Competition, Persuasion, and Search^{*}

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Abstract

An agent engages in sequential search. He does not directly observe the quality of the goods he samples, but he can purchase signals designed by profit maximizing principal(s). We formulate the principal-agent relationship as a repeated contracting problem within a stopping game, and characterize the set of equilibrium payoffs. We show that when the agent's search cost falls below a given threshold, competition does not impact how much surplus is generated in equilibrium nor how the surplus is divided. In contrast, competition benefits the agent at the expense of total surplus when the search cost exceeds that threshold. Our results challenge the view that monopoly decreases market efficiency, and moreover, suggest that it generates the highest value of information for the agent.

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1. Introduction

A key hurdle for agents in search markets is the limited information about the quality (or match value) of available goods. While the prior literature has studied how differential levels of information about the quality of sampled goods affect search outcomes,¹ most of this literature has treated information as exogenous to the search process. In an increasingly data-driven economy, however, information is acquired through endogenous contracting with brokers who gather, process, and sell information. For example, employers hire pre-employment testing agencies to assess job candidates, consumers purchase product reviews from platforms like *Consumer Reports* and *CarFax*, and homebuyers pay for home inspections and appraisals.

These information brokers typically have no intrinsic preferences over the agent's match in the search market but aim instead to maximize their profit. Still, information brokers play a role in shaping search outcomes. The amount of information they reveal to the agent can persuade him to stop or continue his search, and it therefore affects the creation and division of surplus in the market.

Relatively little is known about how search outcomes are affected by a market for information, or how these outcomes vary with the market structure for information brokers. Yet, the answers to such questions may be of practical import. For instance, between 2013 and 2016, *CarFax* was engaged in a class-action lawsuit wherein plaintiffs alleged that *CarFax* engaged in anti-competitive practices and had become a de facto monopoly, controlling 90% of the market for certifying pre-owned vehicles. While *CarFax* admitted to its monopoly status, it argued that its use of short-term contracts (rather than long-term exclusive agreements) did not reduce welfare because it disciplined *CarFax* to always provide high quality information at a low price.² This paper explores the merit of such claims: how does competition, or lack thereof, among information brokers affect search outcomes, market efficiency, and the welfare of the searching agent?

The answers to these questions are not a priori obvious. On one hand, a competitive setting might give rise to "Bertrand-like" outcomes in which brokers provide more information and charge less for it than would a monopolist, leading to more efficient search and a

¹For example, McCall (1970) studies an agent who perfectly observes the quality of sampled goods, while Chade (2006) studies imperfect observation of sampled goods.

²See details on the case *Maxon Hyundai Mazda et al. v Carfax, Inc* and the associated arguments referenced above at https://casetext.com/case/mazda-v-carfax and https://casetext.com/case/mazda-v-carfax-inc-1.

higher payoff for the searching agent. On the other hand, competing brokers might collude. In our setting, collusion can occur along two dimensions: keeping the price of information high, or keeping informativeness low. Therefore, collusion between multiple brokers could instead lead to inefficient search outcomes. Moreover, there may be multiple equilibria that feature different "levels of" collusion. Thus, to satisfactorily answer the question of how competition affects search outcomes and welfare, one must compare across the set of equilibrium outcomes under various market structures.

To this end, we embed a repeated contracting game between an agent and information brokers (henceforth, "principals") into a canonical sequential search model (McCall, 1970). The agent ("he") has a unit demand, and in each period, samples one good after incurring a search flow cost. However, unlike in the canonical search model, he does not observe the good's quality. Instead, each principal ("she") designs a signal of the good's quality and sets a price for it, and the agent chooses which signal to purchase, if any. The agent then observes a signal realization based on his purchase decision, and decides either to terminate his search by matching with the good, or to discard the good and continue searching in the next period. We characterize the set of equilibrium payoffs in this *persuaded search game*, and study how the equilibrium payoff set varies between monopolistic and competitive settings.

Our findings align with conventional wisdom that competition among principals benefits the agent. However, our results challenge the prevailing view that competitive settings are more efficient compared to monopolistic ones; instead, we find that competition actually reduces the total surplus generated from search.

These findings suggest nuanced implications for the regulation of dynamic markets for information. Should a regulator "break up" a monopolist, like *CarFax*, into rival firms? Our results indicate the answer to this question depends on the goals of the regulator: If the regulator seeks to maximize efficiency, it will allow the monopolist to persist. If the regulator instead seeks to maximize the agent's surplus, it will break up the monopolist.

At the heart of our main result is the insight that the agent's willingness-to-pay for a signal is determined not only by the signal's informativeness, as would be the case in a static setting, but also by the informativeness and the prices of future signals if he continues searching. In other words, the agent's per-period value of information depends endogenously on his continuation value. In particular, a signal is less valuable today if the agent's continuation value from not purchasing it exceeds that from purchasing it. Indeed, when the gap between these two continuation values is sufficiently large, the agent's value for the signal becomes negative.

A negative value of information plays a key role in determining the agent's "bargaining

power" in the repeated contracting problem. When contracting with monopolist, the agent has limited bargaining power as he faces a take-it-or-leave-it offer from his only source of information. Consider an agent who seeks to gain leverage by threatening to reject an offer that yields him too small a share of the surplus. Such a threat is credible only if he has a negative value of information in the present. We show this is the case if and only if the gap between his highest and lowest feasible continuation payoffs is sufficiently wide or, equivalently, if the search cost is sufficiently low. Consequently, there exists a cost threshold such that, below the threshold, the agent's threats credibly sustain any surplus split between the agent and the principal as an equilibrium outcome. In contrast, above the threshold, the agent's threats have no credibility. Because of this, we show that the principal-preferred outcome—full surplus extraction by the principal—emerges as the unique equilibrium payoff. In other words, the equilibrium payoff set in a monopolistic setting has a *bang-bang* structure; depending on the search cost, either a unique equilibrium or a folk-theorem obtains.

With competing principals, the agent can once again seek to gain leverage against any one principal by refusing to purchase her signal unless she offers him a large enough share of the surplus. As in the monopolistic setting, the agent's threat is credible here if his value for the principal's signal is negative, but unlike the monopolistic setting, the threat is also credible if his value for the signal is merely lower than his value for signals offered by any other principal. As a result, the agent is more effective in securing himself a high payoff in a competitive setting. Formally, we show that a folk-theorem result obtains under competition whenever it does in a monopolistic setting. Moreover, even when the principal-preferred outcome is the unique equilibrium in a monopolistic setting, competition allows for infinitely many equilibrium outcomes, nearly all of which are less efficient than the unique equilibrium outcome of a monopoly.

Using these characterizations, we compare the total surpluses and agent's payoffs generated in the equilibria of monopolistic and competitive settings, leading us to one of the following conclusions depending on the value of the search cost: (i) monopoly and competition yield the same range of total surpluses and agent's payoffs, or (ii) the highest total surplus generated under competition equals the lowest total surplus generated under monopoly, whereas the lowest agent's payoff generated under competition is no lower than the highest agent's payoff generated under monopoly. Moreover, (iii) for certain search cost values, almost all equilibria under competition yield a strictly higher agent's payoff but a strictly lower total surplus than does a monopoly. These conclusions illustrate the role of competition in reducing efficiency while benefiting the agent.

Our characterization of equilibrium payoff sets for stopping games may be of independent interest. The classical theory of repeated games primarily focuses on infinitely repeated. simultaneous-move stage games in which a player's payoff in the repeated game equals the discounted sum of her stage-game payoffs. However, many of the tools for characterizing equilibrium payoffs and folk-theorem results do not readily apply to our persuaded search game, which diverges from the classical theory in three key ways: First, stopping games are not repeated games because they lack well-defined stage games. This implies that a player's equilibrium payoff cannot be bounded by the usual stage-game minimax payoff. Second, we study a game in which players do not move simultaneously in each period. As a result, equilibrium strategies must satisfy sequential rationality within each period. Finally, as is typical among search models, our game features a search cost rather than discounting. Thus, the well-known finding that the one-shot deviation principle implies sequential rationality does not immediately transfer to our setting. To address these differences, we adapt the set-valued dynamic programming approach of Abreu, Pearce, and Stacchetti (1990) (henceforth, "APS") to our setting, and further reduce their set-valued fixed-point operator to a single-dimensional fixed-point problem, yielding a tractable characterization of equilibrium payoff sets for stopping games.

1.1. Related Literature

The literature examining the role of information on search market outcomes has largely focused on a single principal designing information for a searching agent. Sato and Shi-rakawa (2023) characterize the optimal signal for a principal who influences the search behavior of an agent engaged in ordered search à la Weitzman (1979). Similarly, Dogan and Hu (2022) and Hu (2022) characterize the optimal signal for a principal with intrinsic preferences over the outcomes of a large random search market. Mekonnen et al. (2024) do the same for a profit-maximizing principal; they find a unique stationary equilibrium outcome of a persuaded search game between a single principal and a searching agent. Our work refines theirs by showing that in monopoly market structures, their unique stationary equilibrium outcome is in fact the unique equilibrium outcome for high search costs whereas a folk-theorem result obtains for low search costs.

A smaller body of work explores the role of competitive information provision in search markets. Board and Lu (2018) and He and Li (2023) study how competition among principals affects outcomes in a random search market, and Au and Whitmeyer (2023, 2024) study the case of a directed search market. These papers consider an agent who engages bilaterally with a sequence of principals as in Diamond (1971), making each principal-agent relationship short-lived and effectively monopolistic. Consequently, the symmetric equilibria of these game often feature a "Diamond Paradox" outcome with the agent terminating his search in the first period. In contrast, we consider an agent who engages repeatedly and simultaneously with all principals in each period, producing a richer array of possible outcomes, including a folk-theorem result.

Beyond search markets, the persuasion literature also studies the impact of competition in both static settings (Gentzkow and Kamenica, 2016, 2017; Au and Kawai, 2020) and dynamic ones (Li and Norman, 2018, 2021; Wu, 2023). These papers consider the canonical, non-transferable utility persuasion setting in which "senders" design information so as to influence the action taken by a "receiver." Because monetary payments are not allowed. receiver's welfare is determined by the amount of information senders provide, and the literature characterizes conditions under which competition increases information provision (Kamenica, 2019). In our model, however, welfare depends not just on the level of information provision but also on the monetary transfers from the agent (receiver) to the principals (senders). Indeed, our contracting game can be considered one of Bertrand competition (in the market for information). This familiar setting becomes more complicated within the context of search; the transfers from the agent to the principals may distort the agent's search behavior because the agent wants to match with a high quality good as quickly as possible while paying as little as possible for information, whereas the principals want to keep the agent buying signals for as long as possible. We show that these distortions are amplified when there is competition between principals.

Finally, our work also relates to the literature on the design and pricing of information (Admati and Pfleiderer, 1986, 1990; Eső and Szentes, 2007; Bergemann and Bonatti, 2015; Bergemann et al., 2018; Bonatti et al., 2024; Rodríguez Olivera, 2024). However, all of these papers consider the sale of information by a monopolist in a static market. As far as we are aware, our paper is the first to study the competitive sale of information in a dynamic market.

2. Model

2.1. Notation

We denote the L^1 -norm of $x \in \mathbb{R}^n$ by $||x|| \coloneqq \sum_{i=1}^n |x_i|$. Given a subset $X \subseteq \mathbb{R}^n$, we write 2^X to denote the set of non-empty subsets of X, cl(X) to denote its closure, conv(X) to

denote its convex hull, and ΔX to denote the space of probability measures over X. Given a distribution $G: X \to [0,1]$, we denote its support by $\operatorname{supp}(G)$. Finally, given $x \in \mathbb{R}$, let $x^+ := \max\{x, 0\}$.

2.2. Setup

We study a repeated contracting problem between a risk-neutral agent and $n \ge 1$ risk-neutral principals, with $N \coloneqq \{1, ..., n\}$ denoting the set of principals. The agent faces an optimal stopping problem: He has a unit demand, and he sequentially searches without recall for a good to consume. In each period, the agent incurs a search cost k > 0 and samples a good. The good's quality (or match value), denoted by $\theta \in \Theta = [0,1]$, is identically and independently distributed according to an absolutely continuous distribution F with a prior mean of $m^{\varnothing} \coloneqq \int_{0}^{1} \theta dF(\theta)$. We assume that $m^{\varnothing} > k$ so that the agent has an incentive to search.

Neither the agent nor the principals observe quality, but each principal can design and sell an informative signal about the quality of the currently sampled good. Because the principals and the agent are risk neutral, the only payoff-relevant information is the expected quality of a sampled good. We therefore model a signal simply as a posterior-mean distribution $G: \Theta \rightarrow [0,1]$. For example, a fully informative signal is given by the distribution F while an uninformative signal is given by the distribution G^{\varnothing} that is degenerate on the prior mean. From Blackwell (1953), the set of feasible posterior-mean distributions is equivalent to the set of mean-preserving contractions of F, which we denote by $\mathfrak{C}(F)$.

2.3. Timing

At the beginning of each period t = 1, 2, ... in which the agent is engaged in search, he incurs a search cost of k and samples a good of quality $\theta_t \sim F$. Each principal $i \in N$ then makes a take-it-or-leave-it (TIOLI) offer $o_{it} \coloneqq (p_{it}, G_{it})$ consisting of a price $p_{it} \in \mathbb{R}_+$ and a posterior-mean distribution $G_{it} \in \mathfrak{C}(F)$. Let $O_t \coloneqq (o_{it})_{i \in N}$ denote the profile of period-t offers, and let $O_{-it} \coloneqq (o_{jt})_{j \in N \setminus \{i\}}$. In addition to the offers made by the principals, the agent has access to a null offer $o_{\emptyset t} \coloneqq (p_{\emptyset t}, G_{\emptyset t})$, with $p_{\emptyset t} = 0$ and $G_{\emptyset t} = G^{\varnothing}$ for all t.

Given the profile of offers O_t , the agent purchases either from one of the principals, denoted $w_t \in N$, or takes the null offer, denoted $w_t = \emptyset$. Given purchase decision $w_t \in W := N \cup \{\emptyset\}$, a realization $m_t \sim G_{w_t t}$ is drawn and publicly observed. Finally, the agent decides whether to consume the sampled good and stop searching $(d_t = 1)$, in which case the game ends, or to continue searching $(d_t = 0)$, in which case the game continues on to t+1. Let $\mathfrak{O} := \mathbb{R}_+ \times \mathfrak{G}(F)$ be the space of all TIOLI offers. Given a profile of offers $O_t \in \mathfrak{O}^n$, purchase decision $w_t \in W$, and stopping decision $d_t \in \{0,1\}$, the agent's ex-post period-tpayoff is given by $d_t \theta_t - k - p_{w_t t}$, and Principal *i*'s ex-post period-t payoff is given by $p_{it} \mathbb{1}_{[w_t=i]}$.

Before we proceed with our analysis, let us remark on two assumptions we have made thus far. First, we assume that the agent acquires information from at most one principal in a given period. An alternative assumption would be to let the agent acquire information from any subset of principals. This would allow the agent to combine signals in a similar spirit to Gentzkow and Kamenica (2016) and learn more than what is possible by observing the signal of any one principal. Second, we assume that signal realizations are publicly observable. An alternative assumption would be that signal realizations are privately observed by the agent. This alternative would necessitate keeping track of a public history for the principals, a private history for the agent, and each principal's beliefs over the agent's private history. However, as will become clear in Section 4, such alternative assumptions do not change our main results.

2.4. Histories, strategies, payoffs, and equilibrium

A history at the beginning of period t (for which the agent is still engaged in search) contains all past TIOLI offers, purchase decisions, and signal realizations. Let $h^t := (O_\tau, w_\tau, m_\tau)_{\tau < t}$ denote an arbitrary period-t history, with $h^1 = \emptyset$.³ We denote the set of all period-t histories by H^t , and the set of all histories by $H := \bigcup_{t>1} H^t$.

A behavioral strategy for Principal $i \in N$ is given by $\sigma_i : H \to \Delta \mathbb{O}$, which maps each history $h \in H$ into a distribution over TIOLI offers. Let Σ_P denote the set of all such strategies (all the principals have the same strategy space).

A behavioral strategy for the agent is given by $\sigma_A := (\sigma_A^w, \sigma_A^d)$, where the first mapping $\sigma_A^w : H \times \mathbb{O}^n \to \Delta W$ captures the purchase decision, and the second mapping $\sigma_A^d : H \times \mathbb{O}^n \times W \times \Theta \to [0,1]$ captures the stopping decision. Specifically, following some history $h \in H$ and a profile of offers $O \in \mathbb{O}^n$, the agent purchases a signal from $w \in W$ with probability $\sigma_A^w(w|h,O)$ and continues his search following some signal realization $m \in \Theta$ with probability $\sigma_A^d(h,O,w,m)$. Let Σ_A denote the set of all such strategies for the agent.

Each strategy profile $\sigma \coloneqq ((\sigma_i)_{i \in N}, \sigma_A) \in \Sigma_P^n \times \Sigma_A$ induces a distribution over offers, purchase decisions, and ultimately, a stopping time \tilde{T} which is the (random) time when the agent ends his search. Thus, given a strategy profile σ and a history h^t , Principal *i*'s

³We do not include $\{d_{\tau}\}_{\tau < t}$ in the history as it is redundant; the game reaches Period t if and only if $d_{\tau} = 0$ for all $\tau < t$.

expected payoff in Period t is given by

$$\mathbf{V}_{it}(\sigma|h^{t}) = \mathbb{E}_{\sigma(h^{t})} \left[\sum_{\tau=t}^{\tilde{T}} p_{i\tau} \mathbb{1}_{[i=w_{\tau}]} \right], \tag{1}$$

where the expectation is taken with respect to the distribution induced by σ starting from history h^t . Similarly, the agent's expected payoff is given by

$$\mathbf{U}_{t}(\sigma|h^{t}) = \mathbb{E}_{\sigma(h^{t})} \left[\theta_{\tilde{T}} - \sum_{\tau=t}^{\tilde{T}} \left(p_{w_{\tau}\tau} + k \right) \right].$$

$$\tag{2}$$

When considering payoffs in period t = 1, we simply write $\mathbf{V}_i(\sigma)$ and $\mathbf{U}(\sigma)$ instead of $\mathbf{V}_{i1}(\sigma|h^1)$ and $\mathbf{U}_1(\sigma|h^1)$, respectively.

