Decentralized Learning in Multi-Issue Two-Party Elections with Limited-Attention Voters

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1 Introduction

In democracies with two major political parties, when an election revolves around a single issue, the task for the voters is straightforward: they simply need to determine which party has the advantage on that issue. Even though individual judgements may be occasionally erroneous, as long as they are more likely to correct than wrong, the superior party is likely to be elected with a high probability.

However, elections are rarely about a single issue. In a multi-issue election, given limited attention spans, voters must decide which issues they want to focus on, and their decisions may influence their votes. For example, in the 2020 US presidential election, a voter's support for the Republican or Democratic candidate might hinge on whether they pay closer attention to news about violent crimes or climate change. This raises the question of whether an inferior party will get elected because voters focus on the wrong issues.

We address this question in a model of two-party, two-issue election. In our model, two parties with different positions on two issues compete for the votes of a finite number voters. The party receiving the most votes is elected, and each voter receives a payoff equal to the sum of the winning party's policy payoffs on the two issues. We assume that the issues differ in terms of importance. Specifically, one issue is more important than the other in that the payoff difference between the better and worse party for that issue is larger than the difference for the other issue. Hence, voters prefer the better party on the more important issue even when it is inferior on the other issue.

Before casting her vote, each voter can choose to acquire a free and independent signal about one of the issues. The signal for the important issue is less informative than the one for the other issue. Intuitively, the important issue is more complex. As a result, it is harder for a voter to learn which party is better on that issue. Voters, thus, face a choice between a weaker signal about a more important issue or a stronger signal about a less important one.

We find that voters' information acquisition decisions are connected strategically through an adverse cross-issue inference effect. Ex ante, the issues are uncorrelated, and each party is equally likely to have a superior policy on each issue. However, conditional the vote being tied, it is likely that voters following different issues are supporting different parties. Hence, when a voter learns that one party is likely to be better on one issue, she will infer that, when her vote is pivotal, the other party is likely to be better on the other issue. The negative inference will lower the value of both signals. But importantly, it diminishes the value of the signal about the less important issue to a greater extent, and the effect is stronger when voters are closer to equally spilt between the two issues.

As a result of this adverse inference effect, there may be multiple equilibria. In one equilibrium, a majority of the voters will focus on the important issue. When the number of the voters becomes large, the equilibrium the better party for the more important issue will be elected with near certainty. In another equilibrium, all voters focus on the less important issue. When the number of voters becomes large, the party with an advantage on the less important issue will be elected almost certainly. Nevertheless, the equilibrium outcome is inefficient as there is a fifty percent chance that the losing party is better on the more important issue. Thus, our results show that in a democracy, voters may sometimes focus on more concrete concerns while ignoring issues that may hold greater long-term importance.

This paper is closely related to the literature on endogenous information in collective decision making, eg. Persico (2004) and Martinelli (2006). However, we focus on *what* the voters learn about rather than *how well* the voters learn.

This paper also contributes to the burgeoning literature on attention allocation. In an auction environment, Bobkova (2022) show that bidders' attention allocation between a common-value component and an idiosyncratic private-value component depends on the auction mechanism. They show that, under second-price auction, it is an equilibrium for bidders to focus on the idiosyncratic component, whereas under first-price auction, it is an equilibrium for bidders to focus on the common-value component. Since only the idiosyncratic component matters as to which allocation is efficient, they provide a rationale for the superioity of second-price auctions. In a recent working paper, Bobkova (2023) consider how voting rule affects each committee member's equilibrium attention allocation between a common-value component and an idiosyncratic component. They show that there is free-riding in learning about the common-value component, and that the free-riding is minimized in equilibrium under simple majority rule.

This paper is closely related to the literature on how rational inattention shapes political behavior. Perego and Yuksel (2022) show that fiercer competition among information providers leads to more specialization in the type of information provided so as to soften the ensuing price competition. Since there is no scope for specialization on common issues that everyone cares the same way about, more competition leads to less information provided on common-value issues and more information provided on private issues on which the weight mixture voters disagree on. They assume that probability the alternative is implemented is proportional to approval rate, and hence voters vote as if they alone can change the outcome. Pivotal reasoning thus plays no role. The driving force of their model is the competitive pressure for information providers. In contrast, pivotal reasoning is the driving force in our model. As voters can only choose between learning about the common-value issue or the divisive issue, it is as if there are only two information providers: one provides exclusive information on a unique issue.

2 Model Setup

There are $2n+1$ risk-neutral voters who have to collectively choose between candidate L and R. Voters have identical preference which depends on the candidates' performances on two issues: A and B. For $k \in \{A, B\}$, denote by $\theta_k \in \{L, R\}$ the candidate who is better on issue k. If candidate $c \in \{L, R\}$ is elected, every voter gets payoff $w\mathbf{1}_{\{\theta_A=c\}} + \mathbf{1}_{\{\theta_B=c\}}$, where $\mathbf{1}_E$ is the indicateor function which is equal to 1 if $\theta \in E$ and 0 otherwise. We assume that $w \geq 1$,

		state $\boldsymbol{\theta}_k$	
		L	R
signal $\mathbf{s}_k$	l	$\frac{1}{2}+\delta_k^L$	$\frac{1}{2}-\delta_k^R$
	r	$\frac{1}{2}-\delta_k^L$	$\frac{1}{2}+\delta_k^R$

Table 1: Information Structure

so issue A is weakly more important and receive a weakly higher weight. For example, issue A can be global warming and issue B can be crime.

No one knows the state of the world $\theta = (\theta_A, \theta_B)$. Everyone holds a common prior that all states are equally likely. Every voter receives an informative signal on exactly one issue of her choice. A voter learning about issue $k \in \{A, B\}$ receives signal $s_k \in \{l, r\}$, which is the voter's private information. Each voter's signal is independently distributed according to the conditional probability distribution $\Pr\{s_k|\theta_k\}$ described in Table 1. Define $\overline{\delta}_k = \frac{\delta_k^R+\delta_k^L}{2}$, which measures the average precision of a signal on issue k. More precisely, a signal on issue k matches the candidate with an advantage on issue k with probability $\frac{1}{2}+\overline{\delta}_k$ ex ante.

All voters make their information choice simultaneously. Then each voter receives a private signal on the issue they chose to learn about. All voters then simultaneously cast their vote, either "L" or "R". The candidate with more votes wins the election.

Denote by σ_i a behavioral strategy of voter i. Then $\sigma_i = (\alpha^i, \pi^i)$ where the *attention strategy* α^i is the probability with which voter i learns about issue A, and the *voting strategy* $\pi^i = (\pi_A^i, \pi_B^i)$ describes the probability $\pi_k^i(s_k)$ of voting for candidate R after receiving signal $s_k \in \{l, r\}$ on issue $k \in \{A, B\}$. We say that voter i uses a *pure* attention strategy if she pays attention only to one issue: $\alpha_i \in \{0, 1\}$. We say that voter i votes *for* her signal on issue k if she votes for candidate L after receiving $s_k = l$ and for candidate R after receiving $s_k = r$, i.e. if $\pi_k^i = (\pi_k^i(l), \pi_k^i(r)) = (1, 0)$. We say that voter i votes *against* her signal on issue k if she votes for candidate R after receiving $s_k = l$ and for candidate L after receiving $s_k = r$, i.e. if $\pi_k^i = (\pi_k^i(l), \pi_k^i(r)) = (0, 1)$. If voter i uses a pure responsive voting strategy on issue k, then she either votes for signal on issue k or against signal on issue k. Denote by $\sigma_i = (\alpha^i, \pi^i)$ the strategy of voter i. Denote by σ the strategy profile of all voters.

The solution concept we use is Perfect Bayesian equilibrium.

Better Candidate		on Issue B (θ_B)	
		L	R
on Issue A (θ_A)	L	$-(w+1)$	$-w+1$
	R	$w-1$	$(w+1)$

Table 2: Payoff from the Alternative under Complete Information

3 Benchmark Models

3.1 Complete Information Benchmark

Suppose for this subsection only that the state (θ_A, θ_B) is common knowledge. Table 2, shows the payoff gain from electing candidate R instead of candidate L given every state profile (θ_A, θ_B). It is straightforward to see that the optimal collective action is to elect the candidate better on the more important issue.

3.2 Single-Voter Decision Problem

We next consider another benchmark where $n = 0$, i.e. the single-person decision problem. We show that the single-decision-maker may optimally pay attention to the less important issue.

Suppose the single decision-maker learns about issue $k \in \{A, B\}$. Conditional on receiving signal $s_k = r$, the ex post probability that candidate R is better on issue k ($\theta_k = R$) is $\frac{\frac{1}{2}\left(\frac{1}{2}+\delta_k^R\right)}{\frac{1}{2}\left(\frac{1}{2}+\delta_k^R\right)+\frac{1}{2}\left(\frac{1}{2}-\delta_k^L\right)} = \frac{\frac{1}{2}+\delta_k^R}{1+\delta_k^R-\delta_k^L} > \frac{1}{2}$, whereas conditional on receiving signal $s_k = l$, the expost probability that candidate L is better on issue k ($\theta_k = L$) is $\frac{\frac{1}{2}\left(\frac{1}{2}+\delta_k^L\right)}{\frac{1}{2}\left(\frac{1}{2}-\delta_k^R\right)+\frac{1}{2}\left(\frac{1}{2}+\delta_k^L\right)} = \frac{\frac{1}{2}+\delta_k^L}{1-\left(\delta_k^R-\delta_k^L\right)} > \frac{1}{2}$. So the signle-decision-maker prefers the candidate for whom she has received a favorable signal about. It is optimal for the single-decision-maker to vote according to signal no matter which issue she pays attention to.

If she pays attention to issue k and then vote according signal on issue k, with probability $\frac{1}{2} + \overline{\delta}_k$, she will choose the candidate better on issue k. Since the two issues are ex ante independent, and the distribution of the signal on issue k is independent of θ_k, her signal on issue k says nothing about θ_{-k}. So, regardless of which signal s_k she receives, voter 0 believes that each candidate has an advantage on issue $-k$ with probability $\frac{1}{2}$. So voter 0's expected payoff from paying attention to issue A is

$$\left(\frac{1}{2}+\overline{\delta}_A\right)w+\frac{1}{2}=\frac{1+w}{2}+w\overline{\delta}_A$$

and that from paying attention to issue B is

$$\frac{1}{2}w+\left(\frac{1}{2}+\overline{\delta}_B\right)=\frac{1+w}{2}+\overline{\delta}_B,$$

where $\frac{1+w}{2}$ is the ex ante expected payoff from any candidate.

So single-decision-maker rationally pays attention to the less important issue B if

$$\overline{\delta}_B > w\overline{\delta}_A, \tag{1}$$

i.e. if and only if information on issue B is sufficiently more precise than that on issue A to compensate for lack of importance.

Proposition 1 *Consider $n = 0$. If $\overline{\delta}_B > w\overline{\delta}_A$. the single voter pays attention to issue B and then vote for signal in the unique equilibrium. If $\overline{\delta}_B < w\overline{\delta}_A$, the single voter pays attention to issue B and then vote for signal in the unique equilibrium. If $\overline{\delta}_B = w\overline{\delta}_A$, it is an equilibrium for the single voter to randomize between issues in any way, and then vote for signal.*

We will assume (1) throughout the proposal to focus on the tradeoff between importance and clarity of an issue.

4 Baseline Model

We first focus on the baseline model where information structure on each issue is symmetric in the identity of the candidate better on that issue. That is, we first present results under the following assumption:

Assumption (Symmetry Across Candidates) $\delta_A^R = \delta_A^L = \delta_A$ and $\delta_B^R = \delta_B^L = \delta_B$.

4.1 Equilibrium Attention Allocation May Be Inefficient: Single-Issue Equilibrium

We show that it is an equilibrium for all voters to focus on the less important issue whenever it is optimal for a single-decision-maker to do so.

Proposition 2 *Suppose it is an equilibrium for a single voter to pay attention to issue B. That is, assume $\delta_B \geq w\delta_A$. Then, there is an equilibrium in which all voters pay attention to the less important issue, issue B. This equilibrium is inefficient for large enough electorate.*

Let's consider voter 0's decision problem when the other $2n$ voters all pay attention to issue B and then vote for signal. Then, when voter 0 is pivotal, n voters must have received signal r on issue B, whereas the other n voters have received signal l on issue B. When candidate θ_B is better on issue B, this pivotal event happens with probability $\left(\Pr\{s_B = r|\theta_B\}\right)^n \left(\Pr\{s_B = l|\theta_B\}\right)^n$. Therefore, conditional on being pivotal, the likelihood ratio that the better candidate on issue B is candidate R over candidate L is then

$$\frac{\left(\Pr\{s_B = r|\theta_B = R\}\right)^n \left(\Pr\{s_B = l|\theta_B = R\}\right)^n}{\left(\Pr\{s_B = r|\theta_B = L\}\right)^n \left(\Pr\{s_B = l|\theta_B = L\}\right)^n} = \frac{\left(\frac{1}{2}+\delta_B\right)^n \left(\frac{1}{2}-\delta_B\right)^n}{\left(\frac{1}{2}-\delta_B\right)^n \left(\frac{1}{2}+\delta_B\right)^n} = 1.$$

Because the information structure on issue B is independent of who is better on issue B, equal number of favorable and unfavorable signals cancel out with each other. Thus, when all others pay attention to issue B and vote for signal, being pivotal reveals no information at all about who is better on issue B. It reveals no information about issue A either, since signals on issue B is independent of who is better on issue A.

Therefore, the decision problem facing voter 0 in an election with $2n$ other voters all paying attention to issue B is exactly the same decision problem facing voter 0 as a single decision maker. Since we assume that $\delta_B \geq w\delta_A$, paying attention to issue B is optimal for a single voter, paying attention to issue B and then vote for signal is thus optimal for voter 0 in an election with $2n$ other voters all paying attention to issue B. So, there is an equilibrium for all voters to pay attention to issue B.

Note that, if the other $2n$ voters all pay attention to issue A and vote for signal, the decision problem facing voter 0 is still the same as that facing voter 0 as a single decision maker because the pivotal event still reveals no information. So it is still optimal for voter 0 to pay attention to issue B.

Therefore, there is NO equilibrium in which all voters pay attention to issue A.

It is, however, not an equilibrium for all voters to pay attention to issue A. This is because pivotal event conveys no additional information under simple majority rule when everyone else pays attention to the same issue, and thus what is optimal as a single decision-maker remains optimal conditional on being pivotal in an election.

