Uncertain Costs, Information Design, and Total Surplus*

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Abstract

The paper considers a monopoly uncertain about its costs, and studies the impact of cost-related information on the total surplus, defined as a weighted sum of consumer and producer surpluses. The effect of this information can be positive or negative, depending on the properties of the market demand. We provide the necessary and sufficient conditions on the demand under which: optimal information takes a form of lower censorship regardless of the prior distribution of costs and the weights in the total surplus; and full disclosure is optimal. Finally, we fully characterize the demand functions such that the total surplus is linear in cost, meaning that information about cost has no impact on the total surplus.

JEL classification: D4, D42, D82, D83

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1 Introduction

The main objective of this paper is to investigate the effect of imperfect information of the monopolistic seller about his own costs on the total surplus, i.e., the sum of consumer and producer surpluses with arbitrary weights. The positive effect of sellers' information on the efficiency of competitive markets has been established and investigated a long time ago. By definition, a market is perfectly competitive and maximizes social welfare only if all participants have perfect decision-relevant information. Specifically, this implies that sellers have perfect knowledge of their own costs. While this assumption is necessary in

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perfectly competitive markets, it also has been widely employed in imperfectly competitive markets, selling mechanisms, and auctions. Beginning with seminal works by Akerlof (1972) on selling products with asymmetric information about their quality, Myerson (1981) on optimal mechanism design, and Baron and Myerson (1982) on regulation of the monopolist with private costs, subsequent microeconomic literature has relied on the assumption that each seller perfectly knows his cost of producing the product.

In practice, however, sellers often have only limited information about their costs. For example, a market entrant can learn over time how to minimize costs by using the learning-by-doing process (Lucas, 1988; Yang and Borland, 1991). Similarly, releasing a new or updated product might involve a priori unknown cost. Also, production processes often depend on uncertain or volatile factors, such as weather, natural disasters, labor strikes, political turmoils, changes in the suppliers' networks, or fluctuations in input prices. At the same time, contractual obligations often require the sellers to produce and sell the product at the predetermined price before they infer the production costs.[1] In addition, the sellers' practice of rarely updating their prices—called the *price rigidity*—has been a well-known phenomenon in macroeconomics. In fact, one of the possible explanations for price rigidity is *sticky information*, that is, the tendency of sellers to behave on the basis of old information that does not take into account recent events.[2] Furthermore, recent empirical studies on algorithmic pricing provide the evidence that algorithm adoption allows firms to set prices more efficiently and responsively to changing market conditions compared to traditional price setting by managers.[3] This suggests that firm managers do not effectively use all available information when determine prices. Finally, pricing decisions of publicly traded companies can be affected by activist investors, who generally do not have access to inside information about companies' costs.

This discrepancy between the microeconomic paradigm on one side and the macroeconomic perspective and the empirical evidence on the other side about the precision of sellers' information about their costs and its use for price-setting raises several natural questions. First, what is the effect of the sellers' uncertainty about their costs on market efficiency? Second, how does this effect depend on market demand? Third, if the market regulators have cost-related information, should they provide this information to the sellers? If so, then what is the socially optimal information-disclosure policy?

In addressing the above questions, this paper revisits the role of sellers' perfect information regarding their own costs as a driving force of market efficiency. Specifically, it shows that in the standard monopolistic market with privately informed buyers, the social value of the seller's cost-related information can be negative: a less informed seller—and even completely uninformed sometimes—generates higher total gains from trade than the better informed seller would do. Perhaps surprisingly, the total surplus in markets with certain demand functions is maximized if the seller is completely uninformed about his costs, i.e., providing any cost-related information to the seller reduces the total surplus. Furthermore, even an insignificant update in the seller's information about costs can significantly reduce

[1] For example, Taylor (1980) proposes that firms are prevented from adjusting their prices until a pre-specified contract period expires.

[2] The theory of sticky information was introduced by Fischer (1977) and then intensively developed in the subsequent literature. A notable reference on the relationship between price rigidity and sticky information is Mankiw and Reis (2002).

[3] See, for instance, Calder-Wang and Kim (2023).

the total surplus. In markets with other demand functions, however, it is socially optimal to provide full information to sellers. Finally, sometimes it might be optimal to provide sellers with only partial information. The relationship between market demand, the seller's prior information about his cost, and optimal—from the total surplus's perspective—disclosure policies forms the central question of this paper.

In this light, our paper offers a novel tool for market regulation in addition to common regulatory policies—price caps, output quotas, or monetary transfers (e.g., industry-specific taxes or subsidies).[4] Specifically, it is the disclosure of cost-related information or controlling the seller's access to this information by a third party. As we show, control over this information might be a powerful tool that impacts the outcome and the total surplus generated by a market. Moreover, under the regularity conditions, the optimal information has a simple cutoff form—called lower censorship—meaning no disclosure below some cost threshold and full disclosure above it.

One example of the third party that can provide cost-related information is various government agencies that determine taxes, minimum wage rates, and other input prices (e.g., the electricity or natural gas rates), and, hence, have the information about upcoming changes in their values. In many countries, regional government regulators set the rates of the energy sources and provide information about them. For instance, "The Ontario Energy Board is the provincial regulator of natural gas and electricity utilities in Ontario, Canada. This includes setting rates and licensing all participants in the electricity sector... The Board also provides a broad range of information to energy consumers about electricity and natural gas in Ontario."[5] In addition, local authorities might have more expertise about some factors of production, say, the quality or quantity of natural resources used by the seller. At the same time, knowing this information by firms determines their current decisions regarding the technology, R&D investments, and non-flexible factors of production such as capital stock and land. Thus, the information affects their production costs and the prices.

An alternative example is the data provision by large online market platforms such as Amazon or eBay. These platforms collect a substantial amount of information not only about buyers' preferences, i.e., product valuations, but also about their habits and behavior, for instance, how often buyers return the product, flood sellers with questions about using, maintaining, or repairing the product, and write product reviews. These habits determine the final cost of providing the product, and knowing the information about buyers' habits might impact the seller's pricing decisions. At the same time, the platform's profits depend on the value of transaction fees that it collects from market participants, which stem from the total surplus generated by a market, i.e., the consumer and producer surpluses with some weights. Thus, disclosure of information about buyers' habits affects the platform's profit and, hence, the total surplus.

As a basic setup, we consider a standard model with a monopolistic seller and a continuum of buyers that are privately informed about their valuations of the product. The novelty of our model is the assumption that the seller is imperfectly informed not only about buyers' private valuations, but also about his own costs. Given available information, he updates the posterior mean value of the unit cost by Bayes rule, which is sufficient information

[4] The literature on regulating the monopolist with privately known costs started with a work by Baron and Myerson (1982) and was substantially developed since then. Other notable works include Baron and Besanko (1984), Laffont and Tirole (1986), Alonso and Matouschek (2008), and Amador and Bagwell (2013).

[5] https://en.wikipedia.org/wiki/Ontario_Energy_Board

for price-setting, and selects the optimal price. (Because the posterior mean of a cost is equivalent to the cost equal to this mean, we call it a *posterior cost* hereafter.) Also, there is an information designer, for instance, the market regulator, who has additional cost-related information and can costlessly disclose it to the seller. The goal of the information designer is to maximize the ex-ante total surplus by selecting any disclosure policy.

Before providing the main results, we briefly explain our approach. First, the classical approach in the economic literature considers the demand function represented by the distribution of valuations as a primitive.[6] At the same time, recent studies demonstrate that using other functions associated with demand can provide additional insights about the market outcome, structure, properties, and welfare.[7] Following the latter approach, the role of the primitive in our paper is played by the inverse hazard rate of the distribution of valuations, which has a clear economic meaning. Specifically, it represents the marginal decrease in the revenue, which is caused by a decrease in price in the case of producing an extra unit, and is expressed as a function of price. At the optimal price, it is equal to the profit margin (or the markup) of the monopoly and thus is directly related to the marginal revenue and the demand elasticity. As we demonstrate, the convex properties of the consumer and total surpluses are driven by shape of the inverse hazard rate. Noteworthy, deriving the underlying demand function from the inverse hazard rate is a simple task, since there is a simple one-to-one relationship between these functions.

The first set of results establishes the relationship between the market characteristics and the curvature of the posterior surpluses as functions of the posterior cost. (By the *curvature* of a function we imply the value of its second derivative, which determines the concavity or the convexity of the function.) The shape of the posterior surpluses determines what disclosure policy is locally optimal, since the convexity or the concavity of the posterior total surplus at some cost implies that the optimal disclosure policy is either full information or no information, respectively, regardless of the prior distribution of costs, as long as its support is in the neighborhood of the original cost. In this light, we first provide the necessary and sufficient conditions for the local convexity and concavity of the posterior consumer surplus. We express these conditions in terms of various market characteristics and provide three equivalent—but different from the economic perspective—characterizations. The first one employs only the properties of the optimal price as a function of the posterior cost. This interpretation is especially convenient for small costs: in this case it allows to relate the curvature of the posterior consumer surplus to the log-convexity of the optimal price in cost. The second characterization is based on the properties of the inverse hazard rate and the value of the optimal price. We use this characterization to provide several simple conditions, which guarantee that the posterior consumer surplus and, thus, the total posterior surplus is strictly convex if costs are sufficiently high. Intuitively, it is the case if the upper tail of

[6] Because the approach based on the direct analysis of demand starts with Adam Smith, a comprehensive list of papers that employ it would be excessively long. However, some seminal papers are Johnson and Myatt (2006), who use the rotations of the demand function to analyze the effects of advertising, mass-market and niche-market supply, and product design; and Aguirre et al. (2010), who show that the effects of third-degree price discrimination on welfare and output depend on the curvature of the demand.

[7] A notable work is Weyl and Fabinger (2013), who show that many results about the welfare effects of taxes can be obtained by analyzing the pass-through rate derived from the demand curve. Similarly, Mrázová and Neary (2017) establish that substituting the demand function with its "demand manifold", i.e., a curve that relates the elasticity and convexity of demand, opens the doors to numerous results on comparative statics questions, leads to new families of demand functions, and links demand structure to firm performance.

the distribution of consumers' valuations is thin or smooth enough. We also show how to 'concavify' the posterior consumer surplus around some cost by locally varying the shape of the inverse hazard rate at the optimal price for this cost. Finally, the third characterization utilizes the seller's market power and the concavity of the marginal revenue expressed as a function of price. According to this characterization, the posterior consumer surplus is concave in cost if and only if the monopoly has a sufficiently large market power and its marginal revenue is sufficiently concave in price.[8]

Next, we identify the necessary and sufficient conditions on the inverse hazard rate under which: (i) optimal disclosure takes a simple form of lower censorship; and (ii) full disclosure is optimal. (As a special case, we provide such conditions for the consumer posterior surplus only, which are less restrictive and easier to verify.) The first result says that it is optimal to provide full information above some cutoff cost and not provide any information otherwise. It is independent of the prior distribution of costs and the weights in the total surplus. The second result establishes when the cutoff is equal to the lowest cost, so that full disclosure is optimal. While it is also independent of the prior distribution of costs, it depends on the weights in the total surplus. This is because the posterior consumer surplus is concave in cost and its weight in the total surplus increases, then—which is another result we establish—it is optimal to provide less information to the seller.

Finally, we derive all benchmark demand functions, under which the posterior total surplus is linear in cost for any given weight of the producer surplus. This problem is quite involved and requires solving the non-linear second-order differential equation with respect to the inverse hazard rate under the regularity condition stemming from the uniqueness of the monopoly's optimal price. In order to solve this problem, we introduce a novel parametric representation, which has several economic interpretations. The derived solution establishes a new family of three-parameter distributions. We derive their properties and show that they are similar to those of other common distributions used to model personal and household incomes, such as Dagum and Burr ones, or returns for securities or assets, such as Lévy and inverse-gamma ones.[9] All these distributions are heavy-tailed, have the unimodal density, and the convex inverse hazard rate. At the same time, the inverse hazard rate is generally not decreasing as commonly assumed in the mechanism design literature (Myerson, 1981) and, hence, its density is not log-concave as typically imposed in the differentiated product markets (Ivanov, 2013). In addition, the distribution violates the Marshall's Second Law of Demand, i.e., the inverse relationship between the demand price elasticity and the price, which is commonly used in monopolistic competition (Kokovin et al., 2024). Finally, we provide the closed-form expressions for all major market characteristics—the optimal price and quantity produced, the price elasticity, the profit margin, the pass-through and the marginal pass-through rates, and the producer surplus—as functions of cost.

The linearity of the posterior total surplus in cost has two main implications: the information design and the policy ones. The first implication is that the ex-ante total surplus is invariant to the seller's information about cost. In other words, there are no social benefits or costs from disclosing the cost-related information to the seller. The second implication is the policy one. Consider the case of a gradual increase in unit cost caused, for example, by

[8] We use the standard measures of the market power: the profit margin and the Lerner index. The concavity of the marginal revenue is measured by Arrow-Pratt coefficients of absolute and relative risk-aversion.

[9] Kleiber and Kotz (2003) provide a comprehensive study of these distributions and their applications in economics and actuarial science.

an increase in the minimum wage rate.[10] If the unit cost increases over several periods, then the marginal impact of each wage increase by \$1 on the total surplus remains constant in all periods. In other words, the marginal effect of cost on the total surplus does not depend on the market outcome.

Literature. As a result of recent developments in information design, numerous works have investigated the role of imperfect and endogenously determined information in various models of markets and selling mechanisms. Most of this literature, however, focuses on the impact of information about buyers' valuations rather than sellers' costs on the consumer surplus, producer surplus, and social welfare. For example, Bergemann and Pesendorfer (2007) characterize the seller-optimal structure of buyers' information in selling mechanisms, while Bergemann et al. (2023) consider the same problem from the buyers' perspective. In the monopolist's model, Roesler and Szentes (2019) derive the buyer-optimal structure of the buyer's information. Bergemann et al. (2015) establish the relationship between buyer's information and the buyer's and seller's surpluses in the case of third-degree price discrimination by the monopolist. Lewis and Sappington (1994) and Johnson and Myatt (2006) study disclosure of product-relevant information to consumers by the monopolist. Ivanov (2013) and Hwang et al. (2019) investigate the same question in competitive markets with differentiated products, while Armstrong and Zhou (2021) study the optimal structure of buyers' information from the perspectives of sellers, buyers, and social welfare.

Notably, the effects of buyers' and seller's information on the total surplus in the monopolistic market are qualitatively different. The buyers' information impacts the distribution of their posterior valuations, which forms the market demand function. The demand, in turn, determines the seller's optimal outcome and the total surplus. In contrast, there is no supply function for the monopoly. Instead, the cost-related information induces the distribution over posterior costs. These costs determine the seller's optimal prices and, as a result, the distribution over market outcomes and the resulting posterior total surpluses. Thus, the impact of the seller's information on the ex-ante total surplus is determined by the shape of the posterior total surplus as a function of posterior cost. In turn, posterior costs have a dual effect on the shape of the posterior total surplus. First, they directly enter the seller's profit function and, hence, affect the posterior producer surplus. Second, they determine the seller's optimal prices, which impact both the posterior producer and consumer surpluses.[11]

However, the amount of literature that considers the imperfect information of sellers about their costs is rather limited. Baron and Besanko (1984) study a two-period model in which a regulator faces a monopolist with a privately known marginal cost in the first period but an unknown (to both parties) second-period cost. Laffont and Tirole (1986) consider the problem of regulating a monopolist who is better (but imperfectly) informed about his costs than the regulator. Similarly, Riordan and Sappington (1987) study the problem of awarding a franchise to producers with privately but imperfectly known marginal costs. Christen (2005) focus on the role of competition in acquisition of information about costs by sellers. Routledge and Edwards (2020) consider firms' pricing decisions in competitive markets with unknown costs and market demand. These works, however, do not investigate

[10] In Ontario, Canada, for example, the minimum hourly wage changed from \$14 to \$16.55 between 2018 and 2024 as a result of several increases.

[11] The second effect disappears if the weights of the consumer and producer surpluses in the total surplus are equal, since the price only redistributes the welfare. However, the effect is non-zero for other weights.

the socially optimal level of sellers' uncertainty about their costs and its relationship to the demand function. Recently, Kartik and Zhong (2024) consider the bilateral trade model in which the trade is always efficient ex-post, the buyer's and seller's values are functionally dependent, imperfectly known to the players, and endogenously determined by the information designer.[12] They characterize the set of players' payoff vectors across all their information structures.[13] Our model and the main economic questions are conceptually different. Specifically, we consider the model in which the seller is imperfectly informed about his costs and the buyers' valuations, buyers perfectly know their valuations, the distribution of the buyers' valuations is exogenous, costs and valuations are independent, and there are no limitations on the efficiency of ex-post trade. In this setup, we are interested in deriving the optimal—from the total surplus perspective—information structure of the seller about his costs only, and its relationship to the buyer's exogenous demand function. Finally, the two papers are methodologically distinct. The main results of Kartik and Zhong (2024) are based on the designer's full flexibility over buyer's information, which allows for demand functions with multiple optimal prices. As a result, seller's randomization over these prices (and the buyer's randomization over purchase decisions) play a key role in their characterization. In contrast, our model precludes such demand functions, and, thus, randomizing over prices. Similarly, buyers' randomization over purchase decisions does not play any role in our model. Due to these differences, the analysis and the construction of Kartik and Zhong (2024) are not applicable to our model.

The rest of the paper is organized as follows. Section 2 introduces the general framework. Section 3 provides an illustrative example. Section 4 derives the convex properties of posterior surpluses. Section 5 provides the main results on the characterization of the optimal disclosure policies. Section 6 derives the demand functions under which the total surplus is linear in costs. Finally, Section 7 concludes the paper.

2 Model

We consider the model with the monopolistic seller (he), the unit mass of buyers, and the information designer. The buyers' valuations are distributed according to a cdf $F(x)$ with the density $f(x)$ on the support $\mathbf{X} = [\underline{x}, \bar{x}]$, where $\bar{x} > -\infty$ and $\underline{x} < \bar{x} \leq \infty$. Similarly, the seller's unit cost c is drawn from a continuous cdf $H(c)$ with the support $\mathbf{C} = [\underline{c}, \bar{c}]$, where $\underline{c} > 0$. Buyers' valuations and seller's cost are distributed independently.

Denote the *inverse hazard rate*

$$\phi(x) = \frac{1}{\lambda(x)} = \frac{1 - F(x)}{f(x)} \tag{1}$$

the reciprocal of the hazard rate $\lambda(x) = \frac{f(x)}{1-F(x)}$. We treat $\phi(x)$ as a primitive for the most of the paper. Specifically, we assume that $\phi(x) > 0$ for $x \in (\underline{x}, \bar{x})$, $\lim_{t \downarrow \underline{x}} \int_{\underline{x}}^{t} \frac{1}{\phi(x)} dx = 0$, and

[12] They relax first two assumptions through the paper, however.

[13] Also, Bergemann et al. (2024) consider the variant of the model by Kartik and Zhong (2024) with competing sellers, but without interdependent values. They investigate how the relationship between the consumer and total surpluses is shaped by the sellers' and buyers' information about costs and valuations, respectively.

$\lim_{t\uparrow\bar{x}} \int_t^{\bar{x}} \frac{1}{\phi(x)}dx = \infty$. These conditions guarantee that $F(x)$ is atomless on $\mathbf{X}$, and $\phi(\bar{x}) = 0$ for $\bar{x} < \infty$.[14] If $\bar{x} = \infty$, then $\lim_{x\to\infty} \phi(x)$ can be 0, finite, infinite, or do not exist. Under these conditions, $\phi(x)$ generates the cdf $F(x)$ and the density $f(x) > 0$ for $x \in (\underline{x}, \bar{x})$ as follows:

$$F(x) = 1 - \exp\left(-\int_{\underline{x}}^{x} \frac{1}{\phi(t)}dt\right) \text{ and } f(x) = \frac{1}{\phi(x)} \exp\left(-\int_{\underline{x}}^{x} \frac{1}{\phi(t)}dt\right), \tag{2}$$

We also assume that $\phi(x)$ is twice continuously differentiable and its second derivative $\phi''(x)$ is bounded if $\bar{x} = \infty$. This condition is not purely technical and can qualitatively affect the results as demonstrated below. Next, we assume that

$$\phi(\underline{x}) \geq \underline{x} - \underline{c} \text{ and } \phi(\bar{x}) \leq \bar{x} - \bar{c}. \tag{3}$$

These conditions imply that for any $c \in \mathbf{C}$, there is $p_c \in (c, \bar{x}]$, such that

$$\phi(p_c) = p_c - c. \tag{4}$$

Given a subset $\mathbf{S} \subset \mathbf{C}$, denote

$$\mathbf{P_S} = \{p|\phi(p) = p - c \text{ for } c \in \mathbf{S}\} \tag{5}$$

the set of solutions to (4) induced by costs in $\mathbf{S}$. We also require that

$$\phi'(p) < 1 \text{ for } p \in \mathbf{P_C}, \tag{6}$$

which means that p_c is unique for all $c \in \mathbf{C}$, since the function $p - \phi(p) - c$ is strictly pseudo-monotone in p for $c \in \mathbf{C}$, i.e., it intersects the p-axis only once from below.[15]

Information. Buyers privately know their valuations, while the seller knows only the cdfs F and H. Also, the seller receives a signal s about c whose precision is selected by the information designer. By relabeling signals we can put $s = \mathbf{E}[c|s]$, i.e., s is the seller's *posterior cost* induced by a signal s. Denote $G(x) = \Pr[s \leq x]$ the cdf of posterior costs with the support $\mathbf{S} \subset \mathbf{C}$. Then G is dominated by H by the convex order (Shaked and Shanthikumar, 2007) or, equivalently, G is a mean-preserving contraction of H:

$$G \underset{cx}{\prec} H \Leftrightarrow \int_{\underline{c}}^{x} G(s)\,ds \leq \int_{\underline{c}}^{x} H(s)\,ds \text{ for all } x \in \mathbf{C}, \text{ and} \int_{\underline{c}}^{\bar{c}} G(s)\,ds = \int_{\underline{c}}^{\bar{c}} H(s)\,ds. \tag{7}$$

Surpluses. Given the pair (p, c), the seller's ex-post profit is

$$\pi(p,c) = \Pr[X \geq p](p - c) = (1 - F(p))(p - c) = Q(p)(p - c) = \mathbf{R}(p) - cQ(p),$$

[14] Otherwise, if $\phi(\bar{x}) > 0$, then $\int_{\underline{x}}^{\bar{x}} \frac{1}{\phi(x)}dx < \infty$, which contradicts $\lim_{t\uparrow\bar{x}} \int_{\underline{x}}^{t} \frac{1}{\phi(x)}dx = \infty$.