A strategy profile σ constitutes an *equilibrium* if it is sequentially rational,⁴ i.e., for all $t \ge 1$ and all $h^t \in H$,

(a)
$$\mathbf{V}_{it}(\sigma|h^t) \ge \mathbf{V}_{it}(\tilde{\sigma}_i, (\sigma_j)_{j \in N \setminus \{i\}}, \sigma_A|h^t)$$
 for all $\tilde{\sigma}_i \in \Sigma_P$ and for all $i \in N$, and

(b)
$$\mathbf{U}_t(\sigma|h^t) \ge \mathbf{U}_t((\sigma_i)_{i \in N}, \tilde{\sigma}_A|h^t)$$
 for all $\tilde{\sigma}_A \in \Sigma_A$.

Let \mathscr{C} be the set of equilibrium payoffs, that is,

$$\mathscr{C} := \{ y \in \mathbb{R}^{n+1} : y_i = \mathbf{V}_i(\sigma) \text{ for } i \in N \text{ and } y_{n+1} = \mathbf{U}(\sigma) \text{ for some equilibrium } \sigma \}.$$

In what follows, it will be useful to parameterize the equilibrium payoff set by the number of competing principals and the search cost. Thus, we write $\mathscr{C}(n,k)$ to denote the equilibrium payoff set of a game between *n* principals and an agent who has a search cost of *k*.

⁴More accurately, our solution concept is perfect Bayesian equilibrium comprised of a sequentially rational strategy profile σ , and a belief system that is derived by Bayes rule whenever possible and satisfies the "no signaling what you don't know" restriction (Fudenberg and Tirole, 1991). However, specifying such a belief system is redundant in our setting for two reasons: (1) The only uncertainty in each period is a symmetric uncertainty about a sampled good's quality, so no player holds beliefs about other players. (2) The "no signaling" restriction implies that beliefs about the quality of the sampled good are derived by Bayes rule both on and off path, which we already account for by modeling a signal about a good's quality as a posterior-mean distribution.

3. Main Results

In this section, we present the paper's three main results. After presenting these main results, we discuss the organization of their proofs.

Our first main result compares monopolistic and competitive settings based on the total surplus that can be generated in an equilibrium of each setting. Define the operator $\mathcal{W}: \bigcup_{n\geq 1} 2^{\mathbb{R}^{n+1}} \to 2^{\mathbb{R}}$ such that, given $n\geq 1$ and some set $E\subseteq \mathbb{R}^{n+1}_+$, the corresponding total surplus set is given by

$$\mathcal{W}(E) \coloneqq \{ ||y|| : y \in E \}.$$

Theorem 1 There exists a unique $k^* \in (0, m^{\emptyset})$ such that for all n > 1,

- (i) $\mathcal{W}(\mathcal{E}(1,k)) = \mathcal{W}(\mathcal{E}(n,k))$ for all $k \leq k^*$, and
- (ii) $\inf \mathcal{W}(\mathcal{E}(1,k)) = \sup \mathcal{W}(\mathcal{E}(n,k))$ for all $k > k^*$.

In words, either the equilibrium total surplus set is equivalent under monopolistic and competitive settings, or the *largest* surplus that can be generated in a competitive setting equals the *smallest* surplus that can be generated in a monopolistic setting.⁵

The amount of surplus generated in our setting is determined by the value of information in the search market. Thus, we can also interpret Theorem 1 to mean that the level of information provision in a monopolistic setting is no less efficient than the level of information provision in competitive settings.⁶

Of course, with transferable utility, the fact that there is more efficient information provision in a monopolistic setting than a competitive setting does not imply that the agent is better off under a monopoly. Our second main result compares monopolistic and competitive settings based on the agent's surplus. Define the operator $\mathcal{A}: \bigcup_{n\geq 1} 2^{\mathbb{R}^{n+1}} \to 2^{\mathbb{R}}$ such that, given $n\geq 1$ and some set $E\subseteq \mathbb{R}^{n+1}_+$, the corresponding agent's surplus set is given by

$$\mathscr{A}(E) \coloneqq \{ y_{n+1} \colon y \in E \}.$$

Theorem 2 Let k^* be the threshold in Theorem 1. For all n > 1,

(i) $\mathcal{A}(\mathcal{E}(1,k)) = \mathcal{A}(\mathcal{E}(n,k))$ for all $k \leq k^*$, and

⁵Our comparison is stronger than saying $\mathcal{W}(\mathcal{E}(1,k))$ dominates $\mathcal{W}(\mathcal{E}(n,k))$ in the strong set order.

⁶In this case, a "more efficient" information provision brings the agent's equilibrium search behavior closer to that of an agent in a McCall search setting, i.e., an agent who observes a fully informative signal for free. We discuss the connection to McCall (1970) in Section 4.1.

(*ii*) $\sup \mathcal{A}(\mathcal{E}(1,k)) \leq \inf \mathcal{A}(\mathcal{E}(n,k))$ for all $k > k^*$.

In words, either the equilibrium agent surplus set is equivalent under monopolistic and competitive settings, or even his *largest* payoff in a monopolistic setting is no larger than his *smallest* payoff in a competitive setting.

The contrast in the two theorems suggests that there is a tension between efficiency and the agent's welfare: Competition at best does not improve efficiency, and at worst yields more inefficient outcomes. However, from the agent's perspective, competition at worst does not hurt him, and at best gives him a higher share of the surplus. The following result formalizes that these relationships are strict: there exist cost parameters which admit equilibrium outcomes in a competitive setting that are strictly less efficient, but that yield the agent a strictly higher payoff, than any equilibrium outcome in a monopolistic setting.

Theorem 3 Let k^* be the same threshold as in Theorem 1. There exists a threshold $k^{**} \in (k^*, m^{\emptyset})$ such that for each $k \in (k^*, k^{**})$ and each n > 1, there exists an a non-empty open set $E \subset \mathscr{C}(n,k)$ with $\inf \mathscr{W}(\mathscr{C}(1,k)) > \sup \mathscr{W}(E)$ and $\inf \mathscr{A}(E) > \sup \mathscr{A}(\mathscr{C}(1,k))$.

Indeed, our proof of Theorem 3 reveals that for n > 1 and $k \in (k^*, k^{**})$ the subset of $\mathscr{C}(n,k)$ that yields weakly higher total welfare than $\inf \mathscr{W}(\mathscr{C}(1,k))$ or that yields weakly lower agent surplus than $\sup \mathscr{A}(\mathscr{C}(1,k))$ is a null set. Because the set E described in Theorem 3 is open and non-empty, it is also non-negligible. Therefore, for $k \in (k^*, k^{**})$, "almost all" equilibria under competition feature strictly lower total welfare and strictly higher agent surplus than does any equilibrium under monopoly.

The comparison of welfare across different market structures is naturally of interest in certain applied and policy settings. Here we offer one such application of our main results: consider a regulator (it) which must decide whether to break up a monopoly in the market for information brokers. Specifically, given the agent's search cost $k \in (0, m^{\emptyset})$, the regulator must choose between a market structure (1,k) and (n,k) for some n > 1. Which market structure should the regulator choose if it is interested in the efficiency of the search market equilibrium? If it is interested in agent's welfare?

Theorem 1-Theorem 3 suggest that, in general, a regulator that prioritizes efficiency (weakly) prefers the monopoly to persist while a regulator that prioritizes the agent's welfare (weakly) prefers to break up the monopoly. However, a caveat to this general statement is that equilibrium selection matters for some search cost parameters. Given a market structure (\hat{n},k) , suppose payoff profile $y^*(\hat{n},k)$ is selected from $\mathscr{E}(\hat{n},k)$. We refer to y^* as the equilibrium selection rule equilibrium selection rule. We say that an equilibrium selection rule is *W*-symmetric if, given $n',n'' \ge 1$, the selection rule satisfies $||y^*(n',k)|| = ||y^*(n'',k)||$ whenever $\mathcal{W}(\mathfrak{C}(n',k)) = \mathcal{W}(\mathfrak{C}(n'',k))$.⁷ Similarly, we say an equilibrium selection rule is *A*-symmetric if $y^*_{n'+1}(n',k) = y^*_{n''+1}(n'',k)$ whenever $\mathcal{A}(\mathfrak{C}(n',k)) = \mathcal{A}(\mathfrak{C}(n'',k))$.

Corollary 1 Let k^* be the threshold in Theorem 1, and let n > 1.

- 1. For any $k > k^*$, a regulator that seeks to maximize the total surplus (weakly) prefers market structure (1,k) over (n,k) under any equilibrium selection rule. Moreover, if the selection rule is W-symmetric, the regulator (weakly) prefers market structure (1,k) for any $k \in (0,m^{\varnothing})$.
- 2. For any $k > k^*$, a regulator that seeks to maximize the agent's surplus (weakly) prefers market structure (n,k) over (1,k) under any equilibrium selection rule. Moreover, if the selection rule is A-symmetric, the regulator (weakly) prefers market structure (n,k) for any $k \in (0,m^{\emptyset})$.

The remainder of the paper is organized around proving the main results. Section 4 introduces tools which make the analysis more tractable. Then, Section 5 provides four propositions which characterize the set of equilibrium payoffs as a function of the number of principals, n, and the search cost, k, and shows how these propositions imply our main results. Section 6 concludes. Throughout, we present additional lemmas which are useful in the analysis; we relegate their proofs to the appendix.

4. Preliminary Steps

4.1. Single-agent search problem

Let us consider a hypothetical search problem in which there are no principals but the agent observes a realization from a distribution $G \in \mathcal{G}(F)$ in each period for free. This hypothetical setting is instructive in characterizing the lowest and the highest payoffs possible in the persuaded search game.

⁷For example, $y^*(\hat{n},k) \in \operatorname{argmin}_{y \in \mathscr{C}(\hat{n},k)} ||y||$ is a W-symmetric selection rule. Such a selection rule would be appropriate for a pessimistic regulator who is interested in maximizing total surplus in a "worst-case" scenario. In this case, the regulator would solve $\max_{\hat{n} \in \{1,n\}} ||y^*(\hat{n},k)|| \equiv \max_{\hat{n} \in \{1,n\}} \min_{y \in \mathscr{C}(\hat{n},k)} ||y||$.

Suppose the agent's continuation value in this stationary setting is $u \in \mathbb{R}_+$. The agent then stops his search if the realized posterior mean is m > u and continues searching if m < u.⁸ Thus, his value of search is given by

$$\int_{\Theta} (m-u)^{+} dG(m) + u - k = \int_{u}^{1} (1 - G(m)) dm + u - k = \int_{u}^$$

where the last equality follows from integration by parts. Let $c_G: \mathbb{R}_+ \to \mathbb{R}_+$ be defined as

$$c_G(u) \coloneqq \int_u^1 (1 - G(m)) dm.$$

For any $G \in \mathfrak{G}(F)$, the mapping $u \mapsto c_G(u)$ is continuous, convex, weakly decreasing with $0 = c_G(1) < c_G(0) = m^{\varnothing}$, and has a right derivative given by $\partial_+ c_G(u) = G(u) - 1$. Moreover, the mapping $G \mapsto c_G$ is decreasing in mean-preserving contractions: if G' is a mean-preserving contraction of G'', then $c_{G'} \leq c_{G''}$ pointwise. Hence, for any $G \in \mathfrak{G}(F)$, $c_{G^{\varnothing}} \leq c_G \leq c_F$ pointwise.⁹

The agent's continuation value must be the same as his value of search. Thus, for any given distribution $G \in \mathscr{G}(F)$, the agent's continuation value must be a solution to the equation $u = c_G(u) + u - k$, or equivalently, a solution to

$$k = c_G(u). \tag{3}$$

The expression in (3), seen as a function of u, has a unique solution because c_G is a continuous and decreasing function with $c_G(0) = m^{\emptyset} > k > 0 = c_G(1)$. Given a search cost $k \in (0, m^{\emptyset})$, let $u_k : \mathfrak{C}(F) \to [0, 1]$ be the mapping capturing the solution to (3) so that $k = c_G(u_k(G))$ for all $G \in \mathfrak{C}(F)$.

For a fixed $k \in (0, m^{\varnothing})$, $u_k(G') \leq u_k(G'')$ for any $G', G'' \in \mathfrak{G}(F)$ such that G' is a mean-preserving contraction of G''.¹⁰ Consequently, the lowest payoff the agent can earn is $\underline{u}_k \coloneqq u_k(G^{\varnothing}) = m^{\varnothing} - k$. This is the payoff the agent gets if the signal was completely uninformative and he ends his search in the first period by consuming the first good he samples. In contrast, the highest payoff is $\overline{u}_k \coloneqq u_k(F) > \underline{u}_k$, which is the payoff the agent

⁸If m = u then the agent is indifferent between stopping his search or continuing. We specify how the agent breaks this indifference in equilibrium in Section 4.3.

 $^{^{9}}$ See Mekonnen et al. (2024) for a discussion of these properties.

¹⁰This is because G'' is a Blackwell more informative signal than G', and more informative signals are (weakly) more valuable in single-agent decision problems.



Figure 1: This figure depicts c_F in blue, $c_{G^{\varnothing}}$ in red, and a function c_G in black for some arbitrary $G \in \mathfrak{G}(F)$. The continuation values \bar{u}_k , \underline{u}_k , and $u_k(\cdot)$ correspond to the points where each respective curve intersects with the dashed line representing search cost k.

gets if the signal structure was fully informative. We refer to \bar{u}_k as the *McCall payoff* (the payoff generated by the agent in McCall (1970)), to \underline{u}_k as the *autarky payoff*, and to $\bar{u}_k - \underline{u}_k$ as the *full surplus*. The following lemma describes how the value of search and the full surplus change as a function of the agent's search cost.

Lemma 1 For any $G \in \mathfrak{C}(F)$, the value of search $u_k(G)$ is strictly decreasing in k over the interval $(0, m^{\varnothing})$. Furthermore, the full surplus, $\bar{u}_k - \underline{u}_k$ is strictly decreasing in k over the interval $(0, m^{\varnothing})$.

4.2. Feasible payoffs

We now return to the contracting setting between the agent and the principals. In any equilibrium, each principal's payoff must be non-negative as prices are non-negative, the agent's payoff must weakly exceed his autarky payoff since he can always guarantee this payoff by taking the null offer and immediately ending his search, and the total surplus generated in any equilibrium cannot exceed the McCall payoff as this is the highest possible surplus in the search market. Thus, for $n \ge 1$ principals and a search cost $k \in (0, m^{\emptyset})$, the feasible payoff set is given by

$$\mathscr{F}(n,k) \coloneqq \{ y \in \mathbb{R}^{n+1} : y_i \ge 0 \text{ for } i \in N, y_{n+1} \ge \underline{u}_k, \text{ and } ||y|| \le \overline{u}_k \}.$$

The shaded blue simplex in Figure 2 depicts the feasible set $\mathscr{F}(2,k)$ in a duopoly market for some search cost $k \in (0,m^{\emptyset})$. For any $n \ge 1$ and $k \in (0,m^{\emptyset})$, it is straightforward to see that $\mathscr{F}(n,k)$ is an (n+1)-dimensional simplex. Importantly, it has a non-empty interior because $\bar{u}_k > \underline{u}_k > 0$. Naturally, $\mathscr{C}(n,k) \subseteq \mathscr{F}(n,k)$.



Figure 2: This figure depcits the feasible set $\mathcal{F}(n,k)$ for n=2 and $k \in (0,m^{\varnothing})$. Vertices A,B,C, and D correspond to, respectively, the autarky payoff profile $(0,0,\underline{u}_k)$, the "Bertrand" payoff profile $(0,0,\overline{u}_k)$, and the principal optimal payoff profiles $(\overline{u}_k - \underline{u}_k, 0, \underline{u}_k)$ and $(0, \overline{u}_k - \underline{u}_k, \underline{u}_k)$. The face BDC depicts all efficient outcomes, i.e, payoff profiles with $||y|| = \overline{u}_k$.

4.3. Equilibrium outcomes and self-generating sets

As discussed in the previous subsection, the set of equilibrium outcomes is a subset of the feasible set. In this subsection, we characterize equilibrium payoff sets by taking a dynamic programming approach. This is tractable in our model since each good is drawn i.i.d., making all continuation periods identical. Specifically, we study *self-generating* subsets of $\mathcal{F}(n,k)$ by adapting APS to a sequential-move stopping game with no discounting.

To formally describe our approach, we first introduce simple strategies which are history-independent. A simple strategy for Principal $i \in N$, denoted by $\alpha_i \in \Delta \mathfrak{G}$, is a mixed strategy over offers. Let $\alpha := (\alpha_i)_{i \in N}$ and let $\alpha_{-i} := (\alpha_j)_{j \in N \setminus \{i\}}$. Similarly, the agent's simple strategy is given by $\beta := (\beta^w, \beta^d)$ where $\beta^w : \mathfrak{G}^n \to \Delta W$ is a behavioral strategy over purchase decisions, and $\beta^d : \mathfrak{G}^n \times W \times \Theta \to [0,1]$ is the probability with which the agent continues to search. We also introduce mappings of the form $\psi : \mathfrak{G}^n \times W \times \Theta \to \mathbb{R}^{n+1}$, which we refer to as a *continuation payoff function*, with $\psi(O, w, m)$ representing a continuation payoff profile if the agent continues searching after the principals make TIOLI offers $O \in \mathbb{G}^n$, the agent accepts the offer from $w \in W$, and a signal realization of $m \in \Theta$ is observed. In this case, $\psi_i(O, w, m)$ for $i \in N$ represents Principal *i*'s continuation payoff and $\psi_{n+1}(O, w, m)$ represents the agent's continuation payoff. We denote the image of the mapping ψ by Im_{ψ} .

