Selection Procedures in Competitive Admission

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March, 2025

Abstract

Two identical firms compete to attract and hire from a pool of candidates of unknown productivity. Firms simultaneously post a selection procedure which consists of a test and an acceptance probability for each test outcome. After observing the firms' selection procedures, each candidate can apply to one of them. Both firms have access to a limited set of feasible tests. The firms face two key considerations when choosing their selection procedure: the statistical properties of their test and the selection into the procedure by the candidates. I identify two partial orders on tests that are useful to characterise the equilibrium of this game: the test's accuracy (Lehmann, 1988) and difficulty. I show that in any symmetric equilibrium, the test chosen must be maximal in the accuracy order and minimal in the difficulty order. Intuitively, competition leads to maximal but misguided learning: firms end up having precise knowledge that is not payoff relevant. I also consider the cases where firms face capacity constraints, have the possibility of making a wage offer and the existence of asymmetric equilibria where one firm is more selective than another.

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1 Introduction

An organisation's admission process, whether it is a firm hiring workers or a university admitting students, usually consists of two important parts: recruitment and selection. Recruitment is the process of attracting the most suitable pool of candidates while selection aims at identifying the best candidates from that pool. For candidates, applying to a position is a costly process in terms of time, effort and missed opportunities. Therefore, candidates' application decision depends on their probability of being selected. From the organisation's perspective, it means that when choosing its selection procedure, it needs to take into account two elements: the statistical properties of the selection procedure and its impact on the pool of candidates it attracts. The goal of this paper is to study how these two elements interact in a competitive market for admission and to determine the properties of the selection procedures used in equilibrium.

There are various ways in which an organisation can vary the statistical properties of its selection procedures. One way is to vary how precise its testing is. The organisation could ask the candidates for more information, conduct longer interviews or require additional tests. Another possibility is to vary the direction of learning. The tests can be difficult, making it effective at identifying top candidates, or easy to identify poor candidates. These two dimensions of testing, one vertical, one horizontal, can be captured by the notions of accuracy (Lehmann, 1988) and difficulty (Hancart, 2024).

In this paper, I build a model of competition where firms compete by posting selection procedures. I characterise the properties of the test used in equilibrium and show how it depends on the characteristics of the admission procedure. In the baseline model, admission procedures are simple: firms only need to decide whether to accept candidates. Firms want to accept any candidates with positive productivity while all candidates want to be accepted to any firm. I show that in any symmetric equilibrium, the test used must be maximally accurate and minimally difficult. Intuitively, competition leads firms to use as much information as possible but learn too precisely about poor candidates compared to what would be optimal absent competition. I contrast this result with two modifications of the admission procedure. In the first one, firms face capacity constraints. In the second extension, firms also make a wage offer if they accept the candidate. I show that in both these extensions, firms use more difficult tests in equilibrium. A key mechanism for these results is how the candidates' selection into the selection procedures varies across environments.

Specifically, two identical firms post simultaneously a selection procedure that consists of a test and an acceptance rule. A test is a Blackwell experiment and must come from an exogenously given set of feasible tests. I assume for simplicity that all tests are binary. The acceptance rule is an accept/reject decision based on the signal realisation. There is a continuum of candidates that differ in their privately observed productivity. Each candidate decides where to apply after having observed the selection procedures. Applying is costly for the candidates so after having observed the selection procedures, they apply to only one of the two firms.[1] I study symmetric equilibria of this game.

I first show that competition drives firms' profits to what they would be absent any information: firms accept until they make zero profits or they accept everyone. The intuition for this result is similar to price undercutting in a Bertrand competition model. If firms share the market, one of them can increase their acceptance probability by an arbitrarily small amount and attract all candidates.

The observation that firms make zero profits is important. It implies that selection into the tests is the key driver determining equilibrium tests: if by deviating a firm can only attract candidates with positive productivity, this will be a profitable deviation. This deviation is profitable independently of the optimality of the test in a decision problem.

To characterise the tests used in equilibrium, I identify two natural orders on tests that will allow to interpret the equilibrium choices of the firms: Lehmann's (1988) accuracy and difficulty. Accuracy is a weaker order than Blackwell's (1953) informativeness order for environments satisfying monotonicity assumptions. The difficulty order was first introduced in Hancart (2024). This notion captures that varying the difficulty of a test changes which types are better identified: a more difficult test is informative after a high grade, as only high types are likely to produce a high grade but it is less informative after a low grade.

The main characterisation result is that in any symmetric equilibrium, the test used must be maximal in the accuracy order and minimal in the difficulty order. In equilibrium, the firms have precise information after a low signal but the high signal contains little information compared to what would be optimal.

The mechanism behind these results is the selection of the candidates into the test. I say that selection into a test is positive if whenever one candidate prefers a test over another, then all

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[1]We can also interpret this constraint as a requirement that candidates must put some effort in the selection process and can direct that effort to at most one of the two firms.

candidates with higher productivity also prefer that test.

In the case of tests ordered by accuracy, selection into a more accurate test is positive. Intuitively, higher types benefit more from a more accurate test. Using this observation, I show that the equilibrium test has to be maximally accurate, in the sense that no test can be more accurate than the equilibrium test. I also show that if all tests are ordered by accuracy, there is a unique symmetric equilibrium where both firms use the most accurate test.[2]

To show that in any symmetric equilibrium, firms choose a minimally difficult test, I show that the selection into an *easier* test is positive. The reason is that if firms want to attract any candidate when offering a more difficult test, they must also have a more lenient acceptance rule, otherwise no type would ever want to deviate. Low productivity candidates benefit relatively more from a more lenient acceptance rule as they are more likely to produce a low signal, and even more so in a more difficult test. Therefore, the selection into the more difficult test is negative. Like with tests ordered by accuracy, I also show that if all tests are comparable in the difficulty order, there is a unique symmetric equilibrium where the easiest test is offered as long as it provides some information to the firm. Corollary 1 combines these two results to show existence results when we can link any two tests by a sequence of tests that are comparable in terms of difficulty or accuracy even though the original tests are comparable in neither.

These results show the importance of including orders on tests that are not informativeness orders (Blackwell, 1953; Lehmann, 1988) when studying models with information acquisition. In this case, firms endogenously choose maximally accurate tests. However, if they can also adjust the difficulty of their test, they mostly learn about non-payoff relevant information.

In Section 3.1, I apply the previous results to characterise equilibria in more structured environments. First, I show an equilibrium always exists when types are binary, provided the set of feasible test is closed and convex in an appropriate sense. In the second application, I assume that the set of feasible tests derives from a cost constraint: there is a continuous posterior separable cost function (Caplin et al., 2022) that determines the cost of each test and each firm can design any test less costly than some $\kappa > 0$. I show that in any symmetric equilibrium, the cost constraint binds but at the same time, the expected posterior productivity at the high signal is zero.

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[2]Unless it is explicitly proven, there is no guarantee that an equilibrium exists. This is common issue in competitive markets with adverse selection, see e.g., Rothschild and Stiglitz (1976). One contribution of this paper is to provide economically interpretable conditions under which an equilibrium of this game exists.

In Section 4, I consider two modifications to the admission environment and show how they can change the predictions of the baseline model. I first consider the possibility for firms to make a wage offer after having accepted the candidate. In this case, firms compete both by using their acceptance rule and by making wage offers. I show that the test offered in any symmetric equilibrium is maximally difficult. This is because there is positive selection into harder tests under wage competition. When firms can make wage offers, competition is fierce not on the acceptance rule but on the wages firms offer: firms do not over-accept but they over-pay. Therefore, candidates only receive an offer after a high signal. In this case, higher productivity candidates benefit relatively more from a more difficult test as they are relatively more likely to receive a high signal, leading to positive selection into harder tests.

In the second extension, firms face capacity constraints and they do not make wage offers. Again, I show that capacity constraints lead firms to use more difficult tests in equilibrium. One important factor behind this result is that when firms are at capacity, they cannot benefit from undercutting their competitor. Therefore, firms only accept after a high signal in equilibrium and selection into a more difficult test is positive. At the extreme, when the capacity constraint is severe, the firms use the most difficult feasible test in equilibrium.

Wage competition and capacity constraints show how selection into the selection procedure can vary depending on the type of admissions procedures firms conduct and how it affects the qualitative properties of tests used in equilibrium. When firms are not constrained by capacity and have fixed wages, the test used in a symmetric equilibrium is 'lemon-dropping', i.e., the test is good at identifying low types. If firms face capacity constraints or compete using wages, the model predicts that firms will use 'cherry-picking' tests, i.e., tests good at identifying high types.

Finally, I explore the possibility of asymmetric equilibria in markets with fixed wages. In particular, I show conditions under which a *two-tier structure* can emerge in equilibrium, i.e., an equilibrium where a selective firm only attracts high types and a safe firm attracts lower types. I study this question in the context of binary types. I show that if firms do not face any capacity constraints, there cannot be any two-tier structure equilibria. On the other hand, when there are capacity constraints, it is possible to construct a two-tier structure equilibrium for some parameter values. In that equilibrium, the selective firm chooses a test that is either more accurate or more difficult than the safe firm. Therefore, ex-ante identical firms can become ex-post vertically differentiated endogenously through their choice of selection procedure.

1.1 Relation to the literature

This paper introduces a model of competition where firms compete by posting selection procedures. When choosing their selection procedure, the firms must consider two key channels: the statistical properties of their test and the selection into their selection procedure. I show that natural orders related to the statistical properties of test have a strong implications on selection. Under the assumption of perfect competition, the selection effect determines the nature of tests in equilibrium. I also make predictions on the nature of the tests used depending on the primitive of the model. Finally, I show that ex-ante identical firms can be ex-post differently productive by using different selection procedures.

This paper relates to the literature studying competitive markets with private information (e.g., Rothschild and Stiglitz, 1976; Peters, 1997; Guerrieri et al., 2010; Auster and Gottardi, 2019). This literature typically assumes that the firms can flexibly design a mechanism or a contract subject to incentive-compatibility constraints. Instead, in this paper, the firms have limited set of feasible tests but they do not need to satisfy any incentive-compatibility constraints to reveal information. This approach allows me to study how the statistical properties of the tests interact with the strategic choice of the agents. It is also worth noting that absent any test and with the payoffs assumed, the firms could never elicit any information in an incentive-compatible way. Therefore, the firms need hard information through tests to inform their decision.[3]

There is a small literature that studies the design of selection procedures where strategic choice from applicants play a key role. Chade et al. (2014) study a competitive markets for admission procedures in the university context. They consider a fixed testing technology and analyse a game where universities and students make their decisions simultaneously. In this paper, I endogenise the testing technology and make it an additional instrument for competition. Another important difference is the fact that universities and candidate move simultaneously. This timing changes how selection operates, as students cannot respond to changes in university policies. Adda and Ottaviani (2024) and Alonso (2018) are two papers that study how changing statistical properties of tests changes candidates' application behaviour. Adda and Ottaviani (2024) study how changing the accuracy of the grant evaluation, in the sense of Lehmann (1988), affects participation.[4] Alonso (2018) examines the choice

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[3]Here, communication cannot help the firm even in the presence of tests, see Hancart (2024), Silva (2024) or Weksler and Zik (2022).

[4]Adda and Ottaviani (2024) also study competition between fields as they can adjust the accuracy of grant

of selection procedure in a labour market setting with horizontally differentiated workers and wage bargaining. In his paper, workers differ in their fit for the firms and one firm's selection procedure is fixed while the other can adjust it. Both Adda and Ottaviani (2024) and Alonso (2018) only consider changes to the accuracy of the test and do not consider other statistical properties like difficulty.

Finally, there is a literature on information intermediaries that study models where certifier(s) can disclose the quality of an agent, e.g., Lizzeri (1999); Harbaugh and Rasmusen (2018); Asseyer and Weksler (2024). In a somewhat related context to this paper, there is also a literature that models education system as intermediaries (Ostrovsky and Schwarz, 2010; Boleslavsky and Cotton, 2015; Bizzotto and Vigier, 2024). In the case where the agent is privately informed (e.g., Lizzeri, 1999; Harbaugh and Rasmusen, 2018), selection into the certifier also play a key role in these papers. These information design problems are however different as the intermediaries are not trying to select candidates but to collect a fee or reveal something about them.

