A Simple Model of Two-Stage Maximization*

Sean Horan, Université du Québec à Montréal†

April 23, 2012

Abstract

I study a minimal departure from the standard model of preference maximization where the decision-maker chooses in stages by sequentially maximizing *two* preferences that are asymmetric and transitive. This simple model has a wide variety of applications to individual decision-making – including multi-attribute choice, cognitive bias, and psychological phenomena such as temptation – as well as several models of collective choice.

I provide a number of results on the choice-theoretical foundations of the model including: (i) an axiomatic characterization; (ii) simple revealed preference definitions based on choice from small menus; (iii) identification and uniqueness results; and, (iv) results regarding a *comparative static* that permits meaningful comparisons among decision-makers.

JEL Codes: D01

Keywords: Two-Stage Choice; Shortlisting; Sequentially Rationalizable Choice; Multi-Attribute Choice; Temptation; Consideration Sets; Limited Attention; Choice Overload; Status Quo Bias; Axiomatization; Revealed Preference; Identification; Comparative Statics.

*Preliminary and Incomplete. I am grateful to Jawwad Noor and Kemal Yildiz for helpful discussions on the temptation literature and the use of existential quantification in decision theory. I am also grateful to seminar participants at the University of Minnesota for their feedback. Otherwise, the usual disclaimers apply.

†Contact Information: Département des sciences économiques, ESG UQAM, Case Postale 8888, Succursale Centre-Ville Montréal, QC, H3C 3P8, Canada; tel: 514-987-3000 (ext. 1780); e-mail: horan.sean@uqam.ca.

1 Introduction

This paper provides choice-theoretical foundations for a simple model of two-stage maximization. Formally, this model of *shortlisting* may be understood as a minimal departure from the classical theory of choice. Instead of maximizing a single preference, the decision-maker maximizes in stages by applying a pair of asymmetric and transitive rationales (P_1, P_2). For any menu, the decision-maker first eliminates those alternatives which are not maximal according to the first rationale P_1. From the remaining options, the decision-maker selects the alternative which maximizes the second rationale P_2.

The model has a wide variety of applications to individual decision-making – including multi-attribute choice, cognitive bias, and psychological phenomena such as temptation and willpower – as well as several models of collective choice.

Interpreted as a model of multi-attribute choice, for instance, the rationales reflect the ranking of choice alternatives on two product dimensions (Manzini and Mariotti [2007]). Instead of making trade-offs between the attributes, the decision-maker chooses by sequentially maximizing her preference for each. While the two dimensions may reflect objective features (like those used in the description of the product), they may also reflect aggregate measures of these features.

To illustrate, consider the choice among gambles (x, p) which provide a prize x with probability p.[1] Rather than evaluating the gambles holistically by using the expected utility $p \cdot u(x)$, recent eye-tracking evidence suggests that decision-makers choose by comparing the features of gambles directly (Arieli, Ben-Ami, and Rubinstein [2011]). One possible explanation is that the data is consistent with a two-stage procedure where (i) the first rationale P_1 captures the rough Pareto ordering of gambles that are unambiguously ranked on the prize and probability dimensions, and (ii) the second rationale P_2 reflects the pairwise ranking of gambles according to other considerations (Manzini and Mariotti [2007], Rubinstein [1988]).

In a different vein, shortlisting can be interpreted as a model of constrained choice where the first rationale captures a hidden budget restriction and the second rationale reflects the decision-maker's true preference.

One possibility is that the hidden constraint models a cognitive bias. Examples include biases such as *limited attention* and *choice overload*, which cause the decision-maker to focus on a consideration set of the feasible alternatives (Masatlioglu, Nakajima, and Ozbay [2010]; Lleras, Masatlioglu, Nakajima, and Ozbay [2011]), and the *status quo bias*, which causes the decision-maker to focus on a particular alternative as an initial benchmark for comparison (Masatlioglu and Ok [2005]; Apesteguia and Ballester [2010]).

Another possibility is that the hidden budget restriction models a psychological phenomenon,

[1] Similar comments apply to delayed rewards (x, t) (Rubinstein [2003]; Manzini, Mariotti, and Mittone [2010]).

such as *temptation* (Strotz [1956]; Gul and Pesendorfer [2001]; Dekel and Lipman [2010a]), *compromise* (Chandrasekhar [2010]), or *lack of willpower* (Masatlioglu, Nakajima, and Ozdenoren [2011]), which constrains the decision-maker. Interpreted as a model of overwhelming temptation, for instance, shortlisting describes a decision-maker who lacks self-control and selects the most preferred alternative among those which are tempting.

In terms of collective choice, the two rationales could reflect the competing objectives of Pareto efficiency and equity (Houy and Tadenuma [2009]), the preferences of two sub-committees involved in different stages of decision-making (Manzini and Mariotti [2006]), or the interplay between social norms, such as politeness (Sen [1977]) or repugnance (Roth [2007]), and individual satisfaction.

Contribution: Shortlisting is a simple model of decision-making with a wide variety of applications. While it is not the most general way to model all of the choice phenomena discussed above, there are compelling reasons to study this specialized model.

One important reason is the simplicity of the model itself. In the interest of developing parsimonious models of behavior, it is important to determine whether a minimal departure from the standard model, like shortlisting, can accommodate systematic violations of preference maximization. A succinct axiomatization of the model makes it easier to answer to this question.

A second reason to study a specialized model is to obtain sharper identification of behavior (Dekel and Lipman [2010b]). While the models which generalize shortlisting (discussed at greater length in the next section) accommodate a wider range of behavior, they suffer the drawback that key parameters of interest, such as preferences, may be poorly identified from choice data.

Motivated by these concerns, I provide four results on the foundations of shortlisting behavior. First, I show that shortlisting can be axiomatized with the help of a natural symmetry property proposed by Manzini and Mariotti [2006] (see Horan [2010] for a related property in the context of choice from a list).[2] Intuitively, this property requires that alternatives which are similar in terms of pairwise choice be treated symmetrically on larger menus. The approach differs from the axiomatization of related models given in the literature (Masatlioglu, Nakajima, and Ozbay [2010]; Lleras, Masatlioglu, Nakajima, and Ozbay [2011]; Au and Kawai [2011]; Yildiz [2011]). In particular, it does not impose a condition – à la *Strong Axiom of Revealed Preference* (SARP) – which explicitly requires the acyclicity of a revealed preference relation. Indeed, a basic goal of the characterization is to establish that the acyclicity requirement imbedded in the model has a straightforward interpretation in terms of behavior.

Second, I show that any feature of the representation with implications for behavior can be determined from choice on menus of two and three alternatives. Similar to the classical result for choice that satisfies the *Weak Axiom of Revealed Preference* (WARP) (see Mas-Collel, Whinston,

[2] Manzini and Mariotti show that this property is necessary but do not prove any results about sufficiency.

and Green [1995], Proposition 1.D.2), this shows that relatively little data is required to construct a representation of behavior consistent with the model. Practically, it suggests that the necessary choice information may be easy to obtain in experiments as well as market settings. By way of contrast, more general models can be significantly more data intensive.

Third, I show that it is possible to obtain a restricted form of uniqueness for conservative representations of behavior. Given a representation (P_1, P_2), there exists a unique minimal i^{th} rationale that can be used to represent behavior with P_{-i}. Intuitively, this rationale reflects the most conservative estimate of the other rationale conditional on using P_{-i} to represent behavior. Moreover, there is a sharply defined range of conservative representations consisting of mutually minimal rationales. Intuitively, these representations reflect the range of ways to attribute revealed preference pairs which are *not* revealed to belong to either rationale.

Finally, I show that it is possible to draw meaningful comparisons between decision-makers whose behavior is consistent with the model. In particular, I define a comparative static which captures the notion that the second rationale of decision-maker A is *more decisive* than the second rationale of decision-maker B. Decisiveness may be viewed as a natural generalization of two comparative statics, namely *preference for commitment* (Gul and Pesendorfer [2001]) and *willpower* (Masatlioglu, Nakajima, and Ozdenoren [2011]), first proposed in the context of temptation.

Organization: The remainder of the paper is structured as follows. After presenting the model and discussing related models of choice in the next section, I provide an axiomatic characterization of shortlisting in Section 3, followed by identification and uniqueness results in Section 4. I conclude with a short discussion of comparative statics in Section 5.

2 Shortlisting and Related Models of Choice

A *rationale* is an asymmetric binary relation defined on a finite domain X. A *shortlisting procedure* is a pair (P_1, P_2) of transitive rationales (i.e. quasi-transitive preferences) that determines a choice function $c_{(P_1,P_2)} : 2^X \to X$. For any menu $A \subseteq X$, the choice induced by (P_1, P_2) is given by

$$c_{(P_1,P_2)}(A) \equiv \max(\max(A; P_1); P_2)$$

where $\max(B; P) = \{x \in B : \text{no } y \in B \text{ s.t. } yPx\}$ denotes the set of P-maximal alternatives in B. Conversely, a choice function $c : 2^X \to X$ can be *represented in terms of shortlisting* if there exists a pair of transitive rationales (P_1, P_2) such that $c(A) = c_{(P_1,P_2)}(A)$ for any menu $A \subseteq X$.

Related Models: First proposed by Manzini and Mariotti [2006], shortlisting is closely related to a variety of models studied in the theory literature. One natural departure is to impose less structure on the rationales. In related work, Manzini and Mariotti [2007] (see also Horan [2012] for identification results) characterize a model of *rational shortlist methods* (RSM) that dispenses with transitivity (so that P_1 and P_2 are asymmetric but may be intransitive). Since there exist RSMs which cannot be represented with transitive rationales (as illustrated by Example 1 of the Appendix), the RSM model strictly generalizes shortlisting.[3]

Conversely, one could impose additional structure on the rationales. Working separately, Au and Kawai [2011] and Yildiz [2011] characterize the *transitive RSM* model where both rationales are transitive and the second rationale is also complete.[4] Compared with shortlisting, this model imposes no additional restrictions on behavior. In particular, any shortlisting procedure (P_1, P_2) can be represented by a transitive RSM where the second rationale is any linear order that completes P_2 (see Remark 1 of the Appendix).

While the completeness of the second rationale has no implications for behavior, the transitive RSM representation is nonetheless instructive because it clarifies the connection with two-stage procedures $(\Gamma, \succ)$ where the decision-maker *filters* the feasible alternatives using a choice correspondence $\Gamma : 2^X \to 2^X$ before maximizing the linear order $\succ$ on $\Gamma(A)$.

In recent work, Masatlioglu, Nakajima, and Ozbay [2010] characterize a model of *choice with limited attention* (CLA) where the filter satisfies a contraction property known as Bordes' [1979] *strong superset property*[5] (SSP). In related work with Lleras [2011], they characterize a model of *choice with limited consideration* (CLC) where the filter instead satisfies the standard contraction axiom known as Sen's [1971] property α. It is straightforward to see that shortlisting coincides with the special case where the filter Γ is generated by a transitive rationale. By a standard result in classical choice theory, this is equivalent to the filter Γ satisfying both Sen's α and Bordes' SSP, as well as the standard expansion axiom known as Sen's γ (see Moulin [1985]; Brandt and Harrenstein [2011]). Since there are CLAs and CLCs that cannot be represented with transitive rationales (as illustrated in Example 2 of the Appendix), these models strictly generalize shortlisting.

Finally, the model is related to the temptation literature – the main difference being that shortlisting is concerned with the decision-maker's *ex post* choice from menus (see Sandroni [2011] for a different model of ex post choice related to temptation). In contrast, temptation models generally examine the *ex ante* preference of a decision-maker who takes ex post temptations into account when choosing among menus (see Lipman and Pesendorfer [2011] for a recent survey).

[3]In turn, the RSM model has been generalized to allow for: more than two rationales (Apesteguia and Ballester [2010]; Manzini and Mariotti [2011]); and, less structure on the first-stage decision (Cherepanov, Feddersen, and Sandroni [2010]; Spears [2011]) in a way that generalizes Lleras, Masatlioglu, Nakajima, and Ozbay [2011].

[4]Remark 5 of the Appendix corrects an error in Au and Kawai's proof of the representation theorem.

[5]For convenience, SSP is stated in the Appendix. Unfortunately, there is little consensus on the name of this property. Masatlioglu, Nakajima, and Ozbay [2010], for instance, call it the "attention filter" axiom. See Monjardet [2008] for references to other names that have been used in the literature. I follow Bordes' terminology since he was the first to name (rather than number) the axiom.

Since the decision-maker plans consistently, these models have implications for choice ex post.

In particular, shortlisting provides a link between ex ante preference and ex post choice for a class of models that generalize the Strotz [1956] model of overwhelming temptation. In that model, ex post choice is determined by a pair of weak orders $(\succeq_v, \succeq_u)$. Overwhelmed, the decision-maker maximizes the *true* preference $\succeq_u$ by choosing from the subset of alternatives that maximize the *temptation* preference $\succeq_v$. Provided that choice is single-valued, the ex post Strotz model is a shortlisting procedure where the rationales have more structure (see Remark 3 of the Appendix).[6]

A key feature of the Strotz model (and a variety of other temptation models besides) is that ex post choice cannot distinguish true preference from temptation. The problem is that the behavior induced by $(\succeq_v, \succeq_u)$ can be represented by a weak order (see Remark 2 of the Appendix). To the extent that temptations are less structured however, features of the two preferences can be determined from ex post choice. In a recent paper, Masatlioglu, Nakajima, and Ozdenoren [2011] axiomatize an ex ante model of limited willpower that generalizes the Strotz model (see Chandrasekhar [2010] for a related model of compromise in the face of temptation). In terms of ex post choice, their model suggests choice behavior consistent with a shortlisting procedure where the first rationale has more structure (i.e. it is either an interval order or a semiorder).[7] The characterization of shortlisting given here provides a way to determine whether ex post choice is consistent with their model of ex ante preference.

3 Axiomatic Characterization

As discussed, shortlisting is a special case of the RSM model. In this section, I show that two simple axioms characterize RSMs which can be represented in terms of shortlisting.

The first axiom, *Choice Symmetry*, was suggested by Manzini and Mariotti [2006].[8] Informally, it requires that alternatives which are similar in context be treated symmetrically in terms of choice. Formally, a menu $A \subseteq X$ is a *context* for x and y provided that it does not contain either alternative (i.e. so that $A \cap \{x, y\} = \emptyset$). Then, x and y are *similar with respect to (w.r.t)* the context A whenever $c(a, x) = x$ if and only if $c(a, y) = y$ for all $a \in A$. Intuitively, alternatives are similar in contexts where they play the same role in terms of pairwise preference. Using this notion of similarity, the axiom can be stated as follows:

Choice Symmetry *Suppose x and y are similar w.r.t. A. If $c(A \cup \{x\}) = x$, then $c(A \cup \{y\}) = y$.*

This property captures a natural form of symmetry in choice. Given alternatives x and y which

[6] In particular, they are negatively transitive. A rationale P is negatively transitive if $xPy \Rightarrow (xPz \text{ or } zPy)$.

[7] An interval order is a rationale P such that $(xPy \text{ and } wPz) \Rightarrow (xPz \text{ or } wPy)$. A semiorder is an interval order such that $xPyPz \Rightarrow (xPw \text{ or } wPz)$. A comprehensive reference is Aleskerov, Bouyssou, and Monjardet [2007].

[8] They use the name *Weak Binariness*. At the risk of creating confusion, I depart from their nomenclature.

are similar with respect to A, one of two possibilities can arise: either both alternatives are chosen in the context of A; or, both are discarded. The *Independence of Irrelevant Alternatives* (IIA), which states that $c(B) = x$ for any B such that $\{x\} \subset B \subset A$ and $c(A) = x$, imposes the same requirement. In addition, it imposes the further restriction that $c(A \cup \{x\}) = c(A \cup \{y\})$ when x and y are discarded.

The second axiom, *Difficult Choice*, imposes a weaker form of choice consistency when similar alternatives x and y are discarded. Intuitively, it states that the *more* preferred of the two alternatives can be discarded from $A \cup \{x, y\}$ when there is another menu containing x and y where the *less* preferred alternative gives rise to a difficult choice.

Formally, *y gives rise to a difficult choice* on $B \supset \{x, y\}$ if $c(B) \neq c(B \setminus \{y\})$ and $c(x, y) = x$ for some $x \in B \setminus \{y\}$. Informally, a difficult choice arises when the addition of a less preferred alternative y causes the decision-maker to reverse her choice on $B \setminus \{y\}$. This type of choice reversal is ruled out by IIA. Intuitively, it suggests a particular kind of menu dependence where y plays a special role (not captured by pairwise preference) in menus that contain x.

When x and y are similar with respect to A, they play duplicate roles on $A \cup \{x, y\}$ in terms of pairwise preference. If y gives rise to a difficult choice on another menu B however, part of the role played by y is not duplicated by x. Provided that x is not chosen from $A \cup \{x, y\}$, Difficult Choice states that x adds nothing beyond y. As such, it can be removed without affecting choice:

Difficult Choice *Suppose x and y are similar w.r.t. A, $c(x, y) = x$, and $c(A \cup \{x, y\}) \neq x$:*

$$\textit{If } c(B) \neq c(B \setminus \{y\}) \textit{ for some } B \supset \{x, y\}, \textit{ then } c(A \cup \{x, y\}) = c(A \cup \{y\}).$$

The representation theorem establishes that these two axioms are necessary and sufficient:

Theorem (Representation) *An RSM can be represented in terms of shortlisting* **iff** *it satisfies Choice Symmetry and Difficult Choice.*

Manzini and Mariotti's [2007] establish that c can be represented by an RSM iff it satisfies:

Expansion *If $c(A) = x = c(B)$, then $c(A \cup B) = x$.*
WWARP *If $c(A) = x = c(x, y)$ for $A \supset \{x, y\}$, then $c(B) \neq y$ for any B s.t. $\{x, y\} \subset B \subset A$.*

Their characterization of RSMs gives an immediate corollary.[9] In particular:

Corollary 1 (Representation) *A choice function c can be represented in terms of shortlisting iff it satisfies Choice Symmetry, Difficult Choice, Expansion, and WWARP.*

[9] These properties are discussed at greater length in their paper. Briefly, Expansion is the analog of Sen's γ for choice functions. As its name suggests, WWARP has some of the flavor of WARP. For choice functions, WARP is equivalent to IIA.