Given a profile of offers O, agent's simple strategy β , and a continuation payoff function ψ , let Principal *i*'s payoff be given by

$$V_i(O,\beta,\psi) \coloneqq p_i\beta^w(i|O) + \sum_{w \in W} \beta^w(w|O) \int_{\Theta} \psi_i(O,w,m)\beta^d(O,w,m) dG_w(m)$$

and let the agent's payoff be given by

$$\begin{split} U(O,\beta,\psi) &\coloneqq -k + \sum_{w \in W} \beta^w(w|O) \bigg[-p_w + \int_{\Theta} m \big(1 - \beta^d(O,w,m) \big) dG_w(m) \bigg] \\ &+ \sum_{w \in W} \beta^w(w|O) \int_{\Theta} \psi_{n+1}(O,w,m) \beta^d(O,w,m) dG_w(m). \end{split}$$

Definition 1 We say a payoff profile $y \in \mathcal{F}(n,k)$ is supported by a set $E \subseteq \mathbb{R}^{n+1}$ if there exist simple strategies (α,β) and a continuation payoff function ψ such that

- (a) y is generated by (α, β, ψ) : For all $i \in N$, $y_i = \mathbb{E}_{\alpha}[V_i(O, \beta, \psi)]$, and $y_{n+1} = \mathbb{E}_{\alpha}[U(O, \beta, \psi)]$, where \mathbb{E}_{α} is the expectation over \mathbb{G}^n with respect to the probability induced by α .
- (b) No principal has a profitable deviation: For all $i \in N$, $y_i \ge \mathbb{E}_{\alpha_{-i}} [V_i(\tilde{o}_i, O_{-i}, \beta, \psi)]$ for all $\tilde{o}_i \in \mathbb{G}$, where $\mathbb{E}_{\alpha_{-i}}$ is the expectation over \mathbb{G}^{n-1} with respect to the probability induced by α_{-i} .
- (c) β is sequentially rational with respect to ψ : For all $O \in \mathbb{G}^n$, $U(O,\beta,\psi) \ge U(O,\tilde{\beta},\psi)$ for all agent simple strategies $\tilde{\beta} \coloneqq (\tilde{\beta}^w, \tilde{\beta}^d)$.
- (d) $Im_{\psi} \subseteq E$.

Simply put, a feasible payoff profile y is supported by some set E if, given some continuation payoffs in E, y can arise as an outcome of optimal simple strategies. Optimality here means that taking the continuation payoffs as given, the agent's simple strategy is a best response to any profile of TIOLI offers. Furthermore, taking the continuation payoff function and the agent's simple strategy as given, each Principal i makes a TIOLI offer that is a best response against the simple strategies of the other principals. Notice that our definition of a self-generating set differs from the one defined in APS: we require the agent's strategy to be optimal following any profile of offers whereas APS require optimality on-path. This distinction reflects the fact that our persuaded search game has a sequential-move "stage game."

While Definition 1 does not place restrictions on the continuation payoff set E, for our purposes, we are interested in sets that can "support themselves."

Definition 2 We say a set $E \subseteq \mathcal{F}(n,k)$ is self-generating if each $y \in E$ is supported by E.

As observed by APS, self-generating sets are linked to (subsets of) the equilibrium payoff set. However, their observation does not readily apply here: as noted, our persuaded search game lacks well-defined stage game payoffs, and features no discounting. Nevertheless, the lemma below extends their Theorems 1 and 2 to our setting, and the proof is contained in the Online Appendix.

Lemma 2 For all $n \ge 1$ and $k \in (0, m^{\emptyset})$, $\mathfrak{C}(n,k)$ is self generating. Furthermore, for any set $E \subseteq \mathfrak{F}(n,k)$, if E is self generating, then $E \subseteq \mathfrak{C}(n,k)$.

While we can use Lemma 2 to assess whether a set $E \subseteq \mathcal{F}(n,k)$ is a subset of the equilibrium payoff set, it is nevertheless a daunting task to check that each payoff profile in E satisfies Points (a)-(d) of Definition 1. Let us cut down the difficulty of this task: given a threshold $x \in \Theta$, let $G^x \in \mathcal{G}(F)$ be a posterior-mean distribution given by

$$G^{x}(m) = \begin{cases} 0 & \text{if} \quad m < \mathbb{E}_{F}[\theta|\theta \le x] \\ F(x) & \text{if} \quad \mathbb{E}_{F}[\theta|\theta \le x] \le m < \mathbb{E}_{F}[\theta|\theta \ge x] \\ 1 & \text{if} \quad m \ge \mathbb{E}_{F}[\theta|\theta \ge x] \end{cases}$$
(4)

for any $m \in [0,1]$, and

$$c_{G^{x}}(u) = \max \left\{ c_{G^{\varnothing}}(u), c_{F}(x) + (1 - F(x))(x - u) \right\}$$
(5)

for any $u \in [0,1]$. We refer to G^x as a *pass-fail* signal (with threshold x) because it represents the posterior-mean distribution induced by revealing if a good's quality is above the threshold x-in which case, the good "passes"-or if it is below the threshold-in which case, the good "fails."

Our next two lemmas shows that any surplus $u \in [\underline{u}_k, \overline{u}_k]$ can be generated by some pass-fail signal. Furthermore, we can arbitrarily divide this surplus between the *n* principals

and the agent (subject to feasibility constraints) in order to generate any payoff profile y with ||y|| = u.

Lemma 3 For each $k \in (0, m^{\emptyset})$, there exists a continuous and strictly decreasing function $\mathbf{x}_k : [\underline{u}_k, \overline{u}_k] \to \Theta$ with $\overline{u}_k = \mathbf{x}_k(\overline{u}_k) < \mathbf{x}_k(\underline{u}_k) < 1$ such that for all $u \in [\underline{u}_k, \overline{u}_k]$, $u = u_k(G^{\mathbf{x}_k(u)})$.

Given a payoff profile $y \in \mathcal{F}(n,k)$, define the offer $o^*(y) \coloneqq (P(y), G^{\mathbf{x}_k(||y||}) \in \mathbb{O}$ where

$$P(y) \coloneqq c_{G^{\mathbf{x}_{k}(||y||)}}(y_{n+1}) - c_{G^{\mathbf{x}_{k}(||y||)}}(||y||).$$
(6)

Let $O^*(y) \in \mathbb{O}^n$ be the profile of TIOLI offers in which each Principal $i \in N$ offers $o^*(y)$, and let $O^*_{-i}(y)$ be the profile in which each Principal $j \in N \setminus \{i\}$ offers $o^*(y)$.

Lemma 4 For each $n \ge 1$, $k \in (0, m^{\emptyset})$, and $y \in \mathcal{F}(n,k)$, there exists a tuple (α, β, ψ) with

- (i) $\alpha_i(o^*(y)) = 1$ for all $i \in N$,
- (ii) β is sequentially rational with respect to ψ , and
- (iii) $\psi(O^*(y), w, m) = y$ for all $(w, m) \in W \times \Theta$

such that the tuple (α, β, ψ) generates y.

Lemma 4 has two important implications. First, each payoff profile $y \in E$ for any set $E \subseteq \mathscr{F}(n,k)$ is associated with a tuple (α,β,ψ) that satisfies Points (a) and (c) of Definition 1. Hence, to show that $E \subseteq \mathscr{C}(n,k)$, we need only show that the tuple (α,β,ψ) also satisfies Points (b) and (d) of Definition 1. We will use this approach to characterize (subsets of) $\mathscr{C}(n,k)$. Second, Lemma 4 implies that even if the agent could purchase signals from multiple principals in a given period, he would find it sub-optimal to do so because the principals sell the same deterministic (pass-fail) signal.

Let us now give some intuition for Lemma 3 and Lemma 4 via a geometric argument. In order to generate a payoff profile $y \in \mathcal{F}(n,k)$, we must first show that it is possible to generate a total surplus of ||y|| and give the agent only a payoff of y_{n+1} while the remaining surplus, $||y|| - y_{n+1}$, is captured by the principals (collectively). A priori, it is not clear if this is possible, since an agent who earns a continuation payoff of y_{n+1} would have an incentive to stop his search earlier than an agent who earns a higher continuation payoff of ||y||.

However, such surplus creation and division is indeed possible. In Figure 3, the curves c_F , $c_{G^{\varnothing}}$, and $c_{G^{\varkappa}_k(||y||)}$ are depicted in blue, red, and black respectively. It is clear from the



Figure 3: This figure depicts c_F in blue, $c_{G^{\varnothing}}$ in red, and $c_{G^{\mathbf{x}_k(||y||)}}$ in black. The price P(y) is given by the difference between $c_{G^{\mathbf{x}_k(||y||)}}(y_{n+1})$ and $c_{G^{\mathbf{x}_k(||y||)}}(||y||) = k$. The figure also depicts the support of $G^{\mathbf{x}_k(||y||)}$, $m_1 \coloneqq \mathbb{E}_F[\theta|\theta \leq \mathbf{x}_k(||y||)]$ and $m_2 \coloneqq \mathbb{E}_F[\theta|\theta \geq \mathbf{x}_k(||y||)]$.

figure that $c_{G^{\mathbf{x}_k(||y||)}}(||y||) = k$, which implies that $||y|| = u(G^{\mathbf{x}_k(||y||)})$. In other words, if the agent observes $G^{\mathbf{x}_k(||y||)}$ for free in each period, then his continuation payoff would be ||y||, and he would optimally stop his search whenever the posterior mean quality exceeds ||y||. Notice that the pass-fail signal has binary support: a posterior mean of $m_1 := \mathbb{E}_F[\theta|\theta \le \mathbf{x}_k(||y||)] < ||y||$ ("fail") and a posterior mean of $m_2 := \mathbb{E}_F[\theta|\theta \ge \mathbf{x}_k(||y||)] > ||y||$ ("pass"). Thus, an agent who observes $G^{\mathbf{x}_k(||y||)}$ for free in each period would stop only when sees a "pass" signal realization, which happens with probability $1 - F(\mathbf{x}_k(||y||))$.

Next, suppose the agent's continuation value is lowered to $y_{n+1} \leq ||y||$, in which case he would optimally stop his search whenever the posterior mean quality exceeds y_{n+1} . However, it is still the case that the agent stops his search only when sees a "pass," because $m_1 < y_{n+1} < m_2$. Thus, despite having a higher incentive to quit his search, the agent's probability of stopping remains $1 - F(\mathbf{x}_k(||y||))$.

Finally, the price is specified so that an agent who pays P(y) and observes $G^{\mathbf{x}_k(||y||)}$ in each period earns a payoff of y_{n+1} . Since a surplus of ||y|| is generated, the remaining $||y|| - y_{n+1}$ is captured by the principals. When they all offer $o^*(y)$, the agent is indifferent across all *n* TIOLI offers. Hence, he can randomize his purchase decision in such a way that each principal's payoff is y_j , thereby generating the payoff profile y as desired.

5. Proof of main results

In this section, we offer a characterization of the equilibrium payoff set as a function of n and k (Propositions 1-4), and show that our main results (Theorems 1-3) follow as a

consequence of this characterization.

Our first proposition fully characterizes the equilibrium payoff set for any search cost in a monopolistic setting. In particular, $\mathscr{C}(1,k)$ has a "bang-bang" structure: either there is a unique equilibrium payoff or a folk-theorem result obtains. Furthermore, when a unique equilibrium payoff obtains, the equilibrium is efficient with the monopolist extracting the full surplus.

Proposition 1 There exists a unique $k^* \in (0, m^{\emptyset})$ such that

$$\mathfrak{E}(1,k) = \begin{cases} \{(\bar{u}_k - \underline{u}_k, \underline{u}_k)\} & \text{if } k > k^* \\ \\ \mathfrak{F}(1,k) & \text{if } k \le k^* \end{cases}$$

Proof. Let n=1. Given a non-empty set $E \subseteq \mathbb{R}^2$, define

$$\underline{\mathscr{V}}(E) \coloneqq \inf_{\beta,\psi} \sup_{o \in \mathfrak{G}} V_1(o,\beta,\psi) \tag{7}$$

s.t. β is sequentially rational with respect to ψ , and

$$Im_{\psi} \subseteq E.$$

The monopolist's minimax payoff $\underline{\mathcal{V}}(E)$ diverges from the standard repeated games minimax payoff (Fudenberg and Maskin, 1986) in two key ways. First, unlike repeated simultaneous-move games, here the agent moves after the principals. Thus, the agent's action in the minimax formulation must be sequentially rational, as captured by the first constraint in (7). Second, because the contracting problem we study is a stopping game, it lacks a well-defined stage game. As a result, $\underline{\mathcal{V}}(E)$ is not a minimax of the stage-game payoffs but instead depends on the continuation payoffs, as captured by the second constraint in (7) and the set of possible continuation values E.

Despite these differences, we may still interpret $\underline{\mathcal{V}}(E)$ as the payoff the monopolist can guarantee herself when she faces an adversarial (but sequentially rational) agent and unfavorable continuation payoffs that lie in E. The following lemma shows that there is a close connection between $\underline{\mathcal{V}}(E)$ and payoff profiles that can be supported by E.

Lemma 5 Let $E \subseteq \mathbb{R}^2$ be non-empty. Any payoff profile $y \in \mathcal{F}(1,k)$ such that $y_1 > \underline{\mathcal{V}}(E)$ is supported by the set $E \cup \{y\}$.

Of particular note is the minimax value $\underline{\mathcal{V}}(\mathscr{C}(1,k))$, which we henceforth denote as ν_k for ease of exposition.¹¹ The value ν_k is the profit guarantee that a monopolist can secure for herself in any equilibrium. Hence, if a payoff profile $y \in \mathscr{C}(1,k)$, then $y_1 \geq \nu_k$. We show that this is also a sufficient condition.

Lemma 6 For any $k \in (0, m^{\emptyset})$, $y \in cl(\mathscr{E}(1,k))$ if and only if $y_1 \ge \nu_k$.

Lemma 7 For any $k \in (0, m^{\emptyset}), \nu_k = \underline{\mathcal{V}}(\operatorname{cl}(\mathscr{E}(1, k))).$

The above two lemmas further simplify our analysis. First, the closure of the equilibrium payoff set is characterized by the intersection of three half spaces: two half spaces that determine feasibility of payoffs and a third one given by Lemma 6. In other words,

$$\operatorname{cl}(\mathfrak{E}(1,k)) = \operatorname{conv}(\{(\bar{u}_k - \underline{u}_k, \underline{u}_k), (\nu_k, \underline{u}_k), (\nu_k, \bar{u}_k - \nu_k)\}),$$

which is a convex triangle as depicted by the shaded red set in Figure 4.



Figure 4: This figure depicts the closure of the equilibrium payoff set as characterized by Lemma 6. The shaded simplex (i.e. the union of the red and blue areas) is the feasible set $\mathcal{F}(1,k)$ for some $k \in (0,m^{\varnothing})$, and the red area is $cl(\mathcal{E}(1,k))$, given the appropriate value ν_k .

Additionally, from Lemma 7,

$$\nu_k = \inf \left\{ y_1 \in [0, \bar{u}_k - \underline{u}_k] : y_1 = \underline{\mathcal{V}} \left(\operatorname{conv} \left(\left\{ (\bar{u}_k - \underline{u}_k, \underline{u}_k), (y_1, \underline{u}_k), (y_1, \bar{u}_k - y_1) \right\} \right) \right) \right\},$$

¹¹The value $\underline{\mathcal{V}}(\mathfrak{E}(1,k))$ is well-defined as $\mathfrak{E}(1,k)$ is non-empty; Mekonnen et al. (2024) (Online Appendix B) show the existence of a stationary equilibrium in this setting.

which reduces the task of characterizing the equilibrium payoff set into finding the smallest fixed point in a single-dimensional fixed-point problem. This approach is different from the standard APS construction, which defines a fixed-point operator over subsets of $\mathcal{F}(1,k)$.

Lemma 8 There exists a unique $k^* \in (0, m^{\emptyset})$ such that

$$\nu_k \!=\! \left\{ \begin{array}{rrr} \bar{u}_k \!-\! \underline{u}_k & \!\! \text{if} \!\! k \!>\! k^* \\ 0 & \!\! \text{if} \!\! k \!\leq\! k^* \end{array} \right.$$

In other words, Lemma 8 states that $\operatorname{cl}(\mathfrak{C}(1,k)) = \mathfrak{F}(1,k)$ whenever $k \leq k^*$ and $\operatorname{cl}(\mathfrak{C}(1,k)) = \{(\bar{u}_k - \underline{u}_k, \underline{u}_k)\}$ whenever $k > k^*$. In fact, the proof for Lemma 8 shows that the inf-sup in (7) is always attained, i.e., for any $k \in (0, m^{\varnothing})$, there exists a pair (β, ψ) and an offer $o \in \mathfrak{O}$ such that $V_1(o,\beta,\psi) = \underline{\mathcal{V}}(\mathfrak{C}(1,k))$. Hence, the equilibrium payoff set is closed, and we get a full characterization of the equilibrium payoff set in a monopolistic setting.

We present a graphical depiction of the equilibrium set for different values of k in Figure 5. Intuitively, the agent has little bargaining power in a monopolistic setting: the singular principal is the the only source of information, and she moves first in each period by making a TIOLI offer. The agent can threaten to reject any offer that does not guarantee him a large enough share of the surplus, but such threats are not credible unless they are sequentially rational. In particular, such a threat is credible only if the agent is willing to forgo an informative signal about the currently sampled good in exchange for a high continuation payoff.

When k is large (i.e., $k > k^*$), even a continuation payoff of \bar{u}_k , which is the highest continuation payoff the agent can earn, is still too low to compensate the agent for forgoing an informative signal about the currently sampled good. In this case, the agent's threat is not credible, leaving him in a weak bargaining position relative to the principal. Consequently, there is a unique equilibrium payoff that features full surplus extraction by the principal.