This single-issue equilibrium is inefficient when the number of voters is large enough. As the number of voters becomes large, the candidate better on issue B will be elected with probability approaching 1. Because relative performance on the two issues are independent, there is a fifty percent chance that the losing candidate is better on issue A. Note that every voter prefers the candidate better on issue A. On the other hand, if every voter pays attention to the more important issue, then as the number of voters becomes large, the candidate better on issue A will be elected with probability approaching 1.

4.2 Multi-Issue Equilibria

We show that, when there are enough voters, there is another equilibria, in which both issues garner attention from some voters, with the more important issue getting the majority's attention.

Proposition 3 *Suppose $\delta_B > w\delta_A$, so a single decision-maker optimally pays attention to the less important issue B. For a large enough electorate, there is a pure strategy equilibrium in which a majority pay attention to the more important issue A whereas a minority pay attention to the less important issue B, and all voters vote for signal.*

We first solve a voter's decision problem given other voters' strategy profile. Given a voter's interim belief about the state profile conditional on being pivotal, in Section 4.2.1, we derive a voter's best response: her optimal voting strategy and then her optimal attention choice. We show that a voter's best response depends crucially on the degree of negative correlation across issues: the likelihood ratio that no candidate is better on both issues over the event that ONE candidate is better on both issues. Section 4.2.2 gives a numeric example of a multi-issue equilibrium. In Section 4.2.3, we show how this negative correlation depends on the attention allocation among other voters. In Section 4.2.4, we show existence of a multi-issue equilibrium in which the majority pay attention to issue A.

4.2.1 Voter Best Response

Fix a strategy profile σ. Let Piv denote the event that voter 0 is pivotal. Let $q(\theta_A\theta_B|\sigma^{-0})$ denote the interim probability of state (θ_A, θ_B) conditional on voter 0 being pivotal if all voters other than voter 0 follow the strategy profile σ. With some abuse of notation, we write $q(\theta_A\theta_B|\sigma)$ below.

Before we delve into the analysis, let's consider some extreme cases. Suppose the same candidate is better on both issues conditional on pivotal, i.e. $q(RR|\sigma) + q(LL|\sigma) = 1$. Then it is optimal for voter 0 to pay attention to issue B and then vote according to signal, as signal on issue B is more precise, and the candidate better on issue B is better on both issues, and thus the better candidate. On the other extreme, suppose different candidates are better on different issues conditonal on pivotal, i.e. $q(RL|\sigma)+q(LR|\sigma) = 1$. Then it is optimal for voter 0 to pay attention to issue B but to vote *against* signal, as signal on issue B is more precise and the candidate WORSE on issue B is better on issue A and thus better overall.

Define *cross-issue negative correlation ratio* to be

$$\varrho := \frac{q(LR|\sigma) + q(RL|\sigma)}{q(RR|\sigma) + q(LL|\sigma)}.$$

We will show that voter best response can be characterized by this ratio.

Optimal Voting Strategy Conditional on the pivotal event alone, the payoff gain from candidate R over L is

$$V_\emptyset(\sigma) = (w+1)(q(RR|\sigma) - q(LL|\sigma)) + (w-1)(q(RL|\sigma) - q(LR|\sigma)).$$

We first show that, $V_\emptyset(\sigma) = 0$, i.e. conditional on the pivotal event alone, no candidate is better than the other. It is intuitive as both the information structure and the voting rule are symmetric across candidates. It follows from the following observation on symmetry across the identity of the better candidate on each issue.

Lemma 4 *Assume $\delta_k^L = \delta_k^R$ for $k = A, B$. Consider strategy profile σ where $\pi_k^i(r) + \pi_k^i(l) = 1$ for each issue $k \in \{A, B\}$ and each voter $i \neq 0$. Then $q(RR|\sigma) = q(LL|\sigma)$, $q(RL|\sigma) = q(LR|\sigma)$, $\varrho = \frac{q(LR|\sigma)}{q(RR|\sigma)}$ and $V_\emptyset(\sigma) = 0$.*

Suppose voter 0 has paid attention to issue A. Conditional on being pivotal and receiving signal $s_A = r$ on issue A, voter 0's expected payoff gain from candidate R instead of candidate L is $\Pr\{s_A = r|Piv\}^{-1}$ times

$$\begin{aligned} & q(RR|\sigma)\left(\frac{1}{2}+\delta_A\right)(w+1) - q(LL|\sigma)\left(\frac{1}{2}-\delta_A\right)(w+1) \\ & +q(RL|\sigma)\left(\frac{1}{2}+\delta_A\right)(w-1) + q(LR|\sigma)\left(\frac{1}{2}-\delta_A\right)(1-w) \\ = & \frac{1}{2}V_\emptyset(\sigma) + V_A(\sigma) \end{aligned} \tag{2}$$

where

$$V_A(\sigma) := \delta_A\left[(w+1)(q(RR|\sigma)+q(LL|\sigma)) + (w-1)(q(RL|\sigma)+q(LR|\sigma))\right]$$

is the part of payoff gain from candidate R attributed to the informativeness of signal $s_A = r$. Note that $V_A(\sigma)$ is always positive because $w \geq 1$.

Because there are only two signals, r and l, conditional on being pivotal and receiving signal $s_A = l$ on issue A, voter 0's expected payoff gain from candidate R instead of candidate L is equal to $\Pr\{s_A = l|Piv^0\}^{-1}$ times

$$\frac{1}{2}V_\emptyset(\sigma) - V_A(\sigma)$$

Given that $V_\emptyset(\sigma) = 0$ and $V_A(\sigma) > 0$ for all strategy profile σ, it is always optimal to vote according to signal on issue A.

Suppose voter 0 paid attention to issue B instead. Conditional on being pivotal and reciving signal $s_B = r$ on issue B, voter 0's expected payoff gain from candidate R instead of L is $\Pr\{s_B = r|Piv\}^{-1}$ times

$$\begin{aligned} & q(RR|\sigma)\left(\frac{1}{2}+\delta_B\right)(w+1) - q(LL|\sigma)\left(\frac{1}{2}-\delta_B\right)(w+1) \\ & +q(LR|\sigma)\left(\frac{1}{2}+\delta_B\right)(-w+1) + q(RL|\sigma)\left(\frac{1}{2}-\delta_B\right)(w-1) \\ = & \frac{1}{2}V_\emptyset(\sigma) + V_B(\sigma) \end{aligned} \tag{3}$$

where

$$V_B(\sigma) := \delta_B\left[(w+1)(q(RR|\sigma)+q(LL|\sigma)) - (w-1)(q(LR|\sigma)+q(RL|\sigma))\right]$$

the part of payoff gain from candidate R attributed to the informativeness of signal $s_B = r$. Analogously, conditional on being pivotal and receiving signal $s_B = l$ on issue B, voter 0's expected payoff gain from candidate R instead of candidate L is $\Pr\{s_B = l|Piv^0\}^{-1}$ times

$$\frac{1}{2}V_{\emptyset}(\sigma) - V_B(\sigma). \tag{4}$$

If the cross-issue negative correlation ratio is sufficiently high: $\varrho > \frac{w+1}{w-1}$, the candidate better on issue B is sufficiently likely to be worse on issue A and thus worse over all, voter 0 will optimally vote against signal on issue B.

Lemma 5 *Assume $\delta_k^R = \delta_k^L$ for $k = A, B$. Suppose all other voters either vote according to or against their signals after their equilibrium attention choice. Then after paying attention to issue A, voter 0's optimal voting strategy is to vote for signal, while after paying attention to issue B, voter 0's optimal voting strategy is to vote according to signal if $\varrho < \frac{w+1}{w-1}$ and to vote against signal if $\varrho > \frac{w+1}{w-1}$.*

Optimal Attention Strategy Suppose voter 0 is pivotal. If voter 0 randomizes equally between the two candidates, her payoff is $\frac{w+1}{2}$, as the better candidate on each issue wins with probability $\frac{1}{2}$. If voter 0 follows issue A and then optimally vote according to signal, her payoff would be

$$\begin{aligned}\frac{w+1}{2} + V_A(\sigma) &= \frac{w+1}{2} + \delta_A\left[\left(q(RR|\sigma) + q(LL|\sigma)\right)(w+1) + \left(q(RL|\sigma) + q(LR|\sigma)\right)(w-1)\right] \\ &= \frac{w+1}{2} + V_A(\sigma),\end{aligned}$$

as her vote matches the better candidate on issue A with an additional probability of δ_A, and the payoff gain from the better candidate on issue A is $w+1$ in state (R, R) and (L, L) and $w-1$ in state (R, L) and (L, R). If voter 0 follows issue B and then optimally afterwards, her payoff is

$$\begin{aligned}&\frac{w+1}{2} + \left|\delta_B\left[\left(q(RR|\sigma) + q(LL|\sigma)\right)(w+1) - \left(q(RL|\sigma) + q(LR|\sigma)\right)(w-1)\right]\right| \\ =\ &\frac{w+1}{2} + \left|V_B(\sigma)\right|\end{aligned}$$

Voter 0's optimal attention choice also depends crucially on the cross-issue negative correlation ratio ϱ. Define

$$T_2 \ : \ = \frac{w+1}{w-1}\frac{\delta_B+\delta_A}{\delta_B-\delta_A}$$
$$T_1 \ : \ = \frac{w+1}{w-1}\frac{\delta_B-\delta_A}{\delta_B+\delta_A}$$

Lemma 6 *Suppose $\delta_k^R = \delta_k^L$ for $k = A, B$. Suppose all other voters either vote according to or against their signals after their equilibrium attention choice. Then voter 0's best response is to*

$$\begin{cases} \text{follow issue B then vote against signal} & \text{if } \varrho \geq T_2 \\ \text{follow issue A then vote according to signal} & \text{if } T_1 \leq \varrho \leq T_2 \\ \text{follow issue B then vote according to signal} & \text{if } \varrho \leq T_1 \end{cases}.$$

When cross-issue negative correlation ratio is low, it is sufficiently likely that one candidate is better on both issues. In that case, whoever is better on one issue is likely better on both issues, and hence the correct candidate. It is thus optimal to pay attention to issue B since its information is more precise and choose whoever the signal suggests is better on issue B. When cross-issue negative correlation ratio is very high, it is much more likely that whoever is better on one issue is worse on the other issue. In that case, whoever is worse on issue B is very likely better on issue A, and thus the correct candidate given that issue A is more important. It is thus optimal to pay attention to issue B since its information is more precise and choose whoever the signal suggests is worse on issue B. It is optimal to pay attention to the more important issue when cross-issue negative correlation ratio is at an intermediate level.

4.2.2 Multi-issue Equilibrium: Example

We give a numeric example of existence of a multi-issue equilibrium.

Example 7 *Consider $n = 5$, $w = 1.7$, $\delta_A = 0.1$ and $\delta_B = 0.2$. Then the two thresholds are $T_1 = 1.2857$ and $T_2 = 11.5714$. Consider the symmetric pure strategy profile where, among all voters except voter 0, m_A of them pay attention to issue A and vote according to signal, while the remaining $2n-m_A$ pay attention to issue B and vote according to signal. Figure 1 plots the cross-issue negative correlation ratio as a function of m_A. For $m_A = 0, 1$ or $9, 10$,* *voter 0's best response is to pay attention to issue B and vote according to signal, whereas for all other values of m_A, voter 0's best response is to pay attention to issue A and vote according to signal. It is an equilibrium for all voters to pay attention to the less important issue B because when all other voters do so, i.e. when $m_A = 0$, voter 0's best response is to pay attention to issue B and vote according to signal as well. There is a multi-issue equilibrium in which 9 voters pay attention to the more important issue A and vote according to signal while the remaining 2 voters pay attention to the less important issue B and vote according to signal. It is an equilibrium because, for a voter who is supposed to pay attention to issue B, in total 9 other voters pay attention to issue A and by Figure 1, it is indeed this voter's best response to pay attention to issue B and vote according to signal; whereas for a voter who is supposed to pay attention to issue A, in total only 8 other voters pay attention to issue A, and by 1, it is this voter's best response to pay attention to issue A and vote according to signal.*

Attention allocation is locally substitutes near the equilibrium allocation because for m_A near the equilibrium $m_A^* = 9$, a voter has less incentive to pay attention to issue A as more other voters pay attention to issue A. However, attention allocation is complements globally. If almost no one else pays attention to the more important issue, e.g. $m_A = 0$ or 1, a voter has no incentive to pay attention to it either. However, if enough people pay attention to the more important issue, e.g. $m_A = 8$, then a voter has an incentive to pay attention to it too. Complementarity in attentional allocation causes existence of equilibrium multiplicity.

Note that there is a unique pure multi-issue equilibrium.

