Acceptance Deadlines and Job Offer Design*

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January 30, 2024

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Abstract

This paper studies talent recruiting in an incomplete-information environment with the acceptance deadline of an employer's job offer being a strategic recruiting device. When the terms of employment are invariable, increasing the acceptance deadline raises the chance of the employer hiring candidates with more promising outside options, but reduces the probability of hiring those with less promising alternatives. The employer is more likely to choose extreme deadlines, i.e., extend exploding offers, which require immediate responses, and open offers, which have the longest deadline, when the candidate is more willing to postpone his acceptance decision. Committing herself to a firm deadline is not optimal for the employer; allowing requests for a deadline extension benefits the two parties. When incorporating the acceptance deadline into the design of the job offer, the optimal design for the employer can be implemented using a "bonus-for-early-acceptance" (BFEA) mechanism, which is widely applied in practice. In a BFEA mechanism, the employer (i) specifies a date that her offer expires and (ii) provides a salary bonus for accepting the offer, which is decreasing over time before the offer expires. A candidate anticipating a better outside option takes a longer time to respond and receives a lower bonus. Our result indicates that different BFEA mechanisms adopted in various real-world labor markets reflect the level of competition faced by employers.

Key words: acceptance deadline, exploding offer, open offer, re-negotiation, job offer design

JEL classification: D82, D86, M51, M55

*We are indebted to Benjamin Balzer, Onur Kesten, Shiko Maruyama, Claudio Mezzeti, Matthew Mitchell, Siqi Pan, Xianwen Shi, Toru Suzuki, Emil Temnyalov, Kentaro Tomoeda, Tsz-Ning Wong, Lily Yang, Benjamin Young, Pei-Cheng Yu, Jun Zhang, and participants in the UTS lunch seminar, the Economic Theory and Experiments Seminar at the University of Melbourne, the UNSW Theory Workshop, the School Seminar at the University of Queensland, the 2018 Australasian Economic Theory Workshop, 2019 North American Summer Meeting of the Econometric Society, and 2020 World Congress of the Econometric Society for their invaluable comments and suggestions. The usual disclaimer applies.

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1 Introduction

A job offer is usually attached with a deadline of accepting the offer, especially in labor markets with defined hiring cycles. This is because after receiving an offer, a job candidate often prefers to hold on to the offer and searches for better options. The employer extending the offer, however, prefers to receive a prompt response: even if the quick response is a rejection, it gives the employer plenty of time to recruit other good job candidates. The acceptance deadline, ranging from several weeks to even a couple of minutes, is employed as a strategic recruiting device to manage the conflicting incentives and influence the job choice decision of the potential employee.[1]

This paper is devoted to developing a simple dynamic framework of talent recruiting with the goal of answering the following questions. Fixing the terms of employment stipulated in an offer, what is optimal acceptance deadline optimizing the employer's recruiting outcome? Should the employer commit itself to a firm deadline or allow requests for a deadline extension? When incorporating the acceptance deadline into job offer design, what is the optimal design of a job offer?

We propose a partial-equilibrium framework to study these problems in an incomplete-information environment. In the model, two qualified job candidates are available for an employer to fill an open slot, with one of them being strictly preferred. The secondary candidate always accepts the employer's offer once receiving it, which is not the case for the preferred candidate. The preferred candidate has a privately known outside option, which is an alternative offer from a competing employer. The alternative offer, however, is uncertain: The candidate does not know *whether* and *when* the offer will arrive. The candidate becomes increasingly pessimistic about getting the alternative offer as the time of waiting for the offer elapses. The better the alternative option is, the longer he wants to wait for the option before he is willing to accept the current employer's offer.

The employer (optimally) first extends an offer to the preferred candidate and sets a deadline for the candidate to respond. The choice of acceptance deadline affects not only the chance of the employer recruiting the preferred candidate, but also its chance of hiring the secondary candidate, as the secondary candidate may also switch to other options anytime. The tradeoffs for the employer to choose the deadline are as follows. If the deadline is short, the chance of hiring the preferred candidate is low, because the preferred candidate accepts the offer only if his outside option is not attractive. If the deadline is long, the employer risks giving the preferred candidate too much time to explore the alternative option; even one willing to accept the employer's offer under a shorter deadline may find a better match during a longer period of consideration. Moreover, with a long deadline, the employer has a low probability of recruiting the secondary candidate, as the probability that he switches to other options during the period of waiting becomes high.

[1]The Career Centers of many universities in the U.S. advise the employers to set and the students to expect a two-to-three week response deadline for a job offer. Roth and Xing (1994) document that in the matching market between law clerks and judges, some judges extend offers that are only available during the phone calls of making the offers.

We first study the optimal acceptance deadline that optimizes the employer's recruiting outcome, when the terms of employment in the offer are fixed. In the literature on labor market matching, the two types of offers with extreme deadlines, exploding offers and open offers, are the most studied offer types. An open offer gives a candidate sufficient time to consider his alternative options. An exploding offer is used to deter the receiver from waiting for his outside option in the most aggressive form. In this paper, we show that if the preferred candidate always postpones his acceptance decision until the deadline, the employer is more likely to extend offers with extreme deadlines.

Given that the primary role of the acceptance deadline is to deter the candidate from delay, whether the employer commits itself to a firm deadline potentially affects the recruiting outcome. Many online career platforms (AngelList, The Muse, and JobHero, for example) and human resource practitioners advise job seekers to ask for deadline extensions when they need more time to consider their job offers. Allowing re-negotiation over the acceptance deadline undoubtedly benefits the candidate, but it is not clear how it affects the payoffs of the employers. In this paper, we show that allowing requests for a deadline extension benefits the employer. This result relies on the existence of the secondary candidate. Upon reaching the deadline, if the employer learns that the secondary candidate is gone, extending the deadline is *ex post* optimal. The *ex post* optimality leads to *ex ante* optimality.

In the rest of the analysis, we allow the employer to coordinate the terms of employment specified in the offer with the choice of deadline, that is, employer incorporates the choice of acceptance deadline into the design of a job offer. The design of an offer in our model includes two dimensions, monetary incentives and acceptance deadline. We use the mechanism design approach to study this problem. A mechanism is a menu of job offers targeting the preferred candidate with different outside options (henceforth, "types" of the candidate). When implementing a mechanism, the candidate reports his type, and the employer provides an offer to the candidate based on the report. According to the revelation principle, we focus on *direct incentive feasible mechanisms*, in which the candidate reports his type truthfully. In an incentive feasible mechanism, for sufficiently high types of the candidate, they may not be provided with an offer. For the types of the candidate that the employer would like to recruit, a higher type always receives an offer with a (weakly) longer deadline, but (weakly) less monetary incentives. This feature of the offer design arises from the fact that the time to consider an offer is more valuable to a candidate with a better outside option.

The optimal design of the mechanism depends on the value of the match between the employer and the candidate, and also on the dynamics of the secondary candidate. If the preferred candidate never holds on to an undesirable offer, the optimal mechanism can be implemented as a *bonus-for-early-acceptance (BFEA) mechanism*. Different from a standard mechanism, which is a *menu* of job offers, a BFEA mechanism is a *single* job offer with an *evolving* bonus paid to the candidate for acceptance. Specifically, in a BFEA mechanism, the employer specifies (1) a date when the job offer expires, and (2) a bonus rule describing how the bonus provided to the candidate decreases over time before the expiration date.

We characterize the optimal mechanisms in two cases. In the first case, the secondary candidate never disappears. This case resembles a labor market with abundant supply of qualified job candidates. The payoff of the employer in this market from being rejected by the preferred candidate is not deteriorating significantly over time. In the second case, the secondary candidate may disappear at any time. This case resembles a labor market where there is intensive competition over limited supply of qualified candidates. The payoff of the employer in this market from being rejected by the preferred candidate is deteriorating significantly over time. For the optimal BFEA mechanism in the first case, a salary bonus is provided to the candidate *only* when he accepts the offer immediately, i.e., the salary bonus jumps to zero whenever the candidate postpones his decision. For the optimal BFEA mechanism in the second case, the salary bonus provided to the candidate can be *continuously decreasing* before the expiration date, to create the high pressure for early acceptance.

BFEA mechanisms have been widely adopted by firms in recruitment. However, the BFEA mechanisms adopted in different labor markets differ in their bonus rules. Our finding regarding the optimal mechanisms suggests that the distinctions in the bonus rules are potentially indicative of the different conditions in those labor markets. Roth and Xing (1994) document that the job offers of some big law firms include *signing bonuses* that could only be collected by the potential employees if they accept the offers much earlier before the deadline.[2] According to our finding, it is likely that the labor market of law school graduates around 1994 features an oversupply of good candidates. Lippman and Mamer (2012) point out that some consulting firms make the signing bonuses included in their job offers drop by a certain amount each week until the potential employees make an acceptance decision. Such offers usually have short acceptance deadlines (e.g., three weeks) as well. Neale and Bazerman (1991) describe that when recruiting graduates of management schools, some firms make the salary drop every day before the candidate accepts the offer. For the markets mentioned in these two papers, our finding suggests that the firms may face fierce competition over their potential employees.

The rest of this paper is organized as follows. Section 2 reviews the related literature. Section 3 sets up the model. In Section 4, we consider the case where the employer chooses only the acceptance deadline of a job offer. In Section 5, we allow the employer to vary the monetary incentives and acceptance deadline of a job offer, and study the optimal design of a job offer. Section 6 concludes the paper.

2 Related Literature

This paper is closely related to the literature that studies the acceptance deadlines of offers in search models. In this literature, the offer receiver engages in a continuous-time search for alternative

[2] A signing bonus, which is also known as sign-on bonus, is a lump-sum payment from an employer to its new hire when they sign an employment contract. The signing bonus is a traditional strategic recruitment device for employers to attract talents.

options, and the offer proposer chooses a deadline for the receiver to accept the offer. Different from this paper, the papers in this literature focus on complete information settings, so they do not examine how the proposer screens the receiver using job offer design, and also do not study the impact of deadline re-negotiation. Tang et al. (2009) show that the strategy of the receiver is characterized by a *shortest acceptable deadline*; the receiver accepts the offer if he fails to find a better alternative offer before the deadline that is equal to or longer than the shortest acceptable deadline. When facing uncertainty in the responder's shortest acceptable deadline, the proposer chooses a longer deadline when the uncertainty is larger. Our model setup reduces to that of Tang et al. if we assume out the secondary candidate, which, however, is crucial for our analysis. Hu and Tang (2018) extend the analysis of Tang et al. by (1) allowing the proposer to flexibly adjust the payoff of the receiver from accepting the offer, or the size of the offer, and (2) generalizing the search process of the receiver. They show that under certain conditions, making the search process of the receiver more favorable will induce the proposer to extend an optimal offer with a smaller size and a longer deadline, if the optimal offer is non-exploding. Zorc and Tsetlin (2016) further allow the proposer to engage in a search for alternative options and choose the timing of making an offer to the receiver. They provide the conditions under which an exploding offer is the equilibrium strategy of the proposer. Apart from the papers above, Lippman and Mamer (2012) compare only exploding offers and open offers. They demonstrate that the (exogenous) timing that the proposer makes the offer is crucial for which offer to be optimal. In general, it is indeterminate how the optimal offer type changes with the timing of making the offer.

Since the papers above focus on complete information environments, the bonus-for-early-acceptance mechanisms extensively employed in practice do not arise in these papers. The seminal paper of Armstrong and Zhou (2016) on search deterrence is closest to the current paper in this aspect. Armstrong and Zhou consider a two-period model of consumer search with incomplete information. The optimal selling mechanism characterized in their model is similar in spirit to the optimal offer design in the current paper: the seller endogenously makes it more costly to buy her product later than to buy immediately. However, since there are only two periods in their setting, using only time-contingent prices constrains the ability of the seller to screen the buyer. At optimum, the seller charges a deposit that entitles the buyer to transact later: a higher deposit corresponds to a lower buy-later price. We do not allow the employer to charge the job candidate in the current paper, but by considering a continuous-time model, the employer has great flexibility in designing time-contingent offers to screen the candidate. Moreover, we assume that the proposer has an outside option, the dynamics of which affects the optimal design of the offer.

In our paper, there is no bargaining between the employer and the candidate over how to divide the matching surplus before the acceptance deadline; we focus on the role of an *endogenous* acceptance deadline in deterring a candidate from looking for alternative options. There is a literature on bargaining and negotiation that assumes *exogenous* deadlines for reaching an agreement and investigates how the deadlines affect the timing of agreement and the outcomes of negotiation. Some papers in the literature model negotiation as a war of attrition. Hendricks et al. (1988) analyze a complete-information war of attrition with a deadline. They characterize a set of mixed-strategy equilibria in which the two bargaining parties do not concede at all in a period before the deadline, but concede with positive probabilities at the deadline. Ponsati (1995) obtains a similar result to that of Hendricks et al. under an incomplete information game. Damiano et al. (2012) consider a dynamic collective decision-making problem modeled as a war of attrition. They show that extending the deadline of making the collective decision can facilitate information aggregation. Other papers in the literature model negotiation as an alternating-offer bargaining game. Ma and Manove (1993) explain that the bargainers tend to reach an agreement near the deadline when they can strategically delay their offers and have imperfect control over the offer transmission time. Fershtman and Seidmann (1993) show that if the bargaining parties are sufficiently patient, they delay their agreement until the deadline if they are committed to reject offers that are inferior to the ones that they previously rejected. Yildiz (2004) demonstrates that the certainty of the deadline is crucial for the delay in reaching an agreement; if the deadline is stochastic, an agreement can be reached almost immediately under certain conditions. Unlike three papers above, which impose common deadlines for the bargainers, Sandholm and Vulkan (1999) assume that each bargainer has a private deadline for bargaining, and show that there is a sequential equilibrium in which the players reach their agreement at the first deadline and the player with the longer deadline receives the whole surplus. To our knowledge, Ozyurt (2015) is the only bargaining paper assuming an endogenous deadline. He shows that a bargainer can improve her bargaining position if she controls the deadline, and an agreement between the players can be reached immediately.

