Mechanism Design with Sequential-Move Games: Revelation Principle

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Abstract

Traditionally, mechanism design focuses on simultaneous-move games (e.g., Myerson (1981)). In this paper, we study mechanism design with sequential-move games, and provide two results on revelation principles for general solution concepts (e.g., perfect Bayesian equilibrium, obvious dominance, strong-obvious dominance). First, if a solution concept is additive, implementation in sequential-move games is equivalent to implementation in simultaneous-move games. Second, for any solution concept ρ and any social choice function f, we identify a canonical operator $\gamma^{(\rho,f)}$, which is defined on primitives. We prove that, if ρ is monotonic, f can be implemented by a sequential-move game if and only if $\gamma^{(\rho,f)}$ is achievable, which translates a complicated mechanism design problem into checking some conditions defined on primitives. Most of the existing solution concepts are either additive or monotonic.

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1 Introduction

Traditional mechanism design theory imposes an implicit assumption: only a simultaneous-move mechanism can be adopted. For instance, Myerson (1981) is the first paper that finds the optimal simultaneous-bid mechanism for a general auction setup.

However, sequential-move mechanisms are adopted in many of our usual practices, e.g., English auctions, Dutch auctions, run-off elections, FIFA world cup bids. This immediately begs the question: does the traditional mechanism design theory (which focuses on simultaneous-move mechanisms) suffer loss of generality? Specifically,

$$\left(\begin{array}{c}\text{does the optimal simultaneous-bid mechanism in Myerson (1981) remain the optimal one} \\ \text{when we allow for sequential-bid mechanisms?}\end{array}\right) \tag{1}$$

In sequential-bid mechanisms, payoff-relevant information is partially disclosed at each round of bidding, e.g., we may disclose all of buyers' bids in previous rounds, or we may disclose some or none of them. Do these different rules change bidders' strategic behaviors and the final outcome?

Another critical dimension is solution concept. We may adopt different solution concepts for different problems. For instance, we usually adopt Bayesian Nash equilibrium in auction setups (e.g., Myerson (1981)), while we adopt weak dominance (or equivalently, strategyproofness) in matching problems (e.g., the deferred acceptance mechanism in Gale and Shapley (1962)).[1] Recently, Li (2017) and Pycia and Troyan (2023) propose two new solution concepts: obvious dominance and strong-obvious dominance, and they prove that, for these solution concepts, implementation by sequential-move mechanisms differs substantially from implementation by simultaneous-move mechanisms. This result implies that (1) is a non-trivial question, and begs an answer for it.

Interestingly, when we compare implementation in weak dominance to implementation in obvious dominance, some authors (e.g., Ashlagi and Gonczarowski (2018)) define the previous one on simultaneous-move games only, while define the latter one on

[1] The cooperative game-theory approach is usually adopted in matching problems, and non-cooperative solution concepts (e.g., Bayesian Nash equilibrium) do not apply to such problems.

sequential-move games. Does this suffer loss of generality? If no, in what sense?

We divide solution concepts into two groups: Category I and Category II. Category I includes all of the solution concepts on which implementation by a sequential-move mechanism is equivalent to implementation by a simultaneous-move mechanism, and Category II includes all the others. Li (2017) and Pycia and Troyan (2023) prove that obvious dominance and strong-obvious dominance are in Category II, while folk wisdom seems to suggest that weak dominance and Bayesian Nash equilibrium are in Category I. This leads to an important question: what property the last two solution concepts possess (but the first two do not) makes them in Category I? For instance, consider a new solution concept: max-min equilibrium (see below).[2] Is it in Category I or Category II?

$$\begin{aligned}
\text{Bayesian Nash Equilibrium:}\quad & \int_{\Theta_{-i}} \left(u_i^{\theta_i}\left[f\left(\theta_i,\theta_{-i}\right)\right]-u_i^{\theta_i}\left[f\left(\theta_i',\theta_{-i}\right)\right]\right)\mu^{\theta_i}\left[d\theta_{-i}\right]\geq 0, \forall\left(i,\theta\right)\in\mathcal{I}\times\Theta,\\
\text{obvious dominance:}\quad & \min_{\theta_{-i}\in\Theta_{-i}} u_i^{\theta_i}\left[f\left(\theta_i,\theta_{-i}\right)\right]\geq \max_{\theta_{-i}\in\Theta_{-i}} u_i^{\theta_i}\left[f\left(\theta_i',\theta_{-i}\right)\right], \forall\left(i,\theta\right)\in\mathcal{I}\times\Theta,\\
\text{max-min equilibrium:}\quad & \min_{\theta_{-i}\in\Theta_{-i}} u_i^{\theta_i}\left[f\left(\theta_i,\theta_{-i}\right)\right]\geq \min_{\theta_{-i}\in\Theta_{-i}} u_i^{\theta_i}\left[f\left(\theta_i',\theta_{-i}\right)\right], \forall\left(i,\theta\right)\in\mathcal{I}\times\Theta.
\end{aligned}$$

A powerful tool in traditional mechanism design is revelation principle: a social choice function can be implemented by a general simultaneous-move mechanism if and only if it can be implemented by the induced direct mechanism. General mechanisms are complicated and they are not primitives, while direct mechanisms are simple and they are primitives.[3] *The revelation principle translates a complicated problem involving non-primitives to a simple problem defined on primitives.*

In this paper, we study mechanism design with sequential-move mechanisms for general solution concepts.[4] In this setup, we aim to establish an analogous revelation

[2] An max-min equilibrium describes strategic behaviors of uncertain-averse players, who assess strategies by the worst possible scenarios.

[3] A direct mechanisms is equivalent to a social choice function.

[4] In mechanism design, there are two paradigms: static mechanism design (or equivalently, one-period mechanism design) and dynamic mechanism design (or equivalently, multi-period mechanism design). This paper belongs to static mechanism design. That is, we design a mechanism in one single period, but within this period, our mechanism is a sequential-move game rather than a simultaneous-move game.

principle, which translates a complicated problem involving mechanisms to a simple problem defined on primitives.

Given simultaneous-move games, even though a social choice function could possibly be implemented by different equilibria in different mechanisms, all of them induce the same "outcome": a direct mechanism, and incentive compatibility on this unique direct mechanism defines the simple problem in revelation principle. However, given sequential-move games, players disclose their types sequentially and gradually, and the protocol of disclosing is not *a priori* fixed. Hence, a social choice function could possibly be implemented by different equilibria in different mechanisms, which lead to multiple different "outcomes." We thus face two difficulties: (1) what is the counterpart of "direct mechanism" for each implemented outcome? and (2) which "outcome" should we focus on, when we define the simple problem in revelation principle?

Due to this problem (i.e., indeterminacy of implemented outcomes), current revelation principles for obvious dominance and strong-obvious dominance have the following form: a social choice function f can be implemented by a general sequential-move mechanism if and only if f can be implemented by some mechanism in *a particular set of potential sequential-move mechanisms* (e.g., Bade and Gonczarowski (2017), Ashlagi and Gonczarowski (2018), Mackenzie (2020), Pycia and Troyan (2023)). This translates a complicated mechanism design problem into a simpler mechanism design problem, which is not yet defined on primitives.

In order to achieve our goal, we propose a device called *operator*, which is defined on primitives. If a social choice function is implemented by a general sequential-move mechanism, this mechanism induces a particular operator, which describes the dynamic process of players disclosing their types sequentially. We show that a social choice function can be implemented by a general sequential-move mechanism if and only if the corresponding operator satisfies some properties. In particular, the traditional revelation principle for simultaneous-move mechanisms is a degenerate case of our revelation principle. A conceptual description of our revelation principle is provided in Section 2.

Revelation principle for dynamic mechanism design is studied in Myerson (1986), Sugaya and Wolitzky (2021).

Based on this, we provide two results for general solution concepts. First, we identify a property of solution concepts: additivity (Definition 13). We prove that additive solution concepts (e.g., weak dominance, perfect Bayesian equilibrium, max-min equilibrium) are in Category I, and as a result, the traditional revelation principle remains valid when we allow for sequential-move mechanisms.

Second, we consider non-additive solution concepts, e.g., obvious dominance, strong-obvious dominance. Nevertheless, these solution concepts share another property: monotonicity (Definition 14).

For any social choice function $f : \Theta \longrightarrow \mathcal{X}$, the value of $f(\theta)$ hinges critically on the value of θ, which is privately observed by the players. Thus, if a mechanism implements f, all of the players must reveal their types fully in the end—this is formalized as a condition defined on primitives: the operator induced by the mechanism (which implements f) is *achievable* (Definition 6).

Our second result is that we identify a canonical operator $\gamma^{(\rho,f)}$ for any solution concept ρ and any social choice function f. If ρ is monotonic, we prove that $\gamma^{(\rho,f)}$ is a lower bound for any operator induced by a mechanism which implements f under ρ. This leads to a revelation principle for monotonic solution concepts: f can be implemented by a general sequential-move mechanism if and only if $\gamma^{(\rho,f)}$ is achievable, which is a condition defined on primitives.

The remainder of the paper proceeds as follows. We describe our revelation principle in Section 2. We define the model in Section 3. In Section 4, we identify a set of games and strategy profiles which substantially simplify our analysis. We propose new tools in Section 5. We study additive and monotonic solution concepts in Sections 6 and 7, respectively. We conclude in Section 8.

2 Revelation principle: a conceptual summary

In this section, we describe both the traditional revelation principle and our revelation principle for sequential-move mechanisms. The former is a degenerate case of the latter.

Given simultaneous-move games and Bayesian Nash equilibrium, the traditional revelation principle says that a social choice function $f : \Theta \longrightarrow \mathcal{X}$ is implemented by a mechanism G and an equilibrium S if and only if (i) $f = G \circ S$ is the induced direct mechanism, and (ii) (Bayesian) incentive compatibility holds, i.e.,

$$\int\limits_{\Theta_{-i}} \left(u_i^{\theta_i^*} \left[f\left(\theta_i^*, \theta_{-i}\right) \right] - u_i^{\theta_i^*} \left[f\left(\theta_i', \theta_{-i} \right) \right] \right) \mu^{\theta_i^*} \left[d\theta_{-i} \right] \geq 0, \forall \left(i, \theta^* \right) \in \mathcal{I} \times \Theta, \forall \theta' \in \Theta \diagdown \left\{ \theta^* \right\}. \tag{2}$$

We now consider sequential-move games, and for simplicity, we focus on perfect-information games only in this section. Suppose that $f : \Theta \longrightarrow \mathcal{X}$ is implemented by a sequential-move mechanism G and a strategy profile S. In order to establish a revelation principle, we need to generalize two ideas: (i) what object (defined on primitives) does $G \circ S$ induce? (ii) how should incentive compatibility be defined in such a setup?

Regarding (i), we propose to use an "operator" (Definition 4) to represent $G \circ S$. An operator γ is a function which maps any set $\widehat{\Theta}$ in $\left(\times_{i \in \mathcal{I}} \left[2^{\Theta_i} \diagdown \{\varnothing\} \right] \right)$ to a partition of $\widehat{\Theta}$, or precisely,

$$\gamma : \left(\times_{i \in \mathcal{I}} \left[2^{\Theta_i} \diagdown \{\varnothing\} \right] \right) \times \Theta \longrightarrow \times_{i \in \mathcal{I}} \left[2^{\Theta_i} \diagdown \{\varnothing\} \right] \text{ and } \left(\begin{array}{c} \text{given } \theta, \theta' \in \widehat{\Theta}, \\ \text{we have } \theta \in \gamma\left[\widehat{\Theta}, \theta \right] \text{ and} \\ \theta' \in \gamma\left[\widehat{\Theta}, \theta \right] \Longrightarrow \gamma\left[\widehat{\Theta}, \theta' \right] = \gamma\left[\widehat{\Theta}, \theta \right] \end{array} \right).$$

We use $\gamma^{G \circ S}$ to represent the operator induced by $G \circ S$. Suppose the true state is θ^*. At the beginning of the game, players follow $S\left(\theta^*\right)$ to proceed to the next information set, and $\gamma^{G \circ S}\left[\Theta, \theta^*\right]$ denotes the set of states that take the same strategy (i.e., $S\left(\theta^*\right)$); at the second information set, players follow $S\left(\theta^*\right)$ to proceed to the next information set, and $\gamma^{G \circ S}\left[\gamma^{G \circ S}\left[\Theta, \theta^*\right], \theta^*\right]$ denotes the set of states that take the same strategy (i.e., $S\left(\theta^*\right)$);.... i.e., $G \circ S$ describes how players sequentially disclose their types, which is recorded as:

$$\Theta \longrightarrow \gamma^{G \circ S}\left[\Theta, \theta^*\right] \longrightarrow \gamma^{G \circ S}\left[\gamma^{G \circ S}\left[\Theta, \theta^*\right], \theta^*\right] \longrightarrow \gamma^{G \circ S}\left[\gamma^{G \circ S}\left[\gamma^{G \circ S}\left[\Theta, \theta^*\right], \theta^*\right], \theta^*\right] \ldots \tag{3}$$

Since $f : \Theta \longrightarrow \mathcal{X}$ is implemented by $G \circ S$, players must disclose the true state θ^* in the end, i.e., the process described in (3) must lead to $\{\theta^*\}$. — This condition is called *achievability* (Definition 6).

Regarding (ii), we break the incentive compatibility condition in a sequential-move game into two parts. First, at any particular information set, the adopted solution concept dictates whether it is incentive compatible to falsely report θ' at the true state θ^*. We use an abstract function (called solution notion, see Definition 5) to describe it:

$$\rho : \left(\times_{i \in \mathcal{I}} \left[2^{\Theta_i} \diagdown \{\varnothing\} \right] \right) \times \Theta \times \Theta \times \mathcal{I} \longrightarrow \{0, 1\}.$$

The interpretation is:

$$\rho \left[\widehat{\Theta}, \theta^*, \theta', i \right] = 1 \Longleftrightarrow \left[\begin{array}{c} \text{at a history } h \text{ in a game } G, \\ \text{with } \widehat{\Theta} \text{ being the set of states in which } S \text{ leads to } h, \\ \text{agent } i \text{ prefers truthfully revealing } \theta^* \\ \text{to falsely reporting } \theta' \end{array} \right].$$

For instance, Perfect Bayesian equilibrium could be represented by ρ^{PBE} as follows.

$$\rho^{PBE} \left[\widehat{\Theta}, \theta^*, \theta', i \right] \equiv \begin{cases} 1, & \text{if } \int\limits_{\widehat{\Theta}_{-i}} \left(u_i^{\theta_i^*} \left[f\left(\theta_i^*, \widehat{\theta}_{-i} \right) \right] - u_i^{\theta_i^*} \left[f\left(\theta_i', \widehat{\theta}_{-i} \right) \right] \right) \mu^{\theta_i^*} \left[d\widehat{\theta}_{-i} \right] \geq 0, \\ \\ 0, & \text{otherwise.} \end{cases}.$$

Second, since players disclose their types gradually, we require that, whenever a player discloses her type partially at an information set, she must be incentive compatible to do so.—The contrapositive statement is called (ρ, f)-consistency (Definition 7), which is formalized as follows.

$$\rho \left[\widehat{\Theta}, \theta^*, \theta', i \right] = 0 \Longrightarrow \theta_i' \in \gamma_i^{G \circ S} \left[\widehat{\Theta}, \theta^* \right], \forall i \in \mathcal{I}, \tag{4}$$

That is, if it is not incentive compatible to falsely report θ' (i.e., $\rho \left[\widehat{\Theta}, \theta^*, \theta', i \right] = 0$), then player i must bundle θ_i' with θ_i^* (i.e., $\theta_i' \in \gamma_i^{G \circ S} \left[\widehat{\Theta}, \theta^* \right]$).

Given these new notions, it is easy to show a conceptual revelation principle:

$$\left(\begin{array}{c} f \text{ is implemented by} \\ \text{a sequential-move mechanism } G \\ \text{and a strategy profile } S \\ \text{under the solution concept } \rho \end{array} \right) \Longleftrightarrow \left(\begin{array}{c} \text{the induced operator } \gamma^{G \circ S} \\ \text{is both achievable and } (\rho, f)\text{-consistent} \end{array} \right).$$

This revelation principle generalizes the traditional revelation principle for simultaneous-move games. To see this, suppose that f is implemented by a simultaneous-move mechanism G and a strategy profile S. Thus, the induced operator $\gamma^{G \circ S}$ satisfies

$$\gamma^{G \circ S}[\Theta, \theta^*] = \{\theta^*\}, \forall \theta^* \in \Theta,$$

i.e., players fully disclosed θ^* in G at each true state $\theta^* \in \Theta$, and achievability of $\gamma^{G \circ S}$ holds. Furthermore, this implies

$$\theta_i' \notin \gamma_i^{G \circ S}[\Theta, \theta^*], \forall (i, \theta^*) \in \mathcal{I} \times \Theta, \forall \theta' \in \Theta \diagdown \{\theta^*\},$$

and hence, (ρ, f)-consistency (i.e., (4)) immediately implies

$$\rho\left[\Theta, \theta^*, \theta', i\right] = 1, \forall (i, \theta^*) \in \mathcal{I} \times \Theta, \forall \theta' \in \Theta \diagdown \{\theta^*\},$$

which becomes the usual Baysian incentive compatibility condition (i.e., (2)), if ρ is Bayesian Nash equilibrium.

3 Model

There is a finite set of agents, denoted by $\mathcal{I}$. Let $\Theta \equiv \times_{i \in \mathcal{I}} \Theta_i$ denote a set of states, which could be either finite or infinite. Let $\mathcal{X}$ denote a set of social outcomes. At each state $(\theta_i)_{i \in \mathcal{I}} \in \Theta$, agent i observes θ_i privately, and agent i's utility depends only on θ_i, i.e., her utility function is described $u_i^{\theta_i} : \mathcal{X} \longrightarrow \mathbb{R}$. Our goal is to implement a social choice function (hereafter, SCF) $f : \Theta \longrightarrow \mathcal{X}$.

Let $\mathbb{N}$ denote the set of positive integers. We use $E \subset E'$ to denote that E is a weak subset of E', and use $E \subsetneqq E'$ to denote that E is a strict subset of E'. Throughout the paper, we use $-i$ to denote $\mathcal{I} \diagdown \{i\}$. For any $E \subset \times_{j \in \mathcal{I}} X_j$ and any $i \in \mathcal{I}$, define

$$E_i \equiv \{e_i \in X_i : \exists e_{-i} \in X_{-i}, \ (e_i, e_{-i}) \in E\}.$$

3.1 Mechanisms

A mechanism (or equivalently, a sequential-move game), denoted by G, is a tuple:

$$G \equiv \begin{bmatrix} \mathcal{I}, \mathcal{A}, \mathcal{H}, \mathcal{T}, K \in \mathbb{N}, \phi : \mathcal{H} \diagdown \mathcal{T} \longrightarrow \mathcal{I}, \\ A : \mathcal{H} \diagdown \mathcal{T} \longrightarrow 2^{\mathcal{A}} \diagdown \{\varnothing\}, \ (\zeta_i : \mathcal{H} \longrightarrow \mathcal{H})_{i \in \mathcal{I}}, \ g : \mathcal{T} \longrightarrow \mathcal{X} \end{bmatrix},$$

where $\mathcal{A}$ is a set of actions, and $\mathcal{H}$ is a set of histories such that

$$\sigma \in \mathcal{H} \text{ and } \mathcal{H} \subset \{\sigma\} \cup \left[\bigcup_{k=1}^{K} \mathcal{A}^k \right],$$

where σ denotes the initial history, at which no action has been taken by any player yet. For each $h \in \mathcal{H}$, let $|h|$ denote the length of h, i.e., $|\sigma| = 0$ and $\left| \left(a^1, ..., a^k\right) \right| = k$. We require $K \geq |h|$ for any $h \in \mathcal{H}$, i.e., K is an upper bound for lengths of histories.[5] However, the upper bound (i.e., K) can be arbitrarily large for all of the games we consider.