2 Model

There are two firms and a continuum of agents with mass normalised to one. Each agent has a private type which corresponds to his value to the firms $\theta \in \Theta = [\underline{\theta}, \overline{\theta}]$. I assume that types are distributed independently across agents according to the cdf F admitting a strictly positive density f. There is an exogenous set of binary tests $T \subseteq \Pi := \{\pi : \Theta \to \Delta\{l, h\}\}$. The conditional probabilities of test t are denoted by $\pi_t(\cdot|\theta)$. To simplify notation, I denote by $\pi_t(\theta)$ the probability that type θ sends signal h. The set T captures the restriction on the firms' testing capacity.

Assumption 1. *Each test is monotone in the type: $\pi_t(\theta)$ is increasing.*

Each test is interior almost everywhere: $\pi_t(\theta) \in (0, 1)$ for $\theta \in (\underline{\theta}, \overline{\theta})$.

Assumption 1 guarantees that a high signal is good news about the type. The assumption that each test is interior rules out that any set of types with positive measure can be identified or excluded from observing a signal.

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evaluation.

This model can incorporate discrete types by having the tests measurable only with respect to a discrete set of types. In particular, we will be able to give more complete results for the case of binary types.

Definition 1. *The set of feasible tests T induces binary types if there is θ^* such that $\pi_t(\cdot|\theta) = \pi_t(\cdot|\theta')$ whenever $\theta, \theta' > \theta^*$ or $\theta, \theta' \leq \theta^*$.*

Whenever T induces binary types, then abusing notation, we can set $\underline{\theta} = \mathbb{E}[\theta|\theta \leq \theta^*]$ and $\overline{\theta} = \mathbb{E}[\theta|\theta > \theta^*]$. I will denote by μ the mass of $\theta > \theta^*$.

The firms post simultaneously an admission procedure $s \in S$, which I will describe in details below. After observing the admission process, agents decides whether to apply to firm 1 or 2. Denote by $\sigma : \theta \times S \times S \to [0, 1]$ the probability agent θ chooses firm 1 given the selection procedures.

The firms can decide whether to admit the agents, $a \in \{0, 1\}$. The agents have payoffs $u(a) = a$, i.e., they want to be accepted. An admission procedure is a test $t \in T$ and a decision rule, $\alpha : \{h, l\} \to [0, 1]$, a mapping from the signal to a probability of accepting: $s = (t, \alpha)$.

Firm 1's payoffs are

$$v(s, s', \sigma) = \int_{\Theta} \sigma(s, s', \theta)\theta\Big(\pi_t(\theta)\alpha(h) + (1 - \pi_t(\theta))\alpha(l)\Big)dF.$$

Firm 2's payoffs are defined analogously. The firm cares both about how many agents it attracts and their quality. It also assumes there is no capacity constraint for the firm. This captures the idea that the supply of agents is smaller than the total demand and therefore the two firms must compete to attract them. I introduce capacity constraints in Section 4.1.

Call $T_i = \{t \in T : \int_{\Theta} \theta (1 - \pi_t(\theta))dF \leq 0 \leq \int_{\Theta} \theta \pi_t(\theta)dF\}$ the set of *minimally informative tests*. These are all the test that generate payoff relevant information for the firms.

I consider subgame-perfect equilibria of this game where agents break ties uniformly and firms use pure strategies.

2.1 Orders on tests

I will make use of two natural partial orders on tests that will be useful for the analysis of this model: accuracy (Lehmann, 1988) and difficulty (Hancart, 2024).

Lehmann's (1988) accuracy order captures a notion of informativeness of a test. It allows for the comparison of more experiments than Blackwell's (1953) order when the decision problem has some monotonicity properties. Accuracy is a concept defined on continuous probability distribution. For each test t, we can define an equivalent experiment on the signal space $\tilde{X} = [0, 2]$ with the cdf $\tilde{F}_t(x|\theta) = x(1 - \pi_t(\theta))$ if $x \in [0, 1]$ and $\tilde{F}_t(x|\theta) = 1 - \pi_t(\theta) + (x - 1)\pi_t(\theta)$ otherwise. This is equivalent to having a uniform random variables independently drawn after the discrete signal realisation l, h with corresponding numerical values $0, 1$.

Definition 1 (Lehmann (1988)). *A test t is more accurate than a test d, $t \succeq_a d$, if*

$$x^*(\theta, x) \equiv \tilde{F}_t(x^*|\theta) = \tilde{F}_d(x|\theta)$$

is weakly increasing in θ for each $x \in [0, 2]$.

Lehmann (1988) showed that if payoffs satisfy a monotonicity condition, then a test t being more accurate than d implies that the payoffs of any decision-maker using test t is higher than if he were to use d. This single-crossing property holds in our model. Note that if a test t is more informative than d, in the sense of Blackwell (1953), then t is more accurate than d but not the other way around.

The second notion I will use is the notion of difficulty introduced in Hancart (2024). I define it as follows:

Definition 2. *A test t is more difficult that d, $t \succeq_d d$, if for $x = h, l$ and $\theta > \theta'$,*

$$\pi_t(x|\theta)\pi_d(x|\theta') \geq \pi_d(x|\theta)\pi_t(x|\theta')$$

If t is more difficult than d, I will also say that d is easier than t. Hancart (2024) shows that this definition is equivalent to having the posterior beliefs of test t first-order stochastic dominate the posterior beliefs of test d for any prior. Let $\mu(\cdot|t, x)$ denote the posterior beliefs in test t after signal x and $\succeq_{FOSD}$ the first-order stochastic dominance order.
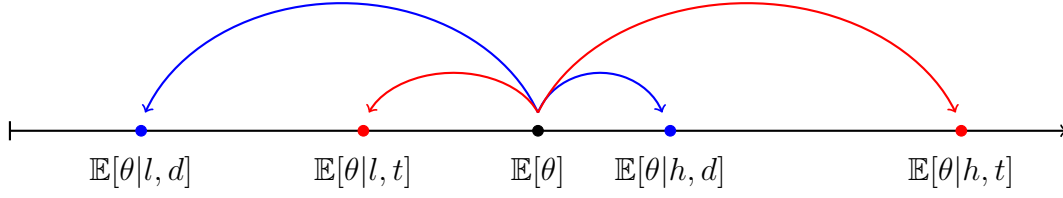
9

Figure 1: Illustration of posterior means for two tests, $t \succeq_d d$. The good news signal h shifts the posterior towards a higher posterior mean in the more difficult test. The bad news signal l shifts the posterior towards lower posterior mean in the easier test.

Proposition 1 (Hancart (2024)). *A test t is more difficult than d if and only if $\mu(\cdot|t, x) \succeq_{FOSD} \mu(\cdot|d, x)$ for $x = h, l$ for any prior (including non full-support).*

Intuitively, the difficulty order captures the following intuitive property of difficulty. When a test is more difficult than another, a high grade is shifts beliefs more towards high type in the more difficult test. This is because receiving a high grade is harder in a difficult test. On the other hand, after a low grade, beliefs are more pessimistic in the easier test as only bad candidates fail an easy test whereas it is 'normal' to fail a difficult test. This is illustrated in Figure 1. Proposition 1 guarantees that this intuition holds for all priors and that the difficulty order as defined is the only one that guarantees this property holds for any prior.

We also record the following result on test comparable in terms of difficulty.

Lemma 1. *Suppose test t is more difficult than d. Then for all $\theta \in \Theta$, $\pi_t(\theta) \leq \pi_d(\theta)$.*

All proofs are in appendix A

Lemma 1 shows that the difficulty order leads to the natural property that high signals are less likely in a more difficult test.

When two tests are comparable in terms of difficulty, they are not in terms of accuracy, except in knife-edge cases where the likelihood ratios are constant. In appendix B, I also show that if we evaluate the cost of a test using continuous and posterior-separable cost function (Caplin et al., 2022), for any given test, we can find another test comparable in the difficulty order with equal cost. I provide examples of tests comparable in terms of accuracy and difficulty below.

Examples (Accuracy). Take a test t: $\pi_t(\theta) \in (0, 1)$.

1. Let $\pi_d(\theta) = \beta\pi_t(\theta) + (1 - \beta)c$ with $\beta, c \in [0, 1]$. Then $t \succeq_a d$.

2. Assume that π_t is differentiable in θ. Define d with $\frac{\partial\pi_d(\theta)}{\partial\theta} = c\frac{\partial\pi_t(\theta)}{\partial\theta}$ with $c \in [0, 1)$ and $\pi_d(\underline{\theta}) = \pi_t(\underline{\theta})$. Then $t \succeq_a d$.

These two examples show instances where one test is more accurate than another. In the first one, the less accurate test is obtained by mixing the original test with an uninformative test. In the second example, the more accurate test is more responsive to the type than the less accurate one.

Examples (Difficulty). Take a test t: $\pi_t(\theta) \in (0, 1)$.

1. Let $\pi_d(\theta) = \pi_t(\theta) + c$ with $c > 0$ and $\pi_d(\theta) \in (0, 1)$. Then $t \succeq_d d$.

2. Let $\pi_d(\theta) = (\pi_t(\theta))^c$ with $c \in (0, 1)$. Then $t \succeq_d d$.

3. Let $\pi_d(\theta) = \mathbb{1}[\theta \geq d]$ for $d \in \Theta$. Then the difficulty of the test is increasing in d.

The two examples of tests ordered by difficulty show how to modify the original test to make all types more likely to obtain the high signal and respecting the definition of difficulty. The last example shows how a deterministic 'certification' test is in line with the difficulty order.

Whenever the type space and the signal space are binary, Blackwell and Girshick (1954, Section 12.5) show that Blackwell informativeness reduces to

$$t \succeq_a d \Leftrightarrow \pi_t(h|\bar{\theta})\pi_d(h|\underline{\theta}) \geq \pi_d(h|\bar{\theta})\pi_t(h|\underline{\theta}) \text{ and } \pi_d(l|\bar{\theta})\pi_t(l|\underline{\theta}) \geq \pi_t(l|\bar{\theta})\pi_d(l|\underline{\theta}).[5]$$

This implies that when T induces binary types, any pair of tests is comparable either in terms of accuracy or in terms of difficulty, or both.

Finally, we can find families of tests that are indexed by both their accuracy and difficulty:

1. Let $\pi_{d,\sigma} = \left(\frac{\sigma}{2} + \frac{1-\sigma}{\bar{\theta}-\underline{\theta}}(\theta - \underline{\theta})\right)^d$ where $d > 0$ and $\sigma \in [0, 1]$. Then the difficulty is increasing in d and the accuracy is decreasing in σ.

2. Let $\pi_{d,\sigma}(\theta) = \sigma\pi(\theta) + (1 - \sigma)d$ where $\pi(\theta) \in (0, 1)$ and is increasing and $\sigma, d \in [0, 1]$. Then the difficulty of the test is decreasing in d and the accuracy is increasing in σ.

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[5]This result is under the assumption that the tests are monotonic.

3. Let $\pi_{d,\sigma}(\theta) = 1 - \frac{d-\theta}{\sigma}$ with $\underline{\theta} > d - \sigma$ and $\overline{\theta} < d$. Then the difficulty of the test is increasing in d and the accuracy is decreasing σ.

The first two examples are combination of the examples above. The last example is obtained in the following way. Suppose that $y = \theta + \sigma\epsilon$ where $\epsilon \sim U[0,1]$. Then $\pi_{d,\sigma}(\theta) = \Pr[y \geq d|\theta]$. That is, we draw a random signal y whose noise is governed by σ. Then the test generates a high signal whenever the signal y is higher than the difficulty level d.[6]

3 Analysis

I first show that competition leads firms to "over-accept" candidates in the sense that they reward the low signal even though it has negative posterior expected productivity. In any symmetric equilibrium, the payoffs of the firms are no different than if they could not observe any signals.

Lemma 2. *Suppose Assumption 1 is satisfied. In any symmetric equilibrium $s = (t, \alpha)$, firms' profits are* $\max\{0, \frac{1}{2}\mathbb{E}[\theta]\}$.

- *If $\mathbb{E}[\theta] \geq 0$, $\alpha(h) = \alpha(l) = 1$.*

- *If $\mathbb{E}[\theta] < 0$ and $\int_{\Theta} \theta\pi_t(\theta)dF > 0$, $\alpha(h) = 1$, $\alpha(l) > 0$.*

The intuition for Lemma 2 is the familiar Bertrand undercutting logic. If firms make positive profits, they can relax their acceptance rule and attract all candidates. If the increased probability of acceptance is small enough the firms profits are larger than when sharing the market with the other firm. Therefore in equilibrium firms accept candidates until they make zero profits or they accept any candidates applying. This means that the firms make exactly the same profits as if they could not collect any information about candidates.

I now turn to the characterisation of the tests used in equilibrium. Specifically, I will use the two orders we have introduced earlier to determine the properties of equilibrium selection procedures.