The literature on unraveling in labor markets is also related to the current paper. Labor-market unraveling is the phenomenon that employment contracts are signed long before pertinent information about the employment is fully available. It results in inefficient firm-worker matchings and reduces labor mobility (Niederle and Roth, 2003). Starting from Roth and Xing (1994), which documents the challenges faced by various matching markets in avoiding unraveling, the literature has been investigating the reasons why some labor markets unravel. They find that market unraveling can result from instability of matching outcomes (Roth, 1984, 1991; Mongell and Roth, 1991; Roth and Xing, 1994; Kagel and Roth, 2000), risk aversion of market participants (Li and Rosen, 1998; Li and Suen, 2000; Suen, 2000), costs for participating in each round of a dynamic matching market (Damiano et al., 2005), information asymmetry about labor supply (McKinney et al., 2005), similarity of market participants' preferences (Halaburda, 2010), and rigidity of monetary transfers between market participants (Du and Livne, 2016).

Unraveling is sometimes associated with exploding offers. Niederle and Roth (2009) and Pan (2017) show through laboratory experiments and theory, respectively, that exploding offers with binding acceptance can be a factor driving the market to unravel. Though exploding offers can also arise in our model, the nature of the timing problem studied in the current paper is different from the unraveling problem. In the models of unraveling, both employers and candidates prefer early matching in equilibrium. In our model, the two parties have heterogeneous preferences over the time of matching, and the deadline and monetary incentives are adopted by the employer to manage the conflicting interests.

3 Setup

Two qualified job candidates are available for an employer A to fill an open slot, with one being preferred and the other being secondary. The payoffs of A from hiring the preferred and the secondary candidates are respectively v and $\underline{v}$, with $v > \underline{v} \geq 0$, so A would like to recruit the preferred candidate whenever possible. Independent of his identify, the payoff of a candidate from being hired by A is $r \in (0, 1]$, which consists of the direct payoff included in a standard job offer of A and various indirect benefits from working with A. The values of r, v, and $\underline{v}$ are public information.

An offer of A to a candidate will not for sure be accepted. To capture the candidates' different likelihood of accepting A's offer, we assume that the secondary candidate always accepts A's offer once receiving it if he is still available. The preferred candidate, however, may reject A's offer because of a possible outside option, which is an offer that he *may* receive from some competing employer. The payoff $\bar{r} \in [0, 1]$ from the outside option is *privately* known to the candidate and has distribution H over $[0, 1]$ and continuously differentiable density h.

The outside option of the preferred candidate is not guaranteed; the candidate does not know *whether* and *when* the competing offer will arrive. It is common knowledge that the preferred candidate receives the competing offer with probability $p_0 \in (0, 1)$ (with probability $(1 - p_0)$, he will never receive the competing offer), and conditional on that he will receive the offer, the offer arrives with constant rate $\lambda > 0$ at each time point $t \geq 0$, namely the arrival time of the competing offer is distributed exponentially with PDF and CDF, respectively,

$$f(t) = \lambda e^{-\lambda t}, \qquad F(t) = 1 - e^{-\lambda t}. \tag{1}$$

The parameter λ captures the labor market condition. A higher λ indicates a more competitive market: the competitors of A in the market tend to make faster recruitment decisions. Both A and the candidates do not discount the future.

When extending an offer to the preferred candidate, A may provide additional benefits, such as salary bonuses and other fringe benefits, beyond those included in a standard offer and give the candidate some time $\bar{t} \geq 0$ to make the acceptance decision. We use $\tau \geq 0$ to denote the monetary equivalent of the additional benefits. When τ is included in the offer, the payoffs of the employer and the preferred candidate from being matched with each other become $v - \tau$ and $r + \tau$, respectively. We call $\bar{t}$ the *acceptance deadline* of the job offer. Upon reaching the acceptance deadline, the offer will be withdrawn if the candidate does not accept it.

The choice of acceptance deadline affects A's chance of hiring the preferred candidate. If the offer is an exploding offer, which has $\bar{t} = 0$, then the candidate accepts it only if

$$\bar{r} \cdot p_0 \leq r + \tau, \tag{2}$$

the left-hand side (LHS) of which is the expected payoff from rejecting A's offer, and the right-hand side (RHS) is the payoff from accepting A's offer. If the offer is a non-exploding offer with $\bar{t} > 0$, then the candidate with an outside option $\bar{r} > r+\tau$ will take the time to wait for the outside option. In the case that he receives the competing offer before $\bar{t}$, he rejects A's offer. In the case that he fails to receive the competing offer before $\bar{t}$, his posterior belief $p_{\bar{t}}$ of receiving the competing offer in the future is

$$p_{\bar{t}} = \frac{(1 - F(\bar{t}))p_0}{(1 - F(\bar{t}))p_0 + (1 - p_0)} < p_0, \tag{3}$$

and he accepts A's offer if and only if

$$\bar{r} \cdot p_{\bar{t}} \leq r + \tau. \tag{4}$$

It is clear that $p_{\bar{t}}$ is decreasing in $\bar{t}$. The comparison between (4) and (2) indicates that setting $\bar{t} > 0$ allows A to recruit candidates with $\bar{r} > (r + \tau)/p_0$. However, the increased probability of recruiting candidates with good market prospects comes with a cost: with a non-exploding offer, the candidates with $\bar{r} \in (r+\tau, (r+\tau)/p_0]$, who accept A's offer for sure when $\bar{t} = 0$, will be recruited with probability less than 1.

The secondary candidate may not be always available. We assume that the secondary candidate becomes unavailable at a constant rate $\delta \geq 0$ at each time point $t \geq 0$. That is, the time t that the secondary candidate becomes unavailable is also distributed exponentially with PDF and CDF, respectively,

$$g(t) = \delta e^{-\delta t}, \qquad G(t) = 1 - e^{-\delta t}. \tag{5}$$

It is obvious that the longer the preferred candidate takes to consider A's offer, the less likely that A matches with the secondary candidate, when the preferred candidate rejects the offer.

In the rest of this paper, we analyze the strategic roles of the acceptance deadline $\bar{t}$ of an offer in two separate recruiting environments. In the first environment, the terms of employment stipulated in the standard offer of A are unchangeable, so A cannot provide additional incentives τ to the preferred candidate. This setting conforms many real world situations, because salaries and other monetary incentives included in job offers are typically determined before recruiting and by considering various factors, such as the income equity with existing employees, the internal policy for the wage level of a specific position, and so on. In the second environment, A can change the monetary incentives included in the offer, along with choosing the acceptance deadline. Equipped with the two strategic recruiting devices, the employer can screen the preferred candidate. We characterize the optimal design of the job offer.

For the convenience of discussion, we sometimes refer the preferred candidate as "the candidate" in the rest of the analysis, as the job offer we are analyzing is just for him; the (non-strategic) secondary candidate always accepts the standard offer when he is approached.

4 Optimal Acceptance Deadline

In this section, we study the optimal acceptance deadline in the environments where the terms of employment stipulated in the employer's offer are invariable, namely τ is restricted to be 0. We first illustrate the trade-offs of the employer in choosing the acceptance deadline and study how the optimal deadline depends on the recruiting environment. Then, we examine how the response of the candidate to the offer shapes the choice of deadline. We conclude this section by showing that allowing for re-negotiating the deadline benefits not only the candidate, but also the employer.

4.1 Information Asymmetry and Optimal Deadline

To begin, we deviate from our setup and consider the simpler setting where the employer also knows the possible outside option $\bar{r}$ of the preferred candidate. Studying this public-information setting serves two purposes: on the one hand, it provides preliminary results that will facilitate analyzing the private-information setting; on the other hand, it helps to clarify how the asymmetry of information changes the employer's incentives in choosing the deadline.

In the public-information setting, the choice of deadline depends on the value of $\bar{r}$. For the convenience of discussion, we define

$$P_N(\bar{t}) = (1 - F(\bar{t}))p_0 + (1 - p_0) \quad \text{and} \quad P_R(\bar{t}) = F(\bar{t})p_0,$$

where $P_N(\bar{t})$ is the probability that the preferred candidate fails to receive the competing offer before the deadline $\bar{t}$, and $P_R(\bar{t}) = 1 - P_N(\bar{t})$ is the probability that the candidate receives the competing offer before $\bar{t}$. (The subscripts N and R denote "Not Receive" and "Receive", respectively.) Let $V_A(\bar{t}|\bar{r})$ denote the expected payoff of A when he chooses deadline $\bar{t}$ for the candidate with $\bar{r}$. It is obvious that for $\bar{r} \le r$, we have $V_A(\bar{t}|\bar{r}) = v$ for any $\bar{t}$, as the candidate always accepts A's offer. For $\bar{r} > r$, the choice of $\bar{t}$ matters. We define $\bar{t}_r(\bar{r}) = \min\{\bar{t} \ge 0 : \bar{r} \cdot p_{\bar{t}} \le r\}$, or equivalently,

$$\bar{t}_r(\bar{r}) = \min\left\{\frac{1}{\lambda} \ln \frac{(\bar{r} - r)p_0}{r(1 - p_0)}, 0\right\}, \tag{6}$$

which is the shortest acceptance deadline that makes the candidate with $\bar{r}$ willing to accept A's offer upon reaching the deadline. Then, we have that for $\bar{r} > r$, $V_A(\bar{t}|\bar{r}) = \underline{v}$ if $\bar{t} < \bar{t}_r(\bar{r})$, and

$$V_A(\bar{t}|\bar{r}) = P_N(\bar{t}) \cdot v + \int_0^{\bar{t}} \frac{dP_R(t)}{dt}(1 - G(t))dt \cdot \underline{v}$$

if $\bar{t} \ge \bar{t}_r(\bar{r})$. Note that $V_A(\bar{t}|\bar{r})$ is decreasing in $\bar{t}$ when $\bar{t} \ge \bar{t}_r(\bar{r})$. For $\bar{r} \le r/p_0$, which is always true when $r \ge p_0$, we have $\bar{t}_r(\bar{r}) = 0$, so A optimally extends an exploding offer to the candidate and gets payoff $V_A(\bar{t}_r(\bar{r}))|\bar{r}) = v$. For $\bar{r} > r/p_0$, A chooses $\bar{t} \in \{0, \bar{t}_r(\bar{r})\}$ and gets expected payoff $\max\{\underline{v}, V_A(\bar{t}_r(\bar{r})|\bar{r})\}$, the result of which depends on the values of $\bar{r}$ and r.[3]

[3]Note that when $\bar{r} > r/p_0$, there is a positive probability $\int_0^{\bar{t}_r(\bar{r})} \frac{dP_R(t)}{dt} G(t)dt > 0$ that the employer fails to recruit anyone to fill the open slot by choosing $\bar{t}_r(\bar{r}) > 0$. Thus, it is possible that $V_A(\bar{t}_r(\bar{r})|\bar{r}) < \underline{v}$.

The proposition below illustrates how the choice of deadline changes with $\bar{r}$ and $r < p_0$. As discussed above, if the preferred candidate has an outside option $\bar{r}$ that is not very attractive, namely $\bar{r} \leq r/p_0$, then A extends an exploding offer to the candidate. If the preferred candidate has a sufficiently attractive outside option, the result depends on the payoff $\underline{v}$ of A from hiring the secondary candidate. Henceforth, we regard $\underline{v}$ and v as the qualities of the secondary and preferred candidates, respectively. When the quality of the secondary candidate is relatively low, in the sense that condition (7) holds, A chooses the deadline $\bar{t}_r(\bar{r}) > 0$ necessary for the preferred candidate to consider the offer, and the deadline is increasing in $\bar{r}$. When the quality of the secondary candidate is high enough so that (7) does not hold, A switches from non-exploding offers to an exploding offer when the preferred candidate is sufficiently hard to recruit due to a high $\bar{r}$.

Proposition 1. *For the recruiting problem with $\bar{r}$ being public information, there exists cutoff $r^P \in [0, p_0)$ for the value of r, with $r^P = 0$ if and only if*

$$\frac{\underline{v}}{v} \leq \frac{1 - p_0}{1 - \frac{\lambda}{\lambda+\delta} p_0}, \tag{7}$$

such that

1. *if $r^P < r < p_0$, then for $\bar{r} \leq r/p_0$, the employer extends to the preferred candidate an exploding offer, while for $\bar{r} > r/p_0$, the employer extends a non-exploding offer with deadline $\bar{t}_r(\bar{r})$;*

2. *if condition (7) is violated and $0 < r < r^P$, then there exists a cutoff $\bar{r}^P(r) \in (r/p_0, 1)$ such that for $\bar{r} \leq r/p_0$ and $\bar{r} > \bar{r}^P(r)$, the employer extends to the candidate an exploding offer, while for $r/p_0 < \bar{r} < \bar{r}^P(r)$, the employer extends a non-exploding offer with deadline $\bar{t}_r(\bar{r})$.*

Now we return to the private-information setting, in which $\bar{r}$ is only observable to the preferred candidate. In this setting, it is without loss to restrict the choice of the optimal deadline to the set $[0, \bar{t}_r^{max}]$, where $\bar{t}_r^{max} = \bar{t}_r(1)$, because a deadline longer than $\bar{t}_r(1)$ offers A no benefit, but reduces the probability of acceptance. In the case $r \geq p_0$, $\bar{t}_r^{max} = 0$, the problem is trivial, so in the rest of our analysis we focus on $r < p_0$.

