Revenue Management by a Patient Seller

Artyom Shneyerov

Concordia University, CIREQ and CIRANO

Abstract

We consider a classic setting as in Myerson (1981), where a seller faces N risk-neutral buyers. However, the seller is assumed to be *more patient* than the buyers. We restrict attention to selling mechanisms where the price can change over time, and buyers can buy at any point in time. A static auction with an optimally chosen reserve at time 0 is included as a special case, but, as we show, in general is not optimal. The optimal solution involves delays, an auction at the initial time as well as the auction at the final time, and sales at posted prices at intermediate times.

Keywords: Revenue management, dynamic pricing, dynamic auctions, optimal control

1. Introduction

Consider a classic setting of Myerson (1981): the seller, who owns a single unit, faces N risk-neutral buyers. Can the seller benefit from dynamic pricing? In this paper we show that dynamic pricing dominates the instanteneous optimal auction if the seller is more patient than the buyers. We study dynamic pricing games (DPG), where the seller commits to a *price schedule* $p(\cdot)$, beginning with a high price, and then reducing it until the item is sold. We consider a perishable good: the time of sale is exogenously constrained by T, the lifespan of the good. The price schedule is the object of seller's design, with the purpose of maximizing the discounted expected revenue.

Buyers draw their valuations from the same distribution, and we restrict attention to symmetric equilibria. Buyers are forward-looking. At any point in time, a buyer can either purchase at the standing price $p(t)$, or wait. If several buyers are willing to purchase at the same time, the seller conducts an auction among those buyers. We assume an open auction, or, equivalently, a Vickrey auction. But all our results generalize to other revenue equivalent formats, including first-price sealed-bid (or Dutch).

A variety of plausible pricing behaviors is allowed, as we put essentially *no* restrictions on the form of the price schedule beyond monotonicity. For example, we allow the price to drop instantaneously at any point in time. We also allow the price to remain constant over an interval, so that the good is effectively removed from the market over that period. A classic, instantaneous auction at time 0 with a reserve price R is also included as a special case, with the price that stays at R until the final time T is reached.

Our main result can be summarized as follows. If the seller is *less* (or equally) patient, the instantaneous auction at time $t = 0$ with a reserve price is optimal, as in Myerson (1981). However, if the seller is *more* patient than the buyers, then he or she can do better by committing to a reserve price that declines over time. Thus, an optimal solution involves *delays*. We use optimal control to characterize the solution. As we show, the seller can equivalently implement the price schedule via a *time of sale* function $t(v)$, prescribing which buyer types should buy at different points in time. Any point of strict monotonicity of this function corresponds to sale at a posted price, while any flat segment corresponds to an auction.

To gain further insights into the structure of the solution, we first consider a relaxed problem, with the monotonicity constraint on $t(v)$ removed. We show that the optimal solution will involve auctions at the initial time $t = 0$ and the final time $t = T$. Otherwise, sales will occur at posted prices. The posted prices are shown to be always above the static optimal reserve price, but at the final time, the price is discontinuously reduced below it. Moreover, the optimal solution is continuous, implying that temporarily removing the good from the market is *not* optimal. A numerically computed example suggests that our optimal dynamic mechanism outperforms the instantaneous optimal auction by about 11%.

We next characterize the solution under the monotonicity constraint on $t(v)$. We show that almost all properties of the relaxed-optimal solution carry over. The only possible exception is that now we cannot rule out auctions at intermediate times, precisely when the monotonicity constraint is binding.

On the technical side, our general analysis is based on the Lebesgue rectification theorem, a new technique that we believe can be applied in other contexts. We parameterize the solution as a curve in the plane, with potential discontinuities filled out by connecting line segments. The aforementioned theorem implies that, for *monotone* functions, the coordinates in such a parameterization can be chosen as *absolutely continuous* functions. This enables us to impose the monotonicity constraint is a standard fashion, in terms of the derivatives of the coordinate functions, and invoke the Maximum Principle as in e.g. Clarke (2005, forthcoming) to characterize the solution. The solution is then shown to be continuous. At all points where it is strictly monotone, we show that the Hamiltonian satisfies the same first-order condition as in the unconstrained case. As the properties of the solution chiefly rely on this first-order condition, we can transfer the results from the unconstrained case to the constrained one with little changes.

Literature. In a classic paper, Myerson (1981) shows that, when the buyers and the seller are risk-neutral and have the same time preferences, the seller can do no better sell through an instantaneous auction with an optimally chosen reserve price.[1] The literature on mechanism design has recently focused on dynamics, but, at least to our knowledge, has not incorporated different discount rates.[2,3]

The most closely related paper is Landsberger and Meilijson (1985), who have studied a durable good monopolist that sells to a population of consumers under commitment. The monopolist is assumed to be more patient than the consumers. This setting corresponds to $N = 1$ in our model. Landsberger and Meilijson (1985) have also shown optimality of delays, although they do not fully characterize the optimal solution. Wang (2001) considers pricing by a durable good monopolist and fully characterizes the optimal price schedule, assuming infinite horizon.

Hörner and Samuelson (2011) consider a setting where the seller of a perishable good faces N strategic buyers, as in our model, but under *noncommitment* and equal patience. They show that the deadline at $t = T$ endows the seller with considerable commitment power.

In addition, there is a substantial literature on revenue management when buyers *arrive over time.* In operations research, it is often assumed that buyers are myopic.[4] Recent contributions in economics include Gershkov and Moldovanu (2009a) and Gershkov and Moldovanu (2009b). With forward-looking buyers as in our paper, but without commitment, Conlisk

[1] Riley and Samuelson (1981) and Harris and Raviv (1981) obtain related results.

[2] Unlike our model, the literature has mainly looked at environments with *new* buyers arriving over time. See, for example, Said (2011), and, for an excellent survey of dynamic mechanism design, Bergemann and Said (2010).

[3] The relevance of different discount rates is well understood in other contexts, e.g. in bargaining, as in Rubinstein (1982). In the Rubinstein model, patience allows a bargainer attain a higher level of profits.

[4] This literature is extensively reviewed in Board and Skrzypacz (2010).

et al. (1984) show that optimal pricing is nonstationary even in a stationary environment. Board (2008) fully characterizes the price sequence. Board and Skrzypacz (2010) consider a setting with a perishable good and commitment as in our model, but with buyers entering at random times. The buyers and the seller are assumed to be equally patient. The optimal solution is shown to involve posted prices for all periods except the last one. Similar to one of our results, Board and Skrzypacz (2010) show that auctions may occur at the last instance, otherwise buyers buy exclusively at posted prices.

Another related paper is Fuchs and Skrzypacz (2010), who find that delays are optimal in a market where the seller bargains with sequentially arriving buyers and frequent offers. As in our model, they find that the seller "slowly screens out buyers with higher valuations".

Our paper's main technical innovation, a novel method of imposing the monotonicity constraint, is related to several previous papers. Vind (1967) proposed a time stopping technique that in some (but not all) respects resembles our method. Vind (1967) only allows for a finite number of jumps in an otherwise piecewise continuously differentiable solution. Unfortunately, the existence of piecewise continuous optimal control is not guaranteed in general.[5] Recently, Hellwig (2008) has proposed a decomposition approach that also allows a fully general form of the monotone solution, paying special attention to discontinuities. Our optimal solution is continuous, and this allows us to obtain a characterization that is simpler to use in our application. We should also mention two recent interesting papers on bunching without the use of optimal control: Noldeke and Samuelson (2007) and Toikka (2011).

The structure of the paper is as follows. Section 2 introduces the model and also contains equilibrium characterization results. Section 3 characterizes the solution of the relaxed revenue maximization problem, and discusses the form of the solution. Section 4 shows that essentially the same characterization carries over to the monotonically-constrained problem. Section 5 concludes.

[5] The results that are obtained under weakest conditions, e.g. the Fillipov-Chesari theorem that we invoke (see Seierstad and Sydsæter (1987)), implies that optimal controls may be merely bounded measurable.

2. The Model

A risk-neutral seller, who owns a single unit of a good and discounts future at the rate $\rho > 0$, faces $N \geq 1$ risk-neutral buyers, all of whom discount at the rate $r \geq 0$, potentially different from ρ. The setting is the one of independent private values (IPV). All buyers draw their valuations for the good simultaneously and independently from the same distribution $F(\cdot)$, with support $[0, 1]$ and density $f(\cdot)$ bounded away from 0 on the support. The seller's cost is normalized to be 0.

The seller commits to a non-increasing price schedule that starts at some high price, followed by continuous or instantaneous price reductions. The time of sale is exogenously constrained by the good's lifespan T. Formally, we define an *admissible* price schedule as any nondecreasing, right-continuous function

$$p : [0, T] \to [0, 1].$$

Refer to Figure 1, which shows an admissible price schedule.[6] There is a continuous initial segment, followed by a downward jump at $t = t_1$. From t_1 to t_2, the price remains constant, and then, over the interval $[t_2, T]$, continuously declines again. As we shall see, the intervals of continuous price decline may only involve sales at posted prices. A downward jump will always involve selling at an auction. Whenever the price is constant over an interval of time, the good is effectively removed from the market over that interval, as no buyer who is impatient will prefer to buy later at the same price.

Given the price schedule p, the game, which will be referred to as the *dynamic pricing game*, or DPG(p) in short, is defined as follows. At each point of time t, a buyer may indicate his or her willingness to purchase the good at the standing price $p(t)$. If there are several buyers who want to buy at that price, they (and only they) participate in an auction. The reserve price in the auction is set at $p(t)$. The auction is assumed to be instantaneous button

[6] The restriction to nonincreasing price schedules is convenient and actually is without loss of generality. This is because no buyer who is impatient will ever prefer to buy at a higher price later, if he can buy at a lower price earlier. It can be shown that any price schedule is (weakly) dominated by a nonincreasing one, at least in the best equilibrium for the seller. Formally, for any $p(\cdot)$, such a nonincreasing schedule $p^*(\cdot)$ can be obtained by ironing, i.e. replacing any increasing segment of the schedule with a constant: $p^*(t) \equiv \inf\{p(t') : t' \in [0, t]\}$.