The proof of the representation result is given in the Appendix. The technical part is to show that the axioms ensure the acyclicity of the second rationale. Effectively, Choice Symmetry ensures that the second rationale is triple-acyclic. Combined with Expansion, Difficult Choice further precludes cycles of order 4 or 5. By an induction argument, it follows that the axioms rule out higher-order preference cycles as well. The result then follows by showing that this rationale can be extended into a transitive rationale in a way that is consistent with behavior.

The axiomatization differs in two important ways from existing characterizations of the model. First, it does not impose an explicit acyclicity requirement on revealed preference. In a recent paper, Au and Kawai [2011] show that an RSM can be represented with transitive rationales iff a particular revealed preference that they define is acyclic. Similarly, Yildiz [2011] requires the acyclicity of a slightly coarser revealed preference relation (see Remark 6 of the Appendix).

It is arguable that conditions of this kind provide little insight into the behavioral implications of the model that cannot be inferred from the representation directly. In contrast, the axiomatization given here shows that the choice acyclicity implicit in the model has a straightforward and natural interpretation in terms of behavior.

As a related matter, the characterization does not rely on existential quantification as in Lleras, Masatlioglu, Nakajima, and Ozbay [2011].[10] In recent work, they establish that an RSM can be represented with transitive rationales iff it satisfies LCA-WARP – a condition which requires the existence of a special alternative $a^* \in A$ for every $A \subseteq X$.

Intuitively, axioms that rely on existential quantification, like LCA-WARP, impose consistency requirements across very large collections of choice problems.[11] In this sense, they make complex statements about the model that can be difficult to interpret in terms of behavior (Beja [1989]). By way of contrast, an axiom like IIA is relatively simple since it describes a straightforward behavioral implication (of preference maximization) that only depends on choice from a menu $A \subseteq X$ and a sub-menu $B \subset A$. While more complex than IIA, Choice Symmetry and Difficult Choice are considerably less complex than LCA-WARP (as formalized in Remark 4 of the Appendix).[12] As such, it can be argued that they provide greater insight into behavioral implications of the model.

4 Identification and Uniqueness

In this section, I address the issues of identification and uniqueness. I first show how menus of two and three alternatives can be used to define *revealed rationales* that are contained in any shortlisting representation of c. Next, I characterize the class of shortlisting representations which

[10] The issue of existential quantification also arises for acyclicity conditions since they implicitly require, for every $A \subseteq X$, the existence of an $a^* \in A$ such that no other alternative in A is revealed preferred to a^*.

[11] See Chambers, Echenique, and Shmaya [2010] for work on the axiomatic structure of economic choice models.

[12] Choice Symmetry and Difficult Choice have complexity $O(|X|)$ while LCA-WARP has complexity $O(2^{|X|})$.

use these revealed rationales. Finally, I show that a restricted form of uniqueness can be obtained for conservative representations of behavior.

4.1 Identification

As a corollary of the representation theorem, it follows that shortlisting behavior is completely determined by choice from small menus. Formally, let $\succ^c$ denote the usual (i.e. pairwise) *revealed preference* defined by $x \succ^c y$ if $c(x,y) = x$. Given the revealed preference, an *n-cycle* is a sequence of alternatives $x_0...x_i...x_{n-1}$ such that $x_{i-1} \succ^c x_i$ for all $1 \leq i \leq n-1$ and $x_{n-1} \succ^c x_0$. Then:

Corollary 2 *Suppose c and $\tilde{c}$ can be represented in terms of shortlisting. Then $c(A) = \tilde{c}(A)$ for all $A \subseteq X$ iff (i) $c(x,y) = \tilde{c}(x,y)$ for any $x \neq y$, and (ii) $c(x,y,z) = \tilde{c}(x,y,z)$ for any 3-cycle xyz.*

For behavior consistent with shortlisting, this establishes that the features common to any representation can be inferred from choice on pairs and 3-cycles. In turn, this suggests that the model is amenable to a simple revealed preference exercise.

To fix ideas, suppose that (P_1, P_2) is a shortlisting representation of c. As a preliminary observation, note that the two rationales taken together (i.e. $P_1 \cup P_2$) must contain the revealed preference pairs in $\succ^c$. Otherwise, the choices induced by the pair (P_1, P_2) will not coincide with c on some two-element set(s).

First, consider the task of attributing the revealed preference pairs from any 3-cycle xyz to the rationales P_1 and P_2. It turns out that the choice from $\{x,y,z\}$ uniquely determines how to divide these pairs between the two rationales. If $c(x,y,z) = z$, it must be that xP_1y while yP_2z and zP_2x. Intuitively, xP_1y and yP_2z follow from the fact that x must eliminate y strictly before y eliminates z. Otherwise, the behavior induced by (P_1, P_2) contradicts $c(x,y,z) = z$. In turn, the fact that zP_2x follows from the transitivity of P_1. To see this, suppose zP_1x. Then, xP_1y implies zP_1y which, in turn, precludes $c(y,z) = y$.

Next, consider the task of attributing the revealed preference $x \succ^c y$ when wxz and wyz are 3-cycles such that $c(w,x,z) = w$ and $c(w,y,z) = z$. From the discussion above, it follows that $yP_2zP_2wP_2x$. Moreover, it must be that xP_1y. If xP_2y, P_2 contains a cycle which violates the fact that it is asymmetric. Collecting these observations yields the following:

Definition 1 *Let R_1^c be defined by xR_1^cy if:*

(i) there exists a 3-cycle xyz s.t. $c(x,y,z) = z$; or,

(ii) $c(x,y) = x$ and there exist 3-cycles wxz and wyz s.t. $c(w,x,z) = w$ and $c(w,y,z) = z$.

Let R_2^c be defined by xR_2^cy if there exists a 3-cycle xyz such that $c(x,y,z) \neq z$.

Define the ***revealed i-rationale*** *$P_{\underline{i}}^c \equiv tc(R_i^c)$ to be the transitive closure of R_i^c for $i = 1, 2$.*[13]

[13] The transitive closure $tc(R)$ of R is defined by $x[tc(R)]y$ if there exists a sequence $\{z_i\}_{i=1}^n$ s.t. $xRz_1R...Rz_nRy$.

Given a choice function c consistent with shortlisting, $(P^c_{\underline{1}}, P^c_{\underline{2}})$ need not represent c. Intuitively, the problem is that $P^c_{\underline{1}} \cup P^c_{\underline{2}}$ may fail to contain all of the revealed preference pairs in $\succ^c$. To construct a representation using $P^c_{\underline{1}}$ as the first rationale, for instance, the second rationale P_2 must, *at a minimum*, contain any revealed preference pair not in $P^c_{\underline{1}}$. Formalizing this idea:

Definition 2 *Given a binary relation P and a choice function c, the* ***c-complement*** *of P is the transitive relation $P' \equiv tc(\succ^c \backslash P)$ that contains any revealed preference pair not in P.*

To simplify notation, denote the c-complement of $P^c_{\underline{1}}$ by $P^c_{\overline{2}}$ and the c-complement of $P^c_{\underline{2}}$ by $P^c_{\overline{1}}$. Provided that c can be represented by some pair $(P_1, P^c_{\underline{2}})$, the c-complement $P^c_{\overline{1}}$ serves as a lower bound for the first rationale in the sense that P_1 must contain $P^c_{\overline{1}}$. Similarly, $P^c_{\overline{2}}$ serves as a lower bound for the second rationale in any representation which uses $P^c_{\underline{1}}$. To state the identification result, it is first necessary to define an analogous upper bound for P_1.

To do so, observe that P_1 must, *at a minimum*, be contained in $\succ^c$ for any shortlisting representation (P_1, P_2) of c. Otherwise, the choices induced by (P_1, P_2) fail to coincide with c on some two-element set(s).

First, consider the task of attributing the revealed preference $x \succ^c y$ when xyz is a 3-cycle such that $c(x, y, z) \neq z$. If $c(x, y, z) = x$ so that yP_1z, it follows that P_1 must exclude the revealed preference $x \succ^c y$. Otherwise, transitivity implies xP_1z which, in turn, contradicts $z \succ^c x$. By similar reasoning, P_1 must exclude the revealed preference $x \succ^c y$ when $c(x, y, z) = y$. Next, consider the task of attributing the revealed preference $x \succ^c y$ when wxz and wyz are 3-cycles such that $c(w, x, z) = z$ and $c(w, y, z) = w$. Since wP_1x and yP_1z, P_1 must exclude the revealed preference $x \succ^c y$. Otherwise, transitivity implies wP_1z which, in turn, contradicts $z \succ^c w$.

These observations are sufficient to define an upper bound for P_1:

Definition 3 *Let $\overline{R}^c_2$ be the binary relation defined by $x\overline{R}^c_2y$ if: (i) xR^c_2y; or, (ii) $c(x, y) = x$ and there exist 3-cycles wxz and wyz such that $c(w, x, z) = z$ and $c(w, y, z) = w$. Define the* ***upper bound*** *$P^c_{\overline{\overline{1}}} \equiv (\succ^c \backslash \overline{R}^c_2)$ to be the revealed preferences in $\succ^c$ that are not in $\overline{R}^c_2$.*

To state the identification result:

Proposition 1 (Identification) *Suppose that c can be represented in terms of shortlisting. Then:*

(I) ***(a)*** *$P^c_{\underline{1}} \subseteq P_1$ and $P^c_{\underline{2}} \subseteq P_2$ for any pair of transitive rationales (P_1, P_2) that represents c;*
(b) *the relations $P^c_{\underline{1}}$, $P^c_{\underline{2}}$, $P^c_{\overline{1}}$, and $P^c_{\overline{\overline{1}}}$ are transitive rationales, and $P^c_{\overline{2}}$ is a linear order;*[14]

(II) ***(a)*** *$(P^c_{\underline{1}}, P^c_{\overline{2}})$ uniquely represents c when the first rationale is $P^c_{\underline{1}}$; and*
(b) *$(P_1, P^c_{\underline{2}})$ represents c for any transitive rationale P_1 such that $P^c_{\overline{1}} \subseteq P_1 \subseteq P^c_{\overline{\overline{1}}}$.*

[14]Technically, $P^c_{\overline{2}}$ is a strict weak order. Since it is irreflexive, it is technically incomplete.

Part (I) establishes that the revealed rationales capture the features common to any shortlisting representation of behavior. If $\{(P_1^i, P_2^i)\}_{i=1}^n$ denotes the collection of transitive pairs which represent c, it follows that $P_{\underline{1}}^c \subseteq \cap_{i=1}^n P_1^i$ and $P_{\underline{2}}^c \subseteq \cap_{i=1}^n P_2^i$. In turn, part (II) establishes that there exist shortlisting representations of c which use the revealed rationales. Consequently, $P_{\underline{1}}^c = \cap_{i=1}^n P_1^i$ and $P_{\underline{2}}^c = \cap_{i=1}^n P_2^i$. In other words, the revealed rationales capture exactly the features common to all representations of behavior.

Part (II) identifies tight bounds for shortlisting representations using one of these rationales. The lower bounds $P_{\overline{1}}^c$ and $P_{\overline{2}}^c$ reflect the fact that any pair representing c must contain the revealed preference $\succ^c$. Since $P_{\overline{2}}^c$ is a linear order by part (I), $(P_{\underline{1}}^c, P_{\overline{2}}^c)$ is the unique representation when the first rationale is $P_{\underline{1}}^c$. For any representation where the second rationale is $P_{\underline{2}}^c$, the upper bound $P_{\overline{\overline{1}}}^c$ reflects the fact that $\succ^c$ must contain P_1. Together, the bounds establish the range $P_{\overline{1}}^c \subseteq P_1 \subseteq P_{\overline{\overline{1}}}^c$ of first rationales P_1 such that $(P_1, P_{\underline{2}}^c)$ represents c. Example 4 of the Appendix shows that this range is generically non-degenerate (i.e. $P_{\overline{1}}^c \neq P_{\overline{\overline{1}}}^c$).

Comment: These results extends Theorem 2 of Au and Kawai [2011]. For one, the revealed preference definitions given here simplify the definitions provided in their paper (see Remarks 6-8 of the Appendix). While they suggest how define the upper bound $P_{\overline{\overline{1}}}^c$ in terms of choice from menus of four or fewer alternatives (in Remark 1 of their paper), they provide no results on the informational requirements of the revealed i-rationales or their c-complements. While Proposition 1 provides identification results similar to Theorem 2 of their paper, it does not impose the additional requirement of completeness on the second rationale.

4.2 Uniqueness

Proposition 1 has mixed implications for uniqueness. On the one hand, it precludes the possibility of a unique representation of behavior consistent with the model. In particular, it ensures that the generically distinct pairs $(P_{\underline{1}}^c, P_{\overline{2}}^c)$ and $(P_{\overline{1}}^c, P_{\underline{2}}^c)$ represent c. On the other, it suggests that a limited form of uniqueness can be obtained for conservative representations of behavior.

To get the basic idea, consider the task of analyzing choice data consistent with the model. First, suppose the analyst is confident about P_1 and wants to draw conservative inferences about the second rationale. (One possible explanation is that the analyst was able to infer P_1 from a different source of choice data.) In that case, the analyst may hesitate to impose more structure on P_2 than is required to generate the choice data. The following definition formalizes this approach:

Definition 4 *A rationale P_2 is P_1-**minimal** if (P_1, P_2) is a shortlisting representation of c and there is no transitive rationale $\widetilde{P}_2 \subset P_2$ s.t. $(P_1, \widetilde{P}_2)$ represents c – i.e. $\widetilde{P}_2 \not\subset P_2$ for any representation $(P_1, \widetilde{P}_2)$. The notion of P_2-**minimality** is defined analogously.*

It is possible to extend this approach to the situation where the first rationale is unknown. In particular, the analyst can focus on pairs of rationales that are *mutually* minimal. Formally:

Definition 5 *A rationale pair (P_1, P_2) is* ***minimal*** *if P_1 is P_2-minimal and P_2 is P_1-minimal.*

Given behavior consistent with shortlisting, Proposition 1 establishes that the c-complement $P^c_{\bar{i}}$ of the revealed i-rationale $P^c_{\underline{i}}$ is $P^c_{\underline{i}}$-minimal. Since the revealed rationales serve as lower bounds for representing behavior consistent with the model, $(P^c_{\underline{1}}, P^c_{\bar{2}})$ and $(P^c_{\bar{1}}, P^c_{\underline{2}})$ are minimal representations of c. Intuitively, these two representations reflect extreme but opposing views about how to attribute the revealed preference pairs in $\succ^c$ that are *not* contained in either of the revealed rationales (i.e. $P^c_{\underline{1}}$ or $P^c_{\underline{2}}$). Whereas $(P^c_{\underline{1}}, P^c_{\bar{2}})$ attributes all of these pairs to the second rationale, $(P^c_{\bar{1}}, P^c_{\underline{2}})$ attributes all of these pairs to the first rationale.

The following proposition generalizes these observations. To state the result more succinctly, let $\mathcal{P}_i(c) \equiv \{P : P^c_{\underline{i}} \subseteq P \subseteq P^c_{\bar{i}}\}$ denote the collection of (not necessarily transitive) i^{th} rationales nested between the revealed i-rationale and the c-complement of the *other* revealed rationale. Moreover, let $\mathcal{P}''_i(c) \equiv \{P'' : P \in \mathcal{P}_i(c)\}$ denote the collection of transitive i^{th} rationales obtained from $\mathcal{P}_i(c)$ by two applications of (element-wise) complementation.

Proposition 2 (Uniqueness) *Suppose that (P_1, P_2) is a shortlisting representation of c. Then:*

(i) *For $i = 1, 2$, the c-complement $P'_i \in \mathcal{P}_{-i}(c)$ is the unique P_i-minimal rationale; and,*

(ii) *(P_1, P_2) is minimal iff $P_1 \in \mathcal{P}''_1(c)$ and $P_2 = P'_1$ (or, equivalently, $P_2 \in \mathcal{P}''_2(c)$ and $P_1 = P'_2$).*

Moreover, any distinct minimal representations $(P_1, P_2), (\widetilde{P}_1, \widetilde{P}_2)$ are unranked: if $(\widetilde{P}_1 \setminus P_1) \neq \emptyset$ so that $\widetilde{P}_1$ contains preference pairs not in P_1, then $(P_2 \setminus \widetilde{P}_2) \neq \emptyset$ so that P_2 contains pairs not in $\widetilde{P}_2$.

For any rationale P_i that can be used to represent c, part (i) establishes that the c-complement P'_i is the most conservative estimate of the *other* rationale. Using this result, part (ii) sharply identifies the collection of minimal representations for behavior consistent with the model.

To get some intuition for part (ii), consider the special case where c can be represented by the linear order $\succ^c$. In that case, c has a variety of minimal shortlisting representations ranging from $(\succ^c, \emptyset)$ to $(\emptyset, \succ^c)$. Between these two extremes, there are different ways to attribute the revealed preference pairs in $\succ^c$. The same type of result holds when c cannot be represented by a linear order. Between $(P^c_{\bar{1}}, P^c_{\underline{2}})$, which places maximal weight on the first rationale, and $(P^c_{\underline{1}}, P^c_{\bar{2}})$, which places maximal weight on the second rationale, there is a range of representations reflecting contrasting views about how to attribute the revealed preference pairs not contained in $P^c_{\underline{1}}$ or $P^c_{\underline{2}}$.