How the preferred candidate responds to the offer affects A's incentives in choosing the deadline. Note that when the deadline is $\bar{t} < \bar{t}_r^{max}$, the candidate may not accept A's offer even if he fails to receive the competing offer upon reaching the deadline. This is the case if $\bar{r} > \bar{r}(\bar{t})$, where

$$\bar{r}(\bar{t}) = \frac{r}{p_{\bar{t}}} \tag{8}$$

is the maximum type of the candidate that A can recruit with a positive probability when the deadline is $\bar{t}$. For all candidates with $\bar{r} > \bar{r}(\bar{t})$, we assume, as a benchmark, that they will decline A's offer immediately. We discuss the consequence of relaxing this assumption later in this section.

Assumption 1. *If a candidate expects that he will never accept a job offer with a known deadline, he rejects the offer immediately.*

Under the above assumption, the expected payoff of A from choosing deadline $\bar{t}$ is

$$U_A(\bar{t}|r) = H(r)v + (H(\bar{r}(\bar{t})) - H(r))V_A(\bar{t}|\bar{r}(\bar{t})) + (1 - H(\bar{r}(\bar{t})))\underline{v}.$$

The first term of $U_A(\bar{t}|r)$ is the payoff of A when the candidate has $\bar{r} \leq r$. The second term is the expected payoff when $\bar{r} \in (r, \bar{r}(\bar{t})]$, as the candidate with such an $\bar{r}$ accepts A's offer only if he fails to receive the alternative upon reaching the deadline $\bar{t}$. The third term is the expected payoff when $\bar{r} > \bar{r}(\bar{t})$, as under Assumption 1 such a candidate rejects A immediately. One objective of this section is to illustrate the trade-offs of an employer in choosing the deadline when it faces multiple qualified candidates. To this end, we take the derivative of $U_A(\bar{t}|r)$ with respect to $\bar{t}$ and obtain

$$\frac{dU_A(\bar{t}|r)}{d\bar{t}} = \underbrace{\frac{dH(\bar{r}(\bar{t}))}{d\bar{t}}\left(V_A(\bar{t}|\bar{r}(\bar{t})) - \underline{v}\right)}_{\textit{marginal benefit}} - \underbrace{(H(\bar{r}(\bar{t})) - H(r))\frac{dP_R(\bar{t})}{d\bar{t}}\left[v - (1 - G(\bar{t}))\underline{v}\right]}_{\textit{marginal cost}}.$$

The first term of $dU_A(\bar{t}|r)/d\bar{t}$ is the marginal benefit of increasing $\bar{t}$. When A increases the deadline from $\bar{t}$ to $\bar{t}+d\bar{t}$, the chance that it hires the preferred candidate with $\bar{r} \in (\bar{r}(\bar{t}), \bar{r}(\bar{t}+d\bar{t})]$ is raised from 0, which corresponds to payoff $\underline{v}$, to $P_N(\bar{t}+d\bar{t})$, i.e., the candidates with better market prospects are now possible to accept A's offer. The second term is the marginal cost of increasing the deadline: increasing $\bar{t}$ to $\bar{t} + d\bar{t}$ reduces the probability of recruiting the types of the candidate in $(r, \bar{r}(\bar{t})]$, as the increased deadline gives such candidates extra time $d\bar{t}$ to explore their outside options. The term $[v - (1 - G(\bar{t}))\underline{v}]$ is the expected payoff gain of A from successfully recruiting the preferred candidate at deadline $\bar{t}$.

What is the optimal deadline maximizing the employer's expected payoff? In the literature on labor market matching, exploding offers, which require candidates to respond immediately, draw must attention, as they deter candidates from looking for alternative options in the most extreme form. In the result below, we provide a sufficient condition under which an exploding offer will *not* arise. The result depends on how the candidate handles offers. We show in the next subsection that if replacing Assumption 1 by assuming that the candidate always holds on to the offer and makes the acceptance decision until the deadline, then exploding offers and another type of offers with extreme deadlines, open offers, will be observed more often. An open offer in our model has deadline $\bar{t}_r^{max}$, and gives the candidate effectively sufficient time to wait for his outside option.

Observation 1. *Under Assumption 1, the employer never extends an exploding offer to the preferred candidate at optimum if*

$$h(\bar{r}(0)) > \frac{H(\bar{r}(0)) - H(r)}{\bar{r}(0) - r} \cdot p_0, \tag{9}$$

which is satisfied when $H(\bar{r})$ is weakly convex in $\bar{r}$.

To have A *not* extend an exploding offer at optimum, a sufficient condition is $dU_A(\bar{t} = 0|r)/d\bar{t} > 0$. Note that at $\bar{t} = 0$, $V_A(\bar{t}|\bar{r}(\bar{t})) - \underline{v} = v - (1 - G(\bar{t}))\underline{v}$ in the expression of $dU_A(\bar{t}|r)/d\bar{t}$. It is then easy to verify that $dU_A(\bar{t}=0|r)/d\bar{t} > 0$ if and only if condition (9) holds. On the RHS of (9), $(H(\bar{r}(0)) - H(r)/(\bar{r}(0) - r)$ measures the "average slope" of H over the interval $[r, \bar{r}(0)]$. According to the mean value theorem, there exists $\bar{r}' \in (r, \bar{r}(0))$ such that $H(\bar{r}(0)) - H(r) = f(\bar{r}')(\bar{r}(0) - r)$. Thus, we can simplify (9) to $h(\bar{r}(0)) \geq h(\bar{r}') \cdot p_0$, which is clearly satisfied when $H(\bar{r})$ is weakly convex in $\bar{r}$.

The (stochastic) outside options of the employer and the candidate shape the conflicting preferences of the two parties over the timing of making the acceptance decision. In the proposition below, we examine how the changes in outside options affect the choice of acceptance deadline. Note that only the candidates with $\bar{r} > r$ care about the deadline, so we will focus on the types $\bar{r} \in (r, 1]$ in the analysis below. For this reason, we define

$$\hat{H}(\bar{r}) = \frac{H(\bar{r}) - H(r)}{1 - H(r)},$$

which is the conditional CDF of $\bar{r} \in (r, 1]$. We say that

Proposition 2. *Under Assumption 1, for the optimal deadline $\bar{t}^*$ maximizing the expected payoff of the employer, we find that*

1. *$\bar{t}^*$ is weakly decreasing in the outside option $\underline{v}$ of the employer;*
2. *if $\hat{H}'(\bar{r})$ dominates $\hat{H}(\bar{r})$ in terms of the likelihood ratio given $H'(r) = H(r)$ fixed, then $\bar{t}^*$ is weakly higher under $\hat{H}'(\bar{r})$ than under $\hat{H}(\bar{r})$.*

How the value of r affects the choice of deadline is indeterminate.

Example 1. *Suppose Assumption 1 holds. When $H(\bar{r})$ is the uniform distribution over $[0, 1]$, the optimal acceptance deadline that maximizes A's expected payoff is decreasing in r.*

1. *If condition (7) holds, then for $r \in (0, p_0)$, A chooses $\bar{t} = \bar{t}_r^{max}$; for $r > p_0$, A chooses $\bar{t} = 0$.*
2. *If condition (7) does not hold, then there exists $\check{r}$, such that when $r \leq \check{r}$, A chooses*

$$\bar{t} = -\frac{1}{\lambda + \delta} \ln \left(1 - \frac{(1 - p_0)(v - \underline{v})}{p_0 \underline{v}} \frac{(\lambda + \delta)}{\delta} \right) \leq \bar{t}_r^{max}, \tag{10}$$

which holds with equality at $r = \check{r}$; when $r \in (\check{r}, p_0)$, A chooses $\bar{t} = \bar{t}_r^{max}$; when $r > p_0$, A chooses $\bar{t} = 0$.

This example shows that the impact of the candidate's behavior on the choice of deadline depends on r of A and how attractive the secondary candidate is. If the secondary candidate is sufficiently unattractive so that condition (7) is satisfied, relaxing Assumption 1 has no impact on the strategy of A. That is, A still chooses $\bar{t}_r^{max}$ to recruit every type of the candidate with a positive probability. If the secondary candidate has a sufficiently high ability violating condition (7), then the strategies of A with $\bar{r} < \check{r}$ are different from those in Example 2. Specifically, for A

with $r \leq \hat{r}$, it switches from an exploding offer to a non-exploding offer with deadline (10), namely it increases the acceptance deadline; for A with $r \in (\hat{r}, \check{r})$, it switches from the longest deadline $\bar{t}_r^{max}$ to a shorter deadline (10). Therefore, the behavior of the candidate in dealing with A's offer has heterogeneous effects across A with different r.

4.2 Candidate Behavior and Optimal Deadline

In the analysis above, we adopted Assumption 1 that the preferred candidate will never immediately decline a non-exploding offer of A, even he will never accept it. If we assume instead that before reaching the acceptance deadline, the candidate declines A's offer only if he receives a strictly better alternative offer, how the acceptance deadline will change?[4] We study this question in this subsection.

The expected payoff of A from choosing deadline $\bar{t}$ thus can be expressed as

$$\begin{aligned}\check{U}_A(\bar{t}|r) = H(r)v + (H(\bar{r}(\bar{t})) - H(r))V_A(\bar{t}|\bar{r}(\bar{t}))\\ + (1 - H(\bar{r}(\bar{t}))) \left[P_N(\bar{t})(1 - G(\bar{t}))\underline{v} + \int_0^{\bar{t}} \frac{dP_R(t)}{dt}(1 - G(t))\underline{v} \right].\end{aligned} \quad (11)$$

The first term of $\check{U}_A(\bar{t}|r)$ is the payoff of A when the candidate has outside option $\bar{r} \leq r$. The second term is the expected payoff when the candidate has $\bar{r} \in (r, \bar{r}(\bar{t})]$ and he fails to receive the alternative offer before $\bar{t}$. The third term is the expected payoff when the candidate has $\bar{r} > \bar{r}(\bar{t})$, in which case he may fails to receive the alternative offer before $\bar{t}$, or receives the alternative offer during $[0, \bar{t}]$.

The only difference between $\check{U}_A(\bar{t}|r)$ and $U_A(\bar{t}|r)$ lies in the expected payoff of A from the types of the candidate in $(\bar{r}(\bar{t}), 1]$. When $\bar{t} = 0$ or $\bar{t} = \bar{t}_r^{max}$, $\check{U}_A(\bar{t}|r) = U_A(\bar{t}|r)$. But when $\bar{t} \in (0, \bar{t}_r^{max})$, $\check{U}_A(\bar{t}|r) < U_A(\bar{t}|r)$. Therefore, compared with the analysis under Assumption 1, employer A under the new assumption is more likely to choose an extreme type of offer, namely an exploding offer or an open offer. We provide a sufficient condition under which the employer chooses extreme deadlines for its job offer.

To proceed, we define $P(\bar{t}|r)$ as the probability that A successfully recruits the preferred candidate when the deadline is $\bar{t}$, and have

$$P(\bar{t}|r) = H(r) + (H(\bar{r}(\bar{t})) - H(r))P_N(\bar{t}).$$

Then we can obtain

$$\frac{d\check{U}_A(\bar{t}|r)}{d\bar{t}} = \frac{dP(\bar{t}|r)}{d\bar{t}} \left[v - (1 - G(\bar{t}))\underline{v} \right] - (1 - H(\bar{r}(\bar{t})))P_N(\bar{t})g(\bar{t})\underline{v}. \quad (12)$$

[4]This behavior of the candidate in the assumption can arise when there is a small probability ϵ that the candidate receives a rejection from the competing employer. The current model corresponds to the limit case that $\epsilon \to 0$.

Proposition 3. *If $P(\bar{t}|r)$ is convex in $\bar{t}$, then the optimal acceptance deadline is either* 0 *or* $\bar{t}_r^{max}$. *That is, the employer provides the preferred candidate with either an exploding offer or an open offer.*

The above proposition provides a simple sufficient condition, which is independent of the outside option of A, for the optimality of exploding offers and open offers. Example 2 below shows that this condition is less restrictive than it seems. However, one point worth mentioning is that the optimality of exploding offers and open offers not only relies on the primitives like $P(\bar{t}|r)$, but also on how the job candidates respond to job offers. We can show that when Assumption 1 holds, some types of A choose interior deadliness even if $P(\bar{t}|r)$ is convex in $\bar{t}$.

We use the example that $H(\bar{r})$ is uniform to illustrate Proposition 3, and also show how the choice of deadline changes with the ranking and outside option of the employer. When $H(\bar{r})$ is uniform, $P(\bar{t}|r)$ is convex in $\bar{t}$. Therefore, according to Proposition 3, the optimal deadline of A is either $\bar{t} = 0$, namely A gives an exploding offer to the candidate, or $\bar{t} = \bar{t}_r^{max}$, i.e., A specifies a long enough acceptance deadline such that even the candidate with the best market prospects will accept its offer when he fails to get the offer from B.