[15] A function $u : \mathbf{X} \to \mathbb{R}$, where $\mathbf{X} \subset \mathbb{R}$ is convex, is *pseudo-monotone* if for any $x, y \in \mathbf{X}$, $u(x)(y - x) > 0$ implies $u(y)(y - x) > 0$. A function $u(.)$ is *strictly pseudo-monotone* if for any $x, y \in \mathbf{X}, y \neq x$, $u(x)(y - x) \geq 0$ implies $u(y)(y - x) > 0$; or equivalently, if $u(x) \geq 0$ implies $u(y) > 0$ for all $y > x$.

where

$$Q(p) = 1 - F(p) \tag{8}$$

is the quantity demanded, and

$$\mathbf{R}(p) = Q(p)\,p = (1 - F(p))\,p$$

is the revenue at price p. Since $\pi(p, c)$ is linear in c, the seller's profit for posterior cost s and price p is equal to

$$\pi(p, s) = \mathbf{E}[\pi(p, c)\,|s] = Q(p)(p - s) = \mathbf{R}(p) - sQ(p). \tag{9}$$

Thus, the posterior cost s contains all decision-relevant information about c for the seller. Furthermore, $s = \mathbf{E}[c|s]$ implies that posterior cost s is equivalent to the actual cost $c = s$ from the seller's perspective. Due to this equivalence, we often call s simply a *cost* hereafter to shorten the notation whenever the nature of cost is clear from the context.

The seller's problem given a posterior cost s is to select the optimal price p_s that yields the maximal posterior profit, which reflects the producer surplus in the monopolistic market,

$$\mathbf{PS}_s = \max_{p \geq 0} \pi(p, s) = \pi(p_s, s) = Q(p_s)(p_s - s). \tag{10}$$

The optimal price exists, unique, and is equal to p_s given by (4). This is because p_s satisfies the first-order condition

$$\pi'_p(p_s, s) = 1 - F(p_s) - f(p_s)(p_s - s) = f(p_s)(\phi(p_s) - (p_s - s)) = 0,$$

where $f(p_s) > 0$. It also satisfies the second-order condition due to (6) and is an interior point of $\mathbf{X}$ due to (3). Hence, $\mathbf{P_S}$ defined by (5) determines the set of optimal prices induced by posterior costs in $\mathbf{S}$.

The posterior consumer surplus is

$$\mathbf{CS}_s = \mathbf{CS}(p_s) = \int_{p_s}^{\bar{x}} x - p_s dF(x) = -\mathbf{R}_{\bar{x}} + \int_{p_s}^{\bar{x}} Q(x)\,dx = \int_{p_s}^{\bar{x}} Q(x)\,dx, \tag{11}$$

where

$$\mathbf{R}_{\bar{x}} = \lim_{p \to \bar{x}} \mathbf{R}(p) = \lim_{p \to \bar{x}} Q(p)\,p = \lim_{p \to \bar{x}} Q(p)(p - s), \tag{12}$$

is the limit of the seller's revenue $\mathbf{R}(p) = Q(p)\,p$ as the price converges to $\bar{x}$.[16] It is well known that $\mathbf{E}_F[X] < \infty$ implies $\mathbf{R}_{\bar{x}} = 0$ (Feller, 1966). Then, the last equality in (11) holds, because $\mathbf{CS}_s < \infty$ if and only if $\mathbf{E}_F[X] < \infty$ and, hence, only if $\mathbf{R}_{\bar{x}} = 0$.[17]

The posterior total surplus is defined as a linear combination of the posterior consumer

[16] Since $\lim_{p \to \bar{x}} Q(p) = 0$, then the limits of the seller's revenue $\mathbf{R}(p)$ and profit $\pi(p, s)$ are the same as $p \to \bar{x}$.

[17] Because $\mathbf{CS}_s = \int_{p_s}^{\bar{x}} x - p_s dF(x) = \int_{p_s}^{\bar{x}} x dF(x) - Q(p_s)\,p_s = \mathbf{E}_F[X] - \int_{\underline{x}}^{p_s} x dF(x) - Q(p_s)\,p_s$, where the last two terms are finite, then $\mathbf{CS}_s < \infty$ if and only if $\mathbf{E}_F[X] < \infty$.

and producer surpluses:

$$\mathbf{W}_s = \mathbf{CS}_s + \beta \mathbf{PS}_s = \int_{p_s}^{\bar{x}} x - p_s dF(x) + \beta (1 - F(p_s))(p_s - s),$$

where $\beta \geq 0$ is the relative weight of $\mathbf{PS}_s$. Thus, $\mathbf{W}_s$ for $\beta = 0, \infty$, and 1 correspond to $\mathbf{CS}_s, \mathbf{PS}_s$, and $\mathbf{SW}_s$, respectively, where $\mathbf{SW}_s$ is the standard social welfare:[18]

$$\mathbf{SW}_s = \mathbf{CS}_s + \mathbf{PS}_s = \int_{p_s}^{\bar{x}} x - s dF(x).$$

Then the ex-ante surpluses are given by the averages of the posterior surpluses over G:

$$\mathbf{PS} = \mathbf{E}[\mathbf{PS}_s] = \int_{\mathbf{S}} Q(p_s)(p_s - s)\, dG(s),$$

$$\mathbf{CS} = \mathbf{E}[\mathbf{CS}_s] = \int_{\mathbf{S}} \int_{p_s}^{\bar{x}} x - p_s dF(x)\, dG(s) = \int_{\mathbf{S}} \int_{p_s}^{\bar{x}} Q(x)\, dx dG(s), \text{ and}$$

$$\mathbf{W} = \mathbf{E}[\mathbf{W}_s] = \mathbf{CS} + \beta \mathbf{PS}.$$

The goal of the information designer is to maximize $\mathbf{W}$ over $G(x)$ subject to the constraint (7), that is, $G(x)$ is a mean-preserving contraction of $H(x)$:

$$\max_{G \prec_{cx} H} \mathbf{W}.$$

3 Example 1

Now we provide a simple stylized example, which shows that disclosing *any* information about cost to the uninformed seller can drastically reduce the total surplus for any $\beta \geq 0$.

Consider a distribution of costs $H(c)$ with the mean value $c^e = \mathbf{E}[c]$. The distribution of buyers valuations is

$$F(x) = \begin{cases} 0 & \text{if } x < \underline{x} \\ 1 - \frac{\underline{x} - c^e}{x - c^e} & \text{if } \underline{x} \leq x < \bar{x} \\ 1 & \text{if } x \geq \bar{x}, \end{cases}$$

where $0 < c^e < \underline{x} < \bar{x} < 1$. That is, $F(x)$ is a Pareto distribution on the interval $[\underline{x}, \bar{x})$ with the mass $\frac{\underline{x} - c^e}{\bar{x} - c^e}$ at $\bar{x}$.[19] It is depicted on the left panel of Fig. 1.

First, consider the case of the uninformed seller. For the expected cost c^e, the seller's

[18] We are interested only in those $\mathbf{W}_s$ that put a positive weight on $\mathbf{CS}_s$; i.e., $\beta < \infty$ across the paper.

[19] The mass at $\bar{x}$ does not play any role and is selected for convenience only.

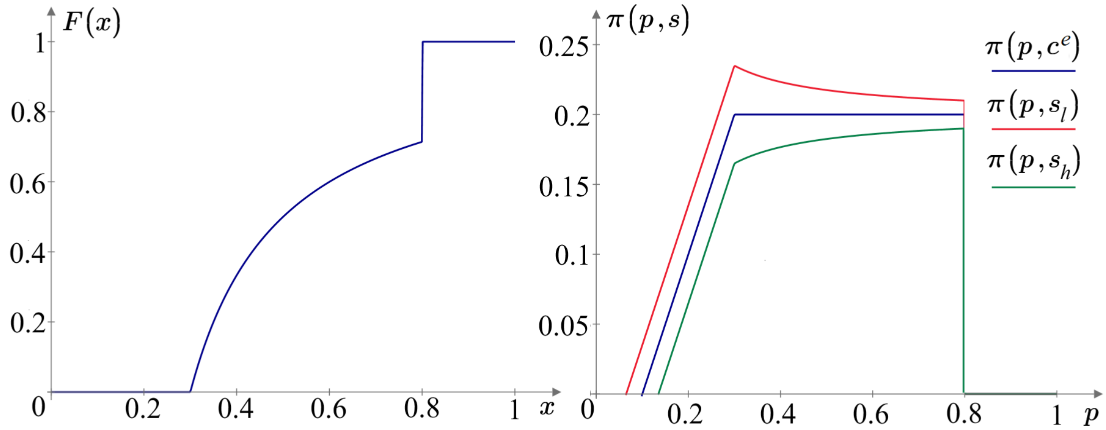

Figure 1: The distribution of buyers' valuations and the seller's profit for different costs

profit is represented by the blue curve on the right panel of Fig. 1 and given by

$$\pi\left(p,c^{e}\right)=\begin{cases}p-c^{e} & \text{if } p<\underline{x}\\ \underline{x}-c^{e} & \text{if } \underline{x}\leq p\leq \bar{x}\\ 0 & \text{if } p\geq \bar{x},\end{cases}$$

Thus, any price $p\in[\underline{x},\bar{x}]$ is optimal and yields the producer surplus $\mathbf{PS}_{c^e}=\underline{x}-c^e$. The ex-ante consumer surplus at price p is given by

$$\mathbf{CS}_p=\int_p^{\bar{x}}(x-p)\,dF\left(x\right)+\left(1-F\left(\bar{x}\right)\right)\left(\bar{x}-p\right).$$

Setting the lowest optimal price $p_{c^e}=\underline{x}$ results in the ex-ante consumer surplus

$$\mathbf{CS}_{\underline{x}}=\int_{\underline{x}}^{\bar{x}}\left(\underline{x}-c^{e}\right)\frac{x-\underline{x}}{\left(x-c^{e}\right)^{2}}dx+\frac{\underline{x}-c^{e}}{\bar{x}-c^{e}}\left(\bar{x}-\underline{x}\right)=\left(\underline{x}-c^{e}\right)\ln\frac{\bar{x}-c^{e}}{\underline{x}-c^{e}},$$

and the social welfare

$$\mathbf{W}_{\underline{x}}=\mathbf{CS}_{\underline{x}}+\mathbf{PS}_{c^e}=\int_{\underline{x}}^{\bar{x}}x-c^{e}dF\left(x\right)+\left(1-F\left(\bar{x}\right)\right)\left(\bar{x}-c^{e}\right)=\left(\underline{x}-c^{e}\right)\left(\ln\frac{\bar{x}-c^{e}}{\underline{x}-c^{e}}+1\right).$$

In this equilibrium, the seller's price is the lowest optimal one for analytical convenience only. Introducing a small decline in the profit function of the uninformed seller for prices above $\underline{x}$ resolves his indifference between prices without affecting any results.[20]

[20] It can be done by using $F\left(x\right)=1-\frac{\underline{x}-c^{e}+\varepsilon}{x-c^{e}+\varepsilon}$ for $x\in[\underline{x},\bar{x})$, where $\varepsilon\downarrow 0$.

Now, consider the case of the informed seller. Given the pair (p, s), his profit is

$$\pi(p,s) = (1 - F(p))(p - s) = \begin{cases} p - s & \text{if } p < \underline{x} \\ (\underline{x} - c^e) \frac{p-s}{p-c^e} & \text{if } \underline{x} \leq p \leq \bar{x} \\ 0 & \text{if } p > \bar{x}. \end{cases}$$

The right side of Fig. 1 depicts $\pi(p,s)$ for three values of s: c^e (the blue curve), $s_l < c^e$ (the red curve), and $s_h > c^e$ (the green curve). Since $\underline{x} > c^e$, then

$$(\underline{x} - c^e) \frac{p - s}{p - c^e} = (\underline{x} - c^e) \left(1 - \frac{s - c^e}{p - c^e}\right)$$

is strictly increasing in p for $p > c^e$ if $s > c^e$ and strictly decreasing in p if $s < c^e$. Thus, the optimal posterior price as a function of cost s is given by

$$p_s = \begin{cases} \underline{x} & \text{if } s < c^e, \\ [\underline{x}, \bar{x}] & \text{if } s = c^e, \\ \bar{x} & \text{if } s > c^e. \end{cases}$$

Despite a substantial difference $\bar{x} - \underline{x}$ in the optimal prices for s above and below c^e, the effect of s on $\mathbf{PS}_s$ is insignificant due to the continuity of the profit function $\pi(p,s)$ in s:

$$\mathbf{PS}_s = \pi(p_s, s) = \begin{cases} \underline{x} - s & \text{if } s < c^e \\ \frac{\underline{x}-c^e}{\bar{x}-c^e}(\bar{x} - s) & \text{if } s > c^e, \end{cases}$$

so that $\lim_{s \uparrow c^e} \mathbf{PS}_s = \lim_{s \downarrow c^e} \mathbf{PS}_s = \underline{x} - c^e$.

In contrast, the effect of the price change on $\mathbf{CS}_s$ caused by a small variation in s is significant. In particular, if $s < c^e$, then the optimal price and the posterior consumer surplus remain the same, those for the uninformed seller, $p_s = \underline{x}$ and $\mathbf{CS}_s = \mathbf{CS}_{\underline{x}}$. However, if $s > c^e$, then the optimal price increases to $p_s = \bar{x}$, and $\mathbf{CS}_s$ falls to 0:

$$\mathbf{CS}_s = \begin{cases} \mathbf{CS}_{\underline{x}} > 0 & \text{if } s < c^e \\ \mathbf{CS}_{\bar{x}} = 0 & \text{if } s > c^e. \end{cases}$$

Suppose that the distribution of posterior costs is binary with the support $\{s_l, s_h\} = \{c^e - \varepsilon, c^e + \varepsilon\}$ and equal probabilities of s_l and s_h, where $\varepsilon \downarrow 0$. The table below illustrates the posterior surpluses and the social welfare for $\underline{x} = 0.3, \bar{x} = 0.7, c^e = 0.1$, and $\varepsilon = 0.01$:

s	$\mathbf{PS}_s$	$\mathbf{CS}_s$	$\mathbf{SW}_s$
$c^e = 0.1$	0.2	0.22	0.42
$s_l = 0.09$	0.21	0.22	0.43
$s_h = 0.11$	0.197	0	0.197

Similarly, the table below represents the ex-ante surpluses and the social welfare of the informed and the uninformed seller:

Seller	**PS**	**CS**	**SW**
Uninformed	0.2	0.22	0.42
Informed	0.204	0.11	0.314

That is, the ex-ante social welfare decreases by more that 25% in the case of the informed seller. Furthermore, as the probability of the higher cost s_h converges to 1, the ex-ante consumer surplus $\mathbf{CS}$ converges to 0 (since consumers receive the surplus $\mathbf{CS}_{s_h}$ almost surely). As a result, the social welfare becomes entirely determined by the producer surplus and decreases by 53% compared to that with the uninformed seller. Also, the negative effect of seller's information holds in the case of replacing the the social welfare with ex-ante total surplus with any weight β.

Intuitively, since providing information to the seller decomposes the prior cost c^e into a distribution over posterior costs with the mean $\mathbf{E}[s] = c^e$, there must be a mass of costs $s > c^e$. These costs result in prices p_s substantially above the one with the uninformed seller, $p_{c^e} = \underline{x}$. As a result, the higher posterior costs transform the mass market, that is, the one in which the product is purchased at a relatively low price by a large fraction of consumers, into the niche market, in which the product is purchased at high price by a small fraction of consumers with high valuations. This transformation reduces the posterior social welfare and, as a consequence, the ex-ante social welfare.

4 Market characteristics

In this section we characterize the relationships between the inverse hazard rate and several key market characteristics, such as the optimal price, the profit margin, the marginal revenue, the demand elasticity, and the pass-through rate. Then we use these relationships in order to establish the convex properties of posterior surpluses.

We start the subsection by relating $\phi(p)$ to the market demand and the marginal revenue. First, consider the *inverse demand* function

$$P(q) = F^{-1}(1-q), \tag{13}$$

which expresses the price as a function of the quantity demanded q. Then $\phi(p)$ is related to $P(q)$ as follows:

$$\phi(p) = -P'(q)\,q|_{q=Q(p)} = -\frac{Q(p)}{Q'(p)}.$$

Thus, $\phi(p)$ represents the marginal decrease in the revenue $R(q) = P(q)\,q$, which is caused by an increase in quantity q and expressed as a function of price p.

Next, we relate $\phi(p)$ to other characteristics of the demand function. One of them is the demand elasticity

$$\varepsilon(p) = \frac{Q'(p)\,p}{Q(p)} = \frac{P(q)}{P'(q)\,q}|_{q=Q(p)} = -\frac{p}{\phi(p)}. \tag{14}$$

Another characteristics of the demand is the *virtual value* function

$$\psi(p) = p - \phi(p), \tag{15}$$

that represents the marginal revenue at price p.[21] The economic meaning of the relationship between $\phi(p), \varepsilon(p)$, and $\psi(p)$ is clear. Given a price p, the more elastic demand $\varepsilon(p)$ implies the lower value of $\phi(p)$, i.e., the smaller decrease in the marginal revenue $\psi(p)|_{p=1-F(q)}$

[21] This is because $MR(q)|_{q=Q(p)} = P(q) + qP'(q)|_{q=Q(p)} = p - \frac{1-F(p)}{f(p)} = p - \phi(p) = \psi(p)$.

stemming from an increase in quantity q. This is because the more elastic demand means the smaller impact of the relative quantity $\frac{dq}{q}$ on the relative price change $\frac{dp}{p}$ and, hence, the marginal revenue $\psi(p)$.

Also, the value of $\phi(p_s)$ at the optimal price p_s determines the *profit margin*, or simply the *margin* $p_s - s$. Because $s > 0$, then $\phi(p_s) = p_s - s < p_s$ yields the classical result $\varepsilon(p_s) = -\frac{p_s}{\phi(p_s)} < -1$, i.e., the demand must be elastic at p_s. Next, using (4) and (15) gives

$$\begin{aligned} \psi(p_s) &= p_s - \phi(p_s) = s, \text{ or} \\ p_s &= \psi^{-1}(s). \end{aligned} \tag{16}$$

Furthermore, using the Implicit Function Theorem provides the relationship between $\phi(x)$ and the *pass-through rate* p'_s:

$$p'_s = \frac{1}{1 - \phi'(p_s)} = \frac{1}{\psi'(p_s)}, \tag{17}$$

where $p'_s > 0$ due to (6).[22] Intuitively, (17) has the standard meaning as the ratio of the slope of the demand function to the slope of the marginal revenue, which are, however, expressed via the price rather than the quantity.[23] Also, because (15) implies

$$\psi'(x) = 1 - \phi'(x),$$

then the condition $\phi'(x) < 1$ is equivalent to $\psi'(x) > 0$. That is, (6) is equivalent to the strict concavity of the revenue $R(q)$ in q.[24] Finally, taking the derivative of p'_s with respect to s gives the marginal pass-through rate:

$$p''_s = \phi''(p_s)(p'_s)^3 = \frac{\phi''(p_s)}{(1 - \phi'(p_s))^3} = -\frac{\psi''(p_s)}{(\psi'(p_s))^3}. \tag{18}$$

As we show below, all these market characteristics—the optimal price p_s, the profit margin $\phi(p_s)$, the pass-through rate p'_s, and the marginal pass-through p''_s—determine the convex properties of posterior surpluses in cost s.

5 Surplus curvature

Before establishing the main results, we characterize the driving forces behind the curvatures of the posterior surpluses and relate them to the inverse hazard rate and market characteristics. For notational simplicity, by the *curvature* of a function we imply the value of its second derivative, which thus determines the local concavity or the convexity of the

[22] The pass-through rate plays an important role in the welfare effects of taxes in imperfectly competitive markets (Weyl and Fabinger, 2013).

[23] Specifically, $p'_s = \frac{P'(q)}{MR'(q)}|_{q=Q(p_s)} = \frac{\frac{dp}{dq}}{\frac{d}{dq}(P(q)q)}|_{q=Q(p_s)} = \frac{1}{\frac{d}{dp}(pQ(p))}|_{p=p_s} = \frac{1}{\frac{d}{dp}(p-\phi(p))}|_{p=p_s} = \frac{1}{1-\phi'(p_s)}$.

[24] Since $\psi(p) = MR(Q(p))$, then $\psi'(p) = -\frac{\partial^2 R(q)}{\partial q^2}|_{q=Q(p)} f(p) > 0$ is equivalent to $\frac{\partial^2 R(q)}{\partial q^2} < 0$.

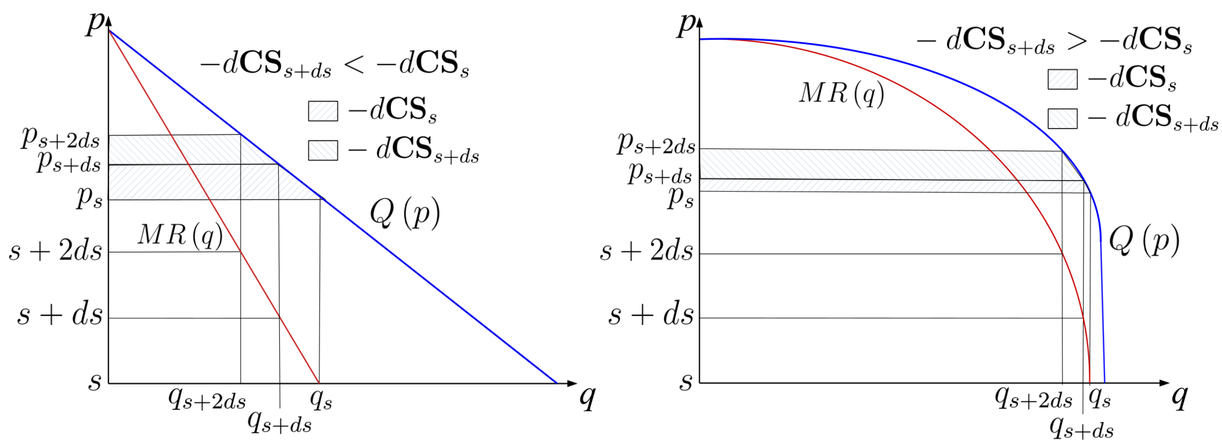

Figure 2: Demand functions with convex (left graph) and concave (right graph) $\mathbf{CS}_s$.

function. Taking the derivatives of (10) and (11) with respect to s results in

$$\mathbf{CS}'_s = -Q(p_s)\,p'_s = -\frac{Q(p_s)}{1-\phi'(p_s)}, \tag{19}$$

$$\mathbf{PS}'_s = -Q(p_s) = -(1-F(p_s)), \text{ and}$$

$$\mathbf{W}'_s = \mathbf{CS}'_s + \beta\mathbf{PS}'_s = -Q(p_s)(p'_s+\beta) = -Q(p_s)\left(\frac{1}{1-\phi'(p_s)}+\beta\right). \tag{20}$$

As can be seen from the first equality in (19), the effect of s on $\mathbf{CS}'_s$ is dual. First, a \$1 increase in cost increases the price by the value of the pass-through rate p'_s. This reduces the ex-post consumer surplus $x - p_s$ of a consumer with valuation x who purchased the product, by p'_s. Because the mass of consumers served is $Q(p_s)$, the overall effect is given by $-Q(p_s)\,p'_s$. The second equality in (19) employs the dependence (17) of the pass-through rate p'_s on the marginal inverse hazard $\phi'(p_s)$. This allows us to express the relationship between $\mathbf{CS}'_s$ and the value of the optimal price p_s only, which we use below.