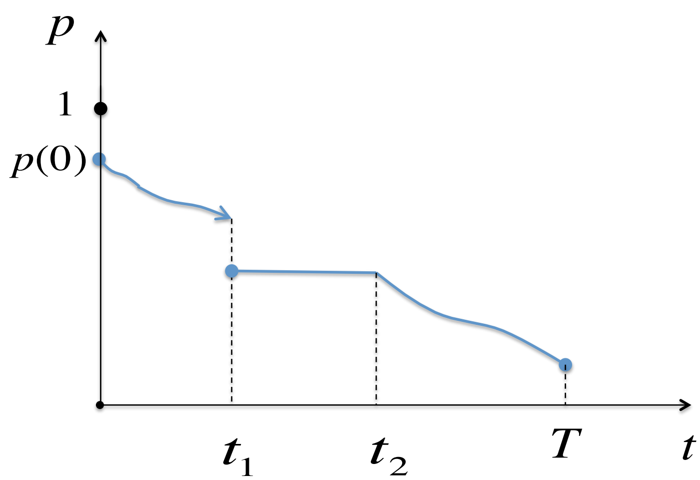


Figure 1: An admissible price schedule.

English (equivalently, second-price sealed-bid).[7]

Remark 1. *A static optimal auction at time* 0 *with a reserve price* R *is a special case of a DPG(p), for* $p(t) = R \ \forall t \in [0, T]$.

A standard argument shows that in the auction, buyers have a dominant strategy to bid their valuations. Each bidder's strategy in the DPG(p) is a pair $(e_i(\cdot), t_i(\cdot))$. The function $e_i(\cdot)$ takes values in $\{0, 1\}$ and prescribes which types of buyer i are *active*: $e_i(v) = 1$ if the buyer is willing to buy the good at some $t \in [0, T]$, and $e_i(v) = 0$ otherwise. Denote the set of active types for buyer i as $\mathcal{A}_i$. The function $t_i(\cdot)$ is defined for active types of buyer i, as the time when he or she wants to buy the good. Whenever the buyer is indifferent between being active or not, we assume that he or she is active. We restrict attention to symmetric equilibria: $e_i(v) = e(v)$, so that $\mathcal{A}_i \equiv \mathcal{A}$, and $t_i(v) = t(v) \ \forall t \in \mathcal{A}$. The equilibrium strategies are denoted as $e_*(\cdot)$ and $t_*(\cdot)$.

The *expected price* paid in equilibrium by a buyer of type v to the seller is denoted as $b(v)$. If there are no other buyers who want to buy at $t_*(v)$, or the set of their types has measure 0, then $b(v) = p(t_*(v))$; otherwise, this payment is determined at auction.

Proposition 1 (Properties of Equilibrium)**.** *Every equilibrium of a DPG(p) has the following properties:*

[7]The Internet allows sellers to conduct auctions with little effort, in many cases almost instantaneously.

(a) *The set of active buyer types $\mathcal{A}$ is an interval, $\mathcal{A} = (\underline{v}, 1]$, where $\underline{v} = p(T)$.*

(b) *The expected price $b(v)$ is given by*

$$b(v) = v - \frac{e^{rt_*(v)}}{F(v)^{N-1}} \int_{\underline{v}}^{v} e^{-rt_*(\tilde{v})} F(\tilde{v})^{N-1} d\tilde{v}, \qquad \forall v \in \mathcal{A}, \tag{1}$$

and is an increasing function.

(c) *The strategy $t_*(\cdot)$ is nonincreasing.*

Proof. See the Appendix. □

The intuition for this result is as follows. First, a standard single-crossing property implies that the set of active buyer types is an interval. Second, the Envelope Theorem applied to the buyer's expected discounted utility implies that the expected price $b(\cdot)$ is given by (1). Third, incentive compatibility implies the monotonicity of the time of sale function $t_*(\cdot)$.

Note also that, if v is an interior point of strict monotonicity of $t_*(\cdot)$, i.e. $t_*(v - \epsilon) > t_*(v) > t_*(v + \epsilon)$, then the sale at $t_*(v)$ occurs at a *posted price* $p(t_*(v))$. Otherwise, if $N \geq 2$, a v-type buyer will participate in an auction with positive probability, where the reserve price is $p(t_*(v))$.

The sale mechanism at time t can also be determined from the local behavior of the price schedule $p(t)$. If $t > 0$ is a point of *continuity* of $p(\cdot)$, then the sale at t may be at a posted price only. (Not all points of continuity may correspond to a sale, as there are no sales for t outside the range of $t_*(\cdot)$.) There can be no auction with a positive probability at a continuity point $t > 0$. Indeed, suppose, to the contrary, there was an auction, with the interval of participating buyer types having non-empty interior $(\underline{v}(t), \bar{v}(t))$, with $\bar{v}(t) > \underline{v}(t)$.[8] Then a buyer with type $v' \in (\underline{v}(t), \bar{v}(t))$ would have a probability of winning the auction strictly less than 1. This buyer would have an incentive to deviate and instead purchase the good at a fixed price $p(t - \epsilon)$ an ϵ instance earlier, for $\epsilon > 0$ sufficiently small.

On the other hand, each point of discontinuity of $p(t)$ must corresponds to an auction. Let t be a discontinuity point of $p(t)$, and assume, to the contrary, that the set of buyer types wanting to buy at $p(t)$ is either a singleton or empty. Consider a buyer with a type slightly

[8]The set of types participating in the auction must be an interval because $t_*(\cdot)$ is nonincreasing.

above v. This buyer is currently buying at a fixed price slightly above $p(t-)$ at time slightly before t, but, since $p(t) < p(t-)$, he or she can improve his expected payoff dramatically by deviating and participating in the auction at time t.[9] Notice that, by our assumption, his expected competition in such an auction would be nil. This deviation proves that every discontinuity point of $p(t)$ necessarily involves an auction.

Proposition 2 below provides a converse of Proposition 1, showing that any nonincreasing function $t : [\underline{v}, 1] \to [0, T]$ can be implemented in equilibrium of DPG(p) with an appropriately chosen price schedule $p(\cdot)$.

Proposition 2 (Implementation)**.** *For any* $\underline{v} \in [0, 1)$*, any nonincreasing, right-continuous function* $t : [\underline{v}, 1] \to [0, T]$ *is an equilibrium time of purchase strategy of the DPG(p) with the price schedule* $p(\cdot) \equiv b(t^{-\mathbb{1}}(\cdot))$*, where* $t^{-\mathbb{1}}(\tau) \equiv \inf\{v : t(v) \geq \tau\}$ *is the generalized inverse of* $t(\cdot)$*, and* $b(\cdot)$ *is determined from* (1).[10]

Proof. See the Appendix. □

Remark 2. *We have assumed that the auction is conducted in either (button) English or second-price format. But it can be shown that the results in Propositions 1 and 2 remain valid for the other popular formats, first-price or Dutch. In the case of the Dutch auction, we may assume that, at any discontinuity point of* $p(\cdot)$*, the price falls instantaneously. This way, the entire price schedule can be thought of as a dynamic Dutch auction clock.*

We are interested in designing a price schedule $p(t)$ that maximizes the seller's expected revenue. In view of Propositions 1 and 2, this problem can be solved by finding a nonincreasing and right-continuous time of sale function $t(\cdot)$; the expected price from a type v buyer can then be found from (1). We will solve this problem using optimal control methods. In order to formulate it as an optimal control problem, we treat $t(\cdot)$ as a control, and the utility

$$U(v) = (v - b(v))e^{-rt(v)}F(v)^{N-1} \tag{2}$$

[9] Here and below we use the notation $g(x_0-)$ to denote the limit of function $g(\cdot)$ as $x \uparrow x_0$.
[10] With $t_*(\cdot)$ replaced by $t(\cdot)$.

as the state variable. Equation (1) implies that a.e. on $[\underline{v}, 1]$

$$U'(v) = e^{-rt(v)} F(v)^{N-1}. \tag{3}$$

From (2),

$$b(v) = v - \frac{e^{rt(v)}}{F(v)^{N-1}} U(v).$$

Substituting this expression for $b(v)$, the seller's expected revenue obtained from any buyer is

$$\Pi = \int_{\underline{v}}^{1} e^{-\rho t(v)} b(v) F(v)^{N-1} dv = \int_{\underline{v}}^{1} \pi(v, U(v), t(v)) f(v) dv, \tag{4}$$

where we denoted

$$\pi(v, U, t) \equiv e^{-\rho t} F(v)^{N-1} v - e^{(r-\rho)t} U.$$

Problem 1. *Find a cutoff type* $\underline{v} \in [0, 1]$ *and a nonincreasing function* $t : [\underline{v}, 1] \to [0, T]$ *that deliver a maximum to* (4)*, where* $U(\cdot)$ *satisfies the first-order differential equation* (3) *with the initial condition* $U(\underline{v}) = 0$.

The presence of the monotonicity constraint on $t(\cdot)$ considerably complicates the analysis of this problem. To get useful insights, in the next section we consider a *relaxed* problem, removing the monotonicity constraint. This relaxed problem can be tackled by standard optimal control methods. The general problem is considered in Section 4, where it is shown that almost all the properties of the optimal solution of the relaxed problem continue to hold. The only new element is that auctions may be optimal at intermediate times.

3. Optimal Solution: Relaxed Problem

In this section, we consider a *relaxed* version of Problem 1. That is, we do not impose the monotonicity constraint on the time of sale function $t(\cdot)$. If the optimal solution to the relaxed problem satisfies this monotonicity condition, it is clear that it solves Problem 1.