Example 2. *When $H(\bar{r})$ is the uniform distribution over* $[0,1]$*, the optimal acceptance deadline that maximizes A's expected payoff weakly is decreasing in $r \in (0,1]$ if condition* (7) *holds. In this case, when $r \in (0, p_0)$, A chooses $\bar{t} = \bar{t}_1(r)$, which is decreasing in r, and when $r \geq p_0$, A chooses $\bar{t} = 0$. However, if condition* (7) *is not satisfied, then the optimal acceptance deadline is not monotonic in r. Specifically, there exists $\hat{r} \in (0, p_0)$, which satisfies*

$$\check{U}_A(0|\hat{r}) = \check{U}_A(\bar{t}_1(\hat{r})|\hat{r}), \tag{13}$$

such that when $r \in (0, \hat{r}] \cup [p_0, 1]$, A chooses to send an exploding offer, while when $r \in (\hat{r}, p_0)$, A chooses $\bar{t} = \bar{t}_r^{max}$.

The results in Example 2 are consistent with intuitions. If condition (7) holds, i.e., the employer does not have a fairly good outside option, so it will give the top candidate a long deadline to consider the offer. If condition (7) fails, employers with different attractiveness (i.e., r) differ in their incentives to choose a short deadline. Less attractive employers are more likely to choose a short deadline when facing a good secondary candidate. One should note that though in this case, both unattractive employers with $r \in (0, \hat{r}]$ and highly attractive employers with $r \in [p_0, 1]$ send exploding offers to the top candidate, their reasons are completely different. A highly attractive employer knows that it is so appealing to the top candidate that the candidate would give up the chance of getting the best alternative offer to accept its exploding offer. However, for an unattractive employer, its unattractiveness makes the cost of recruiting the top candidate prohibitively high when the secondary candidate is very good, so it would rather give the top candidate no time to consider the offer.

4.3 Deadline Renegotiation

As we discussed in the introduction, a primary role of the acceptance deadline of a job offer is to deter the candidate from indefinitely postponing his acceptance decision. Whether the employer commits itself to a firm deadline potentially affects the performance of the deterrence strategy. Until now we have been implicitly assuming that employer A is fully committed to its acceptance deadline, so the preferred candidate cannot negotiate with A for a deadline extension. Open offers are immune to negotiation, as the deadline is long enough for every type of the top candidate to consider the offer. Exploding offers are usually suggestive of no possibility of negotiation. Interior deadlines, however, often leave room for the two parties to revise the initial deadline. Should the employer allow for re-negotiation over the deadline? In the subsection below, we answer this question and provide a main result of this paper.

In practice, it is very common that employers extend their job offer deadlines. Many online career platforms (AngelList, The Muse, and JobHero, for example) and human resource practitioners also advise job seekers to ask for deadline extensions when they need more time to consider their job offers. Allowing re-negotiation over the acceptance deadline undoubtedly benefits the potential employees, but it is not clear how it affects the payoffs of the employers.

In this subsection, we show that the allowing the preferred candidate to ask for a deadline extension benefits the employer. This is because in our model, the employer learns new information over time. Upon reaching the deadline, based on the information that it learns, extending the deadline may be *ex post* optimal. The *ex post* optimality is shown to be *ex ante* optimal. This result relies on the existence of a stochastic outside option for the employer, which is the secondary candidate that can disappear over time. With no secondary candidate or a secondary candidate that never disappears, the possibility of revising its deadline offers no benefit to the employer. With a secondary candidate that can disappear, the employer has incentive to extends its initial deadline if it learns that the secondary candidate is gone before the deadline.

Before stating the result formally, we describe the timing of the game with deadline re-negotiation. At the beginning of the game, same as in the benchmark model, employer A chooses a deadline $\bar{t} \geq 0$ for its offer to the preferred candidate. Before reaching the deadline, employer A learns privately whether the secondary candidate is gone or not. Upon reaching the deadline, if the preferred candidate has not accepted an alternative offer, he can choose either to accept A's offer or to ask A for a deadline extension. The game ends if the candidate accepts A's offer. If the candidate asks for a deadline extension, then A decides whether to decline the request or to extend the deadline by $\Delta\bar{t} > 0$.[5] If the request for a deadline extension is declined, the candidate has to make the acceptance decision immediately. If the deadline is extended, the candidate makes the final decision at time $\bar{t} + \Delta\bar{t}$. We use sequential equilibrium as the solution concept.

Proposition 4. *Allowing the preferred candidate to ask for a deadline extension upon reaching the initial deadline makes employer A (weakly) better off.*

[5] In our setup, choosing $\Delta\bar{t} = 0$ is equivalently to declining the candidate's request for a deadline extension.

The idea of the proof is as follows. Let $\bar{t}^* < \bar{t}_r^{max}$ denote the deadline that maximizes $U_A(\bar{t}|r)$. In the full-commitment case, with deadline $\bar{t}^*$, all the types of the candidate in $[0, \bar{r}(\bar{t}^*)]$ accept A's offer when they fail to get an alternative offer before the deadline, and all the types in $(\bar{r}(\bar{t}^*), 1]$ reject A's offer immediately. If we fix the deadline $\bar{t}^*$ and allow for re-negotiation, A may get worse off. This is because when the deadline is extendable, all the types in $[0, \bar{r}(\bar{t}^*)]$ and a fraction of the types in $(\bar{r}(\bar{t}^*), 1]$ will hold on to the offer and ask for a deadline extension at the end of the initial deadline, which may result in a lower expected payoff to A.

But remember that A has the freedom to choose the initial deadline. The employer can ensure itself the optimal payoff $U_A(\bar{t}^*|r)$ in the full-commitment case by setting 0 deadline at the beginning of the re-negotiation game, and then extend the deadline optimally to $\bar{t}^*$. If there is a re-negotiation equilibrium in which the employer gets an expected payoff that is strictly higher than $U_A(\bar{t}^*|r)$, then it can play according to that equilibrium. Therefore, A gets (weakly) better offer from allowing deadline re-negotiation, even if the candidate does not always hold on to the offer until the deadline. How the candidate responds to an offer, however, does not affect A's incentive to allow for re-negotiation over the deadline. We prove this case in the appendix.

5 Optimal Job Offer Design

In this section, we allow A to discriminate different types of the candidate using two strategic recruiting devices, acceptance deadline and monetary incentives. In our model, different types of the preferred candidate value time differently: a longer acceptance deadline is more valuable to a candidate with a better outside option. Employer A can take advantage of this single-crossing property to customize its offers to different types of the preferred candidate and achieve a higher expected payoff than setting a uniform acceptance deadline. We study the optimal design of job offers that maximizes A's expected payoff from recruitment. Same as in the previous section, we focus on the non-trivial case that $r < p_0$.

We model the screening problem as a mechanism design game, with A having full commitment power. The timing of the game is as follows. At the beginning of the game, A proposes a mechanism $(M, (\bar{t}, \tau))$, in which M is the set of messages that the candidate can report to A, and $\bar{t} : M \to R_+$ and $\tau : M \to R_+$ are respectively the *deadline rule* and *transfer rule*. After observing the proposed mechanism, the candidate chooses a message m in M and reports it to A. Upon receiving the report m of the candidate, A sends a job offer $(\bar{t}(m), \tau(m))$ to the candidate, with $\bar{t}(m)$ being the acceptance deadline of the offer and $\tau(m)$ being the transfer from A to the candidate included in the offer. Then, the candidate decides whether to accept or decline the offer before $\bar{t}(m)$. Note that different from standard mechanism design problems, τ is required to be non-negative, namely we do not allow the employer to charge the candidate.

The revelation principle applies in this environment. Thus, in the rest of the analysis we focus on *direct incentive feasible mechanisms*. A mechanism is *direct* if the message space M of the mechanism is just the set including all possible values of $\bar{r}$, i.e., $M = [0, 1]$. A direct mechanism is *incentive feasible* in our model if the deadline rule and transfer rule $(\bar{t}, \tau)$ satisfy the feasibility constraints

$$\text{(F)} \qquad \bar{t}(\bar{r}) \geq 0 \quad \text{and} \quad \tau(\bar{r}) \geq 0, \qquad \text{for any } \bar{r},$$

and the candidate report his outside option, or type, truthfully.

An important feature of the mechanism design problem is that it involves not only hidden information, but also hidden actions, as A has no control over the acceptance decision of the candidate after it extends the offer. Thus, when designing the mechanism, A should take into account the response of the candidate.

We first formulate the incentive compatibility constraints. Let $U_C(\bar{r}; \bar{t}, \tau)$ denote the expected payoff that a candidate of type $\bar{r}$ obtains from truthfully reporting his type under mechanism $(\bar{t}, \tau)$ and optimally responding to the offer provided by A. When reporting his type truthfully, the type-$\bar{r}$ candidate receives offer $(\bar{t}(\bar{r}), \tau(\bar{r}))$ from A. The candidate holds the offer until he receives an alternative offer from the competing employer B or until the deadline $\bar{t}(\bar{r})$, depending on which comes earlier. If the offer from B arrives before the deadline, which happens with probability $P_R(\bar{t}(\bar{r}))$, the candidate compares A's offer, which gives him payoff $r + \tau(\bar{r})$, with the competing offer, which gives him payoff $\bar{r}$. He accepts the one with the higher payoff and gets payoff $\max\{r + \tau(\bar{r}), \bar{r}\}$. If the offer from B fails to arrive before the deadline, which happens with probability $P_N(\bar{t}(\bar{r}))$, the candidate decides whether to accept A's offer or to keep waiting for the competing offer at the end of the deadline. Thus, the payoff of the candidate when he fails to receive the competing offer until the deadline is $\max\{r + \tau(\bar{r}), \bar{r}p_{\bar{t}(\bar{r})}\}$, where $\bar{r}p_{\bar{t}(\bar{r})}$ is the expected payoff of the candidate from waiting for the competing offer. Therefore, the expected payoff $U_C(\bar{r}; \bar{t}, \tau)$ of the candidate is

$$U_C(\bar{r}; \bar{t}, \tau) = \max\{r + \tau(\bar{r}), \bar{r}p_{\bar{t}(\bar{r})}\} \cdot P_N(\bar{t}(\bar{r})) + \max\{r + \tau(\bar{r}), \bar{r}\} \cdot P_R(\bar{t}(\bar{r})). \tag{14}$$

Since the candidate always can choose to wait for the competing offer, it is easy to verify that

$$U_C(\bar{r}; \bar{t}, \tau) \geq \bar{r}p_0. \tag{15}$$

To formulate the incentive compatibility constraints, we define $U_C(\bar{r}, \bar{r}'; \bar{t}, \tau)$ as the expected payoff that the candidate obtains under mechanism $(\bar{t}, \tau)$ if his true type is $\bar{r}$, but he reports type $\bar{r}'$, so

$$U_C(\bar{r}, \bar{r}'; \bar{t}, \tau) = \max\{r + \tau(\bar{r}'), \bar{r}p_{\bar{t}(\bar{r}')}\} P_N(\bar{t}(\bar{r}')) + \max\{r + \tau(\bar{r}'), \bar{r}\} P_R(\bar{t}(\bar{r}')). \tag{16}$$

Incentive compatibility requires that the candidate have no incentive to misreport his type, that is,

$$\text{(IC)} \qquad U_C(\bar{r}; \bar{t}, \tau) \geq U_C(\bar{r}, \bar{r}'; \bar{t}, \tau), \quad \text{for any } \bar{r}, \bar{r}'.$$

An incentive feasible mechanism is a mechanism that satisfies constraints (F) and (IC).

The (IC) constraints cannot be reformulated using the standard techniques of mechanism design. We show in the following analysis how to simplify the constraints. Depending on the responses of the candidate to the offer provided by A, we can reformulate $U_C(\bar{r};\bar{t},\tau)$ into three different cases:

Case 1: $r+\tau(\bar{r})\geq\bar{r}$

In this case, the type-$\bar{r}$ candidate is provided with an offer that is more attractive than his outside option, so the candidate accepts A's offer regardless of whether he receives the alternative offer or not. Therefore, we have $\max\{r+\tau(\bar{r}),\bar{r}\}=r+\tau(\bar{r})$ and $\max\{r+\tau(\bar{r}),\bar{r}p_{\bar{t}(\bar{r})}\}=r+\tau(\bar{r})$, and $U_C(\bar{r};\bar{t},\tau)$ is reduced to

$$U_C(\bar{r};\bar{t},\tau)=r+\tau(\bar{r}). \tag{17}$$

Note that the deadline $\bar{t}(\bar{r})$ does not affect the payoff of the candidate, because no matter what happens before the deadline, he prefers to accept the offer provided by A. We use $s_1(\bar{t},\tau)$ to denote the set of candidate types that always accept A's offer under mechanism $(\bar{t},\tau)$ when they report truthfully, i.e., $s_1(\bar{t},\tau)=\{\bar{r}:r+\tau(\bar{r})\geq\bar{r}\}$.

Case 2: $\bar{r}>r+\tau(\bar{r})\geq\bar{r}p_{\bar{t}(\bar{r})}$

In this case, A's offer to the type-$\bar{r}$ candidate is not as attractive as his alternative offer, i.e., $\bar{r}>r+\tau(\bar{r})$, so the candidate rejects A's offer once he receives the alternative offer. But A gives the candidate enough time to consider its offer, i.e., $r+\tau(\bar{r})\geq\bar{r}p_{\bar{t}(\bar{r})}$, so the candidate accepts A's offer as long as he receives no alternative offer before the deadline. Therefore, we have $\max\{r+\tau(\bar{r}),\bar{r}\}=\bar{r}$ and $\max\{r+\tau(\bar{r}),\bar{r}p_{\bar{t}(\bar{r})}\}=r+\tau(\bar{r})$, and $U_C(\bar{r};\bar{t},\tau)$ is reduced to

$$U_C(\bar{r};\bar{t},\tau)=(r+\tau(\bar{r}))P_N(\bar{t}(\bar{r}))+\bar{r}P_R(\bar{t}(\bar{r})). \tag{18}$$

We use $s_2(\bar{t},\tau)$ to denote the set of candidate types that fall into the current case under mechanism $(\bar{t},\tau)$, i.e., $s_2(\bar{t},\tau)=\{\bar{r}:\bar{r}>r+\tau(\bar{r})\geq\bar{r}p_{\bar{t}(\bar{r})}\}$.

