# Revealed Invariant Preference*

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**Abstract**

We consider the problem of testing the rationalizability of choice data by a preference satisfying an arbitrary collection of *invariance* axioms. Examples of such axioms include quasilinearity, homotheticity, independence-type axioms for mixture spaces, constant risk and ambiguity aversion axioms, stationarity, separability, and many others. We provide necessary and sufficient conditions for invariant rationalizability via a novel approach which relies on tools from the theoretical computer science literature on automated theorem proving. We also establish a generalization of the Dushnik-Miller theorem, which we use to give a complete description of the counterfactual predictions generated by the data under any such collection of axioms.

## 1 Introduction

Nearly all economic models impose restrictions on the preferences agents are assumed to hold. When these restrictions are at odds with the broad, empirical regularities of individual behavior, this misspecification introduces errors which can lead to unrealistic, or even manifestly incorrect, model predictions (e.g. Mehra and Prescott

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1985). This motivates a basic need to understand not just when observed behavior is consistent with the maximization of an arbitrary preference, but rather with one satisfying the kinds of additional economic axioms commonly employed in practice.

We consider an empiricist, who is able to observe the decisions taken by an agent across a variety of choice problems. Our basic question is to determine precisely which sets of observations are *rationalizable*, i.e. consistent with the maximization of, some preference satisfying a given set of additional axioms. In particular, we focus on a broad class of properties we term *invariance axioms.* Informally, invariance axioms are those which require an individual's preference to remain unchanged under some collection of transformations of the consumption space:

$$x \succsim y \quad \iff \quad f(x) \succsim f(y),$$

for every $f$ in some family $\mathcal{F}$. Despite their abstract nature, many of the most widely studied and commonly assumed economic axioms fall into this category.[1]

Perhaps the most influential approach to testing the rationalizability, absent axiomatic considerations, dates back to Richter (1966) and Afriat (1967), who provide distinct but related 'cyclical consistency' conditions that characterize rational choice.[2] While necessary, as the following example demonstrates, these conditions alone are insufficient to guarantee that behavior is consistent with the types of additional, structural assumptions often imposed in applied work.[3]

**Example 1.** Consider a domain of dated rewards featuring two prizes, a pair of argyle socks, $a$, and a bottle of wine, $b$, that can be delivered to a consumer any number of days into the future. We identify this consumption space with the set

[1]For example, homotheticity, separability, and quasilinearity in consumer theory, various mixture-independence and constant risk and ambiguity aversion axioms in theories of risk and uncertainty, stationarity axioms in dynamic models, and many others, all take this form; see Section 2.3 and Section 4.1 for additional examples.

[2]For a modern treatment, see Nishimura et al. (2017).

[3]There is an extensive literature on modifications to Afriat's theorem to test particular models of preference, including some featuring invariance axioms (e.g. Varian 1982, 1983; Brown and Calsamiglia 2007; Echenique and Saito 2015). We regard this paper as distinct from this literature, as such results generally rely on specialized observations about functional forms which do not generalize from model to model, and remain valid only under the assumption that the data arises from choices on linear budgets in classical consumption space. In contrast, our objective is to obtain general results that do not rely on setting or model-specific details, and which apply to arbitrary data sets.

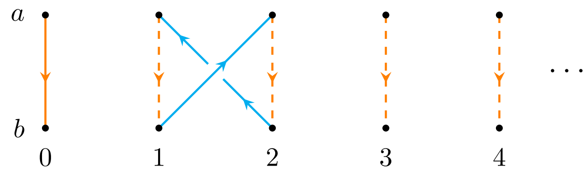



**Figure 1:** The revealed preference of Example 1 is shown in non-dashed lines. The knock-on effects of the orange observation, $(b,0) \succ^R (a,0)$, are shown as dashed arrows. Though the data contains no cycles, there are 'hidden' cycles involving the indirect implications of invariance.

of all pairs $\{a,b\} \times \{0,1,\dots\}$. Suppose that an empiricist observes only that the consumer prefers:

$$(b,0) \succ^R (a,0), \qquad (a,1) \succ^R (b,2), \quad \text{and} \quad (a,2) \succ^R (b,1),$$

where $\succ^R$ denotes the subject's revealed (strict) preference. This set of observations contains no revealed preference cycles, thus by Richter (1966) it is consistent with the maximization of some complete and transitive preference relation.

Suppose, however, we wish to determine whether these data are consistent with the maximization of a *stationary* preference (Fishburn and Rubinstein 1982; Ok and Masatlioglu 2007). Here, stationarity means that the subject's ranking between any two dated rewards remains unchanged if both are delayed by some common, additional duration.

In fact, there is no stationary preference consistent with these observations. Were such a preference to exist, in order to be compatible with the empiricist's observations, it would need to rank $(b,0) \succ (a,0)$.[4] By stationarity, however, it must similarly rank $(b,n) \succ (a,n)$, for all $n = 1,2,\dots$ as well. We term these out-of-sample implications the **knock-on effects** of this observation. In particular, these imply that for any such preference:

$$(b,1) \succ (a,1) \succ (b,2) \succ (a,2) \succ (b,1),$$

and hence that there can be no stationary preference consistent with these observations. In essence, despite their cyclic consistency, the data contain an *indirect* cycle involving the knock-on effects of the observation $(b,0) \succ^R (a,0)$; see Figure 1. ■

[4]And likewise rank $(a,1) \succ (b,2)$ and $(a,2) \succ (b,1)$ to ensure compatibility with the other two observations.

Our primary contribution is to provide a generalized notion of cyclic consistency which is both necessary and sufficient for rationalizability by a preference satisfying a fixed, but otherwise arbitrary, collection of invariance axioms. One strength of our result is that, e.g., unlike Afriat (1967), we require no assumptions on either the consumption space or choice sets faced by the agent. For example, our results characterize consistency with respect to homotheticity or quasilinearity axioms in consumer theory à la Varian (1983) and Brown and Calsamiglia (2007), but are equally applicable (and remain necessary and sufficient) for testing the independence axiom or Savage's (P2) via choices from discrete menus, as in Allais (1953) or Ellsberg (1961).

This allows us to not only unify and extend existing results, but also to obtain new characterizations for a variety of theories whose empirical content was not previously known. For example, our results provide necessary and sufficient conditions for the maximization of a time-stationary preference (Koopmans 1960) over consumption streams, or for an incompletely-observed costliness ordering over Blackwell experiments to be consistent with a weighted sum of Kullback-Leibler divergences (Pomatto et al. 2023).

Our results also enable us, in some instances, to replace complex, existing algebraic tests of consistency for various models with conceptually simpler, no-cycle conditions. For example, in Section 3.1, we obtain a simple, novel characterization of when observations of a subject's betting behavior are consistent with a Bayesian prior (cf. Alon and Lehrer 2014). Similarly, in Section 3.2, we obtain an elementary 'no-cycle' test for additive separability (cf. Tversky 1964; Fishburn 1970).

Finally, we are able to provide new insights into the problem of non-parametric recovery of preferences (Varian 1982). Given observed choices consistent with some collection of axioms, a natural question is to what degree the empiricist able to draw inferences about counterfactual, *unobserved* choices on the basis of the data, without needing to subscribe to any particular choice of rationalization. In general, invariance axioms generate rich, nuanced sets of out-of-sample predictions. We provide a complete characterization of the predictions generated by any data set, under any system of invariance axioms, and on any domain.

A cost of this generality is that, absent constraints on the data or environment, our consistency condition may be difficult to verify in practice.[5] Interestingly, the form our condition takes depends dramatically on the *algebraic* properties of the family of transformations under consideration. When the order with which transformations are applied does not matter (i.e. the transformations *commute*, see Section 3) our condition reduces to a straightforward generalization of the congruence axiom of Richter (1966). However, absent this kind of special structure, invariance axioms can impose intricate systems of simultaneous restrictions on the data.[6] In such cases, our generalization of cyclic consistency applies not to the revealed preference relations themselves, but rather to the system of restriction sets imposed jointly by rationality and invariance.[7]

The paper proceeds as follows. In Section 2, we introduce the basic revealed preference framework, as well as formally define our notion of invariance axioms, and provide numerous examples. Section 3 considers the special case when $\mathcal{M}$ possesses a particularly simple algebraic structure. We find that in this case, a slight generalization of the classical cyclic consistency conditions characterize rationalizability. In contrast, in Section 4, we consider the problem of invariant rationalizability in full generality and provide a novel condition we term 'strong acyclicty.' We show strong acyclicity is both necessary and sufficient for invariant rationalizability, no matter the structure of the data or the problem itself. Section 5 considers the problem of the recoverability of preferences, and provides a complete characterization of the out-of-sample predictions generated by any data set under any collection of invariance axioms. Finally Section 6 concludes.

## 1.1 Related Literature

The revealed preference literature is too large to adequately survey here, see Chambers and Echenique (2016) for an overview.[8] Classically, Richter (1966) was the first

[5] This is in line with a growing literature on the computational difficulty of various revealed preference tests and indices, see, e.g., Smeulders et al. (2013); Echenique (2014); Smeulders et al. (2014); Cherchye et al. (2015); Dean and Martin (2016); De Clippel and Rozen (2021); Smeulders et al. (2021).

[6] We treat the unrestricted case in full generality in Section 4.

[7] Similar systems of constraint sets have also been studied in the context of testing behavioral choice models, see De Clippel and Rozen (2021).

[8] See also Echenique (2020) for a summary of some recent work in this space.

to characterize rationalizability for the abstract choice model. We obtain Richter's original theorem as a special case of our main results (see Section 3). A similarly classic reference in this vein is Duggan (1999) who retains an abstract framework but imposes additional restrictions on the interpretation of 'rationality.'

Other authors have studied the problem of rationalizing choice data via preferences with various general structures. Nishimura et al. (2017) study this problem for continuous and monotone preferences on various spaces. Demuynck (2009) investigates a general class of 'closure operators' on spaces of binary relations that generalize the transitive closure, and obtains a general extension result for algebraic structures satisfying certain properties.[9] While general, applying these tools requires non-trivial effort to establish their conditions are satisfied. In contrast, our results focuses on a smaller mathematical class of algebraic properties, invariance and monotonicity axioms, but are able to derive results that are immediately applicable.

Other authors have considered invariant preferences in various contexts. Ok and Riella (2014, 2021) consider various extension results for invariant preorders on groups. In contrast, we consider both a more general class of primitive relations and more general notion of invariance.[10] Recently Freer and Martinelli (2022), building off the tools of Demuynck (2009), consider the problem of invariant rationalization by incomplete or non-transitive binary relation.[11] Dubra et al. (2004) show that every 'incomplete' expected utility (EU) preference may be completed in such a way as to preserve the EU axioms.

Dushnik and Miller (1941) show that every partial order is equal to the intersection of its linear order extensions. Several authors in economics have taken interest in such unanimity, or Pareto, representation of incomplete preferences. Abstract approaches include Donaldson and Weymark (1998); Bossert (1999); Weymark (2000) and Alcantud (2009). In concrete economic environments, similar representations can be found in, for example, the theory of expected utility preferences (Dubra et al. 2004; Gorno 2017), Krepsian style preferences over menus (Nehring and Puppe 1999), or rankings of accomplishments (Chambers and Miller 2018).

[9]See Ward (1942) for a general theory of closures.

[10]Mathematically, our notion of invariance corresponds to invariance of a preference under an arbitrary semi-group action on the consumption space. For definitions, see Fuchs (2011).

[11]They also establish an invariant rationalizability result in the special case the collection of transformations, under composition, forms a linearly ordered group.

We are not the first paper to exploit the connection between revealed preference and formal logic. Chambers et al. (2014) study the general form of empirical content for theories in first-order logic, relating the syntax of first-order theories to the empirical content via a type of sentence they call "UNCAF" (universal negation of conjunctions of atomic formulae). Chambers et al. (2017) establishes that theoretical relations in theories axiomatizable by universal sentences can be eliminated, resulting in a theory which is itself universally axiomatizable. Such an axiomatization results from enumerating all logical consequences of the original theory without theoretical relations. Thus, these two papers give a r.e. method which could in principle enumerate datasets which are inconsistent with a given theory.[12] In comparison, our results rely only on the simpler framework of propositional logic, and provide a more practical method for understanding inconsistent data. Gonczarowski et al. (2019) show that similar connections with propositional logic obtain in a variety of economic contexts. Galambos (2019); Yildiz (2020) investigate the relation between computational complexity of revealed preference theories and their logical syntax.

Robinson (1965) showed that a certain algorithmic operation on logical clauses called resolution was sound and refutation-complete. This reduced the problem of proving a set of clauses to be inconsistent without constructing a truth table to a discrete search problem. A number of extensions and refinements giving various 'normal forms' for proofs were established in the early artificial intelligence literature to attempt to further reduce the complexity of this search space (see, e.g., Schöning 2008 for an overview).

Finally, our work presupposes no notion of topology, but many works in economics consider topological aspects of the extension problem. Aumann (1962, 1964); Peleg (1970); Levin (1983) are classical references, but the theory has developed much since then (e.g., Ok 2002; Nishimura et al. 2017).

[12] See also Chambers and Echenique (2016), Chapter 13.

# 2 The Model

## 2.1 Preliminaries

Let $X$ denote set of alternatives. A **preference relation** $\succeq$ is a complete and transitive binary relation on $X$. Given a preference, we will use $\succ$ and $\sim$ to denote its asymmetric and symmetric components, respectively.

An **order pair** on $X$ is a pair of binary relations $\langle \geq, > \rangle$, where $>$ is a sub-relation of $\geq$.[13] A pair $\langle \geq', >' \rangle$ **extends** $\langle \geq, > \rangle$ if both (i) $\geq$ is a sub-relation of $\geq'$, and (ii) $>$ is a sub-relation of $>'$. By minor abuse of notation, we will say a binary relation extends an order pair if the relation and its asymmetric component, regarded as an order pair, does so.

The **transitive closure** of an order pair $\langle \geq, > \rangle$ is the pair $\langle \geq_{\mathsf{T}}, >_{\mathsf{T}} \rangle$, defined by $x \geq_{\mathsf{T}} y$ if and only if there exists $x_0, \ldots, x_N \in X$ such that:

$$x = x_0 \geq x_1 \geq \cdots \geq x_N = y, \tag{1}$$

and $x >_{\mathsf{T}} y$ if both (i) $x \geq_{\mathsf{T}} y$, and (ii) for some sequence (1), at least one relation belongs to $>$. We say that $\langle \geq, > \rangle$ is **acyclic** if:

$$x \geq_{\mathsf{T}} y \quad \Longrightarrow \quad y \not>_{\mathsf{T}} x,$$

and refer to a collection $x_0, \ldots, x_N \in X$ such that $x_0 \geq x_1 \geq \cdots x_N > x_0$ as a **cycle**.

## 2.2 Data and Rationalizability

We assume the data in possession of our empiricist takes the form of an order pair of **revealed preference** relations $\langle \succsim^R, \succ^R \rangle$. If $x \succsim^R y$, we say that $x$ is revealed to be at least as preferable as $y$. If, in addition, $x \succ^R y$, we say $x$ is revealed strictly preferable. A preference relation $\succeq$ **rationalizes** the data $\langle \succsim^R, \succ^R \rangle$ if it extends it, i.e. if both (i) $\succsim^R \subseteq \succeq$, and (ii) $\succ^R \subseteq \succ$.[14]

The assumption that the data can be represented as a revealed preference pair is standard (e.g. Chambers and Echenique 2016). However, as the following examples

[13] However we do not require that $>$ is the asymmetric component of $\geq$.

[14] Recall that a preference relation is required to be both complete and transitive.

illustrate, the manner in which $\langle \succsim^R, \succ^R \rangle$ is constructed from primitive choice observations varies by domain, and generally requires additional identifying assumptions. By taking as primitive the revealed preference pair $\langle \succsim^R, \succ^R \rangle$, we obtain a theory that applies to any domain, and any choice of context-specific identification strategy.

### 2.2.1 Classical Consumption Spaces

Suppose there are $L$ commodities, and $X = \mathbb{R}^L_+$ denotes the space of all consumption bundles.[15] In any given choice instance, consumers face a set of prices $p \in \mathbb{R}^L_{++}$, which are observed by the empiricist, and possess a wealth level $w > 0$, which is not. These parameters determine the consumer's feasible choice set:

$$B_{p,w} = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}.$$

However, since consumers' wealth is not directly observed, a typical observation shows only that that the consumer purchased bundle $x$ at prices $p$.

To infer the consumer's choice set, a standard identification strategy is to impose Walras Law, e.g. Samuelson (1938).[16] This not only pins down the choice set of the consumer, but also implies that any bundle on the interior of the consumer's choice set must be strictly dominated. Thus a data set $\{x^t, p^t\}_{t=1}^T$ yields a revealed preference pair by defining $x \succsim^R y$ if either (i) $x = x^t$ and $p^t \cdot x^t \geq p^t \cdot y$, or (ii) $x \gneq y$ as vectors, and $x \succ^R y$ if either (i) $x = x^t$ and $p^t \cdot x^t > p^t \cdot y$, or (ii) $x \gneq y$.[17] When $\langle \succsim^R, \succ^R \rangle$ is obtained in this manner, our notion of rationalizability coincides with that of Afriat (1967).

### 2.2.2 General Choice Environments

In environments featuring bulk discounts, quantity restrictions, monopsony power, or other non-linearities, it is common to assume the choice set faced by the consumer is observed by the empiricist (Matzkin 1991; Chavas and Cox 1993; Forges and Minelli 2009). Likewise, in many laboratory experiments, choices are elicited from subjects

[15] That is, if $x = (x_1, \ldots, x_L)$, the scalar $x_l$ reflects the quantity of the commodity $l$ in the bundle $x$.

[16] Recall Walras' Law is the assumption that the chosen consumption bundle lies on the budget frontier, i.e. $p \cdot x = w$. Under the null hypothesis that choices maximize some locally non-satiated preference on $\mathbb{R}^L_+$, it is well-known this holds as an identity.

[17] For vectors, we write $x \gneq y$ if $x \neq y$ and $x_l \geq y_l$ for all $l = 1, \ldots, L$.

on known choice sets, which are often not of the linear form considered above (e.g. Allais 1953; Ellsberg 1961).

