Team selection problem

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Abstract

We study the optimal selection of a team of individuals whose output is influenced by the presence of others on the team. We assume the characteristics of available agents are known to the principal and the provision of incentives is not an issue.

We restrict attention to two special cases. In case 1, agents produce individually but also contribute to the output of other agents. We provide an algorithm for selecting an optimal set. In case 2, the hired agents divide themselves into smaller teams in which they produce output, just as researchers divide themselves into groups of coauthors. Motivated by this application, we develop a model of the way in which agents are divided into smaller teams. We observe that the principal may increase its payoff by not hiring some agents whose output exceeds the cost of hiring them. For the cases in which this may not happen, we provide an algorithm for selecting an optimal set.

We also provide some comparative statics results. For example, we argue that the composition of the optimal team is more sensitive to the changes in the cost of hiring agents who mainly contribute to the output of others than to the changes in the cost of hiring agents who mainly contribute individually.

1 Introduction

This paper is about the optimal formation of a team of individuals whose individual output is influenced by the presence of others on the team. In our model, different agents have different abilities (or characteristics) and costs, and the task of the principal is to select an optimal set of agents. We are not concerned here with the other important task of the principal which is the provision of incentives to agents in situations when the effort exerted by one agent affects the output of the entire team (see for example, Holmström (1982)).

An example of the kind of organization our model would apply to is one where the provision of tailored incentives may be difficult or unnecessary. For example, it is difficult for a university to substantially

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influence the topics of research conducted by its faculty, or the composition of teams in which its faculty conducts research. On the other hand, employees who are working together typically advise, or discuss their problems with, one another even in the absence of external incentive schemes. In this setting, when making the hiring decisions, universities often take into account potential interactions between researchers. Large consulting firms aim at the "right" composition of experienced and younger workers.

We assume the characteristics of available agents are known to the principal, and study the problem of selecting (or hiring) an optimal set of agents. We restrict attention to two special cases of the problem. In case 1, agents produce individually but also contribute to the output of other agents. That is, each agent is described by two characteristics: individual productivity and their ability to help others. Although we are primarily interested in the optimal selection of agents, case 1 has an immediate, but different interpretation in terms of selecting an optimal portfolio from a discrete set of available assets.

In case 1, we provide a polynomial-time algorithm, which we call the *financing algorithm*, for selecting an optimal set.[1] This algorithm is a version of the Ford-Fulkerson (1956) algorithm adapted to solving project selection problems.

In case 2, the hired agents divide themselves into smaller teams in which they produce output, just as researchers divide themselves into groups of coauthors. Motivated by the application to doing joint research, we develop a model of the way in which agents are divided into smaller teams. We assume that the payoff of a single agent is an equal share of the total output of his or her team. In a *stable partition*, a team of agents with the highest payoff (per agent) forms first. Then, a team from the remaining agents is formed according to the highest payoff (per agent) principle. And so on. We show that a stable partition exists, and is generically unique. We use the concept of stable partition to study case 2.

In case 2, we observe that the principal may increase its payoff by not hiring some agents (or teams of agents) whose output exceeds the cost of hiring them. The reason is that they may divert some other agents from working in other teams to working instead with them, and the loss in terms of the output of these teams may offset the benefit of hiring the agents that divert others. For the cases in which the principal cannot increase its payoff by excluding productive agents, we provide a polynomial-time algorithm for selecting an optimal set. This procedure is a modification of the steepest ascent algorithm.

We are also interested in comparative statics. In particular, we argue that the composition of the optimal team is more sensitive to the changes in the cost of hiring agents who mainly contribute to the output of others than to the changes in the cost of hiring agents who mainly contribute individually, and that an increase in the importance of working in larger teams results in some kind of crowding out effect, according to which some members of the optimal set who were originally working in smaller teams no longer belong to the optimal set.

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[1] The objective of looking for a polynomial-time algorithm should be understood as a rigorous formulation of the problem of describing an optimal set, rather than the suggestion of approaching the problem numerically.

1.1 Related Literature

The most obvious connection between our paper and prior work is team theory as developed by Marschak and Radner (1972). That paper introduces the problem of several agents who must maximize a common objective but have different information when they choose their actions. That problem is used by Prat (2002) to understand the degree of variety in the background of people hired to form a team. In Prat's model the private information of the agent determines the actions she takes and the subsequent output. In our setting, agents do not take actions. Their presence on the team affects total output in a prescribed way as a function of their observable characteristics.

Kremer (1993) (and follow on papers) consider a specific technology that allows the firm to convert agents characteristics into an output function. In his setting, workers can only complement each other. In particular, no workers can generate any output on their own. Our model allows for both possibilities as well as the possibility that a worker can have a negative effect on the productivity of others. Kremer embeds these firms into a perfectly competitive market and draws conclusion about how workers will sort among firms and how wages will behave. We do not consider such questions. For us, wages are fixed and we inquire into the optimal composition of teams. This question is more demanding in our setting than in Kremer's model.

Finally, our case 2 is the same as the model of partnerships of Farrell and Scotchmer (1988). Agents are not explicitly endowed with preferences over other agents but with preferences over output. A production function maps coalitions to the output they can generate and the output is equally split between the agents in the coalition.[2] Farrel and Scotchmer (1988) identify a number of examples where such equal sharing of the outputs is the norm. That paper is concerned with inefficiencies that an equal sharing rule will generate. Our focus is different. Knowing that the gains are to be split equally between the members of the group, and that agents will divide themselves on this basis, how should the principal select the agents. Thus, we are concerned with the selection problem rather than the problem of whether the agents efficiently divide among themselves.

2 Model

Suppose a principal wants to assemble a team of agents. She considers n potential members. Let $N = \{1, ..., n\}$ be the grand set of agents. Suppose it costs c_i to hire agent i. When the principal hires a set of agents S, the principal's payoff, which we will also call the output of S, is $g(S)$. The principal's problem is

$$\max_{S \subset N} \left\{ g(S) - \sum_{i \in S} c_i \right\}. \tag{1}$$

[2] In this sense, it can be interpreted as a contribution to the literature on hedonic coalition formation (see for example Bogomolnaia and Jackson (2002)).

It is assumed that each agent selected by the principal accepts the principal's offer of employment.

The form of function g may be different in different applications. In this paper, we consider only two special cases.