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[6]Note that provided that the noise distribution satisfies increasing hazard rate and decreasing inverse hazard rate, the difficulty is decreasing in d. This is the case for the uniform or exponential distribution. It is however not true that the test's accuracy is decreasing in σ for a large class of noise distributions. The reason is that if the test has $\sigma = 0$, then the test does not discriminate between types that are above or below the threshold d. If the DM's prior is concentrated above or below the threshold then this test does not reveal any useful information to the DM.

First, I show that the test used in equilibrium must be maximal in the accuracy order. Recall that T_i is the set of minimally informative test. In particular, for any test in $t \in T_i$, $\int \theta \pi_t(\theta) dF \geq 0$.

Proposition 2. *Suppose $\mathbb{E}[\theta] < 0$, Assumption 1 is satisfied and there is $t \in T_i$.*

Let $s = (t, \alpha)$ be a strategy used in a symmetric equilibrium. Then there is no test $t' \in T$ such that t' is more accurate than t.

In addition, if t is more accurate than all $t' \in T$, then there is a unique symmetric equilibrium.

The first part of Proposition 2 comes from the positive selection into a more accurate test. This is almost immediate from the definition of accuracy and the formal argument is in the proof Proposition 2. Intuitively, higher productivity agents benefit relatively more from a more accurate test no matter what strategy is employed. To show the second part, I use that the payoff difference between the more and the less accurate test exhibits decreasing differences. This implies that starting from the most accurate test and deviating to a less accurate test the increase in acceptance probability is always larger for low productivity types than higher ones. Combined with the fact that profits are zero at the candidate equilibrium, this implies that there is no profitable deviation to a less accurate test.

Next, I characterise equilibria when tests are ordered by difficulty.

I say that T is rich in the difficulty dimension if for any $t, t' \in T$ such that $t \succeq_d t'$, the set $\{\int \theta \pi_d(\theta) dF : t \succeq_d d \succeq_d t'\}$ is an interval. All examples provided in Section 2.1 satisfy this condition.

Proposition 3. *Suppose $\mathbb{E}[\theta] < 0$ and Assumption 1 is satisfied. Suppose also that any $t, t' \in T$ are not comparable in accuracy and T is rich in the difficulty dimension.*

If $s = (t, \alpha)$ is a strategy used in a symmetric equilibrium, then there is no test $t' \in T_i$ such that t' is easier than t.

If t is easier than all $t' \in T_i$ and more difficult than all $t' \in T \setminus T_i$, then there is a unique symmetric equilibrium where firms use test t.

The assumption that the set of feasible test T is rich is only needed in the case where there is a test in $t \in T \setminus T_i$ that is easier than the tests in T_i. The assumption that tests are not comparable in accuracy will be relaxed in Corollary 1 when we combine the insights of Proposition 2 and Proposition 3.

13

To prove Proposition 3, I show that selection into an easier test in equilibrium is positive. Consider the case where both firms use the easiest minimally informative test in equilibrium and consider a deviation to a harder test. To make this deviation successful, firms must make their acceptance rule more lenient as no candidate would want to choose a more difficult with harsher acceptance rule. But when raising the acceptance probability at low signals in the more difficult test, the agents most likely to benefit from that selection procedure are low productivity agents. Indeed, they are the agents most likely to generate low signals. Therefore, the selection into the more difficult *and* more lenient acceptance rule is negative.

When the set of feasible test is rich and there are tests in $T \setminus T_i$, the easiest test amongst the minimally informative ones has $\mathbb{E}[\theta|t, h] = 0$, i.e., the posterior expectation at the high signal is zero. In equilibrium, the firms set $\alpha(l) = 0$, i.e., a low signal is no longer rewarded. In this case, the selection into an easier test is negative as high productivity candidate are more likely to generate a high signal and are therefore more inclined to choose the harder test.

If in equilibrium $\alpha(l) = 0$, it also means that firms best-reply to their signals in equilibrium. Therefore, in that case, the equilibrium we have found is also an equilibrium of a game with an alternative timing where firms decide whether to accept after the candidate application and having seen the test result.

Proposition 2 and Proposition 3 show existence results when tests are fully ordered by either accuracy or difficulty. This restriction is not necessary. The following definition combines the comparison in terms of difficulty and accuracy.

Definition 2. *Say that $t \succeq_1 d$ if either (i) $t \succeq_a d$ or (ii) $t \preceq_d d$ with $\frac{1-\pi_t(\theta)}{1-\pi_d(\theta)}$ not constant in θ.*

A test t is accuracy/ease connected to d, $t \succeq_1^* d$, if there is sequence $(t_i)_{i=1}^n \in \Pi^n$ such that $t_1 = d$, $t_n = t$ and $t_i \preceq_1 t_{i+1}$ for $i = 1, ..., n - 1$.

Say that $t \succeq_2 d$ if either either (i) $t \succeq_a d$ or (ii) $t \succeq_d d$.

A test t is accuracy/difficulty connected to d, $t \succeq_2^* d$, if there is sequence $(t_i)_{i=1}^n \in \Pi^n$ such that $t_1 = d$, $t_n = t$ and $t_i \preceq_2 t_{i+1}$ for $i = 1, ..., n - 1$.

The notions of accuracy/easy and accuracy/difficulty connectedness allow comparison between tests that are not comparable in neither dimension. In words, a test t is accuracy/ease connected to d if we can find a sequence of tests where along the sequence, tests are either more accurate or easier. Similarly, a test t is accuracy/difficulty connected to d if we can find of tests where along the sequence, tests are either more accurate or more difficult. Note that

the tests in the sequence need not be feasible. Figure 2 illustrating Corollary 1 below can help visualise accuracy/easy and accuracy/difficulty connectedness.

These notions deliver the following existence results. We adapt our notion of richness to the order just defined. I say that T is rich in the accuracy/ease dimension if for any $t, t' \in T$ such that $t \succeq_1^* t'$, the set $\{\int \theta \pi_d(\theta) dF : t \succeq_1^* d \succeq_1^* t'\}$ is an interval. Again all the examples provided in Section 2.1 satisfy this property.

Corollary 1. *Suppose $\mathbb{E}[\theta] < 0$ and Assumption 1 is satisfied. Suppose also that there is $t \in T_i$ and that T is rich in the accuracy/ease dimension.*

Assume there is $t \in T$ such that $t \succeq_1^* d$ for all $d \in T$.

If the test $t \in T_i$, then there exists a symmetric equilibrium where t is used.

If the test $t \notin T_i$ and there is a test $t' \in T_i$ such that for all $d \in T_i$, $t' \succeq_1^* d$ and for all $d \notin T_i$, either $t' \succeq_1^* d$ or $t' \succeq_2^* d$, then there exists a symmetric equilibrium where t' is used.

As in Proposition 3, the assumption that T is rich is only necessary if $t = \max_{\succeq_1^*} T \notin T_i$. Its role is to make sure that the whole span of posterior expectations are possible along the accuracy/ease dimension.

Corollary 1 is illustrated in Figure 2. Corollary 1 combines the ideas of Proposition 2 and Proposition 3 to show existence in this more general setting. It provides conditions under which the test that is maximal in accuracy and minimal in difficulty among the informative tests constitutes an equilibrium. Therefore, equilibrium existence does not depend on tests being fully ordered by accuracy or difficulty. The examples in Section 2.1 where both accuracy and difficulty can be adjusted smoothly satisfy the conditions of Corollary 1.

3.1 Additional structure on the feasible tests

3.1.1 Binary types

In the case where T induces binary types, each test is comparable in terms of either accuracy or difficulty. Therefore, each pair of test is comparable in terms of accuracy/ease. Therefore, provided there exists a least accurate or easiest test, an equilibrium always exists. One sufficient condition is the following. When T induces binary types, each test can be described

15
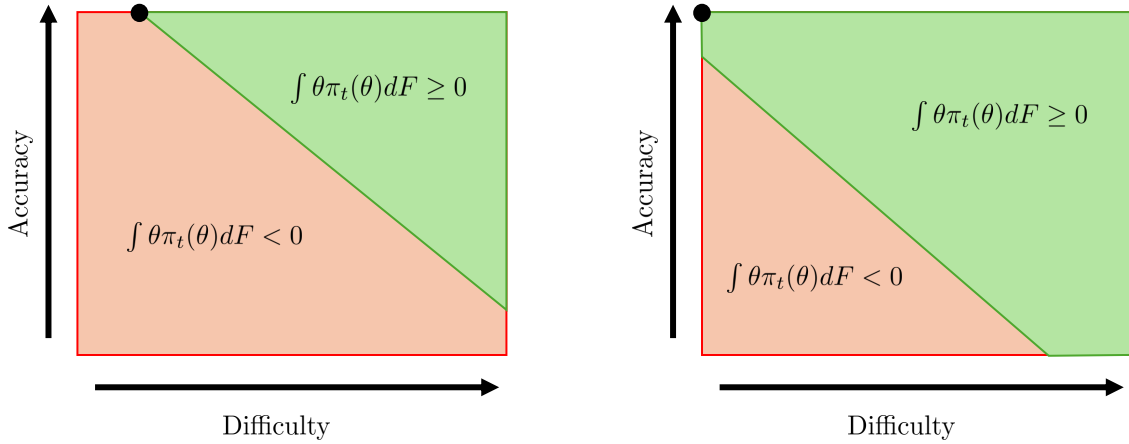
(a) $\succeq_1^*$-maximal test is not minimally informative

(b) $\succeq_1^*$-maximal test is minimally informative

Figure 2: Each point in the rectangle represents a test. A test is minimally informative if it is accurate or difficult enough. A test t is accuracy/ease connected to d if t is north-west of d. A test t is accuracy/difficulty connected to d if t is north-east of d. The black dot indicates the test used in symmetric equilibrium.

as a pair of likelihood ratios: $(\frac{\pi(\bar{\theta})}{\pi(\underline{\theta})}, \frac{1-\pi(\bar{\theta})}{1-\pi(\underline{\theta})}) \in [1, \infty) \times (0, 1]$ with $\pi(\bar{\theta}) \geq \pi(\underline{\theta})$. I say that T is closed and convex if the set of tests interpreted as a subset $[1, \infty) \times (0, 1]$ is closed and convex.

Corollary 2. *Suppose T induces binary types and is closed and convex. Then an equilibrium exists. In that case, the test used in the symmetric equilibrium is*

$$\arg\max\{\frac{\pi_t(\bar{\theta})}{\pi_t(\underline{\theta})} : t \in \arg\min_{d \in T_i} \frac{1-\pi_d(\bar{\theta})}{1-\pi_d(\underline{\theta})}\}.$$

The test used in equilibrium is the one with smallest likelihood ratio at the low signal amongst the minimally informative tests: $\arg\min_{d \in T_i} \frac{1-\pi_d(\bar{\theta})}{1-\pi_d(\underline{\theta})}$. If there are multiple, then we select the one that is maximally accurate amongst them, i.e., with the largest likelihood ratio at the high signal. The test selected will be either more informative or easier than any other test amongst the minimally informative ones.

16

3.1.2 Cost constraint

For a given test t, let $\overline{\pi}_t = \int_\Theta \pi_t(\theta) dF$. We define the cost as follows let $f_{th}(\theta) = f(\theta)\frac{\pi_t(\theta)}{\overline{\pi}_t}$ and $f_{tl}(\theta) = f(\theta)\frac{1-\pi_t(\theta)}{1-\overline{\pi}_t}$. I assume that the cost associated with test t is posterior separable (Caplin et al., 2022):

$$C(t) = \overline{\pi}_t c(f_{th}) + (1 - \overline{\pi}_t)c(f_{tl}),$$

where $c : \Delta\Theta \to \mathbb{R}$ is a strictly convex and continuous function.[7] This class of cost function includes many commonly used cost functions including mutual information cost or log-likelihood ratio cost (Sims, 2003; Pomatto et al., 2023). The representation in terms of posterior beliefs will also be convenient in the proofs. I make the further assumption that the test cannot rule out any state: if $\pi_t(\theta) \in \{0,1\}$ for some θ then $\pi_t(\theta') = \pi_t(\theta)$ for all θ'. Or put differently, if $\pi_t(\theta) \in \{0,1\}$ for some θ and the test is informative then the cost is infinite.

For any $\kappa > 0$, let $T_c(\kappa) = \{\pi : \Delta\Theta \to \Delta\{h, l\} : \pi$ satisfies Assumption 1 and $C(\pi) \leq \kappa\}$. Posterior separable cost functions depend implicitly on the prior belief over types. Here I define the costs at the prior belief f and I will keep that fixed, even though the firm might have additional information through selection prior to administrating the test. The interpretation of this constraint is that the firms have to pay a cost to design the test, captured by C. But once the test is designed, there are no further costs.