5 Comparative Statics

In this section, I define a comparative measure of choice behavior and provide a characterization of this measure in terms of the representation of behavior.

Definition 6 *Given choice functions c_A and c_B, A is* ***more decisive than*** *B if $P_{\underline{1}}^A \subseteq P_{\underline{1}}^B$.*[15]

Informally, A is more decisive than B when the first-stage rationale plays a less significant role in determining A's choice. This generalizes the notion, discussed in the temptation literature, that A has *more willpower* than B (Masatlioglu, Nakajima, and Ozdenoren [2011]) or, conversely, that B has a *greater preference for commitment* than A (Gul and Pesendorfer [2001]).

Underlying these measures is the view that the second rationale reflects the decision-maker's true preference while the first rationale captures a psychological phenomenon (i.e. temptation) that is orthogonal to preference. When shortlisting is viewed in terms of choice overload, limited attention, or the *status quo bias*, decisiveness can be given a similar interpretation. In each case, the first rationale captures a cognitive bias unrelated to second-stage preference.[16]

Decisiveness has a straightforward interpretation in terms of the representation. In particular, it serves as a comparative measure of the conflict between the rationales of two representations. Formally, the rationales of (P_1, P_2) *conflict* whenever xP_1y and yP_2x for some $x, y \in X$.

Proposition 3 (Comparative Static) *If c_A and c_B can be represented in terms of shortlisting by (P_1^A, P_2^A) and (P_1^B, P_2^B), A is more decisive than B iff, for all $x, y \in X$*

$$xP_1^A y \text{ and } y(P_1^A)'x \Rightarrow xP_1^B y \text{ and } y(P_1^B)'x.$$

Intuitively, this establishes that A is more decisive than B iff, from a conservative viewpoint, A exhibits fewer conflicts. Instead of measuring the conflict between the rationales of (P_1, P_2) directly, decisiveness measures the conflict between P_1 and the minimal second rationale that can be used to represent behavior along with P_1 (i.e. the c-complement P_1'). Since Proposition 2 establishes that $P_1' \subseteq P_2$, the conflict between P_1 and P_1' may be understood as a conservative estimate of the conflict between P_1 and P_2. In particular, it shows that there is a direct correspondence between such conflicts and the preference pairs belonging to $P_{\underline{1}}^c$.

Decisiveness permits comparisons between shortlisting representations even when the second stage rationales do not coincide. The next result relates to choice functions c_A and c_B that are *preference-equivalent* in the sense that they admit shortlisting representations (P_1^A, P_2) and (P_1^B, P_2) where the second rationales coincide. In that case, there is a close relationship between

[15]For the sake of notational convenience, I denote $P_i^{c_J}$ by P_i^J for $J = A, B$ and $i = 1, 2$.

[16]For other applications, like multi-criterial choice, the rationales have a different interpretation.

decisiveness and the incidence of cycles in the revealed preference $\succ^c$. To be more formal, say that B is *more cyclic* than A if xyz is a 3-cycle on c_B when xyz is a 3-cycle on c_A. Then:

Proposition 4 *Suppose that c_A and c_B are preference-equivalent choice functions. Then, A is more decisive than B if: (i) B is more cyclic than A; and, (ii) $c_A(w, z) = w$ implies $c_B(w, z) = w$ whenever there are A-cycles wxy and xyz s.t. $c_A(w, x, y) = y$ and $c_A(x, y, z) = x$.*

Au and Kawai [2011] provide an example of preference-equivalent choice functions with the same 3-cycles which are *strictly* ranked in terms of decisiveness (i.e. $P^A_{\underline{1}} \subset P^B_{\underline{1}}$). Proposition 4 shows that this kind of situation occurs when condition (ii) holds in one direction but not the other.

To be more precise, consider two 3-cycles wxy and xyz such that $c_A(w, x, y) = y = c_B(w, x, z)$ and $c_A(x, y, z) = x = c_B(x, y, z)$. If choice on $\{w, z\}$ is given by $c_A(w, z) = z$ and $c_B(w, z) = w$:

$$c_A(w, z) = w \Rightarrow c_B(w, z) = w \quad \textit{but} \quad c_B(w, z) = w \not\Rightarrow c_A(w, z) = w.$$

By definition of the revealed 1-rationale, it is the case that $wP^B_{\underline{1}}z$ whereas $wP^A_{\underline{1}}z$ does not follow. Intuitively, the basic idea is that the choices from the 3-cycles of A fit together more coherently than the choices from the 3-cycles of B. Put somewhat differently, the revealed preference $z \succ^A w$ is more consistent with the choices from xyz and wxy than $z \succ^B w$. Consequently, A is more decisive than B.

6 Conclusion

In this paper, I study a minimal departure from the standard model of preference maximization where the decision-maker chooses in stages by sequentially maximizing *two* preferences that are asymmetric and transitive. This simple model has a wide variety of applications to individual decision-making and collective choice.

The paper provides choice-theoretical foundations for the model. First, I show that it can be axiomatized using a natural symmetry property first proposed by Manzini and Mariotti [2006]. For behavior consistent with the model, I next show that (i) the identifiable features of both rationales can be determined from choice on small menus and, (ii) the range of minimal representations consistent with behavior is sharply defined. I conclude by defining a *comparative static* that permits meaningful comparisons between decision-makers.

7 Appendix

7.1 Guide to the Appendix

Section 7.2 addresses several preliminary points made in Sections 2 and 3 of the text. Section 7.3 details the proof of the representation theorem in Section 3 while Section 7.4 establishes the identification and uniqueness results in Section 4, and Section 7.5 proves the results in Section 5.

7.2 Preliminaries

For convenience, I restate the α, γ and SSP axioms mentioned in Sections 2 and 3. Given a finite domain X, let $C : 2^X \to 2^X$ denote a choice correspondence such that $C(A) \subseteq A$ for any $A \subseteq X$.

Sen's α *If $x \in C(A)$ and $x \in B \subset A$, then $x \in C(B)$.*

Bordes' SSP *If $C(A) \subseteq B \subseteq A$, then $C(A) = C(B)$.*[17]

Sen's γ *If $x \in C(A)$ and $x \in C(B)$, then $x \in C(A \cup B)$.*

To see that there are RSMs that cannot be represented in terms of shortlisting:

Example 1 *Consider a choice function c on $\{w, x, y, z\}$ such that wxy and xyz are 3-cycles where $c(w, x, y) = w$, $c(x, y, z) = x$, and $c(w, x, y, z) = w$.*

While it is not required for the purpose of the example, c can be completed by setting $c(w, y, z) = y$, choosing $c(w, z)$ freely, and requiring $c(w, x, z) = c(w, z)$. Since it does not satisfy Choice Symmetry, c cannot be represented in terms of shortlisting. Intuitively, the problem is that $c(w, x, y) = w$ requires $(x, y) \in P_1$ while $c(x, y, z) = x$ requires $(x, y) \in P_2$. Clearly, this is impossible. However, c can be represented by an RSM (P_1, P_2) where $P_1 = \{(x, y), (y, z)\}$ and $P_2 = \{(w, x), (y, w), (z, x), (a, b)\}$ s.t. $(a, b) \equiv (w, z)$ if $c(w, z) = w$ and $(a, b) \equiv (z, w)$ otherwise.

To see that limited attention and limited consideration are more general than shortlisting:

Example 2 *Consider a choice function c s.t. $c(x, y) = c(x, z) = x$, $c(y, z) = y$, and $c(x, y, z) = y$.*

Since it does not satisfy Expansion, c cannot be represented in terms of shortlisting (or, more generally, by an RSM). Intuitively, the problem is that $c(x, y, z) = y$ requires $(y, x) \in P_1$ (or $(z, x) \in P_1$) while $c(x, y) = x$ (resp. $c(x, z) = x$) requires $(y, x) \notin P_1$ (resp. $(z, x) \notin P_1$). Clearly, this is impossible. However, c can be represented in terms of choice with limited attention or

[17] Masatlioglu, Nakajima, and Ozbay [2010] state the condition as follows: if $x \notin C(A)$, then $C(A) = C(A \setminus \{x\})$. It is easily seen that Bordes' SSP implies their condition. To see the converse, take any $C(A) \subseteq B \subseteq A$, and remove the alternatives in $A \setminus B$ one at a time using their condition. Since everything is finite, this delivers Bordes' SSP.

choice with limited consideration. To see this, first observe that $\succ^c$ is a linear order. Next, let $\Gamma(x, y, z) = \{y, z\}$ and $\Gamma(a, b) = \{a, b\}$ for $\{a, b\} \subset \{x, y, z\}$. It is easy to check that Γ satisfies Bordes' SSP and Sen's α and, moreover, that $(\Gamma, \succ^c)$ represents c.

To address several other remarks made in Section 2 of the text:

Remark 1 *c can be represented by a transitive RSM iff it has a shortlisting representation.*

Proof. ($\Rightarrow$) By definition, any transitive RSM defines a shortlisting procedure. ($\Leftarrow$) Suppose that c can be represented by a shortlisting procedure (P_1, P_2). Since P_2 is asymmetric and transitive, the Szpilrajn extension theorem ensures that P_2 can be completed into a linear order $\succ_2$. By construction, $(P_1, \succ_2)$ is a transitive RSM. To establish the result, it suffices to show that $(P_1, \succ_2)$ represents c. To see this, consider any $A \subseteq X$ and suppose $c_{(P_1,\succ_2)}(A) = x$. To see $c_{(P_1,P_2)}(A) = x$, suppose $\max(A; P_1) = B \subseteq A$. Since $\max(B; \succ_2) = x$, $\neg(b \succ_2 x)$ for all $b \in B \setminus \{x\}$. By definition of $\succ_2$, $\neg(bP_2x)$ for all $b \in B \setminus \{x\}$. Since c is single-valued and (P_1, P_2) represents c, there exists an item $b' \in B$ such that $b'P_2b$ for any $b \in B \setminus \{x\}$. Thus, $\max(B; P_2) = x$ so that $c_{(P_1,P_2)}(A) = \max(B; P_2) = x$. ■

Remark 2 *c has a Strotz representation iff it has a shortlisting representation with negatively transitive rationales.*

Proof. First, define $B(A; R) = \{a \in A : aRb \text{ for any } b \in A\}$. A standard result states that a rationale P is negatively transitive iff $\succeq_P$ is a weak order where $\succeq_P$ is defined by $x \succeq_P y$ if $\neg(yPx)$. Moreover, $\max(A; P) = B(A; \succeq_P)$. Apply this equivalence (twice) to obtain the result. ■

Remark 3 *C has a Strotz representation iff it can be represented by a weak order.*

Proof. ($\Rightarrow$) Suppose C can be represented by a pair of weak orders $(\succeq_v, \succeq_u)$. Define the lexicographic composition $\succeq$ by $x \succeq y$ if (i) $x \succ_v y$ (where the strict preference $x \succ_v y$ denotes $x \succeq_v y$ and $\neg[y \succeq_v x]$) or (ii) $x \sim_v y$ and $x \succeq_u y$ (where the indifference $x \sim_v y$ denotes $x \succeq_v y$ and $y \succeq_v x$). It is easy to see that $\succeq$ is complete and transitive, and that, moreover $C(A) = C_{(\succeq_v, \succeq_u)}(A) = B(A; \succeq)$ for any $A \subseteq X$. ($\Leftarrow$) Suppose C can be represented by $\succeq$. Set $\succeq_v = \succeq$ and define $\succeq_u$ to be total indifference. Then, it follows that $(\succeq_v, \succeq_u)$ is a Strotz representation. ■

For convenience, I state the following condition due to Lleras, Masatlioglu, Nakajima, and Ozbay:

LCA-WARP *For any $A \subseteq X$, there exists an $a^* \in A$ such that, for any $B \supset \{a^*\}$, $c(B) = a^*$ whenever (i) $c(B) \in A$, and (ii) $c(B') \neq c(B' \setminus \{a^*\})$ for some $B' \supset B$.*

The big-O notation used to measure time complexity in computer science can be defined as follows. Given two functions $f, g : X \to \mathbb{R}$ such that $X \subseteq \mathbb{R}$, $f(x) = O(g(x))$ iff there exists a positive real $k \in \mathbb{R}^{++}$ and a real $y \in \mathbb{R}$ such that $|f(x)| \leq k|g(x)|$ for all $x > y$.

Following Beja [1989], the complexity of an axiom is related to the largest collection of menus $W \subseteq 2^X$ instantiated by the axiom. Let $\mathcal{W}_n \subseteq 2^{2^X}$ denote the range of the axiom's universal quantifier for a domain X of size n. The *apparent simultaneity* of an axiom is then defined by the function $r : \mathbb{N} \to \mathbb{N}$ where $r(n) = \max_{W \in \mathcal{W}_n} |W|$. Finally, the complexity of the axiom is defined by $O(r(n))$. To get the basic intuition, consider the complexity of IIA. Since the axiom only addresses choice on two menus (as discussed in the text), it follows that $r(n) = 2$ so that the complexity of IIA is $O(1)$. By similar reasoning, it follows that:

Remark 4 *Choice Symmetry and Difficult Choice are $O(|X|)$, and LCA-WARP is $O(2^{|X|})$.*

Proof. Choice Symmetry and Difficult Choice: Choice Symmetry invariably addresses choice on two menus (i.e. $A \cup \{y\}$ and $A \cup \{x\}$). Given a domain X of size n, the axiom may, in the worst case, address choice on $2(n-2)$ additional menus of two alternatives (which happens when $A = X \setminus \{x, y\}$). Thus, $r(n) = 2(n-1)$ so that Choice Symmetry is $O(n)$. It is easy to carry out a similar accounting exercise for Difficult Choice.

LCA-WARP: Given a domain X of size n, the axiom applies to any $A \subseteq X$. For $A = X$, in particular, condition (i) of LCA-WARP addresses choice on every subset of X. Clearly, this is the worst case. Thus, $r(n) = 2^n - 1$ so that LCA-WARP is $O(2^n)$. ■

7.3 Axiomatic Characterization

The proof relies on results due to Manzini and Mariotti (M&M) [2007] and Au and Kawai (A&K) [2011]. For convenience, they are restated here. M&M give the following characterization of RSMs:

Theorem 1 (M&M) *c can be represented by an RSM iff it satisfies WWARP and Expansion.*

A&K characterize transitive RSMs with the help of the direct revealed preference $\succ_D^c$ – defined by $x \succ_D^c y$ if $c(x, y) = x$ but $c(A) \neq c(A \setminus \{y\})$ for some $A \supset \{x, y\}$ – and following property:

No Binary Chain Cycles (NBCC) *$\succ_D^c$ is acyclic.*

To state their representation theorem:

Theorem 1 (A&K) *An RSM c can be represented by a transitive RSM iff it satisfies NBCC.*

As an intermediate step in their proof, A&K introduce the following property:

Reduction *Suppose that $c(A) = y$ and $c(B) = x$ for $\{x, y\} \subseteq B \subset A$. Then, there exists a $z \in A \setminus B$ such that xyz is a 3-cycle and $c(x, y, z) = y$.*

The proof of the theorem given here uses Reduction as well as the following properties:

Selective IIA *If $c(A) = y$ and $c(x, y) = x$, then $c(A \setminus \{x\}) = y$.*

3-Acyclicity *Given 3-cycles wxy and wyz, $c(w, x, z) = x$ iff $c(w, y, z) = y$.*

4/5-Acyclicity *Suppose that axa' and aya' are 3-cycles s.t. $c(x, a, a') = a$ and $c(y, a, a') = a'$. If $c(x, y) = x$, then $c(x, y, z) = z$ for any 3-cycle xyz.*

The first property, Selective IIA, is a weakening of IIA which states that any alternative that is pairwise chosen over $c(A)$ can be discarded from A without affecting choice. The other two properties, 3-Acyclicity and 4/5-Acyclicity, restrict Choice Symmetry and Difficult Choice, respectively, to the case of 3-cycles.

Lemma 1 *If c satisfies WWARP, Expansion, and 3-Acyclicity, then it satisfies Selective IIA.*

Proof. Let $c(A) = y$ and $c(x, y) = x$. By way of contradiction, suppose $c(A \setminus \{x\}) = z \neq y$. The proof that this generates a contradiction is by induction on $|A| = n \geq 4$. The case $n = 2$ follows from the fact that c is a choice function. For $n = 3$, suppose $c(x, y, z) = y$, $c(x, y) = x$, and $c(y, z) = z$. If $c(x, z) = x$, Expansion implies $c(x, y, z) = c(\{x, y\} \cup \{x, z\}) = x \neq y$ which is a contradiction. If $c(x, z) = z$, Expansion implies $c(x, y, z) = c(\{y, z\} \cup \{x, z\}) = z \neq y$, another contradiction.

As a preliminary observation, note that $c(x, z) = x$ and $c(y, z) = z$. The first point follows by Expansion. Otherwise, $c(A) = c(A \setminus \{x\} \cup \{x, z\}) = z \neq y$. The second point follows by WWARP. Otherwise, $c(A) = y = c(y, z)$, $c(A \setminus \{x\}) = z$, and $\{y, z\} \subset A \setminus \{x\} \subset A$.

Base case $n = 4$: By way of contradiction, suppose $c(w, x, y, z) = y$, $c(x, y) = x$, and $c(w, y, z) = z$. Since $c(x, z) = x$, it follows that $c(w, x) = w$. Otherwise, Expansion delivers $c(x, y, z) = c(\{x, y\} \cup \{x, z\}) = x$ and, consequently, $c(w, x, y, z) = c(\{x, w\} \cup \{x, y, z\}) = x \neq y$. Moreover, $c(y, w) = y$. Otherwise, regardless of $c(w, x, z)$, Expansion implies $c(w, x, y, z) = c(\{a, y\} \cup \{w, x, z\}) = a \neq y$ where $c(w, x, z) = a$. Thus, wxy is a 3-cycle. This leaves two cases: (i) $c(w, z) = w$; and, (ii) $c(w, z) = z$. I show that both lead to contradictions.

