsistent with undominated strategies.

first child to choose in a given cohort can only improve if there is a school chosen by another child in the previous cohort that would make her better off. No child in the previous cohort would engage in such a trade, since all open schools were available to the older cohort and not chosen by them. The child with an index 2 of the young cohort cannot improve by trading with the first child, since the first child is already choosing the best available option for her. A similar argument holds for any other indexed child.

Our analysis suggests that the Sequential Choice Mechanism may be the ideal method to assign children to public daycares. Besides being strategy-proof and Pareto efficient, the benefits of this mechanism over alternatives include fairness, and the resolution of uncertainty. We also believe that the mechanism is sufficiently flexible that it can accommodate the priorities of public authorities. In the remaining of this section, we provide some remarks about the practical benefits of improving the daycare assignment mechanism.

Fairness

A very important, but less obvious, advantage of the sequential choice mechanism is fairness. In the standard static school choice problem, the serial dictatorship mechanism might be considered unfair because parents listed last are at a clear disadvantage than parents listed first. This problem with serial dictatorship is mitigated in a dynamic assignment problem. To illustrate this point, consider the case in which the number of children born at every period is the same. The child who chooses last in her cohort will have at least half of the daycare-spots available to her in period 2, whereas in the static problem, the last child to choose in the serial dictatorship mechanism might have only one option.[14]

Our sequential choice mechanism can be easily adapted to a dynamic model in which children go to school for $m > 2$ consecutive periods. In this case, if the same number of children is born at every period, the last child to choose in her cohort will have her choice set increased over time. For large m, at the last period the

[14]This assumes that there is at least the same number of spots as there are children in a given period. Formally, consider the case in which there are $2n$ children at every period (with n children being born every period) and $2n$ daycare spots available. The last child choosing in her cohort, will have $n+1$ options in her second period. In the static case with $2n$ children and $2n$ spots, the last child to choose in the serial dictatorship mechanism might have only one spot available.

fraction of daycare spots available is close to 1. Therefore, the Sequential Choice Mechanism is perhaps more fair because the difficulties of assignment are spread more or less evenly across all parents.

Transparency and resolution of uncertainty

A central problem of the current daycare assignment mechanism used in Denmark is lack of transparency. In the city of Copenhagen, parents sign up to several wait lists and they have little idea of how these choices will translate into dates of acceptance. For example, it is extremely difficult to guess whether a child of another parent who is listed earlier will drop out of this list in favor of choosing another daycare for which they are also listed. This means that parents are generally forced to make conjectures about the expected behavior of other parents when they choose a daycare for their own child.

The Sequential Choice Mechanism can in principle remove all sources of uncertainty connected with choosing a daycare. The parents look at the set of available options and make the choices of when and where their child will go to daycare well before the child is able to attend daycare. This has obvious advantages of planning. The parent's employers can be given precise plans about parental absence. Consequently parents can better plan their careers and make various beneficial arrangements that require early commitment to a plan.

Priorities

There is some flexibility to incorporate priorities by affecting the order of parental choice. For example, there are several thousand children entering daycare in Copenhagen each month. Clearly, a large advantage in choice can be given to parent with a special need simply by allowing this child to choose first within the set of children in a monthly cohort. Moreover, the mechanism remains strategy-proof, efficient and free of ex-post uncertainty.

7 Conclusion

In this paper we introduced the daycare assignment problem. This problem differs from the school choice problem due to its dynamic structure and the fact that schools' priorities are history dependent. We showed that the Gale-Shapley deferred acceptance algorithm and the Top-Trading Cycles mechanisms– both commonly used in the school choice problem– are not strategy-proof in the daycare assignment problem. These negative results hold even when preferences satisfy consistency across periods, and when schools' priorities are linked across time in only a very weak sense (priorities are history dependent only through currently allocated children; otherwise, they are the same).

The endogeneity of the priorities gives an incentive for manipulation and this motive is indeed strong. We showed that no stable and strategy-proof mechanism exists for this class of dynamic matching. This is particularly important in the context of the school choice problem, where much attention has been given to stability and, in particular, to the Gale-Shapley algorithm (which has been adopted in New York and Boston).

The problem of allocating children to public daycares differs from the school choice problem in two fundamental ways: there is entry and exit of agents over time, and the priorities of schools over children are history dependent. Our lead example is the case of Denmark. We show that no mechanism is strategy-proof and stable. We propose a strategy-proof, and Pareto efficient mechanism in which parents sequentially choose menus of schools, ordered by the childs birth date. Moreover, this mechanism eliminates ex-post uncertainty, and may be considered fair: parents face similar choice sets, which increase over time.

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Appendix A: The Relation between Strong and Weak Stability

Now we will explore under what conditions, the concepts of weakly and strongly stable matchings will coincide. From examples 1 and 2, one could conjecture that weakly and strongly stable matchings may be equivalent if the children's preferences are *separable* and the schools' priority rankings satisfy *IPA*. Indeed this is the case, as we will show in the next two lemmas.

Lemma 2. *Suppose that all schools' priorities satisfy* IPA. *If μ is weakly but not strongly stable, then for some period t and some school-child pair (s,i),*

1. $\mu^t(i) = \mu^{t+1}(i)$,
2. $(s,s) \succ_i (\mu^t(i), \mu^{t+1}(i))$,
3. $|\mu^t(s)| < r_s$ *or/and* $i \rhd_s^t (\mu^{t-1}) j$ *for some* $j \in \mu^t(s)$.

Proof. Since μ is not strongly but weakly stable, for some $t \geq 0$, there must exist (s,s',i) such that $(s,s') \succ_i (\mu^t(i), \mu^{t+1}(i))$, $s \neq \mu^t(i)$, $s' \neq \mu^{t+1}(i)$ and one of the following conditions are satisfied:

1. $|\mu^t(s)| < r_s$ and $\left|\mu^{t+1}(s')\right| < r_{s'}$,
2. $|\mu^t(s)| < r_s$, $\left|\mu^{t+1}(s')\right| = r_{s'}$, and, for some $j' \in \mu^{t+1}(s')$, $i \rhd_{s'}^{t+1} (\bar{\mu}^t) j'$ where $\bar{\mu}^t$ is the period t matching with $\bar{\mu}^t(i) = s$ and $\bar{\mu}^t(i') = \mu^t(i')$ for all $i' \neq i \in I^{t-1} \cup I^t$,
3. $|\mu^t(s)| = r_s$, $\left|\mu^{t+1}(s')\right| < r_{s'}$, and, for some $j \in \mu^t(s)$, $i \rhd_s^t (\mu^{t-1}) j$,
4. $|\mu^t(s)| = r_s$, $\left|\mu^{t+1}(s')\right| = r_{s'}$, for some $j \in \mu^t(s)$, $j' \in \mu^{t+1}(s')$ and for any $\bar{\mu}^t \in M(i,j,\mu)$, $i \rhd_s^t (\mu^{t-1}) j$ and $i \rhd_{s'}^{t+1} (\bar{\mu}^t) j'$.