Taking the second derivatives of $\mathbf{CS}_s$ and $\mathbf{PS}_s$ with respect to s yields

$$\frac{\mathbf{CS}''_s}{f(p_s)} = (p'_s)^2 - \frac{Q(p_s)}{f(p_s)}p''_s = (p'_s)^2 - \phi(p_s)\,p''_s = (p'_s)^2 - (p_s - s)\,p''_s, \tag{21}$$

$$\frac{\mathbf{PS}''_s}{f(p_s)} = p'_s, \text{ and}$$

$$\frac{\mathbf{W}''_s}{f(p_s)} = \frac{\mathbf{CS}''_s}{f(p_s)} + \beta\frac{\mathbf{PS}''_s}{f(p_s)} = (p'_s)^2 - p''_s\phi(p_s) + \beta p'_s. \tag{22}$$

Note that $\mathbf{W}_s$ is concave only if $\mathbf{CS}_s$ is concave and β is sufficiently low. This is because $\mathbf{PS}''_s = f(p_s)\,p'_s > 0$ implies that $\mathbf{PS}_s$ is always strictly convex.[25] In this light, we focus mostly on the curvature of $\mathbf{CS}_s$ hereafter. Figure 2 depicts markets with the convex and concave $\mathbf{CS}_s$. In this picture, the areas of trapezoids $-d\mathbf{CS}_s$ and $-d\mathbf{CS}_{s+ds}$ reflect the absolute values

[25] It also stems from the Blackwell sufficiency, since additional information in the monopolist decision problem can only increase his maximal ex-ante profit.

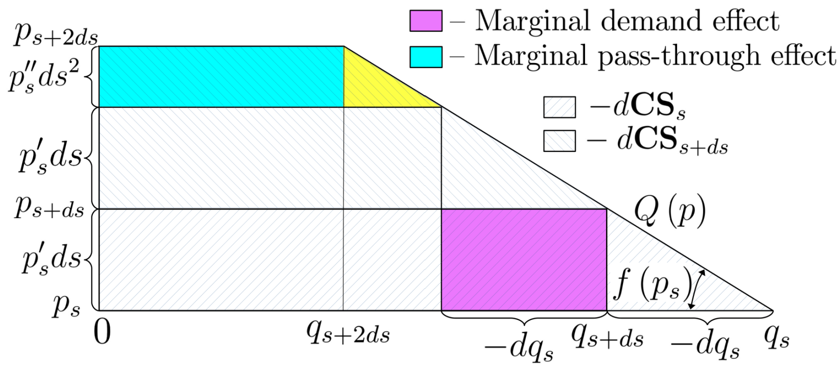

Figure 3: Marginal effect of cost on $\mathbf{CS}'_s$ via optimal prices

of the incremental decreases in $\mathbf{CS}_s$ as posterior cost increases from s to $s + ds$, and then from $s+ds$ to $s+2ds$. Hence, $-d\mathbf{CS}_{s+ds} < (>) - d\mathbf{CS}_s$ implies a smaller (larger) incremental decrease in $\mathbf{CS}_s$ for higher costs, which corresponds to the convex (concave) $\mathbf{CS}_s$.

5.1 Surplus curvature via optimal prices

We start the analysis of $\mathbf{CS}_s$ and $\mathbf{W}_s$ by noting that its curvatures can be entirely expressed in terms of the characteristics of the optimal price: its value p_s, the pass-through rate p'_s, and the marginal pass-through rate p''_s:

$$\begin{aligned} \mathbf{CS}''_s \geq (\leq)\, 0 &\Leftrightarrow (p'_s)^2 - p''_s \phi(p_s) = (p'_s)^2 - p''_s (p_s - s) \geq (\leq)\, 0, \text{ and} \\ \mathbf{W}''_s \geq (\leq)\, 0 &\Leftrightarrow (p'_s)^2 - p''_s \phi(p_s) + \beta p'_s \geq (\leq)\, 0. \end{aligned} \tag{23}$$

The intuition behind (21) and (23) can be explained by considering the marginal impacts of s on $p_s, p'_s, q_s = Q(p_s)$, and, as a consequence, on $\mathbf{CS}'_s$. They are depicted on Figure 3. As cost increases from s to $s + ds$, the price p_s changes by $p'_s ds$, the quantity q_s changes by dq_s, and the posterior surplus $\mathbf{CS}_s$ changes by $d\mathbf{CS}_s$, such that

$$d\mathbf{CS}_s = -\frac{q_{s+ds} + q_s}{2} p'_s ds.$$

Similarly, as cost increases from $s + ds$ to $s + 2ds$, the price p_{s+ds} changes by $p'_s ds + p''_s ds^2$, the quantity q_{s+ds} changes by $q_{s+2ds} - q_{s+ds}$, and $\mathbf{CS}_s$ changes by $d\mathbf{CS}_{s+ds}$, such that[26]

$$d\mathbf{CS}_{s+ds} = -\frac{q_{s+ds} + q_{s+2ds}}{2} \left(p'_s ds + p''_s ds^2\right).$$

In turn, the second-order differential $d^2\mathbf{CS}_s$ is given by the difference $d\mathbf{CS}_{s+2s} - d\mathbf{CS}_{s+ds}$,

[26] Specifically, $-d\mathbf{CS}_s$ is equal to the area of the trapezoid with vertices $(0, p_s), (0, p_{s+ds}), (q_{s+ds}, p_{s+ds})$, and (q_s, p_s). Similarly, $-d\mathbf{CS}_{s+ds}$ is equal to the area of the trapezoid with vertices $(0, p_{s+ds}), (0, p_{s+2ds}), (q_{s+2ds}, p_{s+2ds})$.

which is represented by the difference between the areas of the pink and blue rectangles:[27]

$$d^2\mathbf{CS}_s = d\mathbf{CS}_{s+ds} - d\mathbf{CS}_s = (-dq_s)\, p'_s ds - q_{s+2ds} p''_s ds^2.$$

Because $dq_s = dQ(p_s) = -f(p_s)\, p'_s ds$ and $q_{s+2ds} = Q(p_s) + O(ds)$, we obtain

$$d^2\mathbf{CS}_s = f(p_s)\,(p'_s)^2\, ds^2 - Q(p_s)\, p''_s ds^2 + o\left(ds^2\right) = f(p_s)\left((p'_s)^2 - \phi(p_s)\, p''_s\right) ds^2 + o\left(ds^2\right).$$

As $ds \downarrow 0$, this leads to

$$\frac{\mathbf{CS}''_s}{f(p_s)} = (p'_s)^2 - \phi(p_s)\, p''_s = (p'_s)^2 - (p_s - s)\, p''_s.$$

Intuitively, the convexity of $\mathbf{CS}_s$ is driven by the trade-off between two factors: the *marginal demand effect* and the *marginal pass-through effect*. The first effect is caused by the reduction in the quantity demanded as the price increases in response to the higher cost. It is proportional to $-p'_s q'_s = (p'_s)^2 f(p_s)$, which is represented by the area of the pink rectangle and always positive. The second effect is caused by the impact of cost on the pass-through rate p'_s. It is proportional to $Q(p_s)\, p''_s$, which is represented by the area of the cyan rectangle and can be positive or negative depending on the convexity or the concavity of p_s in s. That is, the relative magnitude of the second effect is proportional to $\frac{Q(p_s)}{f(p_s)} = \phi(p_s) = p_s - s$. Normalizing both effects by $f(p_s)$ gives (21).

Furthermore, the convexity of $\mathbf{CS}_s$ can be related to the *log-convexity* of p_s. In order to see this relationship, we can rewrite (21) as

$$\frac{\mathbf{CS}''_s}{f(p_s)\,(p_s)^2} = \frac{(p'_s)^2 - p''_s\,(p_s - s)}{p_s^2} = \frac{(p'_s)^2 - p''_s p_s}{p_s^2} + \frac{p''_s s}{p_s^2} = -\left(\ln(p_s)\right)'' + \frac{p''_s s}{p_s^2},$$

which gives

$$\mathbf{CS}''_s \leq (\geq)\, 0 \Leftrightarrow \left(\ln(p_s)\right)'' \geq (\leq)\, \frac{p''_s s}{p_s^2}. \tag{24}$$

This inequality provides two important insights about the relationship between the posterior consumer surplus and the optimal price. First, if $s = 0$, then (24) is equivalent to the log-convexity (log-concavity) of the optimal price in cost. Hence, this property can be used to establish a simple sufficient condition of the concavity (or the convexity) of the posterior consumer surplus for sufficiently small costs.

Remark 1 *If p_s is strictly log-convex (strictly log-concave) in s at $s = 0$, then $\mathbf{CS}_s$ is strictly concave (strictly convex) in the neighborhood of $s = 0$.*

Second, (23) implies that $\mathbf{CS}_s$ can be concave only if $p''_s > \frac{(p'_s)^2}{\phi(p_s)} > 0$. That is, the right-hand side of the inequality (24) must be positive. This implies that the optimal price must be sufficiently log-convex, and thus increase in cost faster than exponentially.[28]

[27] Up to terms of order $o\left(ds^2\right)$ resulting from the non-linearity of the demand $Q(p)$ and the area of the yellow triangle.

[28] Otherwise, if p_s is log-linear, i.e., $p_s = ae^{bs}$, where $a, b > 0$, then $\left(\ln(p_s)\right)'' = 0 < \frac{p''_s s}{(p_s)^2} = \frac{b^2}{a} s e^{-bs}$, which results in $\mathbf{CS}''_s > 0$.

5.2 Surplus curvature via inverse hazard rate

Now, we characterize the convexity of the posterior surpluses in terms of the inverse hazard rate. By using (17) and (18), $\mathbf{CS}_s''$ and $\mathbf{W}_s''$ can be also expressed via $\phi(x)$ as follows:

$$\frac{\mathbf{CS}_s''}{f(p_s)} = (p_s')^2 - \phi(p_s)\, p_s'' = (p_s')^2 - \phi(p_s)\, \phi''(p_s)\, (p_s')^3 = (p_s')^2 \left(1 - \phi(p_s)\, \phi''(p_s)\, p_s'\right) \tag{25}$$

$$= \frac{1}{(1-\phi'(p_s))^2}\left(1 - \frac{\phi(p_s)\,\phi''(p_s)}{1-\phi'(p_s)}\right) = \frac{1-\zeta(p_s)}{(1-\phi'(p_s))^2},$$

$$\frac{\mathbf{PS}_s''}{f(p_s)} = p_s' = \frac{1}{1-\phi'(p_s)}, \text{ and} \tag{26}$$

$$\frac{\mathbf{W}_s''}{f(p_s)} = (p_s')^2 - \phi(p_s)\, p_s'' + \beta p_s' = \frac{1-\zeta(p_s) + \beta\left(1-\phi'(p_s)\right)}{(1-\phi'(p_s))^2}, \tag{27}$$

where $\zeta : \mathbf{P_C} \to \mathbb{R}$ is

$$\zeta(x) = \frac{\phi(x)\,\phi''(x)}{1-\phi'(x)}.$$

It is well defined due to $\phi'(x) < 1$, and is continuous in $x \in \mathbf{P_C}$, since $\phi(x)$ is twice continuously differentiable. Then, the convexity of $\mathbf{CS}_s$ and $\mathbf{W}_s$ can be expressed via $\zeta(p_s)$ only:

$$\mathbf{CS}_s'' \geq (\leq)\, 0 \Leftrightarrow \zeta(p_s) = \frac{\phi(p_s)\,\phi''(p_s)}{1-\phi'(p_s)} \leq (\geq)\, 1, \text{ and} \tag{28}$$

$$\mathbf{W}_s'' \geq (\leq)\, 0 \Leftrightarrow 1 - \zeta(p_s) + \beta\left(1-\phi'(p_s)\right) \geq (\leq)\, 0 \Leftrightarrow \zeta(p_s) + \beta\phi'(p_s) \leq 1 + \beta. \tag{29}$$

In this light, the curvatures of $\mathbf{CS}_s$ and $\mathbf{W}_s$ are driven by an interplay between three characteristics of $\phi(x)$ at the optimal price p_s—it value, and the first and the second derivatives. Specifically, ceteris paribus an increase in the curvature of $\phi(x)$ at $x = p_s$ is sufficient to concavify $\mathbf{CS}_s$. However, an increase in the slope of $\phi(x)$ is sufficient to concavify $\mathbf{CS}_s$ only if $\phi''(p_s) > 0$. Finally, it is impossible to isolate the effect of an increase in the profit margin $\phi(x)$ at $x = p_s$ on the curvature of the posterior surpluses. This is because any shift in $\phi(x)$ results in the change of p_s as follows from (4), which in turn affects the values of $\phi'(p_s)$ and $\phi''(p_s)$. In other words, while one can manipulate the values of $\phi'(x)$ and $\phi''(x)$ and, hence the curvatures of $\mathbf{CS}_s$ and $\mathbf{W}_s$ at p_s without affecting it, any change in the value of $\phi(x)$ will change the optimal price p_s.

5.3 Surplus curvature via market power and marginal revenue

We now provide the economic interpretation of the above results in terms of two market characteristics: the profit margin and the marginal revenue. Denote $A_\psi(p)$ and $r_\psi(p)$ the Arrow–Pratt measures of absolute and relative risk aversion of the marginal revenue $\psi(p)$

at price p, respectively:

$$A_\psi(p) = -\frac{\psi''(p)}{\psi'(p)} = -\frac{\psi''(p)}{\psi'(p)} = \frac{\phi''(p_s)}{1-\phi'(p_s)} = \frac{p_s''}{(p_s')^2}, \text{ and}$$
$$r_\psi(p) = pA_\psi(p) = -\frac{p\psi''(p)}{\psi'(p)}.$$

Also, denote

$$L_\phi(s) = \frac{\phi(p_s)}{p_s} = \frac{p_s - s}{p_s}$$

the price-cost margin (also called Lerner index) in the market with the demand induced by $\phi(x)$ and cost s, which is commonly used as a measure of a firm's market power. Employing the relationship (15) between the inverse hazard rate and the virtual value functions gives

$$\psi'(x) = 1 - \phi'(x) \text{ and } \psi''(x) = -\phi''(x).$$

Hence, $\zeta(x)$ is related to $\psi(x)$ as

$$\zeta(x) = \frac{\phi(x)\phi''(x)}{1-\phi'(x)} = -(x-\psi(x))\frac{\psi''(x)}{\psi'(x)},$$

and the value of $\zeta(p_s)$ can be expressed as

$$\zeta(p_s) = -(p_s - \psi(p_s))\frac{\psi''(p_s)}{\psi'(p_s)} = (p_s - s)A_\psi(p_s) = L_\phi(s)r_\psi(p_s), \tag{30}$$

where the last equality holds by dividing and multiplying $\zeta(p_s)$ by p_s. Using this expression, (28) and (29) have the following interpretation.

Remark 2 $\mathbf{CS}_s$ *is concave (convex) at* s *if and only if*

$$\zeta(p_s) = (p_s - s)A_\psi(p_s) = L_\phi(s)r_\psi(p_s) \geq (\leq) 1, \text{ and}$$

$\mathbf{W}_s$ *is concave (convex) at* s *for* $\beta > 0$ *if and only if*

$$1 - \zeta(p_s) + \beta\psi'(p_s) \leq (\geq) 0.$$

Therefore, the convex properties of $\mathbf{CS}_s$ are driven by a combination of two factors resulting from the demand function: the monopoly's market power and the concavity of its marginal revenue in price. Notably, it is not essential whether the magnitudes of these factors are measured in absolute terms, that is, by using the profit margin and the absolute risk-aversion; or relative terms, i.e., using the price-cost margin and the relative risk-aversion. Specifically, the posterior surplus is concave if and only if both factors are strong enough.

5.4 Strict convexity of surpluses

By combining the above characterizations of the curvature of $\mathbf{CS}_s$, we can derive simple criteria for the strict convexity of $\mathbf{CS}_s$ and, thus, $\mathbf{W}_s$. Specifically, it follows that

$$\zeta(p_s) \gtreqless 0 \Leftrightarrow \phi''(p_s) \gtreqless 0 \Leftrightarrow \psi''(p_s) \lesseqgtr 0 \Leftrightarrow p_s'' \gtreqless 0.$$

This yields the following simple but useful result.

Remark 3 *If any of the following equivalent conditions hold for $s \in \mathbf{C}$:*
i) $\phi(x)$ is weakly concave in x at p_s;
ii) the marginal revenue $\psi(p)$ is weakly convex in p at p_s;
iii) the optimal price p_s is weakly concave in s,
then $\mathbf{CS}_s$ is strictly convex in s, and so is $\mathbf{W}_s$ for any $\beta \geq 0$.

However, while the above characterizations of the convex properties of $\mathbf{CS}_s$ have clear economic meanings, they are expressed in terms of the optimal price p_s, which is defined endogenously. In this regard, it is more useful to derive the conditions on the primitive $\phi(x)$ that determine the shape of posterior surpluses without deriving the properties of p_s. The following lemma complements the previous results by providing the sufficient conditions on $\phi(x)$ under which $\mathbf{CS}_s$ and, hence, $\mathbf{W}_s$ are strictly convex if costs are sufficiently high.

Lemma 1 *If any of the following conditions holds:*
(i) $\bar{x} < \infty$;
(ii) $\bar{x} = \infty$, $\lim_{x\to\infty} \phi(x) = 0$, and $\phi'(x)$ is bounded away from 1;
(iii) $\bar{x} = \infty, \phi(x)$ is bounded, $\phi'(x)$ is bounded away from 1, and $\lim_{x\to\infty} \phi''(x) = 0$;
then $\mathbf{CS}_s$ and, hence, $\mathbf{W}_s$ are strictly convex in s for $s \to \bar{x}$ and any $\beta \geq 0$.

The main implication of the Lemma is that if the lower bound on costs $\underline{c}$ is sufficiently high and $\phi(x)$ satisfies the specified conditions, then it is uniquely optimal to fully disclose information about costs. Technically, the conditions in the lemma guarantee that the value of $\zeta(p_s)$ converges to 0 as $s \to \bar{x}$. This leads to $\mathbf{CS}_s'' > 0$, so that both $\mathbf{CS}_s$ and $\mathbf{W}_s$ are convex in s. However, the key properties of the market demand, which lead to the convexity of the posterior consumer surplus, are different. Specifically, the conditions in parts (i) and (ii) of the Lemma hold if the tail of the distribution is light, specifically, vanishes faster than that of the exponential distribution, which has the constant inverse hazard rate. For such distributions, the proportion of buyers with high valuations rapidly decreases as the price goes up in response to a higher cost. The conditions in part (iii) hold if the tail is smooth enough. The intuition behind the first factor—the light-tailedness of the distribution—is straightforward and most easily explained in the case of the bounded support of valuations, i.e., $\bar{x} < \infty$. As the cost s increases and eventually converges to $\bar{x}$, then p_s converges to $\bar{x}$ as well, and $\mathbf{CS}_s$ converges to 0. Due to the vanishing $\mathbf{CS}_s$, it cannot shrink for high costs at the same pace in response to a change in cost—regardless of the convexity of the price—as that for low costs. In other words, the marginal demand effect dominates the marginal pass-through effect, which results in the convex $\mathbf{CS}_s$. (See the left graph in Figure 2.) The same argument holds if $\bar{x} = \infty$, but the upper tail of the distribution is thin. In this case, even though the consumer surplus is positive for all costs, it rapidly vanishes as s

increases. Therefore, the marginal demand effect still dominates the marginal pass-through effect that leads to the convex $\mathbf{CS}_s$.

The intuition behind the effect of the vanishing convexity of the inverse hazard rate on $\mathbf{CS}_s$ is different. As noted above, $\mathbf{CS}_s$ is concave if its incremental decrease due to a marginal increase in posterior cost, increases in the cost value. However, because the demand is shrinking in response to the higher price induced by a higher cost, then the price has to raise at the increasing rate in order to induce a sufficiently large incremental decrease in $\mathbf{CS}_s$.[29] This logic is reflected in the impact of the marginal pass-through rate p_s'' on $\mathbf{CS}_s$ in (21), or equivalently, the impact of the concavity of the marginal revenue $\psi(p)$ —measured by $A_\psi(p_s)$ and $r_\psi(p_s)$—on $\zeta(p_s)$ in (30). However, if $\phi''(x)$ vanishes as s increases, then so does $\phi''(p_s)$, and, hence, p_s'', $A_\psi(p_s)$, and $r_\psi(p_s)$. In this case, the price increase Δp_s is insufficient to compensate a decrease in the demand. Hence, an incremental change in $\mathbf{CS}_s$ decreases with s, which is equivalent to the convexity of $\mathbf{CS}_s$.

Also, two comments about the convexity of $\phi(x)$ for unbounded valuations, i.e., $\bar{x} = \infty$, are important here. First, if $\phi''(x)$ converges to a limit as $x \to \infty$, then the limit can be only zero. By contradiction, if $\phi''(x)$ converges to a positive limit or diverges, then $\phi(x)$ must be bounded from below by a quadratic function for sufficiently large x. This implies, however, that $\phi(x)$ violates the uniqueness of p_s as it would intersect the horizontal axis from above at some point.[30] Also, if $\phi''(x)$ converges to a negative limit, that is, the curvature of $\phi(x)$ converges to that of the quadratic function, then $\phi(x)$ becomes strictly decreasing for sufficiently high x. As a result, it must intersect the horizontal axis, which violates the condition $\phi(x) > 0$. In this regard, the requirement of vanishing $\phi''(x)$ in part (iii) of the Lemma is not substantially restrictive.

On the other hand parts (i) and (ii) of the Lemma rely on the boundedness of $\phi''(x)$. This is because in general $\phi''(x)$ might not have a limit as $x \to \infty$. For instance, it can oscillate around, have irregular peaks and troughs, and frequently change its behavior as x increases unboundedly. Moreover, $\phi''(x)$ can oscillate with an unboundedly increasing magnitude, even though $\phi(x)$ converges to 0, and $\phi'(x)$ is bounded away from 1. In this case, $\zeta(p_s)$ crosses 1 an infinite number of times as s increases, and $\mathbf{CS}_s$ switches from convex to concave an infinite number of times as the cost increases. The example below illustrates this effect.

Example 2. Let $\mathbf{X} = \mathbf{C} = [1, \infty)$, and consider

$$\phi(x) = a\frac{\cos(bx^\alpha)}{x} + \frac{d}{x^{1/2}},$$

where $a, d > 0$ and $\alpha \in \left(\frac{3}{2}, 2\right)$. The left graph on Fig. 4 depicts $\phi(x)$ for $a = \frac{4}{7}, b = \frac{3}{2}, d = 3,$

[29] See the Figure 3. If the parallel sides of the trapezoid $\Delta\mathbf{CS}_s$—reflecting the quantities demanded for prices induced by the original and new costs—shrink as the cost increases, these changes must be compensated by at least proportional increase in the height of the trapezoid Δp_s, in order to preserve the area $\Delta\mathbf{CS}_s$. Hence, Δp_s must increase in s, which implies the convexity of p_s.

[30] In addition, it will result in $\int_{\underline{x}}^{\infty} \frac{1}{\phi(x)} dx < \infty$ that violates the conditions on $\phi(x)$.