Conversely, when k is low (i.e., $k \leq k^*$), \bar{u}_k is large enough to compensate the agent for forgoing an informative signal, making his threats credible. In this case, any surplus split between the agent and the principal can be sustained in equilibrium, which yields a folk-theorem result. The value k^* is the unique value that balances the loss from forgoing information today with the gain from a high continuation payoff.

How does the above intuition change in a competitive setting? When n > 1, the agent can now acquire information from multiple sources, which puts him in a stronger bargaining



Figure 5: Panel (a) presents the feasible payoff set $\mathcal{F}(1,k')$ for $k' > k^*$, shaded in blue, and the equilibrium payoff set $\mathcal{E}(1,k')$ which is given by the unique principal optimal outcome $(\bar{u}_{k'} - \underline{u}_{k'}, \underline{u}_{k'})$, shaded in red. Panel (b) presents the equilibrium payoff set $\mathcal{E}(1,k'')$ for $k'' \leq k^*$, shaded in red, which is equal to the entire feasible set $\mathcal{F}(1,k'')$. Note that because $k'' < k', \mathcal{F}(1,k'') \neq \mathcal{F}(1,k')$ (see Lemma 1 for details).

position. Specifically, rejecting the TIOLI offer from one principal does not entail altogether forgoing the opportunity to learn about the currently sampled good. Hence, the threats the agent uses in a monopolistic setting become even more effective in securing him a higher payoff in a competitive setting. This improved efficacy of the agent's threats has two implications: (i) There exists a Bertrand equilibrium outcome in which each principal offers full information to the agent for free in each period, and (ii) when a folk-theorem result obtains in a monopolistic setting, it also obtains in a competitive setting.

The next two propositions prove these two claims. With some abuse of notation, for each $k \in (0, m^{\emptyset})$ and $n \ge 1$, let $y^A \in \mathcal{F}(n, k)$ denote the *autarky payoff profile* where $y_i^A = 0$ for all $i \in N$ and $y_{n+1}^A = \underline{u}_k$, and let $y^B \in \mathcal{F}(n, k)$ denote the *Bertrand payoff profile* where $y_i^B = 0$ for all $i \in N$ and $y_{n+1}^B = \overline{u}_k$.

Proposition 2 For each $k \in (0, m^{\emptyset})$ and n > 1, the Bertrand payoff profile $y^B \in \mathscr{E}(n,k)$.

Proof. Fix any $k \in (0, m^{\emptyset})$ and n > 1. We prove the proposition by showing that $\{y^B\}$ is a self-generating. To that end, consider the tuple (α, β, ψ) such that

- (i) $\alpha_i((0,F)) = 1$ for all $i \in N$,
- (*ii*) β is sequentially rational with respect to ψ , and

(*iii*) $\psi(O,w,m) = y^B$ for all $(O,w,m) \in \mathbb{G}^n \times W \times \Theta$.

Notice that the specified tuple (α, β, ψ) implies the agent searches in each period by observing full information for free. As such, the tuple (α, β, ψ) generates the payoff profile y^B . Hence, (α, β, ψ) satisfies Points (a) and (c) of Definition 1. Furthermore, by construction, $Im_{\psi} = \{y^B\}$, which implies that Point (d) of Definition 1 is also satisfied. Hence, to show that $\{y^B\}$ is self-generating, it suffices to show that no principal has a profitable deviation. This is trivially true: Principal $i \in N$ gets a payoff of 0 by offering the on-path contract (0,F). If she instead deviates by offering $o_i := (p_i,G_i) \neq (0,F)$ while the other n-1 principals offer (0,F), then it is optimal for the agent to reject Principal i's contract, which again yields her a payoff of 0. Consequently, $\{y^B\}$ is self-generating, which by Lemma 2, implies $y^B \in \mathscr{C}(n,k)$ as desired.

Proposition 3 Let k^* be the same threshold as in Proposition 1. For all $k \le k^*$ and all n > 1, $\mathfrak{C}(n,k) = \mathfrak{F}(n,k)$.

Proof. Fix any $k \leq k^*$ and n > 1. We prove Proposition 3 by showing that $\mathcal{F}(n,k)$ is a self-generating set. To that end, fix a payoff profile $y \in \mathcal{F}(n,k)$, and consider the tuple (α, β, ψ) such that

- (i) $\alpha_i(o^*(y)) = 1$ for all $i \in N$,
- (*ii*) β is sequentially rational with respect to ψ , and
- (*iii*) ψ is given by

$$\psi(O, w, m) = \begin{cases} y & \text{if } o_i = o^*(y) \text{ for all } i \in N \\ y^A & \text{if } o_i \neq o^*(y) \text{ for some } i \in N \text{ and } w = i \\ y^B & \text{if } o_i \neq o^*(y) \text{ for some } i \in N \text{ and } w \neq i \end{cases}$$
(8)

Notice that the specified tuple (α, β, ψ) satisfies the conditions of Lemma 4, and therefore generates the payoff profile y. Hence, Points (a) and (c) of Definition 1 are satisfied. Furthermore, by construction, $Im_{\psi} = \{y, y^A, y^B\} \subseteq \mathcal{F}(n,k)$, which implies that Point (d) of Definition 1 is also satisfied. Hence, to show that $\mathcal{F}(n,k)$ is self-generating, it suffices to show that no principal has a profitable deviation. Suppose Principal *i* deviates by offering $o_i := (p_i, G_i) \neq o^*(y)$ while the other n-1 principals offer $o^*(y)$. Given the continuation payoff function ψ specified in (8), the agent's expected value from accepting o_i is given by $\underline{u}_k + c_{G_i}(\underline{u}_k) - k - p_i$. Conversely, if the agent optimally accepts either the null offer or $o^*(y)$ from one of the non-deviating principals, his payoff is $\max\{\overline{u}_k + c_{G^*k}(||y||)(\overline{u}_k) - P(y) - k, \overline{u}_k + c_{G^{\varnothing}}(\overline{u}_k) - k\}$. Therefore, it is sequentially rational for the agent to accept o_i only if

$$\underline{u}_k + c_{G_i}(\underline{u}_k) - p_i \ge \max\{ \bar{u}_k + c_{G^{\mathbf{x}_k(||y||)}}(\bar{u}_k) - P(y), \bar{u}_k + c_{G^{\varnothing}}(\bar{u}_k) \}$$

Indeed, the agent's willingness-to-pay for signal $G_i \in \mathfrak{G}(F)$ in this setting is given by

$$\mathcal{P}_k(G_i;y) \coloneqq \left(\underline{u}_k - \bar{u}_k + c_{G_i}(\underline{u}_k) - \max\{c_{G^{\mathbf{x}_k(||y||)}}(\bar{u}_k) - P(y), c_{G^{\varnothing}}(\bar{u}_k)\}\right)^+,\tag{9}$$

and it is sequentially rational for the agent to accept $o_i = (p_i, G_i)$ instead of $o^*(y)$ or the null offer only if $p_i \leq \mathscr{P}_k(G_i; y)$. Furthermore, $\mathscr{P}_k(G_i; y)$ serves as an upper bound on Principal *i*'s deviation payoff from offering o_i because $\mathbb{E}_{\alpha_{-i}} \left[V_i \left(o_i, O_{-i}, \beta, \psi \right) \right] = p_i \beta^{\omega}(i | o_i, O^*(y)_{-i}) \leq \mathscr{P}_k(G_i; y)$. The following lemma allows us to further bound $\mathscr{P}_k(G; y)$ for all $G \in \mathscr{G}(F)$, thereby providing a uniform upper bound on any deviation payoff that a principal can attain.

Lemma 9 For any $k \leq k^*$, $\mathcal{P}_k(G;y) = 0$ for all $y \in \mathcal{F}(n,k)$ and all $G \in \mathcal{G}(F)$.

When $k \leq k^*$, Lemma 9 implies that $\mathbb{E}_{\alpha_{-i}} [V_i(o_i, O_{-i}, \beta, \psi)] = 0$. Thus, no principal has a profitable deviation, which implies that Point (b) of Definition 1 is also satisfied. Consequently, $\mathcal{F}(n,k)$ is self-generating, which by Lemma 2, implies $\mathcal{F}(n,k) = \mathcal{E}(n,k)$ as desired.

We have thus far shown the agent's increased bargaining power when $k \leq k^*$ gives rise to a folk-theorem result, regardless of the underlying market structure. The following result considers the effects of the agent's bargaining power when the search cost satisfies $k > k^*$ and n > 1. Even when a monopolistic setting sustains a unique equilibrium payoff, a competitive setting may sustain uncountably many equilibrium outcomes.

Proposition 4 Let k^* be the same threshold as in Proposition 1. There exists a threshold $k^{**} \in (k^*, m^{\emptyset})$ such that for each $k \leq k^{**}$ and each n > 1, $\mathscr{E}(n,k)$ has a non-empty interior.

Proof. Fix n > 1. When $k \le k^*$, Proposition 3 implies that $\mathcal{F}(n,k) = \mathcal{E}(n,k)$. Since $\mathcal{F}(n,k)$ has a non-empty interior, Proposition 4 is immediately obtained for $k \le k^*$. However, our proof applies more generally without distinguishing between $k \le k^*$ and $k > k^*$.

For some $\epsilon > 0$, define the set $\mathcal{F}^{\epsilon}(n,k) \coloneqq \{y \in \mathcal{F}(n,k) : ||y|| - y_{n+1} \leq \epsilon\}$. The subset $\mathcal{F}^{\epsilon}(n,k)$ has a non-empty interior for all values of $\epsilon > 0$. Furthermore, both the autarky payoff profile y^A and the Bertrand payoff profile y^B satisfy $||y^A|| - y^A_{n+1} = ||y^B|| - y^B_{n+1} = 0$, and therefore, $y^A, y^B \in \mathcal{F}^{\epsilon}(n,k)$ for all $\epsilon > 0$.

Our proof proceeds to show there exists a threshold $k^{**} > k^*$ such that for each $k \leq k^{**}$, there exists an $\epsilon > 0$ for which $\mathcal{F}^{\epsilon}(n,k) \subseteq \mathcal{E}(n,k)$. We first make the following observation.

Lemma 10 There exists a $k^{**} \in (k^*, m^{\emptyset})$ such that for all $k \leq k^{**}$, there exists an $\epsilon > 0$ with $\mathcal{P}_k(G;y) = 0$ for all $G \in \mathcal{G}(F)$ and for all $y \in \mathcal{F}^{\epsilon}(n,k)$.

Fix some $k \leq k^{**}$ and some $\epsilon > 0$ so that the conclusion of Lemma 10 holds. We now show that $\mathcal{F}^{\epsilon}(n,k)$ is self-generating, which establishes the desired result that $\mathcal{F}^{\epsilon}(n,k) \subseteq \mathcal{E}(n,k)$. To that end, fix a payoff profile $y \in \mathcal{F}^{\epsilon}(n,k)$, and consider the tuple (α,β,ψ) such that

- (i) $\alpha_i(o^*(y)) = 1$ for all $i \in N$,
- (*ii*) β is sequentially rational with respect to ψ , and
- (*iii*) ψ is given by (8).

Notice that the specified tuple (α, β, ψ) satisfies the conditions of Lemma 4, and therefore generates the payoff profile y. Hence, Points (a) and (c) of Definition 1 are satisfied. Furthermore, by construction, $Im_{\psi} = \{y, y^A, y^B\} \subseteq \mathscr{F}^{\epsilon}(n,k)$, which implies that Point (d) of Definition 1 is also satisfied. Finally, if Principal i deviates by offering $o_i = (p_i, G_i) \neq o^*(y)$ while the other n-1 principals offer $o^*(y)$, then her deviation payoff is

$$\mathbb{E}_{\alpha_{-i}}\left[V_i(o_i, O_{-i}, \beta, \psi)\right] = p_i \beta^{\omega}(i|o_i, O^*(y)_{-i}) \leq \mathcal{P}_k(G_i; y) = 0.$$

Thus, no principal has a profitable deviation, which implies that Point (b) of Definition 1 is also satisfied. Consequently, $\mathcal{F}^{\epsilon}(n,k)$ is self-generating.

Figure 6 presents graphical depictions of the characterizations offered in Proposition 3 and Proposition 4 for n = 2. For some $k'' \leq k^*$, Panel (a) displays $\mathscr{C}(2,k'') = \mathscr{F}(2,k'')$ as the shaded red area. For some $k' \in (k^*, k^{**}]$, Panel (b) displays a subset $E \subseteq \mathscr{C}(2,k')$ with a non-empty interior as the shaded red area. The proof of Proposition 4 offers a characterization for the shape of the red set in Panel (b).

We now use the four propositions to establish our main results (Theorems 1-3). From Proposition 1 and Proposition 3, we can conclude that when $k \leq k^*$, the total surplus set and



(a) $\mathscr{E}(2,k'')$ for $k'' \leq k^*$ (b) Subset of $\mathscr{E}(2,k')$ for $k' \in (k^*,k^{**}]$

Figure 6: Panel (a) presents the equilibrium payoff set $\mathscr{C}(2,k'')$ for $k'' \leq k^*$, shaded in red, which is equal to the entire feasible payoff set $\mathscr{F}(2,k'')$. Panel (b) presents the feasible payoff set $\mathscr{F}(2,k')$ for $k' \in (k^*,k^{**}]$, shaded in blue, and a subset of the the equilibrium payoff set $E \subseteq \mathscr{C}(2,k')$ with a non-empty interior, shaded in red. Note that because k'' < k', $\mathscr{F}(2,k'') \neq \mathscr{F}(2,k')$ (see Lemma 1 for details).

the agent's surplus set are given by $\mathscr{W}(\mathscr{E}(n,k)) = \mathscr{W}(\mathscr{F}(n,k)) = \mathscr{A}(\mathscr{E}(n,k)) = \mathscr{A}(\mathscr{F}(n,k)) = [\underline{u}_k, \overline{u}_k]$ for all $n \ge 1$, which establishes Point (i) in both Theorem 1 and Theorem 2.

In contrast, when $k > k^*$, Proposition 1 implies that $\mathscr{W}(\mathscr{E}(1,k)) = \{\bar{u}_k\}$, while Proposition 2 implies that $\sup \mathscr{W}(\mathscr{E}(n,k)) = \bar{u}_k$ for any n > 1, which establishes Point (*ii*) of Theorem 1. Similarly, when $k > k^*$, Proposition 1 implies that $\mathscr{A}(\mathscr{E}(1,k)) = \{\underline{u}_k\}$, whereas for any $n \ge 1$, inf $\mathscr{A}(\mathscr{E}(n,k)) \ge \inf \mathscr{A}(\mathscr{F}(n,k)) = \underline{u}_k$, which establishes Point (*ii*) of Theorem 2.

Finally, Proposition 4 implies that for all n > 1 and all $k \in (k^*, k^{**}]$, there is a nonempty subset $E \subset \mathcal{F}(n,k)$ such that $E \subseteq \mathcal{E}(n,k)$ and is open, i.e., for all $y \in E$, (a) $||y|| < \bar{u}_k = \inf \mathcal{W}(\mathcal{E}(1,k))$, (b) $y_i > 0$ for all $i \in N$, and (c) $y_{n+1} > \underline{u}_k = \sup \mathcal{A}(\mathcal{E}(1,k))$, which immediately establishes Theorem 3.

6. Conclusion

This paper studies the role competition in persuaded search markets where principal(s) dynamically contract with a searching agent who seeks information on the quality of sampled goods. We find that while competition benefits the agent, it does so, perhaps

counterintuitively, at the expense of efficiency.

These results are driven by the interaction between the value of information and bargaining power in dynamic markets. When the agent is facing a monopolist, he has bargaining power if and only if he can credibly reject a lackluster offer from the principal, which in turn holds if and only if he can have a negative value of information. In contrast, when facing multiple principals, the agent has bargaining power not just when he can credibly reject an offer by any one principal, but also when his value of that principal's signal is lower than his value for signals offered by other principals. This weaker condition allows the agent to secure a larger payoff in competitive settings than he could in a monopolistic one. However, while equilibrium outcomes under monopoly are not in general efficient, we show that the agent's stronger bargaining power in competitive settings can only decrease, and never increase, the total surplus generated across the set of equilibria.

In an increasingly data-driven world, understanding the impact of competition in markets governing the sale of information is itself increasingly important. Many questions remain, both from theoretical and applied lenses: What policies better align the incentives of searching agents and information providers? How can a regulator best redistribute surplus if efficiency and desirable distribution of surplus are at odds? Our paper takes a preliminary step in understanding these market forces.

A. Appendix

A.1. Additional useful lemmas

Lemma 11 For all $k \in (0, m^{\emptyset})$ and $u \in [\underline{u}_k, \overline{u}_k]$,

$$\mathbb{E}_{F}\left[\theta|\theta \leq \mathbf{x}_{k}(u)\right] \leq \underline{u}_{k} < \overline{u}_{k} < \mathbb{E}_{F}\left[\theta|\theta \geq \mathbf{x}_{k}(u)\right].$$

Proof. Fix any $k \in (0, m^{\emptyset})$ and $u \in [\underline{u}_k, \overline{u}_k]$. First, notice that

$$\bar{u}_k \leq \mathbf{x}_k(u) < \mathbb{E}_F [\theta | \theta \geq \mathbf{x}_k(u)],$$

where the first inequality follows from because $\mathbf{x}_k(\cdot)$ is bounded from below by \bar{u}_k as established in Lemma 3, and the second inequality follows because F is absolutely continuous and $\mathbf{x}_k(\cdot)$ is bounded from above by $\mathbf{x}_k(\underline{u}_k) < 1$, also as established in Lemma 3.