Following Nishimura et al. (2017), let $X$ be a general consumption space space, and $\trianglerighteq$ a reflexive, transitive binary relation on $X$ capturing some notion of monotonicity.[18] Suppose the empiricist observes non-empty, set-valued choices $c(B)$ from each choice set $B$ in some collection $\Sigma$. To define a revealed preference pair from these primitives, it is necessary to fix an assumption on how indifference is observed.

One approach is to assume $c(B)$ reflects the set of all jointly most-preferred alternatives in $B$. In this case, if $x$ is chosen in the presence of $y$, but $y$ is not chosen, it may be inferred that $x$ strictly dominates $y$ in the eyes of the consumer. Defining $x \succsim^R y$ if either (i) for some choice set $B \in \Sigma$, $x, y \in B$ and $x \in c(B)$, or (ii) $x \trianglerighteq y$, and likewise $x \succ^R y$ if either (i) for some $B \in \Sigma$, $x, y \in B$, $x \in c(B)$ and $y \notin c(B)$, or (ii) $x \triangleright y$, then defines a revealed preference pair. In this case, our notion of rationalizability coincides with that of Richter (1966).[19]

Alternatively, the empiricist may regard $c(B)$ as only a subset of the full collection of most-preferred alternatives in $B$. Under this interpretation, the only way to infer strict preference is via the dominance relation $\trianglerighteq$. Thus, the revealed preference relations are given by $x \succsim^R y$ if either (i) for some $B \in \Sigma$, $x, y \in B$ and $x \in c(B)$, or (ii) $x \trianglerighteq y$, and $x \succ^R y$ if $x \triangleright y$. Here, our notion of rationalizability coincides with Nishimura et al. (2017).[20]

## 2.3 Invariant Preferences

Let $\mathcal{M}$ denote a set of transformations, each mapping $X \to X$. We say that a preference is **$\mathcal{M}$-invariant** if, for all $x, y \in X$ and all $\omega \in \mathcal{M}$:

$$x \underset{(\succ)}{\succeq} y \quad \implies \quad \omega(x) \underset{(\succ)}{\succeq} \omega(y), \tag{2}$$

[18]For example, when $X = \mathbb{R}^L_+$, $\trianglerighteq$ could represent the pointwise ordering of bundles; when $X$ is a space of monetary lotteries, $\trianglerighteq$ could represent first-order stochastic dominance. In environments without any natural choice of order, $\trianglerighteq$ can always be taken to be the trivial relation containing only comparisons of the form $x \trianglerighteq x$.

[19]Our notion rationalizability here also agrees with the notion of 'strict rationalizability' from an earlier, working paper version of Nishimura et al. (2017).

[20]For other approaches to identifying indifference from choice data, see also Bouacida (2021); Ok and Tserenjigmid (2022).

Note that if $\omega, \omega' \in \mathcal{M}$, then any $\mathcal{M}$-invariant preference also satisfies (2) for both $\omega \circ \omega'$ and $\omega' \circ \omega$. Thus, without loss of generality, we will suppose that $\mathcal{M}$ is (i) closed under composition, and (ii) contains the identity function. If $\mathcal{G} \subseteq \mathcal{M}$ is some sub-collection of transformations such that, for any $\omega \in \mathcal{M}$, there exist $\omega_1, \ldots, \omega_K \in \mathcal{G}$ satisfying:

$$\omega = \omega_K \circ \cdots \circ \omega_1,$$

we say that $\mathcal{G}$ **generates** $\mathcal{M}$.[21] We interpret any such $\mathcal{M}$ as a decision-theoretic axiom, namely the requirement that preferences be $\mathcal{M}$-invariant.

Despite their abstract nature, many of the most economically interesting and widely applied preference axioms fall into this class.[22]

#### 2.3.1 Quasilinearity

Let $X = \mathbb{R}_+ \times Z$. A preference is said to be *quasilinear* if it is invariant under any transformation of form:

$$(t, z) \mapsto (t + \alpha, z),$$

where $\alpha \geq 0$. When the pair $(t, z)$ is interpreted as a dated reward, corresponding to the delivery of a prize $z$ to the consumer $t$ units of time in the future, quasilinearity is also referred to as *stationarity* (e.g. Fishburn and Rubinstein 1982). See also the notion of '$\phi$-additivity' in Caradonna (2023).

#### 2.3.2 Homotheticity

Let $X$ be a cone in a real vector space. A preference is *homothetic* if it is invariant under any transformation of the form:

$$x \mapsto \lambda x,$$

where $\lambda > 0$. The characteristic feature of Cobb-Douglas preferences is their invariance under the related, but more general, family of transformations:

$$(x_1, \ldots, x_L) \mapsto (\lambda_1 x_1, \ldots, \lambda_L x_L),$$

[21] Note that by our assumption that $\mathcal{M}$ is closed under compositions, if $\mathcal{G}$ generates $\mathcal{M}$, then $\mathcal{M}$ is precisely the set of all finite compositions of elements in $\mathcal{G}$.

[22] In fact, our results actually apply to an even broader class of 'generalized' invariance axioms; see Section 4.1.

where $(\lambda_1, \dots, \lambda_L) \in \mathbb{R}^L_{++}$; see Trockel (1989). If $X = \mathcal{L}^\infty(S, \mathcal{S}, \mathbb{P})$ for some probability space $(S, \mathcal{S}, \mathbb{P})$, a homothetic preference on $X$ is said to exhibit *constant relative risk aversion* (e.g. Safra and Segal 1998).[23]

### 2.3.3 Mixture Invariance

Suppose that $X = \Delta(Z)$, the set of all Borel probability measures on a metrizable space $Z$. A preference satisfies the *independence* axiom of Von Neumann and Morgenstern (1947) if it is invariant under the family of transformations:

$$\mu \mapsto \alpha\mu + (1 - \alpha)\nu,$$

where $\alpha \in (0, 1]$ and $\nu \in X$.[24] If instead $X$ denotes the Anscombe-Aumann domain of simple, measurable maps from some measurable space $(S, \mathcal{S})$ into $\Delta(Z)$, the independence axiom corresponds to invariance under transformations of the form:

$$f \mapsto \alpha f + (1 - \alpha)g,$$

where $\alpha \in (0, 1]$ and $g \in X$. Common weakenings of independence such as *certainty independence* (Gilboa and Schmeidler 1989), *weak certainty independence* (Maccheroni et al. 2006), *worst independence* (Chateauneuf and Faro 2009), *risk independence* (Cerreia-Vioglio et al. 2011) and so forth are also all invariance axioms, under appropriate restrictions of this family.

### 2.3.4 Stationarity

Let $X = Z^{\mathbb{N}}$ denote the set of all infinite horizon consumption streams taking values in some set of outcomes $Z$. A preference on $X$ is said to be *stationary* in the sense of Koopmans (1960) if it is invariant under the family generated by the transformations:

$$(x_1, x_2, \dots) \mapsto (z, x_1, x_2, \dots),$$

for each $z \in Z$. See also Epstein (1983).

[23] Such a preference is typically assumed to depend only on the law of each random variable.

[24] More generally, in any mixture space (Herstein and Milnor 1953), the analogous notion of mixture independence remains an invariance axiom.

### 2.3.5 Convolution Invariance

Suppose $X$ consists of all lotteries on $\mathbb{R}$ with bounded support. Mu et al. (2021) consider continuous weak orders on $X$ that are monotone with respect to first-order stochastic dominance, and invariant with respect to convolutions:

$$\mu \mapsto \mu * \nu,$$

with $\nu \in X$.[25] A preference on $X$ is said to exhibit *constant absolute risk aversion* (e.g., Safra and Segal 1998) if it is invariant under the restricted collection of transformations:

$$\mu \mapsto \mu * \delta_\alpha,$$

where $\alpha \in \mathbb{R}$, and $\delta_\alpha$ denotes the Dirac measure centered at $\alpha$.

### 2.3.6 Product & Dilution Invariance

Let $X$ consist of all finite Blackwell experiments on some fixed, finite set of states of the world $\Theta$. Thus elements of $X$ are tuples $\big(S, \{\mu_\theta\}_{\theta\in\Theta}\big)$, where $S$ is some finite set of signals, and each $\mu_\theta$ is a probability measure on $S$. Pomatto et al. (2023) consider complete and transitive 'costliness' orderings over $X$ that are invariant under two families of transformations. The first family is generated by transformations:

$$\big(S, \{\mu_\theta\}_{\theta\in\Theta}\big) \mapsto \big(S \times T, \{\mu_\theta \otimes \nu_\theta\}_{\theta\in\Theta}\big),$$

for each $\big(T, \{\nu_\theta\}_{\theta\in\Theta}\big) \in X$, which map $\big(S, \{\mu_\theta\}_{\theta\in\Theta}\big)$ to the sequence of independent experiments $\big(S \times T, \{\mu_\theta \otimes \nu_\theta\}_{\theta\in\Theta}\big)$. The second family is generated by the maps:

$$\big(S, \{\mu_\theta\}_{\theta\in\Theta}\big) \mapsto \alpha \cdot \big(S, \{\mu_\theta\}_{\theta\in\Theta}\big),$$

where $\alpha \in (0,1]$, and $\alpha \cdot \big(S, \{\mu_\theta\}_{\theta\in\Theta}\big)$ denotes the experiment $\big(S \cup \{*\}, \{\mu'_\theta\}_{\theta\in\Theta}\big)$, where $*$ is an uninformative signal, and the $\{\mu'_\theta\}_{\theta\in\Theta}$ satisfy (i) $\mu'_\theta(A) = \alpha\, \mu_\theta(A)$ for all $A \subseteq S$, and (ii) $\mu'_\theta(\{*\}) = 1 - \alpha$.

[25] The term 'additive' in the paper's title refers to this property when the preference is equivalently regarded as being defined over (bounded) random variables.

# 3 The Commutative Case

We now turn to the problem of characterizing when, given some family of transformations $\mathcal{M}$, the data $\langle \succsim^R, \succ^R \rangle$ can be rationalized by an $\mathcal{M}$-invariant preference. The difficulty of this question turns out to depend dramatically on the algebraic properties of $\mathcal{M}$.

We first focus on the special case in which each pair of transformations in $\mathcal{M}$ commute, i.e.:

$$\omega \circ \omega' = \omega' \circ \omega,$$

for all $\omega, \omega' \in \mathcal{M}$. Every example in Section 2.3.1, Section 2.3.2, and Section 2.3.5 is of this form, as are often families which depend only on a single parameter, such as mixing under various weights with a fixed act or lottery.[26] In such cases, we refer to $\mathcal{M}$ as a **commutative family**.

Given a binary relation $\trianglerighteq$ on $X$, we define its its $\mathcal{M}$-**closure** $\trianglerighteq_{\mathcal{M}}$ as the smallest $\mathcal{M}$-invariant, i.e. satisfying (2), relation containing $\trianglerighteq$. Equivalently:

$$x \trianglerighteq_{\mathcal{M}} y \quad \iff \quad \exists\, \omega \in \mathcal{M} \text{ and } x', y' \in X \text{ s.t. } \begin{cases} x = \omega(x') \\ y = \omega(y') \\ x' \trianglerighteq y'. \end{cases}$$

The $\mathcal{M}$-closure of our revealed preference pair, $\langle \succsim^R_{\mathcal{M}}, \succ^R_{\mathcal{M}} \rangle$, defines an order pair extension of the data which encodes not only the comparisons directly observed by the empiricist, but also their out-of-sample, or indirect, implications under $\mathcal{M}$-invariance.[27]

Our first main result says that, when $\mathcal{M}$ is a commutative family, the data are rationalizable by an $\mathcal{M}$-invariant preference relation if, and only if, their $\mathcal{M}$-closure is acyclic.

**Theorem 1.** *Let $X$ be a set, and $\mathcal{M}$ an arbitrary family of commuting transformations. Then $\langle \succsim^R, \succ^R \rangle$ is rationalizable by an $\mathcal{M}$-invariant preference relation if and only if $\left\langle \succsim^R_{\mathcal{M}}, \succ^R_{\mathcal{M}} \right\rangle$ is acyclic.*

[26]See, for example, worst-independence, Section 2.3.3.

[27]Since we assume the identity function id $\in \mathcal{M}$, it follows that $\langle \succsim^R_{\mathcal{M}}, \succ^R_{\mathcal{M}} \rangle$ extends the revealed preference pair. Our assumption that $\mathcal{M}$ is closed under composition implies that no new relations arise from applying the $\mathcal{M}$-closure repeatedly.

Theorem 1 relies crucially on the assumption that $\mathcal{M}$ is a commutative family. Informally, when the transformations in $\mathcal{M}$ commute, if the data are unable to be extended into an invariant preference, there always exist sequences of transformations which, when carefully applied to the data, yield knock-on effects which form a cycle in the $\mathcal{M}$-closure.[28]

In the trivial case when $\mathcal{M} = \{\mathrm{id}\}$, $\mathcal{M}$ is clearly a commutative family. Moreover, *every* preference is trivially $\mathcal{M}$-invariant. Thus Theorem 1 strictly subsumes the classical characterization of Richter (1966). In Section D.2, we explore connections between Theorem 1 and various well-known modifications of the generalized axiom of revealed preference in the special case of price-consumption data ([**X**]).

Perhaps surprisingly, many problems to which Theorem 1 seemingly does not apply can, in fact, be converted into problems featuring commutative families.

## 3.1 Application: Probabilistic Sophistication

Let $S$ denote a finite set of states of the world, and $X = 2^S$ the power set of $S$. Elements of $X$ correspond to *events*. Consider a complete and transitive order $\succeq$ on $X$, which we interpret as an agent's subjective assessment of the relative likelihood of events (i.e. $A \succeq B$ denotes that the agent subjectively believes that $A$ is more likely than $B$).

Such an ordering is said to be a **qualitative probability** if, for all events $A, B, C \in X$ with $C$ disjoint from $A \cup B$,

$$A \succeq B \quad \Longleftrightarrow \quad A \cup C \ \succeq \ B \cup C,$$

and in addition, $A \subseteq B$ implies $B \succeq A$, and $S \succ \varnothing$. We refer to a qualitative probability as **probabilistically sophisticated** if it can be represented by a probability measure, i.e. a prior over $S$.

Kraft et al. (1959) exhibit a qualitative probability over a five element state space which is not probabilistically sophisticated, disproving a conjecture of de Finetti (1951). Using results on linear inequalities, they obtain an infinite system of 'cancellation' conditions, which jointly characterize probabilistic sophistication. Despite the fact these conditions are not invariance axioms, by regarding $X$ as a subset of

[28] For a formal statement, see [**X**].

a richer domain, we may nonetheless use Theorem 1 to provide a simple test (cf. Epstein 2000).

Let $\mathbb{Z}^S$ denote the set of all integer-valued functions on $S$, and $\mathcal{M}$ the commutative family consisting of the transformations $f \mapsto f + g$, for $g \in \mathbb{Z}^S$. By identifying elements of $X$ with their indicator functions, we may regard $X$ as a subset of $\mathbb{Z}^S$, and hence any order $\succeq$ on $X$ as an (incomplete) order $\succeq^*$ on $\mathbb{Z}^S$. Any probability measure $\mu$ on $S$ defines a (i) complete, (ii) transitive, (iii) increasing, and (iv) $\mathcal{M}$-invariant order $\succeq$ on $\mathbb{Z}^S$ via:

$$f \succeq g \iff \int f \, d\mu \geq \int g \, d\mu.$$

Conversely, however, not every order on $\mathbb{Z}^S$ satisfying (i) - (iv) can be represented by such a functional.[29] Nonetheless, every $\succeq^*$ whose $\mathcal{M}$-closure is acyclic can be extended to a preference on $\mathbb{Z}^S$ admitting such a representation.[30]

**Corollary 1.** *A qualitative probability $\succeq$ on $2^S$ is probabilistically sophisticated if and only if the $\mathcal{M}$-closure of $\succeq^*$ is acyclic.*

In light of Corollary 1, the counterexample of Kraft et al. (1959) must feature some cycle in its $\mathcal{M}$-closure.[31] More generally, Corollary 1 gives a necessary and sufficient condition for arbitrary, possibly incomplete, data on a subject's likelihood assessments to be consistent with some prior over $S$.

**Corollary 2.** *A subject's subjective likelihood assessments $\langle \succsim^R, \succ^R \rangle$ are consistent with a Bayesian prior if and only if:*

$$\Big\langle (\succsim^R \cup \supseteq)^*, \big( \succ^R \cup \{(S, \varnothing)\} \big)^* \Big\rangle$$

*has an acyclic $\mathcal{M}$-closure.*

---

[29]Such orders may fail to be *Archimedean* in the sense of Krantz et al. (1971), p. 73; for example, the lexicographic order on $\mathbb{Z}^2$ satisfies (i) - (iv), but has no representation of this form.

[30]This follows from Theorem 1.4 of Scott (1964). Formally, Scott shows that a necessary and sufficient condition for $\succeq$ to be probabilistically sophisticated is for $\succeq^*$ to be able to be extended into a so-called "strictly monotonic" order (Scott 1964, p. 237). It is straightforward to show any $\mathcal{M}$-invariant preference on $\mathbb{Z}^S$ is strictly monotonic in Scott's sense.

[31]We explicitly exhibit such a cycle in Section D.1.1.

## 3.2 Application: Additive Separability

Suppose that $X = \times_{k=1}^{K} X_k$ is a finite set. We say that a preference $\succeq$ on $X$ is **additively separable** if there exist $K$ real-valued functions $u_k : X_k \to \mathbb{R}$ such that:

$$x \succeq y \quad \text{iff} \quad \sum_{k=1}^{K} u_k(x_k) \geq \sum_{k=1}^{K} u_k(y_k).$$

Call a finite sequences of pairs of tuples $(x^1, y^1), \ldots, (x^N, y^N) \in X \times X$ *admissible* if $(x_k^1, \ldots, x_k^N)$ is a permutation of $(y_k^1, \ldots, y_k^N)$ for all $k = 1, \ldots, K$. A preference turns out to be additively separable if, and only if, for any admissible sequence:

$$\left.\begin{array}{c} x^1 \succeq y^1 \\ \vdots \\ x^{N-1} \succeq y^{N-1} \end{array}\right\} \implies x^N \not\succ y^N, \tag{3}$$

see Tversky (1964); Fishburn (1970).