4.2.3 Cross-Issue Negative Correlation Ratio

From voter 0's perspective, voter i's vote, denoted by $X_i \in \{-1, 1\}$, is a random variable where $X_i = 1$ denotes a vote for candidate R and $X_i = -1$ denotes a vote for candidate L. Let $\mu^i(\theta_A \theta_B | \sigma)$ denote the probability that voter i votes for candidate R, i.e. $X_i = 1$, conditional on state (θ_A, θ_B) under the strategy profile σ. Then

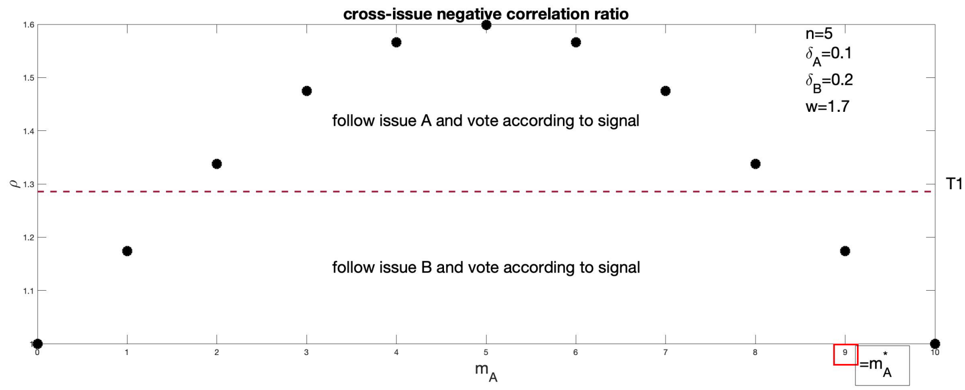

Figure 1: Cross-Issue Negative Correlation Ratio and Best Response

$$\begin{aligned}
\Pr\{X_i = x_i|\theta_A\theta_B,\sigma\} &= \begin{cases} \mu^i(\theta_A\theta_B|\sigma) & \text{if } x_i = 1 \\ 1-\mu^i(\theta_A\theta_B|\sigma) & \text{if } x_i = -1 \end{cases} \\
&= \mu^i(\theta_A\theta_B|\sigma)^{\frac{1+x_i}{2}}\left(1-\mu^i(\theta_A\theta_B|\sigma)\right)^{\frac{1-x_i}{2}} \\
&= \sqrt{\mu^i(\theta_A\theta_B|\sigma)\left(1-\mu^i(\theta_A\theta_B|\sigma)\right)}\left(\frac{\mu^i(\theta_A\theta_B|\sigma)}{1-\mu^i(\theta_A\theta_B|\sigma)}\right)^{\frac{x_i}{2}}.
\end{aligned}$$

Let $\mathbf{m_A}$ denote the strategy profile among other voters where voter 1 to m_A pay attention to issue A, whereas the other $2n - m_A$ voters pay attention to issue B, and all vote according to signal they pay attention to. Then, for voter i who is supposed to pay attention to issue k,

$$\mu^i(\theta_A\theta_B|\mathbf{m_A}) = \begin{cases} \frac{1}{2}+\delta_k & \text{if } \theta_k = R \\ \frac{1}{2}-\delta_k & \text{if } \theta_k = L \end{cases}$$

Let $\rho_k := \frac{\frac{1}{2}+\delta_k}{\frac{1}{2}-\delta_k}$, which is the likelihood ratio that a signal on issue k matches candidate better on issue k v.s. mismatches. By (**??**), we have

$$\Pr\left\{X^i = x_i|\theta_A\theta_B,\sigma\right\} = \begin{cases} \sqrt{\left(\frac{1}{2}+\delta_k\right)\left(\frac{1}{2}-\delta_k\right)}\left(\rho_k\right)^{\frac{x_i}{2}} & \text{if } \theta_k = R \\ \sqrt{\left(\frac{1}{2}+\delta_k\right)\left(\frac{1}{2}-\delta_k\right)}\left(\rho_k\right)^{-\frac{x_i}{2}} & \text{if } \theta_k = L \end{cases} \quad (5)$$

We show the extent to which Figure 1 is representative of how cross-issue negative correlation ratio ϱ changes with the number of other voters paying attention to issue A.

Lemma 8 *Assume that $\delta_k^R = \delta_k^L$ for $k = A, B$. The cross-issue negative correlation ratio ϱ has the following properties:*

1. $\varrho(\mathbf{m_A}) = 1$ *for* $m_A = 0$ *or* $2n$.
2. ϱ *is symmetric around* n, *i.e.* $\varrho(\mathbf{m_A}) = \varrho(\mathbf{2n} - \mathbf{m_A})$
3. $\varrho(\mathbf{n}) \to \infty$ *as* $n \to \infty$.
4. *For* $m_A \leq n - 1$, $\varrho(\mathbf{m_A}) \leq \varrho(\mathbf{m_A{+}2})$, *whereas for* $m_A \geq n - 1$, $\varrho(\mathbf{m_A}) \geq \varrho(\mathbf{m_A{+}2})$.

The first property is easy to establish. Suppose no other voter pays attention to issue A, i.e. $m_A = 0$. Then $\mu(R\theta_B|\mathbf{0}) = \mu(L\theta_B|\mathbf{0})$, thus $\Pr\{X = x|R\theta_B\} = \Pr\{X = x|L\theta_B\}$ for any vote profile x. It follows that $q(L\theta_B|\mathbf{0}) = q(R\theta_B|\mathbf{0})$ regardless of who is better on issue B. Lemma 4, cross-issue negative correlation ratio ϱ is equal to $\frac{q(LR|\mathbf{m_A})}{q(RR|\mathbf{m_A})}$, the likelihood ratio that candidate L v.s. R is better on issue A. Then $\varrho(\mathbf{0}) = 1$. The same argument applies to $m_A = 2n$.

The second property is a immediate from Lemma 4. The proofs of the other two properties are in the Appendix.

4.2.4 Multi-issue Equilibria: Existence

It is an equilibrium where $m_A + 1$ voters pay attention to issue A whereas the other $2n - m_A$ voters pay attention to issue B and all vote according to signal if and only if $\varrho(\mathbf{m_A}) \in [T_1, T_2]$ whereas $\varrho(\mathbf{m_A{+}1}) \leq T_1$. This follows immediately from Lemma **??** because, when all voters follow such a strategy profile, for a voter who is supposed to follow issue A (B), m_A ($m_A + 1$) other voters pay attention to issue A.

By Lemma 8, for n large enough, $\varrho(\mathbf{n}) > T_2$ and $\varrho(\mathbf{2n}) = 1$. So, a multi-issue equilibrium exists if, $\varrho(\mathbf{m_A})$ decreases slow enoough so that it will not jump from above T_2 to below T_1. That is, for all $m_A \geq n$,

$$\frac{\varrho(\mathbf{m_A{+}1})}{\varrho(\mathbf{m_A})} \geq \frac{T_1}{T_2}. \tag{6}$$

Lemma 9 gives an upper bound on rate of change of $\frac{q(LR|\mathbf{m}_A)}{q(RR|\mathbf{m}_A)}$ in m_A.

Lemma 9 *For $m_A \geq n$, $\frac{\varrho(\mathbf{m_A+1})}{\varrho(\mathbf{m_A})} > (\rho_A)^{-1}$.*

Proof. See Appendix. ■

The upper bound on rate of change of $\frac{q(LR|\mathbf{m}_A)}{q(RR|\mathbf{m}_A)}$ derived in Lemma 9 gaurantees that inequality (6) holds because

$$\frac{T_1}{T_2} = \frac{\frac{\delta_B - \delta_A}{\delta_B + \delta_A}}{\frac{\delta_B + \delta_A}{\delta_B - \delta_A}} = \left(\frac{\delta_B - \delta_A}{\delta_B + \delta_A}\right)^2 < \left(\frac{\frac{1}{2} - \delta_A}{\frac{1}{2} + \delta_A}\right)^2 = (\rho_A)^{-2} < (\rho_A)^{-1}$$

4.3 Symmetric Mixed-Attention Equilibria

We characterize the set of all symmetric equilibrium. We show that there are exactly two symmetric mixed attention equilibria: equilibria in which each voter randomizes strictly between paying attention to issue A and issue B.

Let $\boldsymbol{\alpha}$ denote the strategy to follow issue A with probability α and then vote according to signal, and follow issue B with probability $1-\alpha$ and then vote according to signal. As Section 4.2.1 derives voter best response given general voting strategy profile, it suffices to analyze how cross-issue negative correlation ratio changes with α, the probability with which all the other voters pay attention to issue A.

Lemma 10 *Assume that $\delta_k^R = \delta_k^L$ for $k = A, B$. The cross-issue negative correlation ratio ϱ has the following properties:*

1. *$\varrho(\boldsymbol{\alpha}) = 1$ for $\boldsymbol{\alpha} = 0$ or 1,*
2. *for any $\alpha \in (0,1)$, $\varrho(\boldsymbol{\alpha}) \to \infty$ as $n \to \infty$,*
3. *$\varrho(\boldsymbol{\alpha})$ first increases with α and then decreases with α.*

We first observe that $\mu(RL|\boldsymbol{\alpha}) = \alpha\left(\frac{1}{2} + \delta_A\right) + (1-\alpha)\left(\frac{1}{2} - \delta_B\right) = \frac{1}{2} + \alpha\delta_A - (1-\alpha)\delta_B$ and $\mu(LL|\boldsymbol{\alpha}) = \frac{1}{2} - \alpha\delta_A - (1-\alpha)\delta_B$. .

So

$$\begin{aligned}\varrho(\boldsymbol{\alpha}) &= \frac{q(RL|\boldsymbol{\alpha})}{q(LL|\boldsymbol{\alpha})} = \frac{\mu(RL|\boldsymbol{\alpha})^n (1-\mu(RL|\alpha))^n}{\mu(LL|\alpha)^n (1-\mu(LL|\alpha))^n} \\ &= \left(\frac{\frac{1}{4} - (\alpha\delta_A - (1-\alpha)\delta_B)^2}{\frac{1}{4} - (\alpha\delta_A + (1-\alpha)\delta_B)^2}\right)^n.\end{aligned}$$

where the first equality holds by Lemma 4.

It follows that $\varrho(\boldsymbol{\alpha}) = 1$ for $\alpha = 0$ and 1. For any $\alpha \in (0,1)$, $|\alpha\delta_A - (1-\alpha)\delta_B| - |\alpha\delta_A + (1-\alpha)\delta_B| = -2\min\{\alpha\delta_A, (1-\alpha)\delta_B\} < 0$. So $\frac{\frac{1}{4}-(\alpha\delta_A-(1-\alpha)\delta_B)^2}{\frac{1}{4}-(\alpha\delta_A+(1-\alpha)\delta_B)^2} < 1$ and thus $\varrho(\boldsymbol{\alpha}) \to \infty$ as $n \to \infty$. The proof for the last property is in the Appendix.

For voter 0 to optimally randomize between two issues, she must be indifferent between them. So equilibrium α^* must satisfy $\varrho(\boldsymbol{\alpha}^*) = T_1$. The following characterization of the set of symmetric equilibria with $\alpha^* \in (0,1)$ follows immediately from Lemma 10.

Proposition 11 *Assume $\delta_k^R = \delta_k^L$ for $k = A, B$. There are exactly two symmetric mixed-attention equilibria, one converges to $\alpha = 1$ as $n \to \infty$, and one converges to $\alpha = 0$ as $n \to \infty$.*

4.4 Exogenous Information Source

We emphasize that endogenous attention allocation is the source of potential inefficiency.

Suppose each voter exogenously receives a signal on issue A with probability α and a signal on issue B with probability $1-\alpha$. This setup is closest to the classic voting model like Feddersen and Pesendorfer (1998).

We show that the unique symmetric equilibrium in this game is asymptotically efficient. For large enough electorate, there exists an equilibrium in which receiving a signal on the less important issue has zero informational value to a voter, in the sense that the voter is indifferent between the two candidates regardless of which signal she receives on the less important issue.

In this model, a voter's strategy is represented by $\pi = (\pi_A, \pi_B)$ where $\pi_k(s)$ is the probability the voter votes for candidate R upon receiving signal s on issue k. Write $\Delta\pi_k = \pi_k(r) - \pi_k(l)$ and $\overline{\pi}_k = \frac{\pi_k(r)+\pi_k(l)}{2}$.

Suppose every voter uses the same voting strategy π. So $\gamma_k^i(\theta_k) = \gamma_k(\theta_k)$ for all i, where $\gamma_k(\theta_k)$ is the expected probability a voter receiving a signal on issue k votes for candidate R, and

$$\gamma_k(R|\pi) = \left(\frac{1}{2} + \delta_k^R\right)\pi_k(r) + \left(\frac{1}{2} - \delta_k^R\right)\pi_k(l) = \overline{\pi}_k + \delta_k^R\Delta\pi_k$$

and

$$\gamma_k(L|\pi) = \left(\frac{1}{2} - \delta_k^L\right)\pi_k(r) + \left(\frac{1}{2} + \delta_k^L\right)\pi_k(l) = \overline{\pi}_k - \delta_k^L\Delta\pi_k$$

In addition, every voter votes for candidate R with the same expected probability $\mu(\theta_A\theta_B|\pi) = \alpha\gamma_A(\theta_A|\pi) + (1-\alpha)\gamma_B(\theta_B|\pi)$.

Define the cross-issue negative correlation when $(\pi_A, \pi_B) = ((0,1),(0,1))$ to be ϱ_0. Then

$$\varrho_0 = \left[\frac{\frac{1}{4} - \left(-\alpha\overline{\delta}_A + (1-\alpha)\overline{\delta}_B\right)^2}{\frac{1}{4} - \left(\alpha\overline{\delta}_A + (1-\alpha)\overline{\delta}_B\right)^2}\right]^n.$$

Proposition 12 *Suppose $\delta_k^R = \delta_k^L$ for $k = A, B$. If $\varrho_0 \leq \frac{w+1}{w-1}$, then in the unique symmetric equilibrium, each voter votes according to signal on each signal. If $\varrho_0 > \frac{w+1}{w-1}$, in the unique symmetric equilibrium, a voter votes according to signal if the signal is on issue A, whereas a voter is indifferent between both candidates if his signal is on issue B. In addition, if n is large enough, each voter is more likely to vote for the candidate better overall, and thus equilibrium is asymptotically efficient.*

5 Appendix

5.1 Proofs for Section 4.2

Proof of Lemma 4. Consider (θ_A, θ_B) and (θ'_A, θ'_B) where $\theta'_A \neq \theta_A$ and $\theta'_B \neq \theta_B$. Given that $\pi_k^i(r) + \pi_k^i(l) = 1$ for all i and all k, $\mu^i(\theta_A\theta_B|\sigma) = 1 - \mu^i(\theta'_A\theta'_B)$. So $\left(\frac{\mu^i(\theta_A\theta_B|\sigma)}{1-\mu^i(\theta_A\theta_B|\sigma)}\right)^{\frac{x_i}{2}} = \left(\frac{1-\mu^i\left(\theta'_A\theta'_B\right)}{\mu^i\left(\theta'_A\theta'_B\right)}\right)^{\frac{x_i}{2}} = \left(\frac{\mu^i\left(\theta'_A\theta'_B\right)}{1-\mu^i\left(\theta'_A\theta'_B\right)}\right)^{-\frac{x_i}{2}}$. So the probability of vote profile $X^{-0} = x^{-0}$ conditional on state (θ_A, θ_B) is equal to the probability of vote profile $X^{-0} = -x^{-0}$ conditional on state (θ'_A, θ'_B). For every $x \in Piv^0$, we have $-x \in Piv^0$ because $\sum_{i=1}^{2n}(-x^i) = -\sum_{i=1}^{2n} x^i = 0$. It follows that $\Pr\{Piv|\theta_A\theta_B, \sigma\} = \Pr\{Piv|\theta'_A\theta'_B, \sigma\}$. So $q(\theta_A\theta_B|\sigma) = q(\theta'_A\theta'_B|\sigma)$. ■

5.1.1 Cross-Issue Negative Correlation Issue with Evenly Divided Attention

We next show that when attention among other voters is evenly divided, as the electorate becomes larger, conditional on being pivotal, it comes extremely likely that different candidates are better on different issues.