For any history $h \in \mathcal{H} \diagdown \{\sigma\}$, let Path$[h]$ denote the set of sub-histories of h, and rigorously,

$$\text{Path}[h] \equiv \{\sigma\} \cup \left\{ \left(a^1, ..., a^{L'}\right) : L' \leq L \right\}, \forall h = \left(a^1, ..., a^L\right) \in \mathcal{H} \diagdown \{\sigma\}.$$

We require Path$[h] \subset \mathcal{H}$ for any $h \in \mathcal{H}$. A history is terminal if and only if it is not a sub-history of a different history. We use $\mathcal{T}$ to denote the set of terminal histories.

At each non-terminal history $h \in \mathcal{H} \diagdown \mathcal{T}$, $\phi(h) \in \mathcal{I}$ denotes the agent who will choose the next action, and the set of actions available for $\phi(h)$ at h is $A(h)$. For any $a \in A(h)$, let $[h, a]$ denote the history of "a following h," i.e., $[\sigma, a] = (a) \in \mathcal{H}$ and

$$\left[h = \left(a^1, ..., a^k\right), \ a \right] = \left(a^1, ..., a^k, a\right) \in \mathcal{H},$$

$$\text{and } A[h] \equiv \{a \in \mathcal{A} : [h, \ a] \in \mathcal{H}\}, \forall h \in \mathcal{H} \diagdown \mathcal{T}.$$

For each agent $i \in \mathcal{I}$, the function $\zeta_i : \mathcal{H} \longrightarrow \mathcal{H}$ describes i's information sets. For any two of i's histories, $h, h' \in \phi^{-1}(i)$, agent i cannot distinguish them if and only if

[5] One technical difficulty induced by unbounded history lengths is that a strategy profile may result in a terminal history with infinitely-many moves. For simplicity, we impose the upper bound to avoid this technical difficulty.

$\zeta_i(h) = \zeta_i(h')$, which defines an equivalence relation on $\mathcal{H} \diagdown \mathcal{T}$:

$$h \overset{G}{\sim} h' \Longleftrightarrow \begin{pmatrix} \exists i \in \mathcal{I}, \phi(h) = \phi(h') = i, \\ \zeta_i(h) = \zeta_i(h') \end{pmatrix}, \forall (h, h') \in [\mathcal{H} \diagdown \mathcal{T}] \times [\mathcal{H} \diagdown \mathcal{T}].$$

At a history $h \in \phi^{-1}(i)$, agent i's information set is $\left\{h' \in \mathcal{H} \diagdown \mathcal{T} : h \overset{G}{\sim} h'\right\}$. We require

$$h \overset{G}{\sim} h' \Longrightarrow A(h) = A(h'), \forall (h, h') \in [\mathcal{H} \diagdown \mathcal{T}] \times [\mathcal{H} \diagdown \mathcal{T}].$$

Finally, $g : \mathcal{T} \longrightarrow \mathcal{X}$ maps each terminal history $h \in \mathcal{T}$ to a social outcome $g(h) \in \mathcal{X}$.

Let $\mathcal{G}$ denote the set of all such mechanisms. In particular, a mechanism G is a perfect-information game if and only if

$$h \overset{G}{\sim} h' \Longrightarrow h = h', \forall (h, h') \in [\mathcal{H} \diagdown \mathcal{T}] \times [\mathcal{H} \diagdown \mathcal{T}],$$

i.e., every player's information set contains one single history. Let $\mathcal{G}^{\text{PI}}$ denote the set of all perfect-information games.

3.2 Behavior strategies and strategies

Given a mechanism $G \in \mathcal{G}$, we define behavior strategies and strategies in this subsection. A behavior strategy of agent i is a ζ_i-measurable function $B_i : \phi^{-1}(i) \longrightarrow \mathcal{A}$, i.e.,

$$\left[\phi(h) = \phi(h') = i \text{ and } h \overset{G}{\sim} h'\right] \Longrightarrow B_i(h) = B_i(h'), \forall (h, h') \in [\mathcal{H} \diagdown \mathcal{T}] \times [\mathcal{H} \diagdown \mathcal{T}].$$

Let $\mathcal{B}_i$ denote the set of all such behavior strategies of agent i, and $\mathcal{B} \equiv \times_{i \in \mathcal{I}} \mathcal{B}_i$.

Each $B = (B_i)_{i \in \mathcal{I}} \in \mathcal{B}$ determines a unique terminal history, which is described by the function $T^G : \mathcal{B} \longrightarrow \mathcal{T}$, i.e., $\left(\mathcal{B}_{\phi(\sigma)}(\sigma)\right) \in$Path$\left[T^G(B)\right]$, and inductively,

$$h \in \text{Path}\left[T^G(B)\right] \Longrightarrow \left[h,\ \mathcal{B}_{\phi(h)}(h)\right] \in \text{Path}\left[T^G(B)\right], \forall h \in \mathcal{H} \diagdown [\mathcal{T} \cup \{\sigma\}].$$

A strategy of player i in game G is a function, $S_i : \Theta_i \longrightarrow \mathcal{B}_i$. Let $\mathcal{S}_i^G$ denote the set of all such strategies of agent i in G, and $\mathcal{S}^G \equiv \times_{i \in \mathcal{I}} \mathcal{S}_i^G$. We say a strategy profile $(S_i)_{i \in \mathcal{I}} \in \mathcal{S}^G$ in G implements an SCF $f : \Theta \longrightarrow \mathcal{X}$ if and only if

$$g\left[T^G\left([S_i(\theta_i)]_{i \in \mathcal{I}}\right)\right] = f\left[(\theta_i)_{i \in \mathcal{I}}\right], \forall \left[(\theta_i)_{i \in \mathcal{I}}\right] \in \Theta.$$

3.3 Solution concepts

Throughout this subsection, we fix any $G \in \mathcal{G}$, and define six solution concepts.

3.3.1 Obvious dominance

For any agent $i \in \mathcal{I}$, $S_i \in \mathcal{S}_i^G$ is obviously dominant if and only if for any $(\theta, h, h', B, B') \in \Theta \times [\mathcal{H} \diagdown \mathcal{T}] \times [\mathcal{H} \diagdown \mathcal{T}] \times \mathcal{B} \times \mathcal{B}$, we have

$$\begin{pmatrix} \phi(h) = \phi(h') = i \text{ and } h \overset{G}{\sim} h', \\ h \in \text{Path}\left(T^G\left[S_i(\theta_i), B_{-i}\right]\right), \\ h' \in \text{Path}\left[T^G(B')\right] \text{ and } S_i(\theta_i)[h] \neq B_i'(h') \end{pmatrix} \Longrightarrow u_i^{\theta_i}\left[g\left(T^G\left[S_i(\theta_i), B_{-i}\right]\right)\right] \geq u_i^{\theta_i}\left[g\left(T^G\left[B'\right]\right)\right]. \tag{5}$$

Suppose that type θ_i reaches the information set containing both h and h' (i.e., $h \overset{G}{\sim} h'$). Agent i faces two options: (i) sticking to $S_i(\theta_i)$ and (ii) deviating to B_i' with $S_i(\theta_i)[h] \neq B_i'(h')$. Roughly, condition (5) means

$$\min_{B_{-i}} u_i^{\theta_i}\left[g\left(T^G\left[S_i(\theta_i), B_{-i}\right]\right)\right] \geq \max_{B'_{-i}} u_i^{\theta_i}\left[g\left(T^G\left[B_i', B'_{-i}\right]\right)\right], \tag{6}$$

where B_{-i} and B' in (6) are restricted to those leading agent i to reach the information set containing both h and h' (i.e., $h \in$Path$\left(T^G\left[S_i(\theta_i), B_{-i}\right]\right)$, $h' \in$Path$\left(T^G\left[B'\right]\right)$).

A strategy profile $S = (S_i)_{i \in \mathcal{I}} \in \mathcal{S}^G$ is obviously dominant in G if and only if S_i is obviously dominant for every $i \in \mathcal{I}$. Let $\mathcal{S}^{G\text{-OD}}$ denote the set of all such strategy profiles.

3.3.2 Strong-obvious dominance

For any agent $i \in \mathcal{I}$, $S_i \in \mathcal{S}_i^G$ is strong-obviously dominant if and only if for any $(\theta, h, h', B, B') \in \Theta \times [\mathcal{H} \diagdown \mathcal{T}] \times [\mathcal{H} \diagdown \mathcal{T}] \times \mathcal{B} \times \mathcal{B}$, we have

$$\begin{pmatrix} \phi(h) = \phi(h') = i \text{ and } h \overset{G}{\sim} h', \\ h \in \text{Path}\left(T^G[B]\right) \cup \text{Path}\left(T^G\left[S_i(\theta_i), B_{-i}\right]\right), \\ h' \in \text{Path}\left(T^G[B']\right) \text{ and } S_i(\theta_i)[h] = B_i(h) \neq B_i'(h') \end{pmatrix} \Longrightarrow u_i^{\theta_i}\left[g\left(T^G[B]\right)\right] \geq u_i^{\theta_i}\left[g\left(T^G\left[B'\right]\right)\right]. \tag{7}$$

Suppose that type θ_i reaches the information set containing both h and h', and $S_i(\theta_i)[h] \neq B'_i(h')$. Condition (7) means

$$\min_{B} u_i^{\theta_i}\left[g\left(T^G[B]\right)\right] \geq \max_{B'} u_i^{\theta_i}\left[g\left(T^G[B']\right)\right], \tag{8}$$

and we consider all of those B and B' which lead agent i to reach the information set containing h and h', and $S_i(\theta_i)[h] = B_i(h) \neq B'_i(h')$, i.e., $S_i(\theta_i)$ and B_i choose the same action at the information set, though B_i may still deviate from $S_i(\theta_i)$ afterwards. Let $\mathcal{S}^{G\text{-SOD}}$ denote the set of strong-obviously dominant strategy profiles in G.

3.3.3 Weak dominance

For any agent $i \in \mathcal{I}$, $S_i \in \mathcal{S}_i^G$ is weakly dominant if and only if for any $(\theta, h, h', B) \in \Theta \times [\mathcal{H} \diagdown \mathcal{T}] \times [\mathcal{H} \diagdown \mathcal{T}] \times \mathcal{B}$, we have

$$\left(\begin{array}{c} \phi(h) = \phi(h') = i, h \overset{G}{\sim} h', \\ h \in \text{Path}\left(T^G[S_i(\theta_i), B_{-i}]\right), \\ h' \in \text{Path}\left[T^G(B)\right] \end{array}\right) \Longrightarrow u_i^{\theta_i}\left[g\left(T^G[S_i(\theta_i), B_{-i}]\right)\right] \geq u_i^{\theta_i}\left[g\left(T^G[B]\right)\right]. \tag{9}$$

Condition (9) means

$$u_i^{\theta_i}\left[g\left(T^G[S_i(\theta_i), B_{-i}]\right)\right] \geq u_i^{\theta_i}\left[g\left(T^G[B_i, B_{-i}]\right)\right], \tag{10}$$

where B_i and B_{-i} in (10) are restricted to those leading agent i to reach the information set containing h and h' (i.e., $h \in \text{Path}\left(T^G[S_i(\theta_i), B_{-i}]\right)$ and $h' \in \text{Path}\left[T^G(B)\right]$). Let $\mathcal{S}^{G\text{-WD}}$ denote the set of weakly dominant strategy profiles in G.

3.3.4 Perfect Bayesian equilibrium

Throughout the paper, we fix any $\mu \in \triangle(\Theta)$, and we will use it only if we consider Perfect Bayesian equilibrium (hereafter, PBE). For each $(i, \theta) \in \mathcal{I} \times \Theta$, let $\mu^{\theta_i} \in \triangle(\Theta_{-i})$ denote the conditional distribution induced by μ on Θ_{-i} given θ_i. Define

$$\Theta_{-i}^{(S,h)} \equiv \left\{\theta_{-i} \in \Theta_{-i} : \begin{array}{c} \exists (\theta, h') \in \Theta \times [\mathcal{H} \diagdown \mathcal{T}], h \overset{G}{\sim} h', \\ h' \in \text{Path}\left(T^G[S(\theta)]\right) \end{array}\right\}, \forall (i, S, h) \in \mathcal{I} \times \mathcal{S}^G \times [\mathcal{H} \diagdown \mathcal{T}].$$

Suppose that all of the players follow S. The set $\Theta_{-i}^{(S,h)}$ contains all of $\theta_{-i} \in \Theta_{-i}$ that could possibly reach the information set containing h. Thus, upon reaching this information set, player i believes her opponents' types come from $\Theta_{-i}^{(S,h)}$.

$S = (S_i)_{i \in \mathcal{I}} \in \mathcal{S}^G$ is a μ-PBE if and only if for any $(\theta, i, h, B) \in \Theta \times \mathcal{I} \times [\mathcal{H} \diagdown \mathcal{T}] \times \mathcal{B}$,

$$\begin{pmatrix} \phi(h) = i, \\ S_i(\theta_i)[h] \neq B_i(h) \end{pmatrix} \Longrightarrow \int_{\Theta_{-i}^{(S,h)}} \left(u_i^{\theta_i} \left[g\left(T^G [S(\theta)] \right) \right] - u_i^{\theta_i} \left[g\left(T^G [B_i, S_{-i}(\theta_{-i})] \right) \right] \right) \mu^{\theta_i} [d\theta_{-i}] \geq 0.$$

Let $\mathcal{S}^{G\text{-}\mu\text{-PBE}}$ denote the set of μ-PBEs in G.

3.3.5 Ex-post equilibrium

$S = (S_i)_{i \in \mathcal{I}} \in \mathcal{S}^G$ is an ex-post equilibrium if and only if for any $(i, \theta, B, h, h') \in \mathcal{I} \times \Theta \times \mathcal{B} \times [\mathcal{H} \diagdown \mathcal{T}] \times [\mathcal{H} \diagdown \mathcal{T}]$, we have

$$\begin{pmatrix} \phi(h) = \phi(h') = i, h \overset{G}{\sim} h', \\ h \in \text{Path}\left(T^G [S(\theta)]\right), \\ h' \in \text{Path}\left(T^G [B_i, S_{-i}(\theta_{-i})]\right), \\ S_i(\theta_i)[h] \neq B_i(h') \end{pmatrix} \Longrightarrow u_i^{\theta_i} \left[g\left(T^G [S(\theta)] \right) \right] \geq u_i^{\theta_i} \left[g\left(T^G [B_i, S_{-i}(\theta_{-i})] \right) \right].$$

The difference between weak dominance and ex-post equilibrium is illustrated as follows.

$$\begin{aligned} \text{weak dominance:} \quad & u_i^{\theta_i} \left[g\left(T^G [S_i(\theta_i), B_{-i}] \right) \right] \geq u_i^{\theta_i} \left[g\left(T^G [B_i, B_{-i}] \right) \right], \\ \text{ex-post equilibrium:} \quad & u_i^{\theta_i} \left[g\left(T^G [S(\theta)] \right) \right] \geq u_i^{\theta_i} \left[g\left(T^G [B_i, S_{-i}(\theta_{-i})] \right) \right]. \end{aligned}$$

Weak dominance considers all possible deviation B_{-i} such that $h \in \text{Path}\left(T^G [S_i(\theta_i), B_{-i}]\right)$ and $h' \in \text{Path}\left(T^G [B_i, B_{-i}]\right)$, while ex-post equilibrium considers all possible deviation $S_{-i}(\theta_{-i})$ such that $h \in \text{Path}\left(T^G [S(\theta)]\right)$ and $h' \in \text{Path}\left(T^G [B_i, S_{-i}(\theta_{-i})]\right)$ (i.e., a special subset of B_{-i}). Thus, dominance implies ex-post equilibrium. Let $\mathcal{S}^{G\text{-EP}}$ denote the set of weakly dominant strategy profiles in G, and we have

$$\mathcal{S}^{G\text{-WD}} \subset \mathcal{S}^{G\text{-EP}}. \tag{11}$$

It is difficult to directly characterize implementation in weak dominance. Instead, we will fully characterize implementation in ex-post equilibrium, which provides an indirect way to fully characterize implementation in weak dominance.

3.3.6 Max-min equilibrium

A strategy profile $S = (S_i)_{i\in\mathcal{I}} \in \mathcal{S}^G$ is a max-min-equilibrium if and only if for any $(i,\theta,B,h,h') \in \mathcal{I}\times\Theta\times\mathcal{B}\times[\mathcal{H}\diagdown\mathcal{T}]\times[\mathcal{H}\diagdown\mathcal{T}]$, we have

$$\left(\begin{array}{c}\phi(h)=\phi(h')=i,\, h\overset{G}{\sim}h',\\ h\in\text{Path}\left(T^G\left[S(\theta)\right]\right),\\ h'\in\text{Path}\left(T^G\left[B\right]\right),\\ S_i(\theta_i)[h]\neq B_i(h')\end{array}\right)\Longrightarrow\left(\begin{array}{c}\exists\left(\widehat{\theta},\widehat{h}\right)\in\Theta\times[\mathcal{H}\diagdown\mathcal{T}],\, h\overset{G}{\sim}\widehat{h},\\ \widehat{h}\in\text{Path}\left(T^G\left[B_i,S_{-i}\left(\widehat{\theta}_{-i}\right)\right]\right),\\ u_i^{\theta_i}\left[g\left(T^G\left[S(\theta)\right]\right)\right]\geq u_i^{\theta_i}\left[g\left(T^G\left[B_i,S_{-i}\left(\widehat{\theta}_{-i}\right)\right]\right)\right]\end{array}\right). \tag{12}$$

Condition (12) requires

$$\min_{\theta_{-i}} u_i^{\theta_i}\left[g\left(T^G\left[S(\theta)\right]\right)\right]\geq\min_{\widehat{\theta}_{-i}} u_i^{\theta_i}\left[g\left(T^G\left[B_i,S_{-i}\left(\widehat{\theta}_{-i}\right)\right]\right)\right], \tag{13}$$

where θ_{-i} and $\widehat{\theta}_{-i}$ in (13) are restricted to those leading agent i to reach the information set containing both h and $\widehat{h}$ (i.e., $h\in\text{Path}\left(T^G\left[S(\theta)\right]\right)$ and $\widehat{h}\in\text{Path}\left(T^G\left[B_i,S_{-i}\left(\widehat{\theta}_{-i}\right)\right]\right)$). Let $\mathcal{S}^{G\text{-MM}}$ denote the set of max min-equilibria in G.

3.4 Implementation

Definition 1 *Let ϱ denote one of the six solution concepts defined above. An SCF $f:\Theta\longrightarrow\mathcal{X}$ is ϱ-implementable if there exist a mechanism $G\in\mathcal{G}$ and $S\in\mathcal{S}^{G\text{-}\varrho}$ such that S implements f.*

Following the tradition, we say:

$$\left(\begin{array}{c}f\text{ is SP (i.e., strategyproof) if and only if }f\text{ is "weak-dominance"-implementable.}\\ f\text{ is OSP (i.e., obvious SP) if and only if }f\text{ is "obvious-dominance"-implementable.}\\ f\text{ is SOSP (i.e., strong OSP) if and only if }f\text{ is "strong-obvious-dominance"-implementable.}\end{array}\right)$$

For implementation in obvious dominance and strong-obvious dominance, it suffers no loss of generality to focus on perfect-information games only. This is described by the following lemma, which has been proved in Ashlagi and Gonczarowski (2018), Pycia and Troyan (2023), Bade and Gonczarowski (2017), and Mackenzie (2020).

Lemma 1 *Consider a solution concept* $\varrho \in \{$*obvious dominance, strong-obvious dominance*$\}$. *An SCF* $f : \Theta \longrightarrow Z$ *is* ϱ*-implementable if and only if there exist* $G \in \mathcal{G}^{PI}$ *and* $S \in \mathcal{S}^{G\text{-}\varrho}$ *such that* S *implements* f.

The following result is immediately implied by the definitions in Sections 3.3.3 and 3.3.5 (precisely, (11)).

Lemma 2 *An SCF* $f : \Theta \longrightarrow Z$ *is weak-dominance-implemented by a mechanism* G *only if it is ex-post-implemented by* G.

4 Games and strategies simplified

Let $\mathcal{G}^{\text{PC}}$ denote the set of games with perfect recall, which is defined as follows. Clearly, $\mathcal{G}^{\text{PI}} \subsetneqq \mathcal{G}^{\text{PC}}$.