Case 3: $\bar{r}p_{\bar{t}(\bar{r})}>r+\tau(\bar{r})$

In this case, the mechanism specifies an offer that is less attractive than the candidate's outside option and also does not give the candidate enough time of consideration, so the candidate rejects the offer even if he has no alternative offer upon reaching the deadline. Therefore, we have $\max\{r+\tau(\bar{r}),\bar{r}\}=\bar{r}$ and $\max\{r+\tau(\bar{r}),\bar{r}p_{\bar{t}(\bar{r})}\}=\bar{r}p_{\bar{t}(\bar{r})}$, and $U_C(\bar{r};\bar{t},\tau)$ is simplified to

$$U_C(\bar{r};\bar{t},\tau)=\bar{r}p_0, \tag{19}$$

which is exactly the candidate's *ex ante* expected payoff from his outside option $\bar{r}$. Note that same as in the first case, the payoff of the candidate in this case does not depend on deadline $\bar{t}(\bar{r})$. However, the reason is different. In this case it is because that the acceptance deadline is *too short*, while in the first case it is because $r+\tau(\bar r)$ is sufficiently high. We use $s_3(\bar t,\tau)$ to denote the set of candidate types that always reject A's offer under mechanism $(\bar t,\tau)$, i.e., $s_3(\bar t,\tau)=\{\bar r:\bar r p_{\bar t(\bar r)}>r+\tau(\bar r)\}$.

The design of the mechanism shapes the composition of $s_1(\bar t,\tau)$, $s_2(\bar t,\tau)$, and $s_3(\bar t,\tau)$, which reflect how different types of the candidate respond to A's offer differently. The (IC) constraints impose a monotonic structure on the composition of $s_1(\bar t,\tau)$, $s_2(\bar t,\tau)$, and $s_3(\bar t,\tau)$. Specifically, the types of the candidate included in $s_3(\bar t,\tau)$ (if $s_3(\bar t,\tau)\neq\emptyset$) are higher than any type of the candidate included in $s_2(\bar t,\tau)$. The types of the candidates included in $s_2(\bar t,\tau)$ (if $s_2(\bar t,\tau)\neq\emptyset$) are higher than any type of the candidate included in $s_1(\bar t,\tau)$. The proof of this result is included in the appendix. Given the monotonic structure of $\{s_1(\bar t,\tau),s_2(\bar t,\tau),s_3(\bar t,\tau)\}$, the problem of employer A can be formulated as the following constrained optimization problem:

$$\text{(P)}\quad \max_{(\bar t,\tau)}\int_{s_1(\bar t,\tau)}(v-\tau(\bar r))dH(\bar r)+\int_{s_2(\bar t,\tau)}\left\{(v-\tau(\bar r))P_N(\bar t(\bar r))+\int_0^{\bar t(\bar r)}p_0f(t)(1-G(t))\underline{v}dt\right\}dH(\bar r)$$
$$+\int_{s_3(\bar t,\tau)}\left\{(1-G(\bar t(\bar r)))P_N(\bar t(\bar r))\underline{v}+\int_0^{\bar t(\bar r)}p_0f(t)(1-G(t))\underline{v}dt\right\}dH(\bar r),$$

subject to the (F) and (IC) constraints.

To simplify characterization of the optimal mechanism, we define

$$x(\bar r)=1-F(\bar t(\bar r)),$$

which is the probability that the type-$\bar r$ candidate fails to receive the competing offer before the deadline $\bar t(\bar r)$, conditional on that B is interested in the candidate. The function $x(\bar r)$ is a negative monotone transformation of $\bar t(\bar r)$. Thus, we can use the tuple (x,τ) to represent all mechanisms. For all $\bar r\geq r/p_0$, we further define

$$\underline{\chi}(\bar r)=\frac{r(1-p_0)}{(\bar r-r)p_0},$$

which is smaller than or equal to 1. When reformulating a $(\bar t,\tau)$ mechanism into a (x,τ) mechanism, $\bar t(\bar r)=0$ corresponds to $x(\bar r)=1$, and the deadline $\bar t(\bar r)$ satisfying condition $r=\bar r p_{\bar t(\bar r)}$ for $\bar r\geq r/p_0$ corresponds to $x(\bar r)=\underline{\chi}(\bar r)$. Thus, $\underline{\chi}(\bar r)$ represents the deadline at which the type-$\bar r$ candidate is indifferent between rejecting and accepting A's offer when $\tau(\bar r)=0$.

Proposition 5. *An optimal mechanism (x^*,τ^*) features two cutoff values, $\bar r^*_m$ and $\bar r^*_M$. For the candidate with $\bar r\leq\bar r^*_m$, the optimal mechanism specifies $x^*(\bar r)=1$ and $\tau^*(\bar r)=\bar r^*_m-r$, and the candidate always accepts the offer. For the candidate with $\bar r>\bar r^*_M$, he receives no offer from the employer. The values of $\bar r^*_m$ and $\bar r^*_M$, and the function x^* for $\bar r\in[\bar r^*_m,\bar r^*_M]$ solve the following* *constrained maximization problem:*

$$\text{(P}^*\text{)} \qquad \max_{\{\bar{r}_m, \bar{r}_M, x(\cdot)\}} \int_0^{\bar{r}_m} \Pi_A(1;\bar{r}) dH(\bar{r}) + \int_{\bar{r}_m}^{\bar{r}_M} \Pi_A(x(\bar{r});\bar{r}) dH(\bar{r}) + (1 - H(\bar{r}_M))\underline{v},$$

where $\Pi_A(x(\bar{r});\bar{r})$ *is the expected payoff of the employer when the type of the candidate is* $\bar{r}$*, with*

$$\Pi_A(x(\bar{r});\bar{r}) = (v + r)\left[x(\bar{r})p_0 + (1 - p_0)\right] - J(\bar{r})x(\bar{r})p_0 + \frac{\lambda}{\lambda + \delta}\left[1 - x(\bar{r})^{\frac{(\lambda+\delta)}{\lambda}}\right] p_0 \underline{v},$$

and $J(\bar{r}) = \bar{r} + H(\bar{r})/h(\bar{r})$*, subject to constraints*

(C1*) $x(\bar{r})$ *is decreasing in* $\bar{r}$ *and bounded by* 1 *and* $\underline{\chi}(\bar{r}_M)$*, with* $\bar{r}_M \geq r/p_0$*;*

(C2*) $\bar{r}_m(1 - p_0) = \int_{\bar{r}_m}^{\bar{r}_M} x(\bar{r})p_0 d\bar{r}.$

The function $J(\bar{r})$ is the candidate's *virtual valuation* of an outside option $\bar{r}$. The term $H(\bar{r})/h(\bar{r})$ in $J(\bar{r})$ captures the information rent to the type-$\bar{r}$ candidate. Following the literature, we impose the following regularity condition on $J(\bar{r})$.

Assumption 2. *The function* $J(\bar{r})$ *is strictly increasing in* $\bar{r} \in [0, 1]$.

In the function $\Pi_A(x(\bar{r});\bar{r})$, the first two terms are the payoffs of A from successfully recruiting a type-$\bar{r}$ candidate, and can be rewritten in terms of $\bar{t}(\bar{r})$ as

$$(v + r)\left[x(\bar{r})p_0 + (1 - p_0)\right] - J(\bar{r})x(\bar{r})p_0 = \left[(v + r) - J(\bar{r})p_{\bar{t}(\bar{r})}\right] P_N(\bar{t}(\bar{r})), \tag{20}$$

in which $[(v+r) - J(\bar{r})p_{\bar{t}(\bar{r})}]$ is the *virtual matching surplus* of A from recruiting the candidate using deadline $\bar{t}(\bar{r})$. Setting a longer deadline $\bar{t}(\bar{r})$ for the type-$\bar{r}$ candidate increases the virtual surplus of A, but decreases $P_N(\bar{t}(\bar{r}))$, which is the probability that the candidate accepts A's offer. It is not obvious how the choice of $\bar{t}(\bar{r})$ affects the expected payoff of A from recruiting the candidate. Defining the function $x(\bar{r})$ makes the effect transparent: the expected payoff (20) is linear in $x(\bar{r})$.

With the presence of the secondary candidate, the choice of $x(\bar{r})$ also affects the probability that the employer recruits the secondary candidate. Thus, the third term of $\Pi_A(x(\bar{r});\bar{r})$ appears. In the two subsections below, we first characterize the optimal mechanism for the environment $\delta = 0$, that is, the secondary candidate of A is always available. Then, we consider the general environment with $\delta > 0$.

5.1 Deterministic Secondary Candidate

We first study the simple environment in which A faces no competition over the secondary candidate, that is, $\delta = 0$. In this environment, the acceptance deadline of the offer to the preferred candidate does not affect A's expected payoff from recruiting the secondary candidate. The analysis in this case also applies to the environment where $\delta > 0$ and $\underline{v} = 0$, i.e., A has no secondary candidate.

When $\delta = 0$, the expected payoff $\Pi_A(x(\bar{r}); \bar{r})$ of A is reduced to an affine function of $x(\bar{r})$, with the marginal payoff being

$$\frac{d\Pi_A(x(\bar{r}); \bar{r})}{dx(\bar{r})} = [(v + r) - J(\bar{r}) - \underline{v}]\, p_0,$$

which depends on $\bar{r}$ through $J(\bar{r})$. To facilitate the analysis, we ignore constraint (C2*) and the cutoff $\bar{r}_m$, and consider the following relaxed maximization problem:

$$\max_{\{\bar{r}_M, x(\cdot)\}} \int_0^{\bar{r}_M} \Pi_A(x(\bar{r}); \bar{r}) dH(\bar{r}) + (1 - H(\bar{r}_M))\underline{v}, \quad \text{subject to constraint (C1}^*\text{)}.$$

If $(v + r) - \underline{v} > J(\bar{r})$, the marginal payoff $d\Pi_A(x(\bar{r}); \bar{r})/dx(\bar{r})$ is positive; it is optimal for the employer to choose $x(\bar{r}) = 1$, which is the upper bound of $x(\bar{r})$ imposed by constraint (C1*). If $(v + r) - \underline{v} < J(\bar{r})$, the marginal payoff is negative; the employer optimally chooses $x(\bar{r}) = \underline{\chi}(\bar{r}_M)$, which is the lower bound of $x(\bar{r})$ imposed by constraint (C1*). Given Assumption 2, choosing $x(\bar{r})$ according to the marginal payoff ensures that $x(\bar{r})$ satisfies the monotonicity condition of (C1*).

The problem now is to pin point the optimal $\bar{r}_M$. Increasing $\bar{r}_M$, on the one hand, increases $\Pi_A(x(\bar{r}_M); \bar{r})$ for any $\bar{r}$ with $(v+r)-\underline{v} < J(\bar{r})$, but on the other hand, may decrease $\Pi_A(x(\bar{r}_M); \bar{r}_M)$ and make it lower than $\underline{v}$, which is the payoff of A from not recruiting the candidate. The optimal $\bar{r}_M$ is set to balance these two effects.

When taking into account constraint (C2*) and $\bar{r}_m$ in the original problem, the optimal mechanism differs from the solution to the relaxed problem above if and only if the marginal payoff $d\Pi_A(x(\bar{r}); \bar{r})/dx(\bar{r}) \leq 0$ for $\bar{r} \leq r$. We describe the optimal mechanisms formally in the proposition below.

Proposition 6. *The optimal mechanism (x^*, τ^*) of A depends on v, r, and $\underline{v}$. When $\delta = 0$, namely the secondary candidate never disappears,*

1. *if $(v + r) - \underline{v} \geq J(1)$, then $x^*(\bar{r}) = 1$ and $\tau^*(\bar{r}) = p_0 - r$ for all $\bar{r}$;*
2. *if there exists $\hat{r}^* \in (r, 1)$ that satisfies $(v + r) - \underline{v} = J(\hat{r}^*)$, then for $\bar{r} \leq \hat{r}^*$, $x^*(\bar{r}) = 1$ and $\tau^*(\bar{r}) = (\hat{r}^* - r)(1 - \underline{\chi}(\bar{r}^*_M))p_0$; for $\bar{r} \in (\hat{r}^*, \bar{r}^*_M]$, $x^*(\bar{r}) = \underline{\chi}(\bar{r}^*_M)$ and $\tau^*(\bar{r}) = 0$, where*

$$\bar{r}^*_M \in \arg \max_{\bar{r}_M \geq \max\{r/p_0, \hat{r}^*\}} \int_{\hat{r}^*}^{\bar{r}_M} \Pi_A(x(\bar{r}_M); \bar{r}) dH(\bar{r}) + (1 - H(\bar{r}_M))\underline{v}; \tag{21}$$

3. *if $(v + r) - \underline{v} \leq J(r)$, then $\tau^*(\bar{r}) = 0$ for all $\bar{r}$, $x^*(\bar{r}) = 0$ for $\bar{r} \in [0, r]$, and $x^*(\bar{r}) = \underline{\chi}(\bar{r}^*_M)$ for $\bar{r} \in (r, \bar{r}^*_M]$, where*

$$\bar{r}^*_M \in \arg \max_{\bar{r}_M \geq r/p_0} \int_{r}^{\bar{r}_M} \Pi_A(x(\bar{r}_M); \bar{r}) dH(\bar{r}) + (1 - H(\bar{r}_M))\underline{v}. \tag{22}$$

This proposition shows that the optimal mechanism depends on the total value $(v + r)$ of the match between A and the preferred candidate and the (virtual) value $(\underline{v} + J(\bar{r}))$ of their outside options. When their matching value is sufficiently large, compared with the values of their outside options, A provides the candidate with an exploding offer featuring transfer $\tau^*(\bar{r}) = p_0 - r$, regardless of the type of the candidate. All types of the candidate will accept the offer. When their matching value is sufficiently small, A extends an exploding offer with no transfer to the types of the candidate in $[0, r]$ and a non-exploding offer with no transfer but a deadline

$$\bar{t} = -\frac{1}{\lambda} \ln \underline{\chi}(\bar{r}_M^*), \tag{23}$$

to the types of the candidate in $(r, \bar{r}_M^*]$. The non-exploding offer makes the type $\bar{r} = \bar{r}_M^*$ candidate indifferent between accepting and rejecting A's offer when he fails to receive the competing offer upon reaching the deadline. The optimal mechanism in this case is essentially equivalent to an offer with a uniform deadline (23).