3 Advising and mentoring

Suppose, first, each agent i may produce on his own, output $x_i \geq 0$. However, each agent can increase any other member's output by share $\alpha_i \geq 0$. Thus, the output produced by set S is given by

$$g(S) = \sum_{i \in S} x_i + \sum_{i,j \in S,\ j \neq i} \alpha_i x_j. \tag{2}$$

One interpretation is of a team of mechanics working in a garage, in which case x_i may be interpreted as the expected number of repairs per unit of time performed by mechanic i while working on his own. For every single repair, there is a chance that i will not immediately know what to do, which delays the repair. Then, i may ask around for help, and α_j may be interpreted as the chance that mechanic j will be able to help.[3]

Another interpretation is that of researchers helping one another in conducting their research, e.g., by mentoring or discussing their research projects. Of course, formula (2) is in both cases only a simplistic way of capturing the impact of agents on the output of other agents.

3.1 Examples

We begin with two examples which demonstrate that the optimal team may have the form of an advisor or mentor, who may not be particularly productive individually, but who contributes substantially to the output of others, and a number of most productive agents coalesced around her. The optimal team may also have the form of a star, who is individually productive but may not contribute much to the output of others, and a number of agents hired mainly to enhance her output.

Example 1 *Suppose the set N consists of one agent with $\alpha_i = 5$ and $x_i = 0$ and a certain number of agents, say 10, with $\alpha_i = 1/2^n$ and $x_i = 0$, and a certain number of agents with $\alpha_i = 0$ and $x_i = 1/2^n$, $n = 1, ..., 10$. Suppose $c_i = 1$ for every agent. Then, the optimal set will certainly not include any agent with characteristics $\alpha_i = 1/2^n$ and $x_i = 0$. Indeed, their contribution to the output of any set of other agents falls below the cost of hiring them, since*

$$\sum_{n=1}^{10} \frac{1}{2^n} < 1.$$

[3] It is not our intention to model any concrete protocol for eliciting advice. Our simplistic model postulates that each mechanic is characterized by two parameters: x and α, which may not be perfectly correlated.

It will include the agent with characteristics $\alpha_i = 5$ and $x_i = 0$, the agents with characteristics $\alpha_i = 0$ and $x_i = 1/2^n$ for $n = 1, 2$, since

$$5 \cdot \left(\frac{1}{2} + \frac{1}{4}\right) > 3,$$

but no agent with characteristics $\alpha_i = 0$ and $x_i = 1/2^n$ for $n > 2$, since

$$\frac{1}{8} + 5 \cdot \left(\frac{1}{8}\right) < 1.$$

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Example 2 *Suppose the set N consists of one agent with $x_i = 5$ and $\alpha_i = 0$, agents with $\alpha_i = 1/2^n$ and $x_i = 0$ and agents with $\alpha_i = 0$ and $x_i = 1/2^n$, both for $n = 1, 2, ..., 10$. Suppose $c_i = 1$ for every agent. Then, the optimal set will certainly include the agent with characteristics $x_i = 5$ and $\alpha_i = 0$, because this agent produces herself more than the cost of hiring her. It will also include the agents with characteristics $\alpha_i = 1/2$ and $x_i = 0$ and $\alpha_i = 1/4$ and $x_i = 0$, since they contribute to the output of agent with $x_i = 5$ and $\alpha_i = 0$ more than the cost of hiring them.*

It will not include the agents with characteristics $x_i = 1/2^n$ and $\alpha_i = 0$ for any n or the agents with characteristics $x_i = 0$ and $\alpha_i = 1/2^n$ for $n > 2$. The latter agents would contribute to the output of any set of other agents less than the cost of hiring them, since

$$\frac{1}{8} \cdot \left(5 + \sum_{n=1}^{10} \frac{1}{2^n}\right) < 1.$$

In turn, the total contribution of the former agents is at most

$$\frac{1}{2} + \frac{1}{2} \cdot \left(\frac{1}{2} + \frac{1}{4}\right),$$

which falls below the cost of hiring them.

3.2 Financing Algorithm for Optimal Team Selection

The problem of selecting a set of agents which maximizes the output function given by (2) is a special case of the well-known project selection problem (see Rhys (1970)). The existing literature contains several algorithms for solving this problem (see Hochbaum (2004)). Each of them can be adapted to our problem.

Our algorithm for selecting an optimal team is a version of the Ford-Fulkerson (1956) algorithm. We will call it the *financing algorithm*. All agents i such that $x_i - c_i \geq 0$ are included into the optimal team. Each agent such that $x_i - c_i < 0$ is assigned a number $n_i = -(x_i - c_i)$, which will be called agent i's *need of financing*. In turn, each pair ij will be assigned a number $f_{ij} = \alpha_i x_j + \alpha_j x_i$, and will be called a *source of financing*. Source ij can only be used to cover the needs of i and j. It is important to note that we will think of pair ij as the set $\{i, j\}$, that is, the ordering will be inessential, or we will assume that $ij = ji$.

In the first step of the algorithm, we check for a "financing opportunity", i.e., whether there are i and j with $n_i > 0$ and $f_{ij} > 0$. If we find such i and j, we "transfer money" from ij to i up to the maximum possible amount, which is $\min\{n_i, f_{ij}\}$. That is, if $f_{ij} \leq n_i$, then f_{ij} reduces to 0, and n_i becomes $n_i - f_{ij}$; and if $f_{ij} > n_i$, then f_{ij} becomes $f_{ij} - n_i$, and n_i reduces to 0.

In every next step, we first check for a financing opportunity, exactly as in the first step. If there is no financing opportunity, then we check for what we will call an "opportunity of refinancing". That is, we check whether there exists i and j, and a sequence $i, k_0, k_1, ..., , k_m, j$ such that $f_{ik_0} > 0$, $n_j > 0$, $n_i = 0$ and $n_{k_l} = 0$ for $l = 1, ..., m$, $f_{k_m j} = 0$, $f_{ik_1} = 0$ and $f_{k_l k_{l+1}} = 0$ for $l = 1, ..., m-1$; moreover, in the previous steps, agent i was transferred money from source ik_1, agent k_l was transferred money from source $k_l k_{l+1}$ for $l = 1, ..., m-1$, and agent k_m was transferred money from source $k_m j$.

If we find such an opportunity of refinancing, then we transfer money from i back to ik_1, replacing it with money transferred to i from ik_0; we transfer money from k_l back to $k_l k_{l+1}$ replacing it with money transferred to k_l from $k_{l-1}k_l$ for $l = 1, ..., m-1$, and we transfer money from k_m back to $k_m j$ replacing it with money transferred from $k_{m-1}k_m$. Finally, we transfer money from $k_m j$ to agent j. The transfers are made up to the maximum possible amount. We also reduce the amounts transferred to agent i from source ik_1, agent k_l from source $k_l k_{l+1}$ for $m = 1, ..., l-1$, and agent k_m from source $k_m j$ as some money is now, in all these cases, transferred back.