Definition 2 (Myerson (1997)) $G = \left[\mathcal{I}, \mathcal{A}, \mathcal{H}, \mathcal{T}, K, \phi, A, (\zeta_i)_{i\in\mathcal{I}}, g\right] \in \mathcal{G}$ *is a game with perfect recall if for any* $(h, h', h'', a) \in \mathcal{H} \times \mathcal{H} \times \mathcal{H} \times \mathcal{A}$, *we have*

$$\left(\begin{array}{c} \phi(h) = \phi(h') = \phi(h'') \text{ and } h' \overset{G}{\sim} h'', \\ [h, a] \in \text{path}[h'] \end{array}\right) \Longrightarrow \left(\begin{array}{c} \exists \widetilde{h} \in \mathcal{H} \diagdown \mathcal{T},\ \widetilde{h} \overset{G}{\sim} h, \\ \left[\widetilde{h}, a\right] \in \text{path}[h''] \end{array}\right).$$

Suppose $\phi(h) = \phi(h') = \phi(h'') = i$ and $h' \overset{G}{\sim} h''$, i.e., player i cannot distinguish between h' and h''. Upon reaching h', if player i knows that she chooses a at a previous node h to reach h' (i.e., $[h, a]$ $\in$path$[h']$), then i must recall the same information upon reaching h'': she chooses a at a previous node $\widetilde{h}$ (with $\widetilde{h} \overset{G}{\sim} h$) to reach h''.

For any $[G,S]$ and any $h \in \mathcal{H}$, define

$$\begin{aligned} \mathcal{E}^{[G,S]\text{-}h} &\equiv \left\{\theta \in \Theta : h \in \text{path}\left[T^G\left(S\left(\theta\right)\right)\right]\right\}, \\ \mathcal{E}_i^{[G,S]\text{-}h} &\equiv \left\{\theta_i \in \Theta_i : \exists \theta_{-i} \in \Theta_{-i},\ (\theta_i, \theta_{-i}) \in \mathcal{E}^{[G,S]\text{-}h}\right\}, \forall i \in \mathcal{I}. \end{aligned}$$

In the standard revelation principle for simultaneous-move games, each pair of $\left[G \in \mathcal{G},\ S \in \mathcal{S}^G\right]$ induces a *direct mechanism* which is defined on primitives. For sequential-move games, we will translate $[G,S]$ into a new "device" defined on primitives. In order to achieve this, we need to focus on the following special class of pairs of $[G,S]$.

$$\mathcal{GS}^{\text{PC}} = \left\{ [G,S] : \begin{array}{c} \text{(A) } G \in \mathcal{G}^{\text{PC}}, \\ \\ \text{(B) } S \in \mathcal{S}^G \text{ such that } \left\{T^G\left(S\left(\theta\right)\right) : \theta \in \Theta\right\} = \mathcal{T}, \\ \\ \text{(C) } \mathcal{E}^{[G,S]\text{-}h} = \mathcal{E}^{[G,S]\text{-}h'} \Longrightarrow h = h', \forall \left(h, h'\right) \in \mathcal{H} \times \mathcal{H} \end{array} \right\}. \tag{14}$$

As a comparison, Ashlagi and Gonczarowski (2018), Pycia and Troyan (2023), Bade and Gonczarowski (2017), and Mackenzie (2020) prove that it suffers no loss generality to focus on $[G,S]$ in the following set, when we consider OSP and SOSP.

$$\mathcal{GS}^{\text{PI}} = \left\{ [G,S] : \begin{array}{c} \text{(A) } G \in \mathcal{G}^{\text{PI}}, \\ \\ \text{(B) } S \in \mathcal{S}^G \text{ such that } \left\{T^G\left(S\left(\theta\right)\right) : \theta \in \Theta\right\} = \mathcal{T}, \\ \\ \text{(C) } \mathcal{E}^{[G,S]\text{-}h} = \mathcal{E}^{[G,S]\text{-}h'} \Longrightarrow h = h', \forall \left(h, h'\right) \in \mathcal{H} \times \mathcal{H} \end{array} \right\}.$$

The only difference between $\mathcal{GS}^{\text{PC}}$ and $\mathcal{GS}^{\text{PI}}$ lies in condition (A): $G \in \mathcal{G}^{\text{PC}}$ in $\mathcal{GS}^{\text{PC}}$ and $G \in \mathcal{G}^{\text{PI}}$ in $\mathcal{GS}^{\text{PI}}$.[6]

[6]The purposes of $\mathcal{GS}^{\text{PC}}$ and $\mathcal{GS}^{\text{PI}}$ differ. Roughly, "focusing on $\mathcal{GS}^{\text{PI}}$" is the revelation principle established in Ashlagi and Gonczarowski (2018), Pycia and Troyan (2023), Bade and Gonczarowski (2017), and Mackenzie (2020), while "focusing on $\mathcal{GS}^{\text{PC}}$" is an intermediate step for us, and we will establish a sharper revelation principle later.

4.1 Conditions (A)-(C) in $\mathcal{GS}^{\mathbf{PC}}$

The role of perfect recall is shown by Lemma 3 and the proof is relegated to Appendix A.1.

Lemma 3 *For any $G \in \mathcal{G}^{PC}$ and any $S \in \mathcal{S}^{G}$, we have*

$$\left(\begin{array}{c} h \overset{G}{\sim} h' \text{ and } \phi(h) = \phi(h') = i, \\ \mathcal{E}^{[G,S]\text{-}h} \neq \varnothing \text{ and } \mathcal{E}^{[G,S]\text{-}h'} \neq \varnothing \end{array}\right) \Longrightarrow \mathcal{E}_i^{[G,S]\text{-}h} = \mathcal{E}_i^{[G,S]\text{-}h'}, \forall (i,h,h') \in \mathcal{I} \times [\mathcal{H} \diagdown \mathcal{T}] \times [\mathcal{H} \diagdown \mathcal{T}].$$

Condition (A) in (14) says that we focus on games with perfect recall only. By Lemma 1, this suffers no loss of generality, if we consider OSP or SOSP. However, this is not true for weak dominance, μ-PBE and max-min equilibria, which will be reflected accordingly in the revelation principles established later (Theorems 1, 2 and 3).

Consider any $[G \in \mathcal{G},\ S \in \mathcal{S}^{G}]$. Let $\mathcal{H}^{[G,S]} \equiv \cup_{\theta \in \Theta} \text{path}[T^{G}(S(\theta))])$ denote the equilibrium-path histories. Condition (B) says that it suffers no loss generality to delete off-equilibrium histories in G (i.e., histories in $\mathcal{H} \diagdown \mathcal{H}^{[G,S]}$). This may not be true if $G \in \mathcal{G} \diagdown \mathcal{G}^{\text{PC}}$. To see this, consider $(i, \theta, \theta', h, h') \in \mathcal{I} \times \Theta \times \Theta \times \mathcal{H} \times \mathcal{H}$ in G such that

$$\left(\begin{array}{c} h \overset{G}{\sim} h' \text{ and } \phi(h) = \phi(h') = i, \\ \theta \in \mathcal{E}^{[G,S]\text{-}h} \text{ and } \theta' \in \mathcal{E}^{[G,S]\text{-}h'} \end{array}\right), \text{and} \left(\begin{array}{c} \text{if } G \in \mathcal{G}^{\text{PC}}, \text{Lemma 3 implies } \theta_i \in \mathcal{E}_i^{[G,S]\text{-}h'}; \\ \text{if } G \in \mathcal{G} \diagdown \mathcal{G}^{\text{PC}}, \text{we may have } \theta_i \notin \mathcal{E}_i^{[G,S]\text{-}h'} \end{array}\right).$$

Suppose the true state is θ. At the information set containing both h and h', player i believes that θ' may be the true state. If $G \in \mathcal{G} \diagdown \mathcal{G}^{\text{PC}}$, we may have $\theta_i \notin \mathcal{E}_i^{[G,S]\text{-}h'}$, and playing $S_i(\theta)[h']$ may lead to off-equilibrium histories, which may be a crucial reason that $S_i(\theta)$ is a best reply for i at this information set. Therefore, it suffers loss of generality to to delete off-equilibrium histories in G. However, if $G \in \mathcal{G}^{\text{PC}}$, Lemma 3 implies that following $S_i(\theta)[h']$ always leads to equilibrium-path histories (i.e., to $T^{G}(S_i(\theta_i), S_{-i}(\theta'_{-i}))$ for some $\theta'_{-i} \in \mathcal{E}_i^{[G,S]\text{-}h'}$ with $h \overset{G}{\sim} h'$), and by the usual argument of "pruning," it suffers no loss of generality to delete off-equilibrium histories.

Given condition (B), every history is an equilibrium history, which implies no loss of generality to impose condition (C). To see this, consider two distinct histories $(h, h') \in \mathcal{H} \times \mathcal{H}$ such that $\mathcal{E}^{[G,S]\text{-}h} = \mathcal{E}^{[G,S]\text{-}h'}$, which immediately implies either $h \in$path$[h']$ or

$h' \in$path$[h]$. Without loss of generality, suppose $h \in$path$[h']$, i.e., for any $\widetilde{h} \in$path$[h']$ such that $h \in$path$\left[\widetilde{h}\right]$, we have

$$\mathcal{E}^{[G,S]\text{-}h} = \mathcal{E}^{[G,S]\text{-}\widetilde{h}} = \mathcal{E}^{[G,S]\text{-}h'}, \text{ or equivalently, } |A(h)| = \left|A\left(\widetilde{h}\right)\right| = 1,$$

i.e., players take non-strategic actions at h and $\widetilde{h}$. Therefore, we can identify all of such $\widetilde{h}$ with h', until condition (C) holds.

4.2 Solutions simplified

Fix any $[G,S] \in \mathcal{GS}^{\text{PC}}$. Suppose that the true state is θ, and that players have reached history $h \in \mathcal{H} \diagdown \mathcal{T}$ with $\phi(h) = i$. Recall

$$\Theta_{-i}^{(S,h)} \equiv \left\{ \theta'_{-i} \in \Theta_{-i} : \begin{array}{c} \exists \left(\theta', h'\right) \in \Theta \times [\mathcal{H} \diagdown \mathcal{T}], h \overset{G}{\sim} h', \\ h' \in \text{Path}\left(T^G\left[S\left(\theta'\right)\right]\right) \end{array} \right\} = \bigcup_{h' \in \left\{\widetilde{h} \in \mathcal{H} : \widetilde{h} \overset{G}{\sim} h\right\}} \mathcal{E}_{-i}^{[G,S]\text{-}h'}.$$

Consider any $\theta'_i \in \mathcal{E}_i^{[G,S]\text{-}(h)}$. We ignore weak dominance tentatively,[7] and for the other five solution concepts, Lemma 3 implies that "truthfully revealing θ_i being better than falsely reporting θ'_i" can be reduced to the following simple conditions defined on primitives.

$$\mu\text{-PBE: } \int_{\Theta_{-i}^{(S,h)}} \left(u_i^{\theta_i}\left[f\left(\theta_i, \theta_{-i}\right)\right] - u_i^{\theta_i}\left[f\left(\theta'_i, \theta_{-i}\right)\right]\right) \mu^{\theta_i}\left[d\theta_{-i}\right] \geq 0, \tag{15}$$

$$\text{ex-post equilibrium: } u_i^{\theta_i}\left[f\left(\theta_i, \theta_{-i}\right)\right] \geq u_i^{\theta_i}\left[f\left(\theta'_i, \theta_{-i}\right)\right], \forall \theta_{-i} \in \Theta_{-i}^{(S,h)}, \tag{16}$$

$$\text{max-min equilibrium: } \min_{\theta_{-i} \in \Theta_{-i}^{(S,h)}} u_i^{\theta_i}\left[f\left(\theta_i, \theta_{-i}\right)\right] \geq \min_{\theta_{-i} \in \Theta_{-i}^{(S,h)}} u_i^{\theta_i}\left[f\left(\theta'_i, \theta_{-i}\right)\right], \tag{17}$$

$$\text{obvious dominance: } \min_{\theta_{-i} \in \Theta_{-i}^{(S,h)}} u_i^{\theta_i}\left[f\left(\theta_i, \theta_{-i}\right)\right] \geq \max_{\theta_{-i} \in \Theta_{-i}^{(S,h)}} u_i^{\theta_i}\left[f\left(\theta'_i, \theta_{-i}\right)\right], \tag{18}$$

$$\text{strong-obvious dominance: } \min_{\widetilde{\theta}_i \in \mathcal{E}_i^{[G,S]\text{-}[h, S_i(\theta_i)[h]]}} \min_{\theta_{-i} \in \Theta_{-i}^{(S,h)}} u_i^{\theta_i}\left[f\left(\widetilde{\theta}_i, \theta_{-i}\right)\right] \geq \max_{\theta_{-i} \in \Theta_{-i}^{(S,h)}} u_i^{\theta_i}\left[f\left(\theta'_i, \theta_{-i}\right)\right]. \tag{19}$$

[7] It is not clear how to translate weak dominance to a condition similar to (15)-(19).

5 New Tools

In this section, we introduce several new tools to establish revelation principles.

5.1 Semi-operators and operators

A pair of $[G, S]$ describes how players reveal their types sequentially. Specifically, suppose players follow S in G, and they reach a history h with $\phi(h) = j$. Recall $\mathcal{E}^{[G,S]\text{-}h}$ denotes the set of states in which S reaches h. By playing S_j at h, agent j follows the partition below to reveal her types.

$$\left\{\mathcal{E}_j^{[G,S]\text{-}\left[h,\, S_j(\theta_j)[h]\right]} \times \mathcal{E}_{-j}^{[G,S]\text{-}h} : \theta_j \in \mathcal{E}_j^{[G,S]\text{-}h}\right\}. \tag{20}$$

Or equivalently, the evolution of history is described as follows:

$$h \xrightarrow{\theta_j} \left[h,\ S_j\left(\theta_j\right)[h]\right], \forall \theta_j \in \mathcal{E}_j^{[G,S]\text{-}h}, \tag{21}$$

and if we use the set of states $\mathcal{E}^{[G,S]\text{-}h}$ as a proxy for h, (21) becomes

$$\mathcal{E}_j^{[G,S]\text{-}h} \times \mathcal{E}_{-j}^{[G,S]\text{-}h} \xrightarrow{\theta_j} \mathcal{E}_j^{[G,S]\text{-}\left[h,\, S_j(\theta_j)[h]\right]} \times \mathcal{E}_{-j}^{[G,S]\text{-}h}, \forall \theta_j \in \mathcal{E}_j^{[G,S]\text{-}h},$$

i.e., (20). We thus propose an abstract device (called "semi-operator") to describe this. One innovation is that we do not record the agent attached to each history, which substantially simplifies exposition. It will be clear that this suffers no loss of generality.

Definition 3 *A semi-operator is a function* $\gamma : \left(\times_{i\in\mathcal{I}} \left[2^{\Theta_i} \diagdown \{\varnothing\}\right]\right) \times \Theta \longrightarrow \times_{i\in\mathcal{I}} \left[2^{\Theta_i} \diagdown \{\varnothing\}\right]$.

The notion of semi-operator is too general to describe all of the details contained in (21). For instance, (21) describes a partition. We thus refine the idea as follows.

Definition 4 (operator) *An operator is a semi-operator* γ *such that* $\gamma[E, \cdot]\,|_{\theta\in E}$ *forms a partition on* E*, or equivalently, for any* $(\theta, \theta', E) \equiv \Theta \times \Theta \times \left(\times_{i\in\mathcal{I}} \left[2^{\Theta_i} \diagdown \{\varnothing\}\right]\right)$ *with* $\{\theta, \theta'\} \subset E$*, we have*

$$\theta \in \gamma[E, \theta] \subset E \text{ and } \theta' \in \gamma[E, \theta] \Longrightarrow \gamma[E, \theta] = \gamma\left[E, \theta'\right].$$

Throughout the paper, we consider $\gamma[E,\theta]$ only when $\theta \in E$, and hence, the definition of $\gamma[E,\theta]$ is irrelevant if $\theta \notin E$. Let Γ^{Semi} and Γ denote the set of all semi-operators and operators, respectively.

5.2 Operators induced by games and strategy profiles

Fix any $[G,S] \in \mathcal{GS}^{\text{PC}}$. We will show that $[G,S]$ induces a particular operator, denoted by $\gamma^{[G,S]}$. Consider

$$\Omega^{[G,S]} \equiv \left\{E \in \left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i} \diagdown \{\varnothing\}\right]\right) : \exists h \in [\mathcal{H}\diagdown\mathcal{T}],\ E = \mathcal{E}^{[G,S]\text{-}h}\right\}.$$

By condition (C) in (14), each $E \in \Omega^{[G,S]}$ corresponds to a unique $h \in [\mathcal{H}\diagdown\mathcal{T}]$. We now define

$$\gamma^{[G,S]} : \left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i} \diagdown \{\varnothing\}\right]\right) \times \Theta \longrightarrow \left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i} \diagdown \{\varnothing\}\right]\right),$$
$$\vartheta^{[G]} : \left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i} \diagdown \{\varnothing\}\right]\right) \longrightarrow 2^{\Theta},$$

where $\gamma^{[G,S]}$ is the operator induced by $[G,S]$, and $\vartheta^{[G]}$ describes information sets in G.

First, the definition of an operator is irrelevant when $\theta \notin E$, and furthermore, $[G,S]$ does not provide information when $E \notin \Omega^{[G,S]}$. In such cases, we thus define them in a trivial way: $\vartheta^{[G]}[E] = E$, if $E \notin \Omega^{[G,S]}$, and $\gamma^{[G,S]}[E,\theta] = E$, if $\theta \notin E$ or $E \notin \Omega^{[G,S]}$.

Second, suppose $\theta \in E \in \Omega^{[G,S]}$. By condition (C) in (14), there exists a unique $[j,\ h] \in \mathcal{I} \times [\mathcal{H}\diagdown\mathcal{T}]$ such that $E = \mathcal{E}^{[G,S]\text{-}h}$ and $\phi(h) = j$, i.e., h is j's history. Define

$$\gamma^{[G,S]}[E,\theta] = \mathcal{E}^{[G,S]\text{-}\left[h,\ S_j(\theta_j)[h]\right]},$$

i.e., at the information set containing h, agent j takes the action $S_j(\theta_j)[h]$, and $\gamma^{[G,S]}[E,\theta]$ is defined as the set of states in which S leads to the history $\left[h,\ S_j(\theta_j)[h]\right]$. Furthermore, define

$$\vartheta^{[G]}[E] = \bigcup_{h' \in \left\{\tilde{h} \in [\mathcal{H}\diagdown\mathcal{T}] :\ \tilde{h} \overset{\mathcal{G}}{\sim} h\right\}} \mathcal{E}^{[G,S]\text{-}h'},$$

i.e., at the information set containing h, agent j cannot distinguish h from $h' \in \left\{\tilde{h} \in [\mathcal{H}\diagdown\mathcal{T}] :\ \tilde{h} \overset{\mathcal{G}}{\sim} h\right\}$, and hence, $\vartheta^{[G]}[E]$ is the set of states in which agent j believes that S leads to the information set containing h. In particular, consider the degenerate function

$$\vartheta^* : \left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i}\diagdown\{\varnothing\}\right]\right) \longrightarrow 2^{\Theta} \text{ such that } \vartheta^*[E] = E, \forall E \in \left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i}\diagdown\{\varnothing\}\right]\right).$$

Clearly, we have $\vartheta^{[G]} = \vartheta^*$ for any game $G \in \mathcal{G}^{\text{PI}}$.

5.3 Solution notions

We dissect a solution concept into two parts: (i) at each history, it dictates when type θ_i prefers to truthfully revealing θ_i to falsely reporting θ_i' (Definition 5), and (ii) it dictates when agents' sequential revealing is acceptable (Definition 7). We focus on the former here.