In case (i), wzy is a 3-cycle. Since $c(w, y, z) = z$, 3-Acyclicity implies $c(w, x, y) = x$. By Expansion, it follows that $c(w, x, y, z) = c(\{x, z\} \cup \{w, x, y\}) = x \neq y$ which is a contradiction.

In case (ii), wxz is a 3-cycle. Moreover, $c(w, x, y) = y$. If $c(w, x, y) = x$, then Expansion implies $c(w, x, y, z) = c(\{x, z\} \cup \{w, x, y\}) = x \neq y$. If $c(w, x, y) = w$, then WWARP is violated since $c(w, x, y, z) = y = c(y, w)$ and $\{w, y\} \subset \{w, x, y\} \subset \{w, x, y, z\}$. By 3-Acyclicity, $c(w, x, y) = y$ implies $c(w, x, z) = z$. By Expansion, it follows that $c(\{y, z\} \cup \{w, x, z\}) = c(w, x, y, z) = z \neq y$ which is a contradiction.

Induction Step: Suppose Selective IIA holds for $|A| = n$. By way of contradiction, suppose $c(A) = y$, $c(x, y) = x$, and $c(A \setminus \{x\}) = z$ for $|A| = n + 1$. First, observe that there is some

$a \in A \setminus \{y\}$ such that $c(A \setminus \{a\}) = y$. Otherwise, the pigeonhole principle ensures that there exists some $b \in A \setminus \{y\}$ such that $c(A \setminus \{a'\}) = b = c(A \setminus \{a''\})$ for distinct $a', a'' \in A \setminus \{b\}$. By Expansion, this is a contradiction since $c(A) = c([A \setminus \{a'\}] \cup [A \setminus \{a''\}]) = b \neq y$.

Consider any such $a \in A \setminus \{y\}$. By the induction hypothesis, $c(A \setminus \{a\}) = y$ implies $c(A \setminus \{x, a\}) = y$ since $|A \setminus \{a\}| = n$ and $c(x, y) = x$. It follows that $a = z$. Otherwise, $c(A \setminus \{x\}) = z = c(y, z)$, $c(A \setminus \{x, a\}) = y$, and $\{y, z\} \subset A \setminus \{a, x\} \subset A \setminus \{x\}$ contradict WWARP. Moreover, $c(A \setminus \{x'\}) = y$ for any $x' \in A \setminus \{x\}$ such that $c(x', y) = x'$. If $c(A \setminus \{x'\}) = z' \neq y$, a similar line of reasoning implies $a = z'$, which contradicts $a = z$ unless $z' = z$. By Expansion, $z' = z$ is a contradiction since $c(A) = c([A \setminus \{x\}] \cup [A \setminus \{x'\}]) = z \neq y$.

Finally, observe that $c(x', y) = y$ for any $x' \in A \setminus \{x, y, z\}$. Otherwise, $c(A \setminus \{x'\}) = y$ implies $c(A \setminus \{x, x'\}) = y$ by the induction hypothesis since $|A \setminus \{x'\}| = n$. By similar reasoning, $c(A \setminus \{z\}) = y$ implies $c(A \setminus \{x, z\}) = y$. By Expansion, it follows that $c(A \setminus \{x\}) = c([A \setminus \{x, x'\}] \cup [A \setminus \{x, z\}]) = y \neq z$ which is a contradiction.

As such, for any $x' \in A \setminus \{x, y, z\}$: either (i) $c(x, x') = x'$ so that $x'xy$ is a 3-cycle; or, (ii) $c(x, x') = x$. First, note that there must be a type-(i) item x'. Otherwise, $c(x, x') = x$ for all $x' \in A \setminus \{x\}$ so that repeated application of Expansion on $\{x, y\}$ gives $c(A) = x \neq y$. Next, note that $c(x, x', y) = y$ for some x' of type-(i). Otherwise, $c(x, x', y) = x$ for every x' of type-(i). [That $c(x, x', y) \neq x'$ for any x' of type-(i) follows by WWARP since $c(A) = y = c(x', y)$ and $\{x', y\} \subset \{x, x', y\} \subset A$.] Then, repeated application of Expansion on $\{x, y\}$ [using $\{x, x', y\}$ for type-(i) x' and $\{x, x'\}$ for type-(ii) x'] gives $c(A) = x \neq y$.

Now, consider any $x'xy$ such that $c(x, x', y) = y$. There are two separate cases. If $c(x', z) = z$, $x'xz$ is a 3-cycle. By 3-Acyclicity, $c(x, x', y) = y$ implies $c(x, x', z) = z$. By Expansion, it follows that $c(A) = c([A \setminus \{x\}] \cup \{x, x', z\}) = z \neq y$. If $c(x', z) = x'$, $x'zy$ is a 3-cycle. By 3-Acyclicity, $c(x, x', y) = y$ implies $c(x', y, z) \neq z$. But, this generates a contradiction. By WWARP, it follows that: $c(x', y, z) \neq x'$ since $c(A) = y = c(x', y)$ and $\{x', y\} \subset \{x', y, z\} \subset A$; and, $c(x', y, z) \neq y$ since $c(A \setminus \{x\}) = z = c(y, z)$ and $\{y, z\} \subset \{x', y, z\} \subset A \setminus \{x\}$.

This establishes Selective IIA for $|A| = n + 1$. ■

Lemma 2 *If c satisfies WWARP, Expansion, and 3-Acyclicity, then it satisfies Reduction.*

Proof. Suppose $c(A) = y$ and $c(B) = x$ for $\{x, y\} \subseteq B \subset A$. As a preliminary observation, note that $c(x, y) = x$. Otherwise, WWARP is violated since $c(A) = y$, $c(B) = x$, and $\{x, y\} \subseteq B \subset A$.

First, consider any $z \in A \setminus B$ such that $c(x, z) = x$. By Expansion, it follows that $c(B \cup \{x, z\}) = x$. Repeating the same argument, it follows that $c(B') = x$ for $B' = B \cup \{z \in A \setminus B : c(x, z) = x\}$.

Next, consider any $z \in A \setminus B'$ such that xyz is a 3-cycle where $c(x, y, z) = x$. It follows that $c(A \setminus \{z\}) = y$. If $c(A \setminus \{z\}) = x$, Expansion implies $c(A) = c([A \setminus \{z\}] \cup \{x, y, z\}) = x \neq y$. If $c(A \setminus \{z\}) = z' \notin \{x, y\}$, $z'yz$ is a 3-cycle such that $c(y, z, z') = z'$. Then, by Expansion, it follows that $c(A) = c([A \setminus \{z\}] \cup \{y, z, z'\}) = z' \neq y$. To see that $z'yz$ is a 3-cycle, first observe that $c(y, z') = z'$ by WWARP since $c(A \setminus \{z\}) = z'$, $c(A) = y$, and $\{y, z'\} \subset A \setminus \{z\} \subset A$. Next, observe that $c(z, z') = z$ by Expansion. Otherwise, $c(A) = c([A \setminus \{z\}] \cup \{z, z'\}) = z' \neq y$. Then, $z'yz$ is a 3-cycle. Since $c(x, y, z) = x$, $c(y, z, z') = z'$ follows by 3-Acyclicity. Repeating this argument, $c(A') = y$ for $A' = A \setminus \{z \in A \setminus B' : xyz \text{ is a 3-cycle s.t } c(x, y, z) = x\}$.

To complete the proof, it suffices to show that there is some $z \in A' \setminus B'$ s.t. $c(y, z) = y$. Then, xyz is a 3-cycle because $c(x, z) = z$ and $c(x, y) = x$. Since $\{y, z\} \subset \{x, y, z\} \subset A$ and $c(A) = y = c(y, z)$, $c(x, y, z) = z$ violates WWARP. Since $c(x, y, z) \neq x$ by construction, $c(x, y, z) = y$.

To see that there is some $z \in A' \setminus B'$ such that $c(y, z) = y$, suppose otherwise. Then, $c(y, z) = z$ for any $z \in A' \setminus B'$. By Lemma 1, the fact that c satisfies WWARP, Expansion, and 3-Acyclicity implies that it satisfies Selective IIA. By Selective IIA, it follows that $c(A' \setminus \{z\}) = y$ for any $z \in A' \setminus B'$. Repeating this argument, it follows that $c(B') = y \neq x$ which is a contradiction. ■

Remark 5 *Lemma 2 corrects an error in A&K's proof of Claim 5. Along the same lines as Lemma 2, they establish that WWARP, Expansion, and NBCC imply $c(B') = x$ and $c(A') = y$. Then, by way of contradiction, they attempt to show that there is some $z \in A' \setminus B'$ such that $c(y, z) = y$. They claim that WWARP and Expansion imply $c(B' \cup \{z\}) = x$ for some $z \in A' \setminus B'$ such that $c(y, z) = z$. Example 3 below provides a counter-example to this claim. By Lemma 3 below however, A&K's Claim 5 is nonetheless correct by the argument given in Lemma 2.*

Example 3 *Consider the choice function c from Example 1 with $c(w, z) = z$.*

By the discussion following Example 1, c satisfies WWARP and Expansion. However, it does not satisfy the property claimed by Au and Kawai. To see this, note that $c(w, x, z) = z$ and $c(y, z) = y$ but $c(w, x, y, z) = w \neq z$. Interestingly, c also fails to satisfy Selective IIA. To see this, note that $c(w, x, y, z) = w$ and $c(w, y) = y$ but $c(w, x, z) = z$. Up to relabeling, this is this is one of two choice functions on $|X| = 4$ that fails either property but nonetheless satisfies WWARP and Expansion. The other possibility is identical to c except that x is chosen from $\{w, x, y, z\}$.

Lemma 3 *If c satisfies WWARP, Expansion, and NBCC, it satisfies Selective IIA.*

Proof. It suffices to show that NBCC imply 3-Acyclicity. By way of contradiction, suppose $c(x, y, z) = z$ and $c(w, x, y) \neq w$. Then, by definition of $\succ_D^c$, $y \succ_D^c z \succ_D^c x$ and $x \succ_D^c y$. Since $x \succ_D^c y \succ_D^c z \succ_D^c x$ forms a $\succ_D^c$-cycle, this contradicts NBCC. Thus, c satisfies 3-Acyclicity. Since also satisfies c satisfies WWARP and Expansion, the result then follows by Lemma 1. ■

Lemma 4 *If c satisfies WWARP and Expansion, 3-Acyclicity is equivalent to Choice Symmetry.*

Proof. ($\Leftarrow$) Obvious. ($\Rightarrow$) Suppose that x, y are similar w.r.t A. First, note that c satisfies Selective IIA by Lemma 1. The proof is by induction on $|A|$. The case $|A| = 1$ is trivial. The case $|A| = 2$ holds by 3-Acyclicity if $A \cup \{x\}$ and $A \cup \{y\}$ are 3-cycles [and by Expansion otherwise].

So, suppose Choice Symmetry holds for $|A| = n$. By way of contradiction, suppose $c(A \cup \{x\}) = x$ and $c(A \cup \{y\}) = a \neq y$. First, suppose that $c(a, x) = a$. By Expansion, $c(A \cup \{x, y\}) = c([A \cup \{y\}] \cup \{x, a\}) = a$. But, this contradicts WWARP since $c(a, x) = a$, $c(A \cup \{x\}) = x$, and $\{a, x\} \subset A \cup \{x\} \subset A \cup \{x, y\}$.

Next, suppose that $c(a, x) = x$. If $c([A \setminus \{a\}] \cup \{x\}]) = x$, the induction hypothesis implies $c([A \setminus \{a\}] \cup \{y\}]) = y$. By Expansion, $c(A \cup \{y\}) = c([[A \setminus \{a\}] \cup \{y\}] \cup \{a, y\}) = y \neq a$. So, $c([A \setminus \{a\}] \cup \{x\}) = a' \neq x$. By Expansion, it follows that $c(a, a') = a$. Otherwise, $c(A \cup \{x\}) = c([[A \setminus \{a\}] \cup \{x\}] \cup \{a, a'\}) = a' \neq x$. Moreover, $c(a', x) = a'$. Otherwise, $c(A \cup \{x\}) = x = c(a', x)$, $c([A \setminus \{a\}] \cup \{x\}) = a'$, and $\{a', x\} \subset [A \setminus \{a\}] \cup \{x\} \subset A \cup \{x\}$ contradict WWARP.

Thus, xaa' is a 3-cycle. Observe that $c(a, a', x) = x$. If $c(a, a', x) = a'$, Expansion implies $c(A \cup \{x\}) = c([[A \setminus \{a\}] \cup \{x\}] \cup \{a, a', x\}) = a' \neq x$. If $c(a, a', x) = a$, $c(A \cup \{x\}) = x = c(a, x)$ and $\{a, x\} \subset \{a, a', x\} \subset A \cup \{x\}$ contradict WWARP. So, by Selective IIA, $c(a', x) = a'$ and $c(A \cup \{x\}) = x$ imply $c([A \setminus \{a'\}] \cup \{x\}) = x$.

By the base case of the induction, $c(a, a', x) = x$ implies $c(a, a', y) = y$ since x, y are similar w.r.t. $\{a, a'\} \subset A$. By the induction hypothesis, $c([A \setminus \{a'\}] \cup \{x\}) = x$ implies $c([A \setminus \{a'\}] \cup \{y\}) = y$ since x, y are similar w.r.t. $A \setminus \{a'\} \subset A$ and $|A \setminus \{a'\}| = n$. By Expansion, it follows that $c(A \cup \{y\}) = c([[A \setminus \{a'\}] \cup \{y\}] \cup \{a, a', y\}) = y \neq a$ which is a contradiction. ■

Lemma 5 *Suppose c satisfies WWARP, Expansion, and 3-Acyclicity. Then $x \succ_D^c y$ iff $x\overline{R}_2^c y$.*

Proof. Recall that $x\overline{R}_2^c y$ iff (i) there exists a 3-cycle xyz such that $c(x, y, z) \neq z$ or, (ii) $c(x, y) = x$ and there exist 3-cycles $ww'x$ and $ww'y$ s.t. $c(w, w', x) = w$ and $c(w, w', y) = w'$ (Definition 3).

($\Leftarrow$) First consider case (i). Since $c(x, z) = z$ and $c(x, y, z) \neq z$, it follows by definition that $x \succ_D^c y$. Next, consider case (ii). Since $c(w, w', y) = w'$ and $c(w', x) = w'$, Expansion implies $c(w, w', x, y) = c(\{w, w', y\} \cup \{w', x\}) = w'$. Since $c(w, w', x) = w$ and $c(x, y) = x$, it follows by definition that $x \succ_D^c y$.

($\Rightarrow$) Suppose $x \succ_D^c y$ and $c(x, y, z) = z$ for any 3-cycle xyz. By definition of $\succ_D^c$, there exists some $A \supset \{x, y\}$ such that $c(A) \neq c(A \setminus \{y\})$. First, observe that c satisfies Reduction by Lemma 2.

Next, observe that $c(A) \neq x, y$. First suppose $c(A) = x$. Since $c(A \setminus \{y\}) = b \neq x$, Reduction implies bxy a 3-cycle s.t. $c(b, x, y) = x$. But this contradicts the assumption $c(x, y, z) = z$ for any 3-cycle xyz. So, $c(A) \neq x$. Next suppose $c(A) = y$. Since $c(x, y) = x$, Reduction implies that there exists a 3-cycle xyz with $z \in A \setminus \{x, y\}$ such that $c(x, y, z) = x$. Again, this contradicts the assumption $c(x, y, z) = z$ for any 3-cycle xyz. So, $c(A) \neq y$.

Thus, $c(A) = a$ and $c(A \setminus \{y\}) = b$ where $a \neq x, y$ and $b \neq x$. Given that $c(A) \neq x, y$, this is without loss of generality. If $b = x$, Expansion requires $c(A) = c(\{x, y\} \cup (A \setminus \{y\})) = x \neq a$. Since $\{a, b, x, y\} \subset A$ and each of the elements of $\{a, b, x, y\}$ is distinct, it follows that $|A| \geq 4$.

Finally, observe that: **(I)** bay is a 3-cycle s.t $c(a, b, y) = a$; and, **(II)** xbz is a 3-cycle s.t $c(b, x, z) = b$ for $z \in A \setminus \{b, x, y\}$. To establish **(I)**, note that, by Reduction, $c(A) = a$ and $c(A \setminus \{y\}) = b$ imply that bay is a 3-cycle s.t. $c(a, b, y) = a$. To establish **(II)**:

Proof of (II): If $c(b, x) = x$, Reduction implies the desired result since $c(A \setminus y) = b$ and $c(b, x) = x$. To see that $c(b, x) = x$, suppose otherwise. If $c(b, x) = b$, xyb is a 3-cycle. Since $c(b, y) = y$ holds by (I) and $c(x, y) = x$, $c(b, x) = b$ hold by assumption, xyb is a 3-cycle. Then, by assumption, $c(b, x, y) = b$. By Expansion, it follows that $c(A) = c(\{b, x, y\} \cup [A \setminus \{y\}]) = b \neq a$. This contradiction establishes $c(b, x) = x$. $\square$

To complete the proof, consider the following cases: (1) $c(x, a) = x$; (2) $c(x, a) = a$ and $c(y, z) = z$; and, (3) $c(x, a) = a$ and $c(y, z) = y$. [Whether $c(a, z) = a$ or $c(a, z) = z$ is unimportant.] I show that (1) and (2) deliver the implication (ii) and that (3) leads to a contradiction.