Case 1. $s = s'$. Consequently, $(s,s) \succ_i (\mu^t(i), \mu^{t+1}(i))$. In addition, $|\mu^t(s)| < r_s$ (conditions 1 or 2) or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$ (conditions 3 or 4). Combining this with μ being weakly stable, one obtains that $(\mu^t(i), \mu^{t+1}(i)) \succ_i$

$(s,\mu^{t+1}(i))$. Given weak separability, this, in turn, implies that if $\mu^t(i) \neq \mu^{t+1}(i)$ then $(\mu^t(i),\mu^t(i)) \succ_i (s,s)$. Then, by transitivity of preferences, $(\mu^t(i),\mu^t(i)) \succ_i (\mu^t(i),\mu^{t+1}(i))$. This implies that μ is not weakly stable because child i has the highest priority at school s at period $t+1$, hence, at $t+1$, she has a right to attend school s ahead of any other child. Therefore, $\mu^t(i) = \mu^{t+1}(i)$. This is the condition we seek.

Case 2. $s \neq s'$ and $\mu^t(i) = \mu^{t+1}(i)$. Consequently, $(s,s') \succ_i (\mu^t(i),\mu^t(i))$. In addition, $|\mu^t(s)| < r_s$ or/and $i \rhd^t_s (\mu^{t-1}) j$ for some $j \in \mu^t(s)$. Combining this with μ being weakly stable, one obtains $(\mu^t(i),\mu^t(i)) \succ_i (s,\mu^t(i))$. Recall that $(s,s') \succ_i (\mu^t(i),\mu^t(i))$. Hence, by transitivity, $(s,s') \succ_i (s,\mu^t(i))$. Then, by weak separability, $(s',s') \succ_i (\mu^t(i),\mu^t(i))$. Suppose $(s,s) \succ_i (s',s')$. Then $(s,s) \succ_i (\mu^t(i),\mu^t(i))$ and, by assumption, $|\mu^t(s)| < r_s$ or/and $i \rhd^t_s (\mu^{t-1}) j$ for some $j \in \mu^t(s)$. Hence, we have identified a pair (s,i) asked in the lemma.

Now suppose $(s',s') \succ_i (s,s)$. Since μ is weakly stable, at least one of the two conditions must hold: (a) $(\mu^t(i),\mu^t(i)) \succ_i (\mu^t(i),s')$ or/and (b) $|\mu^{t+1}(s')| = r_{s'}$ and there exists no $j' \in \mu^{t+1}(s')$ such that $i \rhd^{t+1}_{s'} (\mu^t) j'$.

Suppose (a) occurs. Recall $(s,s') \succ_i (\mu^t(i),\mu^t(i))$, hence, $(s,s') \succ_i (\mu^t(i),s')$. Then weak separability implies that $(s,s) \succ_i (\mu^t(i),\mu^t(i))$ because $s \neq s'$. Observe that the pair (s,i) is the pair asked in the lemma as we already pointed out that $(s,s) \succ_i (\mu^t(i),\mu^t(i))$, $|\mu^t(s)| < r_s$ or/and $i \rhd^t_s (\mu^{t-1}) j$ for some $j \in \mu^t(s)$.

Suppose now (b) occurs but not (a). Recall that one of the 4 conditions listed in the beginning of the proof must be satisfied. Since $|\mu^{t+1}(s')| = r_{s'}$, 1 and 3 are ruled out. If condition 2 is satisfied, then $i \rhd^{t+1}_{s'} (\bar{\mu}^t) j'$ for some $j' \in \mu^{t+1}(s')$. Furthermore, $\bar{\mu}^t$ differs from μ^t only in that $\bar{\mu}^t(i) = s$. Then, by *IPA*, $i \rhd^{t+1}_{s'} (\mu^t) j'$. This a contradiction with b occurring. If condition 4 is satisfied, then there must exist j, j' such that, for any $\bar{\mu}^t \in M(i,j,\mu)$, $i \rhd^t_s (\mu^{t-1}) j$ and $i \rhd^{t+1}_{s'} (\bar{\mu}^t) j'$. In particular, it must be true for $\bar{\mu}^t$ such that $\bar{\mu}^t(j) = h$. Observe that $\bar{\mu}^t$ differs from μ^t only in that $\bar{\mu}^t(i) = s$ and $\bar{\mu}^t(j) = h$. By *IPA*, $i \rhd^{t+1}_{s'} (\mu^t) j'$. This a contradiction with b occurring.

Case 3. $s \neq s'$ and $\mu^t(i) \neq \mu^{t+1}(i)$. Consequently, $(s,s') \succ_i (\mu^t(i),\mu^{t+1}(i))$. Since μ is weakly stable, one of the two conditions must hold: (a) $(\mu^t(i),\mu^{t+1}(i)) \succ_i (\mu^t(i),s')$ or/and (b) $|\mu^{t+1}(s')| = r_{s'}$ and no $j' \in \mu^{t+1}(s')$ with $i \rhd^{t+1}_{s'} (\mu^t) j'$ exists.

Suppose (a) occurs. Recall that by assumption, in case 3, $(s,s') \succ_i (\mu^t(i),\mu^{t+1}(i))$, hence, $(s,s') \succ_i (\mu^t(i),s')$. Weak separability and this imply $(s,s) \succ_i (\mu^t(i),\mu^t(i))$. Then, $(s,\mu^{t+1}(i)) \succ_i (\mu^t(i),\mu^{t+1}(i))$ by weak separability. Consider the pair (s,i). As pointed out earlier, $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$. This means that μ is not weakly stable which is a contradiction.