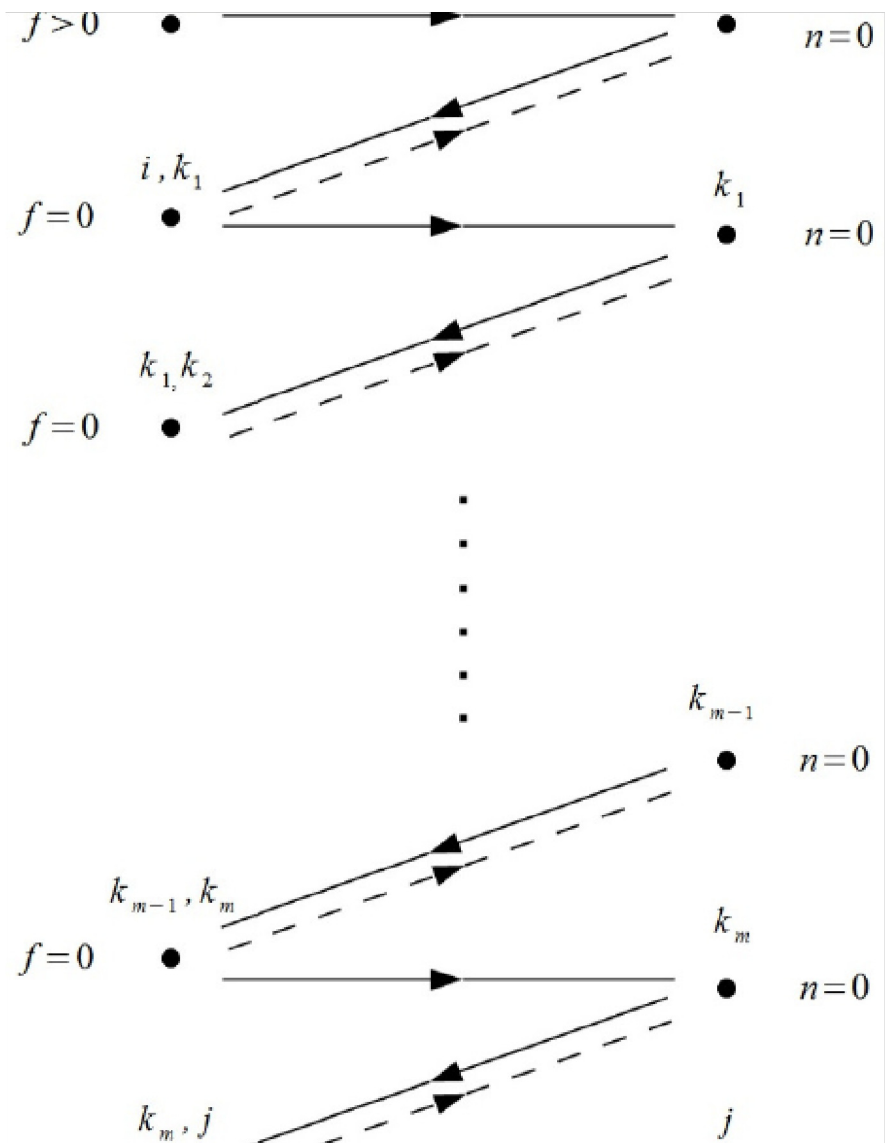
The refinancing algorithm is illustrated in Figure 1.

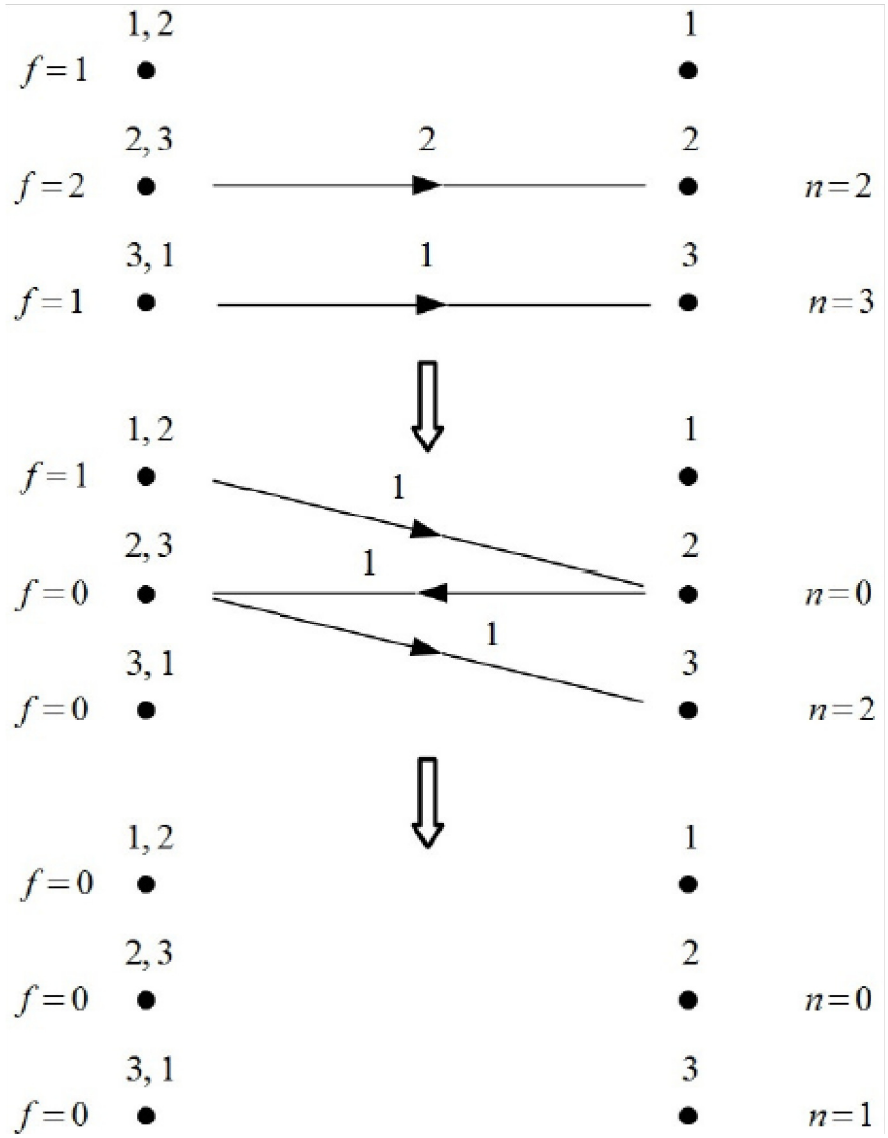
We continue this procedure until financing and refinancing possibilities no longer exist. Then, we recursively remove agents from the grand set N. The set of agents who remain will turn out to maximize (2).