When $(v + r)$ is intermediate, for the types of the candidate in $(\hat{r}^*, \bar{r}_M^*]$, A chooses a non-exploding offer with no transfer but the deadline (23). To the types of the candidate in $[0, \hat{r}^*]$, A extends an exploding offer with a positive transfer that makes the type-$\hat{r}^*$ candidate indifferent between reporting his type truthfully and mimicking the higher types.

Following the revelation principle, we have been formulating a mechanism as a menu of job offers, from which the candidate chooses one that he prefers. The relationship between the deadlines and transfers of any incentive feasible mechanism (Lemma 3) suggests an indirect approach of implementing the optimal mechanisms. In the new approach, instead of proposing a menu of job offers, A extends to the candidate *one* offer that features (1) a date $\bar{t}$ when the offer expires, and (2) a bonus rule $\boldsymbol{\beta} : [0, \bar{t}] \to R_+$ that specifies the transfer (bonus) that the candidate will receive if he accepts the offer at $t \in [0, \bar{t}]$. When using the new approach to implement an optimal direct mechanism, $\boldsymbol{\beta}(t)$ must be decreasing in t, due to incentive compatibility. We name the indirect mechanisms $(\bar{t}, \boldsymbol{\beta})$ with the bonus being decreasing overtime as *bonus-for-early-acceptance (BFEA) mechanisms.*

Corollary 1. *When $\delta = 0$, the optimal mechanisms (x^*, τ^*) can be implemented using the following bonus-for-early-acceptance mechanisms $(\bar{t}^*, \boldsymbol{\beta}^*)$:*

1. *if $(v + r) - \underline{v} \geq J(1)$, then $\bar{t}^* = 0$ and $\boldsymbol{\beta}^*(0) = p_0 - r$, which means that the offer expires immediately if it is not accepted;*

2. *if there exists $\hat{r}^* \in (r, 1)$ that satisfies $(v + r) - \underline{v} = J(\hat{r}^*)$, then $\bar{t}^* = -(1/\lambda) \ln \underline{\chi}(\bar{r}_M^*)$, and*

$$\boldsymbol{\beta}^*(0) = (\hat{r}^* - r) \left[1 - \underline{\chi}(\bar{r}_M^*)\right] p_0 \quad \textit{and} \quad \boldsymbol{\beta}^*(t) = 0 \textit{ for all } t \in (0, \bar{t}^*],$$

   where $\bar{r}_M^*$ satisfies (21);

3. *if $(v + r) - \underline{v} \leq J(r)$, then $\bar{t}^* = -(1/\lambda) \ln \underline{\chi}(\bar{r}_M^*)$ and $\boldsymbol{\beta}^*(t) = 0$ for all $t \in [0, \bar{t}^*]$, where $\bar{r}_M^*$ satisfies (22).*

In the two extreme cases, $(v+r)-\underline{v} \geq J(1)$ and $(v+r)-\underline{v} \leq J(r)$, the bonuses provided to the candidate are constant. In the second case of the corollary, if $\bar{r}_M^* \neq r/p_0$, which is true when $\hat{r}^* > r/p_0$, the candidate receives a strictly positive bonus if and only if he accepts the offer immediately. Learning from Proposition 6, we know that the types of the candidate that accept the offer immediately are the ones with low outside options; the types of the candidate with high outside options choose to postpone their decisions and receive no bonus when they accept the offer.

Formulating an optimal mechanism as a BFEA mechanism has a few advantages. First, the relationship between the transfer and acceptance deadline are transparent in a BFEA mechanism. In particular, the bonus rule $\boldsymbol{\beta}$ gives a direct representation of how the salary bonus decreases over time. Second, it becomes clear that the optimal mechanisms are essentially the search-deterring strategies studied in Armstrong and Zhou (2016). By making the salary bonuses decrease over time, the mechanisms introduce endogenous costs for the candidate to wait for alternative options.

The recruitment strategies adopted by the employers in some labor markets, especially entry-level markets, are BFEA mechanisms. Roth and Xing (1994) document that some big law firms include signing bonuses in their job offers, but the bonuses could be collected by the potential employees only if they accept the offers much earlier before the acceptance deadline. This recruitment strategy is very close to the optimal BFEA mechanisms in the current subsection: there is *no continuous decrease* in the salary bonus over time; a salary bonus is only provided for immediate acceptance.

The BFEA mechanisms adopted in other industries for recruiting, however, feature continuously decreasing salary bonuses. Lippman and Mamer (2012) state that some consulting firms make the *signing bonuses* included in their job offers drop by a certain amount each week until the potential employees make an acceptance decision. Such offers usually have short acceptance deadlines (e.g., three weeks) as well. Neale and Bazerman (1991) describe that when recruiting the graduates of management schools, some firms make their salaries drop every day before the candidates accept the offers. In the next subsection, we show that the competition over the secondary candidates is essential for explaining the continuous decrease in the bonuses.

5.2 Stochastic Secondary Candidate

In this subsection, we study the environment with $\delta > 0$ and $\underline{v} > 0$, that is, A faces competition over the secondary candidate. Compared with the subsection above, having $\delta > 0$ does not affect the constraints in Proposition 5, but renders the payoff function $\Pi_A(x(\bar{r});\bar{r})$ in the objective function (P*) no longer an affine function of $x(\bar{r})$.

It is easy to verify that when $\delta > 0$, the payoff function $\Pi_A(x(\bar{r});\bar{r})$ of A from the candidate is concave in $x(\bar{r})$, with the marginal payoff being

$$\frac{d\Pi_A(x(\bar{r});\bar{r})}{dx(\bar{r})} = \left[(v+r) - J(\bar{r}) - x(\bar{r})^{\frac{\delta}{\lambda}}\underline{v}\right] p_0. \tag{24}$$

When there is no constraint on $x(\bar{r})$, the optimal $x(\bar{r})$ maximizing $\Pi_A(x(\bar{r});\bar{r})$ is

$$\hat{x}(\bar{r}) = \begin{cases} \left[\frac{(v+r)-J(\bar{r})}{\underline{v}}\right]^{\frac{\lambda}{\delta}}, & \text{if } (v+r)-J(\bar{r})>0; \\ 0, & \text{if } (v+r)-J(\bar{r})\leq 0. \end{cases} \tag{25}$$

With the constraints (C1*) and (C2*) in Proposition 5, $x^*(\bar{r})$ in the optimal mechanism in general deviates from $\hat{x}(\bar{r})$ defined in (25). However, similar to the environment with $\delta = 0$, when the value of $(v+r)$ is very large or very small, only (C1*) constraint is critical for characterizing $x^*(\bar{r})$. If $(v+r)$ is sufficiently large such that $(v+r)-\underline{v} > J(1)$, which is similar to the first case in Proposition 6, the marginal payoff (24) is positive for all $x(\bar{r}) \leq 1$. The optimal mechanism (x^*, τ^*) has $\bar{r}^*_M = 1$, and $x^*(\bar{r}) = 1$ for all $\bar{r}$. If $(v+r)$ is sufficiently small such that $(v+r) - \underline{\chi}(1)^{\frac{\delta}{\lambda}}\underline{v} \leq J(r)$, which is similar to the third case in Proposition 6, the marginal payoff (24) is negative for all $\bar{r} \in [r,1]$ when $x(\bar{r}) \geq \underline{\chi}(1)$. The optimal mechanism (x^*, τ^*) has $\bar{r}^*_m = r$, and $x^*(\bar{r}) = \underline{\chi}(\bar{r}^*_M)$ for all $\bar{r} \in (\bar{r}^*_m, \bar{r}^*_M]$. If the value of $(v+r)$ is intermediate, the optimal mechanism in general differs from that in the environment with $\delta = 0$ in that the level of transfer is continuously decreasing in $\bar{r}$ for $\bar{r} > \bar{r}^*_m$. Below we present such a result in a case where (C2*) does not constrain the choice of the optimal mechanism. To proceed, we define a function

$$Q(\bar{r}_m, \bar{r}_M) = \bar{r}_m(1-p_0) - \int_{\bar{r}_m}^{\bar{r}_M} \max\{\hat{x}(\bar{r}), \underline{\chi}(\bar{r}_M)\}p_0 d\bar{r}.$$

This function is to indicate that given a pair of cutoffs $\bar{r}_m, \bar{r}_M$, whether (C2*) holds or not when we choose $x(\bar{r}) = \max\{\hat{x}(\bar{r}), \underline{\chi}(\bar{r}_M)\}$.[6] If $Q(\bar{r}_m, \bar{r}_M) = 0$, (C2*) is satisfied. If $Q(\bar{r}_m, \bar{r}_M) \neq 0$, we can possibly make (C2*) satisfied by adjusting $\bar{r}_m$ or $\bar{r}_M$.

Proposition 7. *Suppose $\delta > 0$ and there exists $\hat{r}^* \in (r,1)$ satisfying $(v+r)-\underline{v} = J(\hat{r}^*)$. Let $\hat{r}^*_M$ be a solution to the following problem:*

$$\max_{\bar{r}_M \geq \max\{r/p_0, \hat{r}^*\}} \int_{\hat{r}^*}^{\bar{r}_M} \Pi_A(x(\bar{r});\bar{r})dH(\bar{r}) + (1-H(\bar{r}_M))\underline{v}, \tag{26}$$

subject to $x(\bar{r}) = \max\{\hat{x}(\bar{r}), \underline{\chi}(\bar{r}_M)\}$.

If $Q(\hat{r}^*, \hat{r}^*_M) \geq 0$, then the optimal mechanism (x^*, τ^*) is characterized by $\bar{r}^*_m = \hat{r}^* - Q(\hat{r}^*, \hat{r}^*_M)$ and $\bar{r}^*_M = \hat{r}^*_M$, and for $\bar{r} \leq \hat{r}^*$, $x^*(\bar{r}) = 1$ and $\tau^*(\bar{r}) = \bar{r}^*_m - r$; for $\bar{r} \in (\hat{r}^*, \bar{r}^*_M]$,

$$x^*(\bar{r}) = \max\{\hat{x}(\bar{r}), \underline{\chi}(\bar{r}^*_M)\} \quad \textit{and} \quad \tau^*(\bar{r}) = \frac{\int_{\bar{r}}^{\bar{r}^*_M} x^*(\tilde{r})p_0 d\tilde{r} + \bar{r}x^*(\bar{r})p_0}{x^*(\bar{r})p_0 + (1-p_0)} - r.$$

In the above proposition, due to the condition $Q(\hat{r}^*, \hat{r}^*_M) \geq 0$, constraint (C2*) imposes no effective restriction on choosing the optimal $x(\bar{r})$ for $\bar{r} \leq \bar{r}^*_M$; (C2*) helps to determine the transfer rule τ^* by pinning down $\bar{r}^*_m$. The optimal $x^*(\bar{r})$ is obtained from $\hat{x}(\bar{r})$ by only incorporating the

[6] Note that the payoff function $\Pi_A(x(\bar{r});\bar{r})$ is concave in $x(\bar{r})$. Under the constraint $x(\bar{r}) \geq \underline{\chi}(\bar{r}^*_M)$, when $\hat{x}(\bar{r}) < \underline{\chi}(\bar{r}^*_M)$, $x(\bar{r}) = \underline{\chi}(\bar{r}^*_M)$ maximizes $\Pi_A(x(\bar{r});\bar{r})$.

boundary conditions in (C1*).