Case (1): Since $c(a, b) = b$ and $c(b, x) = x$ (by (I) and (II) above), bax is a 3-cycle. Moreover, $c(a, b, x) = b$. To see this, note that $c(a, b, x) = x$ contradicts WWARP since $\{a, x\} \subset \{a, b, x\} \subset A$ and $c(a, x) = a = c(A)$. Similarly, $c(a, b, x) = a$ contradicts WWARP since $\{a, b\} \subset \{a, b, x\} \subset A \setminus \{y\}$ and $c(a, b) = b = c(A \setminus \{y\})$. Combined with (I), this delivers the desired implication (ii) with $b = w$ and $a = w'$. **Note:** If $|A| = 4$, $A \setminus \{b, x, y\} = \{a\}$ so that $z = a$. Thus, case (1) holds and (2)-(3) need not be considered.

Case (2): Since $c(b, z) = b$ and $c(b, y) = y$ (by (I) and (II) above), bzy is a 3-cycle. Since bay is a 3-cycle s.t. $c(a, b, y) = a$ (by (I) above), 3-Acyclicity implies $c(b, y, z) = z$. Combined with (II), this delivers the desired implication (ii) with $b = w$ and $z = w'$.

Case (3): Since $c(x, y) = x$ and $c(z, x) = z$ (by (II) above), xyz is a 3-cycle. Since xbz is a 3-cycle s.t. $c(b, x, z) = b$ (by (II) above), 3-Acyclicity implies $c(x, y, z) = y$. But, this contradicts the assumption $c(x, y, z) = z$ for any 3-cycle xyz.

This establishes the desired implication (ii) when (i) does not hold. The result follows. $\blacksquare$

Remark 6 *The previous lemma shows that $\overline{R}_2^c \supseteq R_2^c$ is equivalent to the revealed preference $\succ_D^c$ defined by Au and Kawai [2011] (in the presence of WWARP, Expansion, and 3-Acyclicity). The next lemma shows the equivalence between R_2^c and the revealed preference defined by Yildiz [2011].*

Lemma 6 *Suppose c satisfies WWARP, Expansion, and 3-Acyclicity. Then $xR_2^c y$ iff:*
(i) $c(x, y) = x$ and $c(A) = y$ for some $A \supset \{x, y\}$; or, (ii) $c(B \cup \{y\}) = x \neq c(B)$.

Proof. ($\Rightarrow$) Suppose $xR_2^c y$. By definition of R_2^c, there exists a 3-cycle xyz such that $c(x,y,z) \neq z$. If $c(x,y,z) = x$, then (ii) is satisfied with $B = \{x,z\}$. If $c(x,y,z) = y$, then (i) is satisfied with $A = \{x,y,z\}$. ($\Leftarrow$) First, suppose $c(x,y) = x$ and $c(A) = y$ for some $A \supset \{x,y\}$. By Reduction, there exists a 3-cycle xyz such that $c(x,y,z) = y$. By definition of R_2^c, $xR_2^c y$. Next, suppose that $c(B \cup \{y\}) = x \neq c(B) = b$. By Reduction, bxy is a 3-cycle such that $c(b,x,y) = x$. By definition, it follows that $xR_2^c y$. ■

Lemma 7 *Suppose that c satisfies WWARP, Expansion, and 3-Acyclicity. If $c(A \cup \{x\}) = a$, $c(A \cup \{y\}) = a'$, and x, y are similar w.r.t. A, then $c(x,y) = x$ and $c(A \cup \{x,y\}) \neq x, a'$ imply that axa' and aya' are 3-cycles such that $c(a,a',x) = a$ and $c(a,a',y) = a'$.*

Proof. Note that c satisfies Selective IIA, Reduction, and Choice Symmetry by Lemmas 1-4.

First, observe that $a \neq x$, $a \neq a'$, and $a' \neq y$. If $a = x$, Expansion implies $c(A \cup \{x,y\}) = c([A \cup \{x\}] \cup \{x,y\}) = x$. So, $a \neq x$. By Choice Symmetry, $a = a'$ or $a' \notin \{a,y\}$. If $a = a'$, Expansion implies $c(A \cup \{x,y\}) = c([A \cup \{x\}] \cup [A \cup \{y\}]) = a' = c(A \cup \{y\})$. So, $a' \notin \{a,y\}$.

Next, note that axa' is a 3-cycle s.t. $c(a,a',x) = a$. If $c(a',x) = a'$, $c(A \cup \{x,y\}) = c([A \cup \{y\}] \cup \{a',x\}) = a' = c(A \cup \{y\})$ by Expansion. Thus, $c(a',x) = x$. By symmetry, $c(a',y) = y$. Since $c(A \cup \{y\}) = a'$, $c(A) = a'$ by Selective IIA. Since $c(A \cup \{x\}) = a$ and $c(A) = a'$, Reduction implies that axa' is a 3-cycle and $c(a,a',x) = a$.

By 3-Acyclicity, aya' is a 3-cycle such that $c(a,a',y) \neq y$. Since $\{a,a'\} \subset \{a,a',y\} \subset A \cup \{y\}$ and $c(a,a') = a' = c(A \cup \{y\})$, WWARP implies $c(a,a',y) \neq a$. Thus, $c(a,a',y) = a'$. ■

Lemma 8 *Expansion and Difficult Choice together imply 4/5-Acyclicity.*

Proof. Suppose axa' and aya' are 3-cycles such that $c(a,a',x) = a$, $c(a,a',y) = a'$, and $c(x,y) = x$. As such, x, y are similar w.r.t. $\{a,a'\}$. Moreover, $c(a,a',x,y) = c(\{a,a',x\} \cup \{a,y\}) = a$ by Expansion. So, $c(a,a',x,y) \neq x$. Since $c(a,a',x,y) = a' \neq a = c(a,a',y)$, Difficult Choice implies $c(B) = c(B \setminus \{y\})$ for any $B \supset \{x,y\}$. Now consider any $B = \{x,y,z\}$ such that xyz is a 3-cycle. Since $c(B \setminus \{y\}) = c(x,z) = z$, it follows that $c(B) = c(B \setminus \{y\})$ only when $c(x,y,z) = z$. Thus, $c(x,y,z) = z$ for any 3-cycle xyz. ■

Lemma 9 *If c satisfies WWARP, Expansion, 3-Acyclicity, and 4/5-Acyclicity, then R_2^c is acyclic.*

Proof. Consider any RSM c that satisfies 3-Acyclicity and 4/5-Acyclicity. Suppose that there exists an R_2^c-cycle $x_0 ... x_i ... x_{n-1}$ (i.e. such that $x_0 R_2^c ... R_2^c x_i R_2^c ... R_2^c x_{n-1} R_2^c x_0$). First, observe that R_2^c is asymmetric by definition. Thus, $x_0 R_2^c x_0$ and $x_0 R_2^c x_1 R_2^c x_0$ entails a contradiction. The proof that there is a contradiction for $n \geq 3$ follows by strong induction on the length of the R_2^c-cycle.

Base case $n = 3$: Note that $x_0 x_1 x_2$ is a 3-cycle. Without loss of generality, suppose $c(x_0, x_1, x_2) = x_0$. Then, $x_2 R_2^c x_0 R_2^c x_1$ by definition of R_2^c. Since $x_1 R_2^c x_2$, there exists an item $a \notin \{x_0, x_1, x_2\}$ such that ax_1x_2 is a 3-cycle and $c(a, x_1, x_2) \neq a$. But, this is a contradiction. Since x_0, a are similar w.r.t. $\{x_1, x_2\}$ and $c(x_0, x_1, x_2) = x_0$, 3-Acyclicity requires $c(a, x_1, x_2) = a$.

Induction step: Without loss of generality, suppose $c(x_0, ..., x_{n-1}) = x_0$. In order to establish the result, first observe that: (I) $x_{i-1}x_ix_{i+1}$ is a 3-cycle for $i \neq 1, 2$ [to be understood (mod n) when $i = 0, n-1$]; (II) there are no other 3-cycles $x_jx_ix_k$; and, (III) $c(x_{i-1}, x_i, x_{i+1}) = x_i$. These observations follow from: **(i)** $c(x_0, x_2) = x_0$; and, **(ii)** $c(x_1, x_3) = x_1$.

Proof of (i): By way of contradiction, suppose $c(x_0, x_2) = x_2$. Then, $x_0x_1x_2$ is a 3-cycle. From the base case of the induction, it follows that $c(x_0, x_1, x_2) = x_1$. Otherwise, $x_2R_2^c x_0$ so that $x_0x_1x_2$ is an R_2^c-cycle of length 3. Since $\{x_0, x_1\} \subset \{x_0, x_1, x_2\} \subset \{x_0, ..., x_{n-1}\}$ and $c(x_0, x_1) = x_0 = c(x_0, ..., x_{n-1})$ however, $c(x_0, x_1, x_2) = x_1$ violates WWARP. This is the desired contradiction. $\square$

To establish **(ii)**, observe that, for $n \geq 5$, **(i)** implies:

$$c(x_i, x_j) = x_i \text{ for } [i = 0 \text{ and } 2 \leq j \leq n-2] \text{ or } [i \geq 4 \text{ and } 2 \leq j \leq i-2] \tag{1}$$

To see this, first note that $c(x_0, x_2) = x_0$ implies $c(x_0, x_3) = x_0$. Otherwise, $x_0x_2x_3$ is a 3-cycle. Regardless of $c(x_0, x_2, x_3)$, $x_0R_2^c x_2$ or $x_3R_2^c x_0$. Both contradict the induction hypothesis. In the first case, $x_0x_2...x_{n-1}$ is an R_2^c-cycle of length $n-1$. In the second case, $x_0...x_3$ is an R_2^c-cycle of length 4. By a simple induction argument that follows the same line of reasoning, $c(x_0, x_j) = x_0$ for any j s.t. $2 \leq j \leq n-2$.

Next, observe that, by similar reasoning, $c(x_0, x_2) = x_0$ implies $c(x_{n-1}, x_2) = x_{n-1}$. Then, by a simple induction argument, it follows that $c(x_i, x_2) = x_i$ for any $i \geq 4$. Applying the same type of induction argument to each $i \geq 4$ gives $c(x_i, x_j) = x_i$ for any $2 \leq j \leq i-2$.

Proof of (ii): Suppose $c(x_1, x_3) = x_3$. Consider the cases $n = 4$, $n = 5$, and $n > 5$ separately:

Case $n = 4$**:** Since $c(x_0, x_2) = x_0$, $x_0x_2x_3$ is a 3-cycle. From the base case of the induction, it follows that $c(x_0, x_2, x_3) = x_3$. Otherwise, $x_0R_2^c x_2$ so that $x_0x_2x_3$ is an R_2^c-cycle of length 3. Since $c(x_1, x_3) = x_3$, Expansion implies $c(x_0, x_1, x_2, x_3) = c(\{x_0, x_2, x_3\} \cup \{x_1, x_3\}) = x_3$. But, this contradicts the assumption $c(x_0, x_1, x_2, x_3) = x_0$. So, $c(x_1, x_3) = x_1$.

To establish the result for $n \geq 5$, first observe that $x_{n-1}x_0x_1$ is a 3-cycle s.t. $c(x_0, x_1, x_{n-1}) = x_0$. To see that $c(x_1, x_{n-1}) = x_1$, suppose otherwise. Since $c(x_i, x_{n-1}) = x_{n-1}$ for $i \neq 1$ [by observation (1)] and $c(x_1, x_{n-1}) = x_{n-1}$ [by assumption], repeated application of Expansion on $\{x_1, x_{n-1}\}$ implies $c(x_0, ..., x_{n-1}) = x_{n-1}$. But, this contradicts the assumption that $c(x_0, ..., x_{n-1}) = x_0$. Thus, $c(x_1, x_{n-1}) = x_1$ so that $x_{n-1}x_0x_1$ is a 3-cycle. By the base case of the induction, it follows that $c(x_0, x_1, x_{n-1}) = x_0$. Otherwise, $x_1R_2^c x_{n-1}$ so that $x_{n-1}x_0x_1$ is an R_2^c-cycle of length 3.

Case $n = 5$**:** Since $c(x_4, x_2) = x_4$ [by observation (1)], $x_2x_3x_4$ is a 3-cycle. From the base case of the induction, it follows that $c(x_2, x_3, x_4) = x_3$. Otherwise, $x_4R_2^c x_2$ so that $x_2x_3x_4$ is an R_2^c-cycle of length 3. Since $c(x_1, x_3) = x_3$ [by assumption], $x_1x_2x_3$ is a 3-cycle. From the base case of the induction, it follows that $c(x_1, x_2, x_3) = x_2$. Otherwise, $x_3R_2^cx_1$ so that $x_1x_2x_3$ is an R_2^c-cycle of length 3. Since $c(x_1, x_2, x_3) = x_2$, $c(x_2, x_3, x_4) = x_3$, $c(x_1, x_4) = x_1$, and $x_4x_0x_1$ is a 3-cycle, 4/5-Acyclicity requires $c(x_0, x_1, x_4) \neq x_0$. But, this contradicts $c(x_0, x_1, x_4) = x_0$. So, $c(x_1, x_3) = x_1$.

Case $n > 5$**:** Since $c(x_3, x_{n-1}) = x_{n-1}$ for $i \neq 1$ [by observation (1)] and $c(x_1, x_3) = x_3$ [by assumption], $x_1x_{n-1}x_3$ is a 3-cycle. Since $x_0x_1x_{n-1}$ is a 3-cycle such that $c(x_0, x_1, x_{n-1}) = x_0$, 3-Acyclicity implies $c(x_1, x_3, x_{n-1}) = x_3$ so that $x_3R_2^cx_1$ and $x_1x_2x_3$ is an R_2^c-cycle of length 3. But this contradicts the base case of the induction. So, $c(x_1, x_3) = x_1$. □

For $n = 4$, (i)-(ii) and the fact that $c(x_i, x_{i+1}) = x_i$ directly imply (I)-(III). For $n \geq 5$, observe that (ii) implies $c(x_1, x_i) = x_1$ for any $i \neq 0, 1$. This follows by a simple induction argument [along the same lines as observation (1)]. Together with observation (1) and the fact that $c(x_i, x_{i+1}) = x_i$, this establishes facts (I)-(III) for $n \geq 5$. Since choice from all pairs $\{x_i, x_j\}$ are identified, facts (I) and (II) are immediate. Fact (III) then follows from the base case of the induction. If $c(x_{i-1}, x_i, x_{i+1}) \neq x_i$, $x_{i+1}R_2^cx_{i-1}$ so that $x_{i-1}x_ix_{i+1}$ is an R_2^c-cycle of length 3.

One consequence of facts (I) and (II) is that there exists no 3-cycle $x_1x_2x_i$. In order for $x_1R_2^cx_2$, there must exist an item $a \notin \{x_0, ..., x_{n-1}\}$ such that ax_1x_2 is a 3-cycle and $c(a, x_1, x_2) \neq a$. Now, suppose that such an item a exists. I consider the cases $n = 4$, $n = 5$, and $n > 5$, and show that each entails a contradiction:

Case $n = 4$**:** By facts (I) and (III), $x_0x_1x_3$ and $x_0x_2x_3$ are 3-cycles such that $c(x_0, x_1, x_3) = x_0$ and $c(x_0, x_2, x_3) = x_3$. Since $c(x_1, x_2) = x_1$ by fact (I), 4/5-Acyclicity requires $c(b, x_1, x_2) = b$ for any 3-cycle bx_1x_2. But, this contradicts $c(a, x_1, x_2) \neq a$.

Case $n = 5$**:** Since $c(x_1, x_4) = x_1$ and $c(x_2, x_4) = x_4$, $ax_1x_4x_2$ is a 4-cycle. There are two cases: (5-1) $c(x_4, a) = x_4$ so that ax_1x_4 is a 3-cycle; and, (5-2) $c(x_4, a) = a$ so that ax_4x_2 is a 3-cycle.

Sub-case (1): By 3-Acyclicity, it follows that $c(a, x_1, x_4) = a$. This follows from the fact that $x_0x_1x_4$ is a 3-cycle such that $c(x_0, x_1, x_4) = x_0$ [by facts (I) and (III)]. Since $c(a, x_1, x_4) = a$, 3-Acyclicity also implies $c(a, x_1, x_2) \neq x_2$. Since $c(a, x_1, x_2) \neq a$, it follows that $c(a, x_1, x_2) = x_1$. Since ax_1x_2 and ax_1x_4 are 3-cycles such that $c(a, x_1, x_4) = a$, $c(a, x_1, x_2) = x_1$, and $x_2x_3x_4$ is a 3-cycle such that $c(x_2, x_3, x_4) = x_3$ [which follows from facts (I) and (III)], 4/5-Acyclicity requires $c(a, x_1) = x_1$. But, this contradicts the fact that $c(a, x_1) = a$.

Sub-case (2): The reasoning is similar to sub-case (5-1). First, 3-Acyclicity implies $c(a, x_2, x_4) = a$ [since $x_2x_3x_4$ is a 3-cycle such that $c(x_2, x_3, x_4) = x_3$ by facts (I) and (III)]. Next, 3-Acyclicity implies $c(a, x_1, x_2) = x_2$. Since $x_0x_1x_4$ is a 3-cycle such that $c(x_0, x_1, x_4) = x_0$ [by facts (I) and (III)], 4/5-Acyclicity implies $c(x_1, x_4) = x_1$. But, this contradicts the fact that $c(x_1, x_4) = x_4$.

Case $n > 5$**:** First, observe that $c(a, x_{n-1}) = x_{n-1}$ and $c(a, x_{n-2}) = a$.