Suppose now (b) occurs but not (a), therefore $(\mu^t(i),s') \succ_i (\mu^t(i),\mu^{t+1}(i))$. Recall that $(s,s') \succ_i (\mu^t(i),\mu^{t+1}(i))$. In addition, one of the 4 conditions listed in the beginning of the proof must be satisfied. Since $|\mu^{t+1}(s')| = r_{s'}$, 1 and 3 are ruled out. If condition 2 is satisfied, then $i \rhd_{s'}^{t+1} (\bar{\mu}^t) j'$ for some $j' \in \mu^{t+1}(s')$. Furthermore, $\bar{\mu}^t$ differs from μ^t only in that $\bar{\mu}^t(i) = s$. By *IPA*, $i \rhd_{s'}^{t+1} (\mu^t) j'$. This is a contradiction with (b) occurring. If condition 4 is satisfied, then there must exist j, j' such that, for any $\bar{\mu}^t \in M(i,j,\mu)$, $i \rhd_s^t (\mu^{t-1}) j$ and $i \rhd_{s'}^{t+1} (\bar{\mu}^t) j'$. Fix $\bar{\mu}^t$ such that $\bar{\mu}^t(j) = h$. Observe that $\bar{\mu}^t$ differs from μ^t only in that $\bar{\mu}^t(i) = s$ and $\bar{\mu}^t(j) = h$. By *IPA*, $i \rhd_{s'}^{t+1} (\mu^t) j'$. This is a contradiction with (b) occurring. □

Next we show that the solution concept for the daycare assignment problem, the strong stability, is in fact equivalent to the static concept of weak stability for a large class of problems. Precisely, if the children's preferences are *separable* and the schools' priorities satisfy *IPA*, the two concepts are equivalent.

Theorem 4 (Equivalence of Weak and Strong Stability)**.** *Suppose every child's preferences satisfy* separability *and every school's priorities satisfy* IPA. *Then matching μ is strongly stable if and only if it is weakly stable.*

Proof. By definition, any strongly stable matching is weakly stable. Hence, we need to show that any weakly stable matching is strongly stable. Suppose otherwise, i.e., there exists a weakly stable matching μ which is not strongly stable. By Lemma 2, if μ is weakly but not strongly stable, then for some period t and some school-child pair (s,i),

1. $\mu^t(i) = \mu^{t+1}(i)$,
2. $(s,s) \succ_i (\mu^t(i),\mu^{t+1}(i))$,
3. $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$.

Clearly, $(s,s) \succ_i (\mu^t(i), \mu^t(i))$. In addition, each child's preferences are *separable*, hence, $(s, \mu^t(i)) \succ_i (\mu^t(i), \mu^t(i))$. By combining this with the 3rd condition above, one obtains that μ is not weakly stable. $\square$

8 Appendix B: Properties of the Gale and Shapley Matching

In static settings, one of the most significant results is that the GS matching Pareto dominates all other stable matchings.[15] This result is no longer valid in our daycare assignment problem. In fact, there could be multiple weakly/strongly stable matchings that do not Pareto dominate one another. The following example illustrates this point.

Example 6 (GS matching might not Pareto dominate other stable matchings)**.** *There are 3 schools $\{s, s_1, s_2\}$. All schools have a capacity of one child. There is no school-age child until period $t-1$. At period $t-1$, only one child i is 1 year old. At period t, there are 2 one-year old children $\{i_1, i_2\}$. At period $t+1$, child i' is 1 year old. If children $\bar{\imath} \neq \bar{\imath}' \in \{i, i_1, i_2, i'\}$ have not attended school $\bar{s} = s, s_1, s_2$ in the previous period, then school $\bar{s}$ ranks child $\bar{\imath}$ and child $\bar{\imath}'$ according to the following rankings.*

$$\begin{array}{ccccccc} i & \rhd_s & i_1 & \rhd_s & i_2 & \rhd_s & i' \\ i & \rhd_{s_1} & i' & \rhd_{s_1} & i_2 & \rhd_{s_1} & i_1 \\ i & \rhd_{s_2} & i_1 & \rhd_{s_2} & i_2 & \rhd_{s_2} & i' \end{array}$$

Each child's preferences are separable. *Child i's top choice is (s,s). The preferences of children i_1, i_2 and i' satisfy the following conditions:*

$$\begin{array}{ccccc} (s_1,s_1) & \succ_{i_1} & (s_2,s_2) & \succ_{i_1} & (s,s), \\ (s,s) & \succ_{i_2} & (s_2,s_2) & \succ_{i_2} & (s_1,s_1), \\ (s_1,s_1) & \succ_{i'} & (s_2,s_2) & \succ_{i'} & (s,s). \end{array}$$

The GS matching μ_{GS} is as follows: $\mu_{GS}^{t-1}(i) = \mu_{GS}^t(i) = s$, $\mu_{GS}^t(i_1) = \mu_{GS}^{t+1}(i_1) =$

[15] See Gale and Shapley (1962).

s_1, $\mu^t_{GS}(i_2) = s_2$, $\mu^{t+1}_{GS}(i_2) = s$, $\mu^{t+1}_{GS}(i') = s_2$ and $\mu^{t+2}_{GS}(i') = s_1$. Thanks to Theorem 2, μ_{GS} is weakly stable.

Now let us consider the following matching $\bar{\mu}$: $\bar{\mu}^{t-1}(i) = \bar{\mu}^t(i) = s$, $\bar{\mu}^t(i_1) = \bar{\mu}^{t+1}(i_1) = s_2$, $\bar{\mu}^t(i_2) = s_1$, $\bar{\mu}^{t+1}(i_2) = s$, $\bar{\mu}^{t+1}(i') = s_1$ and $\bar{\mu}^{t+2}(i') = s_1$. It easy to check $\bar{\mu}$ is strongly stable.

Now observe that matching μ_{GS} does not Pareto dominate matching $\bar{\mu}$ because child i' prefers $\bar{\mu}$ to μ. In fact, $\bar{\mu}$ is not Pareto dominated by any strongly stable matching. To see this, observe that the only matching that Pareto dominates $\bar{\mu}$ is the one in which children 1 and 2 switch their matches in period t. But this is not strongly stable because child i_1 justifiably envies child i' at $t+1$. ◇

First observe that in Example 6 both *IPA* and *separability* are satisfied. Hence, the weakly and strongly stable matchings coincide. Hence, the example above shows that there may exist mechanisms that produce strongly/weakly stable matchings not Pareto dominated by the GS matching. This is the first main distinction between the matching produced by the GS algorithm in the school choice problem versus the daycare assignment problem.

Given the importance of this result when compared to the static case, we state the result below.