While (3) is not an invariance axiom, Theorem 1 nonetheless is able to provide a necessary and sufficient test for rationalizability by an additively separable preference. Fix an enumeration of each $X_k = \{x_{k,1}, \ldots, x_{k,|X_k|}\}$. This determines an enumeration of $X = \{x_{1,1}, \ldots, x_{K,|X_K|}\}$. For any $x \in X$, let $\delta_x \in \mathbb{Z}_+^X$ denote the vector whose $(i,j)$-th component is 1 if $x_i$ is the $j$-th element in the enumeration of $X_i$, and zero otherwise. Finally, given data $\langle \succsim^R, \succ^R \rangle$, let $\left\langle (\succsim^R)^*, (\succ^R)^* \right\rangle$ denote the induced orders on $\mathbb{Z}_+^X$ obtained by associating each tuple $x$ with its vector representation $\delta_x$.

Define $\mathcal{M}$ to be the set of transformations $\mathbb{Z}_+^X \to \mathbb{Z}_+^X$ of the form:

$$x \mapsto x + y,$$

for each $y \in \mathbb{Z}_+^X$. By Theorem 1.4 of Scott (1964), a preference on $\{\delta_x : x \in X\} \subset \mathbb{Z}_+^X$ admits an additively separable utility representation if, and only if, it can be extended into an $\mathcal{M}$-invariant preference on $\mathbb{Z}_+^X$. Thus we obtain the following characterization of additively separable rationalizability.

**Corollary 3.** *The data $\langle \succsim^R, \succ^R \rangle$ are rationalizable by an additively separably preference if, and only if:*

$$\left\langle (\succsim^R)^*, (\succ^R)^* \right\rangle$$

*has an acyclic $\mathcal{M}$-closure.*

In particular, Corollary 3 imposes no restrictions on the type of data and hence is applicable to natural experiments involving choice from finite menus, e.g. Harbaugh et al. (2001).

## 3.3 Application: Unanimity Representations

A longstanding strand of economic research has sought to relax the completeness axiom in the study of preferences (Aumann 1962; Bewley 2002; Mandler 2005; Evren and Ok 2011). In such work, preorders (reflexive and transitive binary relations) instead form the basic model of incomplete preferences.

Absent other structure, every preorder can be expressed as a *unanimity* relation, i.e. the set of comparisons agreed upon by every (complete) preference in some family (Dushnik and Miller 1941). A natural question, then, is whether a preorder satisfying additional axioms can always be regarded as the intersection of some family of complete preferences satisfying the same extra conditions (e.g. Bewley 2002; Dubra et al. 2004).

When the additional structure of interest takes the form of *commutative* invariance axioms, Theorem 1 provides an affirmative answer to this question. We illustrate some concrete implications of this below.

### 3.3.1 Additive Preorder Extensions

Let $V$ denote a real vector space, and $C \subseteq V$ a cone. Suppose that $X \subseteq V$ is closed under addition by vectors in $C$, i.e. if $x \in X$ and $c \in C$, then $x + c \in X$. Let $\mathcal{M}$ denote all transformations of the form $x \mapsto x + c$, for $c \in C$. In this circumstance, an $\mathcal{M}$-invariant preference is said to be $C$-**additive**.

**Corollary 4.** *Every $C$-additive preorder $\succeq$ admits a $C$-additive preference extension. Moreover, $\succeq$ is the intersection of a family of $C$-additive preferences if, and only if, for any $c \in C$:*

$$x + c \underset{(\succ)}{\succeq} y + c \quad \implies \quad x \underset{(\succ)}{\succeq} y.$$

Corollary 4 shows that any $C$-additive preorder can always be extended into a $C$-additive preference.[32] Examples of this include:

[32] While we arrive at it through different techniques, the first claim of Corollary 4 can also be

(i) **Quasilinearity**: Let $V = \mathbb{R}^L$, $X = \mathbb{R}^L_+$, and $C = \{(a, 0, \ldots, 0) : a \geq 0\}$. An incomplete preference $\succeq$ is $C$-additive if and only if it is quasilinear, and by Corollary 4, a quasilinear preorder is the intersection of quasilinear preferences on $X$ if and only if it is also invariant under *subtraction* of numeraire, whenever this is well-defined.

(ii) **CARA**: Let $V = X = L^\infty(S, \mathcal{S}, \mathbb{P})$ for some probability space $(S, \mathcal{S}, \mathbb{P})$, and let $C$ denote the subspace spanned by the (equivalence classes of) $\mathbb{P}$-a.e. constant functions. Suppose $\succeq$ is a preorder on $X$ that is indifferent between any two random variables which coincide in law. Then $\succeq$ is $C$-additive if and only if it exhibits constant absolute risk aversion. By Corollary 4, every such preorder can be represented as the intersection of CARA preferences on $X$; moreover these preferences will also depend only on the random variables' laws.

(iii) **Additive Statistics**: Let $V = X = C = L^\infty(S, \mathcal{S}, \mathbb{P})$. In this case, $C$-additive preorders are often referred to simply as 'additive.' Additive orderings have been studied from the perspective of risk measures (Goovaerts et al. 2004) and individual decision-making (Mu et al. 2021). Here, Corollary 4 implies any additive preorder is always the intersection of some collection of additive preferences; as in the case of CARA preferences, if $\succeq$ only depends on the laws of the random variables, then this property will hold for every preference as well.

### 3.3.2 Homothetic preorder extensions

Recall a preorder on a cone $X$ in a real vector space is *homothetic* if it is invariant under transformations of the form $x \mapsto \lambda x$, where $\lambda > 0$. Demuynck (2009) shows that if $X$ is a cone in a Euclidean space, then every montonic and homothetic preorder can be extended by a monotonic and homothetic preference. The following corollary establishes a modest generalization of this result, dispensing with both monotonicity and finite dimensionality:

**Corollary 5.** *Every homothetic preorder has a homothetic preference extension. Moreover, every homothetic preorder is the intersection of a collection of homothetic preferences.*

obtained as a straightforward consequence of Corollary 3.4 in Ok and Riella (2021).

DeMuynck's result follows as any extension of a monotonic preorder is by definition monotonic as well. As an application of Corollary 5, we obtain:

(iv) **CRRA**: Let $X = L^\infty(S, \mathcal{S}, \mathbb{P})$ for some probability space $(S, \mathcal{S}, \mathbb{P})$, and suppose $\succeq$ is a preorder which depends only on the law of the random variables. We say $\succeq$ exhibits constant relative risk aversion if and only if $\succeq$ is homothetic. By Corollary 5, we obtain that every such $\succeq$ is the intersection of a family of CRRA preferences.

#### 3.3.3 An Algebraic Version of Dubra et al. (2004)

Let $\Delta(Y)$ denote the set of countably additive probability measures on some fixed measurable space $(Y, \mathcal{Y})$, and suppose $\succeq$ is a preorder which satisfies the *rational* independence axiom, i.e. for all $p, q, r \in \Delta(Y)$, and all $\lambda \in (0,1] \cap \mathbb{Q}$:

$$p \succeq q \quad \iff \quad \lambda p + (1-\lambda) r \succeq \lambda q + (1-\lambda) r.$$

Let $\Sigma^0$ (resp. $\Sigma^1$) denote the sets of countably additive signed measures that assign zero (resp. unit) mass to $Y$. By minor abuse of notation, we regard $\succeq$ as a relation on $\Sigma^1$. Define on $\Sigma^1$ the binary relation:

$$\mu \succeq' \nu \quad \iff \quad \exists\, \alpha > 0 \text{ s.t. } \mu - \nu = \alpha(p - q) \text{ where } p \succeq q,$$

and respectively $\succ'$, by instead by requiring $p \succ q$. Finally, let $\mathcal{M}$ denote the set of transformations of the form:

$$\mu \mapsto \mu + \theta,$$

for all $\theta \in \Sigma^0$. This turns out to yield an $\mathcal{M}$-invariant extension of $\succeq$.

**Lemma 1.** *The binary relation $\succeq'$ is an $\mathcal{M}$-invariant preorder whose asymmetric component is $\succ'$.*

Since $\mathcal{M}$ is a commutative family, by applying Theorem 1 we obtain an $\mathcal{M}$-invariant preference $\succeq^*$ on $\Sigma^1$ extending $\succeq'$. We claim now the restriction of $\succeq^*$ to $\Delta(Y)$ in fact satisfies rational independence axiom. To prove this, we first establish an intermediate lemma.

**Lemma 2.** *Suppose $p \succeq^* q$. Then for every rational $\gamma \in (0,1] \cap \mathbb{Q}$,*

$$p \succeq^* \gamma p + (1-\gamma) q \succeq^* q.$$

Suppose then that $p, q, r \in \Delta(Y)$ and $\lambda \in (0, 1] \cap \mathbb{Q}$ are arbitrary. Let:

$$\theta = (1 - \lambda)(r - p),$$

and note that $\lambda q + (1 - \lambda)r - \theta = \lambda q + (1 - \lambda)p$. By Lemma 2 and $\mathcal{M}$-invariance, we obtain:

$$\begin{aligned} p \succeq^* q \quad &\iff \quad p \succeq^* \lambda q + (1 - \lambda)p \\ &\iff \quad p + \theta \succeq^* \lambda q + (1 - \lambda)p + \theta \\ &\iff \quad \lambda p + (1 - \lambda)r \succeq^* \lambda q + (1 - \lambda)r, \end{aligned}$$

and hence the restriction of $\succeq^*$ to $\Delta(Y)$ satisfies rational independence.

**Corollary 6.** *Every rationally independent preorder has a rationally independent preference extension.*

# 4 The General Case

Absent the hypothesis of commutativity, the following example shows that Theorem 1 fails dramatically, even in extremely simple cases.

**Example 2.** Suppose $Z$ is a set of prizes, and that our consumption space is $Z^{\mathbb{N}}$, the set of all infinite-horizon consumption streams taking values in $Z$. We are interested in testing whether a given set of observations is not only consistent with a rational preference, but one that is additionally *stationary* in the sense of Koopmans (1960). Recall that stationarity in this context means invariance with respect to the family generated by the transformations:

$$(x_1, x_2, \ldots) \mapsto (z, x_1, x_2, \ldots)$$

for each $z \in Z$.[33]

Suppose the empiricist observes:

$$\begin{aligned} (a, x_1, \ldots) \succ^R (b, y_1, \ldots) \\ (b, x_1, \ldots) \succ^R (a, y_1, \ldots), \end{aligned} \tag{4}$$

[33]Note that these transformations do not commute, as generally $(z, w, x_1, \ldots) \neq (w, z, x_1, \ldots)$ for arbitrary $z, w \in Z$.

and

$$\begin{aligned}(c, y_1, \ldots) &\succ^R (d, x_1, \ldots) \\ (d, y_1, \ldots) &\succ^R (c, x_1, \ldots),\end{aligned} \tag{5}$$

for two fixed, distinct consumption streams $x$ and $y$, and where $a, b, c, d$ are all prizes in $Z$. We assume that neither $x$ nor $y$ ever takes the values $a, b, c$, or $d$.

As in Example 1, the data are transitive, and hence consistent with the maximization of some preference relation. Indeed, their $\mathcal{M}$-closure is transitive, and hence acyclic.[34]

However, there is no stationary preference consistent with both (4) and (5). By completeness, any such preference would need to rank either $y \succeq x$ or $x \succeq y$. In the former case, invariance would imply $(a, y_1, \ldots) \succeq (a, x_1, \ldots)$ and $(b, y_1, \ldots) \succeq (b, x_1, \ldots)$ as knock-on effects, forming a cycle:

$$(a, x_1, \ldots) \succ (b, y_1, \ldots) \succeq (b, x_1, \ldots) \succ (a, y_1, \ldots) \succeq (a, x_1, \ldots)$$

just as in Example 1. However, by an identical argument, the knock-on effects arising from ranking $x \succeq y$ also lead to a transitivity violation, this time involving the comparisons observed in (5). Thus there can be no possible extension of the data that is complete, transitive, and $\mathcal{M}$-invariant. ■

When the transformations in $\mathcal{M}$ commute, the essence of Theorem 1 was that scenarios such as Example 2 could not occur without forcing a cycle somewhere in the $\mathcal{M}$-closure. Without commutativity, this need not be the case: invariant extensions may fail to exist purely on the basis of mutually unsatisfiable *out-of-sample* restrictions imposed by the data.

## 4.1 Generalized Invariance

Before we turn to the problem of characterizing invariant rationalizability, we first wish to provide a more general notion of $\mathcal{M}$-invariance than was given in Section 2.3. Not only does our expanded notion of invariance here cover a number of new and economically interesting examples, it will also be essentially costless to consider this expanded notion when characterizing invariant rationalizability generally.

[34] See Section D.1.2.

A **partial function** from $X \to X$ is a function $\omega$ whose domain is a (possibly proper) subset of $X$, denoted $\mathrm{dom}(\omega)$. We say a preference relation is **invariant** under a partial function $\omega$ if:

$$x \underset{(\succ)}{\succsim} y \quad \implies \quad \omega(x) \underset{(\succ)}{\succsim} \omega(y)$$

whenever $x, y \in \mathrm{dom}(\omega)$ (i.e. whenever the right-hand side is well-defined). The composition of two partial functions $\omega' \circ \omega$ is the obvious partial function whose domain is given by $\omega^{-1}\big(\mathrm{dom}(\omega')\big)$.[35] As before, we say a collection $\mathcal{M}$ of partial functions $X \to X$ defines a set of **generalized invariance axioms** if $\mathcal{M}$ (i) contains the identity function, and (ii) is closed under composition of partial functions.

A number of additional axioms of natural economic interest are of generalized invariance form.

#### 4.1.1 Ordinal Additivity

As in Section 3.1, let $S$ denote a set of states of the world, and let $X = 2^S$ be the collection of all events. A complete and transitive ordering of $X$ is said to satisfy **ordinal additivity** if it is invariant under the collection $\{\omega_A\}_{A \in X}$ where $\omega_A$ denotes the partial function:

$$B \mapsto A \cup B,$$

where for each $A \in X$, $\mathrm{dom}(\omega_A)$ consists of all events disjoint from $A$.

#### 4.1.2 Separability

Let $X = \times_{k=1}^{K} X_k$ denote the set of all bundles taking values in the sets $\{X_k\}_{k=1}^{K}$. For any subset $A \subseteq \{1, \ldots, K\}$, and any $x, y \in X$, we define the bundle $x_A y \in X$ via:

$$(x_A y)_i = \begin{cases} x_i & \text{if } i \in A \\ y_i & \text{if } i \notin A. \end{cases}$$

For any non-empty set $A \subseteq \{1, \ldots, K\}$, and any sub-bundles $z, z' \in \times_{k \in A} X_k$, let $\omega_A^{z,z'}$ denote the partial function:

$$x_A z \mapsto x_A z',$$

[35] If the range of $\omega$ and domain of $\omega'$ do not intersect, their composition is simply the empty partial function.

where $\mathrm{dom}(\omega_A^{z,z'})$ consists of those bundles of the form $x'_A z$. A preference is **separable** (Leontief 1947; Debreu 1959) if it is invariant under all partial transformations of this form.[36]

#### 4.1.3 Savage's (P2)

Let $S$ denote a set of states of the world, $\mathcal{X}$ a set of consequences, and $X = \mathcal{X}^S$ the set of all acts mapping $S \to \mathcal{X}$. For any three acts $f, g, h \in X$, and any $A \subseteq S$, define:

$$f_A h = \begin{cases} f(s) & \text{if } s \in A \\ h(s) & \text{if } s \notin A. \end{cases}$$

Axiom (P2) of Savage (1954) requires that a preference $\succeq$ on $X$ satisfy:

$$f_A h \underset{(\succ)}{\succeq} g_A h \quad \Longrightarrow \quad f_A h' \underset{(\succ)}{\succeq} g_A h',$$

for all $f, g, h, h' \in X$ and $A \subseteq S$. By an analogous construction to Section 4.1.2, this defines a generalized invariance axiom.

#### 4.1.4 Quasi-stationarity

As in Example 2, let $X = Z^{\mathbb{N}}$ denote a space of infinite-horizon consumption streams taking values in some set of prizes $Z$. For all $z, z' \in Z$, define the partial transformation $\omega_z^{z'}$ via:

$$(z, x_1, \ldots) \mapsto (z, z', x_1, \ldots),$$

where $\mathrm{dom}(\omega_z^{z'})$ is the set of consumption streams taking the value $z$ in the first period. A preference $\succeq$ is said to be **quasi-stationary**, in the sense of Olea and Strzalecki (2014), precisely when it is invariant under the family of transformations generated by the partial functions $\{\omega_z^{z'}\}_{z,z' \in Z}$.

#### 4.1.5 Comonotonic Independence

Let $\{1, \ldots, S\}$ denote a finite set of states of the world and $X = \mathbb{R}^S$ the set of all monetary acts. Two acts $f, g \in X$ are said to be *comonotonic* if it is never the case that:

$$f(s) > f(s') \quad \text{and} \quad g(s) < g(s')$$

[36] That is, invariant under every transform $\omega_A^{z,z'}$ for every compatible choice of $A$, $z$, and $z'$.

for any $s, s' \in S$. For each $g \in X$ and $\alpha \in (0,1]$, let $\omega_g^\alpha$ denote the partial function:

$$f \mapsto \alpha f + (1-\alpha)g,$$

where $\mathrm{dom}(\omega_g^\alpha)$ consists of those acts comonotonic with $g$. A preference then satisfies the **comonotonic independence** axiom of Schmeidler (1989) if and only if it is invariant under the family of transformations $\{\omega_g^\alpha\}$, as $g$ ranges over $X$ and $\alpha$ over $(0,1]$.