To see $c(a, x_{n-1}) = x_{n-1}$, suppose otherwise. Then, $ax_{n-1}x_2$ is a 3-cycle. By the induction hypothesis, it follows that $c(a, x_2, x_{n-1}) = a$. Otherwise $x_{n-1} R_2^c x_2$ so that $x_2 ... x_{n-1}$ is an R_2^c-cycle of length $n-3$. Since $c(a, x_2, x_{n-1}) = a$, it follows by definition of R_2^c that $x_2 R_2^c a R_2^c x_{n-1}$. Thus, $x_0 x_1 x_2 a x_{n-1}$ is an R_2^c-cycle of length 5. But, this contradicts the induction hypothesis. So, $c(a, x_{n-1}) = x_{n-1}$.

The reasoning for $c(a, x_{n-2}) = a$ is similar. First, suppose $c(a, x_{n-2}) = x_{n-2}$. Then, $a x_1 x_{n-2}$ is a 3-cycle. By the induction hypothesis, it follows that $c(a, x_1, x_{n-2}) = a$. Otherwise $x_1 R_2^c x_{n-2}$ so that $x_{n-2} ... x_1$ is an R_2^c-cycle of length 4. Since $c(a, x_1, x_{n-2}) = a$, it follows that $x_{n-2} R_2^c a R_2^c x_1$ so that $x_1 ... x_{n-2} a$ is an R_2^c-cycle of length $n-1$. But, this contradicts the induction hypothesis. So, $c(a, x_{n-2}) = a$.

Since $c(a, x_{n-1}) = x_{n-1}$ and $c(a, x_{n-2}) = a$, it follows that $a x_1 x_{n-1}$ and $a x_{n-2} x_2$ are 3-cycles. Since $x_{n-1} x_0 x_1$ is a 3-cycle such that $c(x_{n-1}, x_0, x_1) = x_0$ [by facts (I) and (III)], $c(a, x_{n-1}, x_1) = a$ follows by 3-Acyclicity. Consequently, $c(a, x_1, x_2) \neq x_2$ by 3-Acyclicity. Since $c(a, x_1, x_2) \neq a$, it follows that $c(a, x_1, x_2) = x_1$. Thus, 3-Acyclicity implies $c(a, x_2, x_{n-2}) = x_{n-2}$. By definition of R_2^c, it follows that $x_{n-2} R_2^c x_2$ so that $x_2 ... x_{n-2}$ is an R_2^c-cycle of length $n-3$. But this contradicts the induction hypothesis. ■

Proof of Theorem. (⇒) Suppose c satisfies WWARP, Expansion, Choice Symmetry and Difficult Choice. Then, c satisfies 3-Acyclicity by Lemma 4 and 4/5-Acyclicity by Lemma 8. By Lemma 9, R_2^c is acyclic. Now, by way of contradiction, suppose there exists some $\succ_D^c$-cycle given by the sequence $x_0 ... x_i ... x_{n-1}$. Lemma 5 ensures that any link $x_i \succ_D^c x_{i+1}$ in the sequence can be replaced by a chain $x_i R_2^c ... R_2^c x_{i+1}$ consisting of 1 or 3 links. It follows that R_2^c contains a cycle which contradicts the fact that R_2^c is acyclic and establishes that $\succ_D^c$ is acyclic. In other words, c satisfies NBCC. Since c satisfies WWARP and Expansion, Theorem 1 of A&K establishes that it can be represented by a transitive RSM. Given Remark 1, this establishes that Choice Symmetry and Difficult Choice are sufficient to represent an RSM in terms of shortlisting.

(⇐) Suppose that c is a shortlisting procedure represented by (P_1, P_2). Theorem 1 of M&M establishes that c satisfies WWARP and Expansion. It suffices to show that c satisfies Choice Symmetry and Difficult Choice:

Choice Symmetry:[18] By Lemma 4, it suffices to show that c satisfies 3-Acyclicity. By way of contradiction, suppose $c(x, y, z) = z$ and $c(w, x, y) \neq w$ for 3-cycles xyz and wxy. From $c(x, y, z) = z$, it follows that $x P_1 y$. So, $c(w, x, y) \neq y$. Since $c(w, x, y) \neq w$, it follows that $c(w, x, y) = x$. From $c(w, x, y) = x$, it follows that $y P_1 w$. By transitivity, $x P_1 w$ which contradicts the fact that $c(x, w) = w$.

Difficult Choice: By the argument just above, c satisfies 3-Acyclicity. By Lemmas 1 and 2, c satisfies Selective IIA and Reduction. Now, suppose $c(x, y) = x$ and $c(A \cup \{y\}) \neq c(A \cup$

[18] Proposition 3 of Manzini and Mariotti [2006] provide an alternate proof of necessity.

$\{x,y\}) \neq x$. By Lemma 7, axa' and aya' are 3-cycles such that $c(a,a',x)=a$ and $c(a,a',y)=a'$. From $c(a,a',x)=a$, it follows that $a'P_2aP_2x$. Similarly, $c(a,a',y)=a'$ implies $yP_2a'P_2a$. Thus, $yP_2a'P_2aP_2x$. By transitivity, yP_2x.

It suffices to show $c(B)=c(B\setminus\{y\})$ for any $B\supset\{x,y\}$. To see this, suppose $c(B)\neq c(B\setminus\{y\})$ for some such B. Since $c(x,y)=x$, it follows that $x\succ_D^c y$. By Lemma 5, there are two possibilities: (i) there is a 3-cycle xyz such that $c(x,y,z)\neq z$; or, (ii) there are 3-cycles $ww'x$ and $ww'y$ such that $c(w,w',x)=w$ and $c(w,w',y)=w'$. In either case, xP_2y: (i) $c(x,y,z)\neq z$ implies xP_2y directly; and, (ii) $c(w,w',x)=w$ and $c(w,w',y)=w'$ imply $xP_2wP_2w'P_2y$ so that xP_2y by transitivity. Since yP_2x, P_2 fails to be asymmetric which establishes the desired contradiction. ■

Since the proof of sufficiency only relies on 3-Acyclicity and 4/5-Acyclicity (rather than the more general axioms of Choice Symmetry and Difficult Choice), it follows that:

Corollary 3 *An RSM has a shortlisting representation iff it satisfies 3-Acyclicity and 4/5-Acyclicity.*

7.4 Identification and Uniqueness

7.4.1 Proof of Corollary 2

Remark 7 *The proof of A&K establishes that any RSM which satisfies NBCC can be represented by a pair of rationales* $(\overline{P}_1^c, \succ_2^c)$ *where* $\overline{P}_1^c$ *is defined by* $\overline{P}_1^c \equiv \{(x,y) : c(A)=c(A\setminus y) \text{ for any } A\supset \{x,y\}\}$ *and* $\succ_2^c$ *is any linear order that completes the direct revealed preference* $\succ_D^c$*. The next Lemma establishes the equivalence between* $\overline{P}_1^c$ *and* $P_{\overline{\overline{1}}}^c$*.*

Lemma 10 *If* c *can be represented by a transitive RSM, then* $x\overline{P}_1^c y$ *iff* $xP_{\overline{\overline{1}}}^c y$*.*

Proof. As a preliminary observation, recall that $xP_{\overline{\overline{1}}}^c y$ iff $c(x,y)=x$ and $\neg(x\overline{R}_2^c y)$ (see Definition 3). By Lemma 9, it follows that $xP_{\overline{\overline{1}}}^c y$ iff $c(x,y)=x$ and $\neg(x\succ_D^c y)$. (⇒) If $c(x,y)=y$, then $\neg[x\overline{P}_1^c y]$ by definition of $\overline{P}_1^c$. Next, suppose that $x\succ_D^c y$. Then, by definition of $\succ_D^c$, there exists some $A\supset\{x,y\}$ such that $c(A)\neq c(A\setminus y)$. So, by definition, $\neg[x\overline{P}_1^c y]$. (⇐) Suppose $\neg[x\overline{P}_1^c y]$ so that there is some $A\supset\{x,y\}$ s.t. $c(A)\neq c(A\setminus y)$. One possibility is that $A=\{x,y\}$ so $c(x,y)=y$. If, instead, $A\supsetneq\{x,y\}$, then $c(x,y)=x$ and $x\succ_D^c y$ follows by definition of $\succ_D^c$. ■

In combination with Lemma 5, this implies an important corollary of the main theorem:

Proof of Corollary 2. (⇒) Trivial. (⇐) By the construction of A&K, $(\overline{P}_1^c, \succ_2^c)$ is a transitive RSM representation of c. By Lemma 5, $\succ_2^c$ is determined (up to transitive closure of R_2^c) by choice on pairs and 3-cycles. By Lemma 10, $\overline{P}_1^c$ is also determined by choice on pairs and 3-cycles. Since c and $\tilde{c}$ coincide on pairs and 3-cycles, $(\overline{P}_1^c, \succ_2^c)$ is a transitive RSM representation of $\tilde{c}$. Thus, $c(A)=c_{(\overline{P}_1^c,\succ_2^c)}(A)=\tilde{c}(A)$ for any $A\subseteq X$. ■

Lemma 11 *If c can be represented in terms of shortlisting, then $xP^c_{\overline{1}}y$ implies $x\overline{P}^c_1y$.*

Proof. Lemma 10 establishes $\overline{P}^c_1 = (\succ^c \setminus \succ^c_D)$. By Lemma 5, $\succ^c_D \subseteq P^c_{\underline{2}}$ which implies $(\succ^c \setminus P^c_{\underline{2}}) \subseteq \overline{P}^c_1$. Consequently, $P^c_{\overline{1}} = tc(\succ^c \setminus P^c_{\underline{2}}) \subseteq tc(\overline{P}^c_1)$ because transitive closure preserves set inclusion. Since $\overline{P}^c_1$ is asymmetric and transitive (as shown in the proof of A&K's Theorem 1), $tc(\overline{P}^c_1) = \overline{P}^c_1$ so that $P^c_{\overline{1}} \subseteq \overline{P}^c_1$. ■

The following example illustrates that this implication is strict in some cases:

Example 4 *Consider a choice function c on $X = \{v, w, x, y, z\}$ s.t. $c(v, y) = y$, and $c(w, z) = z = c(v, z)$. Next, suppose vwx, wxy, and xyz are the only 3-cycles so that $vwxy$ and $wxyz$ are the only 4-cycles and $vwxyz$ is a 5-cycle. Finally, suppose $c_3 \equiv [c(v, w, x), c(w, x, y), c(x, y, z)] = [x, x, x]$.*[19]

For c to be consistent with shortlisting, $c(i, j, k) = i$ for any $\{i, j, k\} \subset X$ such that $i \succ^c j \succ^c k$ and $i \succ^c k$. Consistency with shortlisting also requires $c(v, x, y, z) = c(v, w, x, y) = c(w, x, y, z) = c(v, w, x, y, z) = x$, $c(v, w, y, z) = y$, and $c(v, w, x, z) = z$. This specifies c for every subset of X.

Then, $\succ^c_D = R^c_2 = \{(w, x), (x, v), (x, y), (z, x)\}$ and $P^c_{\underline{2}} = R^c_2 \cup \{(w, y), (w, v), (\mathbf{z}, \mathbf{v}), (z, y)\}$. Thus, $\overline{P}^c_1 \equiv (\succ^c \setminus \succ^c_D) = \{(v, w), (y, v), (z, v), (y, w), (z, w), (y, z)\}$ and $P^c_{\overline{1}} \equiv tc(\succ^c \setminus P^c_{\underline{2}}) = \overline{P}^c_1 \setminus \{(z, v)\}$ so that $P^c_{\overline{1}} \subset \overline{P}^c_1$. Since it is straightforward to check that $(\overline{P}^c_1, P^c_{\underline{2}})$ and $(P^c_{\overline{1}}, P^c_{\underline{2}})$ both represent c, this example shows that $P^c_{\overline{1}} \neq \overline{P}^c_1$ for some shortlisting procedures.

7.4.2 Proof of Proposition 1

Lemma 12 *If (P_1, P_2) is a shortlisting representation of c, then xP_1y implies $x \succ^c y$.*

Proof. By way of contradiction, suppose xP_1y and $y \succ^c x$. Then, $c_{(P_1,P_2)}(x, y) \neq y = c(x, y)$. ■

Lemma 13 *If (P_1, P_2) is a shortlisting representation of c, then $P_1 \supseteq P^c_{\underline{1}}$ and $P_2 \supseteq P^c_{\underline{2}}$.*

Proof. The inclusions $R^c_1 \subseteq P_1$ and $R^c_2 \subseteq P_2$ follow from the discussion in Section 3.1 of the text. Since transitive closure preserves set inclusion, it follows that $P^c_{\underline{1}} \subseteq P_1$ and $P^c_{\underline{2}} \subseteq P_2$. ■

Lemma 14 *If c can be represented in terms of shortlisting, xR^c_1y implies $yP^c_{\underline{2}}x$ and $x(\succ^c \setminus P^c_{\underline{2}})y$.*

Proof. Suppose that xR^c_1y. I establish each implication in turn:

– Proof that $yP^c_{\underline{2}}x$: There are two possibilities. Under the first branch of R^c_1, there exists a 3-cycle xyz such that $c(x, y, z) = z$. By definition of R^c_2, it follows that $yR^c_2zR^c_2x$ so that $yP^c_{\underline{2}}x$. Under the second branch of R^c_1, there exist 3-cycles $w'xw$ and $w'yw$ such that $c(w, w', x) = w'$

[19] One could equally construct examples where $c_3 = [v, y, x]$ or $c_3 = [x, w, z]$.

and $c(w, w', y) = w$. By definition of R_2^c, it follows that $yR_2^c wR_2^c w' R_2^c x$ so that $yP_{\underline{2}}^c x$. Thus, $xR_1^c y$ implies $yP_{\underline{2}}^c x$.

– Proof that $x(\succ^c \setminus P_{\underline{2}}^c)y$: Observe that $x \succ^c y$ follows by definition of $xR_1^c y$. By way of contradiction, suppose that $\neg[x(\succ^c \setminus P_{\underline{2}}^c)y]$. Since $x \succ^c y$ and $\neg[x(\succ^c \setminus P_{\underline{2}}^c)y]$, it must be that $xP_{\underline{2}}^c y$. In combination with $yP_{\underline{2}}^c x$ [i.e. the first claim of the lemma], this contradicts the fact that R_2^c is acyclic by Lemma 9. Thus, $xR_1^c y$ implies $x(\succ^c \setminus P_{\underline{2}}^c)y$. ■

Lemma 15 *If c can be represented in terms of shortlisting, $xR_2^c y$ implies $\neg[xP_{\overline{1}}^c y]$ and $x(\succ^c \setminus P_{\underline{1}}^c)y$.*

Proof. Suppose that $xR_2^c y$. I establish each implication in turn:

– Proof that $\neg[xP_{\overline{1}}^c y]$: By way of contradiction, suppose $xR_2^c y$ and $xP_{\overline{1}}^c y$. By definition, $xR_2^c y$ implies that there exists a 3-cycle xyz such that $c(x, y, z) \neq z$. First, suppose $c(x, y, z) = x$. By definition of R_2^c, it follows that $zR_2^c xR_2^c y$. Since R_2^c is acyclic by Lemma 9, it follows that $\neg[yP_{\underline{2}}^c z]$. Since $y \succ^c z$ and $\neg[yP_{\underline{2}}^c z]$, it follows that $y(\succ^c \setminus P_{\underline{2}}^c)z$ so that $yP_{\overline{1}}^c z$. Since $xP_{\overline{1}}^c y$, transitivity implies $xP_{\overline{1}}^c z$. Since $\overline{P}_1^c \subseteq \succ^c$ by Lemma 10 and $P_{\overline{1}}^c \subseteq \overline{P}_1^c$ by Lemma 11, it follows that $x \succ^c z$. But, this contradicts the fact that $z \succ^c x$. Next, suppose $c(x, y, z) = y$. By a similar argument, it follows that $z \succ^c y$ which contradicts the fact that $y \succ^c z$.

– Proof that $x(\succ^c \setminus P_{\underline{1}}^c)y$: First, observe that (i) $R_2^c \subseteq (\succ^c \setminus P_{\overline{1}}^c)$ and (ii) $(\succ^c \setminus P_{\overline{1}}^c) \subseteq (\succ^c \setminus P_{\underline{1}}^c)$. **(i)** Since $R_2^c \subseteq \succ^c$ by definition of R_2^c and $P_{\overline{1}}^c \subseteq \succ^c$ by Lemmas 10 and 11, $R_2 \cup P_{\overline{1}}^c \subseteq \succ^c$. Since $R_2^c \cap P_{\overline{1}}^c = \emptyset$ [implied by the first claim of the lemma], it follows that $R_2^c \subseteq (\succ^c \setminus P_{\overline{1}}^c)$. **(ii)** By Lemma 14, $P_{\underline{1}}^c \subseteq P_{\overline{1}}^c$. Consequently, $(\succ^c \setminus P_{\overline{1}}^c) \subseteq (\succ^c \setminus P_{\underline{1}}^c)$. Combining (i) and (ii), it follows that $R_2^c \subseteq (\succ^c \setminus P_{\underline{1}}^c)$. ■

Lemma 16 *If c can be represented in terms of shortlisting, then $xP_{\underline{1}}^c y$ iff $c(x, y) = x$ and $yP_{\underline{2}}^c x$.*

Proof. ($\Rightarrow$) Suppose that $xP_{\underline{1}}^c y$. By definition, there exists an R_1^c-chain $z_0 ... z_n$ such that $x = z_0$ and $y = z_n$. By Lemma 14, it follows that $z_{i+1} P_{\underline{2}}^c z_i$ for any $0 \leq i \leq n$. As such, $yP_{\underline{2}}^c x$. By Lemmas 12 and 13, it follows that $P_{\underline{1}}^c \subseteq \succ^c$. Since $xP_{\underline{1}}^c y$, it follows that $c(x, y) = x$. This establishes the desired implication.