Theorem 5 (The GS matching does not necessarily Pareto dominate all stable matchings)**.** *The GS matching does not necessarily Pareto dominate all weakly and strongly stable matchings.*

In the light of Example 6, one must explore whether any strongly stable matching Pareto dominates the GS matching. This, indeed, is impossible which we show in the following proposition.

Proposition 3 (The GS matching is not Pareto dominated by any strongly stable matching)**.** *If each school's priority rankings satisfy* IPA*, then the GS matching is not Pareto dominated by any other strongly stable matchings.*

Proof. Recall that time -1 matching μ^{-1} is fixed for all matchings we consider.

On contrary to the proposition, suppose that some strongly stable matching μ Pareto dominates matching μ_{GS}.

Step 1. If $i \in I_{-1}$, then $\mu^0_{GS}(i) = \mu^0(i)$.

Proof of Step 1. For any 2 year old child, her isolated preference is $\succ^2_i (\mu^{-1})$. From Lemma 1, we have that μ^0_{GS} and μ^0 are stable period 0 matchings under isolated preferences and μ^{-1}. Gale and Shapley (1962) show that μ^0_{GS} Pareto dominates every other statically stable period 0 matchings under isolated preferences and μ^{-1} in terms of isolated preferences. This means $\mu^0_{GS}(i) \succ^2_i (\mu^{-1})\mu^0(i)$ if $\mu^0_{GS}(i) \neq \mu^0(i)$. By definition of $\succ^2_i (\mu^{-1})$, $(\mu^{-1}(i), \mu^0_{GS}(i)) \succ_i (\mu^{-1}(i), \mu^0(i))$ if $\mu^0_{GS}(i) \neq \mu^0(i)$. Hence, if μ Pareto dominates μ_{GS}, then $\mu^0_{GS}(i) = \mu^0(i)$.

Step 2. If $i \in I_0$, then $\mu^0_{GS}(i) = \mu^0(i)$.

Proof of Step 2. Suppose $\mu^0_{GS}(i) \neq \mu^0(i)$ for some $i \in I_0$. Then, as in the proof of step 1, we obtain that $\mu^0_{GS}(i) \succ^1_i \mu^0(i)$ or equivalently,

$$(\mu^0_{GS}(i), \mu^0_{GS}(i)) \succ_i (\mu^0(i), \mu^0(i)). \tag{1}$$

The strong stability of μ_{GS} implies $(\mu^0_{GS}(i), \mu^1_{GS}(i)) \succeq_i (\mu^0_{GS}(i), \mu^0_{GS}(i))$; otherwise, μ_{GS} is not even weakly stable as child i is in the highest priority group in time 1. Now weak separability yields $(\mu^1_{GS}(i), \mu^1_{GS}(i)) \succeq_i (\mu^0_{GS}(i), \mu^0_{GS}(i))$. Now it is easy to see that

$$(\mu^1_{GS}(i), \mu^1_{GS}(i)) \succeq_i (\mu^0_{GS}(i), \mu^1_{GS}(i)) \succeq_i (\mu^0_{GS}(i), \mu^0_{GS}(i)). \tag{2}$$

Similarly, as μ is strongly stable, we obtain

$$(\mu^1(i), \mu^1(i)) \succeq_i (\mu^0(i), \mu^1(i)) \succeq_i (\mu^0(i), \mu^0(i)). \tag{3}$$

Now let us show that $\mu^0(i) \neq \mu^1(i)$. Suppose otherwise. Then relations 1 and 2 yield that $(\mu^0_{GS}(i), \mu^1_{GS}(i)) \succ_i (\mu^0(i), \mu^0(i))$. This contradicts with μ Pareto dominating μ_{GS}. Hence, $\mu^0(i) \neq \mu^1(i)$. Consequently, the preference relations in 3 must be strict. Also observe that $\mu^0(i) \neq \mu^1_{GS}(i)$ thanks to relations 1 and 3.

Now let us show that $(\mu^1(i), \mu^1(i)) \succ_i (\mu^1_{GS}(i), \mu^1_{GS}(i))$. If not, weak separability and relation 1 yield that $(\mu^0_{GS}(i), \mu^1_{GS}(i)) \succeq_i (\mu^0(i), \mu^1_{GS}(i))$ and $(\mu^0(i), \mu^1_{GS}(i)) \succeq_i (\mu^0(i), \mu^1(i))$ as $\mu^0(i) \neq \mu^1_{GS}(i)$ and $\mu^0(i) \neq \mu^1(i)$. Consequently, $(\mu^0_{GS}(i), \mu^1_{GS}(i)) \succeq_i (\mu^0(i), \mu^1(i))$ which contradicts that μ Pareto dominates μ_{GS}. Now let us summarize the preference relation we found so far.

$$(\mu^1(i),\mu^1(i)) \succ_i (\mu^1_{GS}(i),\mu^1_{GS}(i)) \succeq_i (\mu^0_{GS}(i),\mu^0_{GS}(i)) \succ_i (\mu^0(i)),\mu^0(i)) \tag{4}$$

From Lemma 1, we know that μ^1 is statically stable under isolated preferences and μ^0. Now suppose we ran the GS algorithm at period 1 under isolated preferences and μ^0. Let us denote the resulting matching $\bar{\mu}^1$. From Gale and Shapley (1962), we know that if $\bar{\mu}^1(i) \neq \mu^1(i)$, then $\bar{\mu}^1(i) \succ^2_i (\mu^0)\mu^1(i)$. In other words, $(\mu^0(i),\bar{\mu}^1(i)) \succeq_i (\mu^0(i),\mu^1(i))$. This along with relation 1 and $\mu^0(i) \neq \mu^1(i)$ implies that $\bar{\mu}^1(i) \neq \mu^0(i)$. Then by weak separability, $(\mu^0(i),\bar{\mu}^1(i)) \succ_i (\mu^0(i),\mu^1(i))$ implies $(\bar{\mu}^1(i),\bar{\mu}^1(i)) \succ_i (\mu^1(i),\mu^1(i))$. Now let us update relation 4.

$$\begin{aligned}(\bar{\mu}^1(i),\bar{\mu}^1(i)) &\succ_i (\mu^1(i),\mu^1(i)) \\ &\succ_i (\mu^1_{GS}(i),\mu^1_{GS}(i)) \succ_i (\mu^0_{GS}(i),\mu^0_{GS}(i)) \succ_i (\mu^0(i)),\mu^0(i))\end{aligned} \tag{5}$$