Definition 5 *A solution notion is a function*

$$\rho : \mathcal{X}^{\Theta} \times \Gamma^{Semi} \times \left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i}\diagdown\{\varnothing\}\right]\right) \times \Theta \times \Theta \times \mathcal{I} \times 2^{\Theta} \longrightarrow \{0,1\},$$

such that

$$\left(\gamma_i(E), \theta_i, \theta_i', \widehat{E}_{-i}\right) = \left(\widetilde{\gamma}_i\left(\widetilde{E}\right), \widetilde{\theta}_i, \widetilde{\theta}_i', \widetilde{E}_{-i}'\right) \Longrightarrow \rho\left[f, (\gamma, E), \theta, \theta', i, \widehat{E}\right] = \rho\left[f, \left(\widetilde{\gamma}, \widetilde{E}\right), \widetilde{\theta}, \widetilde{\theta}', i, \widetilde{E}'\right]. \tag{22}$$

Given $[G,S] \in \mathcal{GS}^{\text{PC}}$ and $\left[f, (\gamma, E), \theta, \theta', i, \widehat{E}\right]$ with $\{\theta, \theta'\} \subset E$, suppose that the true state is θ, and that the players have reached a history h in G with

$$E = \mathcal{E}^{[G,S]\text{-}h} \text{ and } \widehat{E} = \bigcup_{h' \in \left\{\widetilde{h} \in [\mathcal{H}\diagdown\mathcal{T}] : \widetilde{h} \overset{G}{\sim} h\right\}} \mathcal{E}^{[G,S]\text{-}h'}.$$

The interpretation is:

$$\rho\left[f, (\gamma, E), \theta, \theta', i, \widehat{E}\right] = 1 \Longleftrightarrow \left[\begin{array}{l} \text{at a history } h \text{ with } E = \mathcal{E}^{[G,S]\text{-}h} \text{ and the belief of } \widehat{E} \text{ (on } \Theta\text{)}, \\ \text{agent } i \text{ prefers truthfully revealing } \theta \text{ to falsely reporting } \theta' \end{array}\right].$$

(22) requires that the value of $\rho\left[f, (\gamma, E), \theta, \theta', i, \widehat{E}\right]$ does not depend on $\left(\gamma_{-i}(E), \theta_{-i}, \theta_{-i}', \widehat{E}_i\right)$.

Focus on $[G,S] \in \mathcal{GS}^{\text{PC}}$, Definition 5 and (15)-(19) imply that the five solution concepts defined above can be translated to the following solution notions.

$$\rho^{\mu\text{-PBE}}\left[f,(\gamma,E),\theta,\theta',i,\widehat{E}\right] = \begin{cases} 1, & \text{if } \int_{\widehat{E}_{-i}} \left(u_i^{\theta_i}\left[f\left(\theta_i,\widehat{\theta}_{-i}\right)\right] - u_i^{\theta_i}\left[f\left(\theta_i',\widehat{\theta}_{-i}\right)\right]\right)\mu^{\theta_i}\left[d\widehat{\theta}_{-i}\right] \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

$$\rho^{\text{EP}}\left[f,(\gamma,E),\theta,\theta',i,\widehat{E}\right] = \begin{cases} 1, & \text{if } u_i^{\theta_i}\left[f\left(\theta_i,\widehat{\theta}_{-i}\right)\right] \geq u_i^{\theta_i}\left[f\left(\theta_i',\widehat{\theta}_{-i}\right)\right], \forall \widehat{\theta}_{-i} \in \widehat{E}_{-i}, \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

$$\rho^{\text{MM}}\left[f,(\gamma,E),\theta,\theta',i,\widehat{E}\right] = \begin{cases} 1, & \text{if } \min_{\widehat{\theta}_{-i}\in\widehat{E}_{-i}} u_i^{\theta_i}\left[f\left(\theta_i,\widehat{\theta}_{-i}\right)\right] \geq \min_{\widehat{\theta}_{-i}\in\widehat{E}_{-i}} u_i^{\theta_i}\left[f\left(\theta_i',\widehat{\theta}_{-i}\right)\right], \\ 0, & \text{otherwise.} \end{cases} \tag{25}$$

$$\rho^{\text{OD}}\left[f,(\gamma,E),\theta,\theta',i,\widehat{E}\right] = \begin{cases} 1, & \text{if } \min_{\widehat{\theta}_{-i}\in\widehat{E}_{-i}} u_i^{\theta_i}\left[f\left(\theta_i,\widehat{\theta}_{-i}\right)\right] \geq \max_{\widehat{\theta}_{-i}\in\widehat{E}_{-i}} u_i^{\theta_i}\left[f\left(\theta_i',\widehat{\theta}_{-i}\right)\right], \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

$$\rho^{\text{SOD}}\left[f,(\gamma,E),\theta,\theta',i,\widehat{E}\right] = \begin{cases} 1, & \text{if } \left(\begin{array}{c} \min_{\widehat{\theta}_{-i}\in\widehat{E}_{-i}} u_i^{\theta_i}\left[f\left(\widetilde{\theta}_i,\widehat{\theta}_{-i}\right)\right] \geq \max_{\theta_{-i}\in\widehat{E}_{-i}} u_i^{\theta_i}\left[f\left(\theta_i',\widehat{\theta}_{-i}\right)\right], \\ \forall \widetilde{\theta}_i \in \gamma_i\left[E,\theta\right] \end{array} \right), \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

5.4 ρ-implementation

For any $(\theta,\gamma,E) \in \Theta \times \Gamma^{\text{Semi}} \times \left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i} \diagdown \{\varnothing\}\right]\right)$, define

$$\gamma_{(0)}\left[E\,,\theta\right] = E \text{ and inductively, } \gamma_{(n)}\left[E\,,\theta\right] = \gamma\left[\,\gamma_{(n-1)}\left[E\,,\theta\right],\ \theta\,\right], \forall n \in \mathbb{N},$$

Definition 6 (achievability) *An operator $\gamma \in \Gamma$ is achievable if there exists $N \in \mathbb{N}$ such that*

$$\gamma_{(N)}\left[\Theta\,,\theta\right] = \{\theta\}\,, \forall \theta \in \Theta.$$

Furthermore, given an SCF f, an operator $\gamma \in \Gamma$ is f-achievable if there exists $N \in \mathbb{N}$ such that

$$\left\{ f\left(\widetilde{\theta}\right) \in \mathcal{X} : \widetilde{\theta} \in \gamma_{(N)}\left[\Theta, \theta\right] \right\} = \left\{ f\left(\theta\right) \right\}, \forall \theta \in \Theta.$$

Clearly, achievability is stronger than f-achievability.

Definition 7 ((ρ, f, ϑ)-consistent) *Given an SCF f, a solution notion ρ and a function*

$$\vartheta : \left(\times_{i \in \mathcal{I}} \left[2^{\Theta_i} \diagdown \{\varnothing\} \right] \right) \longrightarrow 2^{\Theta},$$

an operator $\gamma \in \Gamma$ is (ρ, f, ϑ)-consistent if

$$\left(\begin{array}{c} \{\theta, \theta'\} \subset E, \\ \rho\left[f, (\gamma, E), \theta, \theta', i, \vartheta(E)\right] = 0 \end{array} \right) \Longrightarrow \theta'_i \in \gamma_i\left[E, \theta\right], \forall \left[E, \theta, \theta', i\right] \in \left(\times_{i \in \mathcal{I}} \left[2^{\Theta_i} \diagdown \{\varnothing\} \right] \right) \times \Theta \times \Theta \times \mathcal{I}. \tag{28}$$

Definition 8 (ρ-implementation) *For any solution notion ρ, an SCF f is ρ-implementable if there exists $\left[G = \left[\mathcal{I}, \mathcal{A}, \mathcal{H}, \mathcal{T}, K, \phi, A, (\zeta_i)_{i \in \mathcal{I}}, g\right], S\right] \in \mathcal{GS}^{PC}$ such that $\gamma^{[G,S]}$ is both f-achievable and $\left(\rho, f, \vartheta^{[G]}\right)$-consistent.*

For any solution concept ϱ defined in Section 3, let ρ denote the induced solution notion (see (23)-(27)). It is straightforward to show that an SCF f is ϱ-implemented by $[G, S] \in \mathcal{GS}^{\text{PC}}$ if and only if f is ρ-implementable. To illustrate this, we adopt the solution concept of obvious dominance. f being ρ^{OD}-implementable means existence of $\left[G = \left[\mathcal{I}, \mathcal{A}, \mathcal{H}, \mathcal{T}, K, \phi, A, (\zeta_i)_{i \in \mathcal{I}}, g\right], S\right] \in \mathcal{GS}^{\text{PC}}$ such that $\gamma^{[G,S]}$ is both f-achievable and $\left(\rho, f, \vartheta^{[G]}\right)$-consistent. First, $\gamma^{[G,S]}$ being f-achievable, which, together with the definition of $\gamma^{[G,S]}$ in Section 5.2, is equivalent to

$$g\left[T^G\left(\left[S_i(\theta_i)\right]_{i \in \mathcal{I}}\right)\right] = f\left[(\theta_i)_{i \in \mathcal{I}}\right], \forall \left[(\theta_i)_{i \in \mathcal{I}}\right] \in \Theta. \tag{29}$$

Second, $\gamma^{[G,S]}$ being $\left(\rho, f, \vartheta^{[G]}\right)$-consistent (together with (26) and Definition 7) is equivalent to

$$\left(\begin{array}{c} \{\theta, \theta'\} \subset \mathcal{E}^{[G,S]\text{-}h}, \\ \phi(h) = i, \\ S_i(\theta_i)[h] \neq S_i(\theta'_i)[h] \end{array} \right) \Longrightarrow \min_{\widehat{\theta}_{-i} \in \vartheta_{-i}^{[G]}\left(\mathcal{E}^{[G,S]\text{-}h}\right)} u_i^{\theta_i}\left[f\left(\theta_i, \widehat{\theta}_{-i}\right)\right] \geq \max_{\widehat{\theta}_{-i} \in \vartheta_{-i}^{[G]}\left(\mathcal{E}^{[G,S]\text{-}h}\right)} u_i^{\theta_i}\left[f\left(\theta'_i, \widehat{\theta}_{-i}\right)\right]. \tag{30}$$

(29) and (30) constitute the definition of f being ϱ^{OD}-implemented by $[G, S]$.

Note that (30) imposes the "consistency" condition only on agent i with $\phi(h) = i$, while (28) in Definition 7 requires it on all agents. The reason is that, with $i \neq \phi(h)$, we have $\gamma_i^{[G,S]}\left[\mathcal{E}^{[G,S]\text{-}h}, \theta\right] = \mathcal{E}_i^{[G,S]\text{-}h}$, and (28) holds automatically. This is is why we do not need to keep track of the agent who is assigned to take an action at history h in G.

5.5 Properties of operators and solution notions

We consider three properties of solution notions.

Definition 9 *A solution notion ρ is regular if for any*

$$\left[f, (\gamma, E), \theta, \theta', i, \widehat{E}\right] \in \mathcal{X}^{\Theta} \times \Gamma^{Semi} \times \left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i} \diagdown \{\varnothing\}\right]\right) \times \Theta \times \Theta \times \mathcal{I} \times 2^{\Theta},$$

and any $(\overline{\gamma}, \overline{E}) \in \Gamma^{Semi} \times \left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i} \diagdown \{\varnothing\}\right]\right)$, we have

$$\rho\left[f, (\gamma, E), \theta, \theta', i, \widehat{E}\right] = \rho\left[f, (\overline{\gamma}, \overline{E}), \theta, \theta', i, \widehat{E}\right].$$

Definition 10 *A solution notion ρ is dissectible if for any*

$$\left[f, (\gamma, E), \theta, \theta', i, \widehat{E}\right] \in \mathcal{X}^{\Theta} \times \Gamma^{Semi} \times \left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i} \diagdown \{\varnothing\}\right]\right) \times \Theta \times \Theta \times \mathcal{I} \times 2^{\Theta},$$

with $\{\theta, \theta'\} \subset E$, we have

$$\rho\left[f, (\gamma, E), \theta, \theta', i, \widehat{E}\right] = 0 \Longleftrightarrow \begin{pmatrix} \exists\widetilde{\theta} \in \gamma[E, \theta], \\ \exists\widetilde{\gamma} \in \Gamma^{Semi}, \widetilde{\gamma}[E, \theta] = \left\{\theta, \widetilde{\theta}\right\}, \\ \rho\left[f, (\widetilde{\gamma}, E), \theta, \theta', i, \widehat{E}\right] = 0 \end{pmatrix},$$

$$\text{or equivalently, } \rho\left[f, (\gamma, E), \theta, \theta', i, \widehat{E}\right] = 1 \Longleftrightarrow \begin{pmatrix} \forall\widetilde{\theta} \in \gamma[E, \theta], \\ \forall\widetilde{\gamma} \in \Gamma^{Semi} \text{ such that } \widetilde{\gamma}[E, \theta] = \left\{\theta, \widetilde{\theta}\right\}, \\ \rho\left[f, (\widetilde{\gamma}, E), \theta, \theta', i, \widehat{E}\right] = 1 \end{pmatrix}.$$

Definition 11 *A solution notion ρ is normal if for any*

$$[f, (\gamma, E), \theta, \theta', i] \in \mathcal{X}^{\Theta} \times \Gamma^{Semi} \times \left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i} \diagdown \{\varnothing\}\right]\right) \times \Theta \times \Theta \times \mathcal{I},$$

$$\textit{we have}\left(\begin{array}{c}\{\theta,\theta'\}\subset E,\\ f\left(\widetilde{\theta}\right)=f(\theta),\forall\widetilde{\theta}\in E,\end{array}\right)\Longrightarrow\rho\left[f,(\gamma,E),\theta,\theta',i,E\right]=1.$$

Given ρ being a regular solution notion, $\rho\left[f,(\gamma,E),\theta,\theta',i,\widehat{E}\right]$ does not depend on (γ,E). Given ρ being a dissectible solution notion, the value of $\rho\left[f,(\gamma,E),\theta,\theta',i,\widehat{E}\right]$ is fully determined by the values of $\rho\left[f,(\widetilde{\gamma},E),\theta,\theta',i,\widehat{E}\right]$ for all $\left(\widetilde{\gamma},\widetilde{\theta}\right)\in\Gamma^{\text{Semi}}\times\gamma[E,\theta]$ with $\widetilde{\gamma}[E,\theta]=\left\{\theta,\widetilde{\theta}\right\}$. Trivially, a regular solution notion is also dissectible. Finally, for a normal solution notion, truthful reporting is always a best reply on E, if f is constant on E. The following table describes the properties of the five solution notions above (see (23)-(27)).[8]

	μ-PBE	MM	EP	OD	SOD
regular	yes	yes	yes	yes	no
dissectible	yes	yes	yes	yes	yes
normal	yes	yes	yes	yes	yes

We consider one property of operators.

Definition 12 (increasing operators) *An operator $\gamma\in\Gamma$ is increasing if*

$$\theta\in E\subset E'\Longrightarrow\gamma[E,\theta]\subset\gamma\left[E',\theta\right],\forall\left(\theta,E,E'\right)\equiv\Theta\times\left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i}\diagdown\{\varnothing\}\right]\right)\times\left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i}\diagdown\{\varnothing\}\right]\right).$$

6 Implementation by simultaneous-move games

In this section, we study when implementation by a sequential-move game is equivalent to implementation by a simultaneous-move game. Before players take an action in a simultaneous-move game, no player has disclosed her type, and hence, each player i believes her opponents come from Θ_{-i}, and the standard revelation principle applies. For instance, f can be μ-PBE-implemented by a simultaneous-move game if and only if

$$\int_{\Theta_{-i}}\left(u_i^{\theta_i}\left[f\left(\theta_i,\widetilde{\theta}_{-i}\right)\right]-u_i^{\theta_i}\left[f\left(\theta_i',\widetilde{\theta}_{-i}\right)\right]\right)\mu^{\theta_i}\left[d\widetilde{\theta}_{-i}\right]\geq0,\forall\left(\theta,\theta',i\right)\in\Theta\times\Theta\times\mathcal{I},$$

[8]For strong-obvious dominance, $\rho^{\text{SOD}}\left[f,(\gamma,E),\theta,\theta',i,\widehat{E}\right]$ depends on γ.

i.e., at every state, it is a best reply for every player to truthfully report her type. Similarly, for any solution concept ϱ, we can translate it to a solution notion ρ, and the standard revelation principle defined on ϱ is translated to the following condition on ρ.[9]

$$\rho\left[f,(\gamma,\Theta),\theta,\theta',i,\Theta\right]=1,\forall\left[\gamma,\theta,\theta',i\right]\in\Gamma^{\text{Semi}}\times\Theta\times\Theta\times\mathcal{I}. \tag{31}$$

In particular, we will consider additive solution notions, which is defined as follows.

Definition 13 (additive solution notion) *A solution notion ρ is additive if for any*

$$\left[f,\gamma,\theta,\theta',i\right]\in\mathcal{X}^{\Theta}\times\Gamma^{Semi}\times\Theta\times\Theta\times\mathcal{I},$$

and any partition $P:\Theta\longrightarrow 2^{\Theta}$, (i.e., $\theta\in P(\theta)$ and $\theta'\in P(\theta)\Longrightarrow P(\theta)=P(\theta')$), we have

$$\left(\begin{array}{c}\forall\widetilde{\theta}\in\Theta,\\ \rho\left[f,(\gamma,\Theta),\theta,\theta',i,P\left(\widetilde{\theta}\right)\right]=1\end{array}\right)\Longrightarrow\rho\left[f,(\gamma,\Theta),\theta,\theta',i,\Theta\right]=1.$$

Given an additive solution notion ρ and a partition P on Θ, if revealing θ_i is better than revealing θ_i' on each $P\left(\widetilde{\theta}\right)$, then revealing θ_i is better than revealing θ_i' on Θ. It is easy to check that μ-PBE, ex-post equilibrium and max-min equilibrium are additive (see (23)-(25)). We use the following result to establish revelation principles for such solution concepts, and the proof is relegated to Appendix A.2.

Proposition 1 *For any solution notion ρ which is regular, normal and additive, an SCF $f:\Theta\longrightarrow\mathcal{X}$ is ρ-implementable if and only if*

$$\rho\left[f,(\gamma,\Theta),\theta,\theta',i,\Theta\right]=1,\forall\left[\gamma,\theta,\theta',i\right]\in\Gamma^{Semi}\times\Theta\times\Theta\times\mathcal{I}.$$

Proposition 1 immediately leads to the following revelation principles.

Theorem 1 *An SCF f can be μ-PBE-implemented by a game with perfect recall if and only if*

$$\int_{\Theta_{-i}}\left(u_i^{\theta_i}\left[f\left(\theta_i,\widetilde{\theta}_{-i}\right)\right]-u_i^{\theta_i}\left[f\left(\theta_i',\widetilde{\theta}_{-i}\right)\right]\right)\mu^{\theta_i}\left[d\widetilde{\theta}_{-i}\right]\geq 0,\forall\left(\theta,\theta',i\right)\in\Theta\times\Theta\times\mathcal{I},$$

or equivalently, f can be μ-PBE-implemented by a simultaneous-move game.

[9]In (31), we will consider regular solution notions only, which does not depends on γ.

Theorem 2 *An SCF f can be max-min-implemented by a game with perfect recall if and only if*

$$\min_{\widetilde{\theta}_{-i}\in\Theta_{-i}} u_i^{\theta_i}\left[f\left(\theta_i,\widetilde{\theta}_{-i}\right)\right] \geq \min_{\widetilde{\theta}_{-i}\in\Theta_{-i}} u_i^{\theta_i}\left[f\left(\theta_i',\widetilde{\theta}_{-i}\right)\right], \forall\left(\theta,\theta',i\right)\in\Theta\times\Theta\times\mathcal{I},$$

or equivalently, f can be max-min-implemented by a simultaneous-move game.

Even though we consider ex-post equilibrium only (but not weak dominance), Lemma 2 and Proposition 1 could help us fully characterize weak-dominance-implementation.

Theorem 3 *The following five statements are equivalent.*

(i)an SCF f can be weak-dominance-implemented (i.e., strategyproof) by a game with perfect recall;

(ii) f can be ex-post-implemented by a game with perfect recall;

(iii)

$$u_i^{\theta_i}\left[f\left(\theta_i,\theta_{-i}\right)\right] \geq u_i^{\theta_i}\left[f\left(\theta_i',\theta_{-i}\right)\right], \forall\theta_{-i}\in\Theta_{-i}; \tag{32}$$

(iv) f can be weak-dominance-implemented by a simultaneous-move game.

(v) f can be ex-post-implemented by a simultaneous-move game.