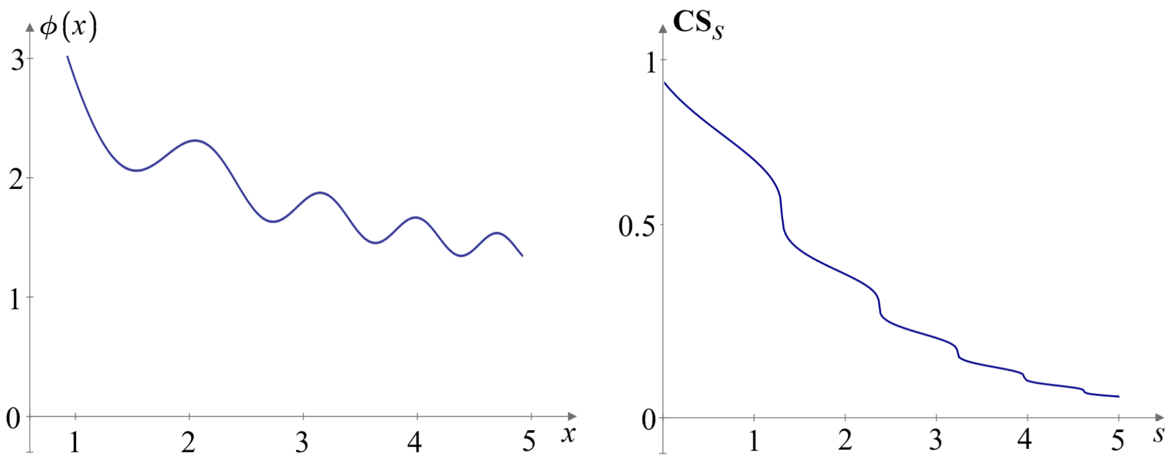

Figure 4: Non-convexity of $\mathbf{CS}_s$ due to an oscillating unbounded $\phi''(x)$.

and $\alpha = \frac{9}{5}$. Then

$$\phi'(x) = -\frac{a}{x^2}\left(\alpha b x^\alpha \sin(bx^\alpha) + \cos(bx^\alpha)\right) - \frac{d}{2}x^{-3/2}, \text{ and}$$
$$\phi''(x) = \frac{a}{x^3}\left(\alpha b(3-\alpha)x^\alpha \sin(bx^\alpha) + \left(2 - \alpha^2 b^2 x^{2\alpha}\right)\cos(bx^\alpha)\right) + \frac{3d}{4}x^{-5/2}.$$

We can select a small enough so that $\phi(x) > 0$ and $\phi'(x) < 1$ for $x \in \mathbf{X}$. Also, the choice of α guarantees that $\phi'(x)$ vanishes while $\phi''(x)$ oscillates unboundedly as $x \to \infty$. The overall effect, however, is that $\zeta(x)$ also oscillates with a large magnitude—substantially bigger than 1—and, hence, crosses 1 an infinite number of times. As a result, the consumer surplus $\mathbf{CS}_s$ changes periodically from convex to concave an infinite number of times as $s \to \infty$. (See the right graph on Fig. 4.) It is worth noting that bounding the set of valuations by some $\bar{x} < \infty$ would still preserve the oscillating behavior of $\mathbf{CS}_s$ as long as the sets $\mathbf{X}$ and $\mathbf{C}$ are large enough relative to $1/b$.

In addition, if valuations are unbounded, and $\phi(x)$ and $\phi''(x)$ do not vanish at infinity, then the convexity of $\mathbf{CS}_s$ for high s can be violated due to the oscillating behavior of $\phi'(x)$ and, thus, the pass-through rate p'_s. The following example demonstrates this effect.

Example 3. Let $\mathbf{X} = \mathbf{C} = [0, \infty)$, and consider the triangle-wave function

$$\phi(x) = d + a - \frac{2a}{\pi}\arccos(\cos(2\pi bx)),$$

where $a, b, d > 0, d > \frac{a}{2}$, and $4ab < 1$.[31] The second condition guarantees that $\phi(x) > 0$.

As x increases, $\phi'(x)$ switches periodically between $4ab$ and $-4ab$, whereas $\phi''(x) = 0$ almost everywhere.[32] Taking (a, b) such that $\frac{2ab}{\pi}$ is close to 1 from below, results in the oscillation of term $\frac{1}{1-\phi'(x)}$ and, hence, the periodic non-convexity of $\mathbf{CS}_s$ as $s \to \infty$. Even though $\mathbf{CS}_s$ is convex on each segment with the linear $\phi(x)$, it is not globally convex. (The

[31] The parameters a and $1/b$ represents the amplitude and the period of $\phi(x)$, respectively.

[32] Formally, $\phi(x)$ is not twice differentiable at peaks and troughs, however, a smooth approximation around these points would work. In this case, $\phi''(x)$ is bounded at these points and does not vanish as $x \to \infty$.

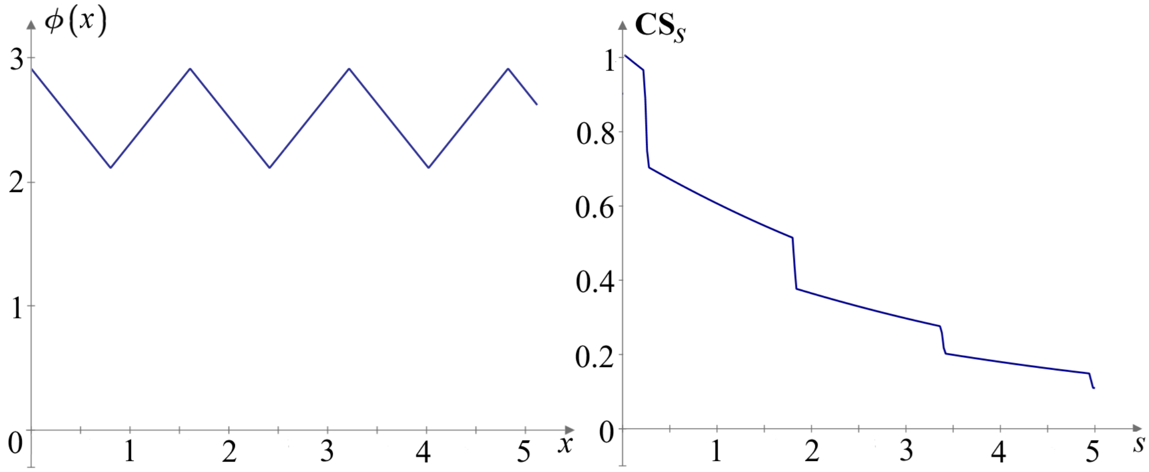

Figure 5: Non-convexity of $\mathbf{CS}_s$ due to an oscillating $\phi'(x)$.

left graph on Figure 5 depicts $\phi(x)$ for $a = \frac{3}{8}, b = \frac{2}{\pi}$, and $d = \frac{5}{2}$.) In fact, $\phi(x)$ in this example is a periodic analogue of Example 1, since in both cases the non-convexity of $\mathbf{CS}_s$ stems from a rapid change in $\phi'(x)$, and, hence, the pass-through rate p'_s.

Examples above demonstrate that the global behavior of $\mathbf{W}_s$ might be rather complex in general. This implies that the derivation of optimal signals and their structure can be quite complicated without imposing additional conditions on the demand function, especially if the set of possible costs is large enough. The following section addresses this issue by providing the conditions, which that substantially simplify the structure of optimal signals.

6 Lower censorship as optimal information disclosure

In this section, we establish the necessary and sufficient conditions on $\phi(x)$ under which optimal signals have a simple form of lower censorship regardless of the prior distribution of costs $H(c)$ and the weight β in the total surplus. That is, it is optimal to pool all costs below some cutoff and fully disclose them above it. Under these conditions, we also provide simple necessary and sufficient conditions for the optimality of full disclosure.

Before providing these conditions and the main results, we introduce some notations. A function $\mathbf{W}_s$ is called *inverted (strictly) S-shaped* on $[\underline{c}, \bar{c}] \subset \mathbb{R}$ if there exists $\hat{s} \in \mathbf{C}$ such that $\mathbf{W}_s$ is (strictly) concave on $[\underline{c}, \hat{s}]$ and (strictly) convex on $[\hat{s}, \bar{c}]$. If $\mathbf{W}_s$ is twice differentiable, then it is (strictly) inverted S-shaped if and only if $\mathbf{W}'_s$ is (strictly) quasi-convex on $[\underline{c}, \bar{c}]$, or equivalently, $\mathbf{W}''_s$ is (strictly) quasi-monotone on $[\underline{c}, \bar{c}]$. Given an inverted (strictly) S-shaped $\mathbf{W}_s$ with the weight β, denote s_β the optimal cutoff for the lower censorship disclosure policy. Specifically, s_0 corresponds to the optimal cutoff for $\mathbf{CS}_s$. Also, $s_\infty = \underline{c}$, since s_∞ corresponds to the optimal cutoff for $\mathbf{PS}_s$, which is strictly convex in s. Therefore, it is optimal to disclose all information about costs. In order to determine the value of s_β in general, denote m_s the expected value of c given that it is below s:

$$m_s = \mathbf{E}[c|c \leq s] = \frac{1}{H(s)} \int\limits_{\underline{c}}^{s} c dH(c), \text{ and}$$

$$m_s' = \frac{h(s)}{H(s)}(s - m_s). \tag{31}$$

The ex-ante total surplus under lower censorship with cutoff s is

$$\mathbf{EW}_s = \int\limits_{\underline{c}}^{s} \mathbf{W}_{m_s} dH(t) + \int\limits_{s}^{\bar{c}} \mathbf{W}_t dH(t) = H(s)\mathbf{W}_{m_s} + \int\limits_{s}^{\bar{c}} \mathbf{W}_t h(t) dt.$$

Taking the derivative of $\mathbf{EW}_s$ results in

$$\begin{aligned} \mathbf{EW}_s' &= H(s)\mathbf{W}_{m_s}' m_s' + h(s)(\mathbf{W}_{m_s} - \mathbf{W}_s) = h(s)\left(\mathbf{W}_{m_s}'(s - m_s) + \mathbf{W}_{m_s} - \mathbf{W}_s\right) \\ &= h(s)(s - m_s)\left(\mathbf{W}_{m_s}' - \frac{\mathbf{W}_s - \mathbf{W}_{m_s}}{s - m_s}\right) = h(s)\mathbf{W}_{m_s}'\left(s - m_s - \int\limits_{m_s}^{s} \frac{\mathbf{W}_z'}{\mathbf{W}_{m_s}'} dz\right), \end{aligned} \tag{32}$$

where the second equality follows from (31), and the last one holds by the fundamental theorem of calculus.

If $\mathbf{W}_s$ is inverted S-shaped, then $\mathbf{EW}_s'$ is quasi-monotone, i.e., it intersects the horizontal axis at most once from above. Specifically, if $s_\beta \in (\underline{c}, \bar{c})$, i.e., it is an interior point, then $\mathbf{EW}_{s_\beta}' = 0$, or equivalent, $\mathbf{W}_{m_{s_\beta}}' = \frac{\mathbf{W}_{s_\beta} - \mathbf{W}_{m_{s_\beta}}}{s_\beta - m_{s_\beta}}$. That is, the tangent line to $\mathbf{W}_s$ at $s = m_{s_\beta}$ coincides with the line that connects points $\left(m_{s_\beta}, \mathbf{W}_{m_{s_\beta}}\right)$ and $\left(s_\beta, \mathbf{W}_{s_\beta}\right)$. Using $m_{\bar{c}} = c^e = E[c]$, if $\mathbf{EW}_{\bar{c}}' \geq 0$, or equivalently, $\mathbf{W}_{c^e}' \geq \frac{\mathbf{W}_{\bar{c}} - \mathbf{W}_{c^e}}{\bar{c} - c^e}$, then $s_\beta = \bar{c}$, that is, no disclosure is optimal.[33] As noted by Kolotilin et al. (2022), no disclosure can be optimal even though $\hat{s} \in (\underline{c}, \bar{c})$, i.e., the inflection point $\hat{s}$ of $\mathbf{W}_s$ is an interior point of $\mathbf{C}$. In other words, $\mathbf{W}_s$ must not be globally concave on $\mathbf{P_C}$ in order to pool all information about costs.[34] It is the case if the portion of $\mathbf{C}$ on which the inverted S-shaped $\mathbf{W}_s$ is convex in s, is small enough, and $H(c)$ is concentrated at low values.

We start the characterization of optimal signals with the case of $\beta = 0$, i.e., $\mathbf{W}_s = \mathbf{CS}_s$.

Proposition 1 *Lower censorship is a (uniquely) optimal for $\mathbf{CS}_s$ with any $H(c)$ supported on $\mathbf{C}$ if and only if $1 - \zeta(x)$ is (strictly) quasi-monotone in $x \in \mathbf{P_C}$. If $1 - \zeta(x)$ is (strictly) quasi-monotone in $x \in \mathbf{P_C}$, then full disclosure is (uniquely) optimal for $\mathbf{CS}_s$ if $\zeta(p_{\underline{c}}) < 1$ and suboptimal if $\zeta(p_{\underline{c}}) > 1$.*[35]

The intuition behind the Proposition is straightforward. First, (25) implies that the quasi-monotonicity of $1 - \zeta(x)$ is equivalent to that of $\mathbf{CS}_s''$, which is equivalent to the

[33] If $\bar{c} = \bar{x}$, then $\mathbf{W}_{\bar{c}} = 0$, and $\mathbf{W}_{c^e}' \geq \frac{\mathbf{W}_{\bar{c}} - \mathbf{W}_{c^e}}{\bar{c} - c^e}$ can be simplified further to $\mathbf{W}_{c^e}' \geq -\frac{\mathbf{W}_{c^e}}{\bar{c} - c^e}$.

[34] See Figure 1(b) in their paper.

[35] If $\zeta(p_{\underline{c}}) = 1$, then full disclosure is (uniquely) optimal if $1 - \zeta(x)$ is (strictly) pseudo-monotone in $x \in \mathbf{P_C}$, which is a slightly stronger condition that quasi-monotonicity.

inverted S-shapedness of $\mathbf{CS}_s$. By Theorem 1 in Kolotilin et al. (2022), the lower censorship is an optimal signal structure for $\mathbf{CS}_s$ with any distribution of costs $H(c)$ if and only if $\mathbf{CS}_s$ is inverted S-shaped. Second, if $1-\zeta(x)$ is quasi-monotone on $\mathbf{P_C}$ and $\zeta(p_{\underline{c}}) < 1$, then $\mathbf{CS}_s$ is convex for all $s \in \mathbf{C}$. Therefore, it is optimal to fully disclose information about costs. Finally, the last part holds because $\zeta(p_{\underline{c}}) > 1$ implies the strict concavity of $\mathbf{W}_s$ in the neighborhood of $\underline{c}$. Using this property and the fact that m_s converges to s as s converges to $\underline{c}$ leads to $\mathbf{W}'_{m_s} > \frac{\mathbf{W}_s - \mathbf{W}_{m_s}}{s-m_s}$ for costs near $\underline{c}$. By (32), it means that the marginal gains in the ex-ante total surplus from pooling low costs in some neighborhood of $\underline{c}$ are positive, $\mathbf{EW}'_{\underline{c}} > 0$. Combining these results yields the simple necessary and sufficient condition for the optimality of full disclosure given that

For $\beta > 0$, however, the quasi-monotonicity of $1-\zeta(x)$ alone does not guarantee that $\mathbf{W}_s$ takes the inverted S-shaped form. This is because a sum of S-shaped functions is not S-shaped in general. It is easy to see this by employing their second derivatives, and noting a sum of quasi-monotone functions is not necessarily quasi-monotone. Also, the strict convexity of $\mathbf{PS}_s$ does not imply that $\mathbf{W}_s$ is inverted S-shaped.

The next result addresses this issue and completely characterizes scenarios in which the optimal signal structure takes a form of lower censorship regardless of the distribution of costs $H(c)$ and the weight β.

Theorem 1 *Suppose* $1-\zeta(x)$ *is (strictly) quasi-monotone in* $x \in \mathbf{P_C}$*. Then:*
(i) lower censorship is (uniquely) optimal for $\mathbf{W}_s$ *with any* $\beta > 0$ *and* $H(c)$ *supported on* $\mathbf{C}$ *if and only if* $\frac{1-\zeta(x)}{1-\phi'(x)}$ *is (strictly) increasing in* $x \in \{\mathbf{P_C} | \zeta(x) > 1\}$*; and*
(ii) if $\frac{1-\zeta(x)}{1-\phi'(x)}$ *is (strictly) increasing in* $x \in \{\mathbf{P_C} | \zeta(x) > 1\}$*, then full disclosure is (uniquely) optimal for* $\mathbf{W}_s$ *if* $\zeta(p_{\underline{c}}) + \beta\phi'(p_{\underline{c}}) < 1+\beta$ *and suboptimal if* $\zeta(p_{\underline{c}}) + \beta\phi'(p_{\underline{c}}) > 1+\beta$.[36]

The proof of the theorem consists of several parts. For the 'if' part of (i), we apply the aggregation theorem by Quah and Strulovici (2012) and their signed-ratio monotonicity condition to functions $1-\zeta(x)$ and $1-\phi'(x)$ in order to guarantee that a linear combination of these functions is quasi-monotone in x.[37] As a result, the second derivative $\mathbf{W}''_s$ of the posterior surplus $\mathbf{W}_s$ is pseudo-monotone and, hence, quasi-monotone, in cost s, which implies that it is inverted S-shaped. Second, we apply the result by Kolotilin et al. (2022) who show that lower censorship is a (uniquely) optimal signal structure for $\mathbf{W}_s$ with any distribution of costs $H(c)$ if and only if $\mathbf{W}_s$ is (strictly) inverted S-shaped.[38] For the 'only-if' of (i), we marginally extend the result of Quah and Strulovici (2012). Specifically, if their signed-ratio monotonicity condition is violated, then a linear combination of pseudo-monotone functions is generally not only not pseudo-monotone, but also not quasi-monotone. The part (ii) of the Theorem relies on the same arguments as those used in Proposition 1.

[36] If $\zeta(p_{\underline{c}}) + \beta\phi'(p_{\underline{c}}) = 1+\beta$, then full disclosure is (uniquely) optimal if $1-\zeta(x)$ is (strictly) pseudo-monotone in $x \in \mathbf{P_C}$.

[37] Formally, Quah and Strulovici (2012) state their proposition for pseudo-monotone functions, which is a subset of quasi-monotone functions. However, Choi and Smith (2017) note that the 'if' part of the result by Quah and Strulovici (2012) can be also applied to aggregate quasi-monotone functions if these functions do not common flat regions at which they are equal to 0.

[38] Kolotilin et al. (2022) show the optimality of upper censorship for an S-shaped function. It is equivalent to the optimality of lower censorship for an inverted S-shaped function.

In order to explain the economic intuition behind the Theorem, we need to delve into the meaning of functions $1-\zeta(x)$ and $1-\phi'(x)$ and the signed-ratio monotonicity condition on them. First, as follows from (25) and (26), the normalized values of these functions at price p_s reflect the curvatures of the posterior consumer and producer surpluses, respectively, at cost s. Second, the signed-ratio monotonicity condition on $1-\zeta(x)$ and $1-\phi'(x)$ is equivalent to the ranking of the growth rates of the functions.[39] That is, the rate of a change in the convexity of the posterior producer surplus $\mathbf{PS}_s$, must always exceed the rate of a change in the concavity of the posterior consumer surplus $\mathbf{CS}_s$. Altogether, this implies that the curvature of the posterior total surplus $\mathbf{WS}_s$ must change monotonically—from concave to convex—at any inflection point $\hat{s}$, i.e., such that $\mathbf{WS}''_{\hat{s}}=0$. This means that there can be at most one inflection point, and $\mathbf{WS}_s$ cannot contain two disjoint intervals of costs on which it is either convex or concave. The example below illustrates Theorem 1.

Example 4. Let $\mathbf{X}=[0,\infty)$ and consider the Burr distribution of consumer valuations.[40] It has two parameters, and its density, cdf, and the inverse hazard rate are

$$f(x)=\frac{abx^{a-1}}{(1+x^a)^{b+1}},\ F(x)=1-(1+x^a)^{-b},\ \text{and}\ \phi(x)=\frac{x^a+1}{abx^{a-1}},$$

where $a,b>0$. Specifically, put $a>1$ and $b=1$.[41] Then $\lim_{x\to 0}\phi(x)=\infty, \phi(x)\to\frac{x}{a}<x$ as $x\to\infty$, and

$$\phi'(x)=\frac{x^a-(a-1)}{ax^a}=\frac{1}{a}-\frac{a-1}{ax^a}<\frac{1}{a}<1 \text{ for all } x>0,$$

imply that for any $s\in[0,\infty)$ there is the unique optimal price p_s. Also, the function $\zeta(x)$ is

$$\zeta(x)=\frac{1}{x^a}.$$

Clearly, $1-\zeta(x)$ is strictly increasing, that is, strictly quasi-monotone in x, and $\zeta(x)\gtrless 1$ if and only if $x\lessgtr 1$.

Altogether, we obtain

$$\frac{1-\zeta(x)}{1-\phi'(x)}=\frac{a}{a-1}-\frac{2a}{(a-1)(x^a+1)}=\frac{a}{a-1}\left(1-\frac{2}{x^a+1}\right),$$

which is strictly increasing in x for all $x>0$. That is, $\phi(x)$ satisfies all conditions of Theorem 1, so that lower censorship is uniquely optimal for $\mathbf{W}_s$ with any $\beta>0$ and $H(c)$ supported on $\mathbf{C}\subset[0,\infty)$. Fig. 6 depicts $\phi(x)$ and $\mathbf{CS}_s$ for $a=7$.

[39] Consider two differentiable functions $\varphi(x)$ and $g(x)$, such that $\varphi(x)<0<g(x)$. The signed ratio monotonicity condition, which says that $\frac{\varphi(x)}{g(x)}$ must be increasing, is equivalent to the ranking of the rates, $\frac{g'(x)}{g(x)}\geq\frac{\varphi'(x)}{\varphi(x)}$. It guarantees that for any point $\hat{x}$, such that $\varphi(\hat{x})+\alpha g(\hat{x})=0$, that is, $\alpha=-\frac{\varphi(\hat{x})}{g(\hat{x})}>0$, we have $\varphi'(\hat{x})+\alpha g'(\hat{x})=\varphi'(\hat{x})-\frac{\varphi(\hat{x})}{g(\hat{x})}g'(\hat{x})=-\varphi(\hat{x})\left(\frac{g'(\hat{x})}{g(\hat{x})}-\frac{\varphi'(\hat{x})}{\varphi(\hat{x})}\right)\geq 0$. That is, a linear combination of functions, $\varphi(x)+\alpha g(x)$, can intersect the x-axis only from below.

[40] This distribution includes other families as special cases, e.g., log-logistic and Lomax distributions. It is also known as Singh–Maddala distribution and often used to model household income (Singh and Maddala, 1976).

[41] The choice of $b=1$ is for analytical and notational simplicity only. The results can be easily extended to the general case.