Proposition 4. *Suppose $\mathbb{E}[\theta] < 0$. Suppose $T_c(\kappa) \cap T_i$ is non-empty. In any symmetric equilibrium, $s = (t, \alpha)$, we have*

$$\int_\Theta \theta \pi_t(\theta) dF = 0.$$

and $C(t) = \kappa$.

The proof works by showing that for any test with $\int_\Theta \theta \pi_t(\theta) dF > 0$, it is possible to find an easier test that still has positive posterior expected productivity at the high signal and is less costly. Therefore by Proposition 3, t cannot be part of a symmetric equilibrium. The budget constraint is binding because C is increasing in the Blackwell order. Therefore, if the budget constraint is not binding, there is test that is more informative (and thus more accurate) and feasible. That contradicts Proposition 2.

---

[7]I endow the set $\Delta\Theta$ with the weak* topology of. In this topology, a sequence $(F_n)_n$ converges to F if for any continuous bounded $\phi : \Theta \to \mathbb{R}$, $\int \phi(\theta) dF_n(\theta) \to \int \phi(\theta) dF$.

Given the flexibility in the design of the test with these constraints, the proof sketched above is not the only one. However, it does illustrate how the results derived above can be useful in a more flexible environment. Moreover, it shows that the results does not rely on the extreme flexibility of the set of feasible tests but rather on the monotonicity in the Blackwell order and the possibility of finding isocost tests comparable in difficulty.

4 Extensions

4.1 Capacity constraint

As in the analysis of markets with Bertrand competition, capacity constraints can radically change the predictions of our model. The reason is that capacity constraints shift the focus of firms from both the quality and the quantity of applicants to the quality only. This gives incentives to use more difficult tests in equilibrium.

I model capacity constraints as follows. Each firm has a capacity $k > 0$, i.e., in equilibrium we must have $\int_{\Theta} \sigma(s, s', \theta) \left(\alpha(l)(1 - \pi_t(\theta)) + \alpha(h)\pi_t(\theta) \right) dF \leq k$.

The timing of the game is the following

1. Firms post selection procedures simultaneously

2. Candidates choose where to apply

3. Candidates applying to firm i form a queue whose order is random. Firm i treats applications sequentially until it exhausts the pool of applicants or hits its capacity constraint.

The solution concept is still symmetric subgame equilibrium where firms use pure strategies and agents break ties symmetrically.

Given the strategy of the agents σ, let $p_1 = \min\{1, \frac{k}{\int_{\Theta} \sigma(s, s', \theta)(\alpha(l)(1 - \pi_t(\theta)) + \alpha(h)\pi_t(\theta))dF}\}$. The payoffs of firm 1 in equilibrium are

$$p_1 \cdot \int_{\Theta} \sigma(s, s', \theta)\theta \left(\alpha(l)(1 - \pi_t(\theta)) + \alpha(h)\pi_t(\theta) \right) dF.$$

18

I illustrate the effect of capacity constraints in the extreme case where firms don't even have the capacity to accept all types that get a high signal in the most difficult test.

Proposition 5. *Assume Assumption 1 is satisfied. Suppose there is a test t such that $t \succeq_d t'$ for all $t' \in T$ and that $k < \frac{1}{2} \int_\Theta \pi_t(\theta)dF$.*

There is a symmetric equilibrium $s = (t, \alpha)$, i.e., both firms offer the most difficult test.

We get the inverse prediction than in the case with no capacity constraints. The reason is that capacity constraints limit the scope for undercutting the competitor: once the firms are at capacity, there is no benefit to increasing the acceptance probability to attract all the candidates. Given the assumption on the capacity constraint, there is no incentive to accept after a low signal. When the low signal is not rewarded, the selection into the easier test is negative. Moreover, only the expected productivity of the accepted candidates matter for the payoffs as the firms are at capacity. From Proposition 1, a more difficult test always has a higher posterior expected value. Therefore a deviation to an easier test results both in negative selection and lower expected value, even absent any selection effect.

Proposition 5 gives us the following (informal) comparative statics result:

Observation 1. *Firms facing a capacity constraint use more difficult tests in equilibrium than firms not facing a capacity constraint.*

This result predicts that whether the firms face capacity constraints or not will affect the qualitative nature of the tests they use. Loosely speaking, a firm facing a tight capacity constraint will use a difficult selection procedure, a 'cherry-picking' type of test whereas a a firm without capacity constraint will use easier tests, a 'lemon-dropping' type of test.

4.2 Asymmetric equilibria

A natural question in this context is whether different selection procedures can generate differentiation amongst firms despite being ex-ante identical. In particular, I will look at whether a two-tier structure can emerge in equilibrium where one firm attracts all candidates above a threshold and the other firm attracts candidates below the threshold. To answer this question, I specialise the setting to a binary type model. I show that a two-tier structure never emerges when there are no capacity constraints but that it can when there are.

As the equilibrium construction in Proposition 6 below will use mixed strategies, I no longer require that agents break ties uniformly. I do require however that any types generating the same distribution over signals use the same strategy. If there are capacity constraints, the timing and rationing is as defined in Section 4.1.

A *two-tier equilibrium* is an equilibrium where only types above a threshold apply to a firm. I call the firm where only types above the threshold apply the selective firm and the firm where types below the threshold apply the safe firm.

Proposition 6. *Suppose T induces binary types. Let $\bar{t} \in \arg\max_{t \in T} \frac{\pi_t(\bar{\theta})}{\pi_t(\underline{\theta})}$ and assume such $\bar{t}$ exists.*

If there are no capacity constraints, then there are no two-tier equilibria.

If there are capacity constraints, there is a two-tier equilibrium where the selective firm chooses $\bar{t}$ and the safe firm chooses some other test $t \in T$ if

$$
\underline{\theta} \geq 0,
$$
$$
(1 - \mu)\pi_{\bar{t}}(\underline{\theta}) \geq \mu\pi_{\bar{t}}(\bar{\theta}),
$$
$$
\frac{\mu\pi_{\bar{t}}(\bar{\theta}) + (1 - \mu)\pi_{\bar{t}}(\underline{\theta})}{2} \geq k,
$$
$$
and \quad \frac{\pi_t(\underline{\theta})}{2} \geq k.
$$

In equilibrium, $\bar{\theta}$ chooses the competitive firm and $\underline{\theta}$ chooses the safe firm with probability $\frac{\mu\pi_{\bar{t}}(\bar{\theta}) + (1-\mu)\pi_{\bar{t}}(\underline{\theta})}{2(1-\mu)\pi_{\bar{t}}(\underline{\theta})}$.

The reason two-tier equilibria cannot exist without capacity constraints is that as long as the safe firms make positive profits, the selective firm has an incentive to lower its standards to attract more candidates. Here decreasing standards corresponds to offering a selection procedure with a lower ratio of likelihood of acceptance between high and low types. On the other hand, if the selective firm is at capacity, the benefits from decreasing standards could be limited.

With capacity constraints, the selective firm uses the test with the highest likelihood ratio at the high signal. That means that the test is either more accurate or more difficult than the other feasible tests. In equilibrium, the selective firm only accepts after a high signal. Under this strategy, the selection into the most difficult test is positive. Moreover, because in equi-

20

librium firms are at capacity, they cannot improve payoffs by simply lowering their standards and attracting more types. For example, when the selective firm decreases its standards, it increases the share of lower quality students applying to it, thereby decreasing its payoffs.

The equilibrium I construct has high types choosing the competitive firm and low types mixing between the competitive and the safe firm. The sufficient conditions in Proposition 6 reflect the equilibrium conditions to maintain that equilibrium. The first one, $\underline{\theta} \geq 0$ is necessary to make sure that the safe firm makes profits in equilibrium. The second condition ensures that the mixed strategy is feasible. The last two conditions guarantee that the capacity constraints of both firms are binding.

4.3   Wage competition

In this subsection, I consider the consequences of wage setting in this context. Formally, a firm can offer a positive transfer to the agent based on the signal it received: $m : \{h, l\} \to \mathbb{R}_+$. An admission procedure is a test, a decision rule and a transfer rule, $s = (t, \alpha, m)$. An agent's payoff is the transfer $a \cdot m$. Firm 1's payoffs are

$$v(s, \sigma) = \int_{\Theta} \sigma(s, s', \theta)\Big(\pi_t(\theta)\alpha(h)(\theta - m(h)) + (1 - \pi_t(\theta))\alpha(l)(\theta - m(l))\Big)dF.$$

For the next result, I will use a stronger notion of richness of the set of feasible tests. I say that T is *extra rich* in the difficulty dimension if for all θ and $t, t' \in T$ such that $t \succeq_d t'$, the set $\{\pi_d(\theta) : t \succeq_d d \succeq_d t'\}$ is an interval. Again, all examples provided in Section 2.1 satisfy this condition.

Proposition 7. *Suppose T is extra rich in the difficulty dimension and T_i is not empty. If there is $t \succeq_d t'$ for all $t' \in T$, then in any symmetric equilibrium $s = (t, \alpha)$ and $\alpha(l) = 0$.*

The main consequence of wage setting is that competition moves from the admission rule dimension to the wage offered. This implies that in any symmetric equilibrium, agents get hired only after a high signal. This changes the selection effect of offering a more difficult test. Now, high productivity agents benefit relatively more from a more difficult test as they are more likely to get a high signal. Because of this positive selection into a more difficult test, firms can always deviate if there is a more difficult test available.

21

Here we cannot establish generally that there is an equilibrium where the most difficult test is chosen. A deviating firm to an easier test faces negative selection but can compensate by offering lower wages. Therefore whether a deviation is profitable depends on the specification of the feasible tests and the prior. For example, one can show that if $\theta \sim U[0, 1]$ and $T = \{t : \pi_t(\theta) = \sigma\theta + (1 - \sigma)d, \ d \in [0, 1]\}$, then an equilibrium exists.

This second extension gives us a second informal comparative statics result:

Observation 2. *If firms compete using wages, they use more difficult tests in equilibrium than when they can only compete using admission probability. Moreover, the hiring probability is lower for almost all types.*

As in the case of capacity constraints, firms offer a more difficult test in equilibrium when they can compete using wage offers. Because in equilibrium they only accept after a high signal, it also implies that the probability of accepting any given type decreases.[8]

5 Conclusion

I have introduced a new model of competition where firms compete by posting selection procedures. They key channel I explored is how statistical properties of the tests imply different strategic choices from the tested agents. In particular, I showed that two natural orders on tests, accuracy and difficulty, create single-crossing utility differences for the agent. This led to positive or negative selection into a test that in turn determined the equilibrium.

The model makes some predictions the qualitative nature of the tests used in equilibrium depending on the primitives of the game. In the absence of capacity constraints, the firms use maximally accurate but the easiest test that is minimally informative. We can interpret this as maximal but misguided learning. In equilibrium, firms are very confident the type is of low quality after a low signal but their posterior expectation is barely high enough to make them accept the agent. On the other hand, when firms face capacity constraints or can compete using wages, they use more difficult tests in equilibrium.

I provide a number of equilibrium existence results. Technically, they rely on the agents' payoffs to have decreasing difference to make sure that any deviation to a another test benefits

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[8]Under capacity constraints, the probability of being accepted decreased as well but that was exogenously imposed by the capacity constraint.

more the low types than the high types. General results are harder to obtain and it is likely that symmetric equilibria in pure strategy do not exist for arbitrary domains of feasible tests. One contribution of this paper to identify families of tests that are both economically interpretable and where an equilibrium exists.

I see this model as a first step towards studying the effect of competition on the choice of tests. There are many natural extensions one would want to consider such as differentiated firms, both horizontally and vertically. Another interesting extensions would be introducing peer effects which is particularly relevant in a university admission context.

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A Proofs

A.1 Preliminary results

I first show the following preliminary lemma.

Lemma 3. *In any symmetric equilibrium, without loss of generality, we can focus on deviations that are cutoff strategies:* $\alpha(l) > 0 \Rightarrow \alpha(h) = 1$ *and* $\alpha(h) < 1 \Rightarrow \alpha(l) = 0$.