Proof of Theorem 3: Clearly, (iii), (iv) and (v) are equivalent. If (32) holds, the traditional direct mechanism weak-dominance-implements f, i.e., (iii)$\Longrightarrow$(i). Furthermore, (i)$\Longrightarrow$(ii) is implied by Lemma 2, and (ii)$\Longrightarrow$(iii) is implied by Proposition 1.■

7 Monotonic solution concepts: revelation principle

In this section, we consider non-additive solution notions (e.g., obvious dominance, strong-obvious dominance), and it suffers loss of generality to focus on simultaneous-move games. Nevertheless, obvious dominance and strong-obvious dominance possess the following property.

Definition 14 (monotonic solution notion) *A solution notion ρ is monotonic if for any*

$$\left[f,(\gamma,E),\theta,\theta',i,\widehat{E}\right]\in\mathcal{X}^{\Theta}\times\Gamma^{Semi}\times\left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i}\diagdown\{\varnothing\}\right]\right)\times\Theta\times\Theta\times\mathcal{I}\times2^{\Theta},$$

and any $\widetilde{E}\in2^{\Theta}$, we have $\left(\begin{array}{c}\widehat{E}\subset\widetilde{E},\\ \rho\left[f,(\gamma,E),\theta,\theta',i,\widehat{E}\right]=0\end{array}\right)\Longrightarrow\rho\left[f,(\gamma,E),\theta,\theta',i,\widetilde{E}\right]=0.$

7.1 Canonical operator and revelation principle

Based on any (ρ,f), we define a canonical operator $\gamma^{(\rho,f)}$. In particular, a profile $[(\rho,f),(\gamma,E)]$ induces a binary relation on Θ_i as follows, which is denoted by $>_i^{[(\rho,f),(\gamma,E)]}$.

$$\theta>_i^{[(\rho,f),(\gamma,E)]}\theta'\Longleftrightarrow\left(\begin{array}{c}\{\theta,\theta'\}\subset E\text{ and}\\ \rho\left[f,(\gamma,E),\theta,\theta',i,E\right]=0\end{array}\right),\forall\left(i,\theta,\theta'\right)\in\mathcal{I}\times\Theta\times\Theta.\tag{33}$$

Furthermore, define

$$\theta\sim_i^{[(\rho,f),(\gamma,E)]}\theta'\Longleftrightarrow\left(\begin{array}{c}\text{either }\theta_i=\theta_i',\\ \text{or }\theta>_i^{[(\rho,f),(\gamma,E)]}\theta',\\ \text{or }\theta'>_i^{[(\rho,f),(\gamma,E)]}\theta\end{array}\right),\forall\left(i,\theta,\theta'\right)\in\mathcal{I}\times\Theta\times\Theta.\tag{34}$$

Clearly, $\sim_i^{[(\rho,f),(\gamma,E)]}$ is symmetric, i.e.,

$$\theta\sim^{[(\rho,f),(\gamma,E)]}\theta'\Longleftrightarrow\theta'\sim^{[(\rho,f),(\gamma,E)]}\theta,\forall\left(i,\theta,\theta'\right)\in\mathcal{I}\times\Theta\times\Theta,\tag{35}$$

For any $[E,\ \theta]\in\left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i}\diagdown\{\varnothing\}\right]\right)\times\Theta$, we inductively define

$$\gamma^{(\rho,f)\text{-}(1)}[E,\ \theta]=\{\theta\},\tag{36}$$

$$\text{and }\gamma^{(\rho,f)\text{-}(n+1)}[E,\ \theta]=\bigcup_{\widetilde{\theta}\in\gamma^{(\rho,f)\text{-}(n)}[E,\ \theta]}\left\{\theta'\in E:\widetilde{\theta}\sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(n)},E\right)\right]}\theta',\forall i\in\mathcal{I}\right\},\forall n\in\mathbb{N}.\tag{37}$$

Finally, define the canonical operator $\gamma^{(\rho,f)}$ as follows.

$$\gamma^{(\rho,f)}[E,\ \theta]=\begin{cases}E, & \text{if }\theta\notin E,\\ \bigcup_{n=1}^{\infty}\gamma^{(\rho,f)\text{-}(n)}[E,\ \theta], & \text{if }\theta\in E.\end{cases},\forall[E,\ \theta]\in\left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i}\diagdown\{\varnothing\}\right]\right)\times\Theta.\tag{38}$$

$\gamma^{(\rho,f)}[E,\ \theta]|_{\theta\in E}$ is aimed to be a partition on E, which must satisfy two necessary conditions: (i) reflexivity: $\theta \in \gamma^{(\rho,f)}[E,\ \theta]$ (formalized in (36)), and (ii) transitivity: $\theta'' \in \gamma^{(\rho,f)}[E,\ \theta']$ and $\theta' \in \gamma^{(\rho,f)}[E,\ \theta]$ imply $\theta'' \in \gamma^{(\rho,f)}[E,\ \theta]$ (formalized in (37)). The following result establishes revelation principles for monotonic solution notions.

Proposition 2 *Consider any SCF f and any solution notion ρ which is dissectible, monotonic and normal. Then, f is ρ-implementable if and only if $\gamma^{(\rho,f)}$ is achievable.*

7.2 Properties of $\gamma^{(\rho,f)}$ and proof of Proposition 2

The following five lemmas describe properties of the canonical (semi-)operator $\gamma^{(\rho,f)}$, which play critical roles in our proof of Proposition 2. Their proofs are relegated to Appendix A.4-A.8.

Lemma 4 *For any SCF f and any dissectible solution notion ρ, $\gamma^{(\rho,f)}$ is an operator which is (ρ, f, ϑ^*)-consistent.*

Lemma 5 *For any SCF f and any dissectible and monotonic solution notion ρ, $\gamma^{(\rho,f)}$ is an increasing operator.*

Lemma 6 *Consider any SCF f and any solution notion ρ which is dissectible and monotonic. For any $[G, S] \in \mathcal{GS}^{PC}$ such that $\gamma^{[G,S]}$ is $\left(\rho, f, \vartheta^{[G]}\right)$-consistent, we have*

$$\gamma^{(\rho,f)}[E,\ \theta] \subset \gamma^{[G,S]}[E,\ \theta], \forall [E,\ \theta] \in \left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i}\diagdown\{\varnothing\}\right]\right) \times \Theta.$$

Lemma 7 *Consider any SCF f and any dissectible solution notion ρ. If $\gamma^{(\rho,f)}$ is achievable, there exists $[G, S] \in \mathcal{GS}^{PC}$ such that $\gamma^{[G,S]}$ is both achievable and $\left(\rho, f, \vartheta^{[G]}\right)$-consistent.*

Lemma 8 *For any SCF f and any normal solution notion ρ, the semi-operator $\gamma^{(\rho,f)}$ is achievable if and only if $\gamma^{(\rho,f)}$ is f-achievable.*

Proof of Proposition 2: Consider any SCF f and any solution notion ρ which is dissectible, monotonic and normal. First, suppose that f is ρ-implementable. By Definition 8, there exists $\left[G = \left[\mathcal{I}, \mathcal{A}, \mathcal{H}, \mathcal{T}, K, \phi, A, \left(\zeta_i\right)_{i\in\mathcal{I}}, g\right], S\right] \in \mathcal{GS}^{\text{PC}}$ such that $\gamma^{[G,S]}$ is both f-achievable and $\left(\rho, f, \vartheta^{[G]}\right)$-consistent.

$\gamma^{[G,S]}$ being f-achievable implies

$$f\left(\widetilde{\theta}\right) = f\left(\theta\right), \forall \theta \in \Theta, \forall \widetilde{\theta} \in \gamma_{(K)}^{[G,S]}\left[\Theta, \theta\right]. \tag{39}$$

We now prove

$$\gamma_{(k)}^{(\rho,f)}\left[\Theta,\, \theta\right] \subset \gamma_{(k)}^{[G,S]}\left[\Theta,\, \theta\right], \forall k \in \{0, 1, ..., K\}, \forall \theta \in \Theta. \tag{40}$$

With $k = 0$, we have $\gamma_{(0)}^{(\rho,f)}\left[\Theta,\, \theta\right] = \Theta = \gamma_{(0)}^{[G,S]}\left[\Theta,\, \theta\right]$, i.e., (40) holds. Suppose (40) holds for $k = n < K$. Consider any $\theta \in \Theta$ and $k = n+1$, and we have

$$\gamma_{(n+1)}^{(\rho,f)}\left[\Theta,\, \theta\right] = \gamma^{(\rho,f)}\left(\gamma_{(n)}^{(\rho,f)}\left[\Theta,\, \theta\right]\right) \subset \gamma^{(\rho,f)}\left(\gamma_{(n)}^{[G,S]}\left[\Theta,\, \theta\right]\right) \subset \gamma^{[G,S]}\left(\gamma_{(n)}^{[G,S]}\left[\Theta,\, \theta\right]\right) = \gamma_{(n+1)}^{[G,S]}\left[\Theta,\, \theta\right],$$

where the first $\subset$ follows from the induction hypothesis and Lemma 5, and the second $\subset$ follows from $\gamma^{[G,S]}$ being $\left(\rho, f, \vartheta^{[G]}\right)$-consistent and Lemma 6. Therefore, (40) holds.

(39) and (40) imply that $\gamma^{(\rho,f)}$ is f-achievable, which, together with Lemma 8, implies that $\gamma^{(\rho,f)}$ is achievable. This proves the "only if" part.

Second, suppose that $\gamma^{(\rho,f)}$ is achievable. By Lemma 7, there exists $[G, S] \in \mathcal{GS}^{\text{PC}}$ such that $\gamma^{[G,S]}$ is both achievable and $\left(\rho, f, \vartheta^{[G]}\right)$-consistent. Therefore, f is ρ-implementable. This proves the "if" part.■

7.3 Obvious dominance

We apply Proposition 2 to obvious dominance which is dissectible, monotonic and normal. Define

$$\theta' >_i^{\text{OD-}f\text{-}E} \theta \Longleftrightarrow \begin{pmatrix} \{\theta, \theta'\} \subset E \text{ and} \\ \min_{\theta_{-i}\in E_{-i}} u_i^{\theta_i}\left[f\left(\theta_i, \theta_{-i}\right)\right] < \max_{\theta_{-i}\in E_{-i}} u_i^{\theta_i}\left[f\left(\theta_i', \theta_{-i}\right)\right] \end{pmatrix}, \forall \left(i, \theta, \theta'\right) \in \mathcal{I} \times \Theta \times \Theta.$$

For any $[E,\ \theta] \in \left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i}\diagdown\{\varnothing\}\right]\right) \times \Theta$, define $\gamma^{\text{OD-}f\text{-}(1)}[E,\ \theta] = \{\theta\}$ and

$$\gamma^{\text{OD-}f\text{-}(n+1)}[E,\ \theta] = \bigcup_{\widetilde{\theta}\in\gamma^{\text{OD-}f\text{-}(n)}[E,\ \theta]} \left\{\theta' \in E : \forall i \in \mathcal{I}, \left(\begin{array}{l}\text{either } \theta'_i = \widetilde{\theta}_i,\\ \text{or } \theta' >_i^{\text{OD-}f\text{-}E} \widetilde{\theta}\\ \text{or } \widetilde{\theta} >_i^{\text{OD-}f\text{-}E} \theta'\end{array}\right)\right\}, \forall n \in \mathbb{N}.$$

Finally,

$$\gamma^{\text{OD-}f}[E,\ \theta] \equiv \begin{cases} E, & \text{if } \theta \notin E, \\ \bigcup_{n=1}^{\infty} \gamma^{\text{OD-}f\text{-}(n)}[E,\ \theta], & \text{if } \theta \in E. \end{cases}.$$

Theorem 4 *An SCF $f : \Theta \longrightarrow \mathcal{X}$ is OSP if and only if $\gamma^{OD\text{-}f}$ is achievable.*

7.4 Strong-obvious dominance

We apply Proposition 2 to strong-obvious dominance which is dissectible, monotonic and normal. Given $[f, (\gamma, E)]$ and $(i, \theta, \theta') \in \mathcal{I} \times \Theta \times \Theta$, define

$$\theta' >_i^{\text{SOD-}f\text{-}(\gamma,E)} \theta \Longleftrightarrow \left(\begin{array}{c}\{\theta, \theta'\} \subset E \text{ and}\\ \min_{\widetilde{\theta}\in\gamma[E,\theta]} \min_{\theta_{-i}\in E_{-i}} u_i^{\theta_i}\left[f\left(\widetilde{\theta}_i, \theta_{-i}\right)\right] < \max_{\theta_{-i}\in E_{-i}} u_i^{\theta_i}\left[f\left(\theta'_i, \theta_{-i}\right)\right]\end{array}\right).$$

For any $[E,\ \theta] \in \left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i}\diagdown\{\varnothing\}\right]\right) \times \Theta$, define $\gamma^{\text{SOD-}f\text{-}(1)}[E,\ \theta] = \{\theta\}$ and

$$\gamma^{\text{SOD-}f\text{-}(n+1)}[E,\ \theta] = \bigcup_{\widetilde{\theta}\in\gamma^{\text{SOD-}f\text{-}(n)}[E,\ \theta]} \left\{\theta' \in E : \forall i \in \mathcal{I}, \left(\begin{array}{l}\text{either } \theta'_i = \widetilde{\theta}_i,\\ \text{or } \theta' >_i^{\text{SOD-}f\text{-}\left(\gamma^{\text{SOD-}f\text{-}(n)},E\right)} \widetilde{\theta}\\ \text{or } \widetilde{\theta} >_i^{\text{SOD-}f\text{-}\left(\gamma^{\text{SOD-}f\text{-}(n)},E\right)} \theta'\end{array}\right)\right\}, \forall n \in \mathbb{N}.$$

Finally,

$$\gamma^{\text{SOD-}f}[E,\ \theta] \equiv \begin{cases} E, & \text{if } \theta \notin E, \\ \bigcup_{n=1}^{\infty} \gamma^{\text{SOD-}f\text{-}(n)}[E,\ \theta], & \text{if } \theta \in E. \end{cases}.$$

Theorem 5 *An SCF $f : \Theta \longrightarrow \mathcal{X}$ is SOSP if and only if $\gamma^{SOD\text{-}f}$ is achievable.*

8 Conclusion

Revelation principle is a pillar in traditional mechanism design which focuses on simultaneous-move games. In this paper, we propose a general approach to establish the analogous revelation principle for mechanism design with sequential-move games.

A Proof

A.1 Proof of Lemma 3

Consider any $G \in \mathcal{G}^{\text{PC}}$, any $S \in \mathcal{S}^{G}$, and any $(i, h, h') \in \mathcal{I} \times [\mathcal{H} \diagdown \mathcal{T}] \times [\mathcal{H} \diagdown \mathcal{T}]$ such that

$$h \overset{G}{\sim} h', \phi(h) = \phi(h') = i, \mathcal{E}^{[G,S]\text{-}h} \neq \varnothing, \text{ and } \mathcal{E}^{[G,S]\text{-}h'} \neq \varnothing.$$

We prove $\mathcal{E}_i^{[G,S]\text{-}h} = \mathcal{E}_i^{[G,S]\text{-}h'}$ by contradiction. Suppose $\mathcal{E}_i^{[G,S]\text{-}h} \neq \mathcal{E}_i^{[G,S]\text{-}h'}$. Without loss of generality, suppose there exists $\theta_i \in \mathcal{E}_i^{[G,S]\text{-}h} \diagdown \mathcal{E}_i^{[G,S]\text{-}h'}$. Pick any $\left[\theta_{-i}, \left(\theta_i', \theta_{-i}'\right)\right] \in \Theta_{-i} \times \Theta$ such that

$$(\theta_i, \theta_{-i}) \in \mathcal{E}^{[G,S]\text{-}h}, \text{ and } \left(\theta_i', \theta_{-i}'\right) \in \mathcal{E}^{[G,S]\text{-}h'}.$$

In particular, $\theta_i \notin \mathcal{E}_i^{[G,S]\text{-}h'}$ implies that, upon reaching h', player i has already revealed that she is not of type θ_i. Thus, there exists $\widehat{h}' \in$path$[h']$ such that $\phi\left(\widehat{h}'\right) = i$, and

$$S_i(\theta_i)\left[\widehat{h}'\right] \neq S_i\left(\theta_i'\right)\left[\widehat{h}'\right]. \tag{41}$$

Since $G \in \mathcal{G}^{\text{PC}}$, Definition 3 implies existence of $\widehat{h} \in$path$[h]$ such that $\widehat{h} \overset{G}{\sim} \widehat{h}'$ and

$$\left[\widehat{h}, \ S_i\left(\theta_i'\right)\left[\widehat{h}'\right]\right] \in \text{path}[h],$$

which, together with $(\theta_i, \theta_{-i}) \in \mathcal{E}^{[G,S]\text{-}h}$, implies

$$S_i(\theta_i)\left[\widehat{h}\right] = S_i\left(\theta_i'\right)\left[\widehat{h}'\right]. \tag{42}$$

Furthermore, $\widehat{h} \overset{G}{\sim} \widehat{h}'$ implies

$$S_i(\theta_i)\left[\widehat{h}\right] = S_i(\theta_i)\left[\widehat{h}'\right]. \tag{43}$$

Thus, (42) and (43) imply $S_i\left(\theta_i'\right)\left[\widehat{h}'\right] = S_i(\theta_i)\left[\widehat{h}'\right]$, contradicting (41).■

A.2 Proof of Proposition 1

Consider any SCF $f : \Theta \longrightarrow \mathcal{X}$ and any solution notion ρ which is regular, normal and additive. First, suppose

$$\rho\left[f,(\gamma,\Theta),\theta,\theta',i,\Theta\right]=1,\forall\left[\gamma,\theta,\theta',i\right]\in\Gamma^{\text{Semi}}\times\Theta\times\Theta\times\mathcal{I}. \tag{44}$$

Clearly, f can be ρ-implemented by the simultaneous-move direct mechanism f, and (44) implies that it is always a best reply for every player to truthfully reveal her type. Therefore, the "if" part of Proposition 1 holds.

Second, suppose f is ρ-implementable, i.e., there exists $[G,S]\in\mathcal{GS}^{\text{PC}}$ such that $\gamma^{[G,S]}$ is both f-achievable and $\left(\rho,f,\vartheta^{[G]}\right)$-consistent. Fix any $\left[\gamma,\theta,\theta',i\right]\in\Gamma^{\text{Semi}}\times\Theta\times\Theta\times\mathcal{I}$, we aim to prove

$$\rho\left[f,(\gamma,\Theta),\theta,\theta',i,\Theta\right]=1, \tag{45}$$

i.e., the "only if" part of Proposition 1 holds.