## 4.2 Characterizing Rationalizability

We now turn to a more in-depth analysis of the types of out-of-sample, or *indirect*, restrictions that can be generated by the data, and the requirements of invariance and rationality. Let $\omega_0, \ldots, \omega_N \in \mathcal{M}$, and let $x_0, y_0, \ldots, x_N, y_N \in X$ be any sequence of $\succsim^R$-unrelated pairs of alternatives.[37] We say these transforms and pairs of alternatives form a **broken cycle**, if, for all $i = 0, \ldots N$,

$$\omega_i(x_i) \succsim_{\mathsf{T}}^R \omega_{i+1}(y_{i+1}), \tag{6}$$

where the subscript $i$ is understood mod-$(N+1)$.[38] Note that as part of this definition we are implicitly requiring that each $x_i, y_i \in \mathrm{dom}(\omega_i)$. We define the **length** of any such broken cycle to be $N+1$.[39] Finally, if for any $i = 0, \ldots, N$ we have:

$$\omega_i(x_i) \succ_{\mathsf{T}}^R \omega_{i+1}(y_{i+1}),$$

we say (6) defines a **strict** broken cycle.

Any broken cycle implies restrictions on the comparisons a rationalizing preference can make. These restrictions can be represented by order pairs. Given a broken cycle (6), let:

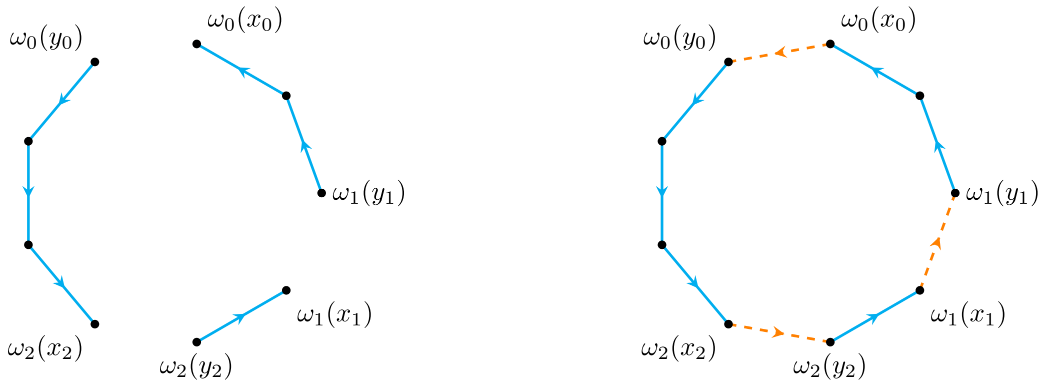
$$W = \big\{(y_i, x_i) : i = 0, \ldots, N\big\},$$

and let $S \subseteq W$. We say the pair $\langle W, S\rangle$ defines a **forbidden subrelation** for (6) if either (i) $S \neq \varnothing$, or (ii) (6) is strict. Forbidden subrelations collect the specific combinations of additional comparisons which, if added to the data, would 'complete' the broken cycle; see Figure 2.

[37] That is, each $x_i$ is unrelated under $\succsim^R$ to $y_i$, for $i = 0, \ldots, N$.

[38] Recall $\langle \succsim_{\mathsf{T}}^R, \succ_{\mathsf{T}}^R \rangle$ denotes the transitive closure of $\langle \succsim^R, \succ^R \rangle$; see Section 2.

[39] In particular, a broken cycle of length one is simply a transform $\omega$ and a $\succsim^R$-unrelated pair of alternatives $x, y \in \mathrm{dom}(\omega)$, such that $\omega(x) \succsim_{\mathsf{T}}^R \omega(y)$.




**(a)** A broken cycle with $N = 3$. **(b)** A forbidden subrelation.

**Figure 2:** A broken cycle (blue) and a forbidden subrelation (dashed orange). Here, the illustrated forbidden subrelation is given by $W = S = \{(y_0, x_0), \dots, (y_2, x_2)\}$. Any relation extending both $\langle \succsim^R, \succ^R \rangle$ and $\langle W, S \rangle$ must contain a cycle.

Informally, forbidden subrelations encode *sets* of constraints on any extension $\succeq$ of the data which must be satisfied as a necessary condition for acyclicity:

$$\text{“Cannot \textit{simultaneously} have } \underbrace{\cdots, y_{i_j} \succeq x_{i_j}, \cdots}_{\text{Relations in } W \setminus S} \text{ and } \underbrace{\cdots, y_{i_k} \succ x_{i_k}, \cdots}_{\text{Relations in } S}\text{.”}$$

Let $\mathcal{F}$ denote the collection of all forbidden subrelations generated by broken cycles in the data. The requirement that an extension of the data not extend any pair in $\mathcal{F}$ generalizes the standard cyclic consistency conditions in, e.g., Richter (1966) or Nishimura et al. (2017). For example, as we assume $\succsim^R$ is reflexive, and the identity function belongs to $\mathcal{M}$, for any collection $x_0, \dots, x_N \in X$, we have a broken cycle $x_i \succsim^R_{\mathsf{T}} x_i$, $i = 1, \dots, N$, whose forbidden subrelations encode the requirement that any rationalizing preference cannot cycle over these alternatives.[40]

Forbidden subrelations also account for knock-on effects in a systematic fashion. In Example 2, the observations (4) define a broken cycle of length two. Here, however, the pair $W = \{(y, x)\}$ and $S = \varnothing$ defines a forbidden subrelation. This is the formal encoding of the earlier observation that no stationary preference extension can rank $y \succeq x$. The fact this subrelation contains *fewer* comparisons than the length of

[40]Similarly, if there is an indirect revealed preference between $x$ and $y$, e.g. $x \succsim^R_{\mathsf{T}} y$, but no direct revealed preference, then this yields a broken cycle of length one, whose (unique) forbidden subrelation encodes the requirement that any extension $\succeq$ must rank $x \succeq y$.

the broken cycle (4) is because the constraint $\langle W, S\rangle$ is implicitly accounting for the knock-on effects that would arise from its violation.

We now seek to determine when any $\mathcal{M}$-invariant preference exists which (i) extends the data, but (ii) does not extend any pair in $\mathcal{F}$. Suppose, based on two broken cycles, the empiricist observes restrictions:

$$\langle W_1, S_1\rangle : \quad \text{“Cannot simultaneously have } x \succeq y \text{ and } x' \succ y', \text{”}$$

and

$$\langle W_2, S_2\rangle : \quad \text{“Cannot simultaneously have } x'' \succeq y'' \text{ and } y \succ x. \text{”}$$

Since any rationalizing preference $\succeq$ must rank either $x \succeq y$ or $y \succ x$, the empiricist can deduce a further, indirect, constraint:

$$\langle \hat{W}, \hat{S}\rangle : \quad \text{“Cannot simultaneously have } x'' \succeq y'' \text{ and } x' \succ y', \text{”}.$$

Should this be violated, then $\langle W_1, S_1\rangle$ or $\langle W_2, S_2\rangle$ must also be violated, and hence the extension must fail to be transitive.[41]

We refer to this operation of reducing compatible constraint sets by appeal to the law of the excluded middle as ‘collapsing’ them. More formally, given finite order pairs $\langle W_1, S_1, \rangle$ and $\langle W_2, S_2\rangle$, we say they are **compatible** if there exists some pair $\omega_1, \omega_2 \in \mathcal{M}$ and $x, y \in \mathrm{dom}(\omega_1) \cap \mathrm{dom}(\omega_2)$ such that:

$$\big(\omega_i(x), \omega_i(y)\big) \in W_i \setminus S_i \quad \text{and} \quad \big(\omega_{-i}(y), \omega_{-i}(x)\big) \in W_{-i}.$$

In other words, $\langle W_1, S_1\rangle$ and $\langle W_2, S_2\rangle$ are compatible if there exist alternatives $x$ and $y$ such that, for *any* choice of ranking of $x$ and $y$, some knock-on effect of that ranking always belongs to one of the pairs. When $\langle W_1, S_1\rangle$ and $\langle W_2, S_2\rangle$ are compatible, their **collapse** is the pair $\langle \hat{W}, \hat{S}\rangle$ given by:

$$\hat{W} = W_1 \cup W_2 \setminus \big\{(\omega_i(x), \omega_i(y)), (\omega_{-i}(y), \omega_{-i}(x))\big\}$$

[41] Moreover, if instead the empiricist observed:

$$\langle W_1', S_1'\rangle : \quad \text{“Cannot simultaneously have } \omega(x) \succeq \omega(y) \text{ and } x' \succ y', \text{”}$$

and

$$\langle W_2', S_2'\rangle : \quad \text{“Cannot simultaneously have } x'' \succeq y'' \text{ and } \omega'(y) \succ \omega'(x), \text{”}$$

the same conclusion would obtain.

and

$$\hat{S} = S_1 \cup S_2 \setminus \big\{(\omega_i(x), \omega_i(y)), (\omega_{-i}(y), \omega_{-i}(x))\big\}.$$

By construction, if any extension of the data also extends the collapse of two forbidden subrelations, it must extend (at least) one of them.

By repeatedly applying the collapse operation, we are able to iteratively reduce constraint sets. This allows us to construct a test for feasibility. Let $\mathcal{F}^1$ denote the collection of all order pairs in $\mathcal{F}$, as well as their collapses and, recursively, let $\mathcal{F}^n$ denote the collection of all order pairs in $\mathcal{F}^{n-1}$ and their collapses. Define:

$$\mathcal{F}^* = \bigcup_{n \in \mathbb{N}} \mathcal{F}^n.$$

We say that the data are **strongly acyclic** if the *empty* order pair $\langle \varnothing, \varnothing \rangle$ does not belong to $\mathcal{F}^*$. Intuitively, if any extension of the data extends any relation in $\mathcal{F}^*$, then it must necessarily extend some forbidden subrelation, somewhere. Thus, if $\langle \varnothing, \varnothing \rangle$ belongs to $\mathcal{F}^*$, it means that *every* complete, $\mathcal{M}$-invariant extension of the data must fail to be transitive, and hence that the data are not rationalizable.

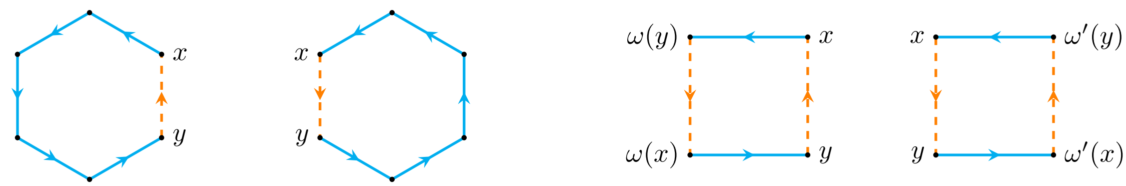
Our next result shows that strong acyclicity is, in fact not only necessary but also sufficient for rationalizability by an $\mathcal{M}$-invariant preference. Notably, this remains true without any assumptions on the domain, data, or structure of $\mathcal{M}$.

**Theorem 2.** *The data $\langle \succsim^R, \succ^R \rangle$ are strongly acyclic if, and only if, they are rationalizable by an $\mathcal{M}$-invariant preference.*

### 4.2.1 Discussion

One of the foundational observations of revealed preference theory is that choice data contains implications beyond those those which are directly observed (Samuelson 1938; Houthakker 1950). For example, if it is observed that $x \succsim^R y \succsim^R z$, the empiricist is able to conclude that *any* rationalization must regard $x$ to be at least as good as $z$.

This form of indirect inference gives rise to the many cyclic consistency conditions in the revealed preference literature (e.g. Afriat 1967; Richter 1966; Nishimura et al. 2017): a revealed preference cycle is simply a collection of observations that indirectly reveal $x$ to be weakly preferred to $y$, while also (directly) revealing $y$ to be strictly preferred to $x$.




**(a)** Without invariances, if two broken cycles define constraint sets whose collapse is $\langle \varnothing, \varnothing \rangle$, they can be 'glued together' into a (complete) cycle.

**(b)** However generally, even if two broken cycles define constraints whose collapse is $\langle \varnothing, \varnothing \rangle$, they need not yield a revealed preference cycle.

**Figure 3:** When $\mathcal{M} = \{\mathrm{id}\}$, strong acyclicity is equivalent to the acyclicity of $\langle \succsim^R, \succ^R \rangle$. Indeed, when $\mathcal{M}$ is commutative, this remains true for the modified data $\langle \succsim^R_{\mathcal{M}}, \succ^R_{\mathcal{M}} \rangle$. However, for general $\mathcal{M}$, failures of rationalizability may take on more complex forms.

Invariance axioms, however, introduce a fundamentally new type of inference: knock-on effects. When $\mathcal{M} = \{\mathrm{id}\}$, it is straightforward to show that strong acyclicity reduces to the classical requirement that the revealed preference be acyclic; see Figure 3. In this case, despite being defined at the level of constraint sets, strong acyclicity admits an equivalent, simpler description in terms of the revealed preference itself. When $\mathcal{M}$ is commutative, Theorem 1 shows that this remains true, when the data are replaced by their $\mathcal{M}$-closure, $\langle \succsim^R_{\mathcal{M}}, \succ^R_{\mathcal{M}} \rangle$.

However, in general, no such simplifications are possible. Despite this, we argue strong acyclicity is simply the *set-valued* analogue of standard, no-cycle conditions. Suppose $\trianglerighteq$ is any binary relation on $X$. To construct the transitive closure $\trianglerighteq_{\mathsf{T}}$, one first considers pairs of compatible comparisons $x \trianglerighteq y$ and $y \trianglerighteq z$. Here, compatibility simply refers to the fact $y$ is the dominated alternative in one pair, and the dominating alternative in the other. The first-order implication of transitivity, $x \trianglerighteq z$ is obtained by collapsing these elements, canceling off the opposing $y$ terms, and collecting what remains. The full transitive closure is then obtained by collecting all those pairs formed by iterating this pairwise collapse operation any finite number of times. This is precisely analogous to our construction of $\mathcal{F}^*$.[42]

[42]Moreover, conventionally a cycle is simply a collection $(x_0, x_1), (x_1, x_2), \ldots, (x_N, x_0) \in \mathcal{R}$ such that every $x_i$ appears as a head of one element, and tail of another. Analogously, a violation of strong acyclity is simply a collection of constraint sets such that every relation, in every set, is collapsed away at some point in the reduction process.

#### 4.2.2 Proof Sketch

Methodologically, the proof of Theorem 2 differs fundamentally from the transfinite induction techniques commonly used in the literature (Richter 1966; Duggan 1999; Demuynck 2009).[43] We first define, for each pair $(x, y) \in X \times X$ two boolean variables, denoted:

$$[\mathtt{x} \succeq \mathtt{y}] \quad \text{and} \quad [\mathtt{x} \succ \mathtt{y}].$$

Denote set of all such variables by $\mathcal{V}$. Formally, a *model* is a map $\mu : \mathcal{V} \to \{\mathtt{True}, \mathtt{False}\}$. Given some logical formula relating a finite collection of variables in $\mathcal{V}$, we say that a model is consistent with the formula if the truth and falsity assignments made by the model make the formula evaluate to true.[44]

We then construct a set $\Phi$ of logical formulas involving the variables in $\mathcal{V}$ in such a manner as to obtain a one-to-one relation between models consistent with $\Phi$, and $\mathcal{M}$-invariant weak orders rationalizing the data $\langle \succsim^R, \succ^R \rangle$.[45] Given $\Phi$, the data are rationalizable by an $\mathcal{M}$-invariant preference relation if and only if there exists *any* model $\mu$ consistent with $\Phi$, i.e. if $\Phi$ is satisfiable.

We first show that any violation of strong acyclicity can be algorithmically transformed to yield a formal proof of the unsatisfiability of $\Phi$. Given any constraint set in $\langle W, S \rangle \in \mathcal{F}^*$, we can uniquely associate it with its **clausal representation**:

$$\langle W, S \rangle \mapsto \left[ \bigvee_{(x,y) \in W \setminus S} \neg [\mathtt{x} \succeq \mathtt{y}] \right] \vee \left[ \bigvee_{(x,y) \in S} \neg [\mathtt{x} \succ \mathtt{y}] \right].$$

The clausal representation of any constraint set has a particular form: it is a disjunction solely of *negated* variables in $\mathcal{V}$.[46] We then show that, given any two compatible constraint sets, there is a formal proof deriving the (clausal representation) of their

[43] As well as in the proof of Theorem 1.

[44] For example, if $\mu([\mathtt{x} \succeq \mathtt{y}]) = \mathtt{True}$ and $\mu([\mathtt{x} \succ \mathtt{y}]) = \mathtt{False}$, $\mu$ is consistent with the formula $[\mathtt{x} \succeq \mathtt{y}] \vee [\mathtt{x} \succ \mathtt{y}]$, as these assignments cause this formula to evaluate to $\mathtt{True}$.

[45] For example, for every $x, y, z \in X$, $\Phi$ includes the formula:

$$\neg [\mathtt{x} \succeq \mathtt{y}] \vee \neg [\mathtt{y} \succeq \mathtt{z}] \vee [\mathtt{x} \succeq \mathtt{z}],$$

which jointly encode the requirement that a rationalization $\succeq$ be transitive; here, we use $\neg$ to denote logical negation, and $\vee$ to denote disjunction. For a complete description of $\Phi$, see Section B.1.

[46] These may be viewed as propositional logic analogues of the 'UNCAF' formulas of Chambers et al. (2014).

collapse from their clausal representations, and by carefully combining these 'sub-proofs,' one can obtain a formal proof of unsatisfiability of $\Phi$.

To prove the converse, we rely on a theorem due to Robinson (1965) which shows that if a *finite* set of clauses is unsatisfiable, then there exists a formal proof of this fact with precise, combinatorial form. By the axiom of choice, if $\Phi$ is unsatisfiable, then so too is some finite subset $\Phi' \subset \Phi$.[47] By Robinson's theorem, there exists a particular form of proof of the unsatisfiability of $\Phi'$; we then show that our method of mapping violations of strong acyclicity into formal proofs of unsatisfiability can be inverted, *precisely* for those proofs of the form Robinson's theorem guarantees, and hence we obtain a violation of strong acyclity from the proof of unsatisfiability of $\Phi'$.

# 5 Out-of-Sample Predictions

In light of Theorem 2, the collapse operation provides a purely algorithmic means of evaluating whether or not an $\mathcal{M}$-invariant rationalizing preference exists for the data. However, the collapse is defined over sets of restrictions, rather than the data itself. In particular, it does not speak to which comparisons *every* $\mathcal{M}$-invariant, rationalizing preference must agree upon. When the data are rationalizable by at least one such preference, we term these comparisons the **out-of-sample predictions** generated by the model and data.