First, remove any agent i with $n_i > 0$. And for any other agent k, take back the amount of financing transferred to k from source ik. We continue recursion by removing any agent i who is not fully financed, and taking back the amount of financing of any other agent k from source ik. Stop when no other agent can be removed. Denote the team selected by our algorithm by S^*.

We illustrate the algorithm by the example depicted in Figure 2. We take $N = \{1, 2, 3\}$, $x_1 - c_1 \geq 0$, $n_2 = 2$, $n_3 = 3$, $f_{12} = 1$ and $f_{23} = 2$, and $f_{13} = 1$. Say, agent 2 is first transferred 2 from source $\{2, 3\}$. This leads to a situation that there is no other financing possibility. However, there is a possibility of refinancing. Namely, agent 2 can be transferred 1 from source $\{1, 2\}$, transferring 1 back to source $\{2, 3\}$, which is transferred to agent 3. This exhausts financing possibilities and possibilities of refinancing. Agent 2 is fully financed, but agent 3 is not. Thus, we remove agent 3. However, we must now remove agent 2 as well, because we take back the amount of 1 transferred from source $\{2, 3\}$. The optimal team consists just of agent 1.

This financing algorithm as described will not run in polynomial time. One needs to define the search order for a refinancing opportunity by adapting the *breadth-first search algorithm*. With this modification our algorithm would be guaranteed to run in polynomial time. This version of the Ford-Fulkerson algorithm was proposed independently by Dinic (1970), and Edmonds and Karp (1972). Because we are interested in a method of selecting an optimal set of agents, rather than its computational properties, we omit the discussion of the breadth-first search algorithm.





Proposition 1 *The financing algorithm selects a set S which maximizes the output function given by (2).*

Proof. First, notice that

$$\sum_{i\in S^*} x_i + \sum_{i,j\in S^*,\ j\neq i} \alpha_i x_j \geq 0,$$

because the financing needs of all agents from S^* are covered from sources ij, $i,j \in S^*$. Thus, the optimal team must contain the team selected by the financing algorithm.

Next, observe that when no financing opportunity and no opportunity of refinancing exist, then $f_{ik} = 0$ for all pairs i and k, except possibly pairs $i,k \in S^*$. Indeed, suppose that $f_{ik} > 0$ for a pair i, k, such that, say, $i \notin S^*$. Consider the following two cases:

(i) If $n_i > 0$, then there would exist a financing opportunity, since agent i could be financed from source ik;

(ii) Suppose that $n_i = 0$, and consider the set of all j with the following property:

(*) there exists a sequence $i, k_0, k_1, ..., , k_m, j$, where $k_0 = k$, such that agent i received a transfer from source ik_1, agent k_l received a transfer from source $k_l k_{l+1}$ for $l = 1, ..., m-1$, and agent k_m received a transfer from source $k_m j$.

Since there is no opportunity of refinancing, $n_j = 0$ for every j with property (*). By definition, every j with property (*) received transfers only from sources jk such that k also has property (*). Thus, since $n_j = 0$ for every j with (*), the set of all j with (*) should be included into S^*. This, however, contradicts the assumption that $i \notin S^*$.

Consider any set S that contains S^*. Denote by S' the set obtained from S by including all agents who were removed from the grand set later than the first removed agent from S; moreover, assume that every agent i included to S' has cost c'_i, possibly lower than c_i, such that his or her need of financing (at the time when the first agent from S is removed) is equal to 0. Observe that $U(S) \leq U(S')$. This follows from the fact that all agents $i \in S' - S$ (when their costs are c'_i) are fully financed from sources ij such that $i, j \in S'$.

Finally, notice that $U(S') = U(S^*)$. This follows because $f_{ij} = 0$ for all pairs i and j except (possibly) pairs $i, j \in S^*$. ■

3.3 Comparative statics

We conclude this section with two examples of comparative statics analysis. Call the number

$$x_i + \sum_{j \in S,\ j \neq i} \alpha_i x_j$$

the *total contribution* of agent i to the output of team S. We will also call the first component x_i agent i's *individual contribution*, and the second component agent i's *contribution to the output of others*. We argue that

Claim 1 *The composition of the optimal team is more sensitive to changes in the cost of hiring agents who mainly contribute to the output of others than to changes in the cost of hiring agents who mainly contribute individually.*

This asymmetry follows from the fact that if x_i is much larger than c_i for an agent i, then small changes in c_i will not affect any hiring decisions; the optimal team is unaltered. In terms of our financing algorithm, we have that $n_i < 0$ both before and after the chance in cost. Only when x_i is not much larger than c_i, or changes in c_i are large, the hiring decisions may be affected by the change in cost.

On the other hand, even when the contribution of agent i to the production of other members of the optimal team is much larger than c_i, small changes in c_i may affect hiring decisions, including the possibility that agent i will no longer be hired. Indeed, when c_i rises, agent i needs more financing in the language of our algorithm, which takes financing from sources ij away from agents j. For some of them, the cost of hiring c_j exceeds x_j, and one may not be able to take financing away from them, without removing them from the optimal team. This, in turn, reduces the contribution of agent i to the production of other members of the team, and may result in even the removal of agent i.

The argument is illustrated in the following example.

Example 3 *Let $N = \{1, 2, 3\}$. Let*

$$a_1 = 1 \text{ and } a_2 = a_3 = 0,$$

and let

$$x_1 = 0 \text{ and } x_2 = x_3 = 1;$$

finally, suppose that $c_1 = 0.5$ and $c_2 = c_3 = 1.5$.

Then, the optimal set consists of all three agents. The contribution of agent 1 to the production of others is 2 which exceed the cost of hiring agent 1 by 1.5. If the cost of hiring agent 1 increases by 1 to 1.5, it might seem that the total contribution of agent 1 still exceeds the cost of hiring her. However, it now becomes optimal not to hire any agent.

Claim 1 suggests that the employment in the professions in which the production is more individual should be less sensitive to the changes in hiring costs, such as salaries, than the employment in the professions in which agents' production depends more substantially on the advising or mentoring of other agents.