($\Leftarrow$) Suppose that $c(x, y) = x$ and $yP_{\underline{2}}^c x$ so that there exists an R_2^c-chain $w_0 ... w_{n+1}$ such that $y = w_0$ and $x = w_{n+1}$. The cases $n = 1$ and $n = 2$ are straightforward.

If $yR_2^c w_1 R_2^c x$, xyw_1 is a 3-cycle. Moreover, $c(w_1, x, y) = w_1$. Otherwise, $xR_2^c y$ so xyw_1 is an R_2^c-cycle which contradicts that R_2^c is acyclic by Lemma 9. So, $xR_1^c y$ under the first branch of R_1^c.

If $yR_2^c w_1 R_2^c w_2 R_2^c x$, then xyw_1w_2 is a 4-cycle. If $w_1 \succ^c x$, then xyw_1 is a 3-cycle. By reasoning similar to the case $n = 1$, it follows that $c(w_1, x, y) = w_1$ so that $xR_1^c y$. Similarly, $xR_1^c y$ if $y \succ^c w_2$. So, it suffices to consider the sub-case where $x \succ^c w_1$ and $w_2 \succ^c y$ so that xw_1w_2 and

xw_1w_2 are 3-cycles. By reasoning similar to the case $n = 1$, it follows that $c(w_1, w_2, x) = w_2$ and $c(w_1, w_2, y) = w_1$. Consequently, $xR_1^c y$ under the second branch of R_1^c.

For $n \geq 3$, the proof that $xP_{\underline{1}}^c y$ is by strong induction on n.

Base case $n = 3$**:** Suppose $yR_2^c w_1 R_2^c w_2 R_2^c w_3 R_2^c x$. By reasoning similar to the case $n = 1$, the two sub-cases $w_1 \succ^c x$ (where w_1xy is a 3-cycle) and $y \succ^c w_3$ (where w_3xy is a 3-cycle) imply $xR_1^c y$. Next, consider the sub-case $w_2 \succ^c x \succ^c w_1$ (where w_1w_2x is a 3-cycle). If $c(w_1, w_2, x) \neq w_2$, it follows that $xR_2^c w_1$. Consequently, $xw_1w_2w_3$ is an R_2^c-cycle which contradicts the fact that R_2^c is acyclic by Lemma 9. Thus, $c(w_1, w_2, x) = w_2$ so that $w_2R_2^c x$. Since $yR_2^c w_1 R_2^c w_2$, this sub-case reduces to the case $n = 2$, establishing that $xR_1^c y$. A similar argument establishes that $xR_1^c y$ in the sub-case $w_3 \succ^c y \succ^c w_2$ (where w_2w_3y is a 3-cycle). Finally, consider the sub-case $x \succ^c w_2 \succ^c y$ (where w_1w_2y and w_2w_3x are 3-cycles). Similar reasoning to the case $n = 1$ establishes $xR_1^c w_2 R_1^c y$. Since $xP_{\underline{1}}^c y$ in all sub-cases, the result follows.

Induction Step: Now, suppose $xP_{\underline{1}}^c y$ if $c(x, y) = x$ and $yR_2^c w_1 ... w_j R_2^c x$ for $j \leq n$. To establish the induction step, consider alternatives x and y such that $c(x, y) = x$ and $yR_2^c w_1 ... w_{n+1} R_2^c x$.

First, suppose that $y \succ^c w_i \succ^c x$ for some $1 \leq i \leq n+1$ so that w_ixy is a 3-cycle. By reasoning similar to the case $n = 1$, $c(w_i, x, y) = w_i$. Consequently, $xR_1^c y$ under the first branch of R_1^c.

Next, suppose that $x \succ^c w_i \succ^c y$ for some $2 \leq i \leq n$. Since $w_i ... w_{n+1} x$ is an R_2^c-chain of length $n + 2 - i \leq n$ and $c(w_i, x) = x$, the induction hypothesis implies $xP_{\underline{1}}^c w_i$. Similarly, the fact that $yw_1 ... w_i$ is an R_2^c-chain of length $i \leq n$ and $c(w_i, y) = y$ implies $w_i P_{\underline{1}}^c y$. Consequently, $xP_{\underline{1}}^c w_i P_{\underline{1}}^c y$ so that $xP_{\underline{1}}^c y$.

To complete the induction, it suffices to consider the case where (i) $x, y \succ^c w_i$ or (ii) $w_i \succ^c x, y$ for any $1 \leq i \leq n+1$. Observe that w_1 is type-(i) while w_{n+1} is type-(ii). By finiteness, there exists a largest $1 \leq i^* \leq n$ such that w_{i^*} is of type-(i) and w_{i^*+1} is of type-(ii). Notice that $w_{i^*}w_{i^*+1}x$ and $w_{i^*}w_{i^*+1}y$ are 3-cycles. By reasoning similar to the case $n = 1$, $c(w_{i^*}, w_{i^*+1}, x) = w_{i^*+1}$ and $c(w_{i^*}, w_{i^*+1}, y) = w_{i^*}$. Consequently, $xR_1^c y$ under the second branch of R_1^c. ■

Remark 8 *Lemma 16 establishes that $P_{\underline{1}}^c$ is equivalent to the lower bound $\underline{P}_1^c$ in Au and Kawai [2011]. Formally, they define $\underline{P}_1^c$ by $x\underline{P}_1^c y$ iff $x\overline{P}_1^c y$ and $y \succ_D^c ... \succ_D^c x$. By Lemma 5, $yP_{\underline{2}}^c x$ is equivalent to $y \succ_D^c ... \succ_D^c x$. To see that $x\overline{P}_1^c y$ can be replaced by $c(x, y) = x$ in their definition:*

Proof. **(⇒)** Suppose $x\overline{P}_1^c y$. By definition of $\overline{P}_1^c$, $c(x, y) = x$. **(⇐)** Suppose that $c(x, y) = x$ and $y \succ_D^c ... \succ_D^c x$. To see that $x\overline{P}_1^c y$, suppose otherwise. Then, $c(x, y) = y$ (which contradicts $c(x, y) = x$) or $x \succ_D^c y$ (which contradicts the acyclicity of $\succ_D^c$). Thus, $x\overline{P}_1^c y$ as required. ■

Lemma 17 *If c can be represented in terms of shortlisting, $\succ^c \setminus P_{\underline{2}}^c$, R_1^c, and $\succ^c \setminus P_{\underline{1}}^c$ are acyclic.*

Proof. I establish the three results in turn.

– Acyclicity of $\succ^c \setminus P^c_{\underline{2}}$:[20] First, observe that $\tilde{R}_1 \equiv (\succ^c \setminus R^c_2)$ is acyclic. To see this, suppose otherwise. Since X is finite, there exists an $\tilde{R}_1$-cycle $x_0 ... x_{n-1}$ of minimal length. First, note $x_{i-1} \succ^c x_{i+1}$. Otherwise, $x_{i-1}x_ix_{i+1}$ is a 3-cycle (i.e. on $\succ^c$) which leads to a contradiction. If $c(x_{i-1}, x_i, x_{i+1}) = x_i$, then $x_{i-1}R^c_2 x_i R^c_2 x_{i+1}$ so that $\neg(x_{i-1}\tilde{R}_1 x_i)$ and $\neg(x_i \tilde{R}_1 x_{i+1})$. Likewise, $c(x_{i-1}, x_i, x_{i+1}) = x_{i-1}$ implies $\neg(x_{i-1}\tilde{R}_1 x_i)$ and $c(x_{i-1}, x_i, x_{i+1}) = x_{i+1}$ implies $\neg(x_i \tilde{R}_1 x_{i+1})$. Thus, $x_{i-1} \succ^c x_{i+1}$.

This observation also establishes that there is no $\tilde{R}_1$-cycle of length $n = 3$ or $n = 4$. The conclusion for $n = 3$ is immediate. For $n = 4$, the fact that $x_0 \succ^c x_2$ (using $i = 1$) contradicts the fact that $x_2 \succ^c x_0$ (using $i = 3$). To establish the result for $n \geq 5$, first note that the minimality of $x_0 ... x_{n-1}$ implies that $x_{i-1}R^c_2 x_{i+1}$. Otherwise, $x_{i-1}\tilde{R}_1 x_{i+1}$ so that $x_{i+1} ... x_{n-1}x_0 ... x_{i-1}$ an $\tilde{R}_1$-cycle of length $n-1 < n$. If $n = 2m$ (so n is even), then $x_0x_2 ... x_{n-4}x_{n-2}$ is an R^c_2-cycle of length $m+1$. But, this contradicts the fact that R^c_2 is acyclic by Lemma 5. If $n = 2m+1$ (so n is odd), then $x_0x_2 ... x_{n-1}x_1 ... x_{n-4}x_{n-2}$ is an R^c_2-cycle of length n. Again, this contradicts the fact that R^c_2 is acyclic.

The result follows from the observation that $x(\succ^c \setminus P^c_{\underline{2}})y$ implies $x\tilde{R}_1 y$. Since $R^c_2 \subseteq tc(R^c_2) = P^c_{\underline{2}}$, it follows that $(\succ^c \setminus P^c_{\underline{2}}) \subseteq (\succ^c \setminus R^c_2) = \tilde{R}_1$. Since $\tilde{R}_1$ is acyclic, $(\succ^c \setminus P^c_{\underline{2}})$ is acyclic.

– Acyclicity of R^c_1:[21] Since $R^c_1 \subseteq (\succ^c \setminus P^c_{\underline{2}})$ by Lemma 14 and $(\succ^c \setminus P^c_{\underline{2}})$ is acyclic, R^c_1 is acyclic.

– Acyclicity of $\succ^c \setminus P^c_{\underline{1}}$:[22] Let $R^c_{\underline{2}} \equiv (P^c_{\underline{2}} \cap \succ^c)$, $R^c_{\overline{1}} \equiv (P^c_{\overline{1}} \setminus P^c_{\underline{1}})$, and define $\tilde{R}_2 \equiv (R^c_{\underline{2}} \cup R^c_{\overline{1}})$. First, observe that $\tilde{R}_2$ is acyclic. To see this, suppose otherwise. Since X is finite, there exists an $\tilde{R}_2$-cycle $x_0 ... x_{n-1}$ of minimal length. I show that this entails a contradiction by induction on n.

As a preliminary point, note that $R^c_{\underline{2}} \subseteq P^c_{\underline{2}}$ and $R^c_{\overline{1}} \subseteq P^c_{\overline{2}}$ are asymmetric by Lemma 9 and the result immediately above. Thus, any $\tilde{R}_2$-cycle must contain links of both kinds. Accordingly, there exists some i such that $x_{i-1}R^c_{\overline{1}} x_i$ and $x_i R^c_{\underline{2}} x_{i+1}$. Without loss of generality, relabel the indices so that $i = 1$.

For the base case $n = 3$, suppose that $x_0 R^c_{\overline{1}} x_1 R^c_{\underline{2}} x_2 \tilde{R}_2 x_0$. There are two sub-cases to consider: (i) $x_2 R^c_{\overline{1}} x_0$; and, (ii) $x_2 R^c_{\underline{2}} x_0$. In sub-case (i), it follows that $x_2 P^c_{\overline{1}} x_0 P^c_{\overline{1}} x_1$ so that $x_2 P^c_{\overline{1}} x_1$. By Lemma 16, $x_2 \succ^c x_1$ which contradicts the assumption that $x_1 R^c_{\underline{2}} x_2$ (i.e. that $x_1 \succ^c x_2$). In sub-case (ii), it follows that $x_1 P^c_{\underline{2}} x_2 P^c_{\underline{2}} x_0$ so that $x_1 P^c_{\underline{2}} x_0$. Since $x_0 \succ^c x_1$ by definition of $\tilde{R}_2$, Lemma 16 implies $x_0 P^c_{\underline{1}} x_1$. But, this contradicts the assumption that $x_0 R^c_{\overline{1}} x_1$ (i.e. that $\neg[x_0 P^c_{\underline{1}} x_1]$).

Now suppose that there are no $\tilde{R}_2$-cycles of length $i \leq n$. By way of contradiction, suppose that $x_0 R^c_{\overline{1}} x_1 R^c_{\underline{2}} x_2 \tilde{R}_2 ... \tilde{R}_2 x_n \tilde{R}_2 x_0$ is an $\tilde{R}_2$-cycle of length $n+1$. There are three cases to consider: (i) $x_2 \succ^c x_0$; (ii) $x_0 \tilde{R}_2 x_2$; and, (iii) $x_0 P^c_{\underline{1}} x_2$. In case (i), the fact that $\succ^c \subseteq (P^c_{\overline{1}} \cup P^c_{\underline{2}})$ implies $x_2 P^c_{\overline{1}} x_0$

[20] Alternatively, the acyclicity of $(\succ^c \setminus P^c_{\underline{2}})$ follows from Lemma 11 and the fact that $(\overline{P}^c_1, \succ^c_2)$ represents c.

[21] Alternatively, the acyclicity of R^c_1 follows from Lemma 13, or Lemmas 9 and 16.

[22] The proof parallels Au and Kawai's [2011] proof of Theorem 2(ii.a).

or $x_2 P^c_{\underline{2}} x_0$ without loss of generality. As a result, an argument similar to the base case $n = 3$ yields a contradiction. In case (ii), it follows that $x_0 x_2 ... x_{n+1}$ is a $\tilde{R}_2$-cycle of length n. In case (iii), Lemma 16 implies $x_2 P^c_{\underline{2}} x_0$. As a result, an argument similar to sub-case (ii) of the base case $n = 3$ yields a contradiction.

The result follows from the fact that $x(\succ^c \setminus P^c_{\underline{1}})y$ implies $x\tilde{R}_2 y$. As noted, $\succ^c \subseteq (P^c_{\overline{1}} \cup P^c_{\underline{2}})$ so that $(P^c_{\overline{1}} \cup R^c_{\underline{2}}) = \succ^c$. As such, $(\succ^c \setminus P^c_{\underline{1}}) = R^c_{\overline{1}} \cup [R^c_{\underline{2}} \setminus P^c_{\underline{1}}] \subseteq \tilde{R}_2$. Since $\tilde{R}_2$ is acyclic, $(\succ^c \setminus P^c_{\underline{1}})$ is acyclic. ■

Lemma 18 *If c can be represented in terms of shortlisting, then $P^c_{\overline{2}}$ is a strict weak order.*

Proof. By Lemma 17, $P^c_{\overline{2}}$ is asymmetric and transitive. It suffices to show completeness (for distinct $x, y \in X$). Consider any alternatives x and y and suppose that $x \succ^c y$. If $\neg[x P^c_{\underline{1}} y]$, then $\succ^c \subseteq P^c_{\underline{1}} \cup P^c_{\overline{2}}$ implies $x P^c_{\overline{2}} y$. If $x P^c_{\underline{1}} y$, then Lemma 16 implies $y P^c_{\underline{2}} x$. Since $P^c_{\underline{2}} \subseteq P^c_{\overline{2}}$ by Lemma 15, it follows that $y P^c_{\overline{2}} x$. Thus, $x \succ^c y$ implies $x P^c_{\overline{2}} y$ or $y P^c_{\overline{2}} x$. Since $\succ^c$ is complete (for distinct $x, y \in X$), it follows that $P^c_{\overline{2}}$ is complete. ■

Proof of Proposition 1. Suppose that c can be represented by a shortlisting procedure.

(I.a) This is established in Lemma 13.

(I.b) Lemmas 9 and 17 establish that R^c_1, R^c_2, and $(\succ^c \setminus P^c_{\underline{2}})$ are acyclic. Consequently, $P^c_{\underline{1}}$, $P^c_{\underline{2}}$, and $P^c_{\overline{1}}$ are transitive rationales. Moreover, $P^c_{\overline{\overline{1}}}$ is a transitive rationale (as shown in the proof of A&K's Theorem 1). Finally, $P^c_{\overline{2}}$ is a linear order by Lemma 18.

(II.a) Given (I.b), $(P^c_{\underline{1}}, P^c_{\overline{2}})$ is a shortlisting procedure. Since $P^c_{\overline{2}}$ is complete, there is no rationale $P_2 \supsetneq P^c_{\overline{2}}$ such that $(P^c_{\underline{1}}, P_2)$ represents c. By Corollary 2, it suffices to show that $(P^c_{\underline{1}}, P^c_{\overline{2}})$ coincides with c on pairs and 3-cycles in order to establish that $(P^c_{\underline{1}}, P^c_{\overline{2}})$ represents c.

First, suppose that $c(x, y) = x$. By Lemmas 12 and 13, it follows that $P^c_{\underline{1}} \subseteq \succ^c$ so that $\neg[y P^c_{\underline{1}} x]$. If $x P^c_{\underline{1}} y$, then $c_{(P^c_{\underline{1}}, P^c_{\overline{2}})}(x, y) = x$ follows directly. If $\neg[x P^c_{\underline{1}} y]$, then $x \succ^c y$ implies $x P^c_{\overline{2}} y$ by definition of $P^c_{\overline{2}}$ [because $\succ^c \subseteq (P^c_{\underline{1}} \cup P^c_{\overline{2}})$]. Since $P^c_{\overline{2}}$ is asymmetric, it follows that $c_{(P^c_{\underline{1}}, P^c_{\overline{2}})}(x, y) = x$. Next, consider some 3-cycle xyz and suppose that $c(x, y, z) = x$. By definition, $y P^c_{\underline{1}} z$ and $z P^c_{\underline{2}} x P^c_{\underline{2}} y$. Since $P^c_{\underline{2}} \subseteq P^c_{\overline{2}}$ by Lemma 15, $z P^c_{\overline{2}} x P^c_{\overline{2}} y$ as well. Observe that $P^c_{\underline{1}} \cap \{x, y, z\}^2 = \{(y, z)\}$ because $P^c_{\underline{1}} \subseteq \succ^c$. Moreover, the fact that $P^c_{\overline{2}}$ is asymmetric implies $P^c_{\overline{2}} \cap \{x, y, z\}^2 = \{(z, x), (x, y), (z, y)\}$. Thus, $c_{(P^c_{\underline{1}}, P^c_{\overline{2}})}(x, y, z) = x$.