Next, we argue that $\mathbb{E}_F[\theta|\theta \leq \mathbf{x}_k(\underline{u}_k)] = \underline{u}_k$, which is sufficient for the desired result because $\mathbb{E}_F[\theta|\theta \leq \mathbf{x}_k(u)] \leq \mathbb{E}_F[\theta|\theta \leq \mathbf{x}_k(\underline{u}_k)]$ (because $\mathbf{x}_k(\cdot)$ is decreasing). To see why $\mathbb{E}_F[\theta|\theta \leq \mathbf{x}_k(\underline{u}_k)] = \underline{u}_k$, notice that

$$c_{G^{\varnothing}}(\underline{u}_{k}) = c_{G^{\mathbf{x}_{k}(\underline{u}_{k})}}(\underline{u}_{k})$$

$$\iff m^{\varnothing} - \underline{u}_{k} = c_{F}(\mathbf{x}_{k}(\underline{u}_{k})) + (1 - F(\mathbf{x}_{k}(\underline{u}_{k})))(\mathbf{x}_{k}(\underline{u}_{k}) - \underline{u}_{k})$$

$$\iff m^{\varnothing} - \underline{u}_{k} = \int_{\mathbf{x}_{k}(\underline{u}_{k})}^{1} (\theta - \mathbf{x}_{k}(\underline{u}_{k})) dF(\theta) + (1 - F(\mathbf{x}_{k}(\underline{u}_{k})))(\mathbf{x}_{k}(\underline{u}_{k}) - \underline{u}_{k})$$

$$\iff \mathbb{E}_{F}[\theta|\theta \leq \mathbf{x}_{k}(\underline{u}_{k})] = \underline{u}_{k},$$

where the first line follows because both terms equal k by construction, the first equivalence follows because $c_{G^{\varnothing}}(u) = (m^{\varnothing} - u)^+$ and from the definition of c_{G^x} in (5), the second equivalence follows from expressing $c_F(\cdot)$ in its integral form, and the last equivalence follows from algebra.

Lemma 12 For all $k \in (0, m^{\emptyset})$ and $u \in [\underline{u}_k, \overline{u}_k]$,

$$c_{G^{\mathbf{x}_{k}(u)}}(u') = c_{F}(\mathbf{x}_{k}(u)) + (1 - F(\mathbf{x}_{k}(u)))(\mathbf{x}_{k}(u) - u')$$

for all $u' \in [\underline{u}_k, \overline{u}_k]$.

Proof. Fix any $k \in (0, m^{\emptyset})$, and $u, u' \in [\underline{u}_k, \overline{u}_k]$. As defined in (5),

$$c_{G^{\mathbf{x}_{k}(u)}}(u') \!=\! \max\{c_{G^{\varnothing}}(u'),\!c_{F}(\mathbf{x}_{k}(u)) \!+\! (1\!-\!F(\mathbf{x}_{k}(u)))(\mathbf{x}_{k}(u)\!-\!u')\} \!>\! 0$$

where the inequality follows because F is absolutely continuous over Θ and because $u' \leq \bar{u}_k \leq \mathbf{x}_k(u) < 1$ as stated in Lemma 3.

Recall that $c_{G^{\varnothing}}(u') = (m^{\varnothing} - u')^+$ by definition. Suppose for the sake of contradiction that $c_{G^{\varnothing}}(u') > c_F(\mathbf{x}_k(u)) + (1 - F(\mathbf{x}_k(u)))(\mathbf{x}_k(u) - u')$, which implies that $c_{G^{\varnothing}}(u') = m^{\varnothing} - u'$. We then have

$$c_{G^{\varnothing}}(u') > c_{F}(\mathbf{x}_{k}(u)) + (1 - F(\mathbf{x}_{k}(u)))(\mathbf{x}_{k}(u) - u')$$
$$\iff m^{\varnothing} - u' > \int_{\mathbf{x}_{k}(u)}^{1} (\theta - \mathbf{x}_{k}(u)) dF(\theta) + (1 - F(\mathbf{x}_{k}(u)))(\mathbf{x}_{k}(u) - u')$$
$$\iff \mathbb{E}_{F}[\theta|\theta \leq \mathbf{x}_{k}(u)] > u',$$

where the first equivalence follows because $c_{G^{\varnothing}}(u') = (m^{\varnothing} - u')^+$ by definition and $c_{G^{\varnothing}}(u') > 0$ by assumption, and by expressing $c_F(\cdot)$ in its integral form, and the last equivalence follows from algebra. However, the last line contradicts Lemma 11. Hence, we must have $c_{G^{\varnothing}}(u') \leq c_F(\mathbf{x}_k(u)) + (1 - F(\mathbf{x}_k(u)))(\mathbf{x}_k(u) - u')$ to avoid the contradiction, establishing the desired result.

Lemma 13 For all $k \in (0, m^{\emptyset})$ and $u' \in [\underline{u}_k, \overline{u}_k]$, the mapping $u \mapsto c_{G^{\mathbf{x}_k(u)}}(u')$ is strictly increasing over the interval $[\underline{u}_k, \overline{u}_k]$.

Proof. Fix any $k \in (0, m^{\emptyset})$ and $u' \in [\underline{u}_k, \overline{u}_k]$. From Lemma 12, we know that for all $u \in [\underline{u}_k, \overline{u}_k]$,

$$c_{G^{\mathbf{x}_{k}(u)}}(u') = c_{F}(\mathbf{x}_{k}(u)) + (1 - F(\mathbf{x}_{k}(u)))(\mathbf{x}_{k}(u) - u'),$$

which is a continuous function of u. Furthermore, $c_F(\cdot)$ is differentiable, while $F(\cdot)$ and $\mathbf{x}_k(\cdot)$ are differentiable almost everywhere—the absolute continuity of $F(\cdot)$ implies that $F(\cdot)$ is differentiable almost everywhere on (0,1) and it also implies that $c_F(\cdot)$ is differentiable everywhere on (0,1), and the a.e. differentiability of $\mathbf{x}_k(\cdot)$ follows via implicit differentiation. The lemma is then established by taking the derivative with respect to u over the open

interval $(\underline{u}_k, \overline{u}_k)$, which yields a.e.

$$\frac{d}{du}c_{G^{\mathbf{x}_k(u)}}(u') = -\frac{\partial \mathbf{x}_k(u)}{\partial u}f(\mathbf{x}_k(u))(\mathbf{x}_k(u) - u') > 0,$$

where the inequality follows from the fact that $\mathbf{x}_k(\cdot)$ is strictly decreasing and that $\mathbf{x}_k(u) > \mathbf{x}_k(\bar{u}_k) = \bar{u}_k \ge u'$ for all $u \in (\underline{u}_k, \bar{u}_k)$ as established in Lemma 3.

A.2. Proofs omitted from the main text

Proof of Lemma 1. To prove the first statement, fix some $G \in \mathscr{G}(F)$ and consider $k',k'' \in (0,m^{\varnothing})$ such that k' < k''. Since $c_G(u_{k'}(G)) = k' < k'' = c_G(u_{k''}(G))$ by construction, and since c_G is a decreasing function, we have that $u_{k'}(G) > u_{k''}(G)$. Thus, for each $G \in \mathscr{G}(F)$, the mapping $k \to u_k(G)$ is continuous and strictly decreasing.

To prove the second statement, note that $\underline{u}_k = m^{\varnothing} - k$, and hence $\partial \underline{u}_k / \partial k = -1$. Additionally, $c_F(\overline{u}_k) = k$ for all $k \in (0, m^{\varnothing})$. Recall that $c_F(\cdot)$ is differentiable since F is assumed to be absolutely continuous. Thus, we can implicitly differentiate to conclude that

$$\frac{\partial \bar{u}_k}{\partial k} \!=\! \frac{-1}{1\!-\!F(\bar{u}_k)} \!<\! -1,$$

where the inequality follows from the fact that $\bar{u}_k > 0$ for all $k \in (0, m^{\emptyset})$; otherwise, we would have $c_F(0) = k$ by (3), and $c_F(0) = m^{\emptyset}$ by construction, which violates the assumption that $k < m^{\emptyset}$.

We can therefore conclude that $\partial \bar{u}_k / \partial k < \partial \underline{u}_k / \partial k$; in other words, $\bar{u}_k - \underline{u}_k$ is decreasing in k.

Proof of Lemma 3. Fix any $k \in (0, m^{\emptyset})$. We show that for each $u \in [\underline{u}_k, \overline{u}_k]$, there exists a unique value $\mathbf{x}_k(u) \in [\overline{u}_k, 1)$ such that $u = u_k(G^{\mathbf{x}_k(u)})$. By (3), this is equivalent to showing that $c_{G^{\mathbf{x}_k(u)}}(u) = k$.

To that end, consider the function $L_k: \Theta \times [\underline{u}_k, \overline{u}_k] \to \mathbb{R}$ given by

$$L_k(x,u) \coloneqq c_F(x) + (1 - F(x))(x - u).$$

Notice that (i) L_k is continuous in both variables, (ii) for each $x \in \Theta$, $L_k(x, \cdot)$ is strictly decreasing in its second argument, and (iii) for each $u \in [\underline{u}_k, \overline{u}_k]$, $L_k(\cdot, u)$ is strictly increasing

in its first argument over the interval (0,u) and strictly decreasing over (u,1).¹²

For each $u \in [\underline{u}_k, \overline{u}_k]$,

$$L_k(\bar{u}_k, u) = c_F(\bar{u}_k) + (1 - F(\bar{u}_k))(\bar{u}_k - u) = k + (1 - F(\bar{u}_k))(\bar{u}_k - u) \ge k,$$

where the second equality follows from the fact that \bar{u}_k solves (3) when $c_G = c_F$. Similarly,

$$L_k(1,u) = c_F(1) + (1 - F(1))(1 - u) = 0 < k.$$

Therefore, there is a unique $\mathbf{x}_k(u) \in [\bar{u}_k, 1)$ such that $L_k(\mathbf{x}_k(u), u) = k$.

Notice that $\mathbf{x}_k(\bar{u}_k) = \bar{u}_k$ because $L(\bar{u}_k, \bar{u}_k) = c_F(\bar{u}_k) = k$. Furthermore, $\mathbf{x}_k(\cdot)$ is strictly decreasing in u because L_k is strictly decreasing in (x, u) over the relevant region $(\bar{u}_k, 1) \times (\underline{u}_k, \bar{u}_k)$. Hence, $\bar{u}_k = \mathbf{x}_k(\bar{u}_k) < \mathbf{x}_k(\underline{u}_k) < 1$.

To conclude the proof, note that for all $u \in [\underline{u}_k, \overline{u}_k]$,

$$k = c_{G^{\varnothing}}(\underline{u}_k) \ge c_{G^{\varnothing}}(u),$$

where the equality follows from the fact that \underline{u}_k solves (3) when $c_G = c_{G^{\varnothing}}$, and the inequality follows from the fact that $c_{G^{\varnothing}}(\cdot)$ is a decreasing function. Hence,

$$c_{G^{\alpha_k(u)}}(u) = \max\{c_{G^{\varnothing}}(u), L_k(\mathbf{x}_k(u), u)\} = \max\{c_{G^{\varnothing}}(u), k\} = k$$

which implies that $u = u_k(G^{\mathbf{x}_k(u)})$, as desired.

Proof of Lemma 4. Fix any $n \ge 1$, $k \in (0, m^{\emptyset})$, and $y \in \mathcal{F}(n,k)$. We first show that we can generate a surplus of ||y|| while giving the agent a payoff of y_{n+1} . We then show that we can distribute the remaining $||y|| - y_{n+1}$ surplus to the principals in such a way that each Principal $i \in N$ earns y_i .

To that end, let (α, β, ψ) be as in the statement of the lemma, i.e., $\alpha_i(o^*(y)) = 1$ for all $i \in N$, $\psi(O^*(y), w, m) = y$ for all $(w, m) \in W \times \Theta$, and $\beta = (\beta^w, \beta^d)$ is sequentially rational

¹²Recalling that $F(\cdot)$ and $c_F(\cdot)$ are absolutely continuous functions, it follows that $L_k(\cdot, u)$ is absolutely continuous in its first argument, and $\frac{\partial L_k(x,u)}{\partial x} = f(x)(u-x)$ almost everywhere. Absolutely continuity of $L_k(\cdot, u)$ coupled with the fact that f(x)(u-x) is strictly positive for x < u and strictly negative for x > u implies the desired claims.

with respect to ψ . The latter necessarily implies that

$$\beta^{d} (O^{*}(y), w, m) = \begin{cases} 1 & \text{if } m < y_{n+1} \\ 0 & \text{if } m > y_{n+1} \end{cases}$$

for all $(w,m) \in W \times \Theta$, and that

$$\operatorname{supp}\beta^{w}(\cdot|O^{*}(y)) \subseteq \operatorname{argmax}_{w \in W} \int_{\Theta} \max\{m, y_{n+1}\} dG_{w}(m) - k - p_{w},$$

where $(p_w, G_w) = (P(y), G^{\mathbf{x}_k(||y||)})$ if $w \in N$ and $(p_w, G_w) = (0, G^{\varnothing})$ if $w = \emptyset$.

Clearly, the agent is indifferent across the *n* identical $o^*(y)$ offers. If he chooses one of the $o^*(y)$ offers (i.e., $w \in N$), his expected payoff is

$$\begin{split} \int_{\Theta} \max\{m, y_{n+1}\} dG^{\mathbf{x}_k(||y||)}(m) - k - P(y) = y_{n+1} + c_{G^{\mathbf{x}_k(||y||)}}(y_{n+1}) - k - P(y) \\ = y_{n+1} + c_{G^{\mathbf{x}_k(||y||)}}(||y||) - k \\ = y_{n+1}, \end{split}$$

where the second equality follows from substituting in the expression for P(y) as defined in (6), and the last equality follows because $||y|| = u_k(G^{\mathbf{x}_k(||y||)})$ by construction (see Lemma 3).

If the agent instead chooses the null offer (i.e., $w = \emptyset$), his expected payoff is

$$\int_{\Theta} \max\{m, u\} dG^{\varnothing}(m) - k = y_{n+1} + c_{G^{\varnothing}}(y_{n+1}) - k$$
$$\leq y_{n+1} + c_{G^{\varnothing}}(\underline{u}_k) - k$$
$$= y_{n+1},$$

where the inequality follows because $c_{G^{\varnothing}}(\cdot)$ is a decreasing function and $y_{n+1} \ge \underline{u}_k$, and the last equality follows because $\underline{u}_k = u_k(G^{\varnothing})$. Therefore, the agent (weakly) prefers one of the *n* identical $o^*(y)$ offers instead of the null offer. Additionally, picking any one of the *n* offers generates the agent a payoff of y_{n+1} .

It remains to show that we can also generate y_i for each principal $i \in N$. Let

 $\beta^d (O^*(y), w, m) = \mathbb{1}_{[m \le y_{n+1}]}$ for all $(w, m) \in W \times \Theta$ so that Principal *i*'s payoff is given by

$$V_i(O^*(y),\beta,\psi) = \beta^w(i|O^*(y))P(y) + G^{\mathbf{x}_k(||y||)}(y_{n+1})y_i.$$
 (A2)

Let us simplify (A2) by making two observations. Our first observation is about the price $P(y) = c_{G^{\mathbf{x}_k(||y||)}}(y_{n+1}) - c_{G^{\mathbf{x}_k(||y||)}}(||y||)$. From Lemma 12, we know that

$$c_{G^{\mathbf{x}_{k}(||y||)}}(||y||) = c_{F}(\mathbf{x}_{k}(||y||)) + (1 - F(\mathbf{x}_{k}(||y||)))(\mathbf{x}_{k}(||y||) - ||y||),$$

and

$$c_{G^{\mathbf{x}_{k}(||y||)}}(y_{n+1}) = c_{F}(\mathbf{x}_{k}(||y||)) + (1 - F(\mathbf{x}_{k}(||y||)))(\mathbf{x}_{k}(||y||) - y_{n+1}).$$

Therefore,

$$P(y) = c_{G^{\mathbf{x}_{k}(||y||)}}(y_{n+1}) - c_{G^{\mathbf{x}_{k}(||y||)}}(||y||)$$

= $(1 - F(\mathbf{x}_{k}(||y||)))(||y|| - y_{n+1})$
= $(1 - F(\mathbf{x}_{k}(||y||)))\sum_{i \in N} y_{i}.$ (A3)

Our second observation is about $G^{\mathbf{x}_k(||y||)}(y_{n+1})$. From Lemma 11, we know that

$$\mathbb{E}_{F}[\theta|\theta \leq \mathbf{x}_{k}(||y||)] \leq y_{n+1} < \mathbb{E}_{F}[\theta|\theta \geq \mathbf{x}_{k}(||y||)],$$

which in turn implies that $G^{\mathbf{x}_k(||y||)}(y_{n+1}) = F(\mathbf{x}_k(||y||))$ by (4).

These two observations $-P(y) = (1 - F(\mathbf{x}_k(||y||))) \sum_{i \in N} y_i$ and $G^{\mathbf{x}_k(||y||)}(y_{n+1}) = F(\mathbf{x}_k(||y||)) - allow us to simplify (A2):$

$$V_{i}(O^{*}(y),\beta,\psi) = \beta^{w}(i|O^{*}(y))P(y) + G^{\mathbf{x}_{k}(||y||)}(y_{n+1})y_{i}$$
$$= \beta^{w}(i|O^{*}(y))\left((1 - F(\mathbf{x}_{k}(||y||)))\sum_{j\in N}y_{j}\right) + F(\mathbf{x}_{k}(||y||))y_{i}.$$
 (A4)

The agent is indifferent across the n identical TIOLI offers made by the principals,

and he weakly prefers these offers to the null offer. Let

$$\beta^{w}(i|O^{*}(y)) = \begin{cases} \frac{y_{i}}{\sum_{j \in N} y_{j}} & \text{if } \sum_{j \in N} y_{j} > 0\\ 1/n & \text{if } \sum_{j \in N} y_{j} = 0 \end{cases}$$

Under this specification, Principal *i*'s payoff in (A4) reduces to $V_i(O^*(y),\beta,\psi) = y_i$. Thus, we have shown that there exists a tuple (α,β,ψ) , as specified in the statement of the lemma, that generates $y \in \mathcal{F}(n,k)$. This completes the proof.