Suppose now that function g has a slightly different form; namely,

$$g(S) = (1 - \gamma) \sum_{i \in S} x_i + \gamma \sum_{i,j \in S,\ j \neq i} \alpha_i x_j \tag{3}$$

is a weighted average of individual contributions and contributions to the output of others. This representation is equivalent to the output function given by (2), since one can rescale x's, α's, and c's. However, representation (3) will be convenient for the following exercise, because γ can be interpreted as the importance of contributing to other agents' output compared to the individual contribution.

Claim 2. *When the importance of contributing to other agents' output γ increases, the optimal team tends to become composed of agents with more extreme characteristics, that is, agents with high x's and agents with high α's, rather than agents with intermediate values of both characteristics.*

The reason is that if set T is obtained from set S by replacing two agents with characteristics (x_1, α_1) and (x_2, α_2) such that $x_1 > x_2$ and $\alpha_1 < \alpha_2$ or $x_1 < x_2$ and $\alpha_1 > \alpha_2$ with other two agents with characteristics $((x_1 + x_2)/2, (\alpha_1 + \alpha_2)/2)$, then $g(T) < g(S)$. Moreover, the replacement does not change the sum of individual contributions. It does not change the contributions of agents 1 and 2 to the output of other agents, or the contribution of other agents to the output of the two agents. But the sum of contributions of agent 1 to the output of agent 2, and agent 2 to the output of agent 1 decreases, because

$$2\frac{(x_1+x_2)}{2}\frac{(\alpha_1+\alpha_2)}{2} < x_1\alpha_2 + x_2\alpha_1.$$

Thus, when the importance of contributing to other agents' output increases, the principal raises its payoff by replacing pairs of agents with characteristics $((x_1+x_2)/2, (\alpha_1+\alpha_2)/2)$ and $((x_1+x_2)/2, (\alpha_1+\alpha_2)/2)$ with pairs of agents with characteristics (x_1, α_1) and (x_2, α_2), or close to thereof, whenever it is possible, and whenever the difference in the costs of the two pairs is close.

We believe that the implication of our model contained in Claim 1 is supported by an intuitive robust argument. In contrast, Claim 2 is an artefact of a specific functional form of output. However, we find both implications to be interesting hypotheses, no matter how appealing is the argument in their favor provided by the present setting.

3.4 Investment problem

The algorithm defined in Section 3.1 can be used not only for selecting an optimal team of workers, but also for solving some other problems with the same structure. For example, it can be used for choosing an optimal portfolio. Suppose an investor chooses among n assets. The investor has a mean-variance utility function, as it is often assumed in finance, which is a linear function of the mean and variance of a portfolio. The mean coefficient in the utility function is positive and the variance coefficient is negative. That is, the utility U of a portfolio with mean return μ and variance σ^2 is $a\mu - b\sigma^2$.

If c_i denotes the cost of acquiring asset i, then the investor's objective function has the form

$$U(S) = a\left(\sum_{i\in S}\mu_i - \sum_{i\in S}c_i\right) - b\sum_{i,j\in S}\sigma_{ij},$$

where $(\sigma_{ij})_{i,j=1}^n$ denotes the covariance matrix of the assets. This problem was introduced by Reiter (1963), but he did not provide a polynomial-time algorithm for finding an optimal portfolio.

Our result imply that this problem is solvable in polynomial time in the special cases in which all covariances σ_{ij}, $i \neq j$, are negative.[4]

3.5 Negative α's

One can imagine cases in which some coefficients α_i are negative. For examples, workers may duplicate one another's work, or disturb one another. We believe that applications of our problem with some negative α's are less natural. However, a solution to such a problem would be very useful for choosing an optimal portfolio. Unfortunately, the problem is in general NP-hard.

[4] This can be relaxed slightly to the case when the matrix of σ_{ij} is sign balanced. See Hansen and Simeone (1986).

We show it by reducing the team selection problem to the following partition problem, which is well-known to be NP-hard. In the partition problem, we are given a set $\{b_1, b_2, \ldots, b_m\}$ of positive integers and we would like to know if there is a subset S of this set such that

$$\sum_{i \in S} b_i = \sum_{i \notin S} b_i.$$

Given an instance of the partition problem, we construct an instance of the team selection problem whose solution will resolve the partition problem:

1. Set $x_i = 0$ for all $i = 1, \ldots, m$.
2. Set $x_i = b_i$ for all $i = m+1, \ldots, 2m$.
3. Set $\alpha_i = b_i$ for all $i = 1, \ldots, m$.
4. Set $\alpha_i = -b_i$ for all $i = m+1, \ldots, 2m$.
5. Finally, set $c_i = x_i^2$ for all $i = 1, \ldots, 2m$.

Let

$$B = \sum_{i=1}^{m} b_i.$$

Since, the principal wants to hire agents $1, ..., m$, no matter what the set of other hired agents, our problem reduces to finding a set $S \subseteq \{m+1, \ldots, 2m\}$ that maximizes

$$\sum_{i \in S} b_i (B - \sum_{i \in S} b_i).$$

Observe that if the answer to the partition problem is 'YES', then the optimal objective function value for this particular instance of team selection problem will be $B^2/4$. If the answer to the partition problem is 'NO', then the optimal objective function value will be strictly less than $B^2/4$.

4 Coauthoring

Suppose now that when a set of agents is hired by a principal, they form smaller teams producing jointly. When the agents in team T work together, they produce output $h(T)$, which is divided equally among them. For example, one can think of a group of scientists doing research jointly, and writing joint papers. In practice, the contribution of all authors is often viewed as equal.[5]

It is natural to assume that:

Assumption 1. Function h is *superadditive*, i.e.,

$$h(S_1 \cup S_2) \geq h(S_1) + h(S_2).$$

[5] It is certainly possible that in practice when a group of scientists write two joint papers, then the benefit of each scientist exceeds that of one paper written by a single author. However, we assume away this kind of economies of scale.

for any disjoint sets S_1 and S_2.

The reason for making this assumption is that if agents $S_1 \cup S_2$ must work as a team and divide the output equally, they want that output to be as high as possible. One possibility is to form two teams S_1 and S_2, which work independently, and which produce output $h(S_1) + h(S_2)$, and then divide this output among agents $S_1 \cup S_2$.

The output $g(S)$ of set a S is equal to the sum of outputs $h(T)$ of some partition of S. The following section describes the manner in which this partition is determined.