When $\mathcal{M} = \{\text{id}\}$, every ($\mathcal{M}$-invariant) rationalizing preference $\succeq^*$ ranks $x \succeq^* y$ if and only if $x \succsim^R_{\mathsf{T}} y$. However, as illustrated by Example 1, when $\mathcal{M}$ is richer, so too are the set of counterfactual predictions generated by the class of $\mathcal{M}$-invariant preferences. Moreover, Example 1 shows that the set of such predictions is richer than either the transitive, or $\mathcal{M}$-invariant closure.

It turns out, however, that the set of out-of-sample predictions generated by the $\mathcal{M}$-invariant rationalizations of a strongly acyclic data set $\langle \succsim^R, \succ^R \rangle$ are straightforwardly described by collapses.

**Theorem 3.** *Suppose $\langle \succsim^R, \succ^R \rangle$ are strongly acyclic. Then $x \succeq y$ for every $\mathcal{M}$-invariant rationalization if, and only if:*

$$\big\langle (y, x), (y, x) \big\rangle \in \mathcal{F}^*,$$

[47] See, for example, Gonczarowski et al. (2019).

*and $x \succ y$ for every such rationalization if, and only if:*

$$\big\langle (y,x), \varnothing \big\rangle \in \mathcal{F}^*.$$

If $\big\langle (y,x),(y,x) \big\rangle \in \mathcal{F}^*$, then there is a constraint on the extension problem requiring no rationalization to rank $y \succ^* x$. In this case, every rationalization must rank $x \succeq^* y$.[48] As such, this condition is clearly necessary. However, the primary content of Theorem 3 is that look only at such 'singleton' restriction sets is also sufficient: the set of comparisons forced by restrictions of this form are, in fact, the *only* comparisons agreed upon by every rationalization.

# 6 Conclusion

This paper studies the problem of characterizing the empirical content of structured families of preferences, satisfying axioms beyond rationality alone. The basic observation underlying our results is that many of the most economically important and widely used decision-theoretic axioms share a common mathematical structure: they are what we have termed 'invariance axioms.' Our main results provide characterizations of the empirical content and out-of-sample predictions generated by *arbitrary* sets of such axioms. The advantage of this abstraction is that it provides a unified theory and framework for studying a wide range of seemingly disparate economic models that had previously only been studied in isolation. By clarifying the common underlying structure at play, we hope that further work may build on the results here to develop further 'universal' revealed preference characterizations.

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[48]Analogously, $\big\langle (y,x), \varnothing \big\rangle \in \mathcal{F}^*$ encodes the restriction that no rationalization can rank $y \succeq^* x$ and hence every rationalization must rank $x \succ^* y$.

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# Appendix

## A Proof of Theorem 1

Recall that a binary relation relation $\succeq \subseteq X \times X$ is $\mathcal{M}$**-invariant** if:

$$x \succeq y \implies \omega(x) \succeq \omega(y)$$

and

$$x \succ y \implies \omega(x) \succ \omega(y),$$

for all $x, y \in X$ and $\omega \in \mathcal{M}$. We say that $\succeq$ is **strongly** $\mathcal{M}$-invariant if:

$$x \succeq y \iff \omega(x) \succeq \omega(y)$$

and

$$x \succ y \iff \omega(x) \succ \omega(y).$$

**Lemma 3.** *Suppose that an acyclic relation $\succsim^R$ is $\mathcal{M}$-invariant. Then so is its transitive closure, $\succsim^R_{\mathsf{T}}$.*

*Proof.* First, let $x, y \in X$ such that $x \succsim^R_{\mathsf{T}} y$. Then there exists $x_1, \ldots, x_K \in X$, $K \geq 2$, such that $x = x_1 \succsim^R \cdots \succsim^R x_K = y$. By $\mathcal{M}$-invariance of $\succsim^R$, for every $\omega \in \mathcal{M}$, we also have that $\omega(x) = \omega(x_1) \succsim^R \cdots \succsim^R \omega(x_k) = \omega(y)$, hence $\omega(x) \succsim^R_{\mathsf{T}} \omega(y)$ as desired.

Now, suppose that $x \succsim^R_{\mathsf{T}} y$ but it is not the case that $y \succsim^R_{\mathsf{T}} x$. We want to show that it is not the case that $\omega(y) \succsim^R_{\mathsf{T}} \omega(x)$ for any $\omega \in \mathcal{M}$. As $x \succsim^R_{\mathsf{T}} y$, there exist $x_1, \ldots, x_K \in X$, $K \geq 2$ such that $x = x_1 \succsim^R \cdots \succsim^R x_K = y$; since additionally it is not the case that $y \succsim^R_{\mathsf{T}} x$, for some $1 \leq i \leq K-1$, we have $x_i \succ x_{i+1}$. Consequently, for any $\omega \in \mathcal{M}$, by $\mathcal{M}$-invariance, we must also have $\omega(x) = \omega(x_1) \succsim^R \cdots \succsim^R \omega(x_K) = \omega(y)$, where $\omega(x_i) \succ \omega(x_{i+1})$. By acyclicity of $\succsim^R$, it is then not the case that $\omega(y) \succsim^R_{\mathsf{T}} \omega(x)$, and hence $\omega(x) \succ^R_{\mathsf{T}} \omega(y)$ as desired. As $\omega \in \mathcal{M}$ was arbitrary, the result follows. $\square$

**Lemma 4.** *Suppose $\mathcal{M}$ is a commutative family. Then every $\mathcal{M}$-invariant preorder has a strongly $\mathcal{M}$-invariant preorder extension.*

*Proof.* Let $\succsim^R$ be a weakly $\mathcal{M}$-invariant preorder. Define $\succeq$ via $x \succeq y$ if and only if there exists $\omega \in \mathcal{M}$ such that $\omega(x) \succsim^R \omega(y)$.[49]

Since the identity function id $\in \mathcal{M}$, it follows immediately that $\succsim^R \subseteq \succeq$. Suppose, now, that $x \succ^R y$ and, for purposes of contradiction that additionally $y \succeq x$. Then there exist $\omega \in \mathcal{M}$ for which $\omega(y) \succsim^R \omega(x)$. Since $\succsim^R$ is $\mathcal{M}$-invariant, this implies that both: $\omega(y) \succsim^R \omega(x)$ and $\omega(x) \succ^R \omega(y)$, a contradiction. Thus $\succ^R \subseteq \succ$ as well, and hence $\succeq$ defines an extension of $\succsim^R$.

We now claim that $\succeq$ is transitive. Suppose that $x \succeq y \succeq z$. As $x \succeq y$, there exist $\omega \in \mathcal{M}$ for which $\omega(x) \succsim^R \omega(y)$. Similarly, as $y \succeq z$, there exist $\omega' \in \mathcal{M}$ for which $\omega'(y) \succeq \omega'(z)$. By the $\mathcal{M}$-invariance of $\succsim^R$ and by commutativity of $\mathcal{M}$, we obtain $(\omega \circ \omega')(x) \succsim^R (\omega \circ \omega')(y)$ and $(\omega \circ \omega')(y) \succsim^R (\omega \circ \omega')(z)$, and hence $(\omega \circ \omega')(x) \succsim^R (\omega \circ \omega')(z)$ by transitivity of $\succsim^R$. But this means $x \succeq z$, thus we conclude $\succeq$ is transitive.

We now show that $\succeq$ is $\mathcal{M}$-invariant. Suppose that $x \succeq y$ and let $\bar{\omega} \in \mathcal{M}$. There exists $\omega \in M$ for which $\omega(x) \succsim^R \omega(y)$. By commutativity of $\mathcal{M}$ and the $\mathcal{M}$-invariance of $\succsim^R$, we have $(\omega \circ \bar{\omega})(x) \succeq (\omega \circ \bar{\omega})(y)$, and hence $\omega(x) \succeq \omega(y)$. Suppose now, additionally, that $x \succ y$ and, for sake of contradiction that for some $\omega' \in \mathcal{M}$, $\omega'(y) \succeq \omega'(x)$. Then there exists $\omega'' \in \mathcal{M}$ for which $(\omega'' \circ \omega')(y) \succsim^R (\omega'' \circ \omega')(x)$, which by definition implies that $y \succeq x$, a contradiction. Hence $\succeq$ is $\mathcal{M}$-invariant.

Finally, we show that $\succeq$ is strongly $\mathcal{M}$-invariant. Suppose that $\omega(x) \succeq \omega(y)$. Then there exist $\omega' \in \mathcal{M}$ for which $(\omega' \circ \omega)(x) \succsim^R (\omega' \circ \omega)(y)$, which implies $x \succeq y$. Suppose further $\omega(y) \succeq \omega(x)$ is false but, for sake of contradiction, that $y \succeq x$. Then there exist $\omega'' \in \mathcal{M}$ such that $\omega''(y) \succeq \omega''(x)$. By $\mathcal{M}$-invariance of $\succsim^R$, $(\omega \circ \omega'')(y) \succsim^R (\omega \circ \omega'')(x)$. But then by the commutativity of $\mathcal{M}$, we conclude $\omega(y) \succeq \omega(x)$, a contradiction. The result follows. $\square$

**Lemma 5.** *Let $\mathcal{M}$ be a commutative family. Let $\succeq$ be an $\mathcal{M}$-invariant preorder, and $w, z \in X$ be $\succeq$-unrelated (and hence distinct) elements of $X$. Then there is an acyclic $\mathcal{M}$-invariant extension $\succeq'$ of $\succeq$ that renders $w$ and $z$ comparable.*

*Proof.* For each $\omega \in \mathcal{M}$, let $e_\omega : \mathcal{M} \to \mathbb{Z}$ denote the function satisfying $\omega \mapsto 1$ and $\omega' \mapsto 0$ for all $\omega \neq \omega'$. By commutativity, any finite string of compositions of

[49]Recall that as $\mathcal{M}$ is closed under composition, this is equivalent to the existence of $\omega_1, \dots, \omega_K \in \mathcal{M}$ such that $(\omega_1 \circ \cdots \circ \omega_K)(x) \succsim^R (\omega_1 \circ \cdots \circ \omega_K)(y)$.

functions in $\mathcal{M}$ may be associated with a finitely-supported, non-negative valued, function $\mathcal{M} \to \mathbb{Z}$ via:

$$\omega_1^{n_1} \circ \omega_2^{n_2} \circ \cdots \circ \omega_K^{n_K} \quad \mapsto \quad n_1 e_{\omega_1} + \cdots + n_K e_{\omega_K},$$

where $\omega^n$ denotes the $n$-fold composition of $\omega$ with itself. Let $\mathcal{M}^*$ denote the set of all such functions; conversely, every element of $\mathcal{M}^*$ clearly corresponds to some (composition of elements in $\mathcal{M}$ and hence) element of $\mathcal{M}$. Note that if $\mathbf{f}, \mathbf{g} \in \mathcal{M}^*$ represent finite strings of transformations in $\mathcal{M}$, then $\mathbf{f} + \mathbf{g}$ represents their composition. For the remainder of this proof, we will freely associate elements of $\mathcal{M}$ with some fixed choice of representative in $\mathcal{M}^*$; the non-uniqueness of this selection will be irrelevant.

Suppose now, for sake of obtaining a contradiction, that no acyclic, $\mathcal{M}$-invariant extension of $\succeq$ exists that compares $w$ and $z$. Thus every $\mathcal{M}$-invariant binary relation that extends $\succeq$ and renders $w$ and $z$ comparable, contains some cycle; in particular, the minimal such extensions obtained either (i) by adding $w \succ' z$ and $\mathbf{f}(w) \succ' \mathbf{f}(z)$ for all $\mathbf{f}$ associated with some finite composition of elements of $\mathcal{M}$, (ii) by adding $z \succ' w$ and all $\mathbf{f}(z) \succ' \mathbf{f}(w)$, or (iii) by adding $z \sim' w$ and all $\mathbf{f}(z) \sim' \mathbf{f}(w)$, must contain some cycle. Consider first $\succeq' = \succeq \cup \succeq^*$, where $\succeq^*$ contains all relations of the form $w \succ^* z$ and $\mathbf{f}(w) \succ^* \mathbf{f}(z)$ for all finite compositions of elements of $\mathcal{M}$, $\mathbf{f}$. Since $\succeq$ is a preorder, it follows there exists a cycle in $\succeq'$ composed of relations of two forms:

$$\begin{array}{rr} \mathbf{a}^1(z) \succeq \mathbf{a}^2(w) & \mathbf{a}^2(w) \succ^*_{\mathsf{T}} \mathbf{a}^2(z) \\ \vdots & \vdots \\ \mathbf{a}^{I-1}(z) \succeq \mathbf{a}^I(w) & \mathbf{a}^I(w) \succ^*_{\mathsf{T}} \mathbf{a}^I(z) \\ \mathbf{a}^I(z) \succeq \mathbf{a}^1(w) & \mathbf{a}^1(w) \succ^*_{\mathsf{T}} \mathbf{a}^1(z) \end{array} \tag{7}$$

for some $x \in X$, where the left column consists of relations in $\succeq$ and the right sequences solely of relations in $\succeq' \setminus \succeq$. Note that $I \geq 2$, and without loss of generality, each $\mathbf{a}^i$ is distinct.[50]

Analogously, if $\succeq' = \succeq \cup \succeq^*$, where $\succeq^*$ contains all relations of the form $z \succ^* w$

[50] If $I = 1$, then we have $\mathbf{a}^1(z) \succeq x$ and $x \succeq \mathbf{a}^1(w)$, hence $\mathbf{a}^1(z) \succeq \mathbf{a}^1(w)$. Since $\succeq$ is $\mathcal{M}$-invariant, this would imply $w$ and $v$ are $\succeq$-related, which is false.

and $\mathbf{f}(z) \succ^* \mathbf{f}(w)$ for finite compositions $\mathbf{f}$, then there exists a cycle of the form:

$$\begin{array}{rr} \mathbf{b}^1(w) \succeq \mathbf{b}^2(z) & \mathbf{b}^2(z) \succ^*_{\mathsf{T}} \mathbf{b}^2(w) \\ \vdots & \vdots \\ \mathbf{b}^{J-1}(w) \succeq \mathbf{b}^J(z) & \mathbf{b}^J(z) \succ^*_{\mathsf{T}} \mathbf{b}^J(w) \\ \mathbf{b}^J(w) \succeq \mathbf{b}^1(z) & \mathbf{b}^1(z) \succ^*_{\mathsf{T}} \mathbf{b}^1(w) \end{array} \tag{8}$$

for some $x' \in X$, where again the left column consists of relations in $\succeq$, the right solely of sequences of relations in $\succeq' \setminus \succeq$, $J \geq 2$, and each $\mathbf{b}^j$ unique.

Finally, suppose $\succeq' = \succeq \ \cup \ \succeq^*$, where $\succeq^*$ contains all relations of the form $z \sim^* w$ and $\mathbf{f}(z) \sim^* \mathbf{f}(w)$ for finite compositions $\mathbf{f}$. By hypothesis, there is a cycle of the form:

$$\begin{array}{rr} \mathbf{c}^1(y_1) \succeq \mathbf{c}^2(x_2) & \mathbf{c}^2(x_2) \sim^*_{\mathsf{T}} \mathbf{c}^2(y_2) \\ \vdots & \vdots \\ \mathbf{c}^{K-1}(y_{K-1}) \succeq \mathbf{c}^K(x_K) & \mathbf{c}^K(a_K) \sim^*_{\mathsf{T}} \mathbf{c}^K(y_K) \\ \mathbf{c}^K(y_K) \succeq \mathbf{c}^1(x_1) & \mathbf{c}^1(x_1) \sim^*_{\mathsf{T}} \mathbf{c}^1(y_1) \end{array} \tag{9}$$

where at least one relation in the left-hand column is strict, $K \geq 2$, each $\mathbf{c}^k$ is unique, and for all $k = 1, \ldots, K$, $\{x_k, y_k\} = \{w, z\}$.

Now, define:

$$\begin{aligned} \mathbf{p}^i &= \mathbf{a}^{i+1} - \mathbf{a}^i \\ \mathbf{q}^j &= \mathbf{b}^{j+1} - \mathbf{b}^j \\ \mathbf{r}^k &= \mathbf{c}^{k+1} - \mathbf{c}^k, \end{aligned}$$

where we interpret indices $I+1, J+1, K+1 \equiv 1$. Note that each $\mathbf{p}^i, \mathbf{q}^j$, and $\mathbf{r}^k$ is not equal to the zero function $\mathbf{0}$ and, by construction:

$$\sum_{i=1}^{I} \mathbf{p}^i = \sum_{j=1}^{J} \mathbf{q}^j = \sum_{k=1}^{K} \mathbf{r}^k = \mathbf{0}.$$

Consider the sets:

$$\begin{aligned} \tilde{A}_{wz} &= \left\{ \mathbf{r}^k \mid y_k = w,\ x_{k+1} = z \right\} \\ \tilde{A}_{zw} &= \left\{ \mathbf{r}^k \mid y_k = z,\ x_{k+1} = w \right\} \\ \tilde{A}_{ww} &= \left\{ \mathbf{r}^k \mid y_k = w,\ x_{k+1} = w \right\} \\ \tilde{A}_{zz} &= \left\{ \mathbf{r}^k \mid y_k = z,\ x_{k+1} = z \right\}. \end{aligned}$$

Clearly these sets cover $\{\mathbf{r}^1, \ldots, \mathbf{r}^K\}$. However, they may not define a partition. Thus let:

$$\begin{aligned} A_{wz} &= \tilde{A}_{wz} \\ A_{zw} &= \tilde{A}_{zw} \setminus \tilde{A}_{wz} \\ A_{ww} &= \tilde{A}_{ww} \setminus \tilde{A}_{zw} \setminus \tilde{A}_{wz} \\ A_{zz} &= \tilde{A}_{zz} \setminus \tilde{A}_{ww} \setminus \tilde{A}_{zw} \setminus \tilde{A}_{wz}, \end{aligned}$$

if these sets are non-empty, and otherwise define them as $\{\mathbf{0}\}$. By hypothesis, at least one of the $A$ sets must contain non-zero elements. Note that each element of $\{\mathbf{r}^1, \ldots, \mathbf{r}^K\}$ is contained in exactly one set in the collection $\{A_{wz}, A_{zw}, A_{ww}, A_{zz}\}$. Let $\{\mathbf{s}^m_{wz}\}_{m=1}^{|A_{wz}|}$ (resp. $\{\mathbf{s}^m_{zw}\}_{m=1}^{|A_{zw}|}$, $\{\mathbf{s}^m_{ww}\}_{m=1}^{|A_{ww}|}$, and $\{\mathbf{s}^m_{zz}\}_{m=1}^{|A_{zz}|}$) denote enumerations of $A_{wz}$ (resp. $A_{zw}, A_{ww}$, and $A_{zz}$).