Proof of Lemma 5. Fix a non-empty set $E \subseteq \mathbb{R}^2$ and a payoff profile $y \in \mathcal{F}(1,k)$ with $y_1 > \underline{\mathcal{V}}(E)$. By construction (from (7)), there exists a pair (β', ψ') , such that

- β' is sequentially rational with respect to ψ' ,
- $Im_{\psi'} \subseteq E$, and
- $y_1 \ge V_1(o,\beta',\psi')$ for all $o \in \mathbb{G}$.

Additionally, by Lemma 4, there exist a TIOLI offer $o^*(y)$ and a pair (β'', ψ'') such that

- β'' is sequentially rational with respect to ψ'' ,
- $\psi''(o^*(y), w, m) = y$ for all $(w, m) \in W \times \Theta$, and
- $y_1 = V_1(o^*(y), \beta'', \psi'')$ and $y_2 = U(o^*(y), \beta'', \psi'')$.

Let us define a new pair $(\hat{\beta}, \hat{\psi})$ so that the agent's simple strategy is given by

$$\hat{\beta}^{w}(\cdot|o) = \begin{cases} \beta^{''w}(\cdot|o) & \text{if } o = o^{*}(y) \\ \beta^{'w}(\cdot|o) & \text{if } o \neq o^{*}(y) \end{cases}$$

and for all $(w,m) \in W \times \Theta$,

$$\hat{\beta}^{d}(\cdot|o,w,m) = \begin{cases} \beta^{''d}(\cdot|o,w,m) & \text{if } o = o^{*}(y) \\ \beta^{'d}(\cdot|o,w,m) & \text{if } o \neq o^{*}(y) \end{cases}$$

,

and the continuation value function is given by

$$\hat{\psi}(o,w,m) = \begin{cases} \psi''(o,w,m) & \text{if } o = o^*(y) \\ \psi'(o,w,m) & \text{if } o \neq o^*(y) \end{cases}$$

Then notice that

- (i) $y_1 = V_1(o^*(y), \hat{\beta}, \hat{\psi})$ and $y_2 = U(o^*(y), \hat{\beta}, \hat{\psi}),$
- (*ii*) $y_1 \ge V_1(o, \hat{\beta}, \hat{\psi})$ for all $o \in \mathfrak{G}$,

(*iii*) $\hat{\beta}$ is sequentially rational with respect to $\hat{\psi}$, and

$$(iv) Im_{\hat{\psi}} \subseteq E \cup \{y\},$$

which are Points (a)-(d) of Definition 1. Hence, y is supported by $E \cup \{y\}$.

Proof of Lemma 6. ("Only-if" direction:) Consider a payoff profile $y \in cl(\mathfrak{C}(1,k))$. There exits a convergent sequence of payoff profiles $(y^{\ell})_{\ell \in \mathbb{N}}$ with $y^{\ell} \in \mathfrak{C}(1,k)$ for each $\ell \in \mathbb{N}$ such that $\lim_{\ell \to \infty} y^{\ell} = y$. Since $y^{\ell} \in \mathfrak{C}(1,k)$ for each $\ell \in \mathbb{N}$, there exists a tuple $(\alpha^{\ell}, \beta^{\ell}, \psi^{\ell})$ such that Points (a)-(d) of Definition 1 are satisfied. Then, Point (b) immediately implies that $y_1^{\ell} \ge \nu_k$ for all $\ell \in \mathbb{N}$, giving the desired conclusion.

("If" direction:) Consider a payoff profile $y \in \mathcal{F}(1,k)$) with $y_1 \ge \nu_k$. First, suppose $\nu_k = \bar{u}_k - \underline{u}_k$. In this case, feasibility implies that $y = (\bar{u}_k - \underline{u}_k, \underline{u}_k)$. From the "only-if" direction, we know that $cl(\mathcal{E}(1,k)) \subseteq \{y' \in \mathcal{F}(1,k) : y'_1 \ge \nu_k\} = \{y\}$. As $\mathcal{E}(1,k)$ is non-empty, we conclude that $cl(\mathcal{E}(1,k)) = \{y\}$.

Next, suppose $\nu_k < \bar{u}_k - \underline{u}_k$. In this case, the set $E' := \{y' \in \mathcal{F}(1,k) : y'_1 > \nu_k\}$ is non-empty. From Lemma 5, any $y' \in E'$ is supported by $\mathscr{C}(1,k) \cup \{y'\}$, which implies that $\mathscr{C}(1,k) \cup E'$ is self-generating. However, Lemma 2 states that $\mathscr{C}(1,k)$ is the largest self-generating set, so $E' \subseteq \mathscr{C}(1,k)$, which further implies that $cl(E') \subseteq cl(\mathscr{C}(1,k))$. Noticing that $y \in cl(E')$, we conclude that $y \in cl(\mathscr{C}(1,k))$ as desired.

Proof of Lemma 7. Notice that $\underline{\mathcal{V}}(\cdot)$ is weakly decreasing in the set-inclusion order, i.e., $\underline{\mathcal{V}}(E) \geq \underline{\mathcal{V}}(E')$ whenever $E \subseteq E'$. Hence, $\nu_k \geq \underline{\mathcal{V}}(\operatorname{cl}(\mathfrak{C}(1,k)))$. For the sake of contradiction, suppose $\nu_k > \underline{\mathcal{V}}(\operatorname{cl}(\mathfrak{C}(1,k)))$. Consider any payoff profile $y \in \operatorname{cl}(\mathfrak{C}(n,k))$.
From Lemma 6, we have $y_1 \ge \nu_k$, and therefore, $y_1 > \underline{\mathcal{V}}(\operatorname{cl}(\mathfrak{C}(1,k)))$. By Lemma 5, y is supported by $\operatorname{cl}(\mathfrak{C}(1,k)) \cup \{y\} = \operatorname{cl}(\mathfrak{C}(1,k))$. Hence, $\operatorname{cl}(\mathfrak{C}(1,k))$ is a self-generating set. However, $\mathfrak{C}(1,k)$ is the largest such set by Lemma 2, so we must have $\mathfrak{C}(1,k) = \operatorname{cl}(\mathfrak{C}(1,k))$, and $\nu_k = \underline{\mathcal{V}}(\operatorname{cl}(\mathfrak{C}(1,k)))$, which contradicts the assumption that $\nu_k > \underline{\mathcal{V}}(\operatorname{cl}(\mathfrak{C}(1,k)))$.

Proof of Lemma 8. Consider the following fixed point problem: find $y_1 \in [0, \bar{u}_k - \underline{u}_k]$ such that

$$y_1 = \underline{\mathcal{V}} \Big(\operatorname{conv} \big(\{ (\bar{u}_k - \underline{u}_k, \underline{u}_k), (y_1, \underline{u}_k), (y_1, \bar{u}_k - y_1) \} \big) \Big).$$
(A5)

To prove the statement of Lemma 8, it suffices to show that $y_1 = 0$ solves (A5) for all $k \le k^*$, and that $y_1 = \bar{u}_k - \underline{u}_k$ is the unique value that solves (A5) for all $k > k^*$. Our proof proceeds in four steps.

Step 1: Fix any $k \in (0, m^{\emptyset})$. We show that $y_1 = \overline{u}_k - \underline{u}_k$ is a fixed point of (A5).

When $y_1 = \bar{u}_k - \underline{u}_k$, the set conv $(\{(\bar{u}_k - \underline{u}_k, \underline{u}_k), (y_1, \underline{u}_k), (y_1, \bar{u}_k - y_1)\})$ collapses into a singleton $\{(\bar{u}_k - \underline{u}_k, \underline{u}_k)\}$. In other words, $\psi(o, w, m) = (\bar{u}_k - \underline{u}_k, \underline{u}_k)$ for all $(o, w, m) \in \mathfrak{G} \times W \times \Theta$. Then, β is sequentially rational with respect to ψ only if for all $o := (p, G) \in \mathfrak{G}$,

$$\beta^{w}(1|o) = \begin{cases} 0 & \text{if } c_{G}(\underline{u}_{k}) - p < c_{G^{\varnothing}}(\underline{u}_{k}) \\ 1 & \text{if } c_{G}(\underline{u}_{k}) - p > c_{G^{\varnothing}}(\underline{u}_{k}) \end{cases}$$

For $\epsilon > 0$, consider the offer $o_{\epsilon} \coloneqq (p_{\epsilon}, G^{\mathbf{x}_k(\bar{u}_k)})$ with $p_{\epsilon} = c_{G^{\mathbf{x}_k(\bar{u}_k)}}(\underline{u}_k) - c_{G^{\varnothing}}(\underline{u}_k) - \epsilon$. Notice

$$c_{G^{\varnothing}}(\underline{u}_k) = k = c_F(\bar{u}_k) = c_{G^{\mathbf{x}_k(\bar{u}_k)}}(\bar{u}_k) < c_{G^{\mathbf{x}_k(\bar{u}_k)}}(\underline{u}_k),$$

where the first equality follows because \underline{u}_k solves (3) when $c_G = c_{G^{\varnothing}}$, the second equality similarly follows as \overline{u}_k solves (3) when $c_G = c_F$, the third equality follows because $\overline{u}_k = u(G^{\mathbf{x}_k(\overline{u}_k)})$ as established in Lemma 3, and the inequality follows because $c_{G^{\mathbf{x}_k(\overline{u}_k)}}(\cdot)$ is strictly decreasing over the interval $(0, \mathbb{E}_F[\theta|\theta \ge \mathbf{x}_k(\overline{u}_k)])$ and $\underline{u}_k < \mathbb{E}_F[\theta|\theta \ge \mathbf{x}_k(\overline{u}_k)]$ by Lemma 11. Thus, for small enough $\epsilon > 0$, $p_\epsilon \in \mathbb{R}_+$ and o_ϵ is well-defined. Furthermore, if the monopolist offers o_ϵ , then sequentially rationality of β with respect to ψ implies that the agent accepts o_ϵ , i.e., $\beta^w(1|o_\epsilon) = 1$. The principal's payoff from offer o_{ϵ} is then given by

$$\begin{split} V_{1}(o_{\epsilon},\beta,\psi) &= p_{\epsilon} + \left(\bar{u}_{k} - \underline{u}_{k}\right) \int_{\Theta} \beta^{d} \left(o_{\epsilon},1,m\right) dG^{\mathbf{x}_{k}(\bar{u}_{k})}(m) \\ &= c_{G^{\mathbf{x}_{k}(\bar{u}_{k})}}(\underline{u}_{k}) - c_{G^{\varnothing}}(\underline{u}_{k}) - \epsilon + \left(\bar{u}_{k} - \underline{u}_{k}\right) \int_{\Theta} \beta^{d} \left(o_{\epsilon},1,m\right) dG^{\mathbf{x}_{k}(\bar{u}_{k})}(m) \\ &\geq c_{G^{\mathbf{x}_{k}(\bar{u}_{k})}}(\underline{u}_{k}) - c_{G^{\varnothing}}(\underline{u}_{k}) - \epsilon + \left(\bar{u}_{k} - \underline{u}_{k}\right) G^{\mathbf{x}_{k}(\bar{u}_{k})}(\underline{u}_{k}) \\ &= c_{G^{\mathbf{x}_{k}(\bar{u}_{k})}}(\underline{u}_{k}) - c_{G^{\varnothing}}(\underline{u}_{k}) - \epsilon + \left(\bar{u}_{k} - \underline{u}_{k}\right) F\left(\mathbf{x}_{k}(\bar{u}_{k})\right) \\ &= c_{F}(\mathbf{x}_{k}(\bar{u}_{k})) + \left(1 - F(\mathbf{x}_{k}(\bar{u}_{k}))\right) \left(\mathbf{x}_{k}(\bar{u}_{k}) - \underline{u}_{k}\right) - c_{G^{\varnothing}}(\underline{u}_{k}) - \epsilon + \left(\bar{u}_{k} - \underline{u}_{k}\right) F\left(\mathbf{x}_{k}(\bar{u}_{k})\right) \\ &= c_{F}(\bar{u}_{k}) + \left(1 - F(\bar{u}_{k})\right) \left(\bar{u}_{k} - \underline{u}_{k}\right) - c_{G^{\varnothing}}(\underline{u}_{k}) - \epsilon + \left(\bar{u}_{k} - \underline{u}_{k}\right) F\left(\bar{u}_{k}\right) \\ &= \bar{u}_{k} - \underline{u}_{k} - \epsilon, \end{split}$$

where the second equality follows by substituting in the expression for p_{ϵ} , the inequality follows because sequential rationality of β implies that $\beta^d(o_{\epsilon}, 1, m) \geq \mathbb{1}_{[m < \underline{u}_k]}$ for all $m \in \Theta$ and because $\underline{u}_k \notin \operatorname{supp}(G^{\mathbf{x}_k(\overline{u}_k)})$ making $G^{\mathbf{x}_k(\overline{u}_k)}(\cdot)$ continuous at \underline{u}_k , the third equality follows from (4) and the fact that $\mathbb{E}_F[\theta|\theta \leq \mathbf{x}_k(\overline{u}_k)] < \underline{u}_k < \mathbb{E}_F[\theta|\theta \geq \mathbf{x}_k(\overline{u}_k)]$ by Lemma 11, the fourth equality follows by Lemma 12, the fifth equality follows by the fact that $\mathbf{x}_k(\overline{u}_k) = \overline{u}_k$ as stated in Lemma 3, and the final equality follows from the fact that $c_F(\overline{u}_k) = c_{G^{\varnothing}}(\underline{u}_k) = k$.

In the limit as ϵ becomes arbitrarily small, we conclude that $\bar{u}_k - \underline{u}_k \leq \underline{\mathcal{V}}(\{(\bar{u}_k - \underline{u}_k, \underline{u}_k)\})$. However, $\underline{\mathcal{V}}(\{(\bar{u}_k - \underline{u}_k, \underline{u}_k)\})$ cannot exceed $\bar{u}_k - \underline{u}_k$, which therefore implies that $\bar{u}_k - \underline{u}_k$ is a fixed-point solution to (A5). This concludes Step 1.

For the remaining steps, we define the continuous mapping $\Phi:(0,m^{\varnothing})\to\mathbb{R}$ given by

$$\Phi(k) \coloneqq c_F(\underline{u}_k) + \underline{u}_k - \left(c_{G^{\varnothing}}(\bar{u}_k) + \bar{u}_k\right). \tag{A6}$$

Step 2: Fix any $k \in (0, m^{\emptyset})$ such that $\Phi(k) > 0$. We show that $y_1 = \bar{u}_k - \underline{u}_k$ is the unique fixed point of (A5).

Our proof proceeds by showing that for all $y_1 \in [0, \bar{u}_k - \underline{u}_k)$,

$$y_1 < \underline{\mathcal{V}} \Big(\operatorname{conv} \big(\{ (\bar{u}_k - \underline{u}_k, \underline{u}_k), (y_1, \underline{u}_k), (y_1, \bar{u}_k - y_1) \} \big) \Big)$$

To that end, fix $y_1 \in [0, \bar{u}_k - \underline{u}_k)$. For some $\epsilon > 0$, consider the offer $o_{\epsilon} \coloneqq (p_{\epsilon}, G^x)$ with



Figure 7: The continuous mapping $\Phi: (0, m^{\emptyset}) \to \mathbb{R}$ is depicted by the red curve. As we will show, the figure also depicts K which is the unique value such that $\bar{u}_K = m^{\emptyset}$, and k^* which is the unique value such that $\Phi(k^*) = 0$. $\Phi(\cdot)$ is strictly increasing over the interval (0, K) and strictly decreasing over the interval (K, m^{\emptyset}) .

$$\begin{split} x = & \underline{u}_k + y_1 \mathbbm{1}_{[\bar{u}_k - y_1 < m^{\varnothing}]} \text{ and } p_{\epsilon} = c_{G^x}(\underline{u}_k) + \underline{u}_k - \left(c_{G^{\varnothing}}(\bar{u}_k - y_1) + \bar{u}_k - y_1\right) - \epsilon. \\ \text{If } \bar{u}_k - y_1 < m^{\varnothing}, \text{ then} \\ c_{G^x}(\underline{u}_k) + \underline{u}_k - \left(c_{G^{\varnothing}}(\bar{u}_k - y_1) + \bar{u}_k - y_1\right) = c_{G^x}(\underline{u}_k) + \underline{u}_k - m^{\varnothing} = c_{G^x}(\underline{u}_k) - k, \end{split}$$

where the equality follows because $c_{G^{\varnothing}}(u) = (m^{\varnothing} - u)^+$ and $\bar{u}_k - y_1 < m^{\varnothing}$ by assumption, and the second equality follows because $\underline{u}_k = m^{\varnothing} - k$. Additionally, $k = c_F(\bar{u}_k) < c_F(x) = c_{G^x}(x) \leq c_{G^x}(\underline{u}_k)$, where the first equality follows because \bar{u}_k solves (3) when $c_G = c_F$, the first inequality follows because c_F is a strictly decreasing function and because $x = \underline{u}_k + y_1 < \bar{u}_k$ (recall that $y_1 < \bar{u}_k - \underline{u}_k$), the second equality follows by the construction of c_{G^x} as in (5), and the last inequality follows because c_{G^x} is a weakly decreasing function. Thus, for small enough $\epsilon > 0$, $p_{\epsilon} \in \mathbb{R}_+$ and o_{ϵ} is well-defined.

Similarly, if $\bar{u}_k - y_1 \ge m^{\varnothing}$, then

$$c_{G^x}(\underline{u}_k) + \underline{u}_k - \left(c_{G^{\varnothing}}(\bar{u}_k - y_1) + \bar{u}_k - y_1\right) = c_F(\underline{u}_k) + \underline{u}_k - \left(c_{G^{\varnothing}}(\bar{u}_k - y_1) + \bar{u}_k - y_1\right) \ge \Phi(k)$$

where the equality follows because $x = \underline{u}_k$ and $c_{G^x}(x) = c_F(x)$ by construction, and the inequality follows because $c_{G^{\varnothing}}(u) + u$ is weakly increasing. As $\Phi(k) > 0$ by assumption, then for small enough $\epsilon > 0$, $p_{\epsilon} \in \mathbb{R}_+$ and o_{ϵ} is once again well-defined.