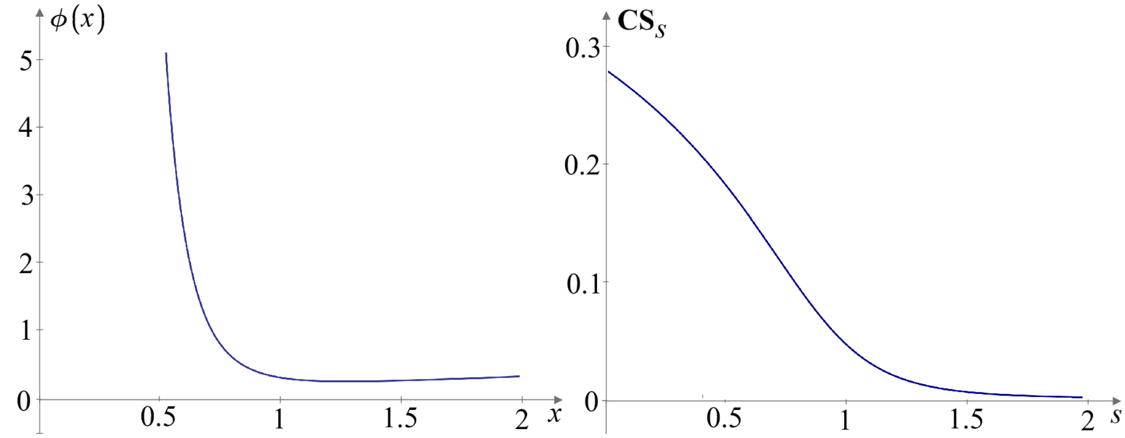

Figure 6: An inverted S-shaped $\mathbf{CS}_s$ for Burr distribution of valuations

For part (ii) of the Theorem, suppose $\mathbf{C} = [0, \infty)$. Because $\underline{c} = 0$, the lowest optimal price is $p_0 = \frac{1}{(a-1)^{\frac{1}{a}}}$. Then

$$\zeta\left(p_{\underline{c}}\right) + \beta\phi'\left(p_{\underline{c}}\right) = a\left(1-\beta\right) + 2\beta - 1 < 1 + \beta,$$

and, hence, full disclosure is optimal if and only if

$$\beta > \beta^* = \frac{a-2}{a-1} = 1 - \frac{1}{a-1}.$$

Thus, if $a \leq 2$, then full disclosure is optimal for any $\beta \geq 0$. As a increases unboundedly, β^* converges to 1; i.e., pooling information about low costs is optimal even for high values of β. Conversely, for a given a, it is optimal to disclose all information as β becomes high enough.

The following result generalizes this observation and shows that it is optimal to disclose more information if the weight of the producer surplus increases.

Proposition 2 *Suppose* $1 - \zeta\left(x\right)$ *is strictly quasi-monotone and* $\frac{1-\zeta(x)}{1-\phi'(x)}$ *is strictly increasing whenever* $\zeta\left(x\right) > 1$ *for* $x \in \mathbf{P_C}$*. Then* s^*_β *is decreasing in* β*.*

The proof follows from Theorem 2 in Curello and Sinander (2024). Because $\mathbf{PS}_s$ is convex in s, then putting a higher weight on it implies that the total surplus $\mathbf{W}_s$ becomes 'coarsely' more convex.[42] As a result, the optimal signal structure(s) become more dispersed. In terms of lower censorship, this implies that the cutoff s^*_β is decreasing in β.

7 Linear surpluses

In this section, we present another main result of the paper. Specifically, for any given weight $\beta \geq 0$ we provide the full characterization of the set of demand functions under which the posterior total surplus $\mathbf{W}_s$ is linear in cost s. In other words, the concavity of the posterior

[42] Following Curello and Sinander (2024), a function $V : \mathbf{C} \to \mathbb{R}$ is *coarsely more convex* than a function $U : \mathbf{C} \to \mathbb{R}$ if the convexity of U on any interval $[x, y] \subset \mathbf{C}$ implies the convexity of V on $[x, y]$.

consumer surplus perfectly counterbalances the convexity of the producer surplus at any cost. Using our characterization, we explicitly derive the main market characteristics as functions of cost: the quantity demanded, the marginal revenue, the demand elasticity, the optimal price, the profit margin, the pass-through rate, the marginal pass-through rate, and the producer surplus.

The linearity of the posterior total surplus in cost has two main implications, one from the information design perspective and the other one from the policy perspective. First, the linearity of $\mathbf{W}_s$ implies the invariance of the ex-ante surplus $\mathbf{W}$ to the seller's information about cost. That is, there are no benefits or losses of controlling the seller's information. The second implication is the policy one. Consider the case of a gradual increase in an input price due to, for instance, the minimum wage rate, which is spread across several periods. If it gradually increases over several periods, then the marginal impact of each wage increase by \$1 on the ex-ante total surplus remains constant in all periods. Thus, the marginal impact of the minimum wage on this surplus does not depend on the current market outcome. Also, by deriving and analyzing the demand functions that induce the linear total surplus, we can gain insight into the key properties of the market demand that induce the convex and concave total surplus without employing the resulting market characteristics.

Another application of our results is analytical. Specifically, even employing the characterization above, it is not easy to obtain the demand functions such for which the posterior total surplus is always concave in cost for an arbitrary bounded interval. Our results below allow to solve this issue easily. For that purpose, it is sufficient to derive a single demand function for some benchmark weight $\beta^* > 0$. Then the total surplus $\mathbf{W}_s$ with the weight β is always strictly concave in s if $\beta < \beta^*$ and strictly convex if $\beta > \beta^*$.

The problem of deriving the demand functions can be approached from several ways, however, their effectiveness and complexity are different. One way is to use the relationship (20) between the negative constant $\mathbf{W}'_s$, the quantity demanded $Q(p_s)$, and the pass-through rate p'_s. Alternatively, one can employ the relationship (22) between $\mathbf{W}''_s$, the pass-through rate p'_s, the marginal pass-though rate p''_s, and the inverse hazard rate $\phi(p_s)$. The main difficulty with these approaches is that the corresponding differential equations are defined in terms of the demand function $Q(x)$ and the optimal price p_s, which is itself determined by $Q(x)$. This 'double-endogeneity' issue substantially complicates the analysis.

We resolve this issue by employing (27) and using the monotone transformation $x = p_s$.[43] In this case, the problem is defined in terms of the inverse hazard rate $\phi(x)$ only. Specifically, given the weight $\beta \geq 0$ and the set of costs $\mathbf{C} = [\underline{c}, \bar{c}]$, it is required to find the optimal prices $p_{\underline{c}}$ and $p_{\bar{c}}$ for the cost bounds, and the function $\phi : [p_{\underline{c}}, p_{\bar{c}}] \to \mathbb{R}_{++}$, such that $p_{\underline{c}}$ and $p_{\bar{c}}$ satisfy (4) for $\phi(x)$, and $\phi(x)$ is a solution to the autonomous second-order differential equation[44]

$$1 - \frac{\phi''(x)\,\phi(x)}{1 - \phi'(x)} + \beta\,(1 - \phi'(x)) = 0, \tag{33}$$

for $x \in [p_{\underline{c}}, p_{\bar{c}}]$, such that $\phi'(x) < 1$ as required by (6).

We start the analysis by transforming (33) into the first-order differential equation.

[43] If $p'_s = 0$ for $s \in \mathbf{C}$, then $\mathbf{CS}_s$ is also linear as follows from (25). It corresponds to the case of the degenerated $F(x)$ such that all valuations are equal to x_0. Then $p_s = x_0$ if $s \leq x_0$. As a result, $p'_s = 0, \phi'(p_s) = \phi'(x_0) = -\infty$, and $\mathbf{CS}_s = 0$ for $s \leq x_0$. I am thankful to Greg Pavlov for providing this example.

[44] If $\bar{c} = \infty$, then $p_{\bar{c}} = \infty$.

Lemma 2 *The equation (33) such that $\phi'(x) < 1$ is equivalent to*

$$C\phi(x) = \begin{cases} \frac{(1+\beta-\beta\phi'(x))^{1+\frac{1}{\beta}}}{1-\phi'(x)} & \text{if } \beta > 0 \\ \frac{e^{1-\phi'(x)}}{1-\phi'(x)} & \text{if } \beta = 0. \end{cases} \tag{34}$$

Solving (34) requires some technicalities. First, it can be expressed as

$$C\phi(x) = y_\beta(z), \tag{35}$$

where

$$y_\beta(z) = \begin{cases} \frac{(1+\beta z)^{1+\frac{1}{\beta}}}{z} & \text{if } \beta > 0 \\ \frac{e^z}{z} & \text{if } \beta = 0, \text{ and} \end{cases} \tag{36}$$

$$z = 1 - \phi'(x). \tag{37}$$

Hence, $\phi'(x) < 1$ if and only if $z > 0$. Also, denote $\mathcal{H}_\beta(t) = y_\beta^{-1}(t)$ the inverse function of $y_\beta(.)$. Thus, the solutions to (35) with respect to z are given by

$$z = \mathcal{H}_\beta(C\phi(x)) \tag{38}$$

For example, for $\beta = 0$ that corresponds to the linear consumer surplus, $\mathbf{W}_s = \mathbf{CS}_s$, we have

$$\mathcal{H}_0(t) = -W\left(-\frac{1}{t}\right),$$

where $W(.)$ is Lambert W function (also called product logarithm). Similarly, for $\beta = 1$ that corresponds to the linear social welfare, $\mathbf{W}_s = \mathbf{CS}_s + \mathbf{PS}_s$, we have

$$\mathcal{H}_1(t) = \frac{t - 2 \pm \sqrt{t^2 - 4t}}{2}.$$

Combining (37) and (38) allows us to transform (34) into the ordinary 1st-order autonomous differential equation:

$$\begin{aligned} 1 - \phi'_\beta(x) &= \mathcal{H}_\beta(C\phi(x)), \text{ or} \\ \phi'_\beta(x) &= 1 - \mathcal{H}_\beta(C\phi(x)). \end{aligned} \tag{39}$$

Next, $y_\beta(z)$ is strictly quasi-convex in z, achieves the minimum

$$\underline{y}_\beta = \begin{cases} (1+\beta)^{1+\frac{1}{\beta}} & \text{if } \beta > 0, \\ e & \text{if } \beta = 0, \end{cases}$$

at $z^* = 1$, and increases unboundedly as z goes to 0 and ∞. This implies that for any $\beta \geq 0$, (35) has no solutions if $C\phi(x) < \underline{y}_\beta$, one solution $z^* = 1$ if $C\phi(x) = \underline{y}_\beta$, and two solutions, $z_{0,\beta}$ and $z_{-1,\beta}$, such that $0 < z_{0,\beta} < 1 < z_{-1,\beta}$ if $C\phi(x) > \underline{y}_\beta$. Equivalently, $\mathcal{H}_\beta(t)$ is defined for $t \geq \underline{y}_\beta$ only, and has two branches, $\mathcal{H}_{0,\beta}(t)$ and $\mathcal{H}_{-1,\beta}(t)$, such that $\mathcal{H}_{0,\beta}\left(\underline{y}_\beta\right) = \mathcal{H}_{-1,\beta}\left(\underline{y}_\beta\right) = 1$, and $0 < \mathcal{H}_{0,\beta}(t) < 1 < \mathcal{H}_{-1,\beta}(t)$ for $t > \underline{y}_\beta$. (Here, we use the notation similar to Lambert W function.) Also, $\mathcal{H}_{0,\beta}(t)$ and $\mathcal{H}_{-1,\beta}(t)$ are continuously differentiable in t for $t > \underline{y}_\beta$ and thus Lipschitz continuous. By Picard–Lindelöf theorem, for any $\beta \geq 0$ and (C, x_0, ϕ_0), such that $C > 0$ and $\phi_0 > \frac{\underline{y}_\beta}{C}$, the equation (39) with the boundary condition $\phi(x_0) = \phi_0$ has the unique solution in a neighborhood of (x_0, ϕ_0).

The final remark is that the solutions are parameterized by two constants, $C > 0$ and $x^* \in \mathbb{R}$, where C is the scale parameter, and x^* is the location parameter and the unique minimizer of $\phi(x)$. Given these preliminaries, we now characterize all inverse hazard rates $\phi_\beta(x)$, such that $\phi'_\beta(x) < 1$ on $\mathbf{P_C}$, and the total surplus $\mathbf{W}_s$ is linear in s for a given $\beta \geq 0$.

Theorem 2 *Given $\beta \geq 0$ and $-\infty < \underline{c} < \bar{c} < \infty$, $\mathbf{W}_s$ is linear in s on $[\underline{c}, \bar{c}]$ subject to $\phi'(x) < 1$ for $x \in [p_{\underline{c},\beta}, p_{\bar{c},\beta}]$ if and only if*[45]

$$\phi_\beta(x) = \frac{y_\beta\left(\chi_\beta^{-1}(x)\right)}{C}, \tag{40}$$

for all $x \in [p_{\underline{c},\beta}, p_{\bar{c},\beta}]$, where $y_\beta : \mathbb{R}_{++} \to \mathbb{R}_{++}$ is given by (36), $\chi_\beta : \mathbb{R}_{++} \to \mathbb{R}$ is

$$\chi_\beta(z) = \begin{cases} \frac{1}{C}\int\limits_z^1 \frac{(1+\beta u)^{\frac{1}{\beta}}}{u^2} du + x^* & \text{if } \beta > 0, \\ \frac{1}{C}\int\limits_z^1 \frac{e^u}{u^2} du + x^* & \text{if } \beta = 0, \end{cases} \tag{41}$$

$C > 0, x^* \in \mathbb{R}$, and $p_{s,\beta}$ exists, unique, and satisfies (4) for $\phi_\beta(x)$ and $s \in [\underline{c}, \bar{c}]$.

It is worth noting that $\chi_\beta(z)$ can be expressed via special functions:

$$\chi_\beta(z) = \begin{cases} \frac{\beta}{C}\left(-\mathcal{B}_{1+\beta z}\left(\frac{1}{\beta}+1, -1\right) + \mathcal{B}_{1+\beta}\left(\frac{1}{\beta}+1, -1\right)\right) + x^* & \text{if } \beta > 0, \\ \frac{1}{C}\left(-\operatorname{Ei}(z) + \frac{e^z}{z} - e + \operatorname{Ei}(1)\right) + x^* & \text{if } \beta = 0, \end{cases}$$

where $\mathcal{B}_x(a, b)$ is the incomplete beta function, and $\operatorname{Ei}(x)$ is the exponential integral.

The key element of the Theorem is the variable $z > 0$, which has several economic interpretations and properties. First, it follows from (17) that z is the reciprocal of the pass-through rate p'_s, i.e., $z = \frac{1}{p'_s}$. Second, z is related to the inverse hazard $\phi(x)$ as

$$z = 1 - \phi'_\beta(x) = \mathcal{H}_\beta(C\phi(x)).$$

Third, expressing both the independent variable $x = \chi_\beta(z)$ and the dependent variable $\phi = y_\beta(z)$ as functions of z provides the parametric representation of the inverse hazard rate via (36) and (41). As we show below, this representation substantially simplifies the analysis and allows us to determine the market outcome and its major characteristics.

Theorem 2 raises natural questions about the market structure: the form of the distribution of consumer valuations $F_\beta(x)$ or equivalently, the demand function $Q_\beta(x)$, its support supp(F_β), and the properties of the density $f_\beta(x)$, the inverse hazard rate $\phi_\beta(x)$, and the price elasticity $\varepsilon_\beta(p)$. The following result addresses these questions.

[45] The results hold for $\bar{c} = \infty$ as well. However, in this case $\mathbf{W}_s = \mathbf{CS}_s = \infty$ as we show below.

Theorem 3 *Consider $\phi_\beta(x)$ given by (40). Then:*
(i) The domain of ϕ_β is $\mathcal{X}_\beta = (\underline{\chi}_\beta, \infty)$, where

$$\underline{\chi}_\beta = -\frac{1}{C}\int\limits_1^\infty \frac{(1+\beta u)^{\frac{1}{\beta}}}{u^2}du + x^* = \begin{cases} -\infty & \text{if } \beta \in [0,1], \\ > -\infty & \text{if } \beta > 1, \end{cases}$$

(ii) $\phi_\beta(x) \geq \phi_\beta(x^*) = \frac{y_\beta}{C}$, $\lim_{x\to\underline{\chi}_\beta} \phi_\beta(x) = \lim_{x\to\infty}\phi_\beta(x) = \infty$, $\lim_{x\to\underline{\chi}_\beta}\phi'_\beta(\underline{x}) = -\infty$, and $\lim_{x\to\infty}\phi'_\beta(x) = 1$;
(iii) For a given $\underline{x} = \inf\, supp(F_\beta) > \underline{\chi}_\beta$ and $x \geq \underline{x}$,

$$Q_\beta(x) = 1 - F_\beta(x) = \frac{\frac{1}{\chi_\beta^{-1}(\underline{x})} + \beta}{\frac{1}{\chi_\beta^{-1}(x)} + \beta} \text{ for } \beta \geq 0, \tag{42}$$

$$f_\beta(x) = \begin{cases} C\left(\frac{1}{\chi_\beta^{-1}(\underline{x})} + \beta\right)\frac{\left(\chi_\beta^{-1}(x)\right)^2}{\left(1+\beta\chi_\beta^{-1}(x)\right)^{2+\frac{1}{\beta}}} & \text{if } \beta > 0, \\ \frac{C}{\chi_0^{-1}(\underline{x})}\left(\chi_0^{-1}(x)\right)^2 e^{-\chi_0^{-1}(x)} & \text{if } \beta = 0. \end{cases}$$

(iv) $\mathbf{E}_{F_\beta}[x] = \mathbf{CS}_{s,\beta} = \infty$ for any $\beta \geq 0$;
(v) $f_\beta(x) > 0$ is strictly pseudo-concave with the maximizer $x^{**}_\beta = \max\{\underline{x}, \chi_\beta(2)\} \leq \max\{\underline{x}, x^*\}$; and
(vi) $\varepsilon_\beta(p)$ is strictly pseudo-convex in p with the minimizer $\hat{p}_\beta < \infty$.

The Theorem provides several important insights about the market properties. First, the convexity of the inverse hazard rate along with the condition $s \leq \underline{x}$ imply that the market is the *mass market*, in which the optimal price is relatively low compared to valuations of most of consumers.[46] Hence, the product is sold to a relatively large portion of consumers, including uninformed consumers whose decisions are based on the prior information only. In our market, in addition, the demand function converges to the unit-elasticity demand. This is because (ii) implies that the inverse hazard rate of a distribution $F_\beta(x)$ converges to the identity function—regardless of β—as x increases unboundedly. This means that the distribution is heavy-tailed or, equivalently, a large mass of consumers have high valuations. In fact, its tail is so heavy that the mean value $\mathbf{E}_{F_\beta}[X]$ and the posterior consumer surplus are unbounded.[47] In this light, the market with the globally linear $\mathbf{W}_s$ can be viewed as the extreme mass one, i.e., the limit of the market in which the product valuation by an average consumer $\mathbf{E}_F[x]$ converges to infinity while the monopoly price p_s remains finite.

Also, the distribution of valuations exhibits several properties similar to those of common distributions with semi-infinite support, for example, Dagum, Burr (Singh–Maddala), Lévy,

[46] We use the definition of a market as a mass and niche one based on the ranking of the mean of consumer valuations $\mathbf{E}_F[x]$ and the optimal monopoly price p_s (see, for example, Ivanov, 2009). According to this definition, the market is mass if $\mathbf{E}_F[x] > p_s$, and niche otherwise.

[47] In fact, $\mathbf{E}_{F_\beta}[X] = \infty$ can be proved by applying solely economic arguments. If $\mathbf{W}_s$ is linear in s, has the negative slope and is positive everywhere, it can be the case only if it is infinitely large. As noted above, however, $\mathbf{CS}_s$ is finite if and only if the mean value $\mathbf{E}_{F_\beta}[X]$ is finite. Because $\mathbf{PS}_s$ is finite, then $\mathbf{CS}_s = \mathbf{E}_{F_\beta}[X] = \infty$.

and inverse-gamma ones. First, all these distributions have strictly convex inverse hazard rates, which are bounded away from 0. Second, the inverse hazard rates of these distributions converge to a linear function as x increases unboundedly, and for some parameter values, to the identity function as $\phi_\beta(x)$ does. That is, all these distributions have heavy tails or, equivalently, the mass of consumers with high valuations is sufficiently large. Finally, their densities are everywhere positive and strictly pseudo-concave and, hence, unimodal. This property implies that $F_\beta(x)$ is S-shaped, that is, it is convex if $x < x^{**}$ and concave if $x > x^{**}$. Hence, the demand $Q_\beta(p) = 1 - F_\beta(p)$ is concave if $p < x^{**}$ and convex otherwise.[48]

At the same time, the distribution violates a few properties commonly used in various models of markets and mechanisms. First, since $\phi(x)$ achieves the minimum at x^*, it is not decreasing as commonly assumed in the mechanism design literature. In turn, this implies that the density $f_\beta(x)$ is not log-concave as typically imposed in differentiated product markets.[49] Finally, the distribution violates the Marshall's Second Law of Demand, which requires the price elasticity of demand be decreasing in price and is often used in models of monopolistic competition. Because the price elasticity $\varepsilon_\beta(p)$ is strictly pseudo-convex in p and achieves the minimum at $\hat{p}_\beta < \infty$, it is either non-monotonic (if $\hat{p}_\beta > \underline{x}$) or strictly increasing in p (if $\hat{p}_\beta = \underline{x}$), i.e., behaves in the opposite way to the Marshall's Law. In other words, the models of markets and mechanisms with these conditions rule out at least a subset of the support of the demand functions under which the total surplus is linear in cost.

Notably, the lowest consumer valuation $\underline{x}$ can be arbitrarily small if β is small enough (i.e., $\beta \leq 1$), but is bounded from below otherwise. Intuitively, if β is high, this requires $\mathbf{CS}_s$ be sufficiently concave in order to compensate for the convexity of the heavy-weighted $\mathbf{PS}_s$. For small costs, however, the optimal price $p_{s,\beta}$ and, thus, the profit margin $\phi_\beta(p_{s,\beta})$ are limited. In addition, the value of the derivative $\phi'_\beta(x)$ is highly negative for small x as follows from part (ii), that is, $1 - \phi'(p_s)$ is substantially above 1. Thus, the only factor that can concavify $\mathbf{CS}_s$ is the curvature $\phi''_\beta(x)$. In this case, however, it has to be so high that $\phi(x)$ accelerates unboundedly for finitely small values of x.

In addition, Theorem 3 establishes another property of the variable z. Specifically, the quantity demanded is proportional to $\frac{z}{1+\beta z} = \frac{1}{\frac{1}{z}+\beta}$. Hence, for $\beta = 0$ the demand is linear in z. This is because using (42) and $z = \chi_\beta^{-1}(x)$ results in

$$q = \mathbf{Q}_\beta(z) = Q_\beta(\chi_\beta(z)) = 1 - F_\beta(\chi_\beta(z)) = \frac{\frac{1}{\bar{z}_\beta} + \beta}{\frac{1}{z} + \beta}, \tag{43}$$

where $\bar{z}_\beta = \chi_\beta^{-1}(\underline{x})$ is the value of z at highest demand $\bar{q} = 1$. The relationship (43) can be also obtained from (20) by recalling that $z = \frac{1}{p'_s}$ is the reciprocal of the pass-through rate, and noting that the marginal posterior total surplus

$$\mathbf{W}'_s = -Q(p_s)(p'_s + \beta) = -q\left(\frac{1}{z} + \beta\right),$$

must be constant for the linear $\mathbf{W}_s$.