Next we will proceed to show that $\bar{\mu}^1$ is statically stable under isolated preferences and μ^0_{GS}. Let us postpone the proof momentarily to discuss its implications. From Lemma 1, we know that μ^1_{GS} is a stable matching under isolated preferences and μ^0_{GS}. In addition, it must Pareto dominate $\bar{\mu}^1$ in terms of the isolated preferences, since $\bar{\mu}^1$ is statically stable and the μ^1_{GS} must Pareto dominate all stable matchings (see Gale and Shapley (1962)). Hence, if $\mu^1_{GS}(i) \neq \bar{\mu}^1(i)$, then $\mu^1_{GS}(i) \succ^2_i (\mu^0_{GS})\bar{\mu}^1(i)$. By the definition of $\succ^2_i (\mu^0_{GS})$, $(\mu^0_{GS}(i),\mu^1_{GS}(i)) \succ_i (\mu^0_{GS}(i),\bar{\mu}^1(i))$. Recalling that $(\mu^0_{GS}(i),\mu^0_{GS}(i)) \succ_i (\mu^0(i),\mu^0(i))$, we find that $(\mu^0_{GS}(i),\bar{\mu}^1(i)) \succ_i (\mu^0(i),\bar{\mu}^1(i))$. Weak separability and $(\bar{\mu}^1(i),\bar{\mu}^1(i)) \succ_i (\mu^1(i),\mu^1(i))$ yield $(\mu^0(i),\bar{\mu}^1(i)) \succ_i (\mu^0(i),\mu^1(i))$. The previous three relations yield $(\mu^0_{GS}(i),\mu^1_{GS}(i)) \succ_i (\mu^0(i),\mu^1(i))$. However, recall that μ Pareto dominates μ_{GS}. This is the contradiction we are looking for. Thus, to complete the proof, it is left to show that $\bar{\mu}^1$ is statically stable under isolated preferences and μ^0_{GS}.

We now proceed to show that $\bar{\mu}^1$ is indeed a stable matching under isolated preferences and μ^0_{GS}. We already know from Assumption 1 and (5) that, for all $i \in I_0$, $\bar{\mu}^1(i) \succ^2_i (\mu^0)\mu^1(i)$ if $\bar{\mu}^1(i) \neq \mu^1(i)$. Also, from Gale and Shapley (1962), we know that, for all $i \in I_1$, $\bar{\mu}^1(i) \succ^1_i \mu^1(i)$ if $\bar{\mu}^1(i) \neq \mu^1(i)$. Recall that $\bar{\mu}^1$ is statically stable matching under isolated preferences and μ^0. Now consider the isolated preferences in period 1 from μ^0_{GS} and suppose, under these isolated preferences, $\bar{\mu}^1$ is not stable. Therefore, there must exist a school-child pair (s,i) such that both conditions are satisfied:

I. – if $i \in I_0$, then $s \succ_i^2 (\mu^0_{GS})\bar{\mu}^1(i)$, or
   – if $i \in I_1$, then $s \succ_i^1 \bar{\mu}^1(i)$;

II. $|\bar{\mu}^1(s)| < |r_s|$ or/and $i \rhd_s^1 (\mu^0_{GS})\, j$ for some $j \in \bar{\mu}^1(s)$.

Because $\bar{\mu}^1$ statically stable under the isolated preferences and μ^0, the conditions 1 and 2 below cannot be satisfied at the same time.

1. (a) if $i \in I_0$, then $s \succ_i^2 (\mu^0)\bar{\mu}^1(i)$, or
   (b) if $i \in I_1$, then $s \succ_i^1 \bar{\mu}^1(i)$.
2. $|\bar{\mu}^1(s)| < r_s$ or/and $i \rhd_s^1 (\mu^0)\, j$ for some $j \in \bar{\mu}^1(s)$.

Suppose $i \in I_0$. Then $s \succ_i^2 (\mu^0_{GS})\bar{\mu}^1(i)$. We show that in this case condition $1\,(a)$ is satisfied. By the definition of $\succ_i^2 (\mu^0_{GS})$,

$$(\mu^0_{GS}(i), s) \succ_i (\mu^0_{GS}(i), \bar{\mu}^1(i)).$$

If $\mu^0(i) = \mu^0_{GS}$, then

$$(\mu^0(i), s) \succ_i (\mu^0(i), \bar{\mu}^1(i)).$$

This means that condition 1a is satisfied. Let $\mu^0(i) \neq \mu^0_{GS}$. Then preference relations given in (5), Assumption 1,

$$(\mu^0_{GS}(i), s) \succ_i (\mu^0_{GS}(i), \bar{\mu}^1(i))$$

and the fact that

$$(s,s) \succ_i \left(\bar{\mu}^1(i), \bar{\mu}^1(i)\right)$$

imply that

$$(\mu^0(i), s) \succ_i (\mu^0(i), \bar{\mu}^1(i)).$$

Hence, condition $1\,(a)$ is satisfied. Suppose $i \in I_1$. Then $s \succ_i^1 \bar{\mu}^1(i)$. Since $\succ^1$ does not depend on the last period's matching, condition $1\,(b)$ is satisfied. Therefore, we find that either $1\,(a)$ or $1\,(b)$ is satisfied. This means that 2 cannot be satisfied. Clearly, it must be that $|\bar{\mu}^1(s)| = r_s$. This implies that school s's priority ranking must satisfy $i \rhd_s^1 (\mu_{GS}^0) j$ and $j \rhd_s^1 (\mu^0) i$, for at least some $j \in \bar{\mu}^1(s)$. There are 2 cases consider:

(i) $i \notin \mu_{GS}^0(s)$, or

(ii) $i \in \mu_{GS}^0(s)$ and $i \in I_0$.

If case (i) happens, this implies that $j \notin \mu_{GS}^0(s)$; otherwise, j would have the highest priority at school s, hence, we reach a contradiction with $i \rhd_s^1 (\mu_{GS}^0) j$. Therefore, $j \notin \mu_{GS}^0(s)$. Since school s's priority ranking satisfies *IPA*, given that $i \rhd_s^1 (\mu_{GS}^0) j$ it must be that $j \in \mu^0(s)$ and $j \in I_0$ to have the required reversal of school s's priority ranking. Then $\mu_{GS}^0(j) \neq \mu^0(j)$. This, as argued earlier in step 1, implies that $(\mu_{GS}^0(j), \mu_{GS}^0(j)) \succ_j (\mu^0(j), \mu^0(j)) = (s,s)$, where the last equality comes from the fact above, that if $j \notin \mu_{GS}^0(s)$, it must be that $j \in \mu^0(s)$. Now recall that $j \in \bar{\mu}^1(s)$. Therefore,

$$(\mu_{GS}^0(j), \mu_{GS}^0(j)) \succ_j (\mu^0(j), \bar{\mu}^1(j))$$

which is a contradiction (see preference relation 5).