For any (β, ψ) that satisfies the constraints of (7), the agent's payoff difference from

accepting o_{ϵ} versus accepting the null offer is given by

$$\underbrace{\int_{\Theta} \max\{m, \psi_2(o_{\epsilon}, 1, m)\} dG^x(m) - k - p_{\epsilon}}_{\text{payoff from accepting } o_{\epsilon}} - \underbrace{\left(\underbrace{\int_{\Theta} \max\{m, \psi_2(o_{\epsilon}, \emptyset, m)\} dG^{\varnothing}(m) - k}_{\text{payoff from accepting null offer}}\right)$$
$$\geq \int_{\Theta} \max\{m, \underline{u}_k\} dG^x(m) - p_{\epsilon} - \left(\int_{\Theta} \max\{m, \overline{u}_k - y_1\} dG^{\varnothing}(m)\right)$$
$$= c_{G^x}(\underline{u}_k) + \underline{u}_k - p_{\epsilon} - \left(c_{G^{\varnothing}}(\overline{u}_k - y_1) + \overline{u}_k - y_1\right)$$
$$= \epsilon,$$

where the inequality follows from the fact that $\underline{u}_k \leq \psi_2(o, w, m) \leq \overline{u}_k - y_1$ for all $(o, w, m) \in \mathbb{O} \times W \times \Theta$. Therefore, the agent strictly prefers to accept the offer o_{ϵ} over the null offer, i.e., $\beta^w(1|o_{\epsilon}) = 1$.

Furthermore, for any (β, ψ) that satisfies the constraints of (7), the principal's payoff from offering o_{ϵ} is given by

$$\begin{aligned} V_1(o_{\epsilon},\beta,\psi) &= p_{\epsilon} + \int_{\Theta} \beta^d (o_{\epsilon},1,m) \psi_1(o_{\epsilon},1,m) dG^x(m) \\ &\geq p_{\epsilon} + y_1 \int_{\Theta} \beta^d (o_{\epsilon},1,m) dG^x(m) \\ &\geq p_{\epsilon} + y_1 G^x(\underline{u}_k) \\ &= c_{G^x}(\underline{u}_k) + \underline{u}_k - \left(c_{G^{\varnothing}}(\bar{u}_k - y_1) + \bar{u}_k - y_1 \right) + y_1 G^x(\underline{u}_k) - \epsilon \end{aligned}$$

where the first equality follows because $\beta^w(1|o_{\epsilon}) = 1$, the first inequality follows from the fact that $\psi_1(o,w,m) \ge y_1$ for all $(o,w,m) \in \mathfrak{G} \times W \times \Theta$, the second inequality follows because sequential rationality of β implies that $\beta^d(o,w,m) \ge \mathbb{1}_{[m < \underline{u}_k]}$ for all $m \in \Theta$, and the last equality follows from substituting in the expression for p_{ϵ} .

If $\bar{u}_k - y_1 < m^{\emptyset}$, then

$$V_1(o_{\epsilon},\beta,\psi) \ge c_{G^x}(\underline{u}_k) + \underline{u}_k - \left(c_{G^{\varnothing}}(\bar{u}_k - y_1) + \bar{u}_k - y_1\right) + y_1 G^x(\underline{u}_k) - \epsilon$$
$$= c_{G^x}(\underline{u}_k) + \underline{u}_k - m^{\varnothing} + y_1 G^x(\underline{u}_k) - \epsilon$$

$$=c_{G^{x}}(\underline{u}_{k})-k+y_{1}G^{x}(\underline{u}_{k})-\epsilon$$
$$=c_{G^{x}}(\underline{u}_{k})-k+y_{1}F(x)-\epsilon$$
$$=c_{F}(x)+(1-F(x))(x-\underline{u}_{k})-k+y_{1}F(x)-\epsilon$$
$$=c_{F}(x)-k+y_{1}-\epsilon$$

where the first equality follows because $c_{G^{\varnothing}}(u) = (m^{\varnothing} - u)^+$ and $\bar{u}_k - y_1 < m^{\varnothing}$ by assumption, the second equality follows because $\underline{u}_k = m^{\varnothing} - k$, the third equality follows because $G^x(m) = F(x)$ for all $m \in (\mathbb{E}_F[\theta|\theta \leq \bar{x}], \mathbb{E}_F[\theta|\theta \geq x])$ by (4) and the fact that $\mathbb{E}_F[\theta|\theta \leq x] \leq \mathbb{E}_F[\theta|\theta \leq \bar{u}_k] < \underline{u}_k < \mathbb{E}_F[\theta|\theta \geq \underline{u}_k] \leq \mathbb{E}_F[\theta|\theta \geq x]$ by Lemma 11, the fourth equality follows from Lemma 12, and the final equality follows because $x = \underline{u}_k + y_1$ by construction. Therefore, for any (β, ψ) that satisfies the constraints of (7), the principal's optimal payoff is bounded below by $c_F(x) - k + y_1$, and we can conclude that

$$\underline{\mathcal{V}}\left(\operatorname{conv}\left(\left\{(\bar{u}_k - \underline{u}_k, \underline{u}_k), (y_1, \underline{u}_k), (y_1, \bar{u}_k - y_1)\right\}\right)\right) \ge c_F(x) - k + y_1 > y_1$$

where the second inequality follows because $c_F(\cdot)$ is strictly decreasing with $c_F(\bar{u}_k) = k$ by (A5), and $x = \underline{u}_k + y_1$ with $y_1 \in [0, \bar{u}_k - \underline{u}_k)$ by assumption.

Similarly, if $\bar{u}_k - y_1 \ge m^{\varnothing}$, then

$$V_{1}(o_{\epsilon},\beta,\psi) \geq c_{G^{x}}(\underline{u}_{k}) + \underline{u}_{k} - \left(c_{G^{\varnothing}}(\bar{u}_{k}-y_{1}) + \bar{u}_{k}-y_{1}\right) + y_{1}G^{x}(\underline{u}_{k}) - \epsilon$$

$$= c_{G^{x}}(\underline{u}_{k}) + \underline{u}_{k} - \left(c_{G^{\varnothing}}(\bar{u}_{k}) + \bar{u}_{k}-y_{1}\right) + y_{1}G^{x}(\underline{u}_{k}) - \epsilon$$

$$= c_{F}(\underline{u}_{k}) + \underline{u}_{k} - \left(c_{G^{\varnothing}}(\bar{u}_{k}) + \bar{u}_{k}-y_{1}\right) + y_{1}G^{x}(\underline{u}_{k}) - \epsilon$$

$$= \Phi(k) + y_{1}\left(1 + G^{x}(\underline{u}_{k})\right) - \epsilon$$

where the first equality follows because $c_{G^{\varnothing}}(u) = (m^{\varnothing} - u)^+$ is a constant for all $u \ge m^{\varnothing}$ and $\bar{u}_k > \underline{u}_k - y_1 \ge m^{\varnothing}$ by assumption, and the second equality follows because $x = \underline{u}_k$ and $c_{G^x}(x) = c_F(x)$ by construction. Therefore, for any (β, ψ) that satisfy the constraints of (7), the principal's optimal payoff is bounded below by $\Phi(k) + y_1(1 + F(\underline{u}_k))$, and we can conclude that

$$\underline{\mathscr{V}}\Big(\operatorname{conv}\big(\{(\bar{u}_k - \underline{u}_k, \underline{u}_k), (y_1, \underline{u}_k), (y_1, \bar{u}_k - y_1)\}\big)\Big) \ge \Phi(k) + y_1\big(1 + F(\underline{u}_k)\big) > y_1$$

where the second inequality follows because $\Phi(k) > 0$ by assumption.

Consequently, when $\Phi(k) > 0$ (with either $\bar{u}_k - y_1 < m^{\varnothing}$ or $\bar{u}_k - y_1 \ge m^{\varnothing}$), there exists no fixed point of (A5) with $y_1 \in [0, \bar{u}_k - \underline{u}_k)$. By Step 1, we conclude that $y_1 = \bar{u}_k - \underline{u}_k$ is the unique fixed point of (A5) for any $k \in (0, m^{\varnothing})$ that satisfies $\Phi(k) > 0$. In other words, $\nu_k = \bar{u}_k - \underline{u}_k$ if $\Phi(k) > 0$. This concludes Step 2.

Step 3: Fix any $k \in (0, m^{\emptyset})$ such that $\Phi(k) \leq 0$. We show $y_1 = 0$ is a fixed point of (A5).

When $y_1 = 0$, the set conv $(\{(\bar{u}_k - \underline{u}_k, \underline{u}_k), (y_1, \underline{u}_k), (y_1, \bar{u}_k - y_1)\})$ is equivalent to the feasible set $\mathcal{F}(1,k)$. Consider a pair (β, ψ) where the continuation payoff function ψ is given by

$$\psi(o,w,m) = \begin{cases} (0,\underline{u}_k) & \text{if } w = 1 \\ (0,\overline{u}_k) & \text{if } w = \emptyset \end{cases},$$

and the agent's simple strategy is given by $\beta^d(o, w, m) = \mathbb{1}_{[m \le \psi_2(o, w, m)]}$, and for all $o := (p, G) \in \mathbb{O}$,

$$\beta^{w}(1|o) = \begin{cases} 0 & \text{if} \quad c_{G}(\underline{u}_{k}) + \underline{u}_{k} - p \leq c_{G^{\varnothing}}(\bar{u}_{k}) + \bar{u}_{k} \\ 1 & \text{if} \quad c_{G}(\underline{u}_{k}) + \underline{u}_{k} - p > c_{G^{\varnothing}}(\bar{u}_{k}) + \bar{u}_{k} \end{cases}$$

Notice $Im_{\psi} \subseteq \mathscr{F}(1,k)$ and β is sequentially rational with respect to ψ . Additionally, notice that for any TIOLI offer $o = (p,G) \in \mathfrak{G}$, $c_G(\underline{u}_k) + \underline{u}_k - p \leq c_F(\underline{u}_k) + \underline{u}_k - p \leq c_{G^{\varnothing}}(\overline{u}_k) + \overline{u}_k$, where the first inequality follows because $c_G \leq c_F$ pointwise for all $G \in \mathscr{G}(F)$, and the second inequality follows by the ongoing assumption of Step 3 that $\Phi(k) \leq 0$. Thus, the agent rejects all TIOLI offers made my the principal in this case, i.e., $\beta^w(1|o)=0$ for all $o \in \mathfrak{G}$.

Given the prescribed (β, ψ) , the principal's payoff from offering any $o \in \mathbb{O}$ is

$$V_1(o,\!\beta,\!\psi) \!=\! \int_{\Theta} \!\beta^d(o,\!\emptyset,\!m) \psi_1(o,\!\emptyset,\!m) dG^{\varnothing}(m) \!=\! 0$$

where the first equality follows since the agent rejects all offers, and the second equality follows because $\psi_1(o,w,m) = 0$ for all $(o,w,m) \in \mathfrak{G} \times W \times \Theta$.

We have thus shown that there exists a pair (β, ψ) satisfying the constraints of (7) with $\sup_{o \in \mathcal{O}} V_1(o, \beta, \psi) = 0$. Therefore, $\underline{\mathcal{V}}(\mathcal{F}(1,k)) \leq 0$. However, the smallest possible value for

 $\underline{\mathcal{V}}(\mathcal{F}(1,k))$ cannot be lower than zero, which therefore implies that $y_1 = 0$ is the smallest fixed-point solution to (A5) when $\Phi(k) \leq 0$. In other words, $\nu_k = 0$ if $\Phi(k) \leq 0$. This concludes Step 3.

Step 4: We finalize the proof by showing that there exists a unique $k^* \in (0, m^{\emptyset})$ such that $\Phi(k) \leq 0$ for all $k \leq k^*$ and $\Phi(k) > 0$ for all $k > k^*$.

We claim that there exists a unique $K \in (0, m^{\emptyset})$ such that $\bar{u}_K = m^{\emptyset}$. To see this, recall from Lemma 1 that the mapping $k \mapsto \bar{u}_k$ is continuous and decreasing. Moreover, $\lim_{k\to 0} \bar{u}_k = 1 > m^{\emptyset}$ (the agent is happy to keep searching until he finds the highest quality good when searching is costless) and $\lim_{k\to m^{\emptyset}} \bar{u}_k = 0 < m^{\emptyset}$ (the agent immediately stops his search when searching is too costly). Therefore, the claim is completed by appealing to the intermediate value theorem.

Recall that $c_{G^{\varnothing}}(u) = (m-u)^+$. Thus, for all $k \ge K$, we have $\bar{u}_k \le m^{\varnothing}$, which implies that $c_{G^{\varnothing}}(\bar{u}_k) + \bar{u}_k = m^{\varnothing}$. In this case $\Phi(k) = c_F(\underline{u}_k) + \underline{u}_k - m^{\varnothing}$. As \underline{u}_k is strictly decreasing in k and $c_F(u) + u$ is strictly increasing in u, $\Phi(k)$ is a strictly decreasing function when $k \in (K, m^{\varnothing})$. Additionally, $\lim_{k\to m^{\varnothing}} \Phi(k) = c_F(0) - m^{\varnothing} = 0$. Therefore, $\Phi(k) > 0$ for all $k \in [K, m^{\varnothing})$.

In contrast, when k < K, then $\bar{u}_k > m^{\emptyset}$, which implies that $c_{G^{\emptyset}}(\bar{u}_k) = 0$. In this case $\Phi(k) = c_F(\underline{u}_k) - (\bar{u}_k - \underline{u}_k)$. As \underline{u}_k is strictly decreasing in k and $c_F(u)$ is strictly decreasing in $u, c_F(\underline{u}_k)$ is strictly increasing in k. Furthermore, by Lemma 1, $\bar{u}_k - \underline{u}_k$ is strictly decreasing in k. Thus $\Phi(k)$ is a strictly increasing function when $k \in (0, K)$. Additionally,

$$\lim_{k \to 0} \Phi(k) = c_F(m^{\varnothing}) - (1 - m^{\varnothing}) = \int_{m^{\varnothing}}^{1} (1 - F(m)) dm - (1 - m^{\varnothing}) < 0.$$

Since Φ is continuous and strictly increasing with $\Phi(K) > 0$ and $\lim_{k\to 0} \Phi(k) < 0$, there exists a unique $k^* \in (0, K)$ such that $\Phi(k^*) = 0$, and $\Phi(k) > 0$ for all $k > k^*$ and $\Phi(k) < 0$ for all $k < k^*$ (see Figure 7). This concludes Step 4, and the proof of the lemma.

Proof of Lemma 9. Recall that $c_{G'} \leq c_{G''}$ pointwise for any $G', G'' \in \mathfrak{G}(F)$ such that G' is a mean-preserving contraction of G''. Consequently, for all $y \in \mathfrak{F}(n,k)$, $\mathfrak{P}_k(G';y) \leq \mathfrak{P}_k(G'';y)$. Thus, to prove the lemma, it suffices to show that $\mathfrak{P}_k(F;y) = 0$ for any $k \leq k^*$ and any $y \in \mathcal{F}(n,k)$. To see that this condition holds, note $\mathcal{P}_k(F;y) \geq 0$ by construction. Additionally,

$$\mathcal{P}_{k}(F;y) = \left(\Phi(k) - \left(c_{G^{\mathbf{x}_{k}(||y||)}}(\bar{u}_{k}) - c_{G^{\varnothing}}(\bar{u}_{k}) - P(y)\right)^{+}\right)^{+} \leq \Phi(k)^{+} = 0$$

where the first equality follows from (A6), and the last equality follows because $\Phi(k) \leq 0$ for all $k \leq k^*$ (see Step 4 in the proof of Lemma 8).

Proof of Lemma 10. For each $k \in (0, m^{\emptyset})$, let $\epsilon_k = c_{G^{\mathbf{x}_k(\underline{u}_k)}}(\overline{u}_k)$. Notice that $\epsilon_k > 0$ because

$$c_{G^{\mathbf{x}_k(\underline{u}_k)}}(\bar{u}_k) \!=\! c_F(\mathbf{x}_k(\underline{u}_k)) \!+\! (1\!-\!F(\mathbf{x}_k(\underline{u}_k)))(\mathbf{x}_k(\underline{u}_k) \!-\! \bar{u}_k) \!>\! 0,$$

where the equality follows from Lemma 12 and the inequality follows because $\bar{u}_k < \mathbf{x}_k(\underline{u}_k) < 1$ as stated in Lemma 3.

Consider search costs $k \leq K$, where K is the unique value of the search cost for which $\bar{u}_K = m^{\emptyset}$. Then, for all $y \in \mathcal{F}^{\epsilon_k}(n,k)$

$$\begin{split} \mathscr{P}_{k}(F;y) &= \left(\Phi(k) - \left(c_{G^{\mathbf{x}_{k}(||y||)}}(\bar{u}_{k}) - c_{G^{\varnothing}}(\bar{u}_{k}) - P(y) \right)^{+} \right)^{+} \\ &= \left(\Phi(k) - \left(c_{G^{\mathbf{x}_{k}(||y||)}}(\bar{u}_{k}) - P(y) \right)^{+} \right)^{+} \\ &\leq \left(\Phi(k) - c_{G^{\mathbf{x}_{k}(||y||)}}(\bar{u}_{k}) + P(y) \right)^{+} \\ &= \left(\Phi(k) - c_{G^{\mathbf{x}_{k}(||y||)}}(\bar{u}_{k}) + \left(1 - F(\mathbf{x}_{k}(||y||)) \right) \left(||y|| - y_{n+1} \right) \right)^{+} \\ &\leq \left(\Phi(k) - c_{G^{\mathbf{x}_{k}(\underline{u}_{k})}}(\bar{u}_{k}) F(\bar{u}_{k}) \right)^{+} \end{split}$$

where the first equality is established in the Proof of Lemma 9, the second equality follows because $\bar{u}_k \ge m^{\varnothing}$ for all $k \le K$, implying that $c_{G^{\varnothing}}(\bar{u}_k) = (m^{\varnothing} - \bar{u}_k)^+ = 0$, the last equality follows by the alternative characterization of P(y) given in (A3), and the last inequality follows because $c_{G^{\mathbf{x}_k(||y||)}}(\bar{u}_k) \ge c_{G^{\mathbf{x}_k(\underline{u}_k)}}(\bar{u}_k)$ by Lemma 13, because $||y|| - y_{n+1} \le \epsilon_k = c_{G^{\mathbf{x}_k(\underline{u}_k)}}(\bar{u}_k)$, and because $\mathbf{x}_k(\cdot)$ is a bounded from below by \bar{u}_k as established in Lemma 3.