Finally, (43) can be used to derive the inverse demand function (13) and the standard

[48] In the case of $\chi_\beta(2) < \underline{x}$, $f_\beta(x)$ is strictly decreasing, that is, $F_\beta(x)$ is concave, and $Q_\beta(p)$ is convex.

[49] Bagnoli and Bergstrom (2005) show that the increasing hazard rate, or equivalently, the decreasing inverse hazard rate, is the necessary condition of the log-concavity of the density.

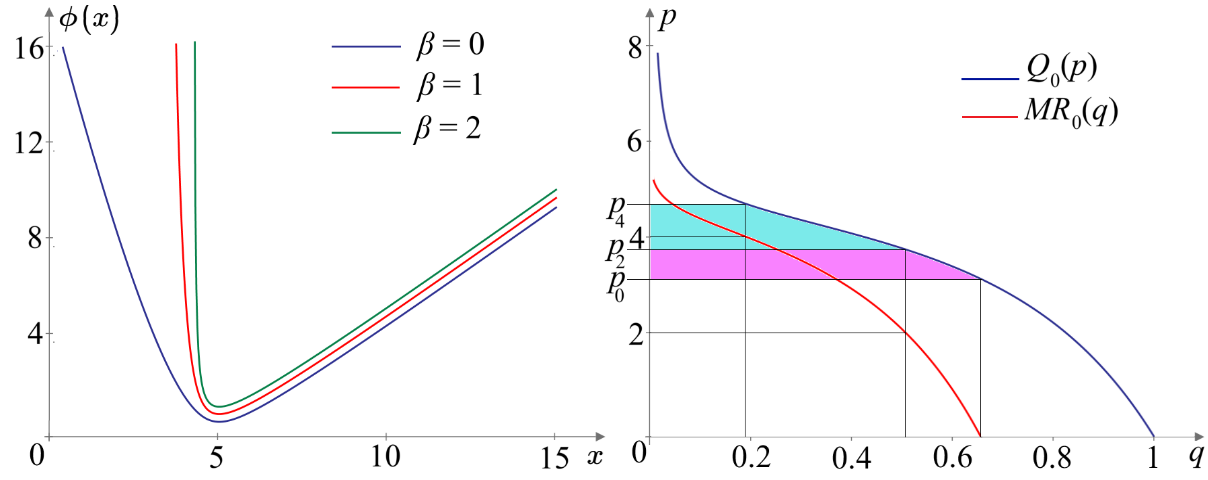

Figure 7: Inverse hazard rates for various β and demand for the linear $\mathbf{CS}_s$ $(\beta = 0)$

marginal revenue. Because the quantity q induces the value of z as

$$z_\beta (q) = \frac{q}{\frac{1}{\bar{z}_\beta} + \beta (1-q)},$$

this results in

$$p = P_\beta (q) = \chi_\beta (z_\beta (q)),$$

and gives the standard marginal revenue

$$MR_\beta (q) = P_\beta (q) + P'_\beta (q) q = \chi_\beta (z_\beta (q)) + \chi'_\beta (z_\beta (q)) z'_\beta (q) q.$$

Fig. 7 depicts the inverse hazard rates $\phi_\beta (x)$ for $\beta = 0, 1$, and 2 (the left panel); and the demand function $Q_0 (p)$ and the marginal revenue $MR_0 (q)$ (the right panel) for $C = 4$ and $x^* = 5$. It also shows that for $\beta = 0$, $\mathbf{CS}_s$ shrinks at the same pace as cost s increases by the increment of 2 from 0 to 2, and from 2 to 4: the pink area is equal to the blue one.

Furthermore, the parametrization via z allows us to derive the market outcome and all major resulting market characteristics as explicit functions of cost s. In order to do so, however, we need to rewrite the virtual value function (15) as a function of z:

$$v_\beta (z) = \psi_\beta (\chi_\beta (z)) = \chi_\beta (z) - \frac{y_\beta (z)}{C} = \begin{cases} \frac{1}{C} \int\limits_z^1 \frac{(1+\beta u)^{\frac{1}{\beta}}}{u} du - \phi_\beta (x^*) + x^* & \text{if } \beta > 0 \\ \frac{1}{C} \int\limits_z^1 \frac{e^u}{u} du - \phi_\beta (x^*) + x^* & \text{if } \beta = 0. \end{cases} \tag{44}$$

As special cases, the function $v_\beta (z)$ for $\beta = 0$ and 1 takes the form:

$$v_0 (z) = \frac{-\operatorname{Ei}(z) + \operatorname{Ei}(1) - e}{C} + x^*, \text{ and } v_1 (z) = -\frac{\ln z + z + 3}{C} + x^*.$$

Employing this function provides the following result.

Theorem 4 *Suppose $\phi_\beta(x)$ is given by (40) for $x \in [p_{\underline{c}}, p_{\bar{c}}] \subset \mathbf{X}$, where $p_{\underline{c}}$ and $p_{\bar{c}}$ satisfy (4). Then for all $s \in [\underline{c}, \bar{c}]$, the following hold:*

$$z_{s,\beta} = v_\beta^{-1}(s), \tag{45}$$

$$p_{s,\beta} = \chi_\beta(z_{s,\beta}) = s + \frac{y_\beta(z_{s,\beta})}{C} \text{ and } \phi_\beta(p_s) = \frac{y_\beta(z_{s,\beta})}{C}, \tag{46}$$

$$p'_{s,\beta} = \frac{1}{z_{s,\beta}} \text{ and } p''_{s,\beta} = \frac{C}{y_\beta(z_{s,\beta})}\left(\frac{1}{z_{s,\beta}^2} + \frac{\beta}{z_{s,\beta}}\right),$$

$$q_{s,\beta} = M_\beta \frac{\frac{1}{z_{\underline{c},\beta}} + \beta}{\frac{1}{z_{s,\beta}} + \beta} \text{ and } \varepsilon_\beta(p_{s,\beta}) = -C\frac{\chi_\beta(z_s)}{y_\beta(z_s)} = -1 - C\frac{v_\beta(z_{s,\beta})}{y_\beta(z_{s,\beta})},$$

$$\mathbf{W}'_s = -M_\beta\left(\frac{1}{z_{\underline{c},\beta}} + \beta\right), \text{ and}$$

$$\mathbf{PS}_s = \begin{cases} \frac{M_\beta}{C}\left(\frac{1}{z_{\underline{c},\beta}} + \beta\right)(1 + \beta z_{s,\beta})^{\frac{1}{\beta}} & \text{if } \beta > 0, \\ \frac{M_0}{C z_{\underline{c},\beta}} e^{z_{s,0}} & \text{if } \beta = 0, \end{cases}$$

where $M_\beta = \exp\left(-\int_{\underline{x}}^{p_{\underline{c},\beta}} \frac{1}{\phi_\beta(t)} dt\right)$ *and* $z_{\underline{c},\beta} = \chi_\beta^{-1}(p_{\underline{c},\beta})$.

Technically, the Theorem relies on two components. The first one is the relationship (45) between the variable z and the cost s. In fact, it is the standard optimality condition (16) in which the marginal revenue is represented by a function $v_\beta(z)$. Solving the equation $v_\beta(z) = s$ provides the expression (45) for $z_{s,\beta}$ as a function of s and β. The second component is expressing all necessary market characteristics as functions of z. Then combining the two components allows us to derive these market characteristics as explicit functions of s.

We complete the analysis by considering the case of bounded costs, i.e., $\bar{c} < \infty$. In this case $\mathbf{W}_s$, which is linear on $\mathbf{C}$, can be constructed as follows. Given β and $\underline{c}$, one can take $C > 0$ and x^*, and derive $p_{\underline{c},\beta}$ and $p_{\bar{c},\beta}$ from (46). Then take any $\underline{x} \le p_{\underline{c}}$ and $\bar{x} > p_{\bar{c}}$. This results in $\underline{x} \le p_{\underline{c}} < p_{\bar{c}} < \bar{x}$. Finally, consider $\tilde{\phi}_\beta(x)$ defined as follows:

$$\tilde{\phi}_\beta(x) = \begin{cases} \phi_L(x) & \text{if } \underline{x} \le x < p_{\underline{c},\beta}, \\ \phi_\beta(x) & \text{if } p_{\underline{c},\beta} \le x \le p_{\bar{c},\beta}, \\ \phi_H(x) & \text{if } p_{\bar{c},\beta} \le x \le \bar{x}, \end{cases}$$

where $\phi_\beta(x)$ is given by (40), $\phi_L(x) > \max\{x - \underline{c}, 0\}, 0 < \phi_H(x) < x - \bar{c}$ for $x < \bar{x}$, and $\lim_{t \uparrow \bar{x}} \int_{\underline{x}}^{t} \frac{1}{\phi(x)} dx = \infty$. The conditions on $\phi_\beta(x), \phi_L(x)$, and $\phi_H(x)$ guarantee the uniqueness of $p_{s,\beta}$ for $s \in \mathbf{C}$. Also, because $\tilde{\phi}_\beta(x)$ satisfies (33) for $x \in [p_{\underline{c},\beta}, p_{\bar{c},\beta}]$, then $\mathbf{W}_s$ is linear on $\mathbf{C}$. Finally, the last two conditions imply that $\tilde{F}_\beta(x)$ induced by $\tilde{\phi}_\beta(x)$ is supported on $[\underline{x}, \bar{x}]$.

8 Conclusion

This paper establishes that the social value of cost-related information in monopolistic markets can be both positive and negative, depending on the properties of market demand. It derives these properties and expresses them in terms of the inverse hazard rate function, the optimal price, the market power, and the concavity of the revenue in the classical model of the monopolist. The paper also provides the necessary and sufficient conditions on market demand under which the optimal disclosure policy takes the form of lower censorship regardless of the prior distribution of costs and the weights in the total surplus. It also establishes the conditions for the optimality of full disclosure. Finally, the paper also provides a complete characterization of demand functions and the major market characteristics, such that the posterior total surplus is linear in marginal (posterior) costs.

Appendix

Proof of Lemma 1 (i) Let $\bar{x} < \infty$, so that $\phi(\bar{x}) = 0$. Because $\phi(x)$ is twice differentiable, then $\phi(x)$ and $\phi'(x)$ are continuous. The continuity of $\phi(x)$ and $p_s \in (s, \bar{x})$ imply $\lim_{s \to \bar{x}} \phi(p_s) = \phi(\bar{x}) = 0$. Because $\phi'(p_s) < 1$ for $s \in \mathbf{C}$,$\phi'(p_s)$ is continuous, and $\mathbf{C}$ is compact, then $\frac{1}{1-\phi'(p_s)}$ is bounded away from 0 for $s \in \mathbf{C}$. Finally, since $\phi''(x)$ is bounded, this leads to $\lim_{s \to \bar{x}} \zeta(p_s) = \lim_{s \to \bar{x}} \frac{\phi''(p_s)\phi(p_s)}{1-\phi'(p_s)} = 0$.

(ii) If $\bar{x} = \infty$, then $p_s > s$ and $\lim_{x \to \infty} \phi(x) = 0$ imply $\lim_{s \to \infty} \phi(p_s) = 0$. Because $\phi'(x)$ is bounded away from 1, and $\phi''(x)$ is bounded, this leads to $\lim_{s \to \infty} \zeta(p_s) = \lim_{s \to \infty} \frac{\phi''(p_s)\phi(p_s)}{1-\phi'(p_s)} = 0$.

(iii) If $\bar{x} = \infty$, $\phi(x)$ is bounded, $\phi'(x)$ is bounded away from 1, and $\lim_{x \to \infty} \phi''(x) = 0$, then $p_s > s$ implies $\lim_{s \to \infty} \zeta(p_s) = \lim_{s \to \infty} \frac{\phi''(p_s)\phi(p_s)}{1-\phi'(p_s)} = 0$.

In either case, $\lim_{s \to \bar{x}} \zeta(p_s) = 0$ and $f(x) > 0$ result in the pointwise convergence of $\mathbf{CS}''_s$ and $\mathbf{W}''_s$ to

$$\mathbf{CS}''_s \underset{s \to \bar{x}}{\rightarrow} \frac{f(p_s)}{(1-\phi'(p_s))^2} > 0, \text{ and}$$

$$\mathbf{W}''_s \underset{s \to \bar{x}}{\rightarrow} \frac{f(p_s)}{(1-\phi'(p_s))^2}(1+\beta(1-\phi'(p_s))) > 0,$$

for any $\beta \geq 0$. That is, $\mathbf{CS}_s$ and, hence, $\mathbf{W}_s$, are strictly convex in s as $s \to \bar{x}$. ■

Proof of Proposition 1 Using (25), $p'_s > 0$, and $\phi'(x) < 1$ for $x \in \mathbf{P_C}$, it follows that $\mathbf{CS}_s$ is (strictly) quasi-monotone in s if and only if $1 - \zeta(x)$ is (strictly) quasi-monotone in x on $\mathbf{P_C}$. Because $\mathbf{CS}''_s$ is (strictly) quasi-monotone if and only if $\mathbf{CS}'_s$ is (strictly) quasi-convex (see, for example, Theorem 9.1 in Hadjisavvas et al., 2005), then the (strict) quasi-monotonicity of $1 - \zeta(x)$ on $\mathbf{P_C}$ is equivalent to the (strict) inverted S-shapedness of $\mathbf{CS}_s$ on $\mathbf{C}$. Next, by Theorem 1 in Kolotilin et al. (2022), lower censorship is (uniquely) optimal for any distribution $H(c)$ if only if $\mathbf{CS}_s$ is (strictly) inverted S-shaped on $\mathbf{C}$.

Now, suppose $1 - \zeta(x)$ is (strictly) quasi-monotone in x on $\mathbf{P_C}$. Then $1 - \zeta(p_{\underline{c}}) > 0$ and

$p_s' > 0$ imply $1-\zeta(p_s) \geq (>)\, 0$ for (almost) all $s \in \mathbf{C}$.[50] This result, (25), and $f(p_s) > 0$ and $\phi'(p_s) < 1$ for $s \in \mathbf{C}$ imply $\mathbf{CS}_s'' \geq (>)\, 0$ for (almost) all $s \in \mathbf{C}$. Because $\mathbf{CS}_s$ is differentiable and, thus, continuous in s, then it is (strictly) convex. Hence, full disclosure is the (uniquely) optimal signal structure for $\mathbf{CS}_s$.

Finally, let $\zeta(p_{\underline{c}}) > 1$. Because there is a neighborhood of $p_{\underline{c}}$, such that $\zeta(x) > 1$ for all x in the neighborhood, and p_s is continuous in s, then $\zeta(p_s) > 1$ and, hence, $\mathbf{CS}_s'' < 0$ for s in some neighborhood of $\underline{c}$. Thus, for any c_1 and $c_2 > c_1$ in this neighborhood, we get

$$\mathbf{CS}_{c_1}' > \frac{\mathbf{CS}_{c_2} - \mathbf{CS}_{c_1}}{c_2 - c_1}.$$

As $s \downarrow \underline{c}$, then $m_s \downarrow \underline{c}$ as well. Since $m_s < s$ for $s > \underline{c}$, and $h(s) > 0$, (32) for $\beta = 0$ yields

$$\mathbf{ECS}_s = h(s)(s - m_s)\left(\mathbf{CS}_{m_s}' - \frac{\mathbf{CS}_s - \mathbf{CS}_{m_s}}{s - m_s}\right) > 0,$$

which implies $s_0 > \underline{c}$; that is, full disclosure is suboptimal. ■

Proof of Theorem 1 (**i-If**) Denote $\mu(x,\beta)$ the linear combination of $1-\zeta(x)$ and $1-\phi'(x)$:

$$\mu(x,\beta) = 1 - \zeta(x) + \beta(1 - \phi'(x)) = 1 + \beta - \zeta(x) - \beta\phi'(x),$$

where $\beta > 0$ and $x \in \mathbf{P_C}$. Because $f(x) > 0$ and $p_s' > 0$, i.e., p_s is strictly increasing in s, then

$$\mathbf{W}_s'' = \frac{f(p_s)\,\mu(p_s,\beta)}{(1-\phi'(p_s))^2}$$

is (strictly) quasi-monotone in s for $s \in \mathbf{C}$ if and only if $\mu(x,\beta)$ is (strictly) quasi-monotone in x for $x \in \mathbf{P_C}$.

Next, we use Proposition 1 in Quah and Strulovici (2012), which states that a linear combination of two (strictly) pseudo-monotone functions $1-\zeta(x)$ and $1-\phi'(x)$ is a (strictly) pseudo-monotone function for all $\beta > 0$ if and only if the ratio $\frac{1-\zeta(x)}{1-\phi'(x)}$ is (strictly) increasing in x for all $x \in \mathbf{P_C}$, such that $1-\zeta(x) < 0$, i.e., $x \in \{\mathbf{P_C}\,|\zeta(x) > 1\}$. Also, Choi and Smith (2017, p.5) note that the 'if' part of Proposition 1 by Quah and Strulovici (2012) can be extended to quasi-monotone functions if both $1-\zeta(x)$ and $1-\phi'(x)$ do not have common flat regions of x at which they are equal to 0. Since $1-\phi'(x) > 0$ implies that this function does not have such regions, $1-\zeta(x)$ is (strictly) quasi-monotone, and $\frac{1-\zeta(x)}{1-\phi'(x)}$ is (strictly) increasing in x for all $x \in \{\mathbf{P_C}\,|\zeta(x) > 1\}$, then $\mu(x,\beta)$ is (strictly) quasi-monotone in x on $\mathbf{P_C}$ for all $\beta > 0$. Thus, $\mathbf{W}_s''$ is (strictly) quasi-monotone in s on $\mathbf{C}$, which is equivalent to the (strict) quasi-convexity of $\mathbf{W}_s'$ (see, for example, Theorem 9.1 in Hadjisavvas et al., 2005). In turn, it is equivalent to the (strict) inverted S-shapedness of $\mathbf{W}_s$. Finally, by Theorem 1 in Kolotilin et al. (2022), lower censorship is (uniquely) optimal for each distribution $H(c)$ if only if $\mathbf{W}_s$ is (strictly) inverted S-shaped on $\mathbf{C}$.

(**i-Only if**) Suppose $\frac{1-\zeta(x_2)}{1-\phi'(x_2)} < \frac{1-\zeta(x_1)}{1-\phi'(x_1)}$ for some $p_{\underline{c}} \leq x_1 < x_2 \leq p_{\bar{c}}$, such that $1-\zeta(x_1) <$

[50] If $1-\zeta(x)$ is (strictly) pseudo-monotone in x on $\mathbf{P_C}$, then $1-\zeta(p_{\underline{c}}) \geq 0$ implies $1-\zeta(p_s) \geq (>)\, 0$ for all $s \in \mathbf{C}$.

0. Take $\beta_0 > 0$, such that

$$\mu(x_1, \beta_0) = 1 - \zeta(x_1) + \beta_0 (1 - \phi'(x_1)) = 0, \text{ or}$$
$$\frac{1 - \zeta(x_1)}{1 - \phi'(x_1)} = -\beta_0.$$

It follows then that

$$\frac{1 - \zeta(x_2)}{1 - \phi'(x_2)} < \frac{1 - \zeta(x_1)}{1 - \phi'(x_1)} = -\beta_0, \text{ or}$$
$$1 - \zeta(x_2) + \beta_0 (1 - \phi'(x_2)) < 0.$$

Next, because $1 - \zeta(x_1) < 0 < 1 - \phi'(x_1)$, then taking $\tilde{\beta} = \beta_0 + \delta$, where $\delta \downarrow 0$ implies

$$h\left(x_1, \tilde{\beta}\right) = 1 - \zeta(x_1) + \tilde{\beta}(1 - \phi'(x_1)) > 0, \text{ and}$$
$$h\left(x_2, \tilde{\beta}\right) = 1 - \zeta(x_2) + \tilde{\beta}(1 - \phi'(x_2)) < 0,$$

that is, $\mu\left(x, \tilde{\beta}\right)$ is not quasi-monotone in x for $\tilde{\beta}$. Thus, $\mu\left(p_s, \tilde{\beta}\right)$ is not quasi-monotone in s, which means $\mathbf{W}''_s$ is not quasi-monotone in s, so that $\mathbf{W}'_s$ is not quasi-convex in s for $\tilde{\beta}$. This implies that $\mathbf{W}_s$ is not inverted S-shaped on $\mathbf{C}$. By Theorem 1 in Kolotilin et al. (2022), there exists a distribution $\tilde{H}(c)$ such that lower censorship is not optimal.

(**ii-If**) Suppose $\frac{1-\zeta(x)}{1-\phi'(x)}$ is (strictly) increasing in $x \in \{\mathbf{P_C} | \zeta(x) > 1\}$, and $\zeta(p_{\underline{c}}) + \beta\phi'(p_{\underline{c}}) < 1+\beta$, or equivalently, $\mu(p_{\underline{c}}, \beta) > 0$. By part (i), $\mu(x, \beta)$ is (strictly) quasi-monotone in x on $\mathbf{P_C}$ for all $\beta > 0$. This implies $\mu(p_s, \beta) \geq (>) \, 0$ for (almost) all $s \in \mathbf{C}$.[51] This result, (27), and $f(p_s) > 0$ and $\phi'(p_s) < 1$ for $s \in \mathbf{C}$ imply $\mathbf{W}''_s \geq (>) \, 0$ for (almost) all $s \in \mathbf{C}$. Because $\mathbf{W}_s$ is continuous in s, then it is (strictly) convex. Hence, full disclosure is (uniquely) optimal for $\mathbf{W}_s$.