In general, the constraint (C2*) restricts the choice of the optimal mechanism. To give an explicit characterization of the optimal design of the job offer in such cases, we consider the environment in which $v + r \geq \underline{v} + \bar{r}$ for all $\bar{r}$, that is, the value of the match between the employer and its preferred candidate is higher than the *nominal* values of their outside options.[7]

Proposition 8. *Suppose $\delta > 0$ and $v + r \geq \underline{v} + \bar{r}$ for all $\bar{r}$. If $H(\bar{r})$ is concave, then there exists μ^* such that in the optimal mechanism (x^*, τ^*) with cutoffs $\bar{r}^*_m, \bar{r}^*_M$, for $\bar{r} \leq \bar{r}^*_M$,*

$$x^*(\bar{r}) = \begin{cases} 1, & \text{if } \bar{r} < \bar{r}^*; \\ \hat{x}(\bar{r};\mu^*), & \text{if } \bar{r}^* < \bar{r} < \bar{r}^{**}; \\ \underline{\chi}(\bar{r}^*_M), & \text{if } \bar{r}^{**} < \bar{r}, \end{cases} \quad \text{where } \hat{x}(\bar{r};\mu^*) = \left[\frac{(v+r) - J(\bar{r}) - \mu^*/h(\bar{r})}{\underline{v}}\right]^{\frac{\lambda}{\delta}}, \tag{27}$$

with $\bar{r}^*, \bar{r}^{**} \in [\bar{r}^*_m, \bar{r}^*_M]$ and $x^*(\bar{r})$ being decreasing in $\bar{r}$, and

$$\tau^*(\bar{r}) = \frac{\int_{\bar{r}}^{\bar{r}^*_M} x^*(\tilde{r}) p_0 d\tilde{r} + \bar{r} x^*(\bar{r}) p_0}{x^*(\bar{r}) p_0 + (1 - p_0)} - r.$$

Similar to Proposition 7, the optimal $x^*(\bar{r})$ in this proposition is continuously decreasing in $\bar{r}$ almost everywhere. The value μ^* is induced by constraint (C2*). As presented in Proposition 7, if (C2*) does not constrain the choice of the optimal mechanism, then $\mu^* = 0$ and $x^*(\bar{r})$ is derived from $\hat{x}(\bar{r})$ by only taking into account the boundaries conditions. If (C2*) constrains the choice of the optimal mechanism, then $\mu^* \neq 0$, and we adjust $\hat{x}(\bar{r})$ using μ^* to derive the optimal $x^*(\bar{r})$.

Same as in the environment with $\delta = 0$, we can transform the optimal direct mechanisms into BFEA mechanisms. The key step of the transformation is to represent the transfer rule $\tau^*(\bar{r})$, which is a function of $\bar{r}$, as a function of $\bar{t}$. Consider the optimal mechanisms in Proposition 8. Given that $x(\bar{r}) = 1 - F(\bar{t}(\bar{r}))$, we know that the optimal deadline rule is $\bar{t}^*(\bar{r}) = -(1/\lambda)\ln x^*(\bar{r})$.[8] For the types of the candidate in $(\bar{r}^*, \bar{r}^{**})$, $\bar{t}^*(\bar{r})$ is strictly increasing. We define

$$\hat{t}(\bar{r}^*) = -\frac{1}{\lambda}\ln \hat{x}(\bar{r}^*;\mu^*),$$

and the inverse function $\bar{r}^{-1} : (\hat{t}(\bar{r}^*), \bar{t}^*(\bar{r}^*_M)] \to (\bar{r}^*, \bar{r}^{**}]$, with

$$t = -\frac{1}{\lambda}\ln \hat{x}(\bar{r}^{-1}(t);\mu^*).$$

Corollary 2. *Suppose $\delta > 0$ and $v + r \geq \underline{v} + \bar{r}$ for all $\bar{r}$. If $H(\bar{r})$ is concave, then the optimal mechanism (x^*, τ^*) with cutoffs $\bar{r}^*, \bar{r}^{**} \in [\bar{r}^*_m, \bar{r}^*_M]$ can be transformed in to the following bonus-for-*

[7] Note that the nominal values $\underline{v} + \bar{r}$ of the outside options are smaller than their virtual values $\underline{v} + J(r)$. Thus, it is possible that even $v + r \geq \underline{v} + \bar{r}$ for all $\bar{r}$, $v + r < \underline{v} + J(r)$.

[8] Regardless of whether $\bar{r}^{**}$ is equal to 0 or not, we always have $x^*(\bar{r}^{**}) = x^*(\bar{r}^*_M)$. This is obvious when $\bar{r}^{**} = \bar{r}^*_M$. When $\bar{r}^{**} < \bar{r}^*_M$, then we have $\hat{x}(\bar{r}^{**};\mu^*) = \underline{\chi}(\bar{r}^*_M)$.

early-acceptance mechanisms $(\bar{t}^*, \boldsymbol{\beta}^*)$: $\bar{t}^* = -(1/\lambda)\ln x^*(\bar{r}_M^*)$, *and*

$$\boldsymbol{\beta}^*(t) = \begin{cases} \bar{r}_m^* - r, & \textit{for } 0 \le t \le \hat{t}(\bar{r}^*); \\ \frac{\int_{\bar{r}^{-1}(t)}^{\bar{t}^*} (1-F(\tilde{t}))p_0 d\bar{r}^{-1}(\tilde{t}) + \bar{r}^{-1}(t)(1-F(t))p_0}{(1-F(t))p_0 + (1-p_0)} - r, & \textit{for } \hat{t}(\bar{r}^*) < t \le \bar{t}^*. \end{cases}$$

In the BFEA mechanism that represents the optimal direct mechanism, it is clear that if the candidate accepts the offer before $\hat{t}(\bar{r}^*)$, he receives bonus $\bar{r}_m^* - r$; if he chooses to accept the offer after $\hat{t}(\bar{r}^*)$, then the bonus is decreasing over time. Note that if $\hat{t}(\bar{r}^*) > 0$, then the BFEA mechanism features a "peaceful period" $(0, \hat{t}(\bar{r}^*)]$, during which the bonus received by the candidate is constantly $\bar{r}_m^* - r$.

Different from the environment with $\delta = 0$, implementing the optimal direct mechanism using a BFEA mechanism in the current environment is *not* always without loss. Consider the scenario that $\bar{r}_M^* < 1$. Under the optimal direct mechanism, if the type of the candidate is in $(\bar{r}_M^*, 1]$, the employer does not extend an offer to the candidate, which allows the employer to get payoff $\underline{v}$ by recruiting the secondary candidate with probability 1. If the employer implements the optimal direct mechanism using a BFEA mechanism and the types of the candidate in $(\bar{r}_M^*, 1]$ may always hold the offer until the expiration date unless they receive an alternative offer, then the employer receives expected payoff

$$\Pi_A^0(x(\bar{r}_M^*)) = x(\bar{r}_M^*)^{\frac{\delta}{\lambda}} \left[x(\bar{r}_M^*)p_0 + (1-p_0)\right] \underline{v} + \frac{\lambda}{\lambda+\delta}\left[1 - x(\bar{r}_M^*)^{\frac{(\lambda+\delta)}{\lambda}}\right] p_0 \underline{v},$$

which is strictly smaller than $\underline{v}$, unless $x(\bar{r}_M^*) = 0$.

The employer can prevent this issue by introducing a communication stage in the BFEA mechanism. That is, A first asks the candidate to report whether his type is below or above $\bar{r}_M^*$. If it is below $\bar{r}_M^*$, then A sends a BEFA offer; if it is above $\bar{r}_M^*$, then A provides no offer to the candidate. It is incentive compatible for the types in $(\bar{r}_M^*, 1]$ to report truthfully. If communication is not possible, and A still wants to use a BFEA mechanism to deter the candidate from delay, then the employer should replace (P*) in Proposition 5 by the following objective function

$$(\text{P}'') \qquad \max_{\{\bar{r}_m, \bar{r}_M, x(\cdot)\}} \int_0^{\bar{r}_m} \Pi_A(1; \bar{r}) dH(\bar{r}) + \int_{\bar{r}_m}^{\bar{r}_M} \Pi_A(x(\bar{r}); \bar{r}) dH(\bar{r}) + (1 - H(\bar{r}_M))\Pi_A^0(x(\bar{r}_M^*)),$$

to characterize the optimal BFEA mechanism.

5.3 Empirical Implication

The differences between the BFEA mechanisms that represent the optimal direct mechanisms in the environment with $\delta > 0$ and in the environment with $\delta = 0$ have important empirical implications. The case with $\delta = 0$ resembles a labor market in which there is an excessive supply of high quality job candidates. An employer that fails to recruit its most preferred candidates can easily find in the market a fairly good alternative candidate to recruit, that is, it has low risk of recruiting no acceptable ones in such a market. The case with $\delta > 0$ represents the opposite labor market condition, i.e., there is limited supply of high quality job candidates and the competition over the good candidates is high. An employer in such a competitive market faces deteriorating outside options.

Remember that when $\delta = 0$, in the optimal BFEA mechanism a salary bonus, if exists, is provided to the candidate only when he accepts the offer immediately. However, when $\delta > 0$, if a salary bonus is included in the optimal BFEA mechanism, the bonus may continuously decrease before the expiration date. We do observe both of these mechanisms in practice. As mentioned in Subsection 5.1, Roth and Xing (1994) provide an example of the optimal BFEA mechanism when $\delta = 0$. In Neale and Bazerman (1991) and Lippman and Mamer (2012), they provide examples of BFEA mechanisms in which the bonuses provided in the job offers drop every week or even everyday. These mechanisms can be taken as the discrete-time version of the optimal BFEA mechanisms in the environment with $\delta > 0$.

The results of this section imply that the types of BFEA offers adopted by the employers in recruitment are informative about the labor market conditions. If one observes that the employers in a market provide offers with fixed signing bonuses that can be collected by candidates in the case of early acceptance, then it is likely that the market has a large pool of good candidates. If one observes that the employers of a market provide offers that are very sensitive to the time that the candidates accept the offer, then it indicates that the employers may face fierce competition over the candidates in the market.

At the end of this section, we give two caveats for understanding the BFEA mechanisms. It is clear that in the optimal BFEA mechanisms, a candidate with a more promising alternative option will accept A's offer later and receive lower bonuses. First, this result does not mean that the more optimistic candidates receive lower payoffs. In fact, as shown in Proposition 9, candidates with better outside options have higher expected payoffs in any incentive feasible mechanism, including the optimal BFEA mechanisms. Second, this result does not imply that a candidate with a higher *quality* tend to receive lower salary bonuses. In our model, the quality of a candidate is public knowledge and is independent of how optimistic the candidate is. When fixing the outside option of a candidate, we can show, using the results of Proposition 6 and Proposition 8, that a candidate with a higher v receives a higher salary bonus.

6 Concluding Remarks

In this paper, we propose a simple model of talent recruiting in which the employer uses the job offer acceptance deadline as a strategic recruiting device. The model is intended to capture the tradeoffs of an employer in extending job offers in the entry-level labor markets with defined hiring cycles, but is also applicable to study sporadic recruitment of seasoned job candidates. When receiving an offer, a job candidate often prefers to postpone his acceptance decision to explore better possible options. The acceptance deadline of the offer plays the role of deterring the candidate from postponing the decision indefinitely. The optimal deterrence can be in the most aggressive form, i.e., exploding offers, or no deterrence at all, i.e., open offers, depending the market conditions. Contrary to intuitions, we find that it is optimal for the employer to keep the deadline re-negotiable, given that the outside option of the employer may get worse over time.

If the monetary incentives included in a job offer are flexible, then the employer can use this traditional recruiting device jointly with the acceptance deadline to screen candidates with different outside options. If a candidate never holds on to an unacceptable offer, the optimal job offer design in this case can be implemented as a bonus-for-early-acceptance mechanism. We characterize the optimal BFEA mechanisms, and show that they are indicative of the labor market conditions: the optimal mechanism features a continuously decreasing bonus rule only if there is competition over the secondary candidate, i.e., the employer has deteriorating outside options.

To make the analysis in this paper tractable, we adopt a partial-equilibrium framework, in which the competition faced by the employer is modeled in a reduced form. This modelling approach allows us to delineate the tradeoffs of an employer in choosing the acceptance deadline and analytically characterize the optimal offer design, but, undoubtedly, ignores the strategic interactions between employers in a market. As a future direction of research, it is interesting to investigate how the employers behave in a general-equilibrium framework. Would all employers rush to request the offer receivers to respond immediately? Is it still optimal for the employers to keep their deadlines soft?

In our model, we also assume away the possibility that new job candidates enter into the market. The consequence of this assumption is that the payoff of the employer from recruiting the outside option is (weakly) decreasing over time; thus, extending an offer immediately is a dominant strategy. With the arrival of new candidates, the employer may prefer to spend some time in searching for better candidates before extending an offer. It is interesting to examine in this environment how the employer chooses his search duration and acceptance deadline when facing competition from other employers in the market.

It is worth mentioning that in this paper, we take a very narrow perspective in examining the impact of the acceptance deadline on a candidate's job choice decisions. In practice, the acceptance deadline may not only affect a candidate's chance of getting an alternative option, but also change the post-hire attitudes of the potential employee, as studied in Lau et al. (2014), the turn-over rate of the organization, and even the future pool of job applicants interested in the organization. A comprehensive examination of the roles of acceptance deadline in recruitment should take these post-hire effects into account.

References

[1] Armstrong, Mark, and Jidong Zhou. 2016. "Search Deterrence." *Review of Economic Studies* 83 (1): 26–57.

[2] Damiano, Ettore, Li, Hao, and Wing Suen. 2005. "Unraveling of Dynamic Sorting." *Review of Economic Studies* 72 (4): 1057–1076.

[3] Damiano, Ettore, Li, Hao, and Wing Suen. 2012. "Optimal Deadlines for Agreements." *Theoretical Economics* 7 (2): 357–393.

[4] Du, Songzi, and Yair Livne. 2016. "Rigidity of Transfers and Unraveling in Matching Markets." Working Paper.

[5] Fershtman, Chaim and Daniel Seidman. 1993. "Deadline Effects and Inefficient Delay in Bargaining with Endogenous Commitment." *Journal of Economic Theory* 60 (2): 306–321.