For any $\widetilde{\theta}_{-i}\in\Theta_{-i}$, consider

$$\mathcal{H}^{\left(\theta_i,\theta'_i,\widetilde{\theta}_{-i}\right)}\equiv\left\{h\in\mathcal{H}:h\in\text{Path}\left(T^G\left[S_i\left(\theta_i\right),S_{-i}\left(\widetilde{\theta}_{-i}\right)\right]\right)\cap\text{Path}\left(T^G\left[S_i\left(\theta'_i\right),S_{-i}\left(\widetilde{\theta}_{-i}\right)\right]\right)\right\}.$$

Clearly, $\mathcal{H}^{\left(\theta_i,\theta'_i,\widetilde{\theta}_{-i}\right)}\neq\varnothing$, because $\sigma\in\mathcal{H}^{\left(\theta_i,\theta'_i,\widetilde{\theta}_{-i}\right)}$. Since $[G,S]\in\mathcal{GS}^{\text{PC}}$, condition (C) in (14) implies existence a unique maximal element in $\mathcal{H}^{\left(\theta_i,\theta'_i,\widetilde{\theta}_{-i}\right)}$, i.e.,

$$\exists!h\in\mathcal{H}^{\left(\theta_i,\theta'_i,\widetilde{\theta}_{-i}\right)},h'\in\mathcal{H}^{\left(\theta_i,\theta'_i,\widetilde{\theta}_{-i}\right)}\Longrightarrow h'\in\text{Path}\left(h\right).$$

We denote this unique maximal element by $h^{\left(\theta_i,\theta'_i,\widetilde{\theta}_{-i}\right)}$. This immediately implies

$$h^{\left(\theta_i,\theta'_i,\widetilde{\theta}_{-i}\right)}\in\mathcal{H}^{\left(\theta_i,\theta'_i,\widetilde{\theta}_{-i}\right)}\diagdown\mathcal{T}\Longrightarrow\left(\begin{array}{c}\phi\left(h^{\left(\theta_i,\theta'_i,\widetilde{\theta}_{-i}\right)}\right)=i,\\ \text{and } S_i\left(\theta_i\right)\left[h^{\left(\theta_i,\theta'_i,\widetilde{\theta}_{-i}\right)}\right]\neq S_i\left(\theta'_i\right)\left[h^{\left(\theta_i,\theta'_i,\widetilde{\theta}_{-i}\right)}\right]\end{array}\right). \tag{46}$$

Define $P_{-i}^{\left(\theta,\theta'\right)}:\Theta_{-i}\longrightarrow 2^{\Theta_{-i}}$ as follows.

$$P_{-i}^{\left(\theta,\theta'\right)}\left(\widetilde{\theta}_{-i}\right)=\begin{cases}\mathcal{E}_{-i}^{[G,S]\text{-}h^{\left(\theta_i,\theta'_i,\widetilde{\theta}_{-i}\right)}}, & \text{if } h^{\left(\theta_i,\theta'_i,\widetilde{\theta}_{-i}\right)}\in\mathcal{T},\\ \vartheta_{-i}^{[G]}\left[\mathcal{E}^{[G,S]\text{-}h^{\left(\theta_i,\theta'_i,\widetilde{\theta}_{-i}\right)}}\right], & \text{if } h^{\left(\theta_i,\theta'_i,\widetilde{\theta}_{-i}\right)}\notin\mathcal{T},\end{cases} \tag{47}$$

where $\vartheta^{[G]}$ is defined in Section 5.2, and

$$\begin{aligned}
\mathcal{E}_{-i}^{[G,S]\text{-}h^{\left(\theta_i,\theta_i',\widetilde{\theta}_{-i}\right)}} &\equiv \left\{\widehat{\theta}_{-i} \in \Theta_{-i} : \exists \overline{\theta}_i \in \Theta_i,\ \left(\overline{\theta}_i, \widehat{\theta}_{-i}\right) \in \mathcal{E}^{[G,S]\text{-}h^{\left(\theta_i,\theta_i',\widetilde{\theta}_{-i}\right)}}\right\}, \\
\vartheta_{-i}^{[G]}\left[\mathcal{E}^{[G,S]\text{-}h^{\left(\theta_i,\theta_i',\widetilde{\theta}_{-i}\right)}}\right] &\equiv \left\{\widehat{\theta}_{-i} \in \Theta_{-i} : \exists \overline{\theta}_i \in \Theta_i,\ \left(\overline{\theta}_i, \widehat{\theta}_{-i}\right) \in \vartheta^{[G]}\left(\mathcal{E}^{[G,S]\text{-}h^{\left(\theta_i,\theta_i',\widetilde{\theta}_{-i}\right)}}\right)\right\}.
\end{aligned}$$

That is, if $h^{\left(\theta_i,\theta_i',\widetilde{\theta}_{-i}\right)}$ is terminal, $P_{-i}^{\left(\theta,\theta'\right)}\left(\widetilde{\theta}_{-i}\right)$ is the set of $\widehat{\theta}_{-i}$ such that $S\left(\theta_i, \widehat{\theta}_{-i}\right)$ leads to $h^{\left(\theta_i,\theta_i',\widetilde{\theta}_{-i}\right)}$, and if $h^{\left(\theta_i,\theta_i',\widetilde{\theta}_{-i}\right)}$ is non-terminal, $P_{-i}^{\left(\theta,\theta'\right)}\left(\widetilde{\theta}_{-i}\right)$ is the set of $\widehat{\theta}_{-i}$ such that $S\left(\theta_i, \widehat{\theta}_{-i}\right)$ leads to the information set containing $h^{\left(\theta_i,\theta_i',\widetilde{\theta}_{-i}\right)}$.

Since $\gamma^{[G,S]}$ is $\left(\rho, f, \vartheta^{[G]}\right)$-consistent, (46) implies

$$h^{\left(\theta_i,\theta_i',\widetilde{\theta}_{-i}\right)} \notin \mathcal{T} \Longrightarrow \rho\left[f, \left(\gamma^{[G,S]}, \mathcal{E}^{[G,S]\text{-}h^{\left(\theta_i,\theta_i',\widetilde{\theta}_{-i}\right)}}\right), \theta, \theta', i,\ \vartheta^{[G]}\left(\mathcal{E}^{[G,S]\text{-}h^{\left(\theta_i,\theta_i',\widetilde{\theta}_{-i}\right)}}\right)\right] = 1, \forall \widetilde{\theta} \in \Theta,$$

which, together with ρ being regular and (22) in Definition 5, implies

$$h^{\left(\theta_i,\theta_i',\widetilde{\theta}_{-i}\right)} \notin \mathcal{T} \Longrightarrow \rho\left[f, (\gamma, \Theta), \theta, \theta', i,\ \Theta_i \times \vartheta_{-i}^{[G]}\left(\mathcal{E}^{[G,S]\text{-}h^{\left(\theta_i,\theta_i',\widetilde{\theta}_{-i}\right)}}\right)\right] = 1, \forall \left(\gamma, \widetilde{\theta}\right) \in \Gamma^{\text{Semi}} \times \Theta. \tag{48}$$

Since $\gamma^{[G,S]}$ is f-achievable, we have

$$h^{\left(\theta_i,\theta_i',\widetilde{\theta}_{-i}\right)} \in \mathcal{T} \Longrightarrow \left(\begin{array}{c} f\left(\widehat{\theta}\right) = f\left(\theta_i, \widetilde{\theta}_{-i}\right), \\ \forall \widehat{\theta} \in \mathcal{E}^{[G,S]\text{-}h^{\left(\theta_i,\theta_i',\widetilde{\theta}_{-i}\right)}} \end{array}\right), \forall \widetilde{\theta} \in \Theta,$$

which, together with ρ being normal and regular, implies

$$h^{\left(\theta_i,\theta_i',\widetilde{\theta}_{-i}\right)} \in \mathcal{T} \Longrightarrow \rho\left[f, (\gamma, \Theta), \theta, \theta', i,\ \Theta_i \times \mathcal{E}_{-i}^{[G,S]\text{-}h^{\left(\theta_i,\theta_i',\widetilde{\theta}_{-i}\right)}}\right] = 1, \forall \left(\gamma, \widetilde{\theta}\right) \in \Gamma^{\text{Semi}} \times \Theta. \tag{49}$$

Define $P^{\left(\theta,\theta'\right)} : \Theta \longrightarrow 2^{\Theta}$ as follows.

$$P^{\left(\theta,\theta'\right)}\left(\widetilde{\theta}\right) = \Theta_i \times P_{-i}^{\left(\theta,\theta'\right)}\left(\widetilde{\theta}_{-i}\right), \forall \widetilde{\theta} \in \Theta. \tag{50}$$

(48), (49) and (50) imply

$$\rho\left[f, (\gamma, \Theta), \theta, \theta', i,\ P^{\left(\theta,\theta'\right)}\left(\widetilde{\theta}\right)\right] = 1, \forall \left(\gamma, \widetilde{\theta}\right) \in \Gamma \times \Theta. \tag{51}$$

We will prove that $P^{\left(\theta,\theta'\right)}$ is a partition on Θ, which, together with ρ being additive, implies (45), i.e., our goal.

Finally, we prove that $P^{(\theta,\theta')}$ is a partition on Θ. Clearly, the definition of $P^{(\theta,\theta')}$ implies $\widetilde{\theta} \in P^{(\theta,\theta')}\left(\widetilde{\theta}\right)$ for any $\widetilde{\theta} \in \Theta$. Consider any $\left(\widetilde{\theta},\widetilde{\theta}'\right) \in \Theta \times \Theta$ such that $\widetilde{\theta}' \in P^{(\theta,\theta')}\left(\widetilde{\theta}\right)$, and we aim to show $P^{(\theta,\theta')}\left(\widetilde{\theta}'\right) = P^{(\theta,\theta')}\left(\widetilde{\theta}\right)$. We consider two cases. First, suppose $h^{(\theta_i,\theta_i',\widetilde{\theta}_{-i})} \in \mathcal{T}$. Then, $\widetilde{\theta}' \in P^{(\theta,\theta')}\left(\widetilde{\theta}\right) = \Theta_i \times P_{-i}^{(\theta,\theta')}\left(\widetilde{\theta}_{-i}\right)$ and (47) implies

$$h^{\left(\theta_i,\theta_i',\widetilde{\theta}'_{-i}\right)} = h^{(\theta_i,\theta_i',\widetilde{\theta}_{-i})} \in \mathcal{T},$$

which further implies

$$P^{(\theta,\theta')}\left(\widetilde{\theta}'\right) = \Theta_i \times \mathcal{E}^{[G,S]\text{-}h^{\left(\theta_i,\theta_i',\widetilde{\theta}'_{-i}\right)}} = \Theta_i \times \mathcal{E}^{[G,S]\text{-}h^{\left(\theta_i,\theta_i',\widetilde{\theta}_{-i}\right)}} = P^{(\theta,\theta')}\left(\widetilde{\theta}\right).$$

Second, suppose $h^{(\theta_i,\theta_i',\widetilde{\theta}_{-i})} \notin \mathcal{T}$. Then, $\widetilde{\theta}' \in P^{(\theta,\theta')}\left(\widetilde{\theta}\right) = \Theta_i \times P_{-i}^{(\theta,\theta')}\left(\widetilde{\theta}_{-i}\right)$ and (47) implies existence of $\widehat{h}' \in \mathcal{H} \diagdown \mathcal{T}$ such that

$$\widehat{h}' \sim^G h^{(\theta_i,\theta_i',\widetilde{\theta}_{-i})}, \tag{52}$$

$$\text{and } \widetilde{\theta}'_{-i} \in \mathcal{E}_{-i}^{[G,S]\text{-}\widehat{h}'}, \tag{53}$$

which, together with $h^{(\theta_i,\theta_i',\widetilde{\theta}_{-i})} \in \mathcal{H}^{(\theta_i,\theta_i',\widetilde{\theta}_{-i})} \diagdown \mathcal{T}$ and (46), implies

$$\{\theta_i, \theta_i'\} \subset \mathcal{E}_i^{[G,S]\text{-}h^{\left(\theta_i,\theta_i',\widetilde{\theta}_{-i}\right)}}, \tag{54}$$

$$S_i(\theta_i)\left[\widehat{h}'\right] \neq S_i(\theta_i')\left[\widehat{h}'\right]. \tag{55}$$

By Lemma 3, (52) and (54) imply

$$\{\theta_i, \theta_i'\} \subset \mathcal{E}_i^{[G,S]\text{-}\widehat{h}'},$$

which, together with (53), imply

$$\left\{\left(\theta_i, \widetilde{\theta}'_{-i}\right), \left(\theta_i', \widetilde{\theta}'_{-i}\right)\right\} \subset \mathcal{E}^{[G,S]\text{-}\widehat{h}'}. \tag{56}$$

(52), (55) and (56) imply $h^{\left(\theta_i,\theta_i',\widetilde{\theta}'_{-i}\right)} = \widehat{h}'$. We thus have

$$h^{\left(\theta_i,\theta_i',\widetilde{\theta}'_{-i}\right)} = \widehat{h}' \sim^G h^{(\theta_i,\theta_i',\widetilde{\theta}_{-i})},$$

and hence, $\vartheta^{[G]}\left[\mathcal{E}^{[G,S]\text{-}h^{\left(\theta_i,\theta_i',\widetilde{\theta}'_{-i}\right)}}\right] = \vartheta^{[G]}\left[\mathcal{E}^{[G,S]\text{-}h^{\left(\theta_i,\theta_i',\widetilde{\theta}_{-i}\right)}}\right]$, which further implies

$$P^{(\theta,\theta')}\left(\widetilde{\theta}'\right) = \Theta_i \times \vartheta_{-i}^{[G]}\left[\mathcal{E}^{[G,S]\text{-}h^{\left(\theta_i,\theta_i',\widetilde{\theta}'_{-i}\right)}}\right] = \Theta_i \times \vartheta_{-i}^{[G]}\left[\mathcal{E}^{[G,S]\text{-}h^{\left(\theta_i,\theta_i',\widetilde{\theta}_{-i}\right)}}\right] = P^{(\theta,\theta')}\left(\widetilde{\theta}\right).$$

That is, we have proved $P^{(\theta,\theta')}\left(\widetilde{\theta}'\right) = P^{(\theta,\theta')}\left(\widetilde{\theta}\right)$ for both cases.■

A.3 Intermediate results

We need the following intermediate results in our proof.

Lemma 9 *For any SCF f and any dissectible solution notion ρ, any*

$$\left[f,(\gamma,E),\theta,\theta',i,\widehat{E}\right]\in\mathcal{X}^{\Theta}\times\Gamma^{Semi}\times\left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i}\diagdown\{\varnothing\}\right]\right)\times\Theta\times\Theta\times\mathcal{I}\times 2^{\Theta},$$

and any $\left(\overline{\gamma},\overline{E}\right)\in\Gamma^{Semi}\times\left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i}\diagdown\{\varnothing\}\right]\right)$, *we have*

$$\left(\begin{array}{c}\gamma\left[E,\ \theta\right]\subset\overline{\gamma}\left[\ \overline{E},\ \theta\right],\\ \rho\left[f,(\gamma,E),\theta,\theta',i,\widehat{E}\right]=0\end{array}\right)\Longrightarrow\rho\left[f,\left(\overline{\gamma},\overline{E}\right),\theta,\theta',i,\widehat{E}\right]=0. \tag{57}$$

Lemma 10 *For any SCF f and any dissectible solution notion ρ, we have*

$$\theta\sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(n)},E\right)\right]}\theta'\Longrightarrow\theta\sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(n+1)},E\right)\right]}\theta',\ \forall\left(i,\theta,\theta',n\right)\in\mathcal{I}\times\Theta\times\Theta\times\mathbb{N}. \tag{58}$$

Lemma 11 *For any SCF f and any dissectible solution notion ρ, we have*

$$\left(\begin{array}{c}\theta\in E,\\ \theta'\in\gamma^{(\rho,f)\text{-}(n+1)}\left[E,\ \theta\right]\end{array}\right)\Longleftrightarrow\left(\begin{array}{c}\theta\in E,\\ \forall i\in\mathcal{I},\exists\left\{\theta^1,...,\theta^{n+1}\right\}\subset E,\\ \theta^1=\theta\text{ and }\theta^{n+1}=\theta',\\ \theta^1\sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(n)},E\right)\right]}\theta^2...\sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(n)},E\right)\right]}\theta^{n+1}\end{array}\right), \tag{59}$$

$$\forall\left(\theta,\theta',E,n\right)\in\Theta\times\Theta\times\left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i}\diagdown\{\varnothing\}\right]\right)\times\mathbb{N}.$$

Proof of Lemma 9: Since ρ is dissectible, Definition 10 implies

$$\rho\left[f,(\gamma,E),\theta,\theta',i,\widehat{E}\right]=0\Longrightarrow\left(\begin{array}{c}\exists\widetilde{\theta}\in\gamma\left[E,\ \theta\right],\\ \exists\widetilde{\gamma}\in\Gamma^{\text{Semi}},\widetilde{\gamma}\left[E,\ \theta\right]=\left\{\theta,\widetilde{\theta}\right\},\\ \rho\left[f,(\widetilde{\gamma},E),\theta,\theta',i,\widehat{E}\right]=0\end{array}\right). \tag{60}$$

Consider any $\widehat{\gamma}\in\Gamma^{\text{Semi}}$ such that $\widehat{\gamma}\left[\overline{E},\ \theta\right]=\widetilde{\gamma}\left[E,\ \theta\right]=\left\{\theta,\widetilde{\theta}\right\}$, and in particular, $\widehat{\gamma}_i\left[\overline{E},\ \theta\right]=\widetilde{\gamma}_i\left[E,\ \theta\right]$. Thus, (22) in Definition 5 implies

$$\widehat{\gamma}\left[\ \overline{E},\ \theta\right]=\widetilde{\gamma}\left[E,\ \theta\right]\Longrightarrow\rho\left[f,\left(\widehat{\gamma},\overline{E}\right),\theta,\theta',i,\widehat{E}\right]=\rho\left[f,(\widetilde{\gamma},E),\theta,\theta',i,\widehat{E}\right]=0. \tag{61}$$

(60) and (61) imply

$$\left(\begin{array}{c}\gamma\left[E,\ \theta\right]\subset\overline{\gamma}\left[\ \overline{E},\ \theta\right],\\ \rho\left[f,\left(\gamma,E\right),\theta,\theta',i,\widehat{E}\right]=0\end{array}\right)\Longrightarrow\left(\begin{array}{c}\exists\widetilde{\theta}\in\gamma\left[E,\ \theta\right]\subset\overline{\gamma}\left[\ \overline{E},\ \theta\right],\\ \exists\widehat{\gamma}\in\Gamma^{\mathrm{Semi}},\widehat{\gamma}\left[\ \overline{E},\ \theta\right]=\widetilde{\gamma}\left[E,\ \theta\right]=\left\{\theta,\widetilde{\theta}\right\},\\ \rho\left[f,\left(\widehat{\gamma},\overline{E}\right),\theta,\theta',i,\widehat{E}\right]=0\end{array}\right).\tag{62}$$

Definition 10 implies

$$\left(\begin{array}{c}\exists\widetilde{\theta}\in\overline{\gamma}\left[\ \overline{E},\ \theta\right],\\ \exists\widehat{\gamma}\in\Gamma^{\mathrm{Semi}},\widehat{\gamma}\left[\ \overline{E},\ \theta\right]=\left\{\theta,\widetilde{\theta}\right\},\\ \rho\left[f,\left(\widehat{\gamma},\overline{E}\right),\theta,\theta',i,\widehat{E}\right]=0\end{array}\right)\Longrightarrow\rho\left[f,\left(\overline{\gamma},\overline{E}\right),\theta,\theta',i,\widehat{E}\right]=0.\tag{63}$$

Thus, (62) and (63) imply (57).■

Proof of Lemma 10: (37) imply that for any $[E,\ \theta]\in\left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i}\diagdown\{\varnothing\}\right]\right)\times\Theta$,

$$\theta\in E\Longrightarrow\{\theta\}=\gamma^{(\rho,f)\text{-}(1)}\left[E,\ \theta\right]\subset\gamma^{(\rho,f)\text{-}(n)}\left[E,\ \theta\right]\subset\gamma^{(\rho,f)\text{-}(n+1)}\left[E,\ \theta\right]\subset E,\forall n\in\mathbb{N}.\tag{64}$$

Since ρ is dissectible, Lemma 9 implies

$$\theta>_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(n)},E\right)\right]}\theta'\Longrightarrow\theta>_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(n+1)},E\right)\right]}\theta',\forall\left(i,\theta,\theta',n\right)\in\mathcal{I}\times\Theta\times\Theta\times\mathbb{N},\tag{65}$$

which, together with (34), implies (58).■

Proof of Lemma 11: First, with $n=1$, (36) and (37) imply

$$\left(\begin{array}{c}\theta\in E,\\ \theta'\in\gamma^{(\rho,f)\text{-}(2)}\left[E,\ \theta\right]\end{array}\right)\Longleftrightarrow\left(\begin{array}{c}\theta\in E,\\ \forall i\in\mathcal{I},\theta=\theta^1\sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(1)},E\right)\right]}\theta^2=\theta'\end{array}\right),$$

i.e., (59) holds for $n=1$. Suppose (59) holds for $n=k\in\mathbb{N}$. Consider $n=(k+1)\in\mathbb{N}$. (37) and implies

$$\left(\begin{array}{c}\theta\in E,\\ \theta'\in\gamma^{(\rho,f)\text{-}(k+2)}\left[E,\ \theta\right]\end{array}\right)\Longleftrightarrow\left(\begin{array}{c}\theta\in E,\\ \exists\widetilde{\theta}\in\gamma^{(\rho,f)\text{-}(k+1)}\left[E,\ \theta\right],\\ \forall i\in\mathcal{I},\widetilde{\theta}\sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(k+1)},E\right)\right]}\theta'\end{array}\right).\tag{66}$$

We thus have

$$\begin{pmatrix} \theta \in E, \\ \exists \widetilde{\theta} \in \gamma^{(\rho,f)\text{-}(k+1)}\left[E,\ \theta\right], \\ \forall i \in \mathcal{I}, \widetilde{\theta} \sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(k+1)},E\right)\right]} \theta' \end{pmatrix} \tag{67}$$

$$\iff \begin{pmatrix} \theta \in E, \exists \widetilde{\theta} \in \gamma^{(\rho,f)\text{-}(k+1)}\left[E,\ \theta\right], \\ \forall i \in \mathcal{I}, \exists \left\{\theta^1, ..., \theta^{k+2}\right\} \subset E, \\ \theta^1 = \theta, \theta^{k+1} = \widetilde{\theta}, \text{ and } \theta^{k+2} = \theta', \\ \theta^1 \sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(k)},E\right)\right]} ... \sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(k)},E\right)\right]} \theta^{k+1} \sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(k+1)},E\right)\right]} \theta^{k+2} \end{pmatrix}$$

$$\iff \begin{pmatrix} \theta \in E, \exists \widetilde{\theta} \in \gamma^{(\rho,f)\text{-}(k+1)}\left[E,\ \theta\right], \\ \forall i \in \mathcal{I}, \exists \left\{\theta^1, ..., \theta^{k+2}\right\} \subset E, \\ \theta^1 = \theta, \theta^{k+1} = \widetilde{\theta}, \text{ and } \theta^{k+2} = \theta', \\ \theta^1 \sim^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(k+1)},E\right)\right]} ... \sim^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(k+1)},E\right)\right]} \theta^{k+1} \sim^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(k+1)},E\right)\right]} \theta^{k+2} \end{pmatrix},$$

where the first $\iff$ follows from the induction hypothesis, and the second $\iff$ follows from Lemma 10. Therefore, (66) and (67) imply that (59) holds for $n = k + 1$.■

A.4 Proof of Lemma 4

Fix any SCF f and any dissectible solution notion ρ.