We now establish a contradiction, by showing that $\succeq$ contains a cycle, contrary to our hypothesis that it is a preorder. Let $\bar{\mathbf{h}}$ denote a sufficiently large vector in $\mathcal{M}^*$.[51] We will consider two cases in turn.

**Case 1: $|A_{wz}| + |A_{zw}| > 0$.**

To build our cycle, we first define two chains in $\succeq$ which will prove important in our construction.[52] By the top-left relation in (7), we have:

$$\mathbf{a^1}(z) \succeq \mathbf{a^2}(w).$$

By $\mathcal{M}$-invariance, this implies:

$$(\bar{\mathbf{h}} + \mathbf{a^1})(z) \succeq (\bar{\mathbf{h}} + \mathbf{a^2})(w),$$

and hence, so long as $\bar{\mathbf{h}}$ is large enough, i.e. $\bar{\mathbf{h}} + \mathbf{p}^1 \geq \mathbf{0}$, we have:

$$(\bar{\mathbf{h}})(z) \succeq (\bar{\mathbf{h}} + \mathbf{p}^1)(w),$$

by (full) $\mathcal{M}$-invariance. By repeating this logic, and also applying it to relations from (8) and (9), we can obtain lengthy chains of $\succeq$-relations. Let us refer to chain one as the sequence:

[51] Sufficiently in the sense only that each vector in the following sequence remain non-negative valued.

[52] The first chain indexes by $|A_{wz}|$ and the second indexes by $|A_{zw}|$; if either of these are zero, these chains are trivial.

$$
\begin{aligned}
\bar{\mathbf{h}}(z) &\succeq (\bar{\mathbf{h}} + \mathbf{p}^1)(w) \\
&\succeq (\bar{\mathbf{h}} + \mathbf{p}^1 + \mathbf{s}^1_{wz})(z) \\
&\qquad\qquad\vdots \\
&\succeq \left(\bar{\mathbf{h}} + |A_{wz}| \sum_{i=1}^{I} \mathbf{p}^i + I \sum_{m=1}^{|A_{wz}|} \mathbf{s}^m_{wz}\right)(z) \\
&\qquad\qquad\vdots \\
&\succeq \left(\bar{\mathbf{h}} + J\,|A_{wz}| \sum_{i=1}^{I} \mathbf{p}^i + IJ \sum_{m=1}^{|A_{wz}|} \mathbf{s}^m_{wz}\right)(z).
\end{aligned}
$$

which follows simply by repeating application of the above observation.[53] Similarly, we refer to chain two as the sequence of relations:

$$
\begin{aligned}
\bar{\mathbf{h}}(z) &\succeq (\bar{\mathbf{h}} + \mathbf{s}^1_{zw})(w) \\
&\succeq (\bar{\mathbf{h}} + \mathbf{s}^1_{zw} + \mathbf{q}^1)(z) \\
&\qquad\qquad\vdots \\
&\succeq \left(\bar{\mathbf{h}} + J \sum_{m=1}^{|A_{zw}|} \mathbf{s}^m_{wz} + |A_{zw}| \sum_{j=1}^{J} \mathbf{q}^j\right)(z) \\
&\qquad\qquad\vdots \\
&\succeq \left(\bar{\mathbf{h}} + IJ \sum_{m=1}^{|A_{zw}|} \mathbf{s}^m_{wz} + I\,|A_{zw}| \sum_{j=1}^{J} \mathbf{q}^j\right)(z).
\end{aligned}
$$

[53]The first part of this chain, up to:

$$\left(\bar{\mathbf{h}} + |A_{wz}| \sum_{i=1}^{I} \mathbf{p}^i + I \sum_{m=1}^{|A_{wz}|} \mathbf{s}^m_{wz}\right)(z)$$

is constructed as follows: for every $l = 1, \dots, I|A_{wz}|$, every term of the form $(\bar{\mathbf{h}} + \dots + \mathbf{p}^l)(w)$ is followed by a term of the form $(\bar{\mathbf{h}} + \dots + \mathbf{p}^l + \mathbf{s}^l_{wz})(z)$, and for every $l = 0, \dots, I|A_{wz}| - 1$, every term of the form $(\bar{\mathbf{h}} + \dots + \mathbf{s}^l_{wz})(z)$ is followed by a term of the form $(\bar{\mathbf{h}} + \dots + \mathbf{p}^l + \mathbf{s}^{l+1}_{wz})(w)$, where indices on $\mathbf{p}$ are to be understood modulo $I$ and on $s_{wz}$ modulo $|A_{wz}|$ as above. The second part of this chain, up through:

$$\left(\bar{\mathbf{h}} + J\,|A_{wz}| \sum_{i=1}^{I} \mathbf{p}^i + IJ \sum_{m=1}^{|A_{wz}|} \mathbf{s}^m_{wz}\right)(z),$$

follows by iterating the first $I|A_{wz}|$ steps of this construction an additional $|J| - 1$ times.

Appending these chains together then yields a chain:

$$\bar{\mathbf{h}}(z) \succeq \cdots \succeq \left(\bar{\mathbf{h}} + I\,|A_{zw}| \sum_{j=1}^{J} \mathbf{q}^j + J\,|A_{wz}| \sum_{i=1}^{I} \mathbf{p}^i + IJ \sum_{m=1}^{|A_{wz}|} \mathbf{s}_{wz}^m + IJ \sum_{m=1}^{|A_{zw}|} \mathbf{s}_{wz}^m\right)(z).$$

Consider now the following modification to this chain: immediately after the first instance of an $\mathbf{f}(z) \succeq \mathbf{g}(w)$ relation, apply $IJ$ applications of each transformation in $A_{ww}$. Similarly, after the first $\mathbf{f}(w) \succeq \mathbf{g}(z)$ relation, insert $IJ$ repetitions of each transformation in $A_{zz}$. The result is a chain:

$$\bar{\mathbf{h}}(z) \succeq \cdots \succeq \left(\bar{\mathbf{h}} + I\,|A_{zw}| \sum_{j=1}^{J} \mathbf{q}^j + J\,|A_{wz}| \sum_{i=1}^{I} \mathbf{p}^i + IJ \sum_{k=1}^{K} \mathbf{r}^k\right)(z).$$

However, since $\sum_i \mathbf{p}^i = \sum_j \mathbf{q}^j = \sum_k \mathbf{r}^k = \mathbf{0}$, the first and last terms in this chain coincide. Moreover, since every relation in the left-hand column of (9) appears in this cycle, the sequence contains at least one strict relation, contradicting the hypothesis that $\succ$ is a preorder.

**Case 2: $|A_{wz}| + |A_{zw}| = 0$.**

Here, we follow a similar construction to the preceding case, except here we first consider a single chain of the form:

$$\begin{aligned}
\bar{\mathbf{h}}(z) &\succeq (\bar{\mathbf{h}} + \mathbf{p}^1)(w) \\
&\succeq (\bar{\mathbf{h}} + \mathbf{p}^1 + \mathbf{q}^1)(z) \\
&\quad\vdots \\
&\succeq \left(\bar{\mathbf{h}} + J \sum_{i=1}^{I} \mathbf{p}^i + I \sum_{j=1}^{J} \mathbf{q}^j\right)(z).
\end{aligned}$$

Consider now the following modification to this chain: immediately after the first instance of an $\mathbf{f}(z) \succeq \mathbf{g}(w)$ relation, insert one application of each transformation in $A_{ww}$. Similarly, after the first $\mathbf{f}(w) \succeq \mathbf{g}(z)$ relation, insert an application of each transformation in $A_{zz}$. The result is a chain:

$$\bar{\mathbf{h}}(z) \succeq \cdots \succeq \left(\bar{\mathbf{h}} + I \sum_{j=1}^{J} \mathbf{q}^j + J \sum_{i=1}^{I} \mathbf{p}^i + \sum_{k=1}^{K} \mathbf{r}^k\right)(z).$$

But by analogous logic to the former case, this also defines a cycle, contradicting the assumption that $\succeq$ is a preorder. Since these cases are exhaustive, we conclude such an extension must exist, which completes the proof. $\square$

We now are in a position to prove Theorem 1.

*Proof.* Suppose $\succsim^R_{\mathcal{M}}$ is acyclic. By Lemma 3, the transitive closure of $\succsim^R_{\mathcal{M}}$ is an $\mathcal{M}$-invariant pre-order, and hence by Lemma 4 admits a strongly $\mathcal{M}$-invariant preorder extension.

The remainder of the proof follows from a standard transfinite induction argument. Let $\mathcal{P}_{\mathcal{M}}$ denote the set of strongly $\mathcal{M}$-invariant preorders on $X$, partially ordered by extension. Given $\succeq_1, \succeq_2 \in \mathcal{P}_{\mathcal{M}}$, we write $\succeq_1 \rhd \succeq_2$ whenever $\succeq_1$ extends $\succeq_2$. Let $\{\succeq_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary $\rhd$-chain of $\mathcal{M}$-invariant preorders. It follows from standard arguments (see, e.g., Richter 1966; Chambers and Echenique 2016) that:

$$\bar{\succeq} = \bigcup_{\lambda \in \Lambda} \succeq_\lambda$$

is a preorder extension of every $\succeq_\lambda$. Similarly, it follows that $\bar{\succeq}$ is strongly $\mathcal{M}$-invariant: if $(x, y) \in \bar{\succeq}$, then there exists some $\lambda \in \Lambda$ such that $x \succeq_\lambda y$, and since $\succeq_\lambda$ is strongly $\mathcal{M}$-invariant, so must be $\bar{\succeq}$ since it extends $\succeq_\lambda$. Hence $\bar{\succeq}$ belongs to $\mathcal{P}_{\mathcal{M}}$, and by Zorn's Lemma, there exists a maximal strongly $\mathcal{M}$-invariant preorder $\succeq^*$ which extends $\succsim^R_{\mathcal{M}}$. Suppose, for purposes of obtaining a contradiction, that $\succeq^*$ is not complete. Then there exist $w, z \in X$ that are $\succeq^*$-unrelated. By Lemma 5 there exists a strongly $\mathcal{M}$-invariant preorder extension of $\succeq^*$ that renders $w$ and $z$ comparable, however, this contradicts the $\rhd$-maximality of $\succeq^*$. Thus $\succeq^*$ is complete and hence is an $\mathcal{M}$-invariant rationalizing preference for $\succsim^R_{\mathcal{M}}$, and hence $\succsim^R$. $\square$

# B Proof of Theorem 2

## B.1 Preliminaries from Propositional Logic

For all $(x, y) \in X \times X$, define two boolean variables:

$$[\mathtt{x} \succeq \mathtt{y}] \quad \text{and} \quad [\mathtt{x} \succ \mathtt{y}].$$

Let $\mathcal{V}$ denote the set of all such variables. A **model** is a mapping $\mu : \mathcal{V} \to \{\top, \bot\}$ assigning a truth value to every variable in $\mathcal{V}$.[54] We may extend any model from boolean variables to well-formed logical formulae in the obvious manner. For a proof of this fact, and an introduction to propositional logic, the interested reader is referred to Schöning (2008).

Every formula in propositional logic is equivalent to one in conjunctive normal form (CNF).[55] A **literal** is an atomic formula, of the form $A$ or $\neg A$, for some $A \in \mathcal{V}$. A finite formula $F$ in conjunctive normal form can be written as:

$$F = (A_{1,1} \vee \cdots \vee A_{1,n_1}) \wedge \cdots \wedge (A_{K,1} \vee \cdots \vee A_{K,n_K}),$$

where each $A_{i,j}$ is a literal. We view the formula $F$ as being formed by the individual **clauses**:

$$C_i = A_{i,1} \vee \cdots \vee A_{i,n_i}.$$

A formula such as $F$ can be compactly expressed in set notation:

$$\Big\{ \underbrace{\{A_{1,1}, \ldots, A_{1,n_1}\}}_{C_1}, \ldots, \underbrace{\{A_{K,1}, \ldots, A_{K,n_K}\}}_{C_K} \Big\},$$

where each $C_i = \{A_{i,1}, \ldots, A_{i,n_i}\}$ is a clause. In other words, within a clause, a comma denotes an OR operation (i.e. $\vee$), and a comma between clauses denotes an AND (i.e. $\wedge$). The formula consisting only of the empty clause $\{\varnothing\}$ is a valid formula; by definition it is unsatisfiable.

Let $C_1$, $C_2$, and $R$ be clauses. We say that $R$ is a **resolvent** of $C_1$ and $C_2$ if there exists some literal $L$ such that $L \in C_1$ and $\neg L \in C_2$, and

$$R = \big(C_1 \setminus \{L\}\big) \;\cup\; \big(C_2 \setminus \{\neg L\}\big).$$

The following property resolution is standard (see, e.g., Schöning 2008 p.32).

**Lemma 6.** *Let $\Theta$ be a set of clauses, and let $R$ be the resolvent of two clauses $C_1$ and $C_2$ in $\Theta$. Then $\Theta$ and $\Theta \cup \{R\}$ are logically equivalent.*

---

[54] In this appendix, we will exclusively use the word 'model' in its logical interpretation, rather than its economic meaning in the main text.

[55] See Schöning 2008.

When speaking of resolvents, we explicitly allow for $R$ to be the empty set. Suppose $\Theta$ is a set of clauses. A **derivation** of $\varnothing$ via resolution is a finite sequence of clauses $\{C_1, \ldots, C_N\}$ such that:

(i) $C_N = \varnothing$; and

(ii) For all $i = 1, \ldots, N$, $C_i$ is either a clause in $\Theta$, or a resolvent some $C_j$ and $C_k$ (the **parents** of $C_i$), where $j, k < i$.

More generally, if we remove condition (i) we speak of a **partial derivation** (of $C_N$). A set of clauses $\Theta$ is said to be **unsatisfiable** if and only if there is no model which evaluates every formula in $\Theta$ to $\top$. Remarkably, by forming a finite number of resolvents, one is always capable of detecting whether any finite set of formulas is unsatisfiable.

**Theorem 4** (Robinson 1965)**.** *Let $\Theta$ be a finite set of clauses. Then $\Theta$ is unsatisfiable if and only if there exists a derivation of $\varnothing$ via resolution.*

The Robinson (1965) paper actually proves stronger analogous result, in the more general setting of first-order logic. For a proof of the above result in propositional logic, the interested reader is referred to Schöning (2008), Chapter 1, Section 5. Many refinements of Theorem 4 exist, intended to further reduce the search space for proofs in the context of machine learning. We will have use of the following modification: say a derivation $\{C_1, \ldots, C_N\}$ of $\varnothing$ is via **negative resolution** if:

(i) $C_N = \varnothing$; and

(ii') For all $i = 1, \ldots, N$, $C_i$ is either a clause in $\Theta$, or a resolvent of some $C_j$ and $C_k$, where $j, k < i$ and either $C_j$ or $C_k$ contains no positive literals.

The following theorem is proven on p.102 in Schöning (2008).

**Theorem 5.** *Let $\Theta$ be a finite set of formulas. Then $\Theta$ is unsatisfiable if and only if there exists a derivation of $\varnothing$ via negative resolution.*

Theorem 5 provides a 'representation theorem' for proofs of inconsistency: while there may be (many) proofs that a given set of clauses is unsatisfiable, Theorem 5 guarantees that at least one can be carried out wholly via resolution where one parent at every step contains no positive literals. Crucially, every order pair in $\langle W, S\rangle \in \mathcal{F}^*$ uniquely defines a clause containing no positive literals:

$$\langle W, S\rangle \mapsto \Big[\bigvee_{(x,y)\in W\setminus S} \neg[\mathtt{x} \succeq \mathtt{y}]\Big] \vee \Big[\bigvee_{(x,y)\in S} \neg[\mathtt{x} \succ \mathtt{y}]\Big]. \tag{10}$$

We term this the **clausal representation** of the order pair $\langle W, S\rangle$. In particular, the clausal representation of $\langle \varnothing, \varnothing\rangle$ is the empty clause.

## B.2 $\mathcal{M}$-invariant Rationalization

Let $\Phi$ denote the collection of all logical formulas of the following form:

(T.1) **Completeness**: For all $x, y \in X$:

$$[\mathtt{x} \succeq \mathtt{y}] \vee [\mathtt{y} \succeq \mathtt{x}].$$

This is in conjunctive normal form (CNF).

(T.2) **Coherency**: For all $x, y \in X$:

$$[\mathtt{x} \succeq \mathtt{y}] \iff \neg[\mathtt{y} \succ \mathtt{x}].$$

In CNF, this may be regarded as two separate clauses,

$$\neg[\mathtt{x} \succeq \mathtt{y}] \vee \neg[\mathtt{y} \succ \mathtt{x}] \tag{T.2.a}$$

and

$$[\mathtt{x} \succeq \mathtt{y}] \vee [\mathtt{y} \succ \mathtt{x}]. \tag{T.2.b}$$

(T.3) **Transitivity**: For all $x, y, z \in X$:

$$[\mathtt{x} \succeq \mathtt{y}] \wedge [\mathtt{y} \succeq \mathtt{z}] \implies [\mathtt{x} \succeq \mathtt{z}],$$

or, in CNF:

$$\neg[\mathtt{x} \succeq \mathtt{y}] \vee \neg[\mathtt{y} \succeq \mathtt{z}] \vee [\mathtt{x} \succeq \mathtt{z}].$$

(T.4) **Extension**: For all $(x, y) \in \succsim^R$,

$$[\mathtt{x} \succeq \mathtt{y}].$$

Moreover, if $(x, y) \in \succ^R$ then:

$$[\mathtt{x} \succ \mathtt{y}].$$

(T.5) **Invariance**: For all $x, y \in X$ and $\omega \in \mathcal{M}$ such that $x, y$ belong to the domain of $\omega$:

$$[\mathtt{x} \succeq \mathtt{y}] \iff [\omega(\mathtt{x}) \succeq \omega(\mathtt{y})],$$

or

$$\neg[\mathtt{x} \succeq \mathtt{y}] \vee [\omega(\mathtt{x}) \succeq \omega(\mathtt{y})] \tag{T.5.a}$$

and

$$[\mathtt{x} \succeq \mathtt{y}] \vee \neg[\omega(\mathtt{x}) \succeq \omega(\mathtt{y})]. \tag{T.5.b}$$

By construction, the set of models which evaluate to $\top$ for every formula in $\Phi$ are in 1-1 correspondence with the $\mathcal{M}$-invariant weak order extensions of $\langle \succsim^R, \succ^R \rangle$.