4.1 Stable partitions

Suppose the principal cannot affect the partition into the smaller teams in which agents produce jointly. This process is decentralized, which seems to resemble the common practice in the case of scientists doing joint research. That is, we study a two-stage game. In period 1, a principal hires a set of agents S; and in period 2, the agents from set S form teams in which they produce. The payoff of the principal is equal to the sum of outputs over all teams, and the payoff of each agent is equal to the output per member of the team this agent belongs to. In order to solve for the equilibrium of the subgame that begins in period 2, we apply a stability concept.

Definition 1 *A partition $\Pi = \{S_1, ...S_m\}$ of set S is stable if*

$$\forall_{T \subset S} \ \exists_{\substack{k=1,\dots,m \\ S_k \cap T \neq \varnothing}} \ \frac{h(T)}{|T|} \leq \frac{h(S_k)}{|S_k|}.$$

Stability excludes the possibility that all members of a subset $T \subset S$ benefit by leaving their teams from the original partition and forming their own team. The intuition is that if this was possible, agents from set T would ultimately realize that they are better off working together, and would leave their teams from the original partition.

Proposition 2 *For every set S, there exists a stable partition $\Pi^* = \{S_1^*, ...S_m^*\}$ of S.*

Proof. Such a partition Π^* can be defined by setting

$$S_k^* = \arg\max \left\{ \frac{h(T)}{|T|} : T \subset S - \bigcup_{l<k} S_l^* \right\}.$$

To show that partition Π^* is stable, choose any $T \subset S$. Let k be the largest index l such that $S_l^* \cap T \neq \varnothing$. Then, by the definition of S_k^*,

$$\frac{h(T)}{|T|} \leq \frac{h(S_k^*)}{|S_k^*|}.$$

Thus, the condition defining stability is satisfied. ■

There may exist several stable partitions of some sets S (see Remark 1); moreover, different stable partitions of set S may result in different values of $g(S)$ defined as[6]

$$g(S) = \sum_{i=1}^{m} h(S_i^*). \tag{4}$$

However, except non-generic cases such as one described in Remark 1, stable partitions have the property that the total output of any set S is the same across all stable partitions of S. Moreover, except non-generic cases, there exists a unique stable partition.

The uniqueness is, for example, guaranteed by the following: For any two sets of agents S_1 and S_2,

$$\frac{h(S_1)}{|S_1|} \neq \frac{h(S_2)}{|S_2|}. \tag{5}$$

Proposition 3 *Under condition (5), for every set S, there exists a unique stable partition $\Pi^* = \{S_1^*, ...S_m^*\}$.*

Proof. Consider any other partition $\Pi = \{S_1, ...S_m\}$. Suppose that sets $S_1, ...S_m$ are arranged so that

$$\frac{h(S_k)}{|S_k|} \geq \frac{h(S_{k+1})}{|S_{k+1}|}, \forall_{k=1,...,m-1}. \tag{6}$$

Let k be the smallest index such that $S_k \neq S_k^*$. Then, by the definition of S_k^*,

$$\frac{h(S_k)}{|S_k|} \leq \frac{h(S_k^*)}{|S_k^*|},$$

and by condition (5), the inequality is strict. By (6), the inequality is valid and strict when we replace S_k with S_l for any $l > k$. This implies that the definition of a stable partition is violated by $T = S_k^*$. ■

We will restrict attention to the cases such that for every set S, the value of $g(S)$ is well defined by (4).

Remark 1 *As in many similar cases in the existing literature, there is an alternative definition of stability, which would require that partitions may be upset by new teams T such that all agents forming the new team obtain weakly higher payoffs than the payoff obtained under the original partition, and at least one agent obtains a strictly higher payoff.*

Context will determine which of the two concepts seem more appropriate. However, both concepts encounters difficulties beyond the situations in which condition (5) is satisfied, in which case the two concepts are equivalent. As we have already pointed out, our original definition allows for the multiplicity of stable partitions. And under the alternative definition, stable partitions may not exist. For example, consider the case of three agents. Suppose that each pair of agents can jointly produce an output of 1*. The output is also* 1 *for the grand team of all three agents, and is* 0 *for each single agent. Then, all two-set partitions are stable under the original definition, and no partition is stable under the alternative definition.*

[6]This, in particular, means that the principal maximizes the total output (or payoff) of its agents. This seems to be a reasonable assumption, for example, in the application to joint research, where the principal is a scientific institution.

In other cases, one can equivalently assume that the output of the agents is divided between them and the principal in fixed proportions.

4.2 Comparison with the first-best outcome

We define the *first-best outcome* as one that solves

$$\max_{S \subset N} \left\{ h(S) - \sum_{i \in S} c_i \right\}; \tag{7}$$

this outcome would yield the highest possible payoff that the principal could attain if the principal was making not only the hiring decisions, but also the decisions regarding the partition into smaller teams that would be working jointly.

There are two kinds of departure from the first-best outcome. First, stable partitions can be inefficient in the sense that set S could produce a higher total output, if it were partitioned differently. Formally, $h(S) \geq g(S)$ for every set S, and it may happen that $h(S) > g(S)$ for some sets S. (Recall that $h(S)$ is the highest output that set S can produce, and $g(S)$ is the output produced in a stable partition of S.)

Second, given that stable partitions could be inefficient, the principal may wish to hire a different set of agents, compared to the set that she would hire if she could choose the partition into smaller teams that would be working jointly. Thus, if we denote by S^{FB} the set that solves problem (7), and by S^* the set that solves problem (1), we can represent the two kinds of departure from the first-best outcome as

$$h(S^{FB}) - g(S^*) = [h(S^{FB}) - h(S^*)] + [h(S^*) - g(S^*)]. \tag{8}$$

The left-hand side of this equation is equal to the total departure from the first-best outcome. The first term in the right-hand side represents the inefficiency caused by hiring a suboptimal set of agents, and the second term represents the inefficiency caused by agents forming suboptimal teams. The following example illustrates the two effects.

Example 4 *(i) Let* $N = \{1, 2, 3, 4, 5\}$ *and* $h(1) = 1.5$, $h(\{2, 3, 4, 5\}) = 5$, $h(\{1, 2\}) = 3.5$, $h(\{3, 4, 5\}) = 2$. *Extend this function to the set of all subsets of* N *by setting* $h(\{i\}) = 0$ *for all* $i \neq 1$, *and for all other sets* S *define* $h(S)$ *to be the maximum of*

$$\sum_{i=1}^{k} h(S_i)$$

over all partitions $\{S_1, ..., S_k\}$ *of* S *that consists of sets* S_i *for which* $h(S_i)$ *has already been defined. It is easy to check that the function* h *as defined is superadditive.*[7]

Then $\{1, 2\}$, $\{3, 4, 5\}$ *is the stable partition of* N. *This partition yields output* $g(\{1, 2, 3, 4, 5\}) = h(\{1, 2\}) + h(\{3, 4, 5\}) = 5.5$. *The principal would prefer partition* $\{1\}$, $\{2, 3, 4, 5\}$, *which would yield*

[7] Notice that this function h violates condition (5). In particular, some sets S have several stable partitions.