Define $X_A = \sum_{i=1}^{m_A} X_i$ and $X_B = \sum_{i=m_A+1}^{2n} X_i$. Then X_A and X_B are the net number of votes for candidate R, i.e. the number of "R" votes minus the number of "L" votes, among voter $i=1,\cdots,2n$ who follow A and B respectively. Let $m_B=2n-m_A$. Then, for $k=A,B$, X_k is a random variable with support $\{-m_k,-m_k+2,\cdots,m_k-2,m_k\}$ where $m_B=2n-m_A$. The event that voter 0 is pivotal, Piv, is exactly the event $X_A+X_B=0$.

Define $g_{m_A}(x_A)$ to be probability $X_A=x_A$ conditional on voter 0 being pivotal and $(\theta_A,\theta_B)=(R,R)$. Since all voters are ex ante identical, we can w.l.o.g. assume that voter $i=1,\cdots,m_A$ follow issue A and voter $i=m_A+1,\cdots,2n$ follow issue B. Conditional on voter 0 being pivotal, $X_A=x_A$ if and only if the total number of "R" votes among voter $i=1,\cdots,m_A$ is $\frac{m_A+x_A}{2}$ and the total number of "R" votes among voter $i=m_A+1,\cdots,2n$ is equal to $\frac{2n-m_A-x_A}{2}$. Then, for $x_A\in\{-\underline{m},-\underline{m}+2,\cdots,\underline{m}-2,\underline{m}\}$ where $\underline{m}=\min\{m_A,m_B\}$, we have

$$
\begin{aligned}
& g_{m_A}(x_A) \\
= & \frac{\Pr\{X_A=x_A,X_B=-x_A|RR\}}{\Pr\{X_A+X_B=0|RR\}} \\
= & \frac{\sum_{x\in Piv^0:\left(\sum_{i=1}^{m_A}x_i\right)=x_A}\Pi_{i=1}^{m_A}\sqrt{\left(\frac{1}{2}+\delta_A\right)\left(\frac{1}{2}-\delta_A\right)}(\rho_A)^{\frac{x_i}{2}}\Pi_{i=m_A+1}^{2n}\sqrt{\left(\frac{1}{2}+\delta_B\right)\left(\frac{1}{2}-\delta_B\right)}(\rho_B)^{\frac{x_i}{2}}}{\sum_{x\in Piv^0}\Pi_{i=1}^{m_A}\sqrt{\left(\frac{1}{2}+\delta_A\right)\left(\frac{1}{2}-\delta_A\right)}(\rho_A)^{\frac{x_i}{2}}\Pi_{i=m_A+1}^{2n}\sqrt{\left(\frac{1}{2}+\delta_B\right)\left(\frac{1}{2}-\delta_B\right)}(\rho_B)^{\frac{x_i}{2}}} \\
= & \frac{\binom{m_A}{\frac{m_A+x_A}{2}}\binom{2n-m_A}{\frac{2n-m_A-x_A}{2}}\left(\left(\frac{1}{2}+\delta_A\right)\left(\frac{1}{2}-\delta_A\right)\right)^{\frac{m_A}{2}}(\rho_A)^{\frac{x_A}{2}}\left(\left(\frac{1}{2}+\delta_B\right)\left(\frac{1}{2}-\delta_B\right)\right)^{\frac{2n-m_A}{2}}(\rho_B)^{-\frac{x_A}{2}}}{\sum_{x'_A\in\{-\underline{m},-\underline{m}+2,\cdots,\underline{m}-2,\underline{m}\}}\left[\begin{array}{c}\binom{m_A}{\frac{m_A+x'_A}{2}}\binom{2n-m_A}{\frac{2n-m_A-x'_A}{2}}\left(\left(\frac{1}{2}+\delta_A\right)\left(\frac{1}{2}-\delta_A\right)\right)^{\frac{m_A}{2}}(\rho_A)^{\frac{x'_A}{2}} \\ \times\left(\left(\frac{1}{2}+\delta_B\right)\left(\frac{1}{2}-\delta_B\right)\right)^{\frac{2n-m_A}{2}}(\rho_B)^{-\frac{x'_A}{2}}\end{array}\right]} \\
= & \frac{\binom{m_A}{\frac{m_A+x_A}{2}}\binom{2n-m_A}{\frac{2n-m_A-x_A}{2}}\left(\frac{\rho_A}{\rho_B}\right)^{\frac{x_A}{2}}}{\sum_{x'_A\in\{-\underline{m},-\underline{m}+2,\cdots,\underline{m}-2,\underline{m}\}}\binom{m_A}{\frac{m_A+x'_A}{2}}\binom{2n-m_A}{\frac{2n-m_A-x'_A}{2}}\left(\frac{\rho_A}{\rho_B}\right)^{\frac{x'_A}{2}}},
\end{aligned}
\tag{7}
$$

where the second equality follows from (5) and the fourth equality follows by dividing both the nominator and the denominator by

$$
\left(\left(\frac{1}{2}+\delta_A\right)\left(\frac{1}{2}-\delta_A\right)\right)^{\frac{m_A}{2}}\left(\left(\frac{1}{2}+\delta_B\right)\left(\frac{1}{2}-\delta_B\right)\right)^{\frac{m_B}{2}}
$$

Proof of Property 4 for Lemma 8. It sufficies to show that, for any $K\in\mathbb{R}^+$, $\Pr\{X_A\geq -K|RR,m_A=n\}\to 0$ as $n\to\infty$. We do so by establishing the following claims:

Claim 13 *$g_n(x) \to 0$ as $n \to \infty$ if $x \in [-K, -1]$.*

Claim 14 *For $x = n \bmod 2 + 2k$ for $k \in \mathbb{N}$, $\frac{g_n(x)}{g_n(-n \bmod 2)} < \left(\frac{\rho_A}{\rho_B}\right)^k$ and thus* $\Pr\{X_A \geq -1 | HH, m_A = n\} \leq \frac{g_n(-n \bmod 2)}{1-\frac{\rho_A}{\rho_B}}$

It then follows that $\Pr\{X_A \geq -K | RR, m_A = n\} \leq \frac{g_n(-n \bmod 2)}{1-\left(\frac{\rho_A}{\rho_B}\right)} + \sum_{x \leq K \text{ and } \frac{x-n \bmod 2}{2} \in \mathbb{N}} g_n(x) \to 0$ as $n \to \infty$.

We now prove the two claims. We show that the density function $g_{m_A=n}(x)$ is single-peaked and that the peak, denoted by x_n^*, is roughly proportional to $-n$ in the sense that $\frac{-x_n^*}{n}$ converges to a positive constant.

We first note that

$$\begin{aligned}
\frac{g_{m_A}(x+2)}{g_{m_A}(x)} &= \frac{\binom{m_A}{\frac{m_A+x+2}{2}}\binom{2n-m_A}{\frac{2n-m-x-2}{2}}\left(\frac{\rho_A}{\rho_B}\right)^{\frac{x}{2}}}{\binom{m_A}{\frac{m_A+x}{2}}\binom{2n-m_A}{\frac{2n-m_A-x}{2}}\left(\frac{\rho_A}{\rho_B}\right)^{\frac{x-2}{2}}} \\
&= \frac{m_A - \frac{m_A+x}{2}}{\frac{m_A+x+2}{2}} \frac{\frac{2n-m_A-x}{2}}{2n-m_A-\frac{2n-m_A-x-2}{2}} \frac{\rho_A}{\rho_B} \\
&= \frac{\left(\frac{m_A}{2} - \frac{x}{2}\right)\left(\frac{2n-m_A}{2} - \frac{x}{2}\right)}{\left(\frac{m_A}{2} + \frac{x}{2} + 1\right)\left(\frac{2n-m_A}{2} + \frac{x}{2} + 1\right)} \frac{\rho_A}{\rho_B}
\end{aligned}$$

So

$$g_n(x+2)/g_n(x) = \left(\frac{\frac{n}{2} - \frac{x}{2}}{\frac{n}{2} + \frac{x}{2} + 1}\right)^2 \frac{\rho_A}{\rho_B}$$

So $g_n(x) \geq g_n(x+2)$ if and only if $\left(1 + \frac{x}{n} + \frac{2}{n}\right)\sqrt{\rho_B/\rho_A} \geq 1 - \frac{x}{n}$, i.e. iff $-\frac{x}{n}\left(1 + \sqrt{\frac{\rho_B}{\rho_A}}\right) \leq \left(1 + \frac{2}{n}\right)\sqrt{\rho_B/\rho_A} - 1$ i.e. iff $-x/n \leq \frac{\left(1+\frac{2}{n}\right)\sqrt{\rho_B/\rho_A}-1}{\sqrt{\rho_B/\rho_A}+1}$. So $g_n(x)$ peaks at $x_n^* = -n \bmod 2 - 2k^*$ where $k^* = \left\lfloor \left(\frac{n}{2} + 1\right)\frac{\sqrt{\frac{\rho_B}{\rho_A}}-1}{\sqrt{\frac{\rho_B}{\rho_A}}+1} - n \bmod 2 \right\rfloor$. So $x_n^*/n \to -\frac{\sqrt{\rho_B/\rho_A}-1}{\sqrt{\rho_B/\rho_A}+1} \in (0,1)$ as $n \to \infty$

For any given x,

$$
\begin{aligned}
\frac{g_n(x)}{g_n(x_n^*)} &= \frac{g_n(x_n^*+2)}{g_n(x_n^*)}\cdots\frac{g_n(x)}{g_n(x-2)} \\
&= \left(\frac{\frac{n}{2}-\frac{x_n^*}{2}}{\frac{n}{2}+\frac{x_n^*}{2}+1}\right)^2\frac{\rho_A}{\rho_B}\cdots\left(\frac{\frac{n}{2}-\frac{x-2}{2}}{\frac{n}{2}+\frac{x}{2}}\right)^2\frac{\rho_A}{\rho_B} \\
&= \frac{\left(\frac{\frac{n}{2}-\frac{x_n^*}{2}}{\frac{n}{2}+\frac{x_n^*}{2}+1}\right)^2\cdots\left(\frac{\frac{n}{2}-\frac{x-2}{2}}{\frac{n}{2}+\frac{x}{2}}\right)^2}{\left(\frac{\frac{n}{2}-\frac{x_n^*}{2}}{\frac{n}{2}+\frac{x_n^*}{2}+1}\right)^{2\left(\frac{x-x_n^*}{2}\right)}}\left(\frac{\frac{n}{2}-\frac{x_n^*}{2}}{\frac{n}{2}+\frac{x_n^*}{2}+1}\right)^{2\left(\frac{x-x_n^*}{2}\right)}\left(\frac{\rho_A}{\rho_B}\right)^{\frac{x-x_n^*}{2}} \\
&= \Pi_{k=0}^{\frac{x-x_n^*}{2}-1}\left(\frac{\frac{\frac{n}{2}-\frac{x_n^*+2k}{2}}{\frac{n}{2}+\frac{x_n^*+2k}{2}+1}}{\frac{\frac{n}{2}-\frac{x_n^*}{2}}{\frac{n}{2}+\frac{x_n^*}{2}+1}}\right)^2\left(\left(\frac{\frac{n}{2}-\frac{x_n^*}{2}}{\frac{n}{2}+\frac{x_n^*}{2}+1}\right)^2\frac{\rho_A}{\rho_B}\right)^{\frac{x-x_n^*}{2}} \\
&\leq \Pi_{k=0}^{\frac{x-x_n^*}{2}-1}\left(\frac{\frac{\frac{n}{2}-\frac{x_n^*+2k}{2}}{\frac{n}{2}-\frac{x_n^*}{2}}}{\frac{\frac{n}{2}+\frac{x_n^*+2k}{2}+1}{\frac{n}{2}+\frac{x_n^*}{2}+1}}\right)^2 \\
&= \Pi_{k=0}^{\frac{x-x_n^*}{2}-1}\left(\frac{1-\frac{k}{\frac{n}{2}-\frac{x_n^*}{2}}}{1+\frac{k}{\frac{n}{2}+\frac{x_n^*}{2}+1}}\right)^2 = \left(\frac{e^{\sum_{k=0}^{\frac{x-x_n^*}{2}-1}\log\left(1-\frac{k}{\frac{n}{2}-\frac{x_n^*}{2}}\right)}}{e^{\sum_{k=0}^{\frac{x-x_n^*}{2}-1}\log\left(1+\frac{k}{\frac{n}{2}+\frac{x_n^*}{2}+1}\right)}}\right)^2 \\
&= \frac{\left(e^{\frac{1}{\frac{n}{2}-\frac{x_n^*}{2}}\sum_{k=0}^{\frac{x-x_n^*}{2}-1}\log\left(1-\frac{k}{\frac{n}{2}-\frac{x_n^*}{2}}\right)}\right)^{2\left(\frac{n}{2}-\frac{x_n^*}{2}\right)}}{\left(e^{\frac{1}{\frac{n}{2}+\frac{x_n^*}{2}+1}\sum_{k=0}^{\frac{x-x_n^*}{2}-1}\log\left(1+\frac{k}{\frac{n}{2}+\frac{x_n^*}{2}+1}\right)}\right)^{2\left(\frac{n}{2}+\frac{x_n^*}{2}+1\right)}} \\
&\leq \left(e^{\int_{\zeta=1-\frac{x-x_n^*-2}{n-x_n^*}}^{1}\log\zeta d\zeta}\right)^{n-x_n^*} \Big/ \left(e^{\int_{\zeta=1}^{1+\frac{x-x_n^*-2}{n+x_n^*+2}}\log\zeta d\zeta}\right)^{n+x_n^*+2} \\
&= \left(\left(e^{\int_{\zeta=1-\frac{x-x_n^*-2}{n-x_n^*}}^{1}\log\zeta d\zeta}\right)^{1-x_n^*/n}\left(e^{-\int_{\zeta=1}^{1+\frac{x-x_n^*-2}{n+x_n^*+2}}\log\zeta d\zeta}\right)^{1+(x_n^*+2)/n}\right)^n \\
&= \left[\left(e^{(z\log z-z)|_{1-\frac{(x-2)/n-x_n^*/n}{1-x_n^*/n}}^{1}}\right)^{1-x_n^*/n}\left(e^{-(z\log z-z)|_1^{1+\frac{(x-2)/n-x_n^*/n}{1+x_n^*/n+2}}}\right)^{1+(x_n^*+2)/n}\right]^n \qquad (8) \\
&\to 0 \text{ as } n\to\infty \qquad (9)
\end{aligned}
$$

because $\frac{(x-2)/n-x_n^*/n}{1-x_n^*/n} \to \frac{\sqrt{\rho_B/\rho_A}-1}{2\sqrt{\rho_B/\rho_A}} \in (0,1)$, $\frac{x/n-(x_n^*+2)/n}{1+(x_n^*+2)/n} \to \frac{\sqrt{\rho_B/\rho_A}-1}{2} > 0$, $1-\frac{x_n^*}{n} \to \frac{2\sqrt{\rho_B/\rho_A}}{\sqrt{\rho_B/\rho_A}+1} > 0$ and $1+(x_n^*+2)/n \to \frac{2}{\sqrt{\rho_B/\rho_A}+1} > 0$ and thus the term inside [] converges to some number in $(0,1)$.