Unfortunately, no closed-form solution will be available even in simple cases. Our approach instead is to characterize a solution implicitly, using the Maximum Principle.

The *Hamiltonian* that corresponds to Problem 1 is

$$H(v,t,U,\lambda) \equiv \left(e^{-\rho t} v F(v)^{N-1} - e^{(r-\rho)t} U\right) f(v) + \lambda e^{-rt} F(v)^{N-1}, \tag{5}$$

where λ is the costate variable associated with the state variable U whose law of motion is given by (3), with the initial condition $U(\underline{v}) = 0$. The function t is viewed as a control variable. Let $t_*(\cdot)$ be the optimal control, and let the corresponding utility function be $U_*(\cdot)$.

The Maximum Principle in the present setting implies the following.[11] For any $v \in [\underline{v}, 1]$,

1. There exist a piecewise continuously differentiable function λ such that, except at the points where $t_*(\cdot)$ is discontinuous,

$$\lambda'(v) = -\frac{\partial H}{\partial U} = e^{(r-\rho)t} f(v), \tag{6}$$

   with the boundary condition

$$\lambda(1) = 0. \tag{7}$$

2. The function $t_*(\cdot)$ maximizes the Hamiltonian,

$$t_*(v) \in \arg\max_{t\in[0,T]} H(v, U_*, \lambda, t).$$

3. At $v = \underline{v}$, a *transversality* condition holds:

$$H(\underline{v}, t_*(\underline{v}), U_*(\underline{v}), \lambda(\underline{v})) \geq 0, \tag{8}$$

   with strict equality if $\underline{v} > 0$.

The derivative of the Hamiltonian with respect to t is, after some re-arrangement,

$$\frac{\partial H}{\partial t} = -\rho F(v)^{N-1} f(v) e^{-\rho t} v - rF(v)^{N-1} \lambda e^{-rt} - (r-\rho) e^{(r-\rho)t} U f(v). \tag{9}$$

---

[11] See e.g. Theorems 2 and 11 in Seierstad and Sydsæter (1987)). Theorem 2 contains the Maximum Principle for a fixed boundary problem. In our case, because $\underline{v}$ is chosen by the seller, we are dealing with a free boundary problem, covered in Theorem 11 in the same book.

We now show that the form on solution will depend on whether or not the seller is more patient than the buyers.

The seller is less patient: $\rho \geq r$**.** Rewrite the F.O.C. (9) as

$$\frac{\partial H}{\partial t} = e^{-\rho t} \left(F(v)^{N-1} f(v) \left(-\rho v - \frac{r\lambda}{f(v)} e^{(\rho - r)t} \right) + (\rho - r) e^{rt} U f(v) \right).$$

Since (6) and (7) imply that $\lambda(v) \leq 0$, the function in the parentheses is monotone increasing in t. Therefore $H(\cdot, t)$ is *strictly quasiconvex* in t. It follows that the maximizer $t_*(v) \in \{0, T\}$. The Hamiltonian (5) converges to 0 as $T \to \infty$. Therefore, in the problem with the constraint $t \leq T$ relaxed, i.e. only imposing $t \geq 0$, the optimizer $t_*(v) = 0$. Because imposing the additional constraint $t < T$ can only lower the seller's expected payoff, $t_*(v) = 0$ is also the solution to the constrained problem. Thus, if the seller is (weakly) less patient, we obtain a standard result that an instantaneous auction at time 0 with a reserve price is an optimal mechanism. See Myerson (1981).

From now on, we make the standard assumption that the Myerson virtual value function is increasing,

Assumption 1 (Increasing Virtual Value)**.** *The function*

$$J(v) \equiv v - \frac{1 - F(v)}{f(v)}$$

is strictly increasing.

Then the optimal reserve price v_0 is given by the (unique) solution to the equation

$$J(v_0) = 0.$$

This is also what we obtain from the transversality condition (8) with $\underline{v} > 0$, by substituting the solution to (6), which, given that $t_*(v) = 0$, is

$$\lambda(v) = -[1 - F(v))].$$

From now on, the focus will be on the case of a more patient seller.

The seller is more patient: $\rho < r$**.** Rewrite (9) as

$$\frac{\partial H}{\partial t} = e^{-rt} h(v, t, U, \lambda),$$

where we denoted

$$h(v, t, U, \lambda) \equiv F(v)^{N-1} f(v) \left(-\rho e^{(r-\rho)t} v - \frac{r\lambda}{f(v)} \right) - (r - \rho) e^{(2r-\rho)t} U f(v).$$

We now consider a solution $t_0(v) \in \mathbb{R}_+$ to the F.O.C. $\frac{\partial H}{\partial t} = 0$,

$$h(v, t, U_*(v), \lambda(v)) = 0, \tag{10}$$

Observe that for $v \in (0, 1)$,

$$\lim_{t \to -\infty} h(v, t, U_*(v), \lambda(v)) = -r\lambda(v) F(v)^{N-1} > 0, \tag{11}$$

$$\lim_{t \to +\infty} h(v, t, U_*(v), \lambda(v)) = -\infty. \tag{12}$$

By continuity of h in t, the above inequalities imply that the solution $t_0(v)$ exists and is unique if $v \in (0, 1)$.

Define as $\bar{v}$ the minimal buyer type that is not delayed,

$$\bar{v} \equiv \inf\{v : t_*(v) = 0\}.$$

The set of non-delayed types is an interval $[\bar{v}, 1]$. In a similar fashion, define $\underline{v}_*$ as the largest v that is delayed until the last instance, i.e. still has $t_*(v) = T$,

$$\underline{v}_* \equiv \sup\{v : t_*(v) = T\}.$$

Our main result in this section is the following proposition. Refer to Figure 2.

Proposition 3 (Optimal Solution)**.** *When the seller is more patient than the buyers, an*

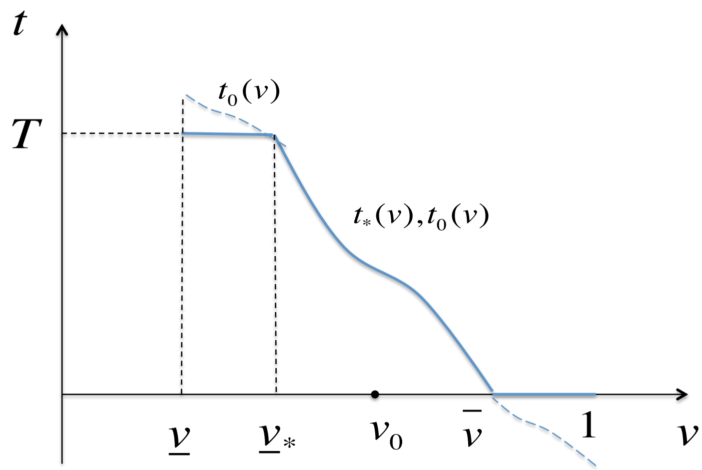


Figure 2: Optimal solution

optimal solution to the relaxed problem is a continuous *function* $t_*(\cdot)$ *of the form*

$$t_*(v) = \begin{cases} T, & v \in [\underline{v}, \underline{v}_*] \\ t_0(v), & v \in [\underline{v}_*, \bar{v}] \\ 0, & v \in [\bar{v}, 1] \end{cases} \tag{13}$$

where $t_0(\cdot)$ *is the unique solution to the F.O.C.* (10)*, and*

$$0 < \underline{v} < v_0 < \underline{v}_* < \bar{v} < 1. \tag{14}$$

Discussion. A number of interesting properties of the optimal DGP can be inferred from Proposition 3. First, in equilibrium, a positive measure of types is delayed. This follows from the fact that $t_*(\underline{v}) = T$. Specifically, all low types $v \in [\underline{v}, \bar{v})$ are delayed. Second, as $\bar{v} < 1$ and $\underline{v} < \underline{v}_*$, there is a positive probability of an auction (for $N \geq 2$) both at the beginning and at the end. Otherwise, for $t \in (0, T)$, the item is sold at a posted price $p(t)$. Third, the seller will never choose to remove the good from the market. This follows from the fact that $t_*(\cdot)$ is a continuous function. Fourth, even if $N = 1$, the posted reserve price at $t = T$ exhibits a downward jump:

$$p(T) = b(\underline{v}) = \underline{v}, \qquad p(T-) = b(\underline{v}_*) > p(T).$$

Proof of Proposition 3. We begin with the following simple observation that will be used for several results in the paper.

Lemma 1 (Quasiconcavity of Hamiltonian)**.** *The function $H(v, t, U, \lambda)$ is strictly quasiconcave in t for $v \in (0, 1]$.*

Proof. This follows from the fact that for $v \in (0, 1]$, $h(v, t, U, \lambda)$ is an increasing function of t. □

The inequalities (14) are proven in a sequence of lemmas below.