However, the stable partitions of any set S have the property that the output $g(S)$ defined by (4) is the same across all stable partitions.

In addition, a straightforward modification of the example yields an example with similar properties, which satisfies condition (5).

output $h(\{1\}) + h(\{2,3,4,5\}) = 6.5$. *If the cost of hiring any agent is zero, the principal cannot obtain a payoff higher than* 5.5. *The best alternative to hiring all agents is to exclude agent* 1*, but this yields only output* $g(\{2,3,4,5\}) = h(\{2,3,4,5\}) = 5$.

Here, the first component of the left-hand side of decomposition (8) is equal to 0*, since* $S^{FB} = S^* = N$*, and the second component is equal to* 1.

(ii) Let now $h(1) = 1$*,* $h(\{2,3,4,5\}) = 5$*,* $h(\{1,2\}) = 3$*,* $h(\{3,4,5\}) = 1$*. (This is the only difference compared to part (i).) The principal prefers to hire the set* $\{2,3,4,5\}$ *which yields output* $g(\{2,3,4,5\}) = 5$ *instead of the set of all agents. Indeed, since* $\{1,2\}$*,* $\{3,4,5\}$ *is the stable partition of* N*,* $g(\{1,2,3,4,5\}) = h(\{1,2\}) + h(\{3,4,5\}) = 4$.

Here, we have that $S^{FB} = N$*,* $S^* = \{2,3,4,5\}$*, the first component of the left-hand side of decomposition (8) is equal to* 1*, and the second component is equal to* 0.

The example illustrates the possibility that a productive agent 1, whose cost of hiring is lower than individual output, may not be hired. The intuition is that some agents (agent 1 in this case) divert other agents (agent 2) from working in other teams (together with agents $3,4,5$), to working with them instead. Because of that, these other teams lose partners who affect their output. This loss may offset the benefit of hiring agents that divert others.

In the case of scientists doing joint research, one may think of, say, an applied theorist who is productive by herself. If this agent, however, joins a research group, she may draw some pure theorists (those who are talented also in doing more applied research) to doing joint research with her. The pure theorists would otherwise be doing research with other pure theorists, who are less talented than them in writing applied theory papers.

4.3 Condition for Efficiency of Stable Partitions

A natural question is under what conditions on h it is the case that a stable partition of any set S coincides with one that is most preferred by the principal. A condition that guarantees no negative externality is caused by internal team formation is that for any pair of disjoint sets S_1 and S_2, function h is either simply additive or "strongly superadditive". More precisely, a superadditive function h satisfies *condition (*)* if for any disjoint sets S_1 and S_2,

$$h(S_1 \cup S_2) > h(S_1) + h(S_2) \;\Rightarrow\; \frac{h(S_1 \cup S_2)}{|S_1| + |S_2|} > \max\left\{\frac{h(S_1)}{|S_1|}, \frac{h(S_2)}{|S_2|}\right\}. \tag{9}$$

Notice that inequality $h(S_1 \cup S_2) > h(S_1) + h(S_2)$ can be expressed as

$$\frac{h(S_1 \cup S_2)}{|S_1| + |S_2|} > \frac{|S_1|}{|S_1| + |S_2|}\frac{h(S_1)}{|S_1|} + \frac{|S_2|}{|S_1| + |S_2|}\frac{h(S_2)}{|S_2|}.$$

Thus, the inequality implies that

$$\frac{h(S_1 \cup S_2)}{|S_1| + |S_2|} > \frac{h(S_i)}{|S_i|}, \tag{10}$$

for one of the two i's, and condition (*) requires that the condition (10) holds for both i's.

Proposition 4 *If condition (*) is satisfied, then for any set S, the stable partition of S maximizes the total output $g(S)$ of set S across all possible partitions.*

Proof. Let $\Pi^* = \{S_1^*, ...S_m^*\}$ be the stable partition of S. We will show by induction that for every $k = m, ..., 1$,

$$g(S_k^* \cup ... \cup S_m^*) = h(S_k^* \cup ... \cup S_m^*) \tag{11}$$

Suppose (11) holds for $k+1$. By condition (*), there are two possibilities: (1) If $h(S_k^* \cup ... \cup S_m^*) = h(S_{k+1}^* \cup ... \cup S_m^*) + h(S_k^*)$, then

$$g(S_k^* \cup ... \cup S_m^*) = g(S_{k+1}^* \cup ... \cup S_m^*) + h(S_k^*) = h(S_{k+1}^* \cup ... \cup S_m^*) + h(S_k^*) = h(S_k^* \cup ... \cup S_m^*),$$

where the first equality follows from the fact that $\{S_k^*, ..., S_m^*\}$ is the stable partition of $S_k^* \cup ... \cup S_m^*$, and $\{S_{k+1}^*, ..., S_m^*\}$ is the stable partition of $S_{k+1}^* \cup ... \cup S_m^*$; and the second equality follows from the induction assumption.

(2) If $h(S_k^* \cup ... \cup S_m^*) > h(S_{k+1}^* \cup ... \cup S_m^*) + h(S_k^*)$. But then, by (*),

$$\frac{h(S_k^* \cup ... \cup S_m^*)}{|S_k^* \cup ... \cup S_m^*|} > \frac{h(S_k^*)}{|S_k^*|}.$$

This last inequality, however, contradicts the definition of S_k^*, according to which $T = S_k^*$ maximizes $h(T)/T$ over all sets $T \subset S_k^* \cup ... \cup S_m^*$.