Proof of $\gamma^{(\rho,f)}$ being (ρ, f, ϑ^*)-consistent: Consider any $\left[i, E, \theta, \theta'\right] \in \mathcal{I} \times \left(\times_{i\in\mathcal{I}} \left[2^{\Theta_i} \diagdown \{\varnothing\}\right]\right) \times \Theta \times \Theta$ such that

$$\left\{\theta, \theta'\right\} \subset E \text{ and } \rho\left[f, \left(\gamma^{(\rho,f)}, E\right), \theta, \theta', i, E\right] = 0, \tag{68}$$

and we aim to prove $\theta'_i \in \gamma_i^{(\rho,f)}\left[E, \theta\right]$.

Since ρ is dissectible, (68) implies

$$\begin{pmatrix} \exists \widetilde{\theta} \in \gamma^{(\rho,f)}\left[E,\ \theta\right], \\ \exists \widetilde{\gamma} \in \Gamma, \widetilde{\gamma}\left[E,\ \theta\right] = \left\{\theta, \widetilde{\theta}\right\}, \\ \rho\left[f, \left(\widetilde{\gamma}, E\right), \theta, \theta', i, E\right] = 0 \end{pmatrix}. \tag{69}$$

Furthermore, $\widetilde{\theta} \in \gamma^{(\rho,f)}\left[E,\ \theta\right]$ and (38) implies

$$\widetilde{\theta} \in \gamma^{(\rho,f)\text{-}(n)}\left[E,\ \theta\right] \text{ for some } n \in \mathbb{N}. \tag{70}$$

Since ρ is dissectible, (69) and (70) imply

$$\rho\left[f,\left(\gamma^{(\rho,f)\text{-}(n)},E\right),\theta,\theta',i,E\right]=0,$$

and hence,

$$\left(\theta_i',\theta_{-i}\right)\in\gamma^{(\rho,f)\text{-}(n+1)}\left[E,\ \theta\right],$$

which, together with (38), implies $\theta_i'\in\gamma_i^{(\rho,f)}\left[E,\ \theta\right]$.■

Proof of $\gamma^{(\rho,f)}$ being an operator: First, (36), (37) and (38) imply

$$\theta\in E\Longrightarrow\theta\in\gamma^{(\rho,f)}\left[E,\ \theta\right]\subset E,\forall\left[\theta,E\right]\in\Theta\times\left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i}\diagdown\{\varnothing\}\right]\right).\tag{71}$$

Second, for any $\left(\theta,\theta',E\right)\in\Theta\times\Theta\times\left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i}\diagdown\{\varnothing\}\right]\right)$, we prove

$$\left[\theta\in E\text{ and }\theta'\in\gamma^{(\rho,f)}\left[E,\theta\right]\right]\Longrightarrow\left[\theta'\in E\text{ and }\theta\in\gamma^{(\rho,f)}\left[E,\theta'\right]\right].\tag{72}$$

Suppose $\theta\in E$ and $\theta'\in\gamma^{(\rho,f)}\left[E,\theta\right]$. If $\theta'=\theta$, (72) holds. Suppose $\theta'\neq\theta$. By (71), we have $\theta'\in\gamma^{(\rho,f)}\left[E,\theta\right]\subset E$. Furthermore, $\theta'\in\gamma^{(\rho,f)}\left[E,\theta\right]$ and (38) imply $\theta'\in\gamma^{(\rho,f)\text{-}(n+1)}\left[E,\ \theta\right]$ for some $n\in\mathbb{N}$. Thus, Lemma 11 implies

$$\left(\begin{array}{c}\forall i\in\mathcal{I},\exists\left\{\theta^1,...,\theta^{n+1}\right\}\subset E,\\ \theta^1=\theta\text{ and }\theta^{n+1}=\theta',\\ \theta^1\sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(n)},E\right)\right]}\theta^2\sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(n)},E\right)\right]}...\sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(n)},E\right)\right]}\theta^{n+1}\end{array}\right),$$

which, together with symmetry of $\sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(n)},E\right)\right]}$ (i.e., (35)), implies

$$\left(\begin{array}{c}\forall i\in\mathcal{I},\exists\left\{\theta^1,...,\theta^{n+1}\right\}\subset E,\\ \theta^1=\theta\text{ and }\theta^{n+1}=\theta',\\ \theta^{n+1}\sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(n)},E\right)\right]}...\sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(n)},E\right)\right]}\theta^2\sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(n)},E\right)\right]}\theta^1\end{array}\right),$$

i.e., $\theta\in\gamma^{(\rho,f)\text{-}(n+1)}\left[E,\ \theta'\right]\subset\gamma^{(\rho,f)}\left[E,\ \theta'\right]$. Therefore, (72) holds.

Third, for any $\left(\theta,\theta',E\right)\in\Theta\times\Theta\times\left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i}\diagdown\{\varnothing\}\right]\right)$, we prove

$$\left[\theta\in E\text{ and }\theta'\in\gamma^{(\rho,f)}\left[E,\theta\right]\right]\Longrightarrow\gamma^{(\rho,f)}\left[E,\theta'\right]\subset\gamma^{(\rho,f)}\left[E,\theta\right].\tag{73}$$

Suppose $\theta\in E$ and $\theta'\in\gamma^{(\rho,f)}\left[E,\theta\right]$. Pick any $\theta''\in\gamma^{(\rho,f)}\left[E,\theta'\right]$, and we aim to prove $\theta''\in\gamma^{(\rho,f)}\left[E,\theta\right]$, i.e., (73) holds.

$\theta' \in \gamma^{(\rho,f)}[E,\theta]$ and $\theta'' \in \gamma^{(\rho,f)}[E,\theta']$ imply $\theta' \in \gamma^{(\rho,f)\text{-}(n+1)}[E,\theta]$ and $\theta'' \in \gamma^{(\rho,f)\text{-}(n'+1)}[E,\theta']$ for some $(n,n') \in \mathbb{N}\times\mathbb{N}$. Thus, Lemmas 10 and 11 imply

$$\left(\begin{array}{c} \forall i \in \mathcal{I}, \exists \left\{\theta^1,...,\theta^{n+1}\right\} \subset E, \\ \theta^1 = \theta \text{ and } \theta^{n+1} = \theta', \\ \theta^1 \sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(n+n')},E\right)\right]} \theta^2 \sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(n+n')},E\right)\right]} ... \sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(n+n')},E\right)\right]} \theta^{n+1} \end{array}\right),$$

$$\text{and} \left(\begin{array}{c} \forall i \in \mathcal{I}, \exists \left\{\theta^1,...,\theta^{n+1}\right\} \subset E, \\ \theta^{n+1} = \theta' \text{ and } \theta^{n+n'+1} = \theta'', \\ \theta^{n+1} \sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(n+n')},E\right)\right]} \theta^{n+2} \sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(n+n')},E\right)\right]} ... \sim_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(n+n')},E\right)\right]} \theta^{n+n'+1} \end{array}\right),$$

which, together with Lemma 11, imply $\theta'' \in \gamma^{(\rho,f)\text{-}(n+n'+1)}[E,\theta] \subset \gamma^{(\rho,f)}[E,\theta]$. Therefore, (73) holds.

Fourth, (72) and (73) imply

$$\left[\theta \in E \text{ and } \theta' \in \gamma^{(\rho,f)}[E,\theta]\right] \Longrightarrow \left[\theta' \in E \text{ and } \theta \in \gamma^{(\rho,f)}[E,\theta']\right] \Longrightarrow \gamma^{(\rho,f)}[E,\theta] \subset \gamma^{(\rho,f)}[E,\theta'],$$

which, together with (73), implies

$$\left[\theta \in E \text{ and } \theta' \in \gamma^{(\rho,f)}[E,\theta]\right] \Longrightarrow \gamma^{(\rho,f)}[E,\theta] = \gamma^{(\rho,f)}[E,\theta']. \tag{74}$$

Finally, (71) and (74) imply that $\gamma^{(\rho,f)}$ is an operator.■

A.5 Proof of Lemma 5

Fix any SCF f and any solution notion ρ which is both dissectible and monotonic. Fix any $(\theta, E, E') \in \Theta \times \left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i}\diagdown\{\varnothing\}\right]\right) \times \left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i}\diagdown\{\varnothing\}\right]\right)$ such that $\theta \in E \subset E'$. Inductively, we prove

$$\left(\begin{array}{c} \gamma^{(\rho,f)\text{-}(n)}[E,\theta] \subset \gamma^{(\rho,f)\text{-}(n)}[E',\theta], \\ \text{and } \forall i \in \mathcal{I}, \\ \theta' >_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(n)},E\right)\right]} \theta'' \Longrightarrow \theta' >_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(n)},E'\right)\right]} \theta'' \end{array}\right), \forall n \in \mathbb{N}, \forall \left(\theta,\theta',\theta''\right) \equiv \Theta\times\Theta\times\Theta. \tag{75}$$

With $n = 1$, (36) implies

$$\gamma^{(\rho,f)\text{-}(1)}[E,\theta] = \{\theta\} \subset \{\theta\} = \gamma^{(\rho,f)\text{-}(1)}[E',\theta].$$

Furthermore, for any $i \in \mathcal{I}$, we have

$$\theta' >_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(1)},E\right)\right]} \theta'' \Longrightarrow \left(\begin{array}{c} \{\theta',\theta''\} \subset E \text{ and} \\ \rho\left[f,\left(\gamma^{(\rho,f)\text{-}(1)},E\right),\theta,\theta',i,E\right]=0 \end{array}\right)$$
$$\Longrightarrow \left(\begin{array}{c} \{\theta',\theta''\} \subset E' \text{ and} \\ \rho\left[f,\left(\gamma^{(\rho,f)\text{-}(1)},E'\right),\theta,\theta',i,E'\right]=0 \end{array}\right)$$
$$\Longrightarrow \theta' >^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(1)},E'\right)\right]} \theta'',$$

where the first and third "$\Longrightarrow$" follow from (33), and the second "$\Longrightarrow$" follow from $E \subset E'$, $\gamma^{(\rho,f)\text{-}(1)}[E,\theta] \subset \gamma^{(\rho,f)\text{-}(1)}[E',\theta]$, Lemma 9 and ρ being monotonic. Therefore (75) holds for $n=1$.

Suppose (75) holds for $n=k \in \mathbb{N}$. Consider $n=(k+1) \in \mathbb{N}$. We have

$$\begin{aligned} \gamma^{(\rho,f)\text{-}(k+1)}[E,\theta] &= \bigcup_{\widetilde{\theta} \in \gamma^{(\rho,f)\text{-}(k)}[E,\,\theta]} \left\{\theta' \in E : \forall i \in \mathcal{I}, \left(\begin{array}{l} \text{either } \theta'_i = \widetilde{\theta}_i, \\ \text{or } \theta' >_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(k)},E\right)\right]} \widetilde{\theta} \\ \text{or } \widetilde{\theta} >_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(k)},E\right)\right]} \theta' \end{array}\right)\right\} \\ &\subset \bigcup_{\widetilde{\theta} \in \gamma^{(\rho,f)\text{-}(k)}[E',\,\theta]} \left\{\theta' \in E : \forall i \in \mathcal{I}, \left(\begin{array}{l} \text{either } \theta'_i = \widetilde{\theta}_i, \\ \text{or } \theta' >_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(k)},E'\right)\right]} \widetilde{\theta} \\ \text{or } \widetilde{\theta} >_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(k)},E'\right)\right]} \theta' \end{array}\right)\right\} \\ &= \gamma^{(\rho,f)\text{-}(k+1)}\left[E',\theta\right] \end{aligned}$$

where the two equalities follow from (37), and "$\subset$" follows from the induction hypothesis.

Furthermore, for any $i \in \mathcal{I}$, we have

$$\theta' >_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(k+1)},E\right)\right]} \theta'' \Longrightarrow \left(\begin{array}{c} \{\theta',\theta''\} \subset E \text{ and} \\ \rho\left[f,\left(\gamma^{(\rho,f)\text{-}(k+1)},E\right),\theta,\theta',i,E\right]=0 \end{array}\right)$$
$$\Longrightarrow \left(\begin{array}{c} \{\theta',\theta''\} \subset E' \text{ and} \\ \rho\left[f,\left(\gamma^{(\rho,f)\text{-}(k+1)},E'\right),\theta,\theta',i,E'\right]=0 \end{array}\right)$$
$$\Longrightarrow \theta' >_i^{\left[(\rho,f),\left(\gamma^{(\rho,f)\text{-}(k+1)},E'\right)\right]} \theta'',$$

where the first and third "$\Longrightarrow$" follow from (33), and the second "$\Longrightarrow$" follow from $E \subset E'$, $\gamma^{(\rho,f)\text{-}(k+1)}[E,\theta] \subset \gamma^{(\rho,f)\text{-}(k+1)}[E',\theta]$, Lemma 9 and ρ being monotonic. Therefore (75) holds for $n = k+1$.

Finally, (75) implies that for any $(\theta, E, E') \equiv \Theta \times \left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i} \diagdown \{\varnothing\}\right]\right) \times \left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i} \diagdown \{\varnothing\}\right]\right)$,

$$\theta \in E \subset E' \Longrightarrow \gamma^{(\rho,f)}[E,\ \theta] = \cup_{n=1}^{\infty}\gamma^{(\rho,f)\text{-}(n)}[E,\ \theta] \subset \cup_{n=1}^{\infty}\gamma^{(\rho,f)\text{-}(n)}[E',\ \theta] = \gamma^{(\rho,f)}[E',\ \theta].$$

Therefore, $\gamma^{(\rho,f)}$ is an increasing operator.■

A.6 Proof of Lemma 6

Fix any SCF f and any solution notion ρ which is dissectible and monotonic. Fix any $[G,S] \in \mathcal{GS}^{\text{PC}}$ such that $\gamma^{[G,S]}$ is $\left(\rho, f, \vartheta^{G}\right)$-consistent, and we aim to show

$$\gamma^{(\rho,f)}[E,\ \theta] \subset \gamma^{[G,S]}[E,\ \theta], \forall [E,\ \theta] \in \left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i} \diagdown \{\varnothing\}\right]\right) \times \Theta.$$

Recall

$$\Omega^{[G,S]} \equiv \left\{E \in \left(\times_{i\in\mathcal{I}}\left[2^{\Theta_i} \diagdown \{\varnothing\}\right]\right) : \exists h \in [\mathcal{H} \diagdown \mathcal{T}],\ E = \mathcal{E}^{[G,S]\text{-}h}\right\}.$$

First, if $\theta \notin E$ or $E \notin \Omega^{[G,S]}$, we have

$$\gamma^{(\rho,f)}[E,\ \theta] \subset E = \gamma^{[G,S]}[E,\ \theta].$$

From now, we assume $\theta \in E \in \Omega^{[G,S]}$. Consider any $h \in [\mathcal{H} \diagdown \mathcal{T}]$, and we aim to show

$$\gamma^{(\rho,f)}\left[\mathcal{E}^{[G,S]\text{-}h},\ \theta\right] \subset \gamma^{[G,S]}\left[\mathcal{E}^{[G,S]\text{-}h},\ \theta\right]. \tag{76}$$

Inductively, we prove

$$\gamma^{(\rho,f)\text{-}(k)}\left[\mathcal{E}^{[G,S]\text{-}h},\ \theta\right] \subset \gamma^{[G,S]}\left[\mathcal{E}^{[G,S]\text{-}h},\ \theta\right], \forall k \in \mathbb{N}, \tag{77}$$

which implies (76) because of the definition of $\gamma^{(\rho,f)}$ (see (38)). With $k = 1$, we have

$$\gamma^{(\rho,f)\text{-}(1)}\left[\mathcal{E}^{[G,S]\text{-}h},\ \theta\right] = \{\theta\} \subset \gamma^{[G,S]}\left[\mathcal{E}^{[G,S]\text{-}h},\ \theta\right],$$

i.e., (77) holds. Suppose (77) holds for any $k = n \in \mathbb{N}$. We show (77) for $k = n+1$. Consider any $\theta' \in \gamma^{(\rho,f)\text{-}(n+1)}\left[\mathcal{E}^{[G,S]\text{-}h},\ \theta\right]$, and we aim to prove $\theta' \in \gamma^{[G,S]}\left[\mathcal{E}^{[G,S]\text{-}h},\ \theta\right]$. By

(33), (34) and (37), $\theta' \in \gamma^{(\rho,f)\text{-}(n+1)}\left[\mathcal{E}^{[G,S]\text{-}h},\ \theta\right]$ implies existence of $\widetilde{\theta} \in \gamma^{(\rho,f)\text{-}(n)}\left[\mathcal{E}^{[G,S]\text{-}h},\ \theta\right]$ such that for any $i \in \mathcal{I}$, one of the following three conditions holds:

$$\theta'_i = \widetilde{\theta}_i, \tag{78}$$

$$\rho\left[f,\left(\gamma^{(\rho,f)\text{-}(n)},\mathcal{E}^{[G,S]\text{-}h}\right),\widetilde{\theta},\theta',i,\mathcal{E}^{[G,S]\text{-}h}\right] = 0, \tag{79}$$

$$\rho\left[f,\left(\gamma^{(\rho,f)\text{-}(n)},\mathcal{E}^{[G,S]\text{-}h}\right),\theta',\widetilde{\theta},i,\mathcal{E}^{[G,S]\text{-}h}\right] = 0. \tag{80}$$

Fix any $i \in \mathcal{I}$, we now prove

$$\theta'_i \in \gamma_i^{[G,S]}\left[\mathcal{E}^{[G,S]\text{-}h},\ \theta\right], \tag{81}$$

and as a result, we have $\theta' \in \gamma^{[G,S]}\left[\mathcal{E}^{[G,S]\text{-}h},\ \theta\right]$.