(**ii-Only if**) Finally, let $\zeta(p_{\underline{c}}) + \beta\phi'(p_{\underline{c}}) > 1 + \beta$, or equivalently, $\mu(p_{\underline{c}}, \beta) < 0$. Because $\mu(x, \beta)$ is continuous in x and p_s is continuous in s, then $\mu(p_s, \beta)$ is continuous in $s \in \mathbf{C}$. Therefore, $\mu(p_s, \beta) < 0$ and, thus, $\mathbf{W}''_s < 0$ for s in some neighborhood of $\underline{c}$. This means that for any c_1 and $c_2 > c_1$ in this neighborhood, we have

$$\mathbf{W}'_{c_1} > \frac{\mathbf{W}_{c_2} - \mathbf{W}_{c_1}}{c_2 - c_1}.$$

As $s \downarrow \underline{c}$, then $m_s \downarrow \underline{c}$ as well. Since $m_s < s$ for $s > \underline{c}$, and $h(s) > 0$, then (32) yields

$$\mathbf{E}\mathbf{W}'_s = h(s)(s - m_s)\left(\mathbf{W}'_{m_s} - \frac{\mathbf{W}_s - \mathbf{W}_{m_s}}{s - m_s}\right) > 0,$$

which implies $s_\beta > \underline{c}$; that is, full disclosure is suboptimal. ■

Proof of Proposition 2 Following Curello and Sinander (2024), a function $U : \mathbf{C} \to \mathbb{R}$

[51] If $1 - \zeta(x)$ is (strictly) pseudo-monotone in x on $\mathbf{P_C}$, then $\mu\left(p_{\underline{c}}, \beta\right) \geq 0$ implies $\mu(p_s, \beta) \geq (>) \, 0$ for all $s \in \mathbf{C}$.

is *regular* if it is differentiable and $\mathbf{C}$ can be partitioned into finitely many intervals, on each of which U is either affine, strictly convex, or strictly concave. Then a regular function U satisfies the *crater property* if for any $x < y < z < w$ in $\mathbf{C}$, such that U is concave on $[x, y]$ and $[z, w]$ and strictly convex on $[y, z]$, the tangents to U at x and at w cross at coordinates $(X, Y) \in \mathbf{C} \times \mathbb{R}$ satisfying $y \leq X \leq z$ and $Y \leq u(X)$. Next, a function $V : \mathbf{C} \to \mathbb{R}$ is *coarsely more convex* than a function U if the convexity of U on $[x, y] \subset \mathbf{C}$ implies the convexity of V on $[x, y]$. That is, for any $x < y$ in $\mathbf{C}$ such that

$$U(\alpha x + (1 - \alpha) y) \leq \alpha U(x) + (1 - \alpha) U(y)$$

holds for all $\alpha \in (0, 1)$, it follows that

$$V(\alpha x + (1 - \alpha) y) \leq \alpha V(x) + (1 - \alpha) V(y)$$

holds for all $\alpha \in (0, 1)$.

By Theorem 2 in Curello and Sinander (2024), if U is regular and satisfies the crater property, and V is regular and coarsely more convex than U, then for any atomless distribution $H(x)$ with convex-support $\mathbf{C}$, we have

$$\arg\max_{G \underset{cx}{\prec} H} \int U(s)\, dG(s) \text{ is lower than } \arg\max_{G \underset{cx}{\prec} H} \int V(s)\, dG(s), \tag{47}$$

where 'lower than' implies the weak set order of optimal distributions.

By the conditions of the Proposition, since $1 - \zeta(x)$ is quasi-monotone and $\frac{1-\zeta(x)}{1-\phi'(x)}$ is increasing in x whenever $\zeta(x) > 1$ for $x \in \mathbf{P_C}$, then $\mathbf{W}_s$ is inverted S-shaped as shown in the proof of Theorem 1. This implies that: i) $\mathbf{W}_s$ is regular and satisfies the crater property; and ii) lower censorship is an optimal signal structure. Consider optimal signal structures of the lower censorship form and put U and V equal to $\mathbf{W}_s$ for $\beta_1 \geq 0$ and $\beta_2 > \beta_1$, respectively. Then (47) is equivalent to $s^*_{\beta_2} \leq s^*_{\beta_1}$ if $\mathbf{W}_s$ is coarsely more convex for β_2 than for β_1. This, however, follows from observing that

$$\mathbf{W}_s^{\beta_2} = \mathbf{CS}_s + \beta_2 \mathbf{PS}_s = \mathbf{CS}_s + \beta_1 \mathbf{PS}_s + (\beta_2 - \beta_1) \mathbf{PS}_s = \mathbf{W}_s^{\beta_1} + (\beta_2 - \beta_1) \mathbf{PS}_s,$$

and using the fact that $(\beta_2 - \beta_1) \mathbf{PS}_s$ is convex in s. By Corollary 1 in Curello and Sinander (2024), $\mathbf{W}_s^{\beta_2}$ is coarsely more convex than $\mathbf{W}_s^{\beta_1}$, which completes the proof. ■

Proof of Lemma 2 Treating $\phi = \phi(x)$ as an independent variable and putting $\vartheta(\phi) = \phi'(x)$ yields

$$\phi''(x) = \frac{d\vartheta}{d\phi}\frac{d\phi}{dx} = \frac{d\vartheta}{d\phi}\phi'(x) = \frac{d\vartheta}{d\phi}\vartheta(\phi).$$

This allows to transform (33) in the equation, which is separable in ϕ and ϑ:

$$1-\vartheta(\phi)-\frac{d\vartheta}{d\phi}\vartheta(\phi)\phi+\beta(1-\vartheta(\phi))^2=0,$$

$$\frac{d\vartheta}{d\phi}=\frac{1-\vartheta(\phi)+\beta(1-\vartheta(\phi))^2}{\vartheta(\phi)\phi}, \text{ and}$$

$$\frac{\vartheta(\phi)}{1-\vartheta(\phi)+\beta(1-\vartheta(\phi))^2}\frac{d\vartheta}{d\phi}=\frac{1}{\phi}.$$

Because

$$\frac{\vartheta}{1-\vartheta+\beta(1-\vartheta)^2}=\frac{\vartheta}{1-\vartheta+\beta-2\beta\vartheta+\beta\vartheta^2}=\frac{\vartheta}{(1-\vartheta)(1+\beta-\beta\vartheta)}$$
$$=-\frac{\beta+1}{1+\beta-\beta\vartheta}+\frac{1}{1-\vartheta},$$

we obtain

$$\left(-\frac{\beta+1}{1+\beta-\beta\vartheta}+\frac{1}{1-\vartheta}\right)\frac{d\vartheta}{d\phi}=\frac{1}{\phi}$$

Integrating both sides of this equation yields

$$\int-\frac{\beta+1}{1+\beta-\beta\vartheta}+\frac{1}{1-\vartheta}d\vartheta=\frac{\beta+1}{\beta}\ln(1+\beta-\beta\vartheta)-\ln(1-\vartheta)$$
$$=\ln\frac{(1+\beta-\beta\vartheta)^{\frac{\beta+1}{\beta}}}{1-\vartheta}=\int\frac{d\phi}{\phi}=\ln\phi+\ln C=\ln(C\phi),$$

for any $C>0$. This gives

$$\frac{(1+\beta-\beta\vartheta)^{1+\frac{1}{\beta}}}{1-\vartheta}=\frac{(1+\beta-\beta\phi'(x))^{1+\frac{1}{\beta}}}{1-\phi'(x)}=C\phi(x).$$

Finally, $\lim_{\beta\downarrow 0}(1+\beta z)^{1+\frac{1}{\beta}}=e^z$ implies $\lim_{\beta\downarrow 0}\frac{(1+\beta-\beta\phi'(x))^{1+\frac{1}{\beta}}}{1-\phi'(x)}=\frac{e^{1-\phi'(x)}}{1-\phi'(x)}$. ■

Proof of Theorem 2 We split the proof of the theorem in several steps.

Step 1: derivation of $\phi(x)$. (i) First, consider $\beta>0$. It is shown above that $\mathbf{W}_s$ is linear in s for $\phi'(x)<1$ if and only if it is a solution to (39), which can be written as

$$\frac{\phi'(x)}{1-\mathcal{H}_\beta(C\phi(x))}=1.$$

The indefinite integration of this equation results in

$$\int\frac{1}{1-\mathcal{H}_\beta(C\phi(x))}d\phi(x)=x+\kappa. \tag{48}$$

Next, replacing the variable $z = 1 - \phi'(x) = \mathcal{H}_\beta(C\phi(x))$ gives

$$\frac{1}{1 - \mathcal{H}_\beta(C\phi(x))} = \frac{1}{1-z}.$$

Employing (35) and (36) yields

$$\begin{aligned}\phi'(x) &= \frac{d}{dx}\phi(x) = \frac{1}{C}y'_\beta(z)\frac{dz}{dx} = \frac{1}{C}\left(\frac{(1+\beta z)^{\frac{1}{\beta}}}{z}\right)'\frac{dz}{dx}\\ &= \frac{-(1+\beta z)^{1+\frac{1}{\beta}} + (1+\beta)z(1+\beta z)^{\frac{1}{\beta}}}{Cz^2}\frac{dz}{dx}\\ &= \frac{(1+\beta z)^{\frac{1}{\beta}}}{Cz^2}(-1-\beta z+z+\beta z)\frac{dz}{dx} = -\frac{(1+\beta z)^{\frac{1}{\beta}}}{Cz^2}(1-z)\frac{dz}{dx}.\end{aligned}$$

This allows us to integrate the left part of (48) by substitution:

$$\begin{aligned}\int \frac{1}{1-\mathcal{H}_\beta(C\phi(x))}d\phi(x) &= \int \frac{1}{1-z}\frac{1}{C}y'_\beta(z)\,dz = -\frac{1}{C}\int \frac{(1+\beta z)^{\frac{1}{\beta}}}{z^2(1-z)}(1-z)\,dz\\ &= -\frac{1}{C}\int \frac{(1+\beta z)^{\frac{1}{\beta}}}{z^2}dz = -\frac{\beta}{C}\int u^{\frac{1}{\beta}}(1-u)^{-2}\,du,\end{aligned}$$

where $u = 1+\beta z$. Using the normalization $z_0 = z^* = 1$ and $\chi_\beta(1) = x^*$, we obtain

$$\begin{aligned}x = \chi_\beta(z) &= -\frac{1}{C}\int_1^z \frac{(1+\beta u)^{\frac{1}{\beta}}}{u^2}du + x^* = \frac{1}{C}\int_z^1 \frac{(1+\beta u)^{\frac{1}{\beta}}}{u^2}du + x^*\\ &= \frac{\beta}{C}\left(-\mathcal{B}_{1+\beta z}\left(\frac{1}{\beta}+1,-1\right) + \mathcal{B}_{1+\beta}\left(\frac{1}{\beta}+1,-1\right)\right) + x^*.\end{aligned}$$

(ii) Let $\beta = 0$. By following the same steps as above, we obtain

$$\mathcal{H}_\beta(C\phi(x)) = -W\left(-\frac{1}{C\phi(x)}\right),$$

$$\begin{aligned}\int \frac{1}{1+W\left(-\frac{1}{C\phi(x)}\right)}d\phi(x) &= \int \frac{1}{1-z}\frac{e^z(z-1)}{Cz^2}dz = -\frac{1}{C}\int \frac{e^z}{z^2}dz\\ &= \frac{1}{C}\int \frac{e^z(z-1)}{z^2}dz - \frac{1}{C}\int \frac{e^z}{z}dz = \frac{1}{C}\frac{e^z}{z} - \frac{\mathrm{Ei}(z)}{C}, \text{ and}\end{aligned}$$

$$x = \chi_0(z) = -\frac{1}{C}\int_1^z \frac{e^u}{u^2}du + x^* = \frac{1}{C}\int_z^1 \frac{e^u}{u^2}du + x^* = \frac{1}{C}\left(-\mathrm{Ei}(z) + \frac{e^z}{z} - e + \mathrm{Ei}(1)\right) + x^*.$$

Using these relationships, $\phi_\beta(x)$ can be expressed as

$$C\phi_\beta(\chi_\beta(z)) = y_\beta(z) = \frac{(1+\beta z)^{\frac{1}{\beta}+1}}{z}, \text{ and}$$

$$\phi_\beta(x) = \frac{y_\beta\left(\chi_\beta^{-1}(x)\right)}{C} = \begin{cases} \frac{\left(1+\beta\chi_\beta^{-1}(x)\right)^{1+\frac{1}{\beta}}}{C\chi_\beta^{-1}(x)} & \text{if } \beta > 0 \\ \frac{e^{\chi_0^{-1}(x)}}{C\chi_0^{-1}(x)} & \text{if } \beta = 0, \end{cases},$$

where $\chi_\beta^{-1}(x)$ is well-defined for all $\beta \geq 0$ as we show below.

Step 2: properties of $\chi_\beta(z), y_\beta(z)$, and $\phi_\beta(x)$. First, consider the limits

$$\begin{aligned} \lim_{z\to\infty} y_\beta(z) &= \lim_{z\to\infty} \frac{(1+\beta z)^{\frac{1}{\beta}+1}}{z} \geq \lim_{z\to\infty} \frac{(\beta z)^{\frac{1}{\beta}+1}}{z} = \lim_{z\to\infty} \beta^{\frac{1}{\beta}+1} z^{\frac{1}{\beta}} = \infty \text{ if } \beta > 0, \\ \lim_{z\to\infty} y_0(z) &= \lim_{z\to\infty} \frac{e^z}{z} = \infty \text{ if } \beta = 0, \text{ and} \\ \lim_{z\to 0} y_\beta(z) &= \lim_{z\to\infty} \frac{1}{z} = \infty \text{ if } \beta \geq 0. \end{aligned}$$

Similarly, consider the limits of $\chi_\beta(z)$ as $z \to 0$ and $z \to \infty$. Because $e^u \geq 1$ and $(1+\beta u)^{\frac{1}{\beta}} \geq 1$ for $u \geq 0$, then

$$\chi_\beta(z) = \begin{cases} \frac{1}{C}\int_z^1 \frac{(1+\beta u)^{\frac{1}{\beta}}}{u^2} du + x^* & \text{if } \beta > 0 \\ \frac{1}{C}\int_z^1 \frac{e^u}{u^2} du + x^* & \text{if } \beta = 0, \end{cases} \geq \frac{1}{C}\int_z^1 \frac{1}{u^2} du + x^* = \frac{1}{C}\left(\frac{1}{z} - 1\right) + x^*.$$

This leads to

$$\lim_{z\downarrow 0} \chi_\beta(z) \geq \lim_{z\downarrow 0} \frac{1}{C}\left(\frac{1}{z} - 1\right) + x^* = \infty.$$

Next, we obtain

$$\underline{\chi}_\beta = \lim_{z\to\infty} \chi_\beta(z) = -\lim_{z\to\infty} \frac{1}{C}\int_1^z \frac{(1+\beta u)^{\frac{1}{\beta}}}{u^2} du + x^* = \begin{cases} -\infty & \text{if } \beta \in (0,1] \\ > -\infty & \text{if } \beta > 1, \end{cases}$$

where the last relation holds because

$$\begin{aligned}
\lim_{z\to\infty}\chi_\beta(z)-x^* &= -\lim_{z\to\infty}\frac{1}{C}\int_1^z\frac{(1+\beta u)^{\frac{1}{\beta}}}{u^2}du \le -\lim_{z\to\infty}\frac{1}{C}\int_1^z\frac{1+\beta u}{u^2}du \\
&= -\lim_{z\to\infty}\frac{1}{C}\int_1^z\frac{1+\beta u}{u^2}du \le -\lim_{z\to\infty}\frac{\beta}{C}\int_1^z\frac{1}{u}du = -\infty \text{ if } \beta\in(0,1]\text{, and} \\
\lim_{z\to\infty}\chi_\beta(z)-x^* &= \lim_{z\to\infty}-\frac{1}{C}\int_1^z\frac{(1+\beta u)^{\frac{1}{\beta}}}{u^2}du > \lim_{z\to\infty}-\frac{1}{C}\int_1^z\frac{(\beta u+\beta u)^{\frac{1}{\beta}}}{u^2}du \\
&= -(2\beta)^{\frac{1}{\beta}}\lim_{z\to\infty}\frac{1}{C}\int_1^z\frac{1}{u^{2-\frac{1}{\beta}}}du > -\infty \text{ if } \beta>1.
\end{aligned}$$

Next, consider $\chi'_\beta(z)$ and $y'_\beta(z)$:

$$y'_\beta(z)=\begin{cases} -\frac{(1+\beta z)^{\frac{1}{\beta}}}{Cz^2}(1-z) & \text{if } \beta>0 \\ -\frac{e^z(1-z)}{Cz^2} & \text{if } \beta=0, \end{cases}\text{, and} \tag{49}$$

$$\chi'_\beta(z)=\begin{cases} -\frac{(1+\beta z)^{\frac{1}{\beta}}}{Cz^2} & \text{if } \beta>0 \\ -\frac{e^z}{Cz^2} & \text{if } \beta=0, \end{cases}<0. \tag{50}$$

These properties imply that the function $\chi_\beta:\mathbb{R}_{++}\to\mathbb{R}$ has the image

$$\mathcal{X}_\beta=\{\chi_\beta(z)\in\mathbb{R}|z>0\}=\left(\underline{\chi}_\beta,\infty\right),$$

and $\chi_\beta(z)$ is strictly decreasing in z due to (50). Therefore, the inverse function $\chi_\beta^{-1}:\mathcal{X}_\beta\to(0,1)$ is well defined, has the image $\mathbb{R}_{++}$, and $\chi_\beta^{-1}(x)$ is strictly decreasing in x. These properties imply that $\phi_\beta(x)=\frac{y_\beta\left(\chi_\beta^{-1}(x)\right)}{C}$ has the domain $\mathcal{X}_\beta$.

Taking the limits

$$\begin{aligned}
\lim_{x\to\infty}\phi_\beta(x) &= \lim_{x\to\infty}\frac{y_\beta\left(\chi_\beta^{-1}(x)\right)}{C} = \lim_{\chi_\beta(z)\to\infty}\frac{y_\beta(z)}{C} = \lim_{z\downarrow 0}\frac{y_\beta(z)}{C} = \infty \text{ for all } \beta\ge 0\text{, and} \\
\lim_{x\to\underline{\chi}_\beta}\phi_\beta(x) &= \lim_{x\to\underline{\chi}_\beta}\frac{y_\beta\left(\chi_\beta^{-1}(x)\right)}{C} = \lim_{\chi_\beta(z)\to\underline{\chi}_\beta}\frac{y_\beta(z)}{C} = \lim_{z\to\infty}\frac{y_\beta(z)}{C} = \infty \text{ for all } \beta\ge 0,
\end{aligned}$$

and using $\phi_\beta(x)=\frac{y_\beta\left(\chi_\beta^{-1}(x)\right)}{C}\ge\phi_\beta(x^*)=\frac{y_\beta}{C}>0$ for $x\in\mathcal{X}_\beta$ means that the image of $\phi_\beta(x)$ is $\left(\frac{y_\beta}{C},\infty\right)\subset\mathbb{R}_{++}$.

Also, using (49)–(50) and taking the derivatives of functions defined parametrically at

$x = \chi_\beta(z)$ yields

$$\phi'_\beta(x)|_{x=\chi_\beta(z)} = \frac{y'_\beta(z)}{\chi'_\beta(z)} = 1 - z, \text{ and} \tag{51}$$

$$\phi''_\beta(x)|_{x=\chi_\beta(z)} = \frac{\frac{d}{dz}\phi'_\beta(x)|_{x=\chi_\beta(z)}}{\chi'_\beta(z)} = \frac{-1}{\chi'_\beta(z)} = \begin{cases} \frac{Cz^2}{(1+\beta z)^{\frac{1}{\beta}}} & \text{if } \beta > 0 \\ \frac{Cz^2}{e^z} & \text{if } \beta = 0. \end{cases} \tag{52}$$

Then (51) results in

$$\lim_{x\to\infty} \phi'(x) = \lim_{z\to 0}(1-z) = 1 \text{ and } \lim_{x\to \underline{\chi}_\beta} \phi'(x) = \lim_{z\to\infty}(1-z) = -\infty.$$

Also, (52) implies $\phi''(x) > 0$ for all $x \in \mathcal{X}_\beta$, i.e., $\phi(x)$ is strictly convex. Finally, we use (51) and (52) to verify that (40) satisfies (33) for all $x \in \mathcal{X}_\beta$:

$$1 - \frac{\phi''_\beta(x)\phi_\beta(x)}{1-\phi'_\beta(x)}|_{x=\chi_\beta(z)} - \beta\left(1 - \phi'_\beta(x)|_{x=\chi_\beta(z)}\right) = 1 - \frac{Cz^2}{(1+\beta z)^{\frac{1}{\beta}}}\frac{(1+\beta z)^{\frac{1}{\beta}+1}}{Cz}\frac{1}{z} - \beta z$$
$$= 1 - (1+\beta z) - \beta z = 0 \text{ if } \beta > 0, \text{ and}$$
$$1 - \frac{\phi''_\beta(x)\phi_\beta(x)}{1-\phi'_\beta(x)}|_{x=\chi_\beta(z)} = 1 - \frac{Cz^2}{e^z}\frac{e^z}{Cz}\frac{1}{z} = 0 \text{ if } \beta = 0.$$

Step 3: existence and uniqueness of $p_{s,\beta}$. The final step is to show that for any $\beta \geq 0, \phi_\beta(x)$ given by (40), and $s \in [\underline{c}, \bar{c}]$, there is unique $p_{s,\beta}$ that satisfies (4).

As shown above, $\chi_\beta^{-1} : \mathcal{X}_\beta \to (0,1)$ is strictly decreasing in x and $\chi^{-1}(x^*) = 1$. Therefore, for any $\underline{x}$ and $\bar{x}$, such that $\inf \mathcal{X}_\beta \leq \underline{x} < x^* = \chi_\beta(1) < \bar{x}$, there exist unique $z_{\underline{x},\beta} = \chi_\beta^{-1}(\underline{x})$ and $z_{\bar{x},\beta} = \chi^{-1}(\bar{x})$, such that $z_{\underline{x},\beta} > z^* = 1 > z_{\bar{x},\beta} > 0, \chi_\beta(z_{\underline{x},\beta}) = \underline{x}$, and $\chi_\beta(z_{\bar{x},\beta}) = \bar{x}$.

Fix $\beta \geq 0$ and $s \geq \underline{c}$. For $x = s$, we have

$$x - s - \phi_\beta(x)|_{x=s} = -\phi_\beta(s) \leq -\phi_\beta(x^*) = -\frac{\underline{y}_\beta}{C} < 0.$$

Now, consider $\psi_\beta(x)$ expressed as a function of z:

$$v_\beta(z) = \psi_\beta(x)|_{x=\chi_\beta(z)} = x - \phi_\beta(x)|_{x=\chi_\beta(z)} = \chi_\beta(z) - \frac{y_\beta(z)}{C}.$$

If $\beta > 0$, then

$$v_\beta(z) - x^* = \frac{1}{C}\left(\int_z^1 \frac{(1+\beta u)^{\frac{1}{\beta}}}{u^2}du - y_\beta(z)\right) = \frac{1}{C}\left(\int_z^1 \frac{(1+\beta u)^{\frac{1}{\beta}}}{u^2}du - \frac{(1+\beta z)^{\frac{1}{\beta}+1}}{z}\right).$$

By the fundamental theorem of calculus and (49), we obtain

$$y_\beta(z) = -\int_z^1 y'_\beta(u)\,du + y_\beta(1) = \int_z^1 \frac{(1+\beta u)^{\frac{1}{\beta}}}{u^2}(1-u)\,du + \underline{y}_\beta.$$

This results in

$$\int_z^1 \frac{(1+\beta u)^{\frac{1}{\beta}}}{u^2}du - \frac{(1+\beta z)^{\frac{1}{\beta}+1}}{z} = \int_z^1 \frac{(1+\beta u)^{\frac{1}{\beta}}}{u^2} - \frac{(1+\beta u)^{\frac{1}{\beta}+1}(1-u)}{u}du - \underline{y}_\beta$$
$$= \int_z^1 \frac{(1+\beta u)^{\frac{1}{\beta}}}{u}du - \underline{y}_\beta,$$

so that

$$v_\beta(z) = \frac{1}{C}\int_z^1 \frac{(1+\beta u)^{\frac{1}{\beta}}}{u}du - \phi_\beta(x^*) + x^*, \text{ and}$$

$$\psi_\beta(x) = v_\beta\left(\chi_\beta^{-1}(x)\right) = \frac{1}{C}\int_{\chi_\beta^{-1}(x)}^1 \frac{(1+\beta u)^{\frac{1}{\beta}}}{u}du - \phi_\beta(x^*) + x^*.$$

Since

$$\lim_{z\downarrow 0}\int_z^1 \frac{(1+\beta u)^{\frac{1}{\beta}}}{u}du \geq \lim_{z\downarrow 0}\int_z^1 \frac{1}{u}du = -\lim_{z\downarrow 0}\ln(z) = \infty,$$

then

$$\lim_{x\to\infty}\psi_\beta(x) = \lim_{x\to\infty} v_\beta\left(\chi_\beta^{-1}(x)\right) = \lim_{\chi_\beta(z)\to\infty} v_\beta(z) = \lim_{z\downarrow 0} v_\beta(z) = \infty,$$

that is, $x - \phi_\beta(x) - s = \psi_\beta(x) - s > 0$ for a sufficiently large $x = \chi_\beta(z)$.