The mapping $k \mapsto \Phi(k) - c_{G^{\mathbf{x}_k(\underline{u}_k)}}(\overline{u}_k)F(\overline{u}_k)$ is continuous. Additionally, for all $k \leq k^*$, $\Phi(k) - c_{G^{\mathbf{x}_k(\underline{u}_k)}}(\overline{u}_k)F(\overline{u}_k) < 0$, since $\Phi(k) \leq 0$ for all $k \leq k^*$ by construction (see Step 4 of Lemma 8) while $c_{G^{\mathbf{x}_k(\underline{u}_k)}}(\overline{u}_k)F(\overline{u}_k) > 0$ for all $k \in (0, m^{\varnothing})$. Hence, there exists a $k^{**} \in (k^*, K]$ such that $\Phi(k) - c_{G^{\mathbf{x}_k(\underline{u}_k)}}(\overline{u}_k)F(\overline{u}_k) \leq 0$ for all $k \leq k^{**}$. Consequently, for all $k \leq k^{**}$, there exists $\epsilon_k > 0$ such that $\mathcal{P}_k(F;y) = 0$ for all $y \in \mathcal{F}^{\epsilon_k}(n,k)$. From the proof of Lemma 9, $\mathcal{P}_k(F;y) = 0$ for any $k \leq k^{**}$ and $y \in \mathcal{F}^{\epsilon_k}(n,k)$ implies the desired conclusion.

B. Online Appendix: Proof of Lemma 2

To see why $\mathscr{C}(n,k)$ is self generating, fix any $y \in \mathscr{C}(n,k)$. By definition of the equilibrium payoff set, there exists some equilibrium strategy profile σ such that $y = ((\mathbf{V}_i(\sigma))_{i \in N}, \mathbf{U}(\sigma))$.

For each Principal $i \in N$, we can decompose her payoff, $\mathbf{V}_i(\sigma)$, into a sum of her payoff in the current period

$$\mathbb{E}_{(\sigma_j|h^1)_{j\in N}}\left[p_i\sigma^w_A(i|h^1,O)\right]$$

and her expected continuation payoff

$$\mathbb{E}_{(\sigma_j|h^1)_{j\in N}}\left[\sum_{w\in W}\sigma^w_A(w|h^1,O)\int_{\Theta}\mathbf{V}_{i2}(\sigma|h^1\cup\{O,w,m\})\sigma^d_A(h^1,O,w,m)dG_w(m)\right],$$

where $\mathbb{E}_{(\sigma_j|h^1)_{j\in N}}$ is the expectation taken over \mathbb{O}^n with respect to the probability induced by $(\sigma_j(\cdot|h^1))_{j\in N}$. Similarly, we can decompose the agent's payoff, $U(\sigma)$, into his payoff in the current period

$$-k + \mathbb{E}_{(\sigma_j|h^1)_{j \in N}} \left[\sum_{w \in W} \sigma^w_A(w|h^1, O) \left(-p_w + \int_{\Theta} m \left(1 - \sigma^d_A(h^1, O, w, m) \right) dG_w(m) \right) \right]$$

and his expected continuation payoff

$$\mathbb{E}_{(\sigma_j|h^1)_{j\in N}}\left[\sum_{w\in W}\sigma^w_A(w|h^1,O)\int_{\Theta}\mathbf{U}_2(\sigma|h^1\cup\{O,w,m\})\sigma^d_A(h^1,O,w,m)dG_w(m)\right].$$

Define simple strategies (α,β) and a continuation payoff function ψ as follows: for all $(O,w,m) \in \mathbb{G}^n \times W \times \Theta$, let

- $\alpha_i(\cdot) = \sigma_i(\cdot|h^1)$ for all $i \in N$,
- $\beta^w(\cdot|O) = \sigma^w_A(\cdot|h^1,O),$
- $\beta^d(\cdot|O,w,m) = \sigma^d_A(h^1,O,w,m),$
- $\psi_i(O, w, m) = \mathbf{V}_{i2}(\sigma | h^1 \cup \{O, w, m)\})$ for all $i \in N$, and

• $\psi_{n+1}(O,w,m) = \mathbf{U}_2(\sigma | h^1 \cup \{O,w,m\}).$

Clearly, $\mathbb{E}_{\alpha}[V_i(O,\beta,\psi)] = \mathbf{V}_i(\sigma)$ for all $i \in N$, and $\mathbb{E}_{\alpha}[U(O,\beta,\psi)] = \mathbf{U}(\sigma)$. Thus, (α,β,ψ) generates the payoff profile $y = ((\mathbf{V}_i(\sigma))_{i \in N}, \mathbf{U}(\sigma))$, and the defined simple strategies and continuation payoff function satisfy Point (a) of Definition 1. Furthermore, taking ψ as a given, the simple strategy profile (α,β) satisfies Points (b) and (c) of the definition; otherwise, there would be a one-shot deviation from σ that improves the payoff of either a principal or the agent, which would contradict the assumption that σ is an equilibrium strategy profile. Finally, since σ is sequentially rational, for any history h_t , the strategy $\sigma|h_t$ is an equilibrium in the game starting at h_t . Thus, following any $(O,w,m) \in \mathbb{O}^n \times W \times \Theta$ and the resulting period-2 history $h^2 = h^1 \cup \{O, w, m\} \in H^2$, the payoff profile $((\mathbf{V}_{i2}(\sigma|h^2))_{i \in N}, \mathbf{U}_2(\sigma|h^2))$ is in $\mathscr{C}(n,k)$. Consequently, ψ satisfies Point (d) of Definition 1, and so, we conclude that $\mathscr{C}(n,k)$ is self generating.

Next, we prove that if E is a self-generating set, then $E \subseteq \mathscr{C}(n,k)$. Since E is self-generating, for each $y \in E$, there exist simple strategies $\boldsymbol{\alpha}_i[y] \in \Delta \mathbb{G}$ for each $i \in N$, $\boldsymbol{\beta}^{\boldsymbol{w}}[y]: \mathbb{G}^n \to \Delta W$, and $\boldsymbol{\beta}^{\boldsymbol{d}}[y]: \mathbb{G}^n \times W \times \Theta \to [0,1]$, and a continuation payoff function $\boldsymbol{\psi}[y]: \mathbb{G}^n \times W \times \Theta \to \mathbb{R}^{n+1}$ such that the tuple $((\boldsymbol{\alpha}_i[y])_{i \in N}, \boldsymbol{\beta}^{\boldsymbol{w}}[y], \boldsymbol{\beta}^{\boldsymbol{d}}[y], \boldsymbol{\psi}[y])$ generates yand satisfies Points (a)-(d) of Definition 1.

Fix any $y^* \in E$. Let $Y(h^1) \coloneqq y^*$. For each t > 1, $h^{t-1} \in H$, and $(O_t, w_t, m_t) \in \mathbb{O}^n \times W \times \Theta$, define $Y(h^t) \coloneqq \psi[Y(h^{t-1})](O_t, w_t, m_t)$ where $h^t = h^{t-1} \cup \{O_t, w_t, m_t\}$. By construction, $Y(h) \in E$ for all $h \in H$ because $Im_{\psi[\tilde{y}]} \subseteq E$ for all $\tilde{y} \in E$.

For each $i \in N$, define $\sigma_i : H \to \Delta \mathfrak{G}$ as $\sigma_i(\cdot|h^t) = \boldsymbol{\alpha}_i[Y(h^t)](\cdot)$. Similarly, define $\sigma_A^w : H \times \mathfrak{G}^n \to \Delta W$ and $\sigma_A^d : H \times \mathfrak{G}^n \times W \times \Theta \to [0,1]$ as $\sigma_A^w(\cdot|h^t, O_t) = \boldsymbol{\beta}^w[Y(h^t)](\cdot|O_t)$ and $\sigma_A^d(h^t, O_t, w_t, m_t) = \boldsymbol{\beta}^d[Y(h^t)](O_t, w_t, m_t)$, respectively. Since each $Y(h^t) \in E$, the tuple $((\boldsymbol{\alpha}_i[Y(h^t)])_{i\in N}, \boldsymbol{\beta}^w[Y(h^t)], \boldsymbol{\beta}^d[Y(h^t)], \boldsymbol{\psi}[Y(h^t)])$ generates $Y(h^t)$ and satisfies Points (a)-(d) of Definition 1. Therefore, the constructed profile of strategies $\sigma \coloneqq ((\sigma_i)_{i\in N}, \sigma_A^w, \sigma_A^d)$ satisfies for all $t \ge 1$ and $h^t \in H$: (i) $Y_i(h^t) = \mathbf{V}_{it}(\sigma|h^t)$ as defined in (1) for all $i \in N$, (ii) $Y_{n+1}(h^t) =$ $\mathbf{U}_t(\sigma|h^t)$ as defined in (2), and (iii) No player can improve her payoff via a one-shot deviation, and thus, no player can improve her payoff by deviating away from σ finitely often.

It remains to argue that, following any history, no player can improve her payoff by deviating infinitely often.¹³ Suppose, for the sake of contradiction, the agent can deviate

¹³The payoffs in our model do not satisfy the usual "continuous at infinity" assumption, so we cannot simply appeal to the one-shot deviation principle.

to strategy $\tilde{\sigma}_A$ following some history $h^t \in H$ and earn a payoff of

$$\mathbf{U}_t((\sigma_i)_{i\in N}, \tilde{\sigma}_A|h^t) - \mathbf{U}_t(\sigma|h^t) > \Delta$$

for some $\Delta > 0$. Let $\tilde{\sigma} := ((\sigma_i)_{i \in N}, \tilde{\sigma}_A)$ denote the deviation strategy profile.

Let $\mathbb{P}_{\tilde{\sigma}}(\tau|h^t)$ represent the expected probability that the game ends in period $t+\tau$ for $\tau \geq 0$, induced by the strategy profile $\tilde{\sigma}$ starting from history h^t . Then,

$$\mathbf{U}_t((\sigma_i)_{i\in N}, \tilde{\sigma}_A | h^t) \leq \sum_{\tau \geq 0} \mathbb{P}_{\tilde{\sigma}}(\tau | h^t) (1 - k(\tau + 1)),$$
(OA1)

where the inequality follows because the highest match value is $\theta = 1$. The right-hand side of the inequality is non-negative only if the game ends in finite time almost surely, so there exists a T > t such that

$$\sum_{\tau \ge T+1} \mathbb{P}_{\tilde{\sigma}}(\tau | h^t) < \frac{\Delta}{2}.$$
 (OA2)

Let σ'_A be the strategy that coincides with $\tilde{\sigma}_A$ until period T and coincides with σ_A thereafter, and let $\sigma' \coloneqq ((\sigma_i)_{i \in N}, \sigma'_A)$. We then have

$$\begin{split} \mathbf{U}_{t}(\tilde{\sigma}|h^{t}) - \mathbf{U}_{t}(\sigma'|h^{t}) &= \mathbb{E}_{\tilde{\sigma}(h^{t})} \left[\mathbf{U}_{T+1}(\tilde{\sigma}|h^{T+1}) \right] - \mathbb{E}_{\sigma'(h^{t})} \left[\mathbf{U}_{T+1}(\sigma|h^{T+1}) \right] \\ &\leq \mathbb{E}_{\tilde{\sigma}(h^{t})} \left[\sum_{\tau \geq 0} \mathbb{P}_{\tilde{\sigma}}\left(\tau|h^{T+1}\right) (1 - k(\tau + 1)) \right] \\ &\leq \mathbb{E}_{\tilde{\sigma}(h^{t})} \left[\sum_{\tau \geq 0} \mathbb{P}_{\tilde{\sigma}}\left(\tau|h^{T+1}\right) \right] \\ &= \sum_{\tau \geq 0} \mathbb{E}_{\tilde{\sigma}(h^{t})} \left[\mathbb{P}_{\tilde{\sigma}}\left(\tau|h^{T+1}\right) \right] \\ &= \sum_{\tau \geq T+1} \mathbb{P}_{\tilde{\sigma}}\left(\tau|h^{t}\right) \\ &< \frac{\Delta}{2}, \end{split}$$

where the first equality follows because σ' coincides with $\tilde{\sigma}$ until period T and with σ from T+1 onward, the first inequality follows from (OA1) and because for any history h^{T+1} , the agent's payoff given σ is $\mathbf{U}_{T+1}(\sigma|h^{T+1}) = Y_{n+1}(h^{T+1}) > 0$ by construction, the last equality

follows because $\mathbb{P}_{\tilde{\sigma}}(\cdot|h)$ is a martingale with respect to h, and the last inequality follows from (OA2). Consequently,

$$\mathbf{U}_t(\sigma'|h^t) - \mathbf{U}_t(\sigma|h^t) > \mathbf{U}_t(\tilde{\sigma}|h^t) - \frac{\Delta}{2} - \mathbf{U}_t(\sigma|h^t) > \frac{\Delta}{2},$$

so σ'_A is a profitable deviation for the agent. However, σ'_A differs from σ finitely often, which contradicts the fact that no one player can improve payoffs by deviating from σ finitely often. A similar argument establishes that no principal can profitable deviate.

We thus conclude that the strategy profile σ is sequentially rational, and therefore, an equilibrium. Finally, noting that $y_i^* = V_i(\sigma)$ for all $i \in N$ and $y_{n+1}^* = U(\sigma)$, we conclude that $y^* \in \mathscr{C}(n,k)$, which proves the desired result that $E \subseteq \mathscr{C}(n,k)$.

References

- D. Abreu, D. Pearce, and E. Stacchetti. Toward a theory of discounted repeated games with imperfect monitoring. *Econometrica: Journal of the Econometric Society*, pages 1041–1063, 1990.
- A. R. Admati and P. Pfleiderer. A monopolistic market for information. Journal of Economic Theory, 39(2):400–438, 1986.
- A. R. Admati and P. Pfleiderer. Direct and indirect sale of information. *Econometrica*, pages 901–928, 1990.
- P. H. Au and K. Kawai. Competitive information disclosure by multiple senders. Games and Economic Behavior, 119:56–78, 2020.
- P. H. Au and M. Whitmeyer. Attraction versus persuasion: Information provision in search markets. *Journal of Political Economy*, 131(1):000–000, 2023.
- P. H. Au and M. Whitmeyer. Attraction via prices and information. arXiv preprint arXiv:2402.11754, 2024.
- D. Bergemann and A. Bonatti. Selling cookies. American Economic Journal: Microeconomics, 7(3):259–294, 2015.
- D. Bergemann, A. Bonatti, and A. Smolin. The design and price of information. American economic review, 108(1):1–48, 2018.
- D. Blackwell. Equivalent comparisons of experiments. The Annals of Mathematical Statistics, pages 265–272, 1953.
- S. Board and J. Lu. Competitive information disclosure in search markets. Journal of Political Economy, 126(5):1965–2010, 2018.
- A. Bonatti, M. Dahleh, T. Horel, and A. Nouripour. Selling information in competitive environments. *Journal of Economic Theory*, 216:105779, 2024.

- H. Chade. Matching with noise and the acceptance curse. Journal of Economic Theory, 129:81–113, 2006.
- P. A. Diamond. A model of price adjustment. Journal of economic theory, 3(2):156–168, 1971.
- M. Dogan and J. Hu. Consumer search and optimal information. The RAND Journal of Economics, 53(2):386–403, 2022.
- P. Eső and B. Szentes. The price of advice. *The Rand Journal of Economics*, 38(4): 863–880, 2007.
- D. Fudenberg and E. Maskin. The folk theorem in repeated games with discounting or with incomplete information. *Econometrica*, 54(3):533–554, 1986. ISSN 00129682, 14680262.
- D. Fudenberg and J. Tirole. Perfect bayesian equilibrium and sequential equilibrium. *journal of Economic Theory*, 53(2):236–260, 1991.
- M. Gentzkow and E. Kamenica. Competition in persuasion. The Review of Economic Studies, 84(1):300–322, 2016.
- M. Gentzkow and E. Kamenica. Bayesian persuasion with multiple senders and rich signal spaces. Games and Economic Behavior, 104:411–429, 2017.
- W. He and J. Li. Competitive information disclosure in random search markets. Games and Economic Behavior, 140:132–153, 2023.
- J. Hu. Industry-optimal information in the search market. SSRN WP 3935299, 2022.
- E. Kamenica. Bayesian persuasion and information design. Annual Review of Economics, 11:249–272, 2019.
- F. Li and P. Norman. On bayesian persuasion with multiple senders. *Economics Letters*, 170:66–70, 2018.
- F. Li and P. Norman. Sequential persuasion. *Theoretical Economics*, 16(2):639–675, 2021.
- J. J. McCall. Economics of information and job search. The Quarterly Journal of Economics, pages 113–126, 1970.
- T. Mekonnen, Z. Murra-Anton, and B. Pakzad-Hurson. Persuaded search. arXiv preprint arXiv:2303.13409, 2024.
- R. Rodríguez Olivera. Strategic incentives and the optimal sale of information. American Economic Journal: Microeconomics, 16(2):296–353, 2024.
- H. Sato and R. Shirakawa. Persuasion in ordered search. SSRN WP 4483732, 2023.
- M. L. Weitzman. Optimal search for the best alternative. *Econometrica*, pages 641–654, 1979.
- W. Wu. Sequential bayesian persuasion. Journal of Economic Theory, 214:105763, 2023.