Proof. Take any selection procedure. Let α^* be the cutoff strategy that solves

$$\alpha(l)\pi_t(l|0) + \alpha(h)\pi_t(h|0) = \alpha^*(l)\pi_t(l|0) + \alpha^*(h)\pi_t(h|0).$$

Such strategy always exists. It is easy to verify that

$$\text{for } \theta < 0, \ \alpha(l)\pi_t(l|\theta) + \alpha(h)\pi_t(h|\theta) \geq \alpha^*(l)\pi_t(l|\theta) + \alpha^*(h)\pi_t(h|\theta),$$
$$\text{for } \theta > 0, \ \alpha(l)\pi_t(l|\theta) + \alpha(h)\pi_t(h|\theta) \leq \alpha^*(l)\pi_t(l|\theta) + \alpha^*(h)\pi_t(h|\theta).$$

This shows that for testing for any profitable deviation, it is without loss to only consider cutoff strategy. ☐

We also have the following lemma that characterise accuracy in terms of likelihood ratios. We will use this characterisation in Corollary 1.

Lemma 4. *A test t is more accurate than d if and only if for all $\theta > \theta'$,*

$$\frac{\pi_t(h|\theta)}{\pi_t(h|\theta')} \geq \frac{\pi_d(h|\theta)}{\pi_d(h|\theta')} \quad \text{and} \quad \frac{\pi_d(l|\theta)}{\pi_d(l|\theta')} \geq \frac{\pi_t(l|\theta)}{\pi_t(l|\theta')}.$$

Proof. A test t is more accurate than d $\Leftrightarrow$ for all θ, θ' with $\theta > \theta'$, the test t is Blackwell more informative than d when restricting attention to $\{\theta, \theta'\}$ (Jewitt, 2007) $\Leftrightarrow$ for all $\theta > \theta'$,

$$\frac{\pi_t(h|\theta)}{\pi_t(h|\theta')} \geq \frac{\pi_d(h|\theta)}{\pi_d(h|\theta')} \quad \text{and} \quad \frac{\pi_d(l|\theta)}{\pi_d(l|\theta')} \geq \frac{\pi_t(l|\theta)}{\pi_t(l|\theta')},$$

by Blackwell and Girshick (1954, Section 12.5). ☐

A.2 Proof Lemma 1

Proof. Suppose there is $\theta' \in \Theta$ such that $\pi_d(\theta') < \pi_t(\theta') \leq 1$. If $t \succeq_d d$, then for all $\theta > \theta'$,

$$\pi_t(\theta)\pi_d(\theta') \geq \pi_t(\theta')\pi_d(\theta).$$

Adding $\pi_d(\theta)\pi_d(\theta')$ on both sides, we obtain

$$\pi_d(\theta')(\pi_t(\theta) - \pi_d(\theta)) > \pi_d(\theta)(\pi_t(\theta') - \pi_d(\theta')) \Leftrightarrow \frac{\pi_t(\theta) - \pi_d(\theta)}{\pi_t(\theta') - \pi_d(\theta')} > \frac{\pi_d(\theta)}{\pi_d(\theta')}, \quad (1)$$

where we have used that $\pi_t(\theta') - \pi_d(\theta') > 0$.

Test t more difficult than d also implies

$$(1 - \pi_t(\theta))(1 - \pi_d(\theta')) \geq (1 - \pi_t(\theta'))(1 - \pi_d(\theta)).$$

Rearranging and adding $\pi_d(\theta)\pi_d(\theta')$ on both sides again, we obtain

$$(1 - \pi_d(\theta))(\pi_t(\theta') - \pi_d(\theta')) > (1 - \pi_d(\theta'))(\pi_t(\theta) - \pi_d(\theta)) \Leftrightarrow \frac{1 - \pi_d(\theta)}{1 - \pi_d(\theta')} > \frac{\pi_t(\theta) - \pi_d(\theta)}{\pi_t(\theta') - \pi_d(\theta')}.$$

Together with inequality (1), we can get

$$\frac{1 - \pi_d(\theta)}{1 - \pi_d(\theta')} > \frac{\pi_t(\theta) - \pi_d(\theta)}{\pi_t(\theta') - \pi_d(\theta')} > \frac{\pi_d(\theta)}{\pi_d(\theta')}.$$

This implies $\pi_d(\theta') > \pi_d(\theta)$, a contradiction. $\qquad\square$

A.3 Proof of Lemma 2

Proof. In any equilibrium, profits must be weakly positive for otherwise the firm can just set $\alpha(x) = 0$ for $x = h, l$ and increase profits.

Suppose first that $\mathbb{E}[\theta] \geq 0$ and suppose that $\alpha(l) < 1$. In any symmetric equilibrium $s = (\alpha, t)$, all agents choose either firm with probability $1/2$ and we have

$$v(s, \sigma) = \frac{1}{2} \int_\Theta \theta \Big(\pi_t(\theta)\alpha(h) + (1 - \pi_t(\theta))\alpha(l) \Big) dF > 0.$$

If one first sets $s' = (\alpha', t)$ with $\alpha'(l) = \alpha(l) + \epsilon$ and leave the test unchanged, almost all types prefer s' to s. The resulting profits are

$$\int_\Theta \theta\Big(\pi_t(\theta)\alpha(h)+(1-\pi_t(\theta))(\alpha(l)+\epsilon)\Big)dF > \frac{1}{2}\int_\Theta \sigma(s,s',\theta)\theta\Big(\pi_t(\theta)\alpha(h)+(1-\pi_t(\theta))\alpha(l)\Big)dF,$$

for ϵ small enough. Therefore $\alpha(l) = 1$. By a similar argument, we can establish that $\alpha(h) = 1$. We thus get that equilibrium profits are $\frac{1}{2}\mathbb{E}[\theta]$.

If $\mathbb{E}[\theta] < 0$, the same argument holds: as long as profits are strictly positive any firm can increase $\alpha(x)$ and have a strictly profitable deviation. As long as $\int_\Theta \theta\pi_t(\theta)dF > 0$, there will also always be an incentive to increase $\alpha(h)$. If $\int_\Theta \theta\pi_t(\theta)dF = 0$ then we must have $\alpha(l) = 0$ and we could have $\alpha(h) = 0$ in equilibrium. $\qquad\square$

A.4   Proof of Proposition 2

Proof. First note that for each strategy α and test d, there is a corresponding cutoff $x \in [0, 2]$ such that $\alpha(h)\pi_d(\theta) + \alpha(l)(1 - \pi_d(\theta)) = 1 - \tilde{F}_d(x|\theta)$. Because each test is monotonic in types, i.e., they have the monotone likelihood ratio property, we have $\tilde{F}_d(x|\theta) \le \tilde{F}_d(x|\theta')$ for any $\theta' < \theta$ and $x \in [0, 2]$.

Suppose there is a symmetric equilibrium with $s = (d, \alpha)$ and $d \prec_a t$ for some $t \in T$. Let x be the corresponding cutoff in the modified test.

From Lemma 2, it must be that firms' profits are zero. Take test t and find α_θ^* such that

$$\alpha(h)\pi_d(\theta) + \alpha(l)(1 - \pi_d(\theta)) = \alpha_\theta^*(h)\pi_t(\theta) + \alpha_\theta^*(l)(1 - \pi_t(\theta)).$$

(Recall that if $\alpha(l) > 0 \Rightarrow \alpha(h) = 1$.) Such α_θ^* exists using the intermediate value theorem and let $x^*(\theta)$ be the corresponding cutoff in the modified test. Observe that

$$1 - \tilde{F}_t(x^*(0)|\theta) \le 1 - \tilde{F}_t(x^*(\theta)|\theta) = 1 - \tilde{F}_d(x|\theta), \text{ for all } \theta \in (0, \bar{\theta})$$

The inequality is reversed for $\theta < 0$. Moreover, there must be some types for whom the inequality is strict as $d \prec_a t$. This implies that if one of the firms deviates to $s' = (t, \alpha_0^*)$, they achieve strictly positive profits.

Now let's show that if t is more accurate than all other tests, then $s = (t, \alpha)$ is a symmetric

28

equilibrium for some strategy α. By Lemma 2, we must have $\alpha(l) \geq 0$. To simplify notation let $\alpha(l) = \alpha$.

Let $d \in T$ such that $t \succ_a d$. Suppose one firm deviates to $s' = (d, \alpha')$. Let $\Delta u(\theta) = \alpha'(h)\pi_d(\theta) + \alpha'(l)(1 - \pi_d(\theta)) - \pi_t(\theta) + \alpha(1 - \pi_t(\theta))$. The argument above shows that this function must be single crossing.

The argument above already shows that we can divide the set of types in 3: $\Theta = \Theta_d \cup \Theta_i \cup \Theta_t$ where for any $\theta \in \Theta_d$, $\Delta u(\theta) > 0$, for any $\theta \in \Theta_i$, $\Delta u(\theta) = 0$ and for any $\theta \in \Theta_t$, $\Delta u(\theta) < 0$. Furthermore we have that if $\theta_d \in \Theta_d$, $\theta_i \in \Theta_i$ and $\theta_t \in \Theta_t$, $\theta_d < \theta_i < \theta_t$. I want to show that $\Delta u(\theta)$ is decreasing on Θ_d.

Assume first that $\alpha'(l) = 0$ and let $\alpha'(h) = \alpha'$ to simplify notation. Because $t \succeq_a d$, we must have that the function $\alpha^*(\theta)$ solving

$$\pi_t(\theta) + \alpha^*(\theta)(1 - \pi_t(\theta)) = \alpha'\pi_d(\theta) \Leftrightarrow \alpha^*(\theta) = \frac{\alpha'\pi_d(\theta) - \pi_t(\theta)}{1 - \pi_t(\theta)}$$

is decreasing.[9] Note that $\alpha^*(\theta)$ defines a well-defined strategy, i.e., $\alpha^*(\theta) \in [0, 1]$ because for any $\theta \in \Theta_d$, we have $0 \leq \alpha < \frac{\alpha'\pi_d(\theta) - \pi_t(\theta)}{1 - \pi_t(\theta)} \leq 1$.

We can rearrange $\Delta u(\theta)$ to get

$$(1 - \pi_t(\theta)) \left(\frac{\alpha'\pi_d(\theta) - \pi_t(\theta)}{1 - \pi_t(\theta)} - \alpha \right).$$

This is the product of two strictly positive decreasing function and is therefore decreasing.

We can make a similar argument for the case where $\alpha'(l) > 0$.

Let $\theta^* = \sup \Theta_i$ or $= \sup \Theta_d$ if $\Theta_i = \emptyset$. For the deviation to be profitable, it must be that $\theta^* > 0$. We can now prove that the deviation is never profitable. We start by adding the

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[9]The definition of accuracy requires the cutoff $x^*(\theta)$ to be increasing. Because the probability of acceptance is $1 - \tilde{F}_t(x|\theta)$, this is equivalent to having α^* decreasing as long as $\alpha^* \in [0, 1]$.

profits from offering s which are equal to zero and then rearranging.

$$v(s', s, \sigma) = \int_{\Theta_d} \theta \left(\alpha'(h) \pi_d(\theta) + \alpha'(l)(1 - \pi_d(\theta)) \right) dF + \frac{1}{2} \int_{\Theta_i} \theta \left(\alpha'(h) \pi_d(\theta) + \alpha'(l)(1 - \pi_d(\theta)) \right) dF$$

$$- \int_{\Theta} \theta \left(\pi_t(\theta) + \alpha(1 - \pi_t(\theta)) \right) dF$$

$$= \int_{\Theta_d} \theta \Delta u(\theta) dF + \frac{1}{2} \int_{\Theta_i} \theta \Delta u(\theta) dF$$

$$- \frac{1}{2} \left(\int_{\Theta_i} \theta \left(\pi_t(\theta) + \alpha(1 - \pi_t(\theta)) \right) dF + \int_{\Theta_t} \theta \left(\pi_t(\theta) + \alpha(1 - \pi_t(\theta)) \right) dF \right)$$

$$- \frac{1}{2} \int_{\Theta_t} \theta \left(\pi_t(\theta) + \alpha(1 - \pi_t(\theta)) \right) dF.$$

Now note that for all $\theta \in \Theta_t$, $\theta > 0$, so the last element is negative. We also have that the second to last element is negative because the original profits are zero and these are the profits from removing negative types. The expression $\int_{\Theta_i} \theta \Delta u(\theta) dF = 0$. Finally, we have that if there is some type $\theta \in \Theta_d$ with $\theta > 0$ (otherwise we are done), we have

$$\int_{\Theta_d} \theta \Delta u(\theta) dF \leq \int_{\Theta_d} \theta \Delta u(0) dF < 0,$$

because $\mathbb{E}[\theta] < 0$. $\square$

A.5   Proof of Proposition 3

Proof. Let $\underline{t} \in T_i$ such that $t \succ_d \underline{t}$ for all $t \in T_i$.