Since $g_n(x)$ is a probability mass function, $g_n(x_n^*) \in (0,1)$. Claim 13 then follows.

For $x \geq -1$ in the support of X_A, $g_n(x+2)/g_n(x) \leq \frac{\rho_A}{\rho_B}$ because $-\frac{x}{2} - \left(\frac{x}{2}+1\right) = -x-1 \leq 0$. Thus, for $x = -n \bmod 2 + 2k$ for any $k \geq 0$, $\frac{g_n(x)}{g_n(-n \bmod 2)} = \frac{g_n(x)}{g_n(x-2)} \cdots \frac{g_n(-n \bmod 2+2)}{g_n(-n \bmod 2)} \leq \left(\frac{\rho_A}{\rho_B}\right)^k$. Claim 14 holds because

$$
\begin{aligned}
\Pr\{X \geq -1\} &= \sum_{k=0}^{\infty} g_n(-n \bmod 2 + 2k) \\
&= \sum_{k=0}^{\infty} \frac{g_n(-n \bmod 2 + 2k)}{g_n(-n \bmod 2)} g_n(-n \bmod 2) \\
&\leq \sum_{k=0}^{\infty} \left(\frac{\rho_A}{\rho_B}\right)^k g_n(-n \bmod 2) \\
&= \frac{g_n(-n \bmod 2)}{1-\rho_A/\rho_B}.
\end{aligned}
$$

■

5.1.2 How Fast Cross-Issue Negative Correlation Ratio Changes with m_A

Proof of Lemma 9. Recall that $\rho_A = \frac{\frac{1}{2}+\delta_A}{\frac{1}{2}-\delta_A}$ and $\rho_B = \frac{\frac{1}{2}+\delta_B}{\frac{1}{2}-\delta_B}$. From (**??**), we have

$$
\begin{aligned}
&\frac{q\left(LR|\mathbf{m}_A\right)}{q\left(RR|\mathbf{m}_A\right)} \\
=\ & \frac{\sum_{x_A\in\{-(2n-m_A),-(2n-m_A)+2,\cdots,(2n-m_A)-2,2n-m_A\}}\binom{m_A}{\frac{m_A+x_A}{2}}\binom{2n-m_A}{\frac{2n-m_A-x_A}{2}}\left(\rho_A\rho_B\right)^{-\frac{x_A}{2}}}{\sum_{x_A\in\{-(2n-m_A),-(2n-m_A)+2,\cdots,(2n-m_A)-2,2n-m_A\}}\binom{m_A}{\frac{m_A+x_A}{2}}\binom{2n-m_A}{\frac{2n-m_A-x_A}{2}}\left(\frac{\rho_A}{\rho_B}\right)^{\frac{x_A}{2}}} \\
=\ & \frac{\sum_{y=-\frac{2n-m_A}{2}}^{\frac{2n-m_A}{2}}\binom{m_A}{\frac{m_A}{2}-y}\binom{2n-m_A}{\frac{2n-m_A}{2}+y}\left(\rho_A\rho_B\right)^{y}}{\sum_{y=-\frac{2n-m_A}{2}}^{\frac{2n-m_A}{2}}\binom{m_A}{\frac{m_A}{2}-y}\binom{2n-m_A}{\frac{2n-m_A}{2}+y}\left(\frac{\rho_B}{\rho_A}\right)^{y}} \\
=\ & \sum_{y=-\frac{2n-m_A}{2}}^{\frac{2n-m_A}{2}}\frac{\binom{m_A}{\frac{m_A}{2}-y}\binom{2n-m_A}{\frac{2n-m_A}{2}+y}\left(\frac{\rho_B}{\rho_A}\right)^{y}}{\sum_{z=-\frac{2n-m_A}{2}}^{\frac{2n-m_A}{2}}\binom{m_A}{\frac{m_A}{2}-z}\binom{2n-m_A}{\frac{2n-m_A}{2}+z}\left(\frac{\rho_B}{\rho_A}\right)^{z}}\left(\rho_A\right)^{2y}
\end{aligned}
$$

where the second equality holds by letting $y = -\frac{x_A}{2}$. The summation is over $y \in \left\{-\frac{2n-m_A}{2}, -\frac{2n-m_A}{2}+1, \cdots, \frac{2n-m_A}{2}-1, \frac{2n-m_A}{2}\right\}$.

Define $f\left(y\right) = \frac{\binom{m_A}{\frac{m_A}{2}-y}\binom{2n-m_A}{\frac{2n-m_A}{2}+y}\left(\frac{\rho_B}{\rho_A}\right)^{y}}{\sum_{z=-\frac{2n-m_A}{2}}^{\frac{2n-m_A}{2}}\binom{m_A}{\frac{m_A}{2}-z}\binom{2n-m_A}{\frac{2n-m_A}{2}+z}\left(\frac{\rho_B}{\rho_A}\right)^{z}}$ for $y \in \left\{-\frac{2n-m_A}{2}, -\frac{2n-m_A}{2}+1, \cdots, \frac{2n-m_A}{2}-1, \right.$ and 0 otherwise. Then

$$
\frac{q\left(LR|\mathbf{m}_A\right)}{q\left(RR|\mathbf{m}_A\right)} = E_{Y\sim f}\left[\left(\rho_A\right)^{2Y}\right].
$$

Similarly,

$$
\begin{aligned}
&\frac{q\left(LR|\mathbf{m}_A\mathbf{+1}\right)}{q\left(RR|\mathbf{m}_A\mathbf{+1}\right)} \\
=\ & \frac{\sum_{x_A\in\{-(2n-m_A)+1,-(2n-m_A)+3,\cdots,(2n-m_A)-3,2n-m_A-1\}}\binom{m_A+1}{\frac{m_A+1+x_A}{2}}\binom{2n-m_A-1}{\frac{2n-m_A-1-x_A}{2}}\left(\rho_A\rho_B\right)^{-\frac{x_A}{2}}}{\sum_{x_A\in\{-(2n-m_A)+1,-(2n-m_A)+3,\cdots,(2n-m_A)-3,2n-m_A-1\}}\binom{m_A+1}{\frac{m_A+1+x_A}{2}}\binom{2n-m_A-1}{\frac{2n-m_A-1-x_A}{2}}\left(\frac{\rho_A}{\rho_B}\right)^{\frac{x_A}{2}}} \\
=\ & \frac{\sum_{x_A\in\{-(2n-m_A)+1,-(2n-m_A)+3,\cdots,(2n-m_A)-3,2n-m_A-1\}}\binom{m_A+1}{\frac{m_A+1-x_A}{2}}\binom{2n-m_A-1}{\frac{2n-m_A-1+x_A}{2}}\left(\rho_A\rho_B\right)^{-\frac{x_A}{2}}}{\sum_{x_A\in\{-(2n-m_A)+1,-(2n-m_A)+3,\cdots,(2n-m_A)-3,2n-m_A-1\}}\binom{m_A+1}{\frac{m_A+1-x_A}{2}}\binom{2n-m_A-1}{\frac{2n-m_A-1+x_A}{2}}\left(\frac{\rho_A}{\rho_B}\right)^{\frac{x_A}{2}}}
\end{aligned}
$$

where the second equality holds because $\binom{m_A+1}{\frac{m_A+1+x_A}{2}} = \binom{m_A+1}{\frac{m_A+1-x_A}{2}}$ and $\binom{2n-m_A-1}{\frac{2n-m_A-1-x_A}{2}} = \binom{2n-m_A-1}{\frac{2n-m_A-1+x_A}{2}}$. Let $y = -\frac{x_A-1}{2}$, then we have

$$\begin{aligned}
&\frac{q\left(LR|\mathbf{m}_A+\mathbf{1}\right)}{q\left(RR|\mathbf{m}_A+\mathbf{1}\right)} \\
=&\ \frac{\sum_{y=-\frac{2n-m_A}{2}+1}^{\frac{2n-m_A}{2}} \binom{m_A+1}{\frac{m_A}{2}+y}\binom{2n-m_A-1}{\frac{2n-m_A}{2}-y}\left(\rho_A\rho_B\right)^{y-\frac{1}{2}}}{\sum_{y=-\frac{2n-m_A}{2}+1}^{\frac{2n-m_A}{2}} \binom{m_A+1}{\frac{m_A}{2}+y}\binom{2n-m_A-1}{\frac{2n-m_A}{2}-y}\left(\frac{\rho_A}{\rho_B}\right)^{-y+\frac{1}{2}}} \\
=&\ \frac{\sum_{y=-\frac{2n-m_A}{2}+1}^{\frac{2n-m_A}{2}} \binom{m_A+1}{\frac{m_A}{2}+y}\binom{2n-m_A-1}{\frac{2n-m_A}{2}-y}\left(\rho_A\right)^{y-1}\left(\rho_B\right)^{y}}{\sum_{y=-\frac{2n-m_A}{2}+1}^{\frac{2n-m_A}{2}} \binom{m_A+1}{\frac{m_A}{2}+y}\binom{2n-m_A-1}{\frac{2n-m_A}{2}-y}\left(\frac{\rho_B}{\rho_A}\right)^{y}} \\
=&\ \left(\rho_A\right)^{-1}\sum_{y=-\frac{2n-m_A}{2}+1}^{\frac{2n-m_A}{2}} \frac{\binom{m_A+1}{\frac{m_A}{2}+y}\binom{2n-m_A-1}{\frac{2n-m_A}{2}-y}\left(\frac{\rho_B}{\rho_A}\right)^{y}}{\sum_{y=-\frac{2n-m_A}{2}+1}^{\frac{2n-m_A}{2}} \binom{m_A+1}{\frac{m_A}{2}+y}\binom{2n-m_A-1}{\frac{2n-m_A}{2}-y}\left(\frac{\rho_B}{\rho_A}\right)^{y}}\left(\rho_A\right)^{2y}.
\end{aligned}$$

Define

$$\tilde{f}\left(y\right) = \frac{\binom{m_A+1}{\frac{m_A}{2}+y}\binom{2n-m_A-1}{\frac{2n-m_A}{2}-y}\left(\frac{\rho_B}{\rho_A}\right)^{y}}{\sum_{y=-\frac{2n-m_A}{2}+1}^{\frac{2n-m_A}{2}} \binom{m_A+1}{\frac{m_A}{2}+y}\binom{2n-m_A-1}{\frac{2n-m_A}{2}-y}\left(\frac{\rho_B}{\rho_A}\right)^{y}}.$$

for $y \in \left\{-\frac{2n-m_A}{2}+1, \cdots, \frac{2n-m_A}{2}-1, \frac{2n-m_A}{2}\right\}$ and 0 otherwise. Then

$$\frac{q\left(LR|\mathbf{m}_A+\mathbf{1}\right)}{q\left(RR|\mathbf{m}_A+\mathbf{1}\right)} = \left(\rho_A\right)^{-1} E_{Y\sim\tilde{f}}\left[\left(\rho_A\right)^{2Y}\right].$$

Note that for $y = \frac{-2n-m_A}{2}$, $\frac{\tilde{h}(y)}{h(y)} = 0$. For $y \in \left\{-\frac{2n-m_A}{2}+1, \cdots, \frac{2n-m_A}{2}-1, \frac{2n-m_A}{2}\right\}$, $\frac{\tilde{h}(y)}{h(y)}$ is equal to a term independent of y times

$$\frac{\binom{m_A+1}{\frac{m_A}{2}+y}\binom{2n-m_A-1}{\frac{2n-m_A}{2}-y}}{\binom{m_A}{\frac{m_A}{2}+y}\binom{2n-m_A}{\frac{2n-m_A}{2}-y}} = \frac{m_A+1}{\frac{m_A}{2}+1-y}\frac{\frac{2n-m_A}{2}+y}{2n-m_A},$$

which is strictly positive and strictly increasing in y. So the distribution of $\tilde{h}$ dominates that of h in the sense of MLRP, and thus

$$\frac{q\left(LR|\mathbf{m}_A+\mathbf{1}\right)}{q\left(RR|\mathbf{m}_A+\mathbf{1}\right)} = \left(\rho_A\right)^{-1} E_{Y\sim\tilde{f}}\left[\left(\rho_A\right)^{2Y}\right] > \left(\rho_A\right)^{-1} E_{Y\sim f}\left[\left(\rho_A\right)^{2Y}\right] = \left(\rho_A\right)^{-1}\frac{q\left(LR|\mathbf{m}_A\right)}{q\left(RR|\mathbf{m}_A\right)}.$$

■

5.2 Proofs for Section 4.3

We now show that $\varrho(\boldsymbol{\alpha})$ first increases with α and then decreases with α.

$$
\begin{aligned}
& \frac{\partial \log \varrho(\boldsymbol{\alpha})}{\partial \alpha} / 2n \\
= \; & \frac{1}{2} \frac{\partial}{\partial \alpha} \log \frac{\frac{1}{4} - \left(\alpha \delta_A - (1-\alpha)\delta_B\right)^2}{\frac{1}{4} - \left(\alpha \delta_A + (1-\alpha)\delta_B\right)^2} \\
= \; & \frac{1}{2} \left(\frac{-2\left(\alpha\delta_A - (1-\alpha)\delta_B\right)\left(\delta_A + \delta_B\right)}{\frac{1}{4} - \left(\alpha\delta_A - (1-\alpha)\delta_B\right)^2} - \frac{-2\left(\alpha\delta_A + (1-\alpha)\delta_B\right)\left(\delta_A - \delta_B\right)}{\frac{1}{4} - \left(\alpha\delta_A + (1-\alpha)\delta_B\right)^2} \right) \\
= \; & \frac{\left(-\alpha\delta_A + (1-\alpha)\delta_B\right)\left(\delta_A + \delta_B\right)}{\frac{1}{4} - \left(\alpha\delta_A - (1-\alpha)\delta_B\right)^2} - \frac{\left(\alpha\delta_A + (1-\alpha)\delta_B\right)\left(\delta_B - \delta_A\right)}{\frac{1}{4} - \left(\alpha\delta_A + (1-\alpha)\delta_B\right)^2}
\end{aligned}
$$