Suppose (ii) happens, $i \in \mu_{GS}^0(s)$, i.e., $s = \mu_{GS}^0(i)$. We know $s \succ_i^2 (\mu_{GS}^0) \bar{\mu}^1(i)$. These conditions yield

$$(\mu_{GS}^0(i), \mu_{GS}^0(i)) \succ_i (\mu_{GS}^0(i), \bar{\mu}^1(i)).$$

This is a contradiction which we are looking for.

This completes the proof of step 2.

Step 3. The GS algorithm yields a strongly stable matching that is not Pareto dominated by any other strongly stable matchings.

Proof of Step 3. Proving step 3 is just a matter of reiterating the arguments of steps 1 and 2 assuming previous periods' matchings are identical with the ones resulted from the GS algorithm. $\square$

Now we study if any strongly stable matching is efficient. The next proposition yields that unless one follows the GS algorithm, then any strongly stable matching is not efficient.

Proposition 4. *Suppose that the priority rankings of all schools satisfy* IPA. *Then any strongly stable matching different from the GS matching is not efficient.*

Proof. Consider any strongly stable matching μ with some period t matching that is different from the one that the GS algorithm under isolated preferences and μ^{t-1} yields. Consider any $i \in I_t$. Then $\mu^t(i) = \mu^{t+1}(i)$ or

$$(\mu^{t+1}(i), \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^t(i));$$

otherwise, μ is not strongly stable because, in this case, child i would have the higher priority at school $\mu^t(i)$ and

$$(\mu^t(i), \mu^t(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$$

by Assumption 1.

For each child $i \in I_{t-1} \cup I_t$, define her preference relation to be $\mathcal{P}_i^t$ such that $s\mathcal{P}_i^t s'$ if and only if

$$(\mu^{t-1}(i), s) \succ_i (\mu^{t-1}(i), s') \text{ whenever } i \in I_{t-1}$$

$$(s, \mu^{t+1}(i)) \succ_i (s', \mu^{t+1}(i)) \text{ whenever } i \in I_t$$

Because μ is strongly stable, there cannot exist any school-child pair (s,i) such that

1. $(\mu^{t-1}(i), s) \succ_i (\mu^{t-1}(i), \mu^t(i))$ or $(s, \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$,

2. $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$.

In terms of $\mathcal{P}$, these conditions mean that there is no school-child pair (s,i) such that

1. $s\mathcal{P}_i^t \mu^t(i)$,

2. $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$.

In other words, μ^t is a statically stable matching under $\mathcal{P}$ and μ^{t-1}.

Consider matching $\bar{\mu}$ such that $\bar{\mu}^\tau = \mu^\tau$ for all $\tau \neq t$ but $\bar{\mu}^t$ is the resulting matching from the GS algorithm under $\mathcal{P}$ and μ^{t-1}.

From Gale and Shapley (1962), we know that $\bar{\mu}^t$ must Pareto dominate every other stable matching under $\mathcal{P}$ and μ^{t-1}. This and that μ^t is a statically stable matching under $\mathcal{P}$ and μ^{t-1} imply that $\bar{\mu}^t(i)\mathcal{P}_i\mu^t(i)$ for all $i \in I_{t-1} \cup I_t$ if $\bar{\mu}^t(i) \neq \mu^t(i)$. Consequently, if $\bar{\mu}^t(i) \neq \mu^t(i)$ for some $i \in I_{t-1}$, then $(\mu^{t-1}(i), \bar{\mu}^t(i)) \succ_i (\mu^{t-1}(i), \mu^t(i))$. Similarly, if $\bar{\mu}^t(i) \neq \mu^t(i)$ for some $i \in I_t$ then

$$(\bar{\mu}^t(i), \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^{t+1}(i)).$$

Now consider $\bar{\mu}$ and μ. Clearly, $\bar{\mu}$ Pareto dominates μ if $\bar{\mu}^t(i) \neq \mu^t(i)$ for some $i \in I_{t-1} \cup I_t$. Hence, it must be that $\bar{\mu}^t(i) = \mu^t(i)$ for all $i \in I_{t-1} \cup I_t$.

Consider $\hat{\mu}$ such that $\hat{\mu}^\tau = \mu^\tau$ for all $\tau \neq t$ but $\hat{\mu}^t$ is the resulting matching from the GS algorithm under isolated preferences and $\hat{\mu}^{t-1}$. Clearly, $\bar{\mu}^{t-1} = \hat{\mu}^{t-1}$, hence, the priority rankings of the schools are the same under both $\bar{\mu}$ and $\hat{\mu}$. In addition, for each $j \in I_{t-1}$, the isolated preference relation $\succ_j^2 (\mu^{t-1})$ is equivalent to $\mathcal{P}_j$. Now consider any child $j \in I_t$. Then under $\mathcal{P}$, the relative ranking of $\mu^{t+1}(j)$ weakly improves from the one under $\succ_j^1$. In all other aspects, $\mathcal{P}_j$ and $\succ_j^1$ are the same. Now recall that $\bar{\mu}^t(i) = \mu^t(i)$ for all $i \in I_{t-1} \cup I_t$. In addition, recall that $\mu^t(i) = \mu^{t+1}(i)$ or

$$(\mu^{t+1}(i), \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^t(i)).$$

Therefore, under both $\mathcal{P}_j$ and $\succ_j^1$, the set of schools that are strictly preferred to $\mu^t(j)$ is the same. Consequently, we obtain that under $\mathcal{P}$ and isolated preferences, for each $j \in I^{t-1} \cup I^t$, the set of schools that are strictly preferred to $\mu^t(j)$ is the same. In addition, because the GS algorithm is used for both cases and $\bar{\mu}^t(j) = \mu^t(j)$ for all $j \in I_{t-1} \cup I_t$, it must be $\bar{\mu}^t = \hat{\mu}^t$ thanks to Theorem 9 in Dubins and Freedman. Consequently, $\mu^t = \hat{\mu}^t$, which contradicts that μ^t differs from the matching that the GS algorithm yields. $\square$

Proposition 4 means that if any strongly stable matching is efficient, then it must be the GS matching. However, from Roth, it is well known that the GS matching (in static settings) is not necessarily Pareto efficient. This is still the case in our setting because the school choice problem is a special case of our problem as we pointed out earlier.