Show that in any symmetric equilibrium, no test $t \succ_d \underline{t}$ is used. Suppose there is a symmetric equilibrium (t, α) with $t \succ_d \underline{t}$. Because t and $\underline{t}$ are not comparable in accuracy, then $\frac{\pi_t(\theta)}{\pi_{\underline{t}}(\theta)}$ is not constant and $\mathbb{E}[\theta|t, h] > \mathbb{E}[\theta|\underline{t}, h] \geq 0$. Therefore, by the richness assumption, there is a test t' with $t \succ_d t'$ and $\mathbb{E}[\theta|t', h] > 0$. Consider the deviation $s' = (t', \alpha')$ with $\alpha' = \alpha'(l)$. Type θ chooses (t', α') if

$$\pi_{t'}(\theta) + \alpha'(1 - \pi_{t'}(\theta)) \geq \pi_t(\theta) + \alpha(1 - \pi_t(\theta)) \Leftrightarrow (\alpha' - 1)(1 - \pi_{t'}(\theta)) \geq (\alpha - 1)(1 - \pi_t(\theta)).$$

Because $\frac{1 - \pi_t(\theta)}{1 - \pi_{t'}(\theta)}$ is increasing in θ, if the inequality is satisfied for θ then it is satisfied for all $\theta' > \theta$.

Suppose $\alpha' = \alpha(l)$, then this inequality is satisfied for all types as $\pi_{t'}(\theta) \geq \pi_t(\theta)$.

If $\alpha' = 0$, there are two possibilities. Either, some types choose t' but then by the positive selection into t' and by t' being minimally informative, the deviating firms makes positive profits. If no type choose t', it means that

$$\pi_{t'}(0) + \alpha'(1 - \pi_{t'}(0)) - \pi_t(0) - \alpha'(1 - \pi_t(0)) < 0 \text{ at } \alpha' = 0$$

Then by the intermediate value theorem, there is $\alpha' \in (0, \alpha)$ such that this holds with equality. At that α', only types with $\theta \geq 0$ choose t' (or if there is a negative type choosing t' they are indifferent between t and t' and therefore do not impact the payoff comparison between the two selection procedures). Therefore (t', α') is a profitable deviation.

Show that $\underline{t}$ is part of a symmetric equilibrium. To examine if there is a profitable deviation from $\underline{t}$, let's first show what types deviate to a selection procedure (t, α') with $t \succ_d \underline{t}$ and $t \in T_i$. First note that because $\pi_{\underline{t}}(\theta) \geq \pi_t(\theta)$ for all θ, we must have that any deviation sets $\alpha'(l) > 0$, otherwise no agent would select (t, α'). To simplify notation, set $\alpha = \alpha(l)$ and $\alpha' = \alpha'(l)$. Type θ deviates if

$$\pi_{\underline{t}}(\theta) + \alpha(1 - \pi_{\underline{t}}(\theta)) \leq \pi_t(\theta) + \alpha'(1 - \pi_t(\theta)).$$

Rearranging, we get, $(1 - \alpha) \geq (1 - \alpha')\frac{1 - \pi_t(\theta)}{1 - \pi_{\underline{t}}(\theta)}$. The right-hand side of this expression is increasing in θ and not constant by assumption. Therefore, we can divide the set of types in 3: $\Theta = \Theta_d \cup \Theta_i \cup \Theta_t$ where for any $\theta \in \Theta_d$, $\Delta u(\theta) > 0$, for any $\theta \in \Theta_i$, $\Delta u(\theta) = 0$ and for any $\theta \in \Theta_t$, $\Delta u(\theta) < 0$. Furthermore we have that if $\theta_d \in \Theta_d$, $\theta_i \in \Theta_i$ and $\theta_t \in \Theta_t$, $\theta_d < \theta_i < \theta_t$.

Denote by $\Delta u(\theta) = \pi_t(\theta) - \alpha'(1 - \pi_t(\theta)) - \pi_{\underline{t}}(\theta) - \alpha(1 - \pi_{\underline{t}}(\theta))$. I will now show that if $\Delta u(\theta) \geq 0$ then $\Delta u(\theta') \geq \Delta u(\theta)$ for all $\theta' < \theta$. In other words, for the set of types choosing to deviate to selection procedure (t, α'), their utility satisfy decreasing differences. We can rearrange Δu to get

$$(1 - \pi_{\underline{t}}(\theta)) \left(1 - \alpha - (1 - \alpha')\frac{1 - \pi_t(\theta)}{1 - \pi_{\underline{t}}(\theta)} \right).$$

This is the product of two positive function decreasing function and is therefore decreasing.

Let $\theta^* = \sup \Theta_i$ or $= \sup \Theta_d$ if $\Theta_d = \emptyset$. For the deviation to be profitable, it must be that

31

$\theta^* > 0$. We can now prove that the deviation is never profitable. We start by adding the profits from offering s which are equal to zero and then rearranging.

$$v(s', s, \sigma) = \int_{\Theta_d} \theta\left(\alpha'(h)\pi_d(\theta) + \alpha'(l)(1 - \pi_d(\theta))\right) dF + \frac{1}{2}\int_{\Theta_i} \theta\left(\alpha'(h)\pi_d(\theta) + \alpha'(l)(1 - \pi_d(\theta))\right) dF$$

$$- \int_{\Theta} \theta\left(\pi_t(\theta) + \alpha(1 - \pi_t(\theta))\right) dF$$

$$= \int_{\Theta_d} \theta\Delta u(\theta) dF + \frac{1}{2}\int_{\Theta_i} \theta\Delta u(\theta) dF$$

$$- \frac{1}{2}\left(\int_{\Theta_i} \theta\left(\pi_t(\theta) + \alpha(1 - \pi_t(\theta))\right) dF + \int_{\Theta_t} \theta\left(\pi_t(\theta) + \alpha(1 - \pi_t(\theta))\right) dF\right)$$

$$- \frac{1}{2}\int_{\Theta_t} \theta\left(\pi_t(\theta) + \alpha(1 - \pi_t(\theta))\right) dF.$$

Now note that for all $\theta \in \Theta_t$, $\theta > 0$, so the last element is negative. We also have that the second to last element is negative because the original profits are zero and these are the profits from removing negative types. The expression $\int_{\Theta_i} \theta\Delta u(\theta) dF = 0$. Finally, we have that if there is some type $\theta \in \Theta_d$ with $\theta > 0$ (otherwise we are done), we have

$$\int_{\Theta_d} \theta\Delta u(\theta) dF \leq \int_{\Theta_d} \theta\Delta u(0) dF < 0,$$

because $\mathbb{E}[\theta] < 0$.

Now suppose that there is $t \notin T_i$ with $t \prec_d \underline{t}$. I first claim that in that case $\int_{\Theta} \theta\pi_{\underline{t}}(\theta) = 0$. This follows from the fact that T is rich in the difficulty dimension. If $\int_{\Theta} \theta\pi_{\underline{t}}(\theta) dF > 0$, then there is a nearby test t' easier than $\underline{t}$ with $\int_{\Theta} \theta\pi_{t'}(\theta) > 0$, a contradiction. In that case, we have $\alpha(l) = 0$ and we set $\alpha(h) = 1$.

Suppose that a firm deviates to $s' = (t, \alpha')$. Let

$$\Delta u(\theta) = \alpha'(h)\pi_t(\theta) + \alpha'(l)(1 - \pi_t(\theta)) - \pi_{\underline{t}}(\theta)$$

$$= \pi_t(\theta)\left(\alpha'(h) - \alpha'(l) + \frac{\alpha'(l)}{\pi_t(\theta)} - \frac{\pi_{\underline{t}}(\theta)}{\pi_t(\theta)}\right).$$

For any $\theta \in (\underline{\theta}, \overline{\theta})$, $\Delta u(\theta) \leq 0$ whenever the term in bracket is negative and the term in bracket is decreasing. Therefore the selection into the easier test is negative. Let Θ_t and Θ_i be the set of types strictly preferring t and types that are indifferent. Since $\int_{\Theta} \theta\pi_t(\theta) dF \leq 0$,

then

$$\int_{\Theta_t} \theta\big(\alpha'(h)\pi_t(\theta) + \alpha'(l)(1 - \pi_t(\theta))\big)dF + \frac{1}{2}\int_{\Theta_i} \theta\big(\alpha'(h)\pi_t(\theta) + \alpha'(l)(1 - \pi_t(\theta))\big)dF \leq 0.$$

$\square$

A.6 Proof of Corollary 1

Suppose $\mathbb{E}[\theta] < 0$. Suppose test $t \in T_i$ such that $t \succeq_1^* d$ for all $d \in T$.

Suppose $s = (t, \alpha)$ is a symmetric equilibrium. Take a deviation to a selection procedure $s' = (d, \alpha')$ such that $d \in T$. Then there is a sequence of tests $(t_i)_{i=1}^n$ such that for $i = 1, ..., n-1$,

$$\frac{1 - \pi_{t_{i+1}}(\theta)}{1 - \pi_{t_{i+1}}(\theta')} \geq \frac{1 - \pi_{t_i}(\theta)}{1 - \pi_{t_i}(\theta')}, \text{ for } \theta > \theta'.$$

Therefore,

$$\frac{1 - \pi_d(\theta)}{1 - \pi_d(\theta')} \geq \frac{1 - \pi_t(\theta)}{1 - \pi_t(\theta')}, \text{ for } \theta > \theta'.$$

We can then apply the same reasoning as in Proposition 2 and Proposition 3 to show that s' is not a profitable deviation.

Suppose there is a test $t \in T_i$ such that $t \succeq_1^* d$ for all $d \in T_i$ and $t \succeq_2^* d$ for all $d \notin T_i$. We first show that $\int \theta \pi_t(\theta)dF = 0$. If not, because there is $t' \notin T_i$ with $t' \succeq_1^* t$, by the richness assumption, there is $t'' \succeq_1^* t$ with $\int \theta \pi_{t''}(\theta)dF = 0$, a contradiction.

In that case, $\alpha(l) = 0$ and we set $\alpha(h) = 1$. If we consider a deviation to $d \notin T_i$ with $t \succeq_2^* d$, we know there is a sequence of tests $(t_i)_{i=1}^n$ such that for $i = 1, ..., n-1$,

$$\frac{\pi_{t_i}(\theta)}{\pi_{t_i}(\theta')} \geq \frac{\pi_{t_{i+1}}(\theta)}{\pi_{t_{i+1}}(\theta')}, \text{ for } \theta > \theta'.$$

Therefore,

$$\frac{\pi_t(\theta)}{\pi_t(\theta')} \geq \frac{\pi_d(\theta)}{\pi_d(\theta')}, \text{ for } \theta > \theta'.$$

We can then apply the same reasoning as in Proposition 2 and Proposition 3 to show that s' is not a profitable deviation.

A.7 Proof of Corollary 2

We need to verify that all the conditions of Corollary 1 are satisfied. Because T is convex, then it is rich in the accuracy/ease dimension. To see this, observe that with binary types, accuracy/ease connectedness reduces to comparability to being either more accurate or easier: $t \succeq_1^* d \Leftrightarrow t \succeq_a d$ or $t \preceq_d d$. Suppose $t \succeq_d d$. Then $t_\beta = \beta t + (1 - \beta)d$ has $t \preceq_d t_\beta \preceq_d d$ where $\beta \in [0, 1]$ and the convex combination is taken with respect to the likelihood ratios:

$$\frac{\pi_d(x|\bar{\theta})}{\pi_d(x|\underline{\theta})} \geq \beta \frac{\pi_t(x|\bar{\theta})}{\pi_t(x|\underline{\theta})} + (1 - \beta)\frac{\pi_d(x|\bar{\theta})}{\pi_d(x|\underline{\theta})} \geq \frac{\pi_t(x|\bar{\theta})}{\pi_t(x|\underline{\theta})}, \text{ for } x = l, h.$$

Let $\pi_\beta(\theta)$ be the induced probability of sending the high signal for type θ. It is easy to verify that $\pi_\beta(\theta)$ is continuous in β. Therefore, the function $\bar{\theta}\pi_\beta(\bar{\theta})\mu + \underline{\theta}\pi_\beta(\underline{\theta})(1 - \mu)$ is continuous in β and T is rich in the accuracy/ease dimension. A similar argument can be made for the case where $t \succeq_a d$.