We first observe that $\frac{\partial \log \varrho(\boldsymbol{\alpha})}{\partial \alpha} > 0$ for $\alpha = 0$ and $\frac{\partial \log \varrho(\boldsymbol{\alpha})}{\partial \alpha} < 0$ for $\alpha = 1$. We now show that $\frac{\partial^2 \log \frac{q(RL|\alpha)}{q(LL|\alpha)}}{\partial \alpha^2} < 0$ whenever $\frac{\partial \log \frac{q(RL|\alpha)}{q(LL|\alpha)}}{\partial \alpha} = 0$. Suppose $\frac{\partial \log \frac{q(RL|\alpha)}{q(LL|\alpha)}}{\partial \alpha} = 0$ at $\alpha = \hat{\alpha}$. Then $(\delta_A + \delta_B) \frac{\frac{1}{2} - \mu(RL|\alpha)}{\frac{1}{4} - \left(\mu(RL|\alpha) - \frac{1}{2}\right)^2} =$

$(\delta_B - \delta_A) \frac{\frac{1}{2} - \mu(LL|\alpha)}{\frac{1}{4} - \left(\mu(LL|\alpha) - \frac{1}{2}\right)^2}$. Then

$$\begin{aligned}
& \frac{\partial^2 \log \frac{q(RL|\alpha)}{q(LL|\alpha)}}{\partial \alpha^2} / n|_{\alpha=\hat{\alpha}} \\
= & \ (\delta_A + \delta_B) \frac{\frac{1}{4} + \left(\frac{1}{2} - \mu(RL|\hat{\alpha})\right)^2}{\left(\frac{1}{4} - \left(\frac{1}{2} - \mu(RL|\alpha)\right)^2\right)^2} \frac{\partial \left(\frac{1}{2} - \mu(RL|\hat{\alpha})\right)}{\partial \hat{\alpha}} - (\delta_B - \delta_A) \frac{\frac{1}{4} + \left(\frac{1}{2} - \mu(LL|\hat{\alpha})\right)^2}{\left(\frac{1}{4} - \left(\frac{1}{2} - \mu(LL|\hat{\alpha})\right)^2\right)^2} \frac{\partial (}{} \\
= & \ (\delta_A + \delta_B) \frac{\frac{1}{4} + \left(\frac{1}{2} - \mu(RL|\hat{\alpha})\right)^2}{\left(\frac{1}{4} - \left(\frac{1}{2} - \mu(RL|\hat{\alpha})\right)^2\right)^2} \frac{\partial \left(-\alpha\delta_A + (1-\alpha)\delta_B\right)}{\partial \alpha}|_{\alpha=\hat{\alpha}} - (\delta_B - \delta_A) \frac{\frac{1}{4} + \left(\frac{1}{2} - \mu(LL|\right.}{\left(\frac{1}{4} - \left(\frac{1}{2} - \mu(LL|\right.\right.} \\
= & \ -(\delta_A + \delta_B) \frac{\frac{1}{4} + \left(\frac{1}{2} - \mu(RL|\hat{\alpha})\right)^2}{\left(\frac{1}{4} - \left(\frac{1}{2} - \mu(RL|\hat{\alpha})\right)^2\right)^2} (\delta_A + \delta_B) - (\delta_B - \delta_A) \frac{\frac{1}{4} + \left(\frac{1}{2} - \mu(LL|\hat{\alpha})\right)^2}{\left(\frac{1}{4} - \left(\frac{1}{2} - \mu(LL|\hat{\alpha})\right)^2\right)^2} (\delta_A - \delta_B \\
= & \ -\left((\delta_A + \delta_B) \frac{\frac{1}{2} - \mu(RL|\hat{\alpha})}{\frac{1}{4} - \left(\frac{1}{2} - \mu(RL|\hat{\alpha})\right)^2}\right)^2 \frac{\frac{1}{4} + \left(\frac{1}{2} - \mu(RL|\hat{\alpha})\right)^2}{\left(\frac{1}{2} - \mu(RL|\hat{\alpha})\right)^2} + \left((\delta_B - \delta_A) \frac{\frac{1}{2} - \mu(LL|\hat{\alpha})}{\frac{1}{4} - \left(\frac{1}{2} - \mu(LL|\hat{\alpha})\right.}\right. \\
= & \ -\left((\delta_A + \delta_B) \frac{\frac{1}{2} - \mu(RL|\hat{\alpha})}{\frac{1}{4} - \left(\frac{1}{2} - \mu(RL|\hat{\alpha})\right)^2}\right)^2 \left(\frac{1}{4}\left(\frac{1}{2} - \mu(RL|\hat{\alpha})\right)^{-2} + 1 - \left(\frac{1}{4}\left(\frac{1}{2} - \mu(LL|\hat{\alpha})\right)^{-2} + \right.\right. \\
= & \ -\frac{1}{4}\left((\delta_A + \delta_B) \frac{\frac{1}{2} - \mu(RL|\hat{\alpha})}{\frac{1}{4} - \left(\frac{1}{2} - \mu(RL|\hat{\alpha})\right)^2}\right)^2 \left(\left(-\hat{\alpha}\delta_A + (1-\hat{\alpha})\delta_B\right)^{-2} + \left(\hat{\alpha}\delta_A + (1-\hat{\alpha})\delta_B\right)^{-2}\right) \\
< & \ 0
\end{aligned}$$

where the second-to-last equallity holds because $\frac{\partial \log \frac{q(RL|\alpha)}{q(LL|\alpha)}}{\partial \alpha} = 0$ at $\alpha = \hat{\alpha}$ and thus $(\delta_A + \delta_B) \frac{\frac{1}{2} - \mu(RL|\hat{\alpha})}{\frac{1}{4} - \left(\frac{1}{2} - \mu(RL|\hat{\alpha})\right)^2} = (\delta_B - \delta_A) \frac{\frac{1}{2} - \mu(LL|\hat{\alpha})}{\frac{1}{4} - \left(\frac{1}{2} - \mu(LL|\hat{\alpha})\right)^2}$, and the inequality holds because $-\hat{\alpha}\delta_A + (1-\hat{\alpha})\delta_B > 0$ given that $\frac{\partial \log \frac{q(RL|\alpha)}{q(LL|\alpha)}}{\partial \alpha} < 0$ if $\frac{1}{2} - \mu(RL|\alpha) = -\alpha\delta_A + (1-\alpha)\delta_B < 0$.

5.3 Analysis for Exogenous Information Source

Lemma 15 *The following equations are equivalent:*

Claim 16 *1.* $q(RR|\pi) = q(LL|\pi)$*;*

2. $q(RL|\pi) = q(LR|\pi)$*;*

3. $\frac{\mu(RR|\pi)+\mu(LL|\pi)}{2} = 0.$

Proof. From (**??**), $q(\theta_A\theta_B|\pi) = q(\theta'_A\theta'_B|\pi)$ if and only if $\left|\mu(\theta_A\theta_B|\pi) - \frac{1}{2}\right| = \left|\mu(\theta'_A\theta'_B|\pi) - \frac{1}{2}\right|$. We observe that

$$\begin{aligned} & \frac{\left(\mu(RR|\pi) - \frac{1}{2}\right) - \left(\frac{1}{2} - \mu(LL|\pi)\right)}{2} \\ = & \frac{\mu(RR|\pi) + \mu(LL|\pi)}{2} - \frac{1}{2} \\ = & \alpha\frac{\gamma_A(R|\pi) + \gamma_A(L|\pi)}{2} + (1-\alpha)\frac{\gamma_B(R|\pi) + \gamma_B(L|\pi)}{2} - \frac{1}{2} \\ = & \frac{\mu(RL|\pi) + \mu(LR|\pi)}{2} - \frac{1}{2} \\ = & \frac{\left(\mu(RL|\pi) - \frac{1}{2}\right) - \left(\frac{1}{2} - \mu(LR|\pi)\right)}{2} \end{aligned}$$

All claims then follow. ■

We can thus define $\overline{\mu}(\pi) = \frac{\mu(RR|\pi)+\mu(LL|\pi)}{2}$. Then $\overline{\mu}(\pi) = \frac{\mu(RL|\pi)+\mu(LR|\pi)}{2}$. It is useful to see that

$$\begin{aligned} & \frac{\mu(RR|\pi) + \mu(LL|\pi)}{2} \\ = & \alpha\overline{\pi}_A + (1-\alpha)\overline{\pi}_B + \alpha\frac{\delta_A^R - \delta_A^L}{2}\Delta\pi_A + (1-\alpha)\frac{\delta_B^R - \delta_B^L}{2}\Delta\pi_B, \end{aligned}$$

We give an expression of the cross-issue negative correlation under exogenous information source. We observe that

$$\begin{aligned} & \mu(LR|\pi) \\ = & \alpha\gamma_A(L) + (1-\alpha)\gamma_B(R) \\ = & \alpha\frac{\gamma_A(L) + \gamma_A(R)}{2} + (1-\alpha)\frac{\gamma_B(L) + \gamma_B(R)}{2} \\ & -\alpha\frac{\gamma_A(R) - \gamma_A(L)}{2} + (1-\alpha)\frac{\gamma_B(R) - \gamma_B(L)}{2} \\ = & \frac{\mu(RR|\pi) + \mu(LL|\pi)}{2} - \alpha\overline{\delta}_A\Delta\pi_A + (1-\alpha)\overline{\delta}_B\Delta\pi_B\,. \end{aligned}$$

Similarly, for any $(\theta_A, \theta_B) \in \{L, R\}^2$,

$$\begin{aligned} \mu(\theta_A\theta_B|\pi) = & \frac{\mu(RR|\pi) + \mu(LL|\pi)}{2} & (10) \\ & +\alpha\overline{\delta}_A\Delta\pi_A\left(\mathbf{1}_{\{\theta_A=R\}} - \mathbf{1}_{\{\theta_A=L\}}\right) + (1-\alpha)\overline{\delta}_B\Delta\pi_B\left(\mathbf{1}_{\{\theta_B=R\}} - \mathbf{1}_{\{\theta_B=L\}}\right) & (11) \end{aligned}$$

In the case where $\frac{\mu(RR|\pi)+\mu(LL|\pi)}{2}=\frac{1}{2}$, cross-issue negative correlation ratio simplifies to

$$\begin{aligned}\varrho &= \frac{q\left(LR|\pi\right)}{q\left(RR|\pi\right)} \\ &= \left[\frac{\frac{1}{4}-\left(-\alpha\overline{\delta}_A\Delta\pi_A+\left(1-\alpha\right)\overline{\delta}_B\Delta\pi_B\right)^2}{\frac{1}{4}-\left(\alpha\overline{\delta}_A\Delta\pi_A+\left(1-\alpha\right)\overline{\delta}_B\Delta\pi_B\right)^2}\right]^n \end{aligned} \tag{12}$$

Define a *symmetric equilibrium to be* an equilibrium in which every voter uses the same voting strategy $\pi^*=(\pi_A^*,\pi_B^*)$.

Lemma 17 **??**$|\Delta\pi_A^*|+|\Delta\pi_B^*|>0.$

Proof. Suppose to the contrary that $\Delta\pi_A^*=0=\Delta\pi_B^*$. Then $q\left(RR|\pi^*\right)=q\left(LL|\pi^*\right)=q\left(RL|\pi^*\right)=q\left(LR|\pi^*\right)$. Then $V_\emptyset\left(\pi^*\right)=0$ and $V_A\left(\pi^*\right)>0$. Thus it is optimal for those receiving signals on issue A to vote according to signal, contradiction. ■

Lemma 18 *Either* $0=V_\emptyset\left(\pi^*\right)=\left(\overline{\mu}\left(\pi^*\right)-\frac{1}{2}\right)$ *or* $\overline{\mu}\left(\pi^*\right)-\frac{1}{2}>0$ *and* $V_\emptyset\left(\pi^*\right)<0$ *or* $\overline{\mu}\left(\pi^*\right)-\frac{1}{2}<0$ *and* $V_\emptyset\left(\pi^*\right)>0$.

Proof. Consider $\overline{\mu}\left(\pi^*\right)-\frac{1}{2}=0$. Then $q\left(RR|\pi^*\right)=q\left(LL|\pi^*\right)$ and $q\left(RL|\pi^*\right)=q\left(LR|\pi^*\right)$ by Lemma 15. So $V_\emptyset\left(\pi^*\right)=0$.

Consider $\overline{\mu}\left(\pi^*\right)-\frac{1}{2}>0$. We first observe that

$$\begin{aligned} & \frac{\frac{1}{4}-\left(\alpha\overline{\delta}_A\left|\Delta\pi_A\right|+\left(1-\alpha\right)\overline{\delta}_B\left|\Delta\pi_B\right|+\overline{\mu}\left(\pi^*\right)-\frac{1}{2}\right)^2}{\frac{1}{4}-\left(-\alpha\overline{\delta}_A\left|\Delta\pi_A\right|-\left(1-\alpha\right)\overline{\delta}_B\left|\Delta\pi_B\right|+\overline{\mu}\left(\pi^*\right)-\frac{1}{2}\right)^2} \\ \leq\ & \frac{\frac{1}{4}-\left(\left|\alpha\overline{\delta}_A\left|\Delta\pi_A\right|-\left(1-\alpha\right)\overline{\delta}_B\left|\Delta\pi_B\right|\right|+\overline{\mu}\left(\pi^*\right)-\frac{1}{2}\right)^2}{\frac{1}{4}-\left(-\left|\alpha\overline{\delta}_A\Delta\pi_A-\left(1-\alpha\right)\overline{\delta}_B\left|\Delta\pi_B\right|\right|+\overline{\mu}\left(\pi^*\right)-\frac{1}{2}\right)^2} && (13) \\ \leq\ & 1. && (14) \end{aligned}$$

because $\frac{\frac{1}{4}-(x+c)^2}{\frac{1}{4}-(-x+c)^2}$ is strictly decreasing in x for $c \in (0,1]$. Note that

$$\begin{aligned}
& \frac{d}{dx}\frac{\frac{1}{4}-(x+c)^2}{\frac{1}{4}-(-x+c)^2} \\
= & \frac{-2(x+c)\left(\frac{1}{4}-(-x+c)^2\right)-(-2)(-x+c)(-1)\left(\frac{1}{4}-(x+c)^2\right)}{\frac{1}{4}-(-x+c)^2} \\
= & \frac{\frac{1}{4}(-2(x+c)-2(-x+c))+2(x+c)(-x+c)[(-x+c)+(x+c)]}{\frac{1}{4}-(-x+c)^2} \\
= & \frac{-4c\left[\frac{1}{4}+x^2-c^2\right]}{\frac{1}{4}-(-x+c)^2}
\end{aligned}$$

has the same sign as $-c$ if $|c| \leq \frac{1}{2}$. Moreover, either (13) or (14) holds with strict inequality because $\alpha\overline{\delta}_A |\Delta\pi_A| + (1-\alpha)\overline{\delta}_B |\Delta\pi_B| > 0$ by Lemma **??**. In addition, (13) holds with strict inequality if $\Delta\pi_B^* \neq 0$.