Similarly, if $\beta = 0$, then

$$v_0(z) = \frac{1}{C}\int_z^1 \frac{e^u}{u}du - \phi_0(x^*) + x^* = \frac{-\operatorname{Ei}(z) + \operatorname{Ei}(1)}{C} - \phi_0(x^*) + x^*,$$

and

$$\psi_0(x) = \frac{-\operatorname{Ei}\left(\chi_0^{-1}(x)\right) + \operatorname{Ei}(1)}{C} - \phi_0(x^*) + x^*.$$

Because $\lim_{z\to 0}\operatorname{Ei}(z) = -\infty$, then $\lim_{x\to\infty}\psi_0(x) = \lim_{z\to 0} v_0(z) = \infty$, that is, $x - s - \phi_0(x) = \psi_0(x) - s > 0$ for a sufficiently large x. Therefore, for any $\beta \geq 0$ there is $p_{s,\beta}$, such that

$$\phi(p_{s,\beta}) - (p_{s,\beta} - s) = 0.$$

Finally, since $z > 0$ yields $\phi_\beta'(x) = 1 - z < 1$ for all $x \in \mathcal{X}_\beta$, then $p_{s,\beta}$ is unique. ■

Proof of Theorem 3 (i)–(ii) were proved in Step 2 in the proof of Theorem 2.

(iii) Let $\beta > 0$, and consider $\underline{x} > \underline{\chi}_\beta$. By using (40) and the integration by substitution

$z=\chi_{\beta}^{-1}(t)$, we get

$$\int_{\underline{x}}^{x}\frac{1}{\phi_{\beta}(t)}dt=\int_{\underline{x}}^{x}\frac{C}{y_{\beta}\left(\chi_{\beta}^{-1}(t)\right)}dt=C\int_{\underline{x}}^{x}\frac{\chi_{\beta}^{-1}(t)}{\left(1+\beta\chi_{\beta}^{-1}(t)\right)^{1+\frac{1}{\beta}}}dt=C\int_{\chi_{\beta}^{-1}(\underline{x})}^{\chi_{\beta}^{-1}(x)}\frac{z}{(1+\beta z)^{1+\frac{1}{\beta}}}\chi_{\beta}'(z)\,dz$$

$$=-C\int_{\chi_{\beta}^{-1}(\underline{x})}^{\chi_{\beta}^{-1}(x)}\frac{z}{(1+\beta z)^{1+\frac{1}{\beta}}}\frac{(1+\beta z)^{\frac{1}{\beta}}}{Cz^{2}}dz=-\int_{\chi_{\beta}^{-1}(\underline{x})}^{\chi_{\beta}^{-1}(x)}\frac{1}{z(1+\beta z)}dz$$

$$=-\left(\ln\left(\frac{\chi_{\beta}^{-1}(x)}{1+\beta\chi_{\beta}^{-1}(x)}\frac{1+\beta\chi_{\beta}^{-1}(\underline{x})}{\chi_{\beta}^{-1}(\underline{x})}\right)\right)=-\ln\left(\frac{\frac{1}{\chi_{\beta}^{-1}(\underline{x})}+\beta}{\frac{1}{\chi_{\beta}^{-1}(x)}+\beta}\right),$$

where $\chi_{\beta}'(z)=-\frac{(1+\beta z)^{\frac{1}{\beta}}}{Cz^{2}}$ by (50). Then employing (2) leads to

$$Q_{\beta}(x)=1-F_{\beta}(x)=\exp\left(-\int_{\underline{x}}^{x}\frac{1}{\phi_{\beta}(t)}dt\right)=\exp\left(\ln\left(\frac{\frac{1}{\chi_{\beta}^{-1}(\underline{x})}+\beta}{\frac{1}{\chi_{\beta}^{-1}(x)}+\beta}\right)\right)=\frac{\frac{1}{\chi_{\beta}^{-1}(\underline{x})}+\beta}{\frac{1}{\chi_{\beta}^{-1}(x)}+\beta}.$$

Similarly, if $\beta=0$, then using $\chi_{0}'(z)=-\frac{1}{C}\frac{e^{z}}{z^{2}}$ yields

$$\int_{\underline{x}}^{x}\frac{1}{\phi_{0}(t)}dt=\int_{\underline{x}}^{x}\frac{1}{y_{0}\left(\chi_{0}^{-1}(t)\right)}dt=\int_{\underline{x}}^{x}C\chi_{0}^{-1}(t)\,e^{\chi_{0}^{-1}(t)}dt=\int_{\chi_{0}^{-1}(\underline{x})}^{\chi_{0}^{-1}(x)}Cze^{-z}\chi_{0}'(z)\,dz$$

$$=-\int_{\chi_{0}^{-1}(\underline{x})}^{\chi_{0}^{-1}(x)}Cze^{-z}\frac{1}{C}\frac{e^{z}}{z^{2}}dz=-\int_{\chi_{0}^{-1}(\underline{x})}^{\chi_{0}^{-1}(x)}\frac{1}{z}dz=-\ln\left(\frac{\chi_{0}^{-1}(x)}{\chi_{0}^{-1}(\underline{x})}\right),\text{ and}$$

$$Q_{0}(x)=1-F_{0}(x)=\exp\left(-\int_{\underline{x}}^{x}\frac{1}{\phi_{0}(t)}dt\right)=\exp\left(\ln\left(\frac{\chi_{0}^{-1}(x)}{\chi_{0}^{-1}(\underline{x})}\right)\right)=\frac{\chi_{0}^{-1}(x)}{\chi_{0}^{-1}(\underline{x})}.$$

Taking the derivative of $F_{\beta}(x)$ and using (50) gives

$$f_{\beta}(x)=-\left(\frac{\frac{1}{\chi_{\beta}^{-1}(\underline{x})}+\beta}{\frac{1}{\chi_{\beta}^{-1}(x)}+\beta}\right)'=\left(\frac{1}{\chi_{\beta}^{-1}(\underline{x})}+\beta\right)\frac{1}{\left(1+\beta\chi_{\beta}^{-1}(x)\right)^{2}}\frac{C\left(\chi_{\beta}^{-1}(x)\right)^{2}}{\left(1+\beta\chi_{\beta}^{-1}(x)\right)^{\frac{1}{\beta}}}$$

$$=C\left(\frac{1}{\chi_{\beta}^{-1}(\underline{x})}+\beta\right)\frac{\left(\chi_{\beta}^{-1}(x)\right)^{2}}{\left(1+\beta\chi_{\beta}^{-1}(x)\right)^{2+\frac{1}{\beta}}}\text{ if }\beta>0,\text{ and}$$

$$f_{0}(x)=-\left(\frac{\chi_{0}^{-1}(x)}{\chi_{0}^{-1}(\underline{x})}\right)'=-\frac{1}{\chi_{0}^{-1}(\underline{x})}\frac{1}{\chi_{0}'\left(\chi_{0}^{-1}(x)\right)}=\frac{C}{\chi_{0}^{-1}(\underline{x})}\left(\chi_{0}^{-1}(x)\right)^{2}e^{-\chi_{0}^{-1}(x)}.$$

(iv) Fix $s \geq \underline{x} > \inf \text{supp}(F_\beta)$ and express $\mathbf{CS}_{s,\beta}$ as follows:

$$\mathbf{CS}_{s,\beta} = \int_{p_{s,\beta}}^{\infty} (x - p_{s,\beta})\, dF_\beta(x) = -\int_{p_{s,\beta}}^{\infty} (x - p_{s,\beta})\, d(1 - F_\beta(x))$$

$$= -(1 - F_\beta(x))(x - p_{s,\beta})|_{p_{s,\beta}}^{\infty} + \int_{p_{s,\beta}}^{\infty} 1 - F_\beta(x)\, dx = -\mathbf{R}_{\infty,\beta} + \int_{p_{s,\beta}}^{\infty} 1 - F_\beta(x)\, dx,$$

where $p_{s,\beta}$ and $\mathbf{R}_{\infty,\beta}$ are given by (4) and (12), respectively. Note that

$$\mathbf{R}_{\infty,\beta} = \lim_{x\to\infty} (1 - F_\beta(x))(x - p_{s,\beta}) = \lim_{x\to\infty} (1 - F_\beta(x))\, x,$$

and $\mathbf{E}_{F_\beta}[x]$ is given by

$$\mathbf{E}_{F_\beta}[x] = \int_{\underline{x}}^{\infty} x dF_\beta(x) = \underline{x} - \mathbf{R}_{\infty,\beta} + \int_{\underline{x}}^{\infty} 1 - F_\beta(x)\, dx.$$

Altogether, these arguments imply

$$\mathbf{E}_{F_\beta}[x] - \underline{x} = -\mathbf{R}_{\infty,\beta} + \int_{p_{s,\beta}}^{\infty} 1 - F_\beta(x)\, dx + \int_{\underline{x}}^{p_{s,\beta}} 1 - F_\beta(x)\, dx = \mathbf{CS}_{s,\beta} + \mathbf{D}_{s,\beta},$$

where $\mathbf{D}_{s,\beta} = \int_{\underline{x}}^{p_{s,\beta}} 1 - F_\beta(x)\, dx$. Next, $\mathbf{D}_{s,\beta} > 0$ due to $p_{s,\beta} > s \geq \underline{x}$. This leads to

$$\mathbf{E}_{F_\beta}[x] - \underline{x} = \mathbf{CS}_{s,\beta} + \mathbf{D}_{s,\beta} > \mathbf{CS}_{s,\beta}.$$

Finally, note that $\mathbf{W}_{s,\beta} = \mathbf{CS}_{s,\beta} + \mathbf{PS}_{s,\beta} = \mu_\beta + \rho_\beta s$, where $\rho_\beta < 0$, and $\mathbf{W}_{s,\beta} > 0$ for all $s > 0$. It can be the case if and only if $\mu_\beta = \infty$. Because $\mathbf{PS}_{s,\beta} < \infty$ for any s and β, then $\mathbf{CS}_{s,\beta} = \mathbf{E}_{F_\beta}[x] = \infty$.

(v) Using (2) and $\phi_\beta(x) \geq \phi_\beta(x^*) = \frac{y_\beta}{C} > 0$ for all $x \geq \underline{x}$, we obtain

$$f_\beta(x) = \frac{1}{\phi_\beta(x)} \exp\left(-\int_{\underline{x}}^{x} \frac{1}{\phi_\beta(t)} dt\right) \geq \frac{1}{\phi_\beta(x)} \exp\left(-\int_{\underline{x}}^{x} \frac{y_\beta}{C} dt\right) = \frac{\exp\left(-\frac{y_\beta}{C}(x - \underline{x})\right)}{\phi_\beta(x)} > 0.$$

Finally, we show that $f_\beta(x)$ is strictly pseudo-concave, i.e., $f'_\beta(x) \gtrless 0$ if and only if $x \lesseqgtr x^{**}_\beta = \max\{\underline{x}, \chi_\beta(2)\}$. This will imply that $f_\beta(x)$ is unimodal with the mode x^{**}_β.

Suppose first that $\underline{x} \leq \chi_\beta(2)$. Using (1), we obtain

$$f'_\beta(x) = -\frac{1+\phi'_\beta(x)}{\phi^2_\beta(x)} \exp\left(-\int_{\underline{x}}^{x} \frac{1}{\phi_\beta(t)} dt\right).$$

Thus, $f'_\beta(x) \gtreqless 0$ if and only if $\phi'_\beta(x) \lesseqgtr -1$. Because $\phi'_\beta(x) = 1 - z = 1 - \chi^{-1}_\beta(x)$, we get

$$f'_\beta(x) \gtreqless 0 \Leftrightarrow 1 - \chi^{-1}_\beta(x) \lesseqgtr -1 \Leftrightarrow z = \chi^{-1}_\beta(x) \gtreqless 2.$$

Because $\chi^{-1}_\beta(x)$ is strictly decreasing in x, then $f'_\beta(x) \gtreqless 0$ if and only if $x \lesseqgtr x^{**}_\beta = \chi_\beta(2)$, where $\chi_\beta(2) < \chi_\beta(1) = x^*$. If $\underline{x} > \chi_\beta(2)$, then $f_\beta(x)$ is strictly decreasing for all $x \geq \underline{x}$, that is, $x^{**} = \underline{x}$.

(vi) The demand elasticity is $\varepsilon_\beta(p) = -\frac{p}{\phi_\beta(p)}$, or $-\frac{1}{\varepsilon_\beta(p)} = \frac{\phi_\beta(p)}{p}$. Because $\phi_\beta(p) \geq \frac{y_\beta}{C} > 0$ and $\lim_{p\to\infty} \phi'_\beta(p) = 1$ for all $\beta \geq 0$, then $\lim_{p\to\infty} \frac{\phi_\beta(p)}{p} = \lim_{p\to\infty} \phi'_\beta(p) = 1$ by L'Hôpital's rule. Thus, there is $\underline{\varepsilon}_\beta < -1$, such that $\frac{\phi_\beta(p)}{p} = -\frac{1}{\varepsilon_\beta(p)} \geq -\frac{1}{\underline{\varepsilon}_\beta}$ or, equivalently, $\varepsilon_\beta(p) \geq \underline{\varepsilon}_\beta = \varepsilon_\beta(\hat{p}_\beta)$ for all $p > 0$. ($\underline{\varepsilon}_\beta < -1$ follows from $\phi_\beta(p_{s,\beta}) = p_{s,\beta} - s < p_{s,\beta}$ for all $s > 0$. Hence, $\frac{p_{s,\beta}}{\phi_\beta(p_{s,\beta})} > 1$, or $\underline{\varepsilon}_\beta \leq \varepsilon_\beta(p_{s,\beta}) = -\frac{p_{s,\beta}}{\phi_{s,\beta}(p_{s,\beta})} < -1$.) Also, since $\frac{\phi_\beta(p)}{p} < 1$ as $p \to \infty$, and $\lim_{p\to\infty} \phi'_\beta(p) = 1$, then $\hat{p}_\beta < \infty$. (Otherwise, if $\hat{p}_\beta = \infty$, then $\lim_{p\to\infty} \frac{\phi_\beta(p)}{p} = 1$ results in the contradiction $\underline{\varepsilon}_\beta = -1$.)

Taking the derivative of $\varepsilon_\beta(p)$ yields

$$\varepsilon'_\beta(p) = -\frac{\phi_\beta(p) - p\phi'_\beta(p)}{\phi^2_\beta(p)} = -\frac{p}{\phi^2_\beta(p)}\left(\frac{\phi_\beta(p)}{p} - \phi'_\beta(p)\right),$$

so that $\varepsilon'_\beta(\hat{p}_\beta) = 0$ implies

$$\frac{\phi_\beta(\hat{p}_\beta)}{\hat{p}_\beta} = \phi'_\beta(\hat{p}_\beta).$$

Also, since $\phi_\beta(x)$ is strictly convex in x for all $\beta \geq 0$, then $\varepsilon_\beta(x) = -\frac{x}{\phi_\beta(x)}$ is strictly pseudo-convex for $x > 0$. To show this, note that $\phi_\beta(x)$ is twice differentiable, and

$$\varepsilon''_\beta(x) = \left(-\frac{x}{\phi_\beta(x)}\right)'' = \frac{\phi''_\beta(x)\phi_\beta(x)x + 2\phi'_\beta(x)\left(\phi_\beta(x) - x\phi'_\beta(x)\right)}{\phi^3_\beta(x)}.$$

Then at any $\hat{x}$, such that $\varepsilon'_\beta(\hat{x}) = 0$, we have $\phi_\beta(\hat{x}) = \hat{x}\phi'_\beta(\hat{x})$, and $\varepsilon''_\beta(\hat{x}) = \frac{\phi''_\beta(\hat{x})\hat{x}}{\phi^2_\beta(\hat{x})} > 0$. That is, the derivative function $\varepsilon'_\beta(x)$ can intersect the horizontal line $x \equiv 0$ only once and from below. That is, $\varepsilon_\beta(x)$ is strictly pseudo-convex, which implies that $\hat{p}_\beta \in [\underline{x}, \infty)$ is unique. ■

Proof of Theorem 4 Suppose $\phi_\beta(x)$ is given by (40) for $x \geq \underline{x}$, and the boundary conditions (3) hold. By Theorem 2, for any $s \in \mathbf{C}$ there is unique p_s that satisfies (4) .

Next, using (16) and (44), we get

$$s = \psi_\beta(p_{s,\beta}) = v_\beta(z_{s,\beta}) \text{ and } z_{s,\beta} = v^{-1}_\beta(s),$$

where $v_\beta^{-1}(s)$ is well defined, since $v_\beta(z)$ is strictly decreasing in z:

$$v'_\beta(z) = \begin{cases} -\frac{1}{C}\frac{(1+\beta z)^{\frac{1}{\beta}}}{z} & \text{if } \beta > 0 \\ -\frac{e^z}{Cz} & \text{if } \beta = 0, \end{cases}$$

that is, $v'_\beta(z) < 0$ for $z > 0$. Next, we have

$$\begin{aligned}
\phi_\beta(p_{s,\beta}) &= \frac{y_\beta(z_{s,\beta})}{C} = \frac{y_\beta\left(v_\beta^{-1}(s)\right)}{C}, \\
p_{s,\beta} &= s + \phi_\beta(p_{s,\beta}) = s + \frac{y_\beta\left(v_\beta^{-1}(s)\right)}{C} = \chi_\beta(z_{s,\beta}) = \chi_\beta\left(v_\beta^{-1}(s)\right), \\
p'_{s,\beta} &= \frac{1}{z_{s,\beta}} = \frac{1}{v_\beta^{-1}(s)}, \text{ and} \\
\varepsilon_\beta(p_{s,\beta}) &= -\frac{p_{s,\beta}}{\phi_\beta(p_{s,\beta})} = -\frac{\chi_\beta(z_s)}{\frac{y_\beta(z_s)}{C}} = -C\frac{\chi_\beta(z_s)}{y_\beta(z_s)} = -1 - C\frac{v_\beta(z_{s,\beta})}{y_\beta(z_{s,\beta})}.
\end{aligned}$$

Also, using $\mathbf{W}''_s = 0$ results in

$$(p'_s)^2 - \phi(p_s)\, p''_s + \beta p'_s = 0,$$

$$p''_s = \frac{(p'_s)^2 + \beta p'_s}{\phi(p_s)} = \frac{C}{y_\beta(z_{s,\beta})}\left(\frac{1}{z_{s,\beta}^2} + \frac{\beta}{z_{s,\beta}}\right) = \frac{C}{y_\beta\left(v_\beta^{-1}(s)\right)}\left(\frac{1}{v_\beta^{-1}(s)^2} + \frac{\beta}{v_\beta^{-1}(s)}\right).$$

Now, consider $Q_\beta(x)$. By using (42) for $\phi_\beta(x)$ on $[p_{\underline{c},\beta}, p_{\bar{c},\beta}]$, we obtain[52]

$$\begin{aligned}
Q_\beta(x) = 1 - F_\beta(x) &= \exp\left(-\int_{\underline{x}}^{p_{\underline{c},\beta}} \frac{1}{\phi_\beta(t)}dt - \int_{p_{\underline{c},\beta}}^{x} \frac{1}{\phi_\beta(t)}dt\right) \\
&= M_\beta \frac{1 + \beta\chi_\beta^{-1}(p_{\underline{c},\beta})}{\chi_\beta^{-1}(p_{\underline{c},\beta})}\frac{\chi_\beta^{-1}(x)}{1 + \beta\chi_\beta^{-1}(x)} = M_\beta \frac{\frac{1}{\chi_\beta^{-1}(p_{\underline{c},\beta})} + \beta}{\frac{1}{\chi_\beta^{-1}(x)} + \beta},
\end{aligned}$$

for $x \in [p_{\underline{c},\beta}, p_{\bar{c},\beta}]$, where $M_\beta = \exp\left(-\int_{\underline{x}}^{p_{\underline{c},\beta}} \frac{1}{\phi_\beta(t)}dt\right)$, and

$$\mathbf{Q}_\beta(z) = Q_\beta(\chi_\beta(z)) = M_\beta \frac{\frac{1}{z_{\underline{c},\beta}} + \beta}{\frac{1}{z} + \beta},$$

where $z_{\underline{c},\beta} = \chi_\beta^{-1}(p_{\underline{c},\beta})$. This gives

$$q_{s,\beta} = Q_\beta(p_s) = \mathbf{Q}_\beta(z_s) = M_\beta \frac{\frac{1}{z_{\underline{c},\beta}} + \beta}{\frac{1}{z_{s,\beta}} + \beta} = M_\beta \frac{\frac{1}{v_\beta^{-1}(\underline{c})} + \beta}{\frac{1}{v_\beta^{-1}(s)} + \beta}.$$

[52] That is, $\phi_\beta(x)$ may not satisfy (40) for $x \in [\underline{x}, p_{\underline{c},\beta})$.

Next, using (20) leads to

$$\mathbf{W}_s' = -Q\left(p_s\right)\left(p_s' + \beta\right) = -q_{s,\beta}\left(\frac{1}{z_{s,\beta}} + \beta\right) = M_\beta\left(\frac{1}{z_{\underline{c},\beta}} + \beta\right).$$

Finally, we have $\mathbf{PS}_{s,0} = \frac{M_0}{Cz_{\underline{c},0}}e^{z_{s,0}}$ and

$$\begin{aligned}\mathbf{PS}_{s,\beta} &= q_{s,\beta}\phi_\beta\left(p_{s,\beta}\right) = M_\beta \frac{\frac{1}{z_{\underline{c},\beta}} + \beta}{\frac{1}{z_{s,\beta}} + \beta}\frac{y_\beta\left(z_{s,\beta}\right)}{C} = \frac{M_\beta}{C}\left(\frac{1}{z_{\underline{c},\beta}} + \beta\right)\frac{z_{s,\beta}}{1+\beta z_{s,\beta}}\frac{\left(1+\beta z_{s,\beta}\right)^{\frac{1}{\beta}+1}}{z_{z.\beta}} \\ &= \frac{M_\beta}{C}\left(\frac{1}{z_{\underline{c},\beta}} + \beta\right)\left(1+\beta z_{s,\beta}\right)^{\frac{1}{\beta}} \text{ if } \beta > 0.\end{aligned}$$

■

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