If T induces binary types and is closed, then $t = \arg\max\{\frac{\pi_t(\bar{\theta})}{\pi_t(\underline{\theta})} : t \in \arg\min_{d \in T_i} \frac{1 - \pi_d(\bar{\theta})}{1 - \pi_d(\underline{\theta})}\}$ exists. If $t \in \arg\min_{d \in T_i} \frac{1 - \pi_d(\bar{\theta})}{1 - \pi_d(\underline{\theta})}\}$, then either $t \succeq_a d$ or $t \preceq_d d$ for all $d \in T_i$.

Any test in $d \in T \setminus T_i$ has negative posterior expectation at the high signal. Therefore it is necessarily either less accurate or easier than t and thus $t \succeq_2^* d$.

A.8 Proof Proposition 4

Proof. Suppose there is a symmetric equilibrium $s = (t, \alpha)$ with $\int_\Theta \theta \pi_t(\theta)dF > 0$. Let $\mu \in [0, \frac{1 - \pi_t(\bar{\theta})}{1 - \bar{\pi}_t})$ (such a μ exists because $\pi_t(\bar{\theta}) < 1$). Define the test d as follows:

$$\pi_d(\theta) = \frac{\lambda\bar{\pi}_t + (1 - \lambda)\pi_t(\theta)}{(1 - \mu)(1 - \lambda(1 - \bar{\pi}_t)) + \mu\bar{\pi}_t}.$$

One can check that with the assumption on μ, we always have $\pi_d(\theta) \leq 1$.

Monotonicity of d follows from the definition.

Let $\bar{\pi}_d = \int_\Theta \pi_d(\theta)dF$. Under our assumption on μ, we get $\bar{\pi}_d = \frac{\bar{\pi}_t}{(1 - \mu)(1 - \lambda(1 - \bar{\pi}_t)) + \mu\bar{\pi}_t}$. Note

as well that for all $\theta \in \Theta$, we have

$$\lambda f(\theta) + (1 - \lambda)f_{th}(\theta) = f_{dh}(\theta),$$
$$\mu f(\theta) + (1 - \mu)f_{dl}(\theta) = f_{tl}(\theta).$$

These expression can be easily verified plugging in the values of π_d and $\overline{\pi}_d$.

This implies that

$$\lambda + (1 - \lambda)\frac{\pi_t(\theta)}{\overline{\pi}_t} = \frac{\pi_d(\theta)}{\overline{\pi}_d}$$
$$\text{and } \mu + (1 - \mu)\frac{1 - \pi_d(\theta)}{1 - \overline{\pi}_d} = \frac{1 - \pi_t(\theta)}{1 - \overline{\pi}_t}.$$

If $\pi_d(\theta) \in (0, 1)$, one can check from these expressions that $\frac{\pi_t(\theta)}{\pi_d(\theta)}$ and $\frac{1-\pi_t(\theta)}{1-\pi_d(\theta)}$ are increasing using that π_d is increasing. Therefore, $t \succeq_d d$. Whenever $\pi_d(\theta) \in \{0, 1\}$, the condition for difficulty is satisfied.

Note also that $\frac{1-\pi_t(\theta)}{1-\pi_d(\theta)}$ is not constant whenever $\mu > 0$.

Set $\mu = 0$. We want to show that for any $\lambda \in [0, 1]$, this test is well-defined, easier than t and $C(d) < C(t)$, i.e.,

$$\overline{\pi}_d c(f_{dh}) + (1 - \overline{\pi}_d)c(f_{dl}) \leq \overline{\pi}_t c(f_{th}) + (1 - \overline{\pi}_t)c(f_{tl}).$$

We can take the LHS and using the strict convexity of c obtain

$$\overline{\pi}_d c(f_{dh}) + (1 - \overline{\pi}_d)c(f_{dl}) < \overline{\pi}_d \left(\lambda c(f) + (1 - \lambda)c(f_{th})\right)(1 - \overline{\pi}_d)c(f_{dl}).$$

Using that $f_{tl} = f_{th}$ if $\mu = 0$, it is enough to verify that

$$\overline{\pi}_d \left(\lambda c(f) + (1 - \lambda)c(f_{th})\right)(1 - \overline{\pi}_d)c(f_{dl}) \leq \overline{\pi}_t c(f_{th}) + (1 - \overline{\pi}_t)c(f_{tl}).$$

Using that $c(f) = 0$, this is equivalent to

$$0 < \left(1 - \frac{\overline{\pi}_t}{1 - \lambda + \lambda\overline{\pi}_t}\right)\left(\overline{\pi}_t c(f_{th}) + (1 - \overline{\pi}_t)c(f_{tl})\right),$$

which is satisfied.

Now observe that because $\int_\Theta \theta \pi_t(\theta) dF > 0$, we can always pick λ such that $\int_\Theta \theta \pi_d(\theta) dF > 0$. We can also increase μ by an arbitrarily small amount and by continuity of the integral and the cost $C(d) < C(t)$ and $\int_\Theta \theta \pi_d(\theta) dF > 0$. By Proposition 3, this contradicts s is an equilibrium.

Budget binding: The budget constraint must bind for otherwise we can find a Blackwell more informative test that is still affordable. By Proposition 2, this contradicts we are in a symmetric equilibrium. $\qquad\square$

A.9 Proof Proposition 5

Proof. We show that the strategy $s = (t, \alpha)$ with $\alpha(h) = 1$, $\alpha(l) = 0$ is an equilibrium. Equilibrium payoffs for both firms are

$$k \cdot \mathbb{E}[\theta|t, h].$$

Consider a deviation of firm 1 to $s' = (d, \alpha')$ with $d \prec_d t$. Let p_i denote the probability of a given type to have its application considered by firm i. Suppose first that $\alpha'(l) = 0$. For simplicity let $\alpha'(h) = \alpha'$. I will first show that selection into firm 1 is negative. The agent's utility difference between firm 1 and 2 is

$$\Delta u(\theta) = p_1 \alpha' \pi_d(\theta) - p_2 \pi_t(\theta).$$

Using that $\frac{\pi_t(\theta)}{\pi_d(\theta)}$ is increasing, we can establish that selection is negative. Let Θ_d and Θ_i be the set of types for which $\Delta u(\theta) > 0$ and $\Delta u(\theta) = 0$. If the capacity constraint is binding, then

$$k \cdot \mathbb{E}[\theta|d, h, \theta \text{ chooses } 1] \le k \cdot \mathbb{E}[\theta|d, h] \le k \cdot \mathbb{E}[\theta|t, h].$$

If the capacity constraint is binding, Equilibrium profits are

$$\int_{\Theta_d} \theta \alpha' \pi_d(\theta) dF + \frac{1}{2} \int_{\Theta_i} \theta \alpha' \pi_d(\theta) dF \le \frac{k}{p_1} \int_{\Theta_d} \theta \alpha' \pi_d(\theta) dF + \frac{1}{2} \int_{\Theta_i} \theta \alpha' \pi_d(\theta) dF$$
$$= k \cdot \mathbb{E}[\theta|d, h, \theta \text{ chooses } 1]$$
$$\le k \cdot \mathbb{E}[\theta|d, h] \le k \cdot \mathbb{E}[\theta|t, h].$$

Now suppose that $\alpha'(l) > 0$. For simplicity let $\alpha'(l) = \alpha'$. First observe that to have a

profitable deviation, it must be $p_2/p_1 > 1$ for otherwise no type would choose firm 2 as $\pi_d(\theta) \geq \pi_t(\theta)$. But in that case firm 1's capacity constraint binds and its profits are lower than under the equilibrium profits as $\mathbb{E}[\theta|t, h] \geq \mathbb{E}[\theta|d, h]$. Let $p = p_2/p_1$ and

$$\Delta u(\theta) = p_1(\pi_d(\theta) + \alpha'(1 - \pi_d(\theta))) - p_2\pi_t(\theta) = \pi_t(\theta) \left(p_1(1 - \alpha') \frac{\pi_d(\theta)}{\pi_t(\theta)} + \frac{p_1\alpha'}{\pi_t(\theta)} - p_2 \right).$$

The term in parenthesis is negative whenever $\Delta u \leq 0$ and it is decreasing while $\pi_t(\theta)$ is increasing and positive. Therefore whenever $\Delta u(\theta) \leq 0$, $\Delta u(\theta)$ is decreasing. This shows there is negative selection into s' and the acceptance probability satsify decreasing differences for the types choosing s'. By a similar argument as above, the deviation cannot be profitable.

$\square$

A.10 Proof Proposition 7

Proof. In any symmetric equilibrium, both firms get zero profits. If it is not the case and firm gets strictly positive profits, then there is at least one signal where firms get positive profits. Then one of them can raise the transfer by ϵ and attract all agents for an arbitrarily small increase.

First, I show that there is no 'cross-subsidisation' in equilibrium, i.e., $\alpha(l) = 0$.

Suppose it is not the case. Let s be the selection procedure in equilibrium. Let $m(h) = m_h$ and $m(l) = m_l$. First note that if $m_l = 0$, then any firm can decrease $\alpha(l)$ and increase its profits. Because $m_l = 0$, this does not change the payoffs of the agents. So $m_l > 0$. We have

$$\int_\Theta \pi_t(\theta)\alpha_h(\theta - m_h) + (1 - \pi_t(\theta))(\theta - m_l)dF = 0.$$

Consider the following deviation s' that leaves all aspects of the selection procedure unchanged but $m'(h) = m_h + \epsilon$ and $m'(l) = m_l - \delta$ with $\epsilon, \delta > 0$ such that

$$\int_\Theta \pi_t(\theta)\alpha_h(\theta - m_h - \epsilon) + (1 - \pi_t(\theta))(\theta - m_l + \delta)dF = 0.$$

We choose ϵ, δ small enough such that $m'(l) > 0$. I will show that some types will choose the deviating firm and that the deviation will exhibit positive selection. This will imply that the deviation is profitable.

Type θ chooses s' if

$$\pi_t(\theta)\alpha_h(\theta - m_h - \epsilon) + (1 - \pi_t(\theta))(\theta - m_l + \delta) \geq \pi_t(\theta)\alpha_h(\theta - m_h) + (1 - \pi_t(\theta))(\theta - m_l)$$
$$\Leftrightarrow \pi_t(\theta)\alpha_h\epsilon \geq (1 - \pi_t(\theta))\alpha_l\delta$$

Plugging in the solution of δ as a function of ϵ and the fact that the original profits are zero, we get that type θ chooses s' if

$$\pi_t(\theta)\int_\Theta (1 - \pi_t(\theta))dF \geq (1 - \pi_t(\theta))\int_\Theta \pi_t(\theta)dF.$$

This has to hold for some types as otherwise we have $\pi_t(\theta)\int_\Theta(1 - \pi_t(\theta))dF < (1 - \pi_t(\theta))\int_\Theta \pi_t(\theta)dF$ for all θ which implies $\int_\Theta \pi_t(\theta)dF \cdot \int_\Theta(1 - \pi_t(\theta))dF < \int_\Theta(1 - \pi_t(\theta))dF \cdot \int_\Theta \pi_t(\theta)dF$, a contradiction. Using the same argument, some types must prefer the original selection procedure s. Therefore, we get positive selection into the new selection procedure. Since absent positive selection, the profits are zero, this must be a strictly profitable deviation.

Suppose there is symmetric equilibrium with $t \prec_d \bar{t}$.

In equilibrium, it must be that $m(l) = 0$ and from the zero profits condition, $m(h) = m = \frac{\int_\Theta \theta\pi_t(\theta)dF}{\int_\Theta \pi_t(\theta)dF}$.

Take some $\epsilon \in (0, \frac{\pi_t(0) - \pi_{\bar{t}}(0)}{\pi_{\bar{t}}(0)})$. We want to find t' with $t \prec_d t' \prec_d \bar{t}$ such that

$$m\pi_t(0) = m(1 + \epsilon)\pi_{t'}(0).$$

For $t' = t$, we have $m\pi_t(0) < m(1+\epsilon)\pi_{t'}(0)$ and for $t' = \bar{t}$ we have $m\pi_t(0) > m(1+\epsilon)\pi_{t'}(0)$, using the bound on ϵ. Because T is extra rich and tests are all comparable in difficulty, by the intermediate value theorem, there is t' with $t \prec_d t' \prec_d \bar{t}$ such that $m\pi_t(0) = m(1 + \epsilon)\pi_{t'}(0)$.