Consider $\Delta\pi_B^* \geq 0$. Then $\frac{q(RR|\pi^*)}{q(LL|\pi^*)} = \left(\frac{\frac{1}{4}-\left(\alpha\overline{\delta}_A|\Delta\pi_A|+(1-\alpha)\overline{\delta}_B|\Delta\pi_B|+\overline{\mu}(\pi^*)-\frac{1}{2}\right)^2}{\frac{1}{4}-\left(-\alpha\overline{\delta}_A|\Delta\pi_A|-(1-\alpha)\overline{\delta}_B|\Delta\pi_B|+\overline{\mu}(\pi^*)-\frac{1}{2}\right)^2}\right)^n < 1$ since $\Delta\pi_A^* \geq 0$. If $\frac{q(RL|\pi^*)}{q(LR|\pi^*)} \leq 1$, then $V_\emptyset(\pi^*) < 0$ and we are done. Suppose $\frac{q(RL|\pi^*)}{q(LR|\pi^*)} > 1$, then $\frac{q(LR|\pi^*)}{q(RL|\pi^*)} = \frac{\frac{1}{4}-\left(\left|\alpha\overline{\delta}_A|\Delta\pi_A|-(1-\alpha)\overline{\delta}_B|\Delta\pi_B|\right|+\overline{\mu}(\pi^*)-\frac{1}{2}\right)^2}{\frac{1}{4}-\left(-\left|\alpha\overline{\delta}_A\Delta\pi_A-(1-\alpha)\overline{\delta}_B|\Delta\pi_B|\right|+\overline{\mu}(\pi^*)-\frac{1}{2}\right)^2}$ because $1 \geq \frac{\frac{1}{4}-\left(\left|\alpha\overline{\delta}_A|\Delta\pi_A|-(1-\alpha)\overline{\delta}_B|\Delta\pi_B|\right|+\overline{\mu}(\pi^*)-\frac{1}{2}\right)^2}{\frac{1}{4}-\left(-\left|\alpha\overline{\delta}_A\Delta\pi_A-(1-\alpha)\overline{\delta}_B|\Delta\pi_B|\right|+\overline{\mu}(\pi^*)-\frac{1}{2}\right)^2}$. Thus $\alpha\overline{\delta}_A\Delta\pi_A-(1-\alpha)\overline{\delta}_B\Delta\pi_B < 0$. So

$\Delta\pi_B > 0$ and $\frac{q(RR|\pi^*)}{q(LL|\pi^*)} < \frac{q(LR|\pi^*)}{q(RL|\pi^*)} < 1$. Then

$$\begin{aligned}
& V_\emptyset(\pi^*) \\
= \; & (w+1)(q(RR|\pi^*) - q(LL|\pi^*)) + (w-1)(q(RL|\pi^*) - q(LR|\pi^*)) \\
= \; & (w+1)\frac{\frac{q(RR|\pi^*)}{q(LL|\pi^*)} - 1}{\delta_B^R \frac{q(RR|\pi^*)}{q(LL|\pi^*)} + \delta_B^L}\left(\delta_B^R q(RR|\pi^*) + \delta_B^L q(LL|\pi^*)\right) \\
& + (w-1)\left(\frac{1 - \frac{q(LR|\pi^*)}{q(RL|\pi^*)}}{\delta_B^L + \delta_B^R \frac{q(LR|\pi^*)}{q(RL|\pi^*)}}\right)\left(\delta_B^L q(RL|\pi^*) + \delta_B^R q(LR|\pi^*)\right) \\
< \; & (w+1)\frac{\frac{q(LR|\pi^*)}{q(RL|\pi^*)} - 1}{\delta_B^R \frac{q(LR|\pi^*)}{q(RL|\pi^*)} + \delta_B^L}\left(\delta_B^R q(RR|\pi^*) + \delta_B^L q(LL|\pi^*)\right) \\
& + (w-1)\left(\frac{1 - \frac{q(LR|\pi^*)}{q(RL|\pi^*)}}{\delta_B^L + \delta_B^R \frac{q(LR|\pi^*)}{q(RL|\pi^*)}}\right)\left(\delta_B^L q(RL|\pi^*) + \delta_B^R q(LR|\pi^*)\right) \\
= \; & \frac{1 - \frac{q(LR|\pi^*)}{q(RL|\pi^*)}}{\delta_B^L + \delta_B^R \frac{q(LR|\pi^*)}{q(RL|\pi^*)}}\left[-(w+1)\left(\delta_B^R q(RR|\pi^*) + \delta_B^L q(LL|\pi^*)\right) + (w-1)\left(\delta_B^L q(RL|\pi^*) + \delta_B^R q(LR|\pi
\end{aligned}$$

$$= -\frac{1 - \frac{q(LR|\pi^*)}{q(RL|\pi^*)}}{1 + \frac{q(LR|\pi^*)}{q(RL|\pi^*)}} V_B(\pi^*)$$

where the strict inequality holds because $\frac{q(RR|\pi^*)}{q(LL|\pi^*)} < \frac{q(LR|\pi^*)}{q(RL|\pi^*)}$. Since $\Delta\pi_B^* > 0$, $\pi_B^*(r) > 0$ and $\pi_B^*(l) < 1$. For π_B^* to be part of voter 0's best response, we must have $V_\emptyset(\pi^*) + V_B(\pi^*) \geq 0$ and $V_\emptyset(\pi^*) - V_B(\pi^*) \leq 0$. So $V_B(\pi^*) \geq 0$. Since $\frac{q(LR|\pi^*)}{q(RL|\pi^*)} < 1$, it follows that $V_\emptyset(\pi^*) < -\frac{1-\frac{q(LR|\pi^*)}{q(RL|\pi^*)}}{1+\frac{q(LR|\pi^*)}{q(RL|\pi^*)}} V_B(\pi^*) \leq 0$.

Consider $\pi_B^* < 0$. Then, the previous inequalities apply with $q(\theta_A R|\pi^*)$

and $q(\theta_A L|\pi^*)$ reversed. So $\frac{q(RL|\pi^*)}{q(LR|\pi^*)} \leq \frac{q(LL|\pi^*)}{q(RR|\pi^*)} < 1$. Then,

$$\begin{aligned}
& V_\emptyset(\pi^*) \\
= \;& (w+1)\left(q(RR|\pi^*) - q(LL|\pi^*)\right) + (w-1)\left(q(RL|\pi^*) - q(LR|\pi^*)\right) \\
= \;& (w+1)\frac{\frac{q(RR|\pi^*)}{q(LL|\pi^*)} - 1}{\delta_B^R \frac{q(RR|\pi^*)}{q(LL|\pi^*)} + \delta_B^L}\left(\delta_B^R q(RR|\pi^*) + \delta_B^L q(LL|\pi^*)\right) \\
& + (w-1)\left(\frac{1 - \frac{q(LR|\pi^*)}{q(RL|\pi^*)}}{\delta_B^L + \delta_B^R \frac{q(LR|\pi^*)}{q(RL|\pi^*)}}\right)\left(\delta_B^L q(RL|\pi^*) + \delta_B^R q(LR|\pi^*)\right) \\
< \;& (w+1)\frac{\frac{q(LR|\pi^*)}{q(RL|\pi^*)} - 1}{\delta_B^R \frac{q(LR|\pi^*)}{q(RL|\pi^*)} + \delta_B^L}\left(\delta_B^R q(RR|\pi^*) + \delta_B^L q(LL|\pi^*)\right) \\
& + (w-1)\left(\frac{1 - \frac{q(LR|\pi^*)}{q(RL|\pi^*)}}{1 + \frac{q(LR|\pi^*)}{q(RL|\pi^*)}}\right)\left(\delta_B^L q(RL|\pi^*) + \delta_B^R q(LR|\pi^*)\right) \\
= \;& \frac{\frac{q(LR|\pi^*)}{q(RL|\pi^*)} - 1}{\delta_B^R \frac{q(LR|\pi^*)}{q(RL|\pi^*)} + \delta_B^L}\left[(w+1)\left(\delta_B^R q(RR|\pi^*) + \delta_B^L q(LL|\pi^*)\right) - (w-1)\left(\delta_B^L q(RL|\pi^*) + \delta_B^R q(LR|\pi^*\right.\right. \\
= \;& \frac{\frac{q(LR|\pi^*)}{q(RL|\pi^*)} - 1}{\delta_B^R \frac{q(LR|\pi^*)}{q(RL|\pi^*)} + \delta_B^L} V_B(\pi^*).
\end{aligned}$$

So $V_\emptyset(\pi^*) - V_B(\pi^*) < \left(1 - \frac{\frac{q(LR|\pi^*)}{q(RL|\pi^*)} + 1}{\frac{q(LR|\pi^*)}{q(RL|\pi^*)} - 1}\right) V_\emptyset(\pi^*) = -\frac{2}{\frac{q(LR|\pi^*)}{q(RL|\pi^*)} - 1} V_\emptyset(\pi^*)$. Since $\Delta\pi_B^* < 0$, we have $\pi_B^*(r) < 1$ and $\pi_B^*(l) > 0$. For π_B^* to be part of voter 0's best response, $V_\emptyset(\pi^*) + V_B(\pi^*) \leq 0$ and $V_\emptyset(\pi^*) - V_B(\pi^*) \geq 0$. So $V_B(p^*) \leq 0$. Since $\frac{q(LR|\pi^*)}{q(RL|\pi^*)} > 1$, $V_\emptyset(\pi^*) < \frac{\frac{q(LR|\pi^*)}{q(RL|\pi^*)} - 1}{\delta_B^R \frac{q(LR|\pi^*)}{q(RL|\pi^*)} + \delta_B^L} V_B(\pi^*) \leq 0$.

The case where $\overline{\mu}(\pi^*) - \frac{1}{2} < 0$ is analogous. ■

Define the cross-issue negative correlation when $(\pi_A, \pi_B) = ((0,1),(0,1))$ to be ϱ_0. Then

$$\varrho_0 = \left[\frac{\frac{1}{4} - \left(-\alpha\overline{\delta}_A + (1-\alpha)\overline{\delta}_B\right)^2}{\frac{1}{4} - \left(\alpha\overline{\delta}_A + (1-\alpha)\overline{\delta}_B\right)^2}\right]^n.$$

Proposition 19 *Suppose $\delta_k^R = \delta_k^L$ for $k = A, B$. If $\varrho_0 \leq \frac{w+1}{w-1}$, then in the unique symmetric equilibrium, each voter votes according to signal on each signal. If $\varrho_0 > \frac{w+1}{w-1}$, in the unique symmetric equilibrium, a voter votes* *according to signal if the signal is on issue A, whereas a voter is indifferent between both candidates if his signal is on issue B. In addition, if n is large enough, each voter is more likely to vote for the candidate better overall, and thus equilibrium is asymptotically efficient.*

Proof. We first show the necessary conditions of an equilibrium:

1. $\pi_A^*(r) = 1$ and $\pi_A^*(l) = 0$.
2. $\overline{\pi}_B^* = \frac{1}{2}$ and thus $\overline{\mu}(\pi^*) = \frac{1}{2}$.

Suppose to the contrary that $(\pi_A^*(l), \pi_A^*(r)) \neq (0,1)$. Then, by Lemma **??**, either $\pi_A^*(r) = 1 \geq \pi_A^*(l) > 0$ or $1 > \pi_A^*(r) \geq \pi_A^*(l) = 0$. It is w.l.o.g. to just consider the former case. For $\pi_A^*(l) > 0$ to be voter 0's best response, voter 0 must weakly prefer candidate R after receiving signal $s_A = l$ on issue A. So $V_\emptyset(\pi^*) - V_A(\pi^*) \geq 0$. Since $V_A(\pi) > 0$ for any π, we have $V_\emptyset(\pi^*) > 0$. Then $\overline{\pi}_B \geq \frac{1}{2}$ by Lemma 18. By hypothesis, $\overline{\pi}_A > \frac{1}{2}$. So $0 < \overline{\pi} - \frac{1}{2} = \overline{\mu}(\pi^*) - \frac{1}{2}$ where the equality holds by symmetry. Contradiction to Lemma 18.

So $\overline{\pi}_A^* = \frac{1}{2}$. If $\overline{\pi}_B^* > \frac{1}{2}$, then $\overline{\mu} > \frac{1}{2}$. Then $V_\emptyset(\pi^*) < 0$ by Lemma 18. Suppose to the contrary that $\overline{\pi}_B^* \neq \frac{1}{2}$. W.l.o.g. consider $\overline{\pi}_B^* > \frac{1}{2}$. Since $\overline{\pi}_A^* > \frac{1}{2}$, then $\overline{\mu}(\pi^*) = \overline{\pi}^* > \frac{1}{2}$. Then $V_\emptyset(\pi^*) < 0$ by Lemma 18. Then $\min\{\pi_B^*(r), \pi_B^*(l)\} = 0$ by Lemma **??**. So $\overline{\pi}_B^* < \frac{1}{2}$. Contradiction to they hypothesis that $\overline{\pi}_B^* > \frac{1}{2}$. Analogously, we can obtain a contradiction for the hypothesis that $\overline{\pi}_B^* < \frac{1}{2}$. So $\overline{\pi}_B^* = \frac{1}{2}$ and thus $\overline{\mu}(\pi^*) = \frac{1}{2}$. Then $V_\emptyset(\pi^*) = 0$ by Lemma 18.

We now pin down π_B^*.