Lemma 2. *All sufficiently low types are screened out: $\underline{v} > 0$. The transversality condition is satisfied as equality, and can be stated as*

$$\underline{v}e^{(r-\rho)T} - \frac{\int_{\underline{v}}^{1} e^{(r-\rho)t(x)} f(x)dx}{f(\underline{v})} = 0. \tag{15}$$

Moreover, under Assumption 1, the lowest participating type is below the static reserve v_0,

$$\underline{v} < v_0. \tag{16}$$

Proof. Assume, by the way of contradiction, that $\underline{v} = 0$. We first perform a transformation of types to facilitate the use of the transversality condition in order to show that $\underline{v} > 0$. The expected (discounted) profit can be written as

$$\begin{aligned}\Pi &= \frac{1}{N}\int_{\underline{v}}^{1}\left(e^{-\rho t_*(v)}Q(y) - e^{(r-\rho)t_*(v)}\frac{U(v)}{F(v)^N}\right)dF^N(v)\\ &= \frac{1}{N}\int_{\underline{v}}^{1}\left(e^{-\rho t_*(Q(y))}Q(y) - e^{(r-\rho)t_*(Q(y))}\frac{U(Q(y))}{F(Q(y))^{N-1}}\right)dy.\end{aligned}$$

where $Q(\cdot)$ is the inverse of $F^N(\cdot)$. Observe that the function $\frac{U(Q(y))}{F(Q(y))^{N-1}}$ has a finite limit as

$y \downarrow 0$:

$$\begin{aligned}
\lim_{y\downarrow 0} \frac{U(Q(y))}{F(Q(y))^{N-1}} &= \lim_{v\downarrow 0} \frac{U(v)}{F(v)^{N-1}} \\
&= \lim_{v\downarrow 0} \frac{U'(v)}{(N-1)f(v)F(v)^{N-2}} \\
&= \lim_{v\downarrow 0} \frac{F(v)^{N-1}e^{-rt_*(v)}}{(N-1)f(v)F(v)^{N-2}} \\
&= 0,
\end{aligned}$$

where the first equality follows by L'Hopital's rule, the second – from the substitution of (3), and the last one – by our assumption that the density $f(v)$ is bounded from below. Therefore

$$\frac{d\Pi}{d\underline{v}}\Big|_{\underline{v}=0} = \frac{1}{N} e^{(r-\rho)t_*(0)} \lim_{v\downarrow 0} \frac{U(Q(y))}{F(Q(y))^{N-1}} > 0.$$

This is a violation of the first-order condition of the optimality of $\underline{v}$, proving that the optimal solution involves $\underline{v} > 0$, and the transversality condition (8) holds with equality. An algebraic manipulation of (8) then yields (15). Finally, in order to prove (16), notice

$$\begin{aligned}
J(\underline{v}) &= \underline{v} - \frac{1}{f(\underline{v})} \int_{\underline{v}}^{1} f(x)dx \\
&\leq \underline{v} - \frac{1}{f(\underline{v})} \int_{\underline{v}}^{1} e^{(r-\rho)(t_*(x)-t_*(\underline{v}))} f(x)dx \\
&= 0,
\end{aligned}$$

where the inequality in the second line follows because $e^{(r-\rho)(t_*(x)-t_*(\underline{v}))} \leq 1$, and the last line follows from the transversality condition (15). As $J(v_0) = 0$ and $J(\cdot)$ is an increasing by Assumption 1, we must have (16). □

Lemma 3. *The function $t_*(\cdot)$ is continuous.*

Proof. Because the Hamiltonian H is quasiconcave in t, the optimal solution $t_*(v)$, which is monotone nonincreasing by assumption, must be a *truncation* of $t_0(v)$ as in (13). By Lemma

2, we have $\underline{v} > 0$. Then

$$\begin{aligned}\frac{\partial h(v, t_0(v), U_*(v), \lambda(v))}{\partial t} &= -(r-\rho)F(v)^{N-1}f(v)e^{(r-\rho)t_0(v)} \\ &- (2r-\rho)(r-\rho)e^{(2r-\rho)t_0(v)}U_*(v) \\ &< 0\end{aligned}$$

for $v \in [\underline{v}, 1]$. The Implicit Function Theorem ensures that $t_0(\cdot)$ is differentiable, and therefore continuous. It follows that $t_*(\cdot)$ is a continuous function. □

Lemma 4. *The buyers who buy before T have types bounded from below by the static optimal reserve, $\underline{v}_* > v_0$.*

Proof. At $v = \underline{v}_*$, the solution satisfies the F.O.C.

$$h(\underline{v}_*, T, U_*(\underline{v}_*), \lambda(\underline{v}_*)) = 0.$$

Upon an algebraic manipulation, this implies

$$\underline{v}_* - \frac{r}{\rho}\frac{\int_{\underline{v}_*}^1 e^{(r-\rho)(t(x)-T)}f(x)dx}{f(\underline{v}_*)} + \frac{r-\rho}{\rho}\frac{U_*(\underline{v}_*)}{F(\underline{v}_*)^{N-1}} = 0.$$

Substituting inequalities

$$\begin{aligned}\frac{\int_{\underline{v}_*}^1 e^{(r-\rho)(t(x)-T)}f(x)dx}{f(\underline{v}_*)} &\leq \frac{1-F(\underline{v}_*)}{f(\underline{v}_*)}, \\ \frac{U_*(\underline{v}_*)}{F(\underline{v}_*)^{N-1}} &= \underline{v}_* - b(\underline{v}_*) \\ &< \underline{v}_*,\end{aligned}$$

we get

$$\underline{v}_* - \frac{r}{\rho}\frac{1-F(\underline{v}_*)}{f(\underline{v}_*)} + \frac{r-\rho}{\rho}\underline{v}_* \geq 0.$$

This inequality is equivalent to

$$\underline{v}_* - \frac{1 - F(\underline{v}_*)}{f(\underline{v}_*)} = J(\underline{v}_*) \geq 0,$$

which in turn implies $\underline{v}_* > v_0$ because, once again, $J(\cdot)$ is an increasing function by Assumption 1. $\square$

Lemma 5. *The optimal solution must involve delays:* $t_*(\underline{v}) = T$.

Proof. Suppose not, i.e. $\bar{t} \equiv t_*(\underline{v}) < T$. Then either $\bar{t} = 0$ or we have an interior maximum. Since h is decreasing in t, in any case we have $h(\underline{v}, \bar{t}, U_*(\underline{v}), \lambda(\underline{v})) \geq 0$. The last statement is equivalent to

$$0 \leq \underline{v}e^{(r-\rho)\bar{t}} - \frac{r}{\rho}\frac{\int_{\underline{v}}^{1} e^{(r-\rho)t(x)} f(x)dx}{f(\underline{v})} \tag{17}$$

However, since $r/\rho > 1$, (17) also implies

$$0 \leq \underline{v}e^{(r-\rho)\bar{t}} - \frac{\int_{\underline{v}}^{1} e^{(r-\rho)t(x)} f(x)dx}{f(\underline{v})}$$

but this contradicts the transversality condition (15). $\square$

Lemma 6. *The optimal solution must involve an auction at time* 0 *with a positive probability:* $\bar{v} < 1$.

Proof. In view of continuity of $h(v, t, U_*, \lambda)$ in t, it is sufficient to demonstrate that the slope of the Hamiltonian is negative at $t = 0$. It is sufficient to show that $h(1, t, U_*(1), 0) < 0$ at $t = 0$. But, since $\lambda_*(1) = 0$, this is immediate:

$$h(1, 0, U_*(1), 0) = -\rho e^{(r-\rho)t} f(1) - (r - \rho)e^{(2r-\rho)t} U_*(1) f(v) < 0. \tag{18}$$

$\square$

This last lemma completes the proof of Proposition 3. $\square$

The analysis in this section has assumed that the solution to the relaxed problem is monotone decreasing. If there is only one buyer, Landsberger and Meilijson (1985) show that the optimal solution $t_*(\cdot)$ is nonincreasing. Landsberger and Meilijson (1985) restrict attention to *differentiable* price schedules. But, a revealed-preference argument easily shows that $t_*(\cdot)$ must be nonincreasing for arbitrary nonincreasing price schedules. Also, our analysis implies that the optimal price schedule is not differentiable at $t = T$, as the posted reserve price exhibits a downward jump there: $p(T) = b(\underline{v}) = \underline{v}$, $p(T-) = b(\underline{v}_*) > p(T)$.

Even though we have not been able to obtain conditions on the model primitives that would guarantee monotonicity for $N > 1$, the following numerical example below suggests that there are cases when the solution to the relaxed problem is monotone.

Example 1. *The type distribution is assumed to be uniform* $[0, 1]$*,* $f(v) = 1$*. We compute the solution to the system of differential equations* (3) *and* (6) *on* $[\underline{v}, \bar{v}]$*, subject boundary conditions. As* $t_*(v) = 0$ *for* $v \in [\bar{v}, 1]$*, we have the (upper) boundary condition for* λ*:*

$$\begin{aligned}\lambda(\bar{v}) &= -\left(1 - F(\bar{v})\right) \\ &= -\left(1 - \bar{v}\right). \end{aligned} \tag{19}$$

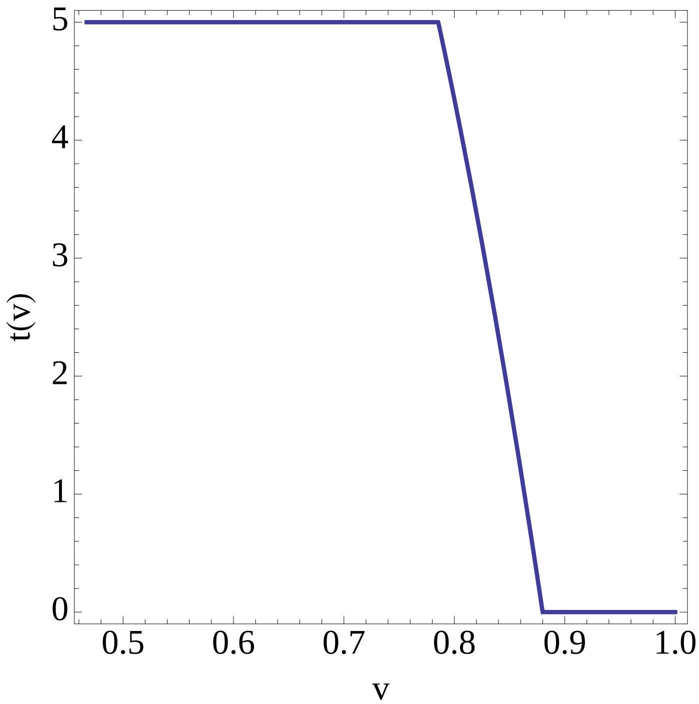
The (lower) boundary condition for $U(\cdot)$ *is*

$$\begin{aligned} U(\underline{v}_*) &= e^{-rT} \int_{\underline{v}}^{\underline{v}_*} F(x)^{N-1} dx \\ &= e^{-rT} \times \frac{\underline{v}_*^N - \underline{v}^N}{N}. \end{aligned} \tag{20}$$