By the induction hypothesis, $\widetilde{\theta} \in \gamma^{(\rho,f)\text{-}(n)}\left[\mathcal{E}^{[G,S]\text{-}h},\ \theta\right]$ implies

$$\widetilde{\theta} \in \gamma^{[G,S]}\left[\mathcal{E}^{[G,S]\text{-}h},\ \theta\right], \tag{82}$$

which, together with $\gamma^{[G,S]}\left[\mathcal{E}^{[G,S]\text{-}h},\ \cdot\right]$ being a partition on $\mathcal{E}^{[G,S]\text{-}h}$ (due to $\gamma^{[G,S]}$ being an operator), implies

$$\gamma^{[G,S]}\left[\mathcal{E}^{[G,S]\text{-}h},\ \widetilde{\theta}\right] = \gamma^{[G,S]}\left[\mathcal{E}^{[G,S]\text{-}h},\ \theta\right]. \tag{83}$$

First, if (78) holds, (82) implies (81). Second, if (79) holds, Lemma 9 and the induction hypothesis imply

$$\rho\left[f,\left(\gamma^{[G,S]},\mathcal{E}^{[G,S]\text{-}h}\right),\widetilde{\theta},\theta',i,\mathcal{E}^{[G,S]\text{-}h}\right] = 0,$$

which, together with ρ being monotonic, implies

$$\rho\left[f,\left(\gamma^{[G,S]},\mathcal{E}^{[G,S]\text{-}h}\right),\widetilde{\theta},\theta',i,\vartheta^{[G]}\left(\mathcal{E}^{[G,S]\text{-}h}\right)\right] = 0. \tag{84}$$

Since $\gamma^{[G,S]}$ is $\left(\rho,f,\vartheta^G\right)$-consistent, (84) implies $\theta'_i \in \gamma_i^{[G,S]}\left[\mathcal{E}^{[G,S]\text{-}h},\ \widetilde{\theta}\right]$, which, together with (83), implies (81). Third, if (80) holds, a similar argument as above shows

$$\rho\left[f,\left(\gamma^{[G,S]},\mathcal{E}^{[G,S]\text{-}h}\right),\theta',\widetilde{\theta},i,\vartheta^{[G]}\left(\mathcal{E}^{[G,S]\text{-}h}\right)\right] = 0,$$

which, together with (22) in Definition 5, implies

$$\rho\left[f,\left(\gamma^{[G,S]},\mathcal{E}^{[G,S]\text{-}h}\right),\left(\theta'_i,\widetilde{\theta}_{-i}\right),\widetilde{\theta},i,\vartheta^{[G]}\left(\mathcal{E}^{[G,S]\text{-}h}\right)\right] = 0. \tag{85}$$

Since $\gamma^{[G,S]}$ is $\left(\rho, f, \vartheta^{G}\right)$-consistent, (85) implies

$$\widetilde{\theta}_i \in \gamma_i^{[G,S]}\left[\mathcal{E}^{[G,S]\text{-}h},\ \left(\theta_i', \widetilde{\theta}_{-i}\right)\right]. \tag{86}$$

Since $\left(\theta_i', \widetilde{\theta}_{-i}\right) \in \gamma^{[G,S]}\left[\mathcal{E}^{[G,S]\text{-}h},\ \left(\theta_i', \widetilde{\theta}_{-i}\right)\right]$, we have

$$\widetilde{\theta}_j \in \gamma_j^{[G,S]}\left[\mathcal{E}^{[G,S]\text{-}h},\ \left(\theta_i', \widetilde{\theta}_{-i}\right)\right], \forall j \in \mathcal{I} \diagdown \{i\}. \tag{87}$$

(86) and (87) imply

$$\widetilde{\theta} \in \gamma^{[G,S]}\left[\mathcal{E}^{[G,S]\text{-}h},\ \left(\theta_i', \widetilde{\theta}_{-i}\right)\right]. \tag{88}$$

We thus have

$$\left(\theta_i', \widetilde{\theta}_{-i}\right) \in \gamma^{[G,S]}\left[\mathcal{E}^{[G,S]\text{-}h},\ \left(\theta_i', \widetilde{\theta}_{-i}\right)\right] = \gamma^{[G,S]}\left[\mathcal{E}^{[G,S]\text{-}h},\ \widetilde{\theta}\right] = \gamma^{[G,S]}\left[\mathcal{E}^{[G,S]\text{-}h},\ \theta\right], \tag{89}$$

where $\in$ follows from $\gamma^{[G,S]}\left[\mathcal{E}^{[G,S]\text{-}h},\ \cdot\right]$ being a partition on $\mathcal{E}^{[G,S]\text{-}h}$, and the first equality follows from (88) and $\gamma^{[G,S]}\left[\mathcal{E}^{[G,S]\text{-}h},\ \cdot\right]$ being a partition on $\mathcal{E}^{[G,S]\text{-}h}$, and the second equality follows from (83). Finally, (89) implies (81).■

A.7 Proofs of Lemma 7

Consider any SCF f and any solution notion ρ which is dissectible. Suppose $\gamma^{(\rho,f)}$ is achievable, i.e., there exists $N \in \mathbb{N}$ such that

$$\gamma_{(N)}^{(\rho,f)}[\Theta, \theta] = \{\theta\}, \forall \theta \in \Theta. \tag{90}$$

For any $\theta \in \Theta$ and any $n \in \mathbb{N}$, recall

$$\begin{aligned}
\gamma_{(0)}^{(\rho,f)}[\Theta, \theta] &= \Theta, \\
\gamma_{(n)}^{(\rho,f)}[\Theta, \theta] &= \gamma^{(\rho,f)}\left[\gamma_{(n-1)}^{(\rho,f)}[\Theta, \theta], \theta\right].
\end{aligned}$$

For any $\theta \in \Theta$ and any $n \in \mathbb{N}$, define

$$\gamma_{(n)\text{-}i}^{(\rho,f)}[\Theta, \theta] = \gamma_i^{(\rho,f)}\left[\gamma_{(n-1)}^{(\rho,f)}[\Theta, \theta], \theta\right], \forall \theta \in \Theta, \forall n \in \mathbb{N}.$$

and

$$
\begin{aligned}
\mathcal{I}_{(\theta,n)} &= \left\{ i \in \mathcal{I} : \begin{array}{l} \exists \theta' \in \gamma^{(\rho,f)}_{(n-1)} [\Theta\,,\theta], \\ \gamma^{(\rho,f)}_{(n)\text{-}i} [\Theta\,,\theta'] \neq \gamma^{(\rho,f)}_{(n)\text{-}i} [\Theta\,,\theta] \end{array} \right\}, \\
\mathcal{I} \diagdown \mathcal{I}_{(\theta,n)} &= \left\{ i \in \mathcal{I} : \begin{array}{l} \forall \theta' \in \gamma^{(\rho,f)}_{(n-1)} [\Theta\,,\theta], \\ \gamma^{(\rho,f)}_{(n)\text{-}i} [\Theta\,,\theta'] = \gamma^{(\rho,f)}_{(n)\text{-}i} [\Theta\,,\theta] \end{array} \right\}, \\
N_{(\theta)} &= \max \left\{ n \in \mathbb{N} : \mathcal{I}_{(\theta,n)} \neq \varnothing \right\}. \qquad (91)
\end{aligned}
$$

Since ρ is dissectible, Lemma 4 implies that $\gamma^{(\rho,f)}$ is an operator, i.e., given any E, every $\gamma^{(\rho,f)}_i [E, \cdot]$ is a partition on E_i. The set $\mathcal{I}_{(\theta,n)}$ includes all of the player i such that $\gamma^{(\rho,f)}_i \left[\gamma^{(\rho,f)}_{(n-1)} [\Theta\,,\theta]\,, \cdot \right]$ is a non-trivial partition of $\gamma^{(\rho,f)}_{(n-1)} [\Theta\,,\theta]$, i.e., player i reveals non-trivial information at the n-th round. The set $\mathcal{I} \diagdown \mathcal{I}_{(\theta,n)}$ includes all of the players who reveal no additional information at the n-th round.

We are ready to define $[G, S] \in \mathcal{GS}^{\text{PC}}$ such that $\gamma^{[G,S]}$ is both achievable and $\left(\rho, f, \vartheta^{[G]}\right)$-consistent. In G, there are at most N rounds. At each round, we invite a group of players to partially disclose their types simultaneously, and the disclosure is made public at the end of each round.

At the first period, we invite players in $\mathcal{I}_{(\theta,1)}$, and each $i \in \mathcal{I}_{(\theta,1)}$ chooses one element in the following partition of Θ_i:

$$
\left\{ \gamma^{(\rho,f)}_i [\Theta, \theta] : \theta \in \Theta \right\}.
$$

S denotes the strategy of (partially) truth revealing, i.e., at the true state θ, player $i \in \mathcal{I}_{(\theta,1)}$ chooses $\gamma^{(\rho,f)}_i [\Theta, \theta]$ in the first round. By following S, players disclose that the true state is in the following set at the end of the first round:

$$
\begin{aligned}
\gamma^{(\rho,f)}_{(1)} [\Theta\,,\theta] &= \left(\times_{i \in \mathcal{I}_{(\theta,1)}} \gamma^{(\rho,f)}_i [\Theta, \theta] \right) \times \left(\times_{i \in \mathcal{I} \diagdown \mathcal{I}_{(\theta,1)}} \gamma^{(\rho,f)}_i [\Theta, \theta] \right) \\
&= \left(\times_{i \in \mathcal{I}_{(\theta,1)}} \gamma^{(\rho,f)}_i [\Theta, \theta] \right) \times \left(\times_{i \in \mathcal{I} \diagdown \mathcal{I}_{(\theta,1)}} \Theta_i \right).
\end{aligned}
$$

Inductively, suppose that players have disclosed $\gamma^{(\rho,f)}_{(k)} [\Theta\,,\theta]$ at the end of k-th round with $k \leq N$. If either $k = N$ or $\mathcal{I}_{(\theta,k+1)} = \varnothing$, the game ends. If $k < N$ and $\mathcal{I}_{(\theta,k+1)} \neq \varnothing$, we proceed to the $(k+1)$-th round. We invite players in $\mathcal{I}_{(\theta,k+1)}$ and each $i \in \mathcal{I}_{(\theta,k+1)}$ chooses one element in the following partition of $\gamma^{(\rho,f)}_{(k)} [\Theta\,,\theta]$:

$$
\left\{ \gamma^{(\rho,f)}_i \left[\gamma^{(\rho,f)}_{(k)} [\Theta\,,\theta]\,,\ \theta \right] : \theta \in \gamma^{(\rho,f)}_{(k)} [\Theta\,,\theta] \right\}.
$$

Recall that S denotes the strategy of (partially) truth revealing, i.e., at the true state θ, player $i \in \mathcal{I}_{(\theta,k+1)}$ chooses $\gamma_i^{(\rho,f)}\left[\gamma_{(k)}^{(\rho,f)}[\Theta,\theta],\theta\right]$ in the $(k+1)$-th round. By following S, players disclose that the state is in the following set at the end of the $(k+1)$-th round:

$$\begin{aligned}\gamma_{(k+1)}^{(\rho,f)}[\Theta,\theta] &= \left(\times_{i\in\mathcal{I}_{(\theta,k+1)}}\gamma_i^{(\rho,f)}\left[\gamma_{(k)}^{(\rho,f)}[\Theta,\theta],\theta\right]\right)\times\left(\times_{i\in\mathcal{I}\setminus\mathcal{I}_{(\theta,k+1)}}\gamma_i^{(\rho,f)}\left[\gamma_{(k)}^{(\rho,f)}[\Theta,\theta],\theta\right]\right)\\ &= \left(\times_{i\in\mathcal{I}_{(\theta,k+1)}}\gamma_i^{(\rho,f)}\left[\gamma_{(k)}^{(\rho,f)}[\Theta,\theta],\theta\right]\right)\times\left(\times_{i\in\mathcal{I}\setminus\mathcal{I}_{(\theta,k+1)}}\gamma_{(k)\text{-}i}^{(\rho,f)}[\Theta,\theta]\right).\end{aligned}$$

At each round, player move simultaneously, and as usual, we pick any order of the players and translate it into a traditional extensive-form game (i.e., players move sequentially following the order, but does not observe previous moves in this round).

We will show that $\gamma^{[G,S]}$ is both achievable and $\left(\rho,f,\vartheta^{[G]}\right)$-consistent. Consider any true state $\theta\in\Theta$. By following S, we reach the following history:

$$T^G[S(\theta)] = \left(a^1,...,a^{n^\theta}\right), \text{ where } n^\theta = \sum_{n=1}^{N_{(\theta)}}\left|\mathcal{I}_{(\theta,n)}\right| < N\times|\mathcal{I}|,$$

where $N_{(\theta)}$ (as defined in (91)) is the largest k such that $\gamma^{(\rho,f)}\left[\gamma_{(k-1)}^{(\rho,f)}[\Theta,\theta],\cdot\right]$ is a non-trivial partition, which implies

$$\gamma_{(n)}^{(\rho,f)}[\Theta,\cdot] = \gamma_{\left(N_{(\theta)}\right)}^{(\rho,f)}[\Theta,\cdot], \forall n\geq N_{(\theta)}. \tag{92}$$

In particular, for each $k\leq N_{(\theta)}$,

$$h_{(k)} = \left(a^1,...,a^{k^\theta}\right)\in \text{path}\left(T^G[S(\theta)]\right), \text{ where } k^\theta = \sum_{n=1}^{k}\left|\mathcal{I}_{(\theta,n)}\right|,$$

is the history at the end of k-th round, and we have

$$\mathcal{E}^{[G,S]\text{-}h_{(k)}} = \gamma_{(k)}^{(\rho,f)}[\Theta,\theta], \forall k\leq N_{(\theta)},$$

By (90), we have $N_{(\theta)}\leq N$, and in particular, we have

$$\gamma_{\left(n^\theta\right)}^{[G,S]}[\Theta,\theta] = \mathcal{E}^{[G,S]\text{-}T^G[S(\theta)]} = \mathcal{E}^{[G,S]\text{-}h_{\left(N_{(\theta)}\right)}} = \gamma_{\left(N_{(\theta)}\right)}^{(\rho,f)}[\Theta,\theta],$$

which, together with (90), (92) and $N_{(\theta)}\leq N$, implies

$$\gamma_{\left(n^\theta\right)}^{[G,S]}[\Theta,\theta] = \gamma_{\left(N_{(\theta)}\right)}^{(\rho,f)}[\Theta,\theta] = \gamma_{(N)}^{(\rho,f)}[\Theta,\theta] = \{\theta\}.$$

Since $n^{\theta} = \sum_{n=1}^{N_{(\theta)}} \left| \mathcal{I}_{(\theta,n)} \right| < N \times |\mathcal{I}|$ for every $\theta \in \Theta$, we thus have

$$\gamma_{(N\times|\mathcal{I}|)}^{[G,S]} [\Theta, \theta] = \{\theta\}, \forall \theta \in \Theta,$$

i.e., $\gamma^{[G,S]}$ is achievable.

Finally, we show $\gamma^{[G,S]}$ is $\left(\rho, f, \vartheta^{[G]}\right)$-consistent. Consider any non-terminal history h. Suppose that h is player j's history at the k-th round. Furthermore, suppose that players have disclosed $\gamma_{(k-1)}^{(\rho,f)} [\Theta, \theta]$ at the end of $(k-1)$-th round for some $\theta \in \Theta$. We thus have

$$\gamma_j^{[G,S]} \left[\mathcal{E}^{[G,S]\text{-}h}, \theta\right] = \gamma_j^{(\rho,f)} \left[\gamma_{(k-1)}^{(\rho,f)} [\Theta, \theta], \theta\right], \tag{93}$$

$$\vartheta^{[G]} \left(\mathcal{E}^{[G,S]\text{-}h}\right) = \gamma_{(k-1)}^{(\rho,f)} [\Theta, \theta]. \tag{94}$$

We now prove

$$\left(\begin{array}{c} \{\theta, \theta'\} \subset \mathcal{E}^{[G,S]\text{-}h}, \\ \rho\left[f, \left(\gamma^{[G,S]}, \mathcal{E}^{[G,S]\text{-}h}\right), \theta, \theta', j, \vartheta^{[G]}\left(\mathcal{E}^{[G,S]\text{-}h}\right)\right] = 0 \end{array} \right) \Longrightarrow \theta'_j \in \gamma_j^{[G,S]} \left[\mathcal{E}^{[G,S]\text{-}h}, \theta\right], \tag{95}$$

i.e., $\gamma^{[G,S]}$ is $\left(\rho, f, \vartheta^{G}\right)$-consistent. Suppose $\{\theta, \theta'\} \subset \mathcal{E}^{[G,S]\text{-}h}$ and

$$\rho\left[f, \left(\gamma^{[G,S]}, \mathcal{E}^{[G,S]\text{-}h}\right), \theta, \theta', j, \vartheta^{[G]}\left(\mathcal{E}^{[G,S]\text{-}h}\right)\right] = 0. \tag{96}$$

We thus have

$$\begin{aligned} & \rho\left[f, \left(\gamma^{(\rho,f)}, \gamma_{(k-1)}^{(\rho,f)} [\Theta, \theta]\right), \theta, \theta', j, \gamma_{(k-1)}^{(\rho,f)} [\Theta, \theta]\right] \\ = \;& \rho\left[f, \left(\gamma^{(\rho,f)}, \gamma_{(k-1)}^{(\rho,f)} [\Theta, \theta]\right), \theta, \theta', j, \vartheta^{[G]}\left(\mathcal{E}^{[G,S]\text{-}h}\right)\right] \\ = \;& \rho\left[f, \left(\gamma^{[G,S]}, \mathcal{E}^{[G,S]\text{-}h}\right), \theta, \theta', j, \vartheta^{[G]}\left(\mathcal{E}^{[G,S]\text{-}h}\right)\right] \\ = \;& 0, \end{aligned} \tag{97}$$

where the first equality follows from (94), and the second equality follows from (93) and (22) in Definition 5, and the third equality follows from (96). By Lemma 4, $\gamma^{(\rho,f)}$ is (ρ, f, ϑ^{*})-consistent, which, together with (97), implies

$$\theta'_j \in \gamma_j^{(\rho,f)} \left[\gamma_{(k-1)}^{(\rho,f)} [\Theta, \theta], \theta\right]. \tag{98}$$

(93) and (98) imply $\theta'_j \in \gamma_j^{[G,S]} \left[\mathcal{E}^{[G,S]\text{-}h}, \theta\right]$, i.e., (95) holds.■

A.8 Proofs of Lemma 8

Consider any SCF f and any normal solution notion ρ. The "only if" part of Lemma 8 is implied by Definition 6. To prove the "if" part, suppose $\gamma^{(\rho,f)}$ is f-achievable. i.e., there exists $N \in \mathbb{N}$,

$$f\left(\widetilde{\theta}\right) = f(\theta), \forall \theta \in \Theta, \forall \widetilde{\theta} \in \gamma^{(\rho,f)}_{(N)}[\Theta, \theta], \tag{99}$$

and we aim to $\gamma^{(\rho,f)}$ is achievable. Since ρ is normal, (99) and Definition 11 imply

$$\rho\left[f, \left(\gamma^{(\rho,f)}, \gamma^{(\rho,f)}_{(N)}[\Theta, \theta]\right), \theta, \theta', i, \gamma^{(\rho,f)}_{(N)}[\Theta, \theta]\right] = 1, \forall \theta \in \Theta, \forall \theta' \in \gamma^{(\rho,f)}_{(N)}[\Theta, \theta],$$

which, together with (36) and (37), implies

$$\gamma^{(\rho,f)\text{-}(n)}\left[\gamma^{(\rho,f)}_{(N)}[\Theta, \theta], \theta\right] = \{\theta\}, \forall \theta \in \Theta, \forall n \in \mathbb{N},$$

and hence,

$$\gamma^{(\rho,f)}\left[\gamma^{(\rho,f)}_{(N)}[\Theta, \theta], \theta\right] = \bigcup_{n=1}^{\infty}\left(\gamma^{(\rho,f)\text{-}(n)}\left[\gamma^{(\rho,f)}_{(N)}[\Theta, \theta], \theta\right]\right) = \{\theta\}, \forall \theta \in \Theta.$$

Therefore,

$$\gamma^{(\rho,f)}_{(N+1)}[\Theta, \theta] = \gamma^{(\rho,f)}\left[\gamma^{(\rho,f)}_{(N)}[\Theta, \theta], \theta\right] = \{\theta\}, \forall \theta \in \Theta,$$

i.e., $\gamma^{(\rho,f)}$ is achievable.■

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