We want to show that for ϵ small enough, this constitutes a profitable deviation, i.e.,

$$\int_0^{\bar{\theta}} \pi_{t'}(\theta)(\theta - m(1 + \epsilon))dF > 0.$$

Because t' is more difficult than t, we have $\mathbb{E}[\theta|t, \theta \in [0, \bar{\theta}]] < \mathbb{E}[\theta|t', \theta \in [0, \bar{\theta}]]$. Moreover, we have that for ϵ small enough, $\mathbb{E}[\theta|t, \theta \in [0, \bar{\theta}]] > \mathbb{E}[\theta|t](1 + \epsilon)$. Combining these facts, we

get

$$\frac{\int_0^{\bar{\theta}} \pi_{t'}(\theta)\theta dF}{\int_0^{\bar{\theta}} \pi_{t'}(\theta)dF} > \frac{\int_0^{\bar{\theta}} \pi_t(\theta)\theta dF}{\int_0^{\bar{\theta}} \pi_t(\theta)dF} > \frac{\int_{\underline{\theta}}^{\bar{\theta}} \pi_t(\theta)\theta dF}{\int_{\underline{\theta}}^{\bar{\theta}} \pi_t(\theta)dF}(1+\epsilon).$$

Recalling that $m = \frac{\int_{\underline{\theta}}^{\bar{\theta}} \pi_t(\theta)\theta dF}{\int_{\underline{\theta}}^{\bar{\theta}} \pi_t(\theta)dF}$, this is what we needed to show. $\qquad\square$

A.11 Proof of Proposition 6

I show that there is an equilibrium where firm 1 chooses selection procedure $(\bar{t}, \alpha(h) = 1, \alpha(l) = 0)$ and firm 2 chooses $(t, \alpha(h) = 1, \alpha(l) = 0)$. To simplify notation, I denote by $\overline{\pi}_i$ and $\underline{\pi}_i$ the probability of $\bar{\theta}$ and $\underline{\theta}$ to generate the high signal in the test chosen by firm i.

In the suggested equilibrium, type $\bar{\theta}$ chooses firm 1 and type $\underline{\theta}$ mixes between firm 1 and firm 2. Denote by σ the probability that type $\underline{\theta}$ chooses firm 2. Both firms' capacities are binding.

Type $\underline{\theta}$ is willing to mix between firm 1 and firm 2 if

$$\frac{k}{(1-\mu)\sigma\underline{\pi}_2}\underline{\pi}_2 = \frac{k}{\mu\overline{\pi}_1 + (1-\mu)(1-\sigma)\underline{\pi}_1}\underline{\pi}_1.$$

Solving for σ, we get $\sigma = \frac{\mu\overline{\pi}_1 + (1-\mu)\underline{\pi}_1}{2(1-\mu)\underline{\pi}_1}$. We have $\sigma \leq 1 \Leftrightarrow \mu\overline{\pi}_1 \leq (1-\mu)\underline{\pi}_1$. This inequality corresponds to the second condition in Proposition 6. Note also that we have $\sigma \geq \frac{1}{2(1-\mu)}$.

Type $\bar{\theta}$ prefers firm 1 if

$$\frac{k}{\mu\overline{\pi}_1 + (1-\mu)(1-\sigma)\underline{\pi}_1}\overline{\pi}_1 \geq \frac{k}{(1-\mu)\sigma\underline{\pi}_2}\overline{\pi}_2 \Leftrightarrow \frac{\overline{\pi}_1}{\underline{\pi}_1} \geq \frac{\overline{\pi}_2}{\underline{\pi}_2}. \tag{2}$$

This inequality is always satisfied by definition of $\bar{t}$, the test chosen by firm 1.

The capacity constraints bind if

$$\text{Firm 1: } \mu\overline{\pi}_1 + (1-\mu)(1-\sigma)\underline{\pi}_1 \geq k,$$
$$\text{Firm 2: } (1-\mu)\sigma\underline{\pi}_2 \geq k.$$

Plugging in the value of σ for the first inequality, we obtain the third condition in Proposition 6. For the second inequality, we obtain the fourth inequality of Proposition 6 using the minimal value of σ.

Let's now consider deviations of firms. First, firm 2 must make positive profits in equilibrium. This is the case only if $\underline{\theta} \geq 0$ (first condition in Proposition 6).

Consider a deviation to $s' = (t', \alpha')$. If firm 2 deviates, there are two possibilities. Either its capacity constraint binds following the deviation or it does not.

If firm 2's capacity constraint still binds in the continuation equilibrium, then we still have that $\underline{\theta}$ mixes between firm 1 and 2 using the same strategy (that only depended on the strategy of firm 1). Type $\overline{\theta}$ would want to deviate to firm 2 if

$$\frac{\overline{\pi}_1}{\underline{\pi}_1} < \frac{\alpha'(h)\pi_{t'}(\overline{\theta}) + \alpha'(l)(1 - \pi_{t'}(\overline{\theta}))}{\alpha'(h)\pi_{t'}(\underline{\theta}) + \alpha'(l)(1 - \pi_{t'}(\underline{\theta}))},$$

using the same calculations as (2). The inequality above is never satisfied because

$$\frac{\overline{\pi}_1}{\underline{\pi}_1} \geq \frac{\pi_{t'}(\overline{\theta})}{\pi_{t'}(\underline{\theta})} \geq \frac{\alpha'(h)\pi_{t'}(\overline{\theta}) + \alpha'(l)(1 - \pi_{t'}(\overline{\theta}))}{\alpha'(h)\pi_{t'}(\underline{\theta}) + \alpha'(l)(1 - \pi_{t'}(\underline{\theta}))},$$

where the first inequality follows from the definition of $\overline{t}$, the test used by firm 1. The inequality above also shows that there is negative selection into firm 2's selection procedure. Therefore, firm 2's profits following such deviation are still $k \cdot \underline{\theta}$, the same as the equilibrium profits.

If capacity constraint of firm 2 does not bind following the deviation, we consider continuation equilibria where if indifferent, type $\overline{\theta}$ chooses firm 1.[10] Therefore, firm 2 benefits from a deviation only if both types strictly prefer firm 2. This is because there is negative selection into firm 2's strategy. In this case, no one is applying to firm 1 and the IC constraints are

$$\alpha'(h)\pi_{t'}(\overline{\theta}) + \alpha'(l)(1 - \pi_{t'}(\overline{\theta})) \geq \overline{\pi}_1,$$
$$\alpha'(h)\pi_{t'}(\underline{\theta}) + \alpha'(l)(1 - \pi_{t'}(\underline{\theta})) \geq \underline{\pi}_2.$$

Combining these inequalities, we have

$$\mu\big[\alpha'(h)\pi_{t'}(\overline{\theta}) + \alpha'(l)(1 - \pi_{t'}(\overline{\theta}))\big] + (1 - \mu)\big[\alpha'(h)\pi_{t'}(\underline{\theta}) + \alpha'(l)(1 - \pi_{t'}(\underline{\theta}))\big]$$
$$\geq \mu\overline{\pi}_1 + (1 - \mu)\underline{\pi}_1.$$

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[10]This assumption on the continuation equilibrium is not necessary to check for profitable deviations but makes the proof shorter.

The RHS of this equation is larger than k by the last condition in Proposition 6, contradicting that the capacity constraint is not binding.

We now need to verify that firm 1 does not have any profitable deviation. Again, we need to distinguish the cases where the capacity constraint is binding or not in the continuation equilibrium.

Let's first consider the case where the capacity constraint binds in equilibrium. Consider a deviation to $s' = (t', \alpha')$. Let $\overline{q} = \alpha'(h)\pi_{t'}(\overline{\theta}) + \alpha'(l)(1 - \pi_{t'}(\overline{\theta}))$, the acceptance probability under s' of $\overline{\theta}$ and define $\underline{q}$ similarly.

Suppose it is still true that

$$\frac{\mu\overline{q} + (1-\mu)\underline{q}}{2} \geq k.$$

Then the same equilibrium as above holds as long as $\frac{\overline{q}}{\underline{q}} \geq \frac{\overline{\pi}_2}{\underline{\pi}_2}$. Moreover, we necessarily have that

$$\frac{\overline{\pi}_1}{\underline{\pi}_1} \geq \frac{\overline{q}}{\underline{q}}.$$

Let σ_q be the probability of type $\underline{\theta}$ to choose firm 2. The strategy σ_q is decreasing in the likelihood ratio $\frac{\overline{q}}{\underline{q}}$ and therefore this deviation is not profitable for firm 1 as it attracts more $\underline{\theta}$. If $\frac{\overline{q}}{\underline{q}} < \frac{\overline{\pi}_2}{\underline{\pi}_2}$ we can construct an equilibrium where the roles of firms are reversed and firm 1 does not benefit from a deviation either.

Suppose now that

$$\mu\overline{q} + (1-\mu)(1-\sigma_q)\underline{q} = \frac{\mu\overline{q} + (1-\mu)\underline{q}}{2} < k.$$

In that case, the equilibrium we constructed is not feasible. The best case scenario for firm 1 is that it attracts only high types. Given the inequality above, the capacity constraint cannot bind. The deviation is profitable if

$$\mu\overline{q}\overline{\theta} > \frac{k}{\mu\overline{\pi}_1 + (1-\mu)(1-\sigma)\underline{\pi}_1}\left(\mu\overline{\pi}_1\overline{\theta} + (1-\mu)(1-\sigma)\underline{\pi}_1\underline{\theta}\right).$$

We get,

$$\frac{k}{\mu\overline{\pi}_1 + (1-\mu)(1-\sigma)\underline{\pi}_1}\left(\mu\overline{\pi}_1\overline{\theta} + (1-\mu)(1-\sigma)\underline{\pi}_1\underline{\theta}\right)$$

$$< \mu\overline{q}\overline{\theta}$$

$$\leq \mu\overline{q}\overline{\theta} + (1-\mu)(1-\sigma_q)\underline{q}\underline{\theta}$$

$$< \frac{k}{\mu\overline{q} + (1-\mu)(1-\sigma_q)\underline{q}}\left(\mu\overline{q}\overline{\theta} + (1-\mu)(1-\sigma_q)\underline{q}\underline{\theta}\right).$$

It is easy to verify that this chain of inequality cannot hold.

B Isocost tests with difficulty comparison

Proposition 8. *Let $\pi_t(\theta) \in (0,1)$ for all $\theta \in \Theta$. Then there exists a test d with $t \succ_d d$ and $C(t) = C(d)$.*

Proof. Let $\mu \in (0, \frac{1-\pi_t(\overline{\theta})}{1-\overline{\pi}_t}))$, $\lambda \in [0,1]$ and $\pi_d(\theta) = \frac{\lambda\overline{\pi}_t + (1-\lambda)\pi_t(\theta)}{(1-\mu)(1-\lambda(1-\overline{\pi}_t))+\mu\overline{\pi}_t}$. We have already shown in the proof of Proposition 4 that this test is well-defined.

We want to show there exists $\lambda \in (0,1)$ such that

$$\overline{\pi}_d c(f_{dh}) + (1-\overline{\pi}_d)c(f_{dl}) = \overline{\pi}_t c(f_{th}) + (1-\overline{\pi}_t)c(f_{tl}).$$

Note that if $\lambda = 1$, then $\pi_d(\theta) = 1$ for all θ and $f_{dh} = f$. Therefore, d is uninformative and $C(d) < C(t)$. If on the other hand, $\lambda = 0$, we obtain $\overline{\pi}_d = \frac{\overline{\pi}_t}{1-\mu+\mu\overline{\pi}_t}$ and $f_{dh} = f_{th}$. To apply the intermediate value theorem and prove our claim, we want to show that

$$\frac{\overline{\pi}_t}{1-\mu+\mu\overline{\pi}_t}c(f_{th}) + (1 - \frac{\overline{\pi}_t}{1-\mu+\mu\overline{\pi}_t})c(f_{dl}) > \overline{\pi}_t c(f_{th}) + (1-\overline{\pi}_t)c(f_{tl}).$$

We can use the fact c is convex and $f_{tl} = \mu f + (1-\mu)f_{dl}$, to strengthen this inequality to

$$\frac{\overline{\pi}_t}{1-\mu+\mu\overline{\pi}_t}c(f_{th}) + (1 - \frac{\overline{\pi}_t}{1-\mu+\mu\overline{\pi}_t})c(f_{dl}) > \overline{\pi}_t c(f_{th}) + (1-\overline{\pi}_t)(\mu c(f) + (1-\mu)c(f_{dl})).$$

Using that $C(f) = 0$ and rearranging, we obtain

$$\overline{\pi}_t c(f_{th}) + (1-\mu)(1-\overline{\pi}_t)c(f_{dl}) > 0,$$

which is satisfied. Therefore, by continuity of c, there is $\lambda \in (0,1)$ that delivers $C(d) = C(t)$. $\qquad\square$