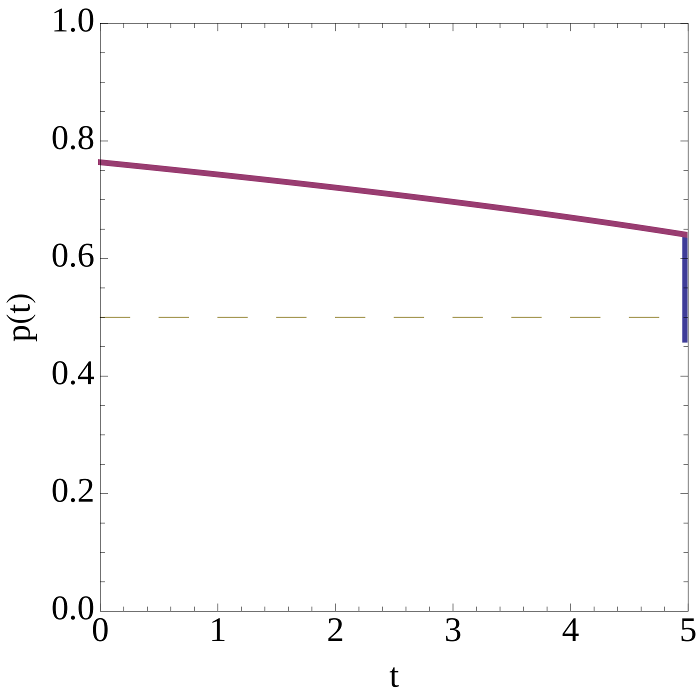
As we have 3 *unknown parameters* $\underline{v}$*,* $\underline{v}_*$ *and* $\bar{v}$*, we need one more equation, the transversality condition* (15)*. Here, it reduces to*

$$e^{(r-\rho)T} \underline{v} - e^{(r-\rho)T} (\underline{v} - \underline{v}_*) - \lambda(\underline{v}_*) = 0. \tag{21}$$

Instead of attempting to solve the F.O.C. (10)*, we impose it directly. By so doing, we must*



(a) The optimal time of sale function $t(v)$



(b) The optimal reserve price function $p(t)$

Figure 3: Computed example

also impose the boundary conditions for $t(\cdot)$*:*

$$t_*(\bar{v}) = 0, \qquad t_*(\underline{v}_*) = T. \tag{22}$$

The numerical solution is computed by using a projection method.[12] *In our problem, the projection step consists of approximating the functions* $t_*(\cdot)$*,* $U_*(\cdot)$ *and* $\lambda(\cdot)$ *by* m*th degree polynomials,*

$$t_*(v) = \sum_{k=0}^{m} t_k v^k, \quad U_*(v) = \sum_{k=0}^{m} U_k v^k, \quad \text{and} \quad \lambda(v) = \sum_{k=0}^{m} \lambda_k v^k,$$

substituting these approximations in the system (3) *and* (6)*, and then discretizing it on a finite grid of* m_0 *points. This gives us* $3m_0$ *equations, and upon adding the 5 equations for the boundary conditions,* (19) - (22)*, we get a total of* $3m_0 + 5$ *equations for* $2m + (m-1) + 5 = 3m + 5$ *variables*

$$(t_k)_{k=0}^{m-1}, (U_k)_{k=0}^{m-1}, (\lambda_k)_{k=0}^{m-1}, \quad \text{and} \quad \underline{v}, \underline{v}_*, \bar{v}$$

To balance the number of equations and variables, we let $m_0 = m$*, and we chose* $m = 5$ *in the example we computed. Other parameters were set as follows:* $r = 0.1$*,* $\rho = 0.005$*,* $T = 5$*,* $N = 5$*. The results of the numerical computations are shown graphically in Figure 3. The computed solution* $t(\cdot)$ *is shown in Figure 3a. It is in fact monotone decreasing. Figure 3b contains the corresponding reserve price schedule, computed according to 3b. The computed solution has the features implied by Proposition 3. The continuously declining part of the schedule, which corresponds to sales at the posted price* $p(t)$*, is located above the static optimal reserve price* $v_0 = 1/2$*. There are two instantaneous auctions at times* $t = 0$ *and* $t = T$*. The expected profit is computed as* $\Pi_{total} = 0.745489$*. The profit in the static optimal auction in this example is* $\Pi_0 = 0.671875$*.* So the dynamic auction yields an improvement of about 11%.

[12] See Judd (1998) for a description of the projection method and its applications in economics.

4. Optimal Solution: General Case

The results in the previous section were obtained assuming that the solution to the relaxed problem is in fact monotone. In this section, we show that essentially all the results continue to hold without assuming monotonicity.

We will continue to use optimal control to characterize a solution. A standard technique for imposing a monotonicity constraint is to differentiate the control function, and impose the non-negativity constraint on the derivative.[13]

We will use this method in a form that allows for *any* monotone decreasing function $t(v)$. Our approach will require some preliminaries. Since $t_*(v)$ is monotone, the Lebesgue decomposition implies $t_*(v) = t_*^a(v) + t_*^j(v) + t_*^s(v)$, where $t^a(v)$ is absolutely continuous, $t_*^j(v)$ is a pure jump function (piecewise constant), and $t_*^s(v)$ is *singular,* i.e. has derivative equal 0 almost everywhere. A potential presence of the singular component creates a difficulty.[14]

In this section, we develop a re-parameterization method that allows us to work exclusively with absolutely continuous functions. The method consists in viewing any admissible function $t(v)$ as a curve in the plane. Let

$$\mathbb{G} = \bigcup_{v \in [\underline{v}, 1]} \{v\} \times [t(v), t(v-)].$$

be a planar curve that corresponds to the graph of the function $\{(v, t(v) : v \in [0, 1]\}$, but with the gaps that correspond to the discontinuities filled in by vertical line segments. That is, if say v_1 is a point of discontinuity, then we connect the points $(v_1, t(v_1-))$ and $(v_1, t(v_1+))$ by a linear segment.

Each such curve $\mathbb{G}$ admits a parametrization,

$$\mathbb{G} \equiv \{(V(w), t(w)) : w \in [0, 1]\},$$

where $V : [0, 1] \to \mathbb{R}_+$ is a continuous increasing function, and $t : [0, 1] \to \mathbb{R}_+$ is a continuous

[13] In the mechanism design literature, this approach was initiated in Guesnerie and Laffont (1984). For a textbook discussion, see the example on p. 244 in Kamien and Schwartz (1991).

[14] Of course, one way out is to limit attention to functions that do not have a singular component. But this would leave open the possibility that the seller could do better with more general price schedules.

decreasing function.[15] Note than many such parameterizations exist for a given $t(v)$, depending on the choice of function V. Our main tool in this section will be the following lemma, which shows that $V(\cdot), t(\cdot)$ can be chosen absolutely continuous, which will allow us to apply the Maximum Principle.

Lemma 7 (Lebesgue). *There exists a parametrization of* $\mathbb{G}$ *such that a.e. on* $[0,1]$,

$$V'(w) = u(w) \geq 0, \qquad t'(w) = -q(w) \leq 0$$

for some measurable non-negative functions $u : [0,1] \to \mathbb{R}_+$ *and* $q : [0,1] \to \mathbb{R}_+$.[16]

Proof. We will choose the *length* of the curve segment as a parameter. For any $w \in [0,1]$, the length of the curve $\ell(w)$ is defined as the supremum of the lengths of polygonal line approximations,

$$\ell(w) \equiv \sup_{0=w_0<w_2<\ldots<w_M=1} \sum_{i=1}^{M} ||z(w_i) - z(w_{i-1})||,$$

where $z(w) \equiv (V(w), t(w))$ and $||\cdot||$ denotes the Euclidean norm. For our *monotone* curve, such a bound always exists because the sum under the supremum is bounded from above by $T+1$. Now consider a parametrization

$$\hat{w}(w) \equiv \frac{\ell(w)}{\ell(1)},$$

i.e. $\hat{w}(w)$ is the normalized length of the curve segment. Define $\hat{z}$ according to $\hat{z}(\hat{w}) = z(w)$. Since

$$||\hat{z}(\hat{w}_1) - \hat{z}(\hat{w}_2)|| \leq |\ell(w_1) - \ell(w_2)| = \ell(1) \cdot |\hat{w}_1 - \hat{w}_2|$$

(the length of a curve segment connecting any two points is weakly greater than the distance between the points), the function $\hat{z} \equiv (\hat{V}, \hat{t})$ is Lipschitz continuous, and therefore absolutely continuous, i.e. $\hat{V}, \hat{t}$ have derivatives a.e. on $[0,1]$, and are equal to the integrals of their derivatives. The result follows. □

[15] To minimize new notation, we keep the same letter t do denote this new function of w.

[16] This result follows from a theorem attributed to Lebesgue, see Theorem 2.2.2 on p.34 and the discussion immediately after in Aleksandrov and Reshetniak (1989). We present a proof for completeness, by following the approach in the proof of Theorem 4.3 on p.136 in Stein and Shakarchi (2005).

Any interval $[w_1, w_2]$ such that $u(w) = 0$ a.e., so that $V(w_1) = V(w_2)$, but $t(w_1) < t(w_2)$, corresponds to a *discontinuity* of the function $t(v)$ at $v = V(w_1)$ $(= V(w_2))$. As w moves along this interval, t changes (in a continuous fashion as a function of w), but v stays the same. Any interval $[w_3, w_4]$ such that $q(w) = 0$ a.e., but $V(w_1) < V(w_2)$, corresponds to an interval $[V(w_3), V(w_4)]$ where the monotonicity constraint on $t(v)$ may be binding.