By taking $k = 1$ in (11), we obtain the result. ■

Remark 2 *Condition (*) is not a necessary condition for Proposition 3 to hold, as demonstrated by the following example. However, it is in a sense close. Inspection of the proof shows that a necessary and sufficient condition requires (9) to be satisfied when one of the sets S_i maximizes $h(T)/|T|$ over all sets $T \subset S_1 \cup S_2$.*

Example 5 *Let $N = \{1, 2, 3, 4\}$, and let*

$$h(\{1,2\}) = h(\{1,2,3\}) = h(\{1,2,4\}) = 1,$$

$$h(\{3,4\}) = 0.25,\ h(\{2,3,4\}) = 1$$

$$h(N) = 1.25$$

and $g(S) = 0$ for any other $S \subset N$.[8]

Intuitively, there are two teams: $\{1, 2\}$ and $\{3, 4\}$ that are productive by themselves; in addition, agent 2 can substantially increase the production of team $\{3, 4\}$. All other teams' output is produced by these three teams if they are included as a subset.

[8] The comment from footnote 3 applies also to this example.

Then, condition (*) is violated, because

$$h(N) > h(\{2,3,4\}) + h(\{1\})$$

and

$$\frac{h(\{2,3,4\})}{|\{2,3,4\}|} = \frac{1}{3} > \frac{5}{16} = \frac{h(N)}{|N|}.$$

However, it is easy to check that for any set $S \subset N$, stable partitions of S coincide with the partitions that would be most preferred by the principal. For example, the stable partition of the grand set N consist of sets $\{1,2\}$ and $\{3,4\}$, not of sets $\{1\}$ and $\{2,3,4\}$.

4.4 Steepest Ascent Algorithm

The results of the previous section suggest that the principal may not hire some productive agents (or some productive teams of agents), because they will have a negative impact on the total output of the hired set. In this section, we show that in the absence of this effect, a simple algorithm selects a set S which maximizes the output function given by (2).

A *group steepest ascent algorithm* procedure is defined by recursion as follows. First, choose a set $S \subset N$ that maximizes $h(S)/|S|$. If

$$h(S) - \sum_{i \in S} c_i > 0, \tag{12}$$

then include S into the set S^*. Otherwise, choose a set S for which the value $h(S)/|S|$ is the second highest. Check if condition (12) is satisfied. Continue in this fashion to choose the set S with the highest value of $h(S)/|S|$ such that condition (12) is satisfied. Denote this set by S_1^*.

Given set $S^* = S_1^* \cup ... \cup S_l^*$, choose the set $S \subset N \setminus S^*$ with the highest value of $h(S)/|S|$ such that condition (12) is satisfied. Denote this set by S_{l+1}^*. Stop when no set $S \subset N \setminus S^*$ satisfies condition (12). Take $S^* = S_1^* \cup ... \cup S_l^*$ at the time of stopping.

The group steepest ascent algorithm may not, in general, be a polynomial-time procedure, because it requires maximization of some objective functions defined on all subsets of some sets. However, its running time is polynomial under the following assumption, which seems to be satisfied in many applications.

Assumption 2. There exists a number k, independent of the number of agents n, such that for any set of agents S,

$$h(S) = h(S_1) + ... + h(S_m),$$

where $\{S_1, ..., S_m\}$ is a partition of S such that $|S_l| \leq k$ for $l = 1, ..., m$.

This assumption says that there exist no gains from working in too large teams (or more precisely, teams of size larger than k).

Definition 2 *The principal does not exclude agents for strategic reasons if any set S which satisfies condition (12) is included into any optimal set.*

Proposition 5 *If the principal does not exclude agents for strategic reasons, then the group steepest ascent algorithm selects a set S^* which maximizes the principal's payoff.*

Proof. Let S be an optimal set. By Definition 2 and the definition of the group steepest ascent algorithm, we have that $S^* \subset S$. Take a smallest optimal set S which contains S^*. Suppose that inclusion is strict.

Let $\Pi = \{S_1, ...S_m\}$ be the stable partition of S. Then, condition (12) is satisfied for all $S = S_1, ...S_m$. If it were not satisfied for a set S_i, then the set $S - S_i$ would be a smaller optimal set; and it would contain S^* by Definition 2.

By this last observation, it is straightforward to show by induction that $S_1, ..., S_m$ must coincide with sets $S_1^*, ..., S_l^*$ (in particular, $m = l$). ■

The group steepest ascent algorithm may not select a set S which maximizes the objective function (2) when the principal may exclude agents for strategic reasons. We conjecture that there is no polynomial-time algorithm in the general case when stable partitions may be inefficient and the principal may exclude agents for strategic reasons. It remains, however, an open question whether there exists a polynomial-time algorithm when stable partitions are efficient but the principal may exclude agents for strategic reasons.

4.5 Comparative statics

We conclude this section and the paper with a comparative statics analysis. Namely, we wish to see the effect on the composition of the optimal set of agents of an increase in the importance of working in teams. One way to model this kind of increase is to multiply $h(S)$ by some factor $\gamma > 1$ for all sets S whose size exceeds a certain critical number k. This analysis leads to the following claim:

Claim 3. *An increase in the importance of working in larger teams results in a crowding out effect, according to which some members of the optimal set who were originally working in smaller teams may no longer belong to the optimal set.*

Obviously, some agents who originally were not members of the optimal set may become hired after the increase. Therefore the increase is "good news" for the agents who were not originally hired, and bad news for some agents who were originally hired, but were working in smaller teams.

The reason is that the increase raises the output per agent of larger teams. Some of the larger teams include the members of originally smaller teams that were a part of the optimal set. They abandon their original teams, and if their original partners are not absorbed by other teams, they become less productive, and may no longer belong to the optimal set. The argument is illustrated in the following example.

Example 6 *Let $N = \{1, 2, 3, 4\}$, and let*

$$h(\{1, 2\}) = 2 = h(S)$$

for any three-agent set S that contains $\{1, 2\}$, and

$$h(\{3, 4\}) = 0.5 = h(S)$$

for any three-agent set S that contains $\{3,4\}$. Let $h(S)=0$ for any other two-agent set S, and let

$$h(N)=2.5 \quad \textit{and} \quad h(\{i\})=0$$

for any $i\in N$.[9] *Suppose that $c_i=c\in(0,0.25)$ for $i=1,2,3,4$. Then, the optimal set consists of all agents, and the stable partition of this set consists of two teams: $\{1,2\}$ and $\{3,4\}$.*

If we multiply $h(S)$ by factor 2 for every set S that contains 3 or 4 elements, then the optimal set will consist only of agents $1,2,3$ or agents $1,2,4$. Indeed, if the principal hired all four agents, they would be working in two teams: $\{1,2,3\}$ and $\{4\}$, or $\{1,2,4\}$ and $\{3\}$. This would yield the same output as that when only agents $1,2,3$ or agents $1,2,4$ were hired.

[9] The comment from footnote 3 applies also to this example.

5 Reference

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