## B.3 Proofs

We proceed in the proof of Theorem 2 via several lemmas.

**Lemma 7.** *Suppose $\langle \varnothing, \varnothing \rangle \in \mathcal{F}^*$. Then there does not exist any $\mathcal{M}$-invariant preference relation extending $\langle \succsim^R, \succ^R \rangle$.*

*Proof.* By minor abuse of notation, we identify every order pair in $\mathcal{F}^*$ with its clausal representation under (10). Let $\Theta$ denote the collection of all clauses of the form (T.1) - (T.5), as well as all clauses in $\mathcal{F}^0$; recall, $\mathcal{F}^0$ consists of the (clausal representations of) forbidden subrelations generated by broken cycles in the data.

Suppose $\succeq$ is an $\mathcal{M}$-invariant weak order extension of $\langle \succsim^R, \succ^R \rangle$. By construction, the rule of assignment (i) $[\mathtt{x} \succeq \mathtt{y}] = \top$ if and only if $x \succeq y$ and (ii) $[\mathtt{x} \succ \mathtt{y}]$ if and only if $x \succ y$, defines a valid model for $\Theta$, i.e. under these assignments, every clause in $\Theta$ evaluates to $\top$.[56] Thus $\Phi$ is satisfiable if and only if $\Theta$ is.

Let $C, C' \in \mathcal{F}^0$ denote clausal representations of two forbidden subrelations, derived from broken cycles (either strict or weak) in the data, and suppose $D \in \mathcal{F}^1$ is the (clausal representation of the) collapse of $C$ and $C'$. Regarding these as sets of negative literals, there exists (negative) literals $L \in C$ and $L' \in C'$ such that:

$$D = \big(C \setminus \{L\}\big) \ \cup \ \big(C' \setminus \{L'\}\big)$$

[56]The clauses of the form (T.1)-(T.5) are clearly necessary as they define the basic properties of an invariant weak order extension. Clauses in $\mathcal{F}^0$ must also hold lest $\succeq$ contain a cycle. Every clause in $\mathcal{F}^0$ can be logically deduced from (T.1) - (T.5), however we do not need this fact in light of standard order-theoretic arguments.

and either:

$$L = \neg[\omega(\mathtt{y}) \succeq \omega(\mathtt{x})] \quad \text{and} \quad L' = \neg[\omega'(\mathtt{x}) \succeq \omega'(\mathtt{y})]$$

or

$$L = \neg[\omega(\mathtt{y}) \succeq \omega(\mathtt{x})] \quad \text{and} \quad L' = \neg[\omega'(\mathtt{x}) \succ \omega'(\mathtt{y})]$$

for some $x, y \in X$, $\omega, \omega' \in \mathcal{M}$. Suppose $L$ and $L'$ are of the former type. Then $\neg[\omega(\mathtt{y}) \succeq \omega(\mathtt{x})] \in C$, hence we may form $C_1$ by resolving $C$ with the (T.5.a) clause $\neg[\mathtt{y} \succeq \mathtt{x}] \vee [\omega(\mathtt{y}) \succeq \omega(\mathtt{x})]$. Then form $C_2$ by resolving $C_1$ with the (T.1) clause $[\mathtt{x} \succeq \mathtt{y}] \vee [\mathtt{y} \succeq \mathtt{x}]$, and finally form $C_3$ by resolving $C_2$ with the (T.5.a) clause $\neg[\mathtt{x} \succeq \mathtt{y}] \vee [\omega'(\mathtt{x}) \succeq \omega'(\mathtt{y})]$. Thus:

$$C_3 = \big(C \setminus \{L\}\big) \ \cup \ \big\{[\omega'(\mathtt{x}) \succeq \omega'(\mathtt{y})]\big\}.$$

Then $C_3$ and $C'$ can be resolved to form $D$. By Lemma 6, $\Theta$ and $\Theta \cup \big\{D\big\}$ are logically equivalent.

Proceeding, suppose now instead that $L$ and $L'$ are of the latter type. Again form $C_1$ via resolving $C$ with the (T.5.a) clause $\neg[\mathtt{y} \succeq \mathtt{x}] \vee [\omega(\mathtt{y}) \succeq \omega(\mathtt{x})]$, and then $C_2$ by resolving $C_1$ and the (T.5.b) clause $[\mathtt{y} \succeq \mathtt{x}] \vee \neg[\omega(\mathtt{y}) \succeq \omega(\mathtt{x})]$. Finally, form $C_3$ by resolving $C_2$ with the type (T.2.b) $[\omega(\mathtt{y}) \succeq \omega(\mathtt{x})] \vee [\omega(\mathtt{x}) \succ \omega(\mathtt{y})]$. Then $D$ is the resolvent of $C_3$ and $C'$, and hence by an analogous argument, $\Theta$ and $\Theta \cup \big\{D\big\}$ are again logically equivalent.

Since $C, C'$ and $D$ were arbitrary, we have shown that $\Theta$ and $\Theta \cup \mathcal{F}^1$ are logically equivalent. However, nothing in the preceding argument relied on $C, C'$ belonging to $\mathcal{F}^0$, rather than any other $\mathcal{F}^n$. Hence by an identical argument, $\Theta \cup \mathcal{F}^n$ and $\Theta \cup \mathcal{F}^{n+1}$ are logically equivalent, implying so too are $\Theta$ and $\Theta \cup \mathcal{F}^*$. Since any model evaluates the empty clause $\varnothing$ to $\bot$, the fact $\langle \varnothing, \varnothing \rangle \in \mathcal{F}^*$, implies $\mathcal{F}^*$ is unsatisfiable by soundness of resolution, and hence so too is $\Theta$. Thus no $\mathcal{M}$-invariant weak order extension of $\langle \succsim^R, \succ^R \rangle$ can exist. $\square$

**Lemma 8.** *Let $C$ be a disjunction of negative literals such that $C \in \Phi$ or $C$ is the resolvent of two elements of $\Phi$, one of which contains no positive literals. Then $C \in \mathcal{F}^1$.*

*Proof.* Suppose first that $C \in \Phi$. Since $C$ is a disjunction of negative literals, it must be of the form (T.2.a), i.e.:

$$C = \neg[\mathtt{x} \succeq \mathtt{y}] \vee \neg[\mathtt{y} \succ \mathtt{x}].$$

Then $C$ corresponds to (the clausal representation of) the forbidden subrelation associated with:

$$x \succsim^R x$$
$$y \succsim^R y,$$

and hence $C \in \mathcal{F}^0 \subseteq \mathcal{F}^1$. Suppose instead then that $C$ is the resolvent of $C', D \in \Phi$, where $D$ is a disjunction of negative literals and hence $D = \neg[\mathtt{x} \succeq \mathtt{y}] \vee \neg[\mathtt{y} \succ \mathtt{x}]$. Since $C$ also contains no positive literals, it must be the case that $C' \in \Phi$ contains exactly one positive literal. Therefore it must be of the form (T.3), (T.4) or (T.5).

**Case**: $C' = \neg[\mathtt{x} \succeq \mathtt{z}] \vee \neg[\mathtt{z} \succeq \mathtt{y}] \vee [\mathtt{x} \succeq \mathtt{y}]$. Then:

$$x \succsim_c x$$
$$z \succsim_c z$$
$$y \succsim_c y$$

defines a broken cycle for which which $C$ is a forbidden subrelation, and hence again $C \in \mathcal{F}^0 \subseteq \mathcal{F}^1$.

**Case**: $C' = [\mathtt{x} \succeq \mathtt{y}]$ or $C' = [\mathtt{y} \succ \mathtt{x}]$. If the former is true, then $C = \neg[\mathtt{y} \succ \mathtt{x}]$. But since $C'$ must be a type (T.4) clause, this implies we must have $x \succsim_c y$ in the data, and hence:

$$x \succsim_c y$$

is a broken cycle with forbidden subrelation $C = \neg[\mathtt{y} \succ \mathtt{x}]$ as desired. If instead the latter is true, by an analogous argument $x \succ_c y$ and:

$$x \succ_c y$$

is a broken cycle which admits forbidden subrelation $C = \neg[\mathtt{y} \succeq \mathtt{x}]$. In either case, we again find $C \in \mathcal{F}^0 \subseteq \mathcal{F}^1$.

**Case**: $C' = [\mathtt{x} \succeq \mathtt{y}] \vee \neg[\omega(\mathtt{x}) \succeq \omega(\mathtt{y})]$. Then:

$$C = \neg[\mathtt{y} \succ \mathtt{x}] \vee \neg[\omega(\mathtt{x}) \succeq \omega(\mathtt{y})].$$

However, the broken cycles:

$$x \succsim^R x \qquad \omega(x) \succsim^R \omega(x)$$
$$y \succsim^R y \qquad \omega(y) \succsim^R \omega(y)$$

yield forbidden subrelations:

$$\neg[\mathtt{x} \succeq \mathtt{y}] \vee \neg[\mathtt{y} \succ \mathtt{x}]$$

and

$$\neg[\omega(\mathtt{x}) \succeq \omega(\mathtt{y})] \vee \neg[\omega(\mathtt{y}) \succ \omega(\mathtt{x})].$$

Thus letting $L = [\mathtt{x} \succeq \mathtt{y}]$ and $L' = [\omega(\mathtt{y}) \succ \omega(\mathtt{x})]$, $C$ is simply the collapse of these two forbidden subrelations and hence belongs to $\mathcal{F}^1$. An analogous argument obtains if instead $C' = \neg[x' \succeq y'] \vee [\mathtt{x} \succeq \mathtt{y}]$ where for some $\omega \in \mathcal{M}$ we have $\omega(x') = x$ and $\omega(y') = y$. □

**Lemma 9.** *Suppose there does not exist an $\mathcal{M}$-invariant weak order extension of $\langle \succsim^R, \succ^R \rangle$. Then $\varnothing \in \mathcal{F}^*$.*

*Proof.* By construction, there is a one-to-one correspondence between $\mathcal{M}$-invariant preference relations extending $\langle \succsim^R, \succ^R \rangle$ and models for $\Phi$. Thus if no such extension exists, $\Phi$ is unsatisfiable. By Propositional Compactness (see Schöning 2008 Chapter I.4), there exists a finite unsatisfiable subset $\Phi^* \subseteq \Phi$.

By Theorem 5, there exists a derivation of the empty set via negative resolution, i.e. there exists a sequence of clauses $C_1, \ldots, C_N$ such that (i) $C_N = \varnothing$, (ii) for all $1 \leq n \leq N-1$ the clause $C_n$ either belongs to $\Phi^*$ or is the resolvent of two clauses $C_i$ and $C_j$, with $i, j < n$, one of which contains no positive literals.

Let $\{D_1, \ldots, D_K\}$ denote those clauses in $\{C_1, \ldots, C_N\}$ which contain no positive literals. For each $D_k$, if $D_k$ is the resolvent of some $C_i$ and $D_j$, define $D_j$ to be its **negative parent** (if $D_k$ is not a resolvent, then we say $D_k$ has no negative parent). Furthermore, if $C_i$ itself is the resolvent of some $C_{i'}$ and $D_{j'}$, then we say $D_{j'}$ is the **negative grandparent** of $D_k$ (similarly, if $C_i \in \Phi^*$, i.e. $C_i$ is not a resolvent, then we say $D_k$ has no negative grandparent). Define $\mathcal{NP}(D_k)$, the **negative predecessors** of $D_k$, as the set consisting of $D_k$'s negative parent and grandparent (if these exist).

Let $\mathcal{D}^0 \subseteq \{D_1, \ldots, D_K\}$ denote the subset of all $D_k$ which belong to $\mathcal{F}^0$.[57] For each $n \geq 1$, define inductively:

$$\mathcal{D}^n = \left\{ D_k : \mathcal{NP}(D_k) \subseteq \mathcal{D}^{n-1} \right\} \,\cup\, \mathcal{D}^{n-1}.$$

[57]Note $\mathcal{D}^0$ is non-empty as it contains at least $D_1 \in \Phi^*$.

In other words, $\mathcal{D}^n$ consists of those positive-literal-free clauses $D_k$ all of whose negative predecessors (if these exist) belong to $\mathcal{D}^{n-1}$ or lower. Viewing $\{C_1,\ldots,C_N\}$ as a binary tree (Schöning 2008, Chapter I.5), by Lemma 8 the sets $\{\mathcal{D}^n\}_{n=0}^{\infty}$ cover $\{D_1,\ldots,D_K\}$.[58] We now wish to show that for all $n\geq 1$, $\mathcal{D}^n\subseteq\mathcal{F}^n\subseteq\mathcal{C}^*$. By definition, $\mathcal{D}^0\subseteq\mathcal{F}^0$. Thus suppose now that for all $n\leq M$, we have $\mathcal{D}^n\subseteq\mathcal{F}^n$, and consider $n=M+1$. Let $D_k\in\mathcal{D}^{M+1}$. We consider three cases.

**Case 1**: $D_k$ has negative parent $D_j$ and negative grandparent $D_{j'}$, both of which belong to $\mathcal{D}^M$ and hence $\mathcal{F}^M$ by the inductive hypothesis. Then $D_k$ is the resolvent of $D_j$ and some $C_i$, and $C_i$ the resolvent of $D_{j'}$ and some $C_{i'}$. Since $D_k$ and $D_j$ contain no positive literals, this means $C_i$ must contain exactly one positive literal. In turn, since $D_{j'}$ contains no positive literals, this implies $C_{i'}$ must contain exactly two positive literals. Since $\Phi^*$ contains no clauses with more than two positive literals, and since every resolvent in $\{C_1,\ldots,C_N\}$ has a parent containing no positive literals, no resolvent in $\{C_1,\ldots,C_N\}$ can have more than 1 positive literal. This means that $C_{i'}\in\Phi^*$ and hence is either of the form $C_{i'}=[\mathtt{x}\succeq\mathtt{y}]\vee[\mathtt{y}\succeq\mathtt{x}]$ or $C_{i'}=[\mathtt{x}\succeq\mathtt{y}]\vee[\mathtt{y}\succ\mathtt{x}]$. Suppose first that $C_{i'}$ is of the former form. Then $C_i$ consists of $D_{j'}$ but with one literal reversed (i.e. the swapping the positions of the two alternatives featuring in it) and made positive. Since this is $C_i$'s only positive literal, it must be the cancelling literal when it is resolved with $D_j$, thus $D_k$ is precisely the collapse of $D_{j'}$ and $D_j$, where the collapse comes from cancelling a pair of reversed weak relations. If, instead, $C_{i'}$ is of the latter form, then once again $C_i$ consists of $D_{j'}$ but now the one literal is reversed, made positive, and made strict if it was weak, or vice-versa. This is then cancelled by resolving with $D_j$ and hence $D_k$ consists of the collapse of $D_{j'}$ and $D_j$ where the collapse occurs between weak and strict opposing negative literals. In either case, we find that $D_k$ is the collapse of two elements of $\mathcal{F}^M$ and hence belongs to $\mathcal{F}^{M+1}$ as desired.

**Case 2**: $D_k$ has a negative parent $D_j$ but no negative grandparent, i.e. $C_i\in\Phi^*$. Since $D_j$ and $D_k$ contain no positive literals, it must be that $C_i$ contains exactly one positive literal. Thus $C_i$ is either of the form:

[58] Viewing the resolution proof as a finite binary tree, Lemma 8 shows that (i) every leaf that belongs to $\{D_1,\ldots,D_K\}$ belongs to $\mathcal{D}^0$, and (ii) every element of $\{D_1,\ldots,D_K\}$ that two leaves for parents belongs to $\mathcal{D}^1$. The claim then follows by inducting on how many generations of ancestors an element of $\{D_1,\ldots,D_K\}$ has in the tree.

(i) $C_i = \neg[\mathtt{x} \succeq \mathtt{z}] \vee \neg[\mathtt{z} \succeq \mathtt{y}] \vee [\mathtt{x} \succeq \mathtt{y}]$

(ii) $C_i = [\mathtt{x} \succeq \mathtt{y}]$ or $C_i = [\mathtt{y} \succ \mathtt{x}]$

(iii) $C_i = \neg[\mathtt{x} \succeq \mathtt{y}] \vee [\omega(\mathtt{x}) \succeq \omega(\mathtt{y})]$ or $C_i = [\mathtt{x} \succeq \mathtt{y}] \vee \neg[\omega(\mathtt{x}) \succeq \omega(\mathtt{y})]$ for some $\omega \in \mathcal{M}$.

Suppose first $C_i$ is of form (i). Then the cancelling literal must be $[\mathtt{x} \succeq \mathtt{y}]$. However, since:

$$x \succsim_c x$$
$$z \succsim_c z$$
$$y \succsim_c y$$

is a broken cycle, we know $\neg[\mathtt{x} \succsim \mathtt{z}] \vee [\mathtt{z} \succsim \mathtt{y}] \vee \neg[\mathtt{y} \succ \mathtt{x}]$ belongs to $\mathcal{F}^0$. Therefore $D_k$ can be formed from collapsing $\neg[\mathtt{x} \succeq \mathtt{z}] \vee [\mathtt{z} \succeq \mathtt{y}] \vee \neg[\mathtt{y} \succ \mathtt{x}] \in \mathcal{F}^0$ with $D_j$. Since $D_j \in \mathcal{F}^M$, this means $D_k \in \mathcal{F}^{M+1}$ as desired.