(II.b) Given (I.b), $(P^c_{\overline{1}}, P^c_{\underline{2}})$ and $(P^c_{\overline{\overline{1}}}, P^c_{\underline{2}})$ are shortlisting procedures. Recall that $P^c_{\overline{1}} \subseteq P^c_{\overline{\overline{1}}}$ by Lemma 11. If $(P^c_{\overline{1}}, P^c_{\underline{2}})$ and $(P^c_{\overline{\overline{1}}}, P^c_{\underline{2}})$ represent c, then $(P_1, P^c_{\underline{2}})$ represents c for any transitive rationale $P^c_{\overline{1}} \subset P_1 \subset P^c_{\overline{\overline{1}}}$. This follows from the fact that $\underline{A} = \max(A; P^c_{\overline{\overline{1}}}) \subseteq \max(A; P^c_1) \subseteq \max(A; P^c_{\overline{1}}) = \overline{A}$ and $\max(\underline{A}; P^c_{\underline{2}}) = \max(\overline{A}; P^c_{\underline{2}})$ for any $A \subseteq X$. By Corollary 2, it suffices to show that $(P^c_{\overline{1}}, P^c_{\underline{2}})$ and $(P^c_{\overline{\overline{1}}}, P^c_{\underline{2}})$ coincide with c on pairs and 3-cycles.

First, suppose that $c(x,y)=x$. By definition of $P^c_{\bar{\bar{1}}}$, it follows that $\neg[yP^c_{\bar{\bar{1}}}x]$. By Lemmas 10 and 11, it follows that $\neg[yP^c_{\bar{1}}x]$. By reasoning similar to (II.a), $c_{(P^c_{\bar{1}},P^c_{\underline{2}})}(x,y)=c_{(P^c_{\bar{\bar{1}}},P^c_{\underline{2}})}(x,y)=x$. Next, suppose that $c(x,y,z)=x$ for some 3-cycle xyz. By definition, $yP^c_{\underline{1}}z$ and $zP^c_{\underline{2}}xP^c_{\underline{2}}y$. Since $P^c_{\underline{1}}\subseteq P^c_{\bar{1}}\subseteq P^c_{\bar{\bar{1}}}$, it follows that $yP^c_{\bar{1}}z$ and $yP^c_{\bar{\bar{1}}}z$ as well. Then, by reasoning similar to (II.a), $c_{(P^c_{\bar{1}},P^c_{\underline{2}})}(x,y,z)=c_{(P^c_{\bar{\bar{1}}},P^c_{\underline{2}})}(x,y,z)=x$. ■

7.4.3 Proof of Proposition 2

Lemma 19 *Given a choice function c and two rationales P and $\widetilde{P}$, $P\subseteq\widetilde{P}$ implies $\widetilde{P}'\subseteq P'$.*

Proof. From $P\subseteq\widetilde{P}$, it follows that $(\succ^c\setminus\widetilde{P})\subseteq(\succ^c\setminus P)$, and, consequently, $\widetilde{P}'\subseteq P'$. ■

Lemma 20 *Suppose c can be represented in terms of shortlisting with i^{th} rationale P_i. Then, $P'_i\in\mathcal{P}_{-i}(c)$ is the unique P_i-minimal rationale that represents c.*

Proof. Suppose $i=1$. [Except as indicated, the proof is similar for $i=2$.] Suppose c is represented by (P_1,P_2). I establish: (i) $P'_1\in\mathcal{P}_2(c)$; (ii) (P_1,P'_1) represents c; and, (iii) $P'_1\subseteq P_2$.

(i) By Proposition 1, $P^c_{\underline{1}}\subseteq P_1$. By Lemma 19, it follows that $P'_1\subseteq P^c_{\bar{2}}$. Next, observe that $R^c_2\cap P_1=\emptyset$. Consequently, $R^c_2\subseteq(\succ^c\setminus P_1)$. By transitive closure, $P^c_{\underline{2}}\subseteq P'_1$ so that $P'_1\in\mathcal{P}_2(c)$.

To see that $R^c_2\cap P_1=\emptyset$, suppose xR^c_2y and xP_1y.[23] By definition, xR^c_2y implies that there is a 3-cycle xyz such that $c(x,y,z)\neq z$. First, suppose $c(x,y,z)=x$ so that yR^c_1z by definition. Since $P^c_{\underline{1}}\subseteq P_1$, it follows that yP_1z. Given that xP_1y, xP_1z follows by transitivity of P_1. Since (P_1,P_2) represents c, $c(x,z)\neq z$. A similar contradiction arises if $c(x,y,z)=y$. So, $R^c_2\cap P_1=\emptyset$.

(ii) Observe that P'_1 is a transitive rationale. This follows from the fact that $P'_1\in\mathcal{P}^c_2$ by (i) above. Thus, (P_1,P'_1) is a shortlisting procedure. By Corollary 2, the result follows by establishing that (P_1,P'_1) coincides with c on pairs and 3-cycles.

First, suppose $c(x,y)=x$. Since (P_1,P_2) represents c, $\neg[yP_1x]$. If xP_1y, then $c_{(P_1,P'_1)}(x,y)=x$ directly. Otherwise, $\neg[xP_1y]$ and $x\succ^c y$ imply xP'_1y. Since P'_1 is asymmetric, $c_{(P_1,P'_1)}(x,y)=x$.[24]

Next, suppose xyz is a 3-cycle such that $c(x,y,z)=x$. Since (P_1,P_2) represents c, Proposition 1(I) implies $P^c_{\underline{1}}\subseteq P_1$. So, $(y,z)\in P_1$. Since P_1 is a transitive rationale and (P_1,P_2) represents c, $P_1\cap\{x,y,z\}^2=\{(y,z)\}$. By (i) above, $P^c_{\underline{2}}\subseteq P'_1\subseteq P^c_{\bar{2}}$. By the argument given in the

[23]For $i=2$, the proof that xR^c_1y and xP_2y generate a contradiction is different. By definition, xR^c_1y implies (i) that there exists a 3-cycle xyz such that $c(x,y,z)=z$ or (ii) a pair of 3-cycles $ww'x$ and $ww'y$ such that $c(w,w',x)=w'$, $c(w,w',y)=w$, and $c(x,y)=x$. In sub-case (i), $yR^c_2zR^c_2x$ by definition. Since $P^c_{\underline{2}}\subseteq P_2$, it follows that yP_2x which contradicts the fact that P_2 is asymmetric. A similar contradiction arises in sub-case (ii).

[24]If $i=2$, the argument is different. From $P'_2\subseteq P^c_{\bar{1}}$ and $P^c_{\bar{1}}\subseteq\succ^c$, it follows that $\neg[yP'_2x]$. If xP'_2y, then $c_{(P'_2,P_2)}(x,y)=x$ directly. Otherwise, $\neg[xP'_2y]$ and $x\succ^c y$ imply xP_2y by definition of P'_2. Since P^c_2 is asymmetric, $c_{(P'_2,P_2)}(x,y)=x$.

proof of Proposition 1(II), $P_{\underline{2}}^c \cap \{x,y,z\}^2 = \{(z,x),(x,y),(z,y)\} = P_{\overline{2}}^c \cap \{x,y,z\}^2$. As such, $P_1' \cap \{x,y,z\}^2 = \{(z,x),(x,y),(z,y)\}$. It follows that $c_{(P_1,P_1')}(x,y,z) = x$.

(iii) Suppose $xP_1'y$. By definition, there exists a $(\succ^c \setminus P_1)$-chain $x_0...x_n$ such that $x_0 = x$ and $x_n = y$. Observe that $x_iP_2x_{i+1}$ for any link in the chain. This follows from $\neg[x_iP_1x_{i+1}]$, $x_i \succ^c x_{i+1}$, and the fact that (P_1,P_2) represents c. Thus, $xP_2...P_2y$. By transitivity, xP_2y. ■

Lemma 21 *Given a choice function c and a rationale P, $xP''y$ implies xPy.*

Proof. Suppose $xP''y$. First, consider the case where $x(\succ^c \setminus P')y$. Then, $\neg[xP'y]$. By definition of P', it follows that $\neg[x(tc(\succ^c \setminus P))y]$. Since $x \succ^c y$, it must be that xPy. Now, consider the general case where there is a $(\succ^c \setminus P')$-chain $x_0...x_n$ with $x_0 = x$ and $x_n = y$. From the first case, x_iPx_{i+1} for any link in the chain. As such, $xPx_1P...Px_{n-1}Py$. Since P is transitive, xPy. ■

Lemma 22 *Suppose (P_1,P_2) and $(\tilde{P}_1,\tilde{P}_2)$ are two minimal representations s.t. $(P_1,P_2) \neq (\tilde{P}_1,\tilde{P}_2)$. Then, $(\tilde{P}_1 \setminus P_1) \neq \emptyset$ implies $(P_2 \setminus \tilde{P}_2) \neq \emptyset$.*

Proof. By way of contradiction, suppose $\tilde{P}_1 \setminus P_1 \neq \emptyset$ and $P_2 \subseteq \tilde{P}_2$. First, observe that $(P_1,\tilde{P}_2)$ represents c. Since $(\tilde{P}_1,\tilde{P}_2)$ is a minimal representation of c, $\tilde{P}_1$ is uniquely $\tilde{P}_2$-minimal so that $\tilde{P}_1 \subseteq P_1$. But, this contradicts the assumption that $\tilde{P}_1 \setminus P_1 \neq \emptyset$. ■

Lemma 23 *If c can be represented by (P_1,P_2), then (P_1'',P_1') and (P_2',P_2'') are minimal.*

Proof. I show that (P_1'',P_1') is minimal. The proof for (P_2',P_2'') is similar.

By Lemma 20, (P_1,P_1') represents c and P_1' is P_1-minimal. Similarly: (a) (P_1'',P_1') represents c and P_1'' is P_1'-minimal; and, (b) (P_1'',P_1''') represents c and P_1''' is P_1''-minimal. Given (a), it suffices to show that P_1' is P_1''-minimal. By Lemma 21, $P_1'' \subseteq P_1$ (using $P = P_1$) and $P_1''' \subseteq P_1'$ (using $P = P_1'$). By Lemma 19, the first inclusion implies $P_1' \subseteq P_1'''$. Since $P_1''' \subseteq P_1'$, it follows that $P_1''' = P_1'$. Given (b), P_1' is P_1''-minimal. ■

Proof of Proposition 2. Part **(i)** is established by Lemma 20. **(ii)** The second part of the statement is established by Lemma 22. I show that (P_1,P_2) is minimal iff $P_1 \in \mathcal{P}_1''(c)$ and $P_2 = P_1'$. The proof for $P_2 \in \mathcal{P}_2''(c)$ and $P_1 = P_2'$ is similar.

($\Rightarrow$) Suppose that (P_1,P_2) is minimal. First observe that $P_2 = P_1'$ by definition. Applying the same reasoning, $P_1 = P_2' = P_1''$. Thus, $P_1 = P_1'' \in \mathcal{P}_1''(c)$. Moreover, $P_2 = P_1' = P_1'''$ as required.

($\Leftarrow$) Suppose that (P_1,P_2) represents c and $\tilde{P}_1 \in \mathcal{P}_1(c)$ is a rationale s.t. $P_1 = \tilde{P}_1''$ and $P_2 = \tilde{P}_1'''$. It suffices to show that $(P_1,\tilde{P}_1')$ represents c. By Lemma 23, it then follows that $(\tilde{P}_1'',\tilde{P}_1''') = (P_1,P_2)$ is minimal. To see that $(P_1,\tilde{P}_1')$ represents c, first observe that $P_1 \in \mathcal{P}_1(c)$ and $\tilde{P}_1' \in \mathcal{P}_2(c)$ are transitive rationales. So, $(P_1,\tilde{P}_1')$ is a shortlisting procedure. To see that it represents c, observe that $P_2 = \tilde{P}_1''' \subseteq \tilde{P}_1'$ by Lemma 21. Since $\tilde{P}_1' \subseteq P_{\overline{2}}^c$, an argument along the lines of that given in the proof of Proposition 1(II.b) establishes that $(P_1,\tilde{P}_1')$ represents c. ■

7.5 Comparative Statics

Lemma 24 *If c can be represented in terms of shortlisting, then $P^c_{\underline{1}} = (\succ^c \setminus P^c_{\overline{2}})$.*

Proof. First observe that $\succ^c \subseteq P^c_{\underline{1}} \cup (P^c_{\overline{2}} \cap \succ^c)$ by definition of $P^c_{\overline{2}}$, and $P^c_{\underline{1}} \cup (P^c_{\overline{2}} \cap \succ^c) \subseteq \succ^c$ since $P^c_{\underline{1}} \subseteq \succ^c$ by Lemmas 12 and 13. So, $P^c_{\underline{1}} \cup (P^c_{\overline{2}} \cap \succ^c) = \succ^c$. By way of contradiction, suppose that $xP^c_{\underline{1}}y$ and $xP^c_{\overline{2}}y$. By Lemma 16, $xP^c_{\underline{1}}y$ implies $yP^c_{\underline{2}}x$. Since $P^c_{\underline{2}} \subseteq P^c_{\overline{2}}$ by Lemma 15, it follows that $yP^c_{\overline{2}}x$. Since $(\succ^c \setminus P^c_{\underline{1}})$ is acyclic by Lemma 17 however, this contradicts the fact that $xP^c_{\overline{2}}y$. Consequently, $P^c_{\underline{1}} \cap P^c_{\overline{2}} = \emptyset$ which establishes the desired result. ■

Corollary 4 *Suppose c can be represented by (P_1, P_2). Then, $xP^c_{\underline{1}}y$ iff xP_1y and yP'_1x.*

Proof. By Lemma 16, $xP^c_{\underline{1}}y$ iff $[c(x,y) = x$ and $yP^c_{\underline{2}}x]$. By Lemma 24, it follows that $xP^c_{\underline{1}}y$ iff $[c(x,y) = x$ and $yP^c_{\overline{2}}x]$. To see this, first suppose that $xP^c_{\underline{1}}y$. By Lemma 16, $c(x,y) = x$ and $yP^c_{\underline{2}}x$. Since $P^c_{\underline{2}} \subseteq P^c_{\overline{2}}$ (by Lemma 13), it follows that $c(x,y) = x$ and $yP^c_{\overline{2}}x$. Next, suppose that $c(x,y) = x$ and $yP^c_{\overline{2}}x$. By Lemma 24, it follows that $xP^c_{\underline{1}}y$.

Since $P^c_{\underline{2}} \subseteq P'_1 \subseteq P^c_{\overline{2}}$ by Proposition 2, $xP^c_{\underline{1}}y$ iff $[c(x,y) = x$ and $yP'_1x]$. Since (P_1, P'_1) represents c by Proposition 2, it then follows that P'_1 is asymmetric, $\succ^c \subseteq P_1 \cup P'_1$, and $P_1 \subseteq \succ^c$. Consequently, $[c(x,y) = x$ and $yP'_1x]$ iff $[xP_1y$ and $yP'_1x]$. ■

Proof of Proposition 3. Fix choice functions c_A, c_B that can be represented by (P^A_1, P^A_2) and (P^B_1, P^B_2). First, suppose that $P^A_{\underline{1}} \subseteq P^B_{\underline{1}}$. By Corollary 4, $[xP^A_1y$ and $y(P^A_1)'x]$ implies $[xP^B_1y$ and $y(P^B_1)'x]$. Next, suppose that, for any $x, y \in X$, $[xP^A_1y$ and $y(P^A_1)'x]$ implies $[xP^B_1y$ and $y(P^B_1)'x]$. By Corollary 4, it follows that $P^A_{\underline{1}} \subseteq P^B_{\underline{1}}$. ■

Proof of Proposition 4. Suppose that c_A and c_B can be represented by (P^A_1, P^A_2) and (P^B_1, P^B_2) with $P^A_2 = P_2 = P^B_2$. As a preliminary point, observe that if stu is an A-cycle such that $c_A(s,t,u) = s$, then $c_B(s,t,u) = s$. To see this, note that $c_A(s,t,u) = s$ implies $uR^A_2sR^A_2t$ so that uP_2sP_2t. Since stu is a B-cycle by (i), it must be that $c_B(s,t,u) = s$. If $c_B(s,t,u) = t$, then $tR^B_{\underline{2}}u$ so that tP_2u. As such, P_2 contains a cycle – which violates the assumption that P_2 is asymmetric. Similar reasoning establishes a contradiction if $c_B(s,t,u) = u$. Consequently, $c_B(s,t,u) = s$.

To establish the result, it suffices to show that $R^A_1 \subseteq R^B_1$. By transitive closure, it follows that $P^A_{\underline{1}} \subseteq P^B_{\underline{1}}$. To show that $R^A_1 \subseteq R^B_1$, fix any pair xy such that xR^A_1y. By definition of R^A_1, there are two possibilities: (a) there exists an alternative z s.t. xyz is an A-cycle and $c_A(x,y,z) = z$; or, (b) there exist z, w where wxz, wyz are A-cycles s.t. $c_A(w,x,z) = w$, $c_A(w,y,z) = z$, and $c_A(x,y) = x$. In case (a), (i) establishes that xyz is a B-cycle while the observation in the first paragraph establishes that $c_B(x,y,z) = z$. So, xR^B_1y by definition. In case (b), (i) establishes that wxz, wyz are B-cycles while the observation in the first paragraph establishes that $c_B(w,x,z) = w$ and $c_B(w,y,z) = z$. Finally, (ii) ensures that $c_B(x,y) = x$. So, xR^B_1y by definition. ■

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