Appendix C: Proofs

Proof of Lemma 1. Necessity. Assume μ is strongly stable. We need to show that for all t, μ^t is statically stable under isolated preferences and μ^{t-1}. Suppose otherwise. Then there must exist, t, and a school-child pair (s,i) such that

1. if $i \in I_t$, then $s \succ_i^1 \mu^t(i)$ and at least one of the following is satisfied: $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$,
2. if $i \in I_{t-1}$, then $s \succ_i^2 \left(\mu^{t-1}\right) \mu^t(i)$ and at least one of the following is satisfied: $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$.

Suppose $i \in I_t$. Then we are in case 1. Since μ is weakly stable, the following 2 conditions cannot be satisfied at the same time: (a) $(s, \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$ and (b) $|\mu^t(s)| < r_s$ and/or $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$. If (b) is not true, then this is a contradiction because (s,i) must satisfy the conditions given in case 1. Hence, assume that (b) is satisfied but (a) is not, i.e., $(\mu^t(i), \mu^{t+1}(i)) \succ_i (s, \mu^{t+1}(i))$. If $\mu^t(i) \neq \mu^{t+1}(i)$, Assumption 1 implies that $(\mu^t(i), \mu^t(i)) \succ_i (s,s)$. By the definition of $\succ^1$, $\mu^t(i) \succ_i^1 s$ which contradicts with the assumption that $s \succ_i^1 \mu^t(i)$. Suppose $\mu^t(i) = \mu^{t+1}(i)$. Recall that $s \succ_i^1 \mu^t(i)$, hence, $(s,s) \succ_i (\mu^t(i), \mu^{t+1}(i))$. Recall that (b) is satisfied. Thus, by moving to school s in period t, child i would have the highest priority at school s at time $t+1$. Hence, μ is not strongly stable. Hence, $i \notin I_t$.

Suppose $i \in I_{t-1}$. Then we are in case 2. Because μ is weakly stable, the following 2 conditions cannot be satisfied at the same time: (a) $(\mu^{t-1}(i), s) \succ_i (\mu^{t-1}(i), \mu^t(i))$ and (b) $|\mu^t(s)| < r_s$ and/or $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$. If (b) is not true, then this is a contradiction because (s,i) must satisfy the conditions given in case 2. Hence, (b) must be satisfied but (a) is not, i.e., $(\mu^{t-1}(i), \mu^t(i)) \succ_i (\mu^{t-1}(i), s)$. By the definition of $\succ_i^2 (\mu^{t-1})$, we have that $\mu^t(i) \succ_i^2 (\mu^{t-1}) s$ which contradicts with the assumption that $s \succ_i^2 \left(\mu^{t-1}\right) \mu^t(i)$. Hence, $i \notin I_{t-1}$. Therefore, for all t, μ^t is statically stable under isolated preferences and μ^{t-1}.

Sufficiency. For any t, μ^t is statically stable under isolated preferences and μ^{t-1}. First let us show that μ is weakly stable. Suppose otherwise. Then, at some period t, there must exist a pair (s,i) such that one of the two conditions below is satisfied:

1. (a) $(s,\mu^{t+1}(i)) \succ_i (\mu^t(i),\mu^{t+1}(i))$, and

   (b) $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$.

   or

2. (a) $(\mu^{t-1}(i),s) \succ_i (\mu^{t-1}(i),\mu^t(i))$, and

   (b) $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$.

Suppose case 1 occurs. If $s \neq \mu^{t+1}(i)$, then weak separability and

$$(s,\mu^{t+1}(i)) \succ_i \left(\mu^t(i),\mu^{t+1}(i)\right)$$

yield $(s,s) \succ_i (\mu^t(i),\mu^t(i))$. By definition of $\succ_i^1$, we have that $s \succ_i^1 \mu^t(i)$. This and $1b$ mean that μ^t is not statically stable under isolated preferences and μ^{t-1}. This is a contradiction. Suppose, on the other hand, that $s = \mu^{t+1}(i)$. If

$$(\mu^{t+1}(i),\mu^{t+1}(i)) \succ_i (\mu^t(i),\mu^t(i)),$$

then the definition of $\succ_i^1$ yields $\mu^{t+1}(i) \succ_i^1 \mu^t(i)$. This and 1b mean that μ^t is not statically stable under isolated preferences and μ^{t-1}.

Suppose $(\mu^t(i),\mu^t(i)) \succ_i (\mu^{t+1}(i),\mu^{t+1}(i))$. This and Assumption 1 yield

$$(\mu^t(i),\mu^t(i)) \succ_i \left(\mu^t(i),\mu^{t+1}(i)\right).$$

Consider period $t+1$. Then by the definition of $\succ_i^2(\mu^t)$, we have that $\mu^t(i) \succ_i^2 (\mu^t)\mu^{t+1}(i)$. In addition, observe that child i has the highest priority at school $\mu^t(i)$. The last 2 conditions contradict that μ^{t+1} is statically stable under isolated preferences and μ^t.

Suppose case 2 occurs. By the definition of $\succ_i^2 (\mu^{t-1})$, we have that $s \succ_i^2 (\mu^{t-1})\mu^t(i)$ since $(\mu^{t-1}(i),s) \succ_i (\mu^{t-1}(i),\mu^t(i))$. But this and $2b$ directly imply that μ^t is not statically stable under isolated preferences and μ^{t-1}. This is a contradiction.

We have shown that μ is weakly stable. Now we are left to show that μ is strongly stable if *IPA* is satisfied. Suppose otherwise. Then by Lemma 2, for some period t and some school-child pair (s,i),

1. $\mu^t(i) = \mu^{t+1}(i)$
2. $(s,s) \succ_i (\mu^t(i),\mu^{t+1}(i))$
3. $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$

The first 2 conditions and the definition of $\succ_i^1$ yield $s \succ_i^1 \mu^t(i)$. This and the third condition imply that μ^t is not statically stable under isolated preferences and μ^{t-1}. $\square$

Proof of Theorem 3. Here we provide the formal proof, using the example given in the text. All we need for this proof is to show that weakly stable matchings under profile 1 and 2 are unique and the ones shown in steps 1 and 2 respectively.