In this parameterized setup, the seller's optimization problem can be formulated as the following optimal control problem.

Problem 2. *Find non-negative piecewise continuous control functions* $u, q : [0,1] \to \mathbb{R}_+$ *that deliver a maximum to the expected profit of the seller*

$$\Pi = \int_{\underline{v}}^{1} u(w)\pi(V(w), U(w), t(w)) f(V(w)) dw,$$

subject to the constraints that the associated state variables U, V, t *evolve according to*

$$U'(w) = u(w) \cdot e^{-rt(w)} F(V(w))^{N-1}, \tag{23}$$

$$V'(w) = u(w) \geq 0, \tag{24}$$

$$t'(w) = -q(w) \leq 0 \tag{25}$$

a.e. on $[0,1]$, *and obey the boundary conditions* $V(1) = 1, V(0) \leq T$ *and* $t(1) = 0, t(0) \geq 0$.

Standard results in optimal control show that this problem always has a solution in the set of bounded measurable (and non-negative) controls u, q. For example, one can verify the conditions of the Filippov-Cesari Theorem.[17]

In this new setup, we have two new state variables V, t, whose laws of motion are governed by (24) and (25), subject to the boundary conditions. The Hamiltonian is given by

$$\begin{aligned} \mathcal{H}(w, U, V, t, \lambda, \mu, \chi) &= u \cdot \pi(w, U, V, t) + \lambda \cdot u \cdot e^{-rt} F(V)^{N-1} + \mu \cdot u - \chi \cdot q \\ &= u \cdot (H(V, U, t, \lambda) + \mu) - \chi \cdot q, \end{aligned} \tag{26}$$

where H is the Hamiltonian defined in the previous section, and μ and χ are the costate

[17]See e.g. Theorem 8 on p. 132 in Seierstad and Sydsæter (1987).

variables that correspond to (24) and (25). We will employ the version of the Maximum principle given by Theorem 2 on p. 85 in Seierstad and Sydsæter (1987).[18] At the optimal solution (u_*, q_*), the costate variables must be absolutely continuous, and satisfy the adjoint system

$$\lambda'(w) = -\frac{\partial \mathcal{H}}{\partial U} = -u_*(w)\frac{\partial H}{\partial U}, \tag{27}$$

$$\mu'(w) = -\frac{\partial \mathcal{H}}{\partial V} = -u_*(w)\frac{\partial H}{\partial V}, \tag{28}$$

$$\chi'(w) = -\frac{\partial \mathcal{H}}{\partial t} = -u_*(w)\frac{\partial H}{\partial t}, \tag{29}$$

a.e. on $[0, 1]$, with the boundary conditions $\lambda(1) = 0$ and

$$\mu(0) \le 0, \qquad \mu(0)V(0) = 0, \tag{30}$$

$$\chi(0) \ge 0, \qquad \chi(0)(T - t(0)) = 0, \tag{31}$$

where the equalities are the complementary slackness conditions that correspond to the constraints $V(0) \ge 0$ and $t(0) \le T$.

The Hamiltonian $\mathcal{H}$ must attain the maximal value $\mathcal{H}_*$ at the optimal solution (u_*, q_*). As we are allowing for arbitrary non-negative values for u, q, it follows that

$$\mathcal{H}_*(w) = 0,$$

and, furthermore, we must have a.e. on $[0, 1]$

$$H_*(w) + \mu(w) \le 0, \tag{32}$$

$$u_*(w) \cdot (H_*(w) + \mu(w)) = 0, \tag{33}$$

[18] Theorem 2 on p. 85 in in Seierstad and Sydsæter (1987) assumes piecewise continuous controls. An extension to bounded measurable controls is given in the (detailed) footnote 9 on p. 132 of the same book, where additional references are provided. In particular, we can either apply a very general Theorem 5.3.1 in Clarke (2005), or a more special Theorem 2 in Clarke (forthcoming).

and

$$\chi(w) \geq 0, \tag{34}$$

$$\chi(w) \cdot q_*(w) = 0. \tag{35}$$

Equations (33) and (35) are the complementary slackness conditions that correspond to the monotonicity constraints $u(w), q(w) \geq 0$.

The following lemma is parallel to Lemma 3 from the previous section.

Lemma 8. *The function $t_*(v)$ is continuous on $[\underline{v}, 1]$.*

Proof. First, observe that an optimal solution $t_*(v)$ can always be constructed without discontinuities at $v = \underline{v}$ and $v = 1$. Clearly, there is no discontinuity at $v = 1$, as it can never be optimal to delay trade for all buyer types. Second, a discontinuous downward jump in $t_*(v)$ at $v = \underline{v}$ occurs on the set of measure 0 and can be eliminated without affecting the expected profit, at the same time resulting in a function that continues to be monotone.

The rest of the proof is by contradiction: suppose there is a discontinuity at some $v_1 \in (0, 1)$. In view of the above observation, this implies that

1. $\exists w_1, w_2$ such that $w_2 > w_1$ and $V_*(w_1) = V_*(w_2) = v_1 \in (\underline{v}, 1)$, so that $u_*(w) = 0$ a.e. on $[w_1, w_2]$.

2. The function V_* is strictly increasing on some open left neighborhood $\mathcal{N}_L(w_1)$ of w_1 and on some open right neighborhood of w_2, so that $u_*(w) > 0$ a.e. on both $\mathcal{N}_L(w_1)$ and $\mathcal{N}_R(w_2)$.

3. $t_*(w_1) > t_*(w_2)$.

Equation (33) and item 2 above imply

$$\lim_{w \uparrow w_1} H_*(w) + \mu(w) = 0, \qquad \lim_{w \downarrow w_2} H_*(w) + \mu(w) = 0.$$

But the maximized Hamiltonian $H_*(w)$ is a continuous function, as is the adjoint $\mu(w)$, so we obtain

$$H_*(w_1) + \mu(w_1) = 0, \qquad H_*(w_2) + \mu(w_2) = 0.$$

Next, observe that (28) implies $\mu(w)$ does not change during the jump: $\mu_0 \equiv \mu(w_1) = \mu(w_2)$, therefore $H(w_1)+\mu_0 = H(w_2)+\mu_0$. But $H_*(w)$ is a *strictly* quasiconcave function on $[w_1, w_2]$. Indeed, as $u_*(w) = 0$ a.e. during the jump (item 1 above), (23) and (27) imply that U_* and λ do not change over the jump, so $H_*(w)$ will change only through the change in $t_*(w)$. As $H(V, t, U, \lambda)$ is a strictly quasiconcave function of t by Lemma 1, and $t_*(w)$ is a non-increasing continuous function that is not constant on $[w_1, w_2]$ (item 3 above), $H_*(w) + \mu_0$ must also be strictly quasiconcave on $[w_1, w_2]$. Since it is equal to 0 at the boundaries w_1, w_2, $H_*(w) + \mu_0$ must take on a positive value in $[w_1, w_2]$. But this contradicts (32). □

Remark 3. *The continuity of $t_*(v)$ implies that the parameterizing function $V_*(w)$ can be chosen as* increasing, *and therefore it will have an inverse $V_*^{-\mathbb{1}}(\cdot)$.*[19]

The proof of our main result in the previous section, Proposition 3, relies on the first-order condition for the Hamiltonian. The following lemma shows that, even with a monotonicity constraint, this first-order condition remains valid at any v where $t_*(v)$ is *srictly* decreasing, i.e. either $t_*(v - \epsilon) > t_*(v)$ or $t_*(v + \epsilon) < t_*(v)$, or both, for all (sufficiently small) $\epsilon > 0$. Subsequently, we use this result to prove Proposition 4, the analogue of Proposition 3. In this lemma, we treat the Hamiltonian as a function of v. That is, along the optimal path, all functions in H are treated as functions of v, i.e. $V_*^{-\mathbb{1}}(v)$ is substituted for w. We still keep the same notation for this functions, but is is important to keep in mind that they may be different from the ones in the previous section.

Lemma 9 (F.O.C.)**.** *Let $v \in [\underline{v}, 1]$ be a point of strict monotonicity of the optimal solution $t_*(v)$. Then the Hamiltonian H satisfies the first-order condition at v,*

$$\frac{\partial H(v, U_*(v), t_*(v), \lambda(v))}{\partial t} = 0.$$

[19]However, in general we do not make the identification $w = v$. We have ruled out the pure jump component, but not the singular component of $t_*(v)$. At the same time, as the function of w, $t_*(w)$ is absolutely continuous.

Proof. First, consider the case when $t_*(v)$ is strictly monotone both from left and right, i.e. $t_*(v-\epsilon) < t_*(v) < t_*(v+\epsilon)$ for all sufficiently small $\epsilon > 0$. Let w be the corresponding value of w, which is unique because $V_*(\cdot)$ is chosen to be increasing. Then there exist two sequences $\underline{w}_n \uparrow w$ and $\overline{w}_n \downarrow w$ such that $V_*(\underline{w}_n) < v < V_*(\overline{w}_n)$ and $t'_*(\underline{w}_n), t'_*(\overline{w}_n) > 0$. (The latter follows from the absolute continuity of $t_*(w)$.) Since the monotonicity constraint is not binding at $\underline{w}_n, \overline{w}_n$, the complementary slackness condition (31) implies $\chi(\underline{w}_n), \chi(\overline{w}_n) = 0$, and therefore (29) implies

$$0 = \chi(\overline{w}_n) - \chi(\underline{w}_n) = -\int_{V_*(\underline{w}_n)}^{V_*(\overline{w}_n)} \frac{\partial H}{\partial t} dv,$$

where in the last equality we have used a change of variables in (29), which is justifiable because $V_*(\cdot)$ is absolutely continuous. Dividing through by $V_*(\overline{w}_n) - V_*(\underline{w}_n) > 0$ and taking the limit as $n \to \infty$, we obtain

$$0 = \lim_{n\to\infty} \frac{1}{V_*(\overline{w}_n) - V_*(\underline{w}_n)} \int_{V_*(\underline{w}_n)}^{V_*(\overline{w}_n)} \frac{\partial H}{\partial t} dv = \frac{\partial H(v, U_*(v), t_*(v), \lambda(v))}{\partial t},$$

where the last equality follows by continuity of $\partial H/\partial t$ in v. This verifies the result in the lemma for any point v such that $t_*(v)$ is strictly monotone from both sides.