Suppose now that $C_i$ is of type (ii). In the first case,

$$D_k = D_j \setminus \{\neg[\mathtt{x} \succeq \mathtt{y}]\}.$$

Note however that if $[\mathtt{x} \succeq \mathtt{y}] \in \Phi^*$, then $x \succsim_c y$, and thus:

$$x \succsim_c y$$

is a forcing collection for $\neg[\mathtt{y} \succ \mathtt{x}]$ and hence this clause belongs to $\mathcal{F}^0$. Thus $D_k$ may be obtained as the collapse of $\neg[\mathtt{y} \succ \mathtt{x}]$ and $D_j$ and hence belongs to $\mathtt{D}^{M+1}$. On the other hand, if $C_i$ equals $[\mathtt{y} \succ \mathtt{x}]$ then $y \succ_c x$ and hence:

$$y \succ_c x$$

is a strict broken cycle for $\neg[\mathtt{x} \succeq \mathtt{y}]$ and since:

$$D_k = D_j \setminus \{\neg[\mathtt{y} \succ \mathtt{x}]\},$$

$D_k$ is just the collapse of $D_j$ and $\neg[\mathtt{x} \succeq \mathtt{y}]$, and hence once again belongs to $\mathcal{D}^{M+1}$.

Finally, suppose that $C_i$ is of the former type (iii). Then the cancelling literal must be $[\omega(\mathtt{x}) \succeq \omega(\mathtt{y})]$. Thus $D_k$ is equal to $D_j$ but with the literal $\neg[\omega(\mathtt{x}) \succeq \omega(\mathtt{y})] \in D_j$ becoming $\neg[\mathtt{x} \succeq \mathtt{y}] \in D_k$. Now,

$$x \succsim_c x$$
$$y \succsim_c y$$

is a broken cycle hence $\neg[\mathtt{x} \succeq \mathtt{y}] \vee \neg[\mathtt{y} \succ \mathtt{x}]$ belongs to $\mathcal{F}^0$. Then $D_k$ arises as the collapse of $D_j \in \mathcal{D}^M$ and $\neg[\mathtt{x} \succeq \mathtt{y}] \vee \neg[\mathtt{y} \succ \mathtt{x}] \in \mathcal{F}^0 \subseteq \mathcal{D}^M$ along the pair $\neg[\mathtt{y} \succ \mathtt{x}]$ and $\neg[\omega(\mathtt{x}) \succeq \omega(\mathtt{y})]$, and hence belongs to $\mathcal{D}^{M+1}$ as desired. If instead $C_i$ is of the latter type (iii), an analogous argument suffices.

**Case 3**: $D_k$ has no negative parent. In this case, $D_k$ cannot be a resolvent at all, and hence belongs to $\Phi^*$. The only clauses in $\Phi^*$ which contain no positive literals are of the form $\neg[\mathtt{x} \succeq \mathtt{y}] \vee \neg[\mathtt{y} \succ \mathtt{x}]$. If $D_k$ is of this form, then it belongs to $\mathcal{F}^0$ as

$$x \succsim_c x$$
$$y \succsim_c y$$

is a broken cycle for it, and hence it belongs to $\mathcal{F}^{M+1}$ as well.

Now, as $D_k$ cannot have a negative grandparent without a negative parent (as our proof of inconsistency is by *negative* resolution), these cases are exhaustive, and we find that for all $1 \leq k \leq K$, the clause $D_k \in \mathcal{F}^*$. Since $D_K = \varnothing$, this implies that $\varnothing \in \mathcal{F}^*$ as desired. $\square$

The proof of Theorem 2 follows from these lemmas.

# C Proof of Theorem 3

*Proof.* Suppose first that $\{[\mathtt{y} \succ \mathtt{x}]\} \in \mathcal{F}^*$. By an identical argument to that in the proof of Theorem 2, $\Phi \cup \{[\mathtt{y} \succ \mathtt{x}]\}$ is unsatisfiable. Thus no model $\mu$ for $\Phi$ evaluates $\mu([\mathtt{y} \succ \mathtt{x}]) = \top$. Since the set of models for $\Phi$ are in 1-1 correspondence with the set of $\mathcal{M}$-invariant rationalizing preferences of $\langle \succsim^R, \succ^R \rangle$ (which is non-empty by hypothesis), we conclude every such rationalizing preference must weakly rank $x$ above $y$. An identical argument holds for the case of $\{[\mathtt{y} \succeq \mathtt{x}]\} \in \mathcal{F}^*$ case.

Conversely, suppose every $\mathcal{M}$-invariant rationalizing preference $\succeq^*$ ranks $x \succeq^* y$. Then no model for $\Phi$ evaluates $[\mathtt{y} \succ \mathtt{x}]$ to $\top$, and hence $\Phi \cup \{[\mathtt{y} \succ \mathtt{x}]\}$ is unsatisfiable. Define $\Phi'$ as follows. First, remove from $\Phi$ any clause containing the literal $[\mathtt{y} \succ \mathtt{x}]$; then for every remaining clause that contains the negative literal $\neg[\mathtt{y} \succ \mathtt{x}]$, delete this literal from it. By construction, any model $\mu'$ for $\Phi'$ uniquely extends to a model $\mu$ for $\Phi$ which evaluates $\mu([\mathtt{y} \succ \mathtt{x}]) = \top$. Since no such models $\mu$ exist, $\Phi'$ must be unsatisfiable. By Propositional Compactness (see Schöning (2008) Chapter

I.4), there exists a finite subset of $\Phi'' \subseteq \Phi'$ that is unsatisfiable; by Theorem 5, there exists a derivation $\{C_1, \ldots, C_N\}$ of $\varnothing$ from $\Phi''$ via negative resolution. Let $\{D_1, \ldots, D_K\} \subset \{C_1, \ldots, C_N\}$ denote the elements of $\{C_1, \ldots, C_N\}$ belonging to $\Phi''$. Note that each $D_k$ either (i) belongs to $\Phi$ as well, or (ii) $D_k \cup \{\neg[\mathtt{y} \succ \mathtt{x}]\}$ belongs to $\Phi$. Moreover, since $\Phi$ is satisfiable by hypothesis, at least one $D_k$ must be of the latter type. Define:

$$\bar{D}_k = \begin{cases} C_i & \text{if } D_k \in \Phi \\ D_k \cup \{\neg[\mathtt{y} \succ \mathtt{x}]\} & \text{else.} \end{cases}$$

Then resolving the $\{\bar{D}_1, \ldots, \bar{D}_K\}$ in the same order as in the derivation $\{C_1, \ldots C_N\}$ generates a partial derivation $\{\bar{C}_1, \ldots, \bar{C}_N\}$ of $\neg[\mathtt{y} \succ \mathtt{x}]$ from $\Phi$ via negative resolution, and hence by an identical argument to Lemma 9 $[\mathtt{y} \succ \mathtt{x}] \in \mathcal{F}^*$. An identical argument again works for the case in which every extension ranks $x \succ^* y$. $\square$

# D Results and Derivations Omitted From Text

## D.1 Omitted Derivations

### D.1.1 $\mathcal{M}$-closure Cycle in Kraft et al. (1959)

The counter-example of a qualitative probability that cannot be represented by any probability measure in Kraft et al. (1959) includes the following relations:

$$\mathbb{1}_{14} \prec^* \mathbb{1}_{235}, \quad \mathbb{1}_{23} \prec^* \mathbb{1}_{15}, \quad \mathbb{1}_{25} \prec^* \mathbb{1}_{34}, \quad \text{and} \quad \mathbb{1}_{35} \prec^* \mathbb{1}_{2}$$

on $\mathbb{Z}^S$, where $S = \{1, \ldots, 5\}$. Thus in the $\mathcal{M}$ closure of $\succeq^*$, we obtain:

$$\mathbb{1}_{14} \prec^{*\mathcal{M}} \mathbb{1}_{235} \prec^{*\mathcal{M}} \mathbb{1}_{15} + \mathbb{1}_{5} \prec^{*\mathcal{M}} \mathbb{1}_{1345} - \mathbb{1}_{2} \prec^{*\mathcal{M}} \mathbb{1}_{124} - \mathbb{1}_{2} = \mathbb{1}_{14},$$

the desired cycle. By Theorem 1, $\succeq$ then cannot be represented by any measure.

### D.1.2 Acyclic $\mathcal{M}$-closure in Example 2

We claim the $\mathcal{M}$-closure of the revealed preference pair in Example 2 is acyclic. To see this, note that two alternatives are comparable in the $\mathcal{M}$-closure if and only if they are of the form:

$$(z_1, \ldots, z_K, x_1, \ldots) \succ^R_{\mathcal{M}} (z_1, \ldots, z_K, y_1, \ldots),$$

where $z_K$ equals $a$ or $b$, or:

$$(z'_1, \ldots, z'_L, y_1, \ldots) \succ^R_{\mathcal{M}} (z'_1, \ldots, z'_L, x_1, \ldots),$$

with $z'_L$ equal to either $c$ or $d$. In particular, since by hypothesis neither $x$ nor $y$ take the values $a, b, c$, or $d$, it can never be the case that:

$$(z_1, \ldots, z_K, y_1, \ldots) = (z'_1, \ldots, z'_L, y_1, \ldots)$$

or

$$(z_1, \ldots, z_K, y_1, \ldots) = (z'_1, \ldots, z'_L, x_1, \ldots),$$

and hence $\succ^R_{\mathcal{M}}$ is vacuously transitive and thus acyclic.

## D.2 Relating Theorem 1 and GARP Variations

In this section, we consider the special case in which our relations $\langle \succsim^R, \succ^R \rangle$ are generated by some price-consumption data set $\{(p_1, x_1), \ldots, (p_K, x_K)\}$. Here, we assume $\langle \succsim^R, \succ^R \rangle$ are the revealed preference relations associated with this data set, via:

$$x \succsim^R y \quad \iff \quad x = x^k \text{ for some } k, \text{ and } p_k \cdot x \geq p_k \cdot y$$

(respectively $\succ^R$ and $>$). We show that for various common choices of $\mathcal{M}$, the acyclicity of $\langle \succsim^R_{\mathcal{M}}, \succ^R_{\mathcal{M}} \rangle$ straightforwardly reduces to the standard, model-specific revealed preference axioms.

### D.2.1 Quasilinearity

Suppose that $X = Y \times \mathbb{R}_+$, and $\mathcal{M}$ consists of all transformations of the form $(y, t) \mapsto (y, t + \alpha)$ for $\alpha \geq 0$. Let $\langle \succsim^R, \succ^R \rangle$ be an arbitrary data set. Then the $\mathcal{M}$-closure $\langle \succsim^R_{\mathcal{M}}, \succ^R_{\mathcal{M}} \rangle$ is defined by:

$$(y, t + \alpha) \succsim^R_{\mathcal{M}} (y', t' + \alpha) \quad \iff \quad (y, t) \succsim^R (y', t')$$

for some $\alpha \geq 0$, and analogously for $\succ^R_{\mathcal{M}}$.

Suppose now that $Y = \mathbb{R}^{L-1}_+$, and $\langle \succsim^R, \succ^R \rangle$ is the revealed preference relation arising from some price-consumption data set; without loss of generality, we normalize each $p_k = (\tilde{p}_k, 1)$. Then a $\langle \succsim^R_{\mathcal{M}}, \succ^R_{\mathcal{M}} \rangle$ cycle is equivalent to the existence of

$(y_{k_0}, t_0), \ldots, (y_{k_N}, t_N) \in X$, and $\alpha_0, \ldots, \alpha_N \geq 0$ such that:

$$\begin{aligned} p_{k_0} \cdot (y_{k_0}, t_{k_0}) &\geq p_{k_0} \cdot (y_{k_1}, t_1 + \alpha_0) = p_{k_0} \cdot (y_{k_1}, t_{k_1} + \alpha_0 - \alpha_1) \\ p_{k_1} \cdot (y_{k_1}, t_{k_1}) &\geq p_{k_1} \cdot (y_{k_2}, t_2 + \alpha_1) = p_{k_1} \cdot (y_{k_2}, t_{k_2} + \alpha_1 - \alpha_2) \\ &\vdots \\ p_{k_N} \cdot (y_{k_N}, t_{k_N}) &> p_{k_N} \cdot (y_{k_0}, t_0 + \alpha_N) = p_{k_N} \cdot (y_{k_0}, t_{k_0} + \alpha_N - \alpha_0) \end{aligned} \tag{11}$$

where $t_{k_i} = t_i + \alpha_i$ for all $i = 1, \ldots N$.[59] Summing over (11):

$$\sum_{i=0}^{N} \tilde{p}_{k_i} \cdot (y_{k_{i+1}} - y_{k_i}) < 0,$$

which precisely corresponds precisely to a negative cycle à la Brown and Calsamiglia (2007).[60]

### D.2.2 Homotheticity

Let $X$ be a cone in a real vector space, and let $\mathcal{M}$ consist of all transformations of the form $x \mapsto \alpha x$, for $\alpha > 0$. The particular case of $X = \mathbb{R}^n_+$ is treated in Chambers and Echenique (2016), Theorem 4.2, but we reproduce the ideas here.

Here, the $\mathcal{M}$-closure of the data set $\langle \succsim^R, \succ^R \rangle$ is given by:

$$x \succsim^R_{\mathcal{M}} y \quad \Longleftrightarrow \quad \alpha x \succsim^R \alpha y$$

for some $\alpha > 0$, with a similar definition for $\succ^R_{\mathcal{M}}$. In Chambers and Echenique (2016), $\langle \succsim^R_{\mathcal{M}}, \succ^R_{\mathcal{M}} \rangle$ is referred to as $\langle \succeq^H, \succ^H \rangle$. The $\mathcal{M}$-closure is acyclic if and only if there do not exist $x_0, \ldots, x_N \in X$ and $\alpha_0, \ldots, \alpha_N > 0$ such that:

$$\begin{aligned} \alpha_0 x_0 &\succsim^R \alpha_0 x_1 \\ \alpha_1 x_1 &\succsim^R \alpha_1 x_2 \\ &\vdots \\ \alpha_N x_N &\succ^R \alpha_N x_0. \end{aligned}$$

Suppose again that $\langle \succsim^R, \succ^R \rangle$ is the revealed preference relation arising from some set of price-consumption observations; without loss of generality, we normalize each

[59] In other words, $x \succsim^R_{\mathcal{M}} y$ if and only if there is some fixed translation along the numeraire axis that brings $x$ equal to some chosen $x_k$, and which leaves $y$ within the budget defined by $p_k$ and $x_k$.

[60] Here, the $i$ indices are understood to satisfy $N + 1 \equiv 0$.

price so $p_k \cdot x_k = 1$. Then (12) is equivalent to the existence of $x_{k_0}, \ldots, x_K \in X$ and $\alpha_0, \ldots, \alpha_K > 0$ such that:

$$\begin{aligned} p_{k_0} \cdot x_{k_0} &\geq p_{k_0} \cdot (\alpha_0 x_1) = p_{k_0} \cdot \left(\frac{\alpha_0 x_{k_1}}{\alpha_1}\right) \\ p_{k_1} \cdot x_{k_1} &\geq p_{k_0} \cdot (\alpha_1 x_2) = p_{k_0} \cdot \left(\frac{\alpha_1 x_{k_2}}{\alpha_2}\right) \\ &\vdots \\ p_{k_N} \cdot x_{k_N} &> p_{k_N} \cdot (\alpha_N x_0) = p_{k_N} \cdot \left(\frac{\alpha_N x_{k_0}}{\alpha_0}\right) \end{aligned} \tag{12}$$

where $\alpha_i x_i = x_{k_i}$ for all $i = 1, \ldots, N$. Taking products of (12) leads to the cancellations of all $\alpha_i/\alpha_{i+1}$ terms, resulting in:

$$\prod_{i=0}^{N} p_{k_i} x_{k_{i+1}} < 1,$$

which is precisely a violation of the homothetic axiom of revealed preference of Varian (1983). As mentioned previously, in the case of general $\langle \succsim^R, \succ^R \rangle$, not necessarily arising from price-consumption observations, Demuynck (2009) obtains a similar characterization, in the special case of monotone and homothetic preferences, via a different approach.

#### D.2.3 Translation-Invariance

Let $S$ be some finite set of states of the world, and let $X = \mathbb{R}^S$ denote the space of portfolios of Arrow securities. Let $\mathcal{M}$ denote the collection of transformations of the form $x \mapsto x + \vec{\alpha}$, where $\vec{\alpha} := (\alpha, \ldots, \alpha)$, for each $\alpha \in \mathbb{R}$. We refer to an $\mathcal{M}$-invariant preference as *translation invariant*. By Theorem 1, the data $\langle \succsim^R, \succ^R \rangle$ are rationalizable by a translation-invariant preference if and only if there does not exist $x_0, \ldots, x_N \in X$ and $\alpha_0, \ldots, \alpha_N \in \mathbb{R}$ such that:

$$\begin{aligned} p_{k_0} \cdot x_{k_0} &\geq p_{k_0} \cdot \big(x_1 + \vec{\alpha}_0\big) = p_{k_0} \cdot \big(x_{k_1} + \vec{\alpha}_0 - \vec{\alpha}_1\big) \\ p_{k_1} \cdot x_{k_1} &\geq p_{k_1} \cdot \big(x_2 + \vec{\alpha}_1\big) = p_{k_1} \cdot \big(x_{k_2} + \vec{\alpha}_1 - \vec{\alpha}_2\big) \\ &\vdots \\ p_{k_N} \cdot x_{k_N} &\geq p_{k_N} \cdot \big(x_0 + \vec{\alpha}_N\big) = p_{k_N} \cdot \big(x_{k_0} + \vec{\alpha}_N - \vec{\alpha}_0\big). \end{aligned} \tag{13}$$

Summing over (13) we obtain:

$$\sum_{i=0}^{N} p_{k_i} \cdot (x_{k_{i+1}} - x_{k_i}) - \sum_{i=0}^{N} (\alpha_i - \alpha_{i+1}) \|p_{k_i}\|_1 < 0,$$

or, normalizing each $p_{k_i}$ by $\|p_{k_i}\|_1$ without loss of generality:

$$\sum_{i=0}^{N} \frac{p_{k_i}}{\|p_{k_i}\|_1} \cdot (x_{k_{i+1}} - x_{k_i}) < 0,$$

precisely the same condition obtained in Chambers et al. (2016).[61]

[61] Economically, this normalization may be regarded as treating *bonds* as a numeraire commodity.