Step 1. Under profile 1, the only weakly stable matching μ is: $\mu^{t-1}(i) = \mu^t(i) = s$, $\mu^{t-1}(\bar{\imath}) = \mu^t(\bar{\imath}) = \bar{s}$, $\mu^t(i_1) = \mu^{t+1}(i_1) = s_1$, $\mu^t(i_2) = \mu^{t+1}(i_2) = s_2$, $\mu^{t+1}(i') = s$ and $\mu^{t+2}(i') = s_2$.

Proof of Step 1. Let $\hat{\mu}$ be weakly stable. It is clear that $\hat{\mu}^{t-1}(i) = \hat{\mu}^t(i) = s$, $\hat{\mu}^{t-1}(\bar{\imath}) = \hat{\mu}^t(\bar{\imath}) = \bar{s}$ and $\hat{\mu}^{t+2}(i') = s_2$. Consequently, we obtain that $\hat{\mu}^t(i_1) = s_1$ because child i_1 has higher priority in school s_1 at period t than anyone but i. However, i must match with s at period t. Hence, $\hat{\mu}^t(i_1) = s_1$. This implies that $\hat{\mu}^t(i_2) = s_2$. Then i_2 has the highest priority at school s_2 at period $t+1$. Since s_2 is the top choice for i_2, $\hat{\mu}^{t+1}(i_2) = s_2$. Consequently, $\hat{\mu}^{t+1}(i') = s$ which means $\hat{\mu}^{t+1}(i_1) = s_1$. Now we have shown that $\hat{\mu} = \mu$.

Step 2. Under profile 2, the only weakly stable matching $\bar{\mu}$ is as follows: $\bar{\mu}^{t-1}(i) = \bar{\mu}^t(i) = s$, $\bar{\mu}^{t-1}(\bar{\imath}) = \bar{\mu}^t(\bar{\imath}) = \bar{s}$, $\bar{\mu}^t(i_1) = s_2$, $\bar{\mu}^t(i_2) = s_1$, $\bar{\mu}^{t+1}(i_1) = s$, $\bar{\mu}^{t+1}(i_2) = s_1$, $\bar{\mu}^{t+1}(i') = s_2$ and $\bar{\mu}^{t+2}(i') = s_2$.

Proof of Step 2. Let $\hat{\mu}$ be a weakly stable matching. It is clear that $\hat{\mu}^{t-1}(i) = \hat{\mu}^t(i) = s$, $\hat{\mu}^{t-1}(\bar{\imath}) = \hat{\mu}^t(\bar{\imath}) = \bar{s}$ and $\hat{\mu}^{t+2}(i') = s_2$. Consequently, we obtain that $\hat{\mu}^t(i_1) = s_2$ because child i_1 has higher priority in school s_2 at period t than i_2. This means that $\hat{\mu}^t(i_2) = s_1$.

Now let us argue that $\hat{\mu}^{t+1}(i') = s_2$. If not, $\hat{\mu}^{t+1}(i_1) = s_2$; otherwise, child i' has higher priority than child i_2 at school s_2 and s_2 is the top choice of child i'. Hence, this contradicts with $\hat{\mu}$ being weakly stable. Thus, $\hat{\mu}^{t+1}(i_1) = s_2$. But because $(s_2, \bar{s}) \succ_{i_1}^2 (s_2, s_2)$ and child i_1 has higher priority at school $\bar{s}$ than anyone but $\bar{\imath}$, $\hat{\mu}$ is weakly stable. This is a contradiction. Hence, $\hat{\mu}^{t+1}(i') = s_2$.

Because $\hat{\mu}^{t+1}(i') = s_2$, $\hat{\mu}^{t+1}(i_1) = s$ as i_1 has higher priority at school s than i_2. Consequently, $\hat{\mu}^{t+1}(i_2) = s_1$. This means $\hat{\mu} = \bar{\mu}$. □

Following the discussion of Section 6.1, we present an example in which a matching satisfies Autarkic efficiency but fails to be efficient because of the potential trades within a generation.

Example 7 (Pareto Improving Trade Within Cohort)**.** *Suppose in period* 0*, two children i_1 and i_2 are two years old and two children j_1 and j_2 are one year old. There are 4 schools s_1, s_2, s_3 and s_4 and each school has a capacity of 1 child. The schools' priorities are given as follows under the assumption that the children have not attended any school in the previous period:*

$$i_1 \rhd_{s_1} i_2 \rhd_{s_1} j_1 \rhd_{s_1} j_2$$
$$i_2 \rhd_{s_2} i_1 \rhd_{s_2} j_2 \rhd_{s_2} j_1$$
$$i_1 \rhd_{s_3} i_2 \rhd_{s_3} j_1 \rhd_{s_3} j_2$$
$$i_1 \rhd_{s_4} i_2 \rhd_{s_4} j_2 \rhd_{s_4} j_1$$

Child i_1's top choice is s_1 while child i_2's is s_2. The other two children's preferences satisfy the following conditions:

$$(s_2, s_2) \succ_{j_1} (s_1, s_1) \succ_{j_1} (s_4, s_2) \succ_{j_1} (s_3, s_1) \succ_{j_1} (s_3, s_3) \succ_{j_1} (s_4, s_4)$$
$$(s_2, s_2) \succ_{j_2} (s_1, s_1) \succ_{j_2} (s_3, s_1) \succ_{j_2} (s_4, s_2) \succ_{j_2} (s_3, s_3) \succ_{j_2} (s_4, s_4)$$

Now consider the following matching μ: $\mu^0(i_1) = s_1$, $\mu^0(i_2) = s_2$, $\mu^0(j_1) = s_3$, $\mu^0(j_2) = s_4$, $\mu^1(j_1) = s_1$, $\mu^1(j_2) = s_2$. Matching μ satisfies Autarkic efficiency. However, μ is not Pareto efficient as it is dominated by the matching $\bar{\mu}$: $\bar{\mu}^0(i_1) = s_1$, $\bar{\mu}^0(i_2) = s_2$, $\bar{\mu}^0(j_1) = s_4$, $\bar{\mu}^0(j_2) = s_3$, $\bar{\mu}^1(j_1) = s_2$, $\bar{\mu}^1(j_2) = s_1$.

Loosely speaking, in Example 7, children j_1 and j_2 are assigned "extreme" allocations under matching μ. Hence, these children j_1 and j_2 can hedge against the extreme allocations by "trading" their allocations. This is one reason why Autarkic efficiency is not equivalent to efficiency. One should observe that in this case trade happens between the children from the same generation. Hence, the infiniteness of time does not play any significant role in Example 7.