Since any point v where $t_*(v)$ decreasing from one side and is constant from the other can be approached by a sequence of points v_n such that $t_*(v_n)$ is strictly monotone from both sides at v_n, the continuity of $\partial H/\partial t$ in v implies that $\partial H/\partial t = 0$ at all points v where $t_*(v)$ is decreasing. □

In order to show the equivalents of Lemmas 4–6 from the previous section, we first establish the transversality condition (15).[20] Since $V_*(\cdot)$ is chosen as an increasing function, we can pick a sequence $w_n \downarrow 0$ such that $u_*(w_n) > 0$. Passing along this sequence to the limit in (33) and invoking the continuity of $H_*(w)$ gives us $H_*(0) = -\mu(0)$. But, $V_*(0) = \underline{v} > 0$, and the complementary slackness condition (30) therefore implies $\mu(0) = 0$. Hence $H_*(0) = 0$, as desired.

Lemma 4 (price posting above the static reserve, $\underline{v}_* > v_0$) only relies on the first-order

[20] Lemma 2 does not use optimal control.

condition for the Hamiltonian at $v = \underline{v}_*$, which holds by Lemma (9) because $\underline{v}_*$ is a point of strict monotonicity of $t_*(\cdot)$. Lemma 5 (optimality of delays) also follows by a straightforward modification of the argument. Suppose, to the contrary, that $t_*(0) < T$. Then the complementary slackness condition (31) implies $\chi(0) = 0$, therefore

$$0 < \chi(w) = \int_0^w \chi'(\tilde{w})d\tilde{w} = -\int_0^w \frac{\partial H}{\partial t} u_*(w)dw,$$

therefore, b continuity, $\partial H/\partial t < 0$ in some open (right) neighborhood of $v = \underline{v}$. But, as in the proof of Lemma 4, this contradicts the transversality condition (15) which implies that for $t < T$,

$$\frac{\partial H(\underline{v}, t, 0, \lambda(\underline{w}))}{\partial t} > 0.$$

Finally, the result in Lemma 6 ($\bar{v} < 1$) is also straightforward to obtain. If, to the contrary, we had $\bar{v} = 1$, then this would be a point of strict monotonicity of $t_*(v)$, therefore $\frac{\partial H}{\partial t} = 0$ by Lemma 9. But, as in the proof of Lemma 6,

$$\left.\frac{\partial H}{\partial t}\right|_{t=0,v=1} < 0,$$

so we have a contradiction.

To summarize, we have shown the following analogue of Proposition 3.

Proposition 4. *A solution to Problem 2 involves a* continuous *function $t_*(v)$ and has the following form: there exist $\underline{v}, \underline{v}_*, \bar{v} \in [0,1]$, $0 < \underline{v} < v_0 < \underline{v}_* < \bar{v} < 1$, such that $t_*(v) = T$ for $v \in [\underline{v}, \underline{v}_*]$ and $t_*(v) = 0$ for $v \in [\bar{v}, 1]$.*

5. Concluding Remarks

In this paper, we have studied the revenue management problem of a seller who has a perishable good and is more patient that his or her forward-looking buyers. Allowing for an essentially arbitrary monotone price schedule, we provide a characterization of the optimal solution and find, in particular, that it always involves delays. This implies that an auction with an optimally chosen reserve price at $t = 0$, is not optimal in this setting. A numerically computed example shows that it may yield significantly smaller revenue. The optimal price schedule $p(t)$ starts out at a price below 1, stays above the static optimal reserve price until time T, and then ends at the final time with an auction fire-sale. So auctions necessarily occur at $t = 0$ and $t = T$, and, in general, may also occur at intermediate times.

Optimal Mechanisms. We have restricted attention to a specific selling mechanism (DPG), it would be interesting to consider the optimal mechanism design problem in greater generality. One could specify a direct revelation mechanism that solicits type reports and assigns the price and time of purchase on the basis of these reports. Indeed, one can show that any equilibrium of DPG corresponds to an incentive compatible and individually rational direct mechanism, with the additional restriction that the time of purchase only depends on *own* report. Without this restriction, we have a difficult nonlinear optimization problem that involves multidimensional functions, and therefore falls outside the realm of standard optimal control. We leave this for future research.

Noncommitment. In our analysis, we have assumed full commitment. This assumption may be a natural in several applications as discussed in Board and Skrzypacz (2010); it is also made in Myerson (1981), which is a natural comparison benchmark for our DPG. But clearly there are applications where the seller cannot commit. In a recent paper, Skreta (2010), using the methods in Skreta (2006), completely solves a two-period optimal auction design problem under noncommitment and shows that the optimal solution involves two auctions: with a reserve price in the first period, auction in the first period, and an auction without a reserve in the second period. Restricting attention to dynamic pricing as we do here, Hörner and Samuelson (2011) have shown that the seller retains a degree of monopoly power even without commitment. Given this result, it is natural to expect that the seller will have a degree of monopoly power in our model even under noncommitment. This is left for future research.

Appendix

Proof of Proposition 1. Proof of (a). In any equilibrium we let $P(v)$ to be the expected price paid by a buyer of type v, and let $X(v)$ be the discounted probability of getting the good. The equilibrium expected utility of a type v buyer then is $U(v) = e_*(v) \cdot \big(e^{-rt_*(v)} v X(v) - P(v)\big)$, and the expected utility of type v buyer if he mimics a type $\hat{v}$ buyer is $\tilde{U}(v, \hat{v}) = e_*(\hat{v}) \cdot \big(e^{-rt_*(\hat{v})} v X(\hat{v}) - P(\hat{v})\big)$. The Envelope Theorem implies $U(v) = \int_0^v e_*(\tilde{v}) X(\tilde{v}) d\tilde{v}$, so $U(v)$ is a nondecreasing, continuous function. This implies that the set of buyer types who have a positive utility is an interval $(\underline{v}, 1]$ for some $\underline{v}$. Moreover, we must have $\underline{v} = \underline{p}$. Otherwise, if $\underline{v} < \underline{p}$, buyers with types slightly above $\underline{v}$ cannot get a positive utility. If $\underline{v} > \underline{p}$, then a buyer with type $\underline{v} - \epsilon$ slightly below $\underline{v}$ will make a positive utility by entering and waiting until T. In that case, he will have no rivals with probability at least $F(\underline{v})^{N-1}$, and will then be able to buy the good at price $\underline{p} < \underline{v} - \epsilon$.

Proof of (b). Observe first that for $v \in (\underline{v}, 1]$, the expected price paid by a type v buyer is

$$b(v) = \frac{P(v)}{X(v)}.$$

Now consider any $v_1, v_2 \in (\underline{v}, 1]$. As $\tilde{U}(v_1, v_1) \geq \tilde{U}(v_1, v_2)$, or, equivalently,

$$X(v_1)(v_1 - b(v_1)) \geq X(v_2)(v_1 - b(v_2)),$$

we have

$$\begin{aligned} b(v_2) &\geq \frac{1}{X(v_2)}(v_1(X(v_2) - X(v_1)) + X(v_1)b(v_1)) \\ &\geq \frac{1}{X(v_2)}(b(v_1)(X(v_2) - X(v_1)) + X(v_1)b(v_1)) \\ &= b(v_1), \end{aligned}$$

where the inequality in the second line follows from $v_1 \geq b(v_1)$. It follows that $b(v)$ is a nondecreasing function. A standard tie-breaking argument implies that it must be increasing. The functional equation (1) follows from the Envelope Theorem.

Proof of (c). If $t(v)$ is *not* nonincreasing, $\exists v_1, v_2$ such that $v_2 > v_1$, and $t_*(v_2) > t_*(v_1)$.

By definition, the reserve price is set at $p(t_*(v_1))$ at time $t_*(v_1)$, so the expected price is bounded from below by the reserve, $b(v_1) \geq p(t_*(v_1))$. Also, $p(t_*(v_2)-) \geq b(v_2)$. Otherwise, a type v_2 buyer could obtain a greater expected payoff by deviating and buying a little earlier, say at time $t_*(v_2) - \epsilon$, at a fixed price that is a notch higher, say $p(t_*(v)-) + \delta$, where, by continuity, an $\epsilon > 0$ can be chosen small enough so that $p(t_*(v_2)-) + \delta < b(v_2)$. So we must have

$$b(v_1) \geq p(t_*(v_1)) \geq p(t_*(v_2)-) \geq b(v_2),$$

in contradiction to $b(v_2) > b(v_1)$ as follows from item 2 of the proposition. □

Proof of Proposition 2. The strategy will be first to show that the expected utility is given by

$$(v - b(v))e^{-rt(v)}F(v)^{N-1} \tag{36}$$

and then to rule out both within-range deviations, i.e. to $t(\hat{v})$ for some $\hat{v} \in [\underline{v}, 1]$, and outside-range deviations.

If $N = 1$, the buyer of type v buys at a posted price $b(v)$ and the expected utility is given by (36). From now on, we assume $N \geq 2$. If v is a continuity point of $t(\cdot)$ such that $t(\cdot)$ is *strictly decreasing* to the left, i.e. $t(v - \epsilon) > t(v)$ for all sufficiently small $\epsilon > 0$, then there are no rival buyers willing to buy at the price $b(v)$, so $b(v)$ is in fact the price paid by the type v buyer conditional on purchasing the good and the expected utility is given by (36).