Since $\overline{\mu}(\pi^*) = \frac{1}{2}$ and by symmetry in information structure within an issue, $q(RL|\pi^*) = q(LR|\pi^*)$ and $q(RR|\pi^*) = q(LL|\pi^*)$. So

$$\varrho^{\frac{1}{n}} = \left(\frac{q(RL|\pi^*)}{q(RR|\pi^*)}\right)^{\frac{1}{n}} = \frac{\frac{1}{4} - (\alpha\delta_A - (1-\alpha)\delta_B \Delta\pi_B^*)^2}{\frac{1}{4} - (\alpha\delta_A + (1-\alpha)\delta_B \Delta\pi_B^*)^2}. \tag{15}$$

We first argue that $\Delta\pi_B^* > 0$. Suppose to the contrary that $\Delta\pi_B^* \leq 0$. Then $\varrho \leq 1$, but then $V_B(\pi^*) > 0$ and thus voter 0's best response must satisfy $\pi_B^*(1) = 1$ and $\pi_B^*(0) = 0$. So $\Delta\pi_B^* > 0$, contradiction.

Since $\delta_A > 0$ and $\delta_B > 0$, the RHS of (15) increases stritly with $\Delta\pi_B$ and is equal to 1 for $\Delta\pi_B = 0$. So $\Delta\pi_B^* > 0$ because otherwise, $\varrho \leq 1$ but then $V_B(\pi^*) > 0$ and thus $(\pi_B^*(0), \pi_B^*(1)) = (0,1)$ since $V_\emptyset(\pi^*) = 0$. Contradiction.

Consider the case where $\varrho_0 \le \frac{w+1}{w-1}$. Then the RHS of (15) is no bigger than $\left(\frac{w+1}{w-1}\right)^{\frac{1}{n+1}}$ for all $\Delta\pi_B \in [-1,1]$, and voter 0's best response to any symmetric strategy profile must involve $\pi_B(r) = 1$ and $\pi_B(l) = 0$. So $\pi_B^*(r) = 1$ and $\pi_B^*(l) = 0$ in the unique equilibrium. Conversely, it is an equilibrium if every voter uses π^* where $\pi_A^*(l) = 0, \pi_A^*(r) = 1, \pi_B^*(r) = 0$ and $\pi_B^*(l) = 1$. Since $\overline{\mu}(\pi^*) = \overline{\pi}^* = \frac{1}{2}$, we have $V_\emptyset(\pi^*) = 0$. Then $\pi_A^* = (0,1)$ is optimal for voter 0 after receiving a signal on issue A. Since $\varrho = \varrho_0 \le \frac{w+1}{w-1}$, $V_B^* \ge 0$. Since $V_\emptyset(\pi^*) = 0$, $\pi_B^* = (0,1)$ is part of voter 0's best response.

Consider the case where $\varrho_0 > \frac{w+1}{w-1}$. Then there exists a unique $\Delta^* \in (0,1)$ such that (15)$= \frac{w+1}{w-1}$ at $\Delta\pi_B = \Delta^*$. We show that $\Delta\pi_B^* = \Delta^*$. If $\Delta\pi_B^* > \Delta^*$, then $\varrho > \frac{w+1}{w-1}$ and thus $V_B(\pi^*) < 0$. Since $V_\emptyset(\pi^*) = 0$, for π^* to be voter 0's best response, we must have $\pi_B^*(0) = 1$ and $\pi_B^*(1) = 0$, contradiction. If $\Delta\pi_B^* < \Delta^*$, then $\varrho < \frac{w+1}{w-1}$ and thus $V_B(\pi^*) > 0$. Since $V_\emptyset(\pi^*) = 0$, for π^* to be voter 0's best response, we must have $\pi_B^*(0) = 0$ and $\pi_B^*(1) = 1$, contradiction.

So $\Delta\pi_B^* = \Delta^*$. Since $\overline{\pi}_B^* = \frac{1}{2}$, we must have $\pi_B^*(r) = \frac{1}{2} + \frac{\Delta^*}{2}$ and $\pi_B^*(l) = \frac{1}{2} - \frac{\Delta^*}{2}$.

We now show that it is an equilibrium if every voter uses π^* where $\pi_A^*(l) = 0, \pi_A^*(r) = 1, \pi_B^*(r) = \frac{1}{2} + \frac{\Delta^*}{2}$ and $\pi_B^*(l) = \frac{1}{2} - \frac{\Delta^*}{2}$. Since $\overline{\mu}(\pi^*) = \overline{\pi}^* = \frac{1}{2}$, we have $V_\emptyset(\pi^*) = 0$. Then $\pi_A^* = (0,1)$ is optimal for voter 0 after receiving a signal on issue A. Since $\Delta\pi_B^* = \Delta^*$, $\varrho = \frac{w+1}{w-1}$ by construction. Thus $V_B(\pi^*) = 0$. So any $\pi_B \in [0,1]^2$ is part of voter 0's best response. So π_B^* is part of voter 0's best response.

As $n \to \infty$, $\left(\frac{w+1}{w-1}\right)^{\frac{1}{n}} \to 1^+$. Thus, for large enough n, $\varrho_0 > \frac{w+1}{w-1}$, and $\Delta^* \to 0^+$. So $\Delta\pi_B^* \to 0^+$. So $\mu(RL|\pi^*) = \frac{1}{2} + \alpha\delta_A - (1-\alpha)\delta_B\Delta^* \to \frac{1}{2} + \alpha\delta_A > \frac{1}{2}$ and $\mu(LR|\pi^*) = \frac{1}{2} - \alpha\delta_A + (1-\alpha)\delta_B\Delta^* \to \frac{1}{2} - \alpha\delta_A < \frac{1}{2}$. It is straightforward that $\mu(RR|\pi^*) \to \frac{1}{2} + \alpha\delta_A > \frac{1}{2}$ and $\mu(LL|\pi^*) \to \frac{1}{2} - \alpha\delta_A < \frac{1}{2}$. So the equilibrium outcome is asymptotically efficient. ■

Proposition 20 *If $\left|\delta_A^R - \delta_A^L\right| < \frac{1-\alpha}{\alpha}$, then for n large enough, there exists an equilibrium where $(\pi_A(l), \pi_A(r)) = (0,1)$ and a voter is indifferent between candidate L and R upon receiving both $s_B = l$ and r. If this equilibrium is played for all n large enough and if $\alpha > 0$, then the probability the optimal candidate is elected goes to 1 as $n \to \infty$.*

Proof. From (**??**) and (**??**), we can see that voter 0 is indifferent between candidate L and R upon receiving both $s_B = l$ and r if and only if

$$[(w+1)(q(RR|\pi) - q(LL|\pi)) + (w-1)(q(RL|\pi) - q(LR|\pi))] = 0 \quad (16)$$

$$(w+1)\left[\delta_B^R q(RR|\pi) + \delta_B^L q(LL|\pi)\right] - (w-1)\left[\delta_B^L q(RL|\pi) + \delta_B^R q(LR|\pi)\right] = 0. \quad (17)$$

The following two equations are sufficient:

$$\begin{cases} q(RR|\pi) = q(LL|\pi) \\ \frac{q(LR|\pi)}{q(RR|\pi)} = \frac{w+1}{w-1} \end{cases} \quad (18)$$

because (16) holds if $q(RR|\pi) = q(LL|\pi)$ by Claim **??** and (17) hold if (18) holds.

Suppose $(\pi_A(l), \pi_A(r)) = (0,1)$. By (12),

$$\begin{aligned} \frac{q(LR|\pi)}{q(RR|\pi)} &= \left[\frac{\frac{1}{4} - \left(-\frac{\alpha}{2}\left(\delta_A^L + \delta_A^R\right) + \frac{1-\alpha}{2}\left(\delta_B^L + \delta_B^R\right)(\pi_B(r) - \pi_B(l))\right)^2}{\frac{1}{4} - \left(\frac{\alpha}{2}\left(\delta_A^L + \delta_A^R\right) + \frac{1-\alpha}{2}\left(\delta_B^L + \delta_B^R\right)(\pi_B(r) - \pi_B(l))\right)^2}\right]^n \\ &= \left[\frac{\frac{1}{4} - \left(\frac{\alpha}{2}\left(\delta_A^L + \delta_A^R\right) - \frac{1-\alpha}{2}\left(\delta_B^L + \delta_B^R\right)(\pi_B(r) - \pi_B(l))\right)^2}{\frac{1}{4} - \left(\frac{\alpha}{2}\left(\delta_A^L + \delta_A^R\right) + \frac{1-\alpha}{2}\left(\delta_B^L + \delta_B^R\right)(\pi_B(r) - \pi_B(l))\right)^2}\right]^n. \end{aligned}$$

Note that $\frac{\frac{1}{4} - \left(\frac{\alpha}{2}\left(\delta_A^L + \delta_A^R\right) - \Delta\right)^2}{\frac{1}{4} - \left(\frac{\alpha}{2}\left(\delta_A^L + \delta_A^R\right) + \Delta\right)^2}$ is equal to 1 for $\Delta = 0$, goes to ∞ as $\Delta \to \left(\frac{1}{2} - \frac{\alpha}{2}\left(\delta_A^L + \delta_A^R\right)\right)^-$, and is continuous and strictly increasing in Δ for $|\Delta| < \frac{1}{2} - \frac{\alpha}{2}\left(\delta_A^L + \delta_A^R\right)$ because

$$\begin{aligned} & \frac{d \log \frac{\frac{1}{4} - \left(\frac{\alpha}{2}\left(\delta_A^L + \delta_A^R\right) - \Delta\right)^2}{\frac{1}{4} - \left(\frac{\alpha}{2}\left(\delta_A^L + \delta_A^R\right) + \Delta\right)^2}}{d\Delta} \\ = & \frac{2\left(\frac{\alpha}{2}\left(\delta_A^L + \delta_A^R\right) - \Delta\right)}{\frac{1}{4} - \left(\frac{\alpha}{2}\left(\delta_A^L + \delta_A^R\right) - \Delta\right)^2} + \frac{2\left(\frac{\alpha}{2}\left(\delta_A^L + \delta_A^R\right) + \Delta\right)}{\frac{1}{4} - \left(\frac{\alpha}{2}\left(\delta_A^L + \delta_A^R\right) + \Delta\right)^2} \\ = & \alpha\left(\delta_A^L + \delta_A^R\right)\left(\frac{1}{\frac{1}{4} - \left(\frac{\alpha}{2}\left(\delta_A^L + \delta_A^R\right) - \Delta\right)^2} + \frac{1}{\frac{1}{4} - \left(\frac{\alpha}{2}\left(\delta_A^L + \delta_A^R\right) + \Delta\right)^2}\right) \\ & +2\Delta\left(-\frac{1}{\frac{1}{4} - \left(\frac{\alpha}{2}\left(\delta_A^L + \delta_A^R\right) - \Delta\right)^2} + \frac{1}{\frac{1}{4} - \left(\frac{\alpha}{2}\left(\delta_A^L + \delta_A^R\right) + \Delta\right)^2}\right) \\ > & 0 \end{aligned}$$

where the inequality holds because

$$\left(\frac{1}{4}-\left(\frac{\alpha}{2}\left(\delta_A^L+\delta_A^R\right)-\Delta\right)^2\right)-\left(\frac{1}{4}-\left(\frac{\alpha}{2}\left(\delta_A^L+\delta_A^R\right)+\Delta\right)^2\right)=\alpha\left(\delta_A^L+\delta_A^R\right)\Delta>0$$

Therefore, there exists $\Delta^*(n)>0$ such that

$$\frac{\frac{1}{4}-\left(\frac{\alpha}{2}\left(\delta_A^L+\delta_A^R\right)-\Delta^*(n)\right)^2}{\frac{1}{4}-\left(\frac{\alpha}{2}\left(\delta_A^L+\delta_A^R\right)+\Delta^*(n)\right)^2}=\left(\frac{w+1}{w-1}\right)^{\frac{1}{n}}.$$

It follows that

$$\frac{q(LR|\pi)}{q(RR|\pi)}=\frac{w+1}{w-1}$$

if $(\pi_A(l),\pi_A(r))=(0,1)$ and

$$\frac{1-\alpha}{2}\left(\delta_B^L+\delta_B^R\right)\left(\pi_B(r)-\pi_B(l)\right)=\Delta^*(n). \tag{19}$$

We now show that, for n large enough, there exists (π_A,π_B) where $(\pi_A(l),\pi_A(r))=(0,1)$ and $(\pi_B(l),\pi_B(r))\in[0,1]^2$ such that (18) holds. So, given that $(\pi_A(l),\pi_A(r))=(0,1)$ and π_B satisfies (19), by Claim **??**, $q(RR|\pi)=q(LL|\pi)$ iff

$$0=\alpha\left(\delta_A^R-\delta_A^L\right)+(1-\alpha)\left(\pi_B(r)+\pi_B(l)-1\right)+2\frac{\delta_B^R-\delta_B^L}{\delta_B^R+\delta_B^L}\Delta^*(n) \tag{20}$$

By assumption $\alpha\left(\delta_A^R-\delta_A^L\right)-(1-\alpha)<0<\alpha\left(\delta_A^R-\delta_A^L\right)+(1-\alpha)$. So, for n small enough, $\Delta^*(n)$ is close enough to 0 and thus there exists $(\pi_B^*(l),\pi_B^*(r))\in[0,1]^2$ that simultaneously satisfy (19) and (20).

Then, given $(\pi_A(l),\pi_A(r))=(0,1)$ and $\pi_B=\pi_B^*$, we have $q(RR|\pi)=q(LL|\pi)$ and $\frac{q(LR|\pi)}{q(RR|\pi)}=\frac{w+1}{w-1}$. Thus, given that all other voters vote according to π, voter 0 is indifferent between candidate R and L upone receiving any signal on issue B. Moreover, because $q(RR|\pi)=q(LL|\pi)$ and hence $q(RL|\pi)=q(LR|\pi)$ by Claim **??**, (**??**) and (**??**) are both equal to 0. Thus it is optimal for voter 0 to vote according to signal on issue A given that all other voters use strategy π.

We now show asymptotic efficiency for $\alpha>0$. Since $\Delta^*(n)>0$, $\mu(RR|\pi)>\frac{1}{2}>\mu(LL|\pi)$. Since $\Delta^*(n)\to 0$ as $n\to\infty$, $\mu(RL|\pi)>\frac{1}{2}>\mu(LR|\pi)$. Thus, in every state, each voter votes for the candidate better overall with probability converging to $\frac{1}{2}+\alpha\frac{\delta_A^R+\delta_A^L}{2}>\frac{1}{2}$. Thus the candidate better overall wins the election with probability converging to 1 as $n\to\infty$. ■

6 References

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