Next, if v is a continuity point of $t(\cdot)$ such that $t(\cdot)$ is *constant* in a left neighborhood of v, $t(v - \epsilon) = t(v) \equiv t_0$ for all sufficiently small $\epsilon > 0$, then the type v buyer will be bidding in the auction with positive probability, as all buyers in the interval $[\underline{v}(v), v]$, where $\underline{v}' \equiv \inf\{\tilde{v} : t(\tilde{v}) = t(v)\}$, will also want to buy the good at the same time. We now show that, in this case also, the expected profit is given by (36). Since we assume a button English (equivalently, sealed-bid second-price) auction, the expected price for type v buyer in the auction is equal to $E[\max\{r, v_{2:N}\}\mathbb{1}(v_{2:N} \leq v)]$, so his or her expected discounted payment is equal to

$$
\begin{aligned}
e^{-rt_0}E[\max\{r, v_{2:N}\}\mathbb{1}(v_{2:N} \leq v)] =& e^{-rt_0}\int_{\underline{v}'}^{v} \tilde{v} dF(\tilde{v})^{N-1} + e^{-rt_0}p(t_0)F(\bar{v}')^{N-1} \\
=& e^{-rt_0}vF(v)^{N-1} - e^{-rt_0}\int_{\underline{v}'}^{v} F(\tilde{v})^{N-1}d\tilde{v} \\
& + e^{-rt_0}\left(p(t_0) - \underline{v}'\right)F(\underline{v}')^{N-1},
\end{aligned}
$$

where the equality in the last line follows by integration by parts. The expected utility is therefore equal to

$$
\begin{aligned}
U(v) &= e^{-rt_0}\int_{\underline{v}'}^{v} F(\tilde{v})^{N-1}d\tilde{v} + e^{-rt_0}\left(\underline{v}' - p(t_0)\right)F(\underline{v}')^{N-1} \\
&= e^{-rt_0}\int_{\underline{v}'}^{v} F(\tilde{v})^{N-1}d\tilde{v} + U(\underline{v}'). \qquad (37)
\end{aligned}
$$

Being the lowest type buyer willing to buy at time $t(v)$, type $\underline{v}'$ can never win the auction, and so will only be able to buy if all other buyers have values less or equal to $\underline{v}'$, and will then be paying the price $b(v)$. Hence, $U(\underline{v}') = (\underline{v}' - b(v))e^{-rt(\underline{v}')}F(\underline{v}')^{N-1}$ and therefore, by (1),

$$
U(\underline{v}') = e^{-rt_0}\int_{\underline{v}}^{\underline{v}'} F(\tilde{v})^{N-1}d\tilde{v}.
$$

Substituting $U(\underline{v}')$ above into (37), and noting that $\int_{\underline{v}'}^{v} + \int_{\underline{v}}^{\underline{v}'} = \int_{\underline{v}}^{v}$, we can see that (36) holds even for a type v that will buy at an auction with positive probability.

We now show that there are no profitable deviations, so that we in fact have an equilibrium. First, we show that a within-range deviation, from $t(v)$ to $t(\hat{v})$, where $\hat{v} \in [\underline{v}, 1]$, is not profitable. Assume $\hat{v} > v$; for $\hat{v} < v$, the argument is parallel. Then

$$\begin{aligned}
U(v,\hat{v}) - U(v) =&(v - b(\hat{v}))e^{-rt(\hat{v})}F(\hat{v})^{N-1} - \int_{\underline{v}}^{v} e^{-rt(\tilde{v})}F(\tilde{v})^{N-1}d\tilde{v} \\
&= (\hat{v} - b(\hat{v}))\, e^{-rt(\hat{v})}F(\hat{v})^{N-1} - \int_{\underline{v}}^{v} e^{-rt(\tilde{v})}F(\tilde{v})^{N-1}d\tilde{v} \\
&\quad + (v - \hat{v})e^{-rt(\hat{v})}F(\hat{v})^{N-1} \\
&= \int_{\underline{v}}^{\hat{v}} e^{-rt(\tilde{v}))}F(\tilde{v})^{N-1}d\tilde{v} - \int_{\underline{v}}^{v} e^{-rt(\hat{v})}F(\tilde{v})^{N-1}d\tilde{v} \\
&\quad + (v - \hat{v})e^{-rt(\hat{v})}F(\hat{v})^{N-1} \\
&= \int_{v}^{\hat{v}} e^{-rt(\tilde{v})}F(\tilde{v})^{N-1}d\tilde{v} - (\hat{v} - v)e^{-rt(\hat{v})}F(\hat{v})^{N-1} \\
&= \int_{v}^{\hat{v}} \left(e^{-rt(\tilde{v})} - e^{-rt(\hat{v})}\right) F(\tilde{v})^{N-1}d\tilde{v} \leq 0,
\end{aligned}$$

where the second equality follows from (1) for $\hat{v}$, and the last inequality follows because $t(\cdot)$ is assumed to be nonincreasing.

To complete the proof, we need to rule out outside-range deviations. There are two types of such deviations. First, consider the case when v is a discontinuity point of $t(\cdot)$, so that the interval $(t(v), t(v-)]$ is outside the range of $t(\cdot)$. For $t' \in (t(v), t(v-))$, $t^{-\mathbb{1}}(t') = v$, so $p(t') = p(t(v))$. No buyer will want to deviate to t' because doing so will involve buying at the same price, but later. Next, either (i) $t(v-) < t(v - \epsilon)$ for all sufficiently small $\epsilon > 0$, or (ii) $t(v-) = t(v - \epsilon)$ for all sufficiently small $\epsilon > 0$. In case (i), we have $p(t(v-) = p(t(v))$ as before, so a deviation to $t(v-)$ is not profitable. In case (ii), we the deviation to $t(v-)$ yields the same expected utility as the deviation to $t(v - \epsilon)$, but $t(v - \epsilon)$ is within the range, so this deviation is not profitable either. Finally, as $p(t(\underline{v})) = \underline{v}$, no active buyer will prefer to become nonactive, and vise versa, no buyer of type $v < \underline{v}$ will prefer to become active. □

References

Aleksandrov, A., Reshetniak, I., 1989. General theory of irregular curves. Vol. 29. Springer.

Bergemann, D., Said, M., 2010. Dynamic auctions: A survey. Cowles Foundation Discussion Paper No. 1757R.

Board, S., 2008. Durable-goods monopoly with varying demand. Review of Economic Studies 75 (2), 391–413.

Board, S., Skrzypacz, A., 2010. Revenue management with forward-looking buyers. Working paper, Stanford University.

Clarke, F., 2005. Necessary conditions in dynamic optimization. No. 816. Amer Mathematical Society.

Clarke, F., forthcoming. Necessary conditions in optimal control and in the calculus of variations. In: Proceedings of Aviero Conference American Mathematical Society.

Conlisk, J., Gerstner, E., Sobel, J., 1984. Cyclic pricing by a durable goods monopolist. The Quarterly Journal of Economics 99 (3), 489.

Fuchs, W., Skrzypacz, A., 2010. Bargaining with arrival of new traders. American Economic Review 100 (3), 802–836.

Gershkov, A., Moldovanu, B., 2009a. Dynamic revenue maximization with heterogeneous objects: A mechanism design approach. American Economic Journal: Microeconomics 1 (2), 168–198.

Gershkov, A., Moldovanu, B., 2009b. Learning about the future and dynamic efficiency. The American Economic Review 99 (4), 1576–1587.

Guesnerie, R., Laffont, J., 1984. A complete solution to a class of principal-agent problems with an application to the control of a self-managed firm. Journal of Public Economics 25 (3), 329–369.

Harris, M., Raviv, A., 1981. Allocation mechanisms and the design of auctions. Econometrica: Journal of the Econometric Society, 1477–1499.

Hellwig, M., 2008. A maximum principle for control problems with monotonicity constraints. Preprint of the Max Planck Institute for Research on Collective Goods, Bonn.

Hörner, J., Samuelson, L., 2011. Managing strategic buyers. The Journal of Political Economy 119 (3), 379–425.

Judd, K., 1998. Numerical methods in economics. The MIT press.

Kamien, M., Schwartz, N., 1991. Dynamic optimization: the calculus of variations and optimal control in economics and management. Advanced textbooks in economics. North-Holland.

Landsberger, M., Meilijson, I., 1985. Intertemporal price discrimination and sales strategy under incomplete information. The RAND Journal of Economics, 424–430.

Myerson, R., 1981. Optimal auction design. Mathematics of operations research 6 (1), 58.

Noldeke, G., Samuelson, L., 2007. Optimal bunching without optimal control. Journal of Economic Theory 134 (1), 405–420.

Riley, J., Samuelson, W., 1981. Optimal auctions. The American Economic Review 71 (3), 381–392.

Rubinstein, A., 1982. Perfect equilibrium in a bargaining model. Econometrica: Journal of the Econometric Society, 97–109.

Said, M., 2011. Sequential auctions with randomly arriving buyers. Games and Economic Behavior.

Seierstad, A., Sydsæter, K., 1987. Optimal control theory with economic applications. Advanced textbooks in economics. North-Holland.

Skreta, V., 2006. Sequentially optimal mechanisms. Review of Economic Studies 73 (4), 1085–1111.

Skreta, V., 2010. Optimal auction design under non-commitment. Working paper, New York University.

Stein, E., Shakarchi, R., 2005. Real analysis. Princeton University Press.

Toikka, J., 2011. Ironing without control. Journal of Economic Theory 146 (6), 2510 – 2526.

Vind, K., 1967. Control systems with jumps in the state variables. Econometrica: Journal of the Econometric Society, 273–277.

Wang, R., 2001. Optimal pricing strategy for durable-goods monopoly. Journal of Economic Dynamics and Control 25 (5), 789–804.