[6] Halaburda, Hanna. 2010. "Unravelling in Two-Sided Matching Markets and Similarity of Preferences." *Games and Economic Behavior* 69 (2): 365–393.

[7] Hendricks, Ken, Andrew Weiss, and Charles Wilson. 1988. "The War of Attrition in Continuous Time with Complete Information." *International Economic Review* 29 (4): 663–680.

[8] Kagel, John, and Alvin Roth. 2000. "The Dynamics of Reorganization in Matching Markets: A Laboratory Experiment Motivated by a Natural Experiment." *Quarterly Journal of Economics* 115 (1): 201-235.

[9] Lau, Nelson, Yakov Bart, J. Neil Bearden, and Ilia Tsetlin. 2014. "Exploding Offers Can Blow Up in More Than One Way." *Decision Analysis* 11 (3): 171–188.

[10] Li, Hao, and Sherwin Rosen. 1998. "Unraveling in Matching Markets." *American Economic Review* 88 (3): 371–387.

[11] Li, Hao, and Wing Suen. 2000. "Risk Sharing, Sorting, and Early Contracting." *Journal of Political Economy* 108 (5): 1058–1091.

[12] Lippman, Steven, and John Mamer. 2012. "Exploding Offers." *Decision Analysis* 9 (1): 6–21.

[13] Ma, Ching-to A., and Michael Manove. 1993. "Bargaining with Deadlines and Imperfect Player Control." *Econometrica* 61 (6): 1313–1339

[14] McKinney, Nicholas, Muriel Niederle, and Alvin Roth. 2005. "The Collapse of a Medical Labor Clearinghouse (and Why Such Failures are Rare)." *American Economic Review* 95 (3): 878–889.

[15] Mongell, Susan, and Alvin Roth. 1991. "Sorority Rush as a Two-Sided Matching Mechanism." *American Economic Review* 81 (3): 441–464.

[16] Neale, Margaret, and Max Bazerman. 1991. *Cognition and Rationality in Negotiation.* Free Press, New York.

[17] Niederle, Muriel, and Alvin Roth. 2003. "Unraveling Reduces Mobility in a Labor Market: Gastroenterology with and without a Centralized Match." *Journal of Political Economy* 111 (6): 1342–1352.

[18] Niederle, Muriel, and Alvin Roth. 2009. "Market Culture: How Rules Governing Exploding Offers Affect Market Performance." *American Economic Journal: Microeconomics* 1 (2): 199–219.

[19] Özyurt, Selçuk. 2015. " 'Take It or Leave It' Offers: Obstinacy and Endogenous Deadlines in Negotiations." Working Paper.

[20] Pan, Siqi. 2017. "Exploding Offers and Unraveling in Two-Sided Matching Markets." *International Journal of Game Theory* 47 (1): 351–373.

[21] Ponsati, Clara. 1995. "The Deadline Effect: A Theoretical Note." *Economics Letters* 48 (3-4): 281–285.

[22] Roth, Alvin. 1984. "The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory." *Journal of Political Economy* 92 (6): 991–1016.

[23] Roth, Alvin. 1991. "A Natural Experiment in the Organization of Entry-Level Labor Markets: Regional Markets for New Physicians and Surgeons in the United Kingdom." *American Economic Review* 81 (3): 415–440.

[24] Roth, Alvin, Keith Murnighan, and Françise Shoumaker. 1988. "The Deadline Effect in Bargaining: Some Experimental Evidence." *American Economic Review* 78 (4): 806–823.

[25] Roth, Alvin, and Xiaolin Xing. 1994. "Jumping the Gun: Imperfections and Institutions Related to the Timing of Market Transaction." *American Economic Review* 84 (4): 992–1044.

[26] Sandholm, Tuomas, and Nir Vulkan. 1999. "Bargaining with Deadlines." *Proceedings of the Sixteenth National Conference on Artificial Intelligence*: 44–51. American Association for Artificial Intelligence, Menlo Park.

[27] Suen, Wing. 2000. "A Competitive Theory of Equilibrium and Disequilibrium Unraveling in Two-Sided Matching." *RAND Journal of Economics* 31 (1): 101-120.

[28] Tang, Wenjie, Neil Bearden, and Ilia Tsetlin. 2009. "Ultimatum Deadlines." *Management Science* 55 (8):1423–1437.

[29] Yildiz, Muhamet. 2004. "Optimism, Deadline Effect, and Stochastic Deadlines." Unpublished Paper, Massachusetts Institute of Technology.

[30] Zorc, Sasa, and Ilia Tsetlin. 2016. "Be Patient yet Firm: Offer Timing, Deadlines, and the Search for Alternatives." Working Paper.

7 Appendix

Proof of Proposition 1

To begin, we fix the value of r and investigate how the optimal deadline changes with $\bar{r}$. For $\bar{r} \le r/p_0$, the optimal deadline is 0, i.e., the employer extends an exploding offer to the candidate. When $\bar{r} > r/p_0$, A may choose 0 or $\bar{t}_r(\bar{r})$ at optimum. If A chooses 0, then its offer will be rejected by the preferred candidate and it gets payoff $\underline{v}$. If A chooses $\bar{t}_r(\bar{r})$, then it gets expected payoff

$$\begin{aligned} V_A(\bar{t}_r(\bar{r})|\bar{r}) &= P_N(\bar{t}_r(\bar{r})) \cdot v + \int_0^{\bar{t}_r(\bar{r})} \frac{dP_R(t)}{dt}(1 - G(t))dt \cdot \underline{v} \\ &= (1 - p_0)\frac{\bar{r}}{\bar{r} - r}v + \frac{\lambda}{\lambda + \delta}\left\{1 - \left[\frac{(1 - p_0)r}{p_0(\bar{r} - r)}\right]^{\frac{\lambda+\delta}{\lambda}}\right\} p_0\underline{v}. \end{aligned}$$

It is easy to verify that $V_A(\bar{t}_r(\bar{r})|\bar{r})$ is decreasing in $\bar{r}$, so $\min_{\bar{r}}\{V_A(\bar{t}_r(\bar{r})|\bar{r})\} = V_A(\bar{t}_r(1)|1)$. Therefore, if $V_A(\bar{t}_r(1)|1) > \underline{v}$, A chooses $\bar{t}_r(\bar{r})$ for any $\bar{r} > r/p_0$. But if $V_A(\bar{t}_r(1)|1) < \underline{v}$, given that $\lim_{\bar{r}\to r/p_0} V_A(\bar{t}_r(\bar{r})|\bar{r}) = v > \underline{v}$, there exists a cutoff $\bar{r}^P(r) \in (r/p_0, 1)$ satisfying $V_A(\bar{t}_r(\bar{r}^P)|\bar{r}^P) = \underline{v}$, such that for $\bar{r} < \bar{r}^P(r)$, A chooses deadline $\bar{t}_r(\bar{r})$, while for $\bar{r} > \bar{r}^P(r)$, A chooses 0 deadline.

The comparison between $V_A(\bar{t}_r(1)|1)$ and $\underline{v}$ depends on the value of r. It is easy to verify that $V_A(\bar{t}_r(1)|1)$ is increasing in r and

$$\lim_{r\to 0^+} V_A(\bar{t}_r(1)|1) = (1 - p_0)v + \frac{\lambda}{\lambda + \delta}p_0\underline{v} \ge \underline{v}$$

if and only if (7) holds. We define $r^P = \min\{r|\lim_{r\to 0^+} V_A(\bar{t}_r(1)|1) \ge \underline{v}\}$, with $r^P = 0$ if and only if (7) is satisfied. When (7) is violated, for $r < r^P$, we have $V_A(\bar{t}_r(1)|1) < \underline{v}$, then according to our discussion above, there exists a cutoff $\bar{r}^P(r) \in (r/p_0, 1)$ such that A chooses 0 if $\bar{r} > \bar{r}^P(r)$.

Proof of Proposition 3

Suppose that $P(\bar{t}|r)$ is convex in $\bar{t}$, so $dP(\bar{t}|r)/d\bar{t}$ is increasing in $\bar{t}$. We do the proof separately for two different cases: (1) $dP(\bar{t}|r)/d\bar{t} \ge 0$ at $\bar{t} = 0$; (2) $dP(\bar{t}|r)/d\bar{t} < 0$ at $\bar{t} = 0$.

Consider $dP(\bar{t}|r)/d\bar{t} \ge 0$ at $\bar{t} = 0$. Taking the derivative of $dU(\bar{t}|r)/d\bar{t}$ in (12) with respective to $\bar{t}$, we obtain

$$\frac{d^2U(\bar{t}|r)}{d\bar{t}^2} = \frac{d^2P(\bar{t}|r)}{d\bar{t}^2}\left[v - (1 - G(\bar{t}))\underline{v}\right] + \frac{dP(\bar{t}|r)}{d\bar{t}}g(\bar{t})\underline{v} - \frac{d}{d\bar{t}}(1 - H(\bar{r}(\bar{t})))P_N(\bar{t})g(\bar{t})\underline{v}.$$

The last term of $d^2U(\bar{t}|r)/d\bar{t}^2$ is always negative, because $(1 - H(\bar{r}(\bar{t})))$, $P_N(\bar{t})$, and $g(\bar{t})$ are decreasing in $\bar{t}$. Given that $P(\bar{t}|r)$ is convex in $\bar{t}$, the first term is always non-negative. The second term is also non-negative, given that $dP(\bar{t}|r)/d\bar{t} \ge 0$ at $\bar{t} = 0$, which implies $dP(\bar{t}|r)/d\bar{t} \ge 0$ for all $\bar{t}$ given the convexity of $P(\bar{t}|r)$. Therefore, $U(\bar{t}|r)$ is convex in $\bar{t}$ in this case. The optimal deadline is thus either 0 or $\bar{t}_r^{max}$.

Consider $dP(\bar{t}|r)/d\bar{t} < 0$ at $\bar{t} = 0$. We prove by contradiction. Suppose that the optimal deadline $\bar{t}^*$ is interior. Then $\bar{t}^*$ should satisfy $dU(\bar{t}^*|r)/d\bar{t} = 0$, which implies that $dP(\bar{t}^*|r)/d\bar{t} > 0$. If so, however, $d^2U(\bar{t}^*|r)/d\bar{t}^2 > 0$, which means that the second order necessary condition for $\bar{t}^*$ to be a global max is violated. A contradiction. Thus, the optimal deadline must be corner solutions.

Proof of Example 1

For the convenience of analysis, we use $\bar{r}_{max} \in [r/p_0, 1]$ to denote $\bar{r}(\bar{t})$, which is the highest type of the candidate that A can recruit with a positive probability using deadline $\bar{t}$. Note that according to the definition of $\bar{r}(\bar{t})$ in (8), there is a one-to-one relationship between $\bar{r}_{max}$ and an acceptance deadline, with $\bar{r}_{max} = r/p_0$ corresponding to $\bar{t} = 0$ and $\bar{r}_{max} = 1$ corresponding to $\bar{t} = \bar{t}_r^{max}$. Using this one-to-one relationship, we can reformulate $\check{U}_A(\bar{t}|r)$ as a function of $\bar{r}_{max}$ as follows,

$$\begin{aligned}\check{U}_A(\bar{r}_{max}|r) &= rv + (\bar{r}_{max} - r)\left[\chi(\bar{r}_{max}|r)p_0 + (1-p_0)\right] v \\ &\quad + (\bar{r}_{max} - r)\frac{\lambda}{(\lambda+\delta)}\left[1 - \chi(\bar{r}_{max}|r)^{\frac{(\lambda+\delta)}{\lambda}}\right] p_0 \underline{v} + (1-\bar{r}_{max})\underline{v},\end{aligned} \tag{28}$$

where $\chi(\bar{r}_{max}|r) = r(1-p_0)/(\bar{r}_{max} - r)p_0$. The function $\check{U}_A(\bar{r}_{max}|r)$ is concave in $\bar{r}_{max}$ and has derivative

$$\frac{d\check{U}_A(\bar{r}_{max}|r)}{d\bar{r}_{max}} = (1-p_0)(v - \underline{v}) - \frac{\delta}{\lambda+\delta}\left[1 - \chi(\bar{r}_{max}|r)^{\frac{(\lambda+\delta)}{\lambda}}\right] p_0 \underline{v}. \tag{29}$$

Note that $d\check{U}_A(\bar{r}_{max}|r)/d\bar{r}_{max}$ is increasing in $\chi(\bar{r}_{max}|r)$. When fixing r, the value $\chi(\bar{r}_{max}|r)$ is decreasing in $\bar{r}_{max}$ and takes values in $[\chi(1|r), 1]$, and $\chi(1|r)$ is increasing in r, with $\chi(1|r=0) = 0$ and $\chi(1|r = p_0) = 1$.

If condition (7) is satisfied, we have $d\check{U}_A(1|0)/d\bar{r}_{max} \geq 0$. Therefore, given that $d\check{U}_A(\bar{r}_{max}|r)/d\bar{r}_{max}$ is decreasing in $\bar{r}_{max}$ and increasing in r for all $r > 0$, we have

$$\frac{d\check{U}_A(\bar{r}_{max}|r)}{d\bar{r}_{max}} > \frac{d\check{U}_A(1|0)}{d\bar{r}_{max}} \geq 0.$$

Thus, it is optimal for A with any r to choose $\bar{r}_{max} = 1$, which corresponds to the acceptance deadline $\bar{t}_r^{max}$. If condition (7) is not satisfied, given the monotonicity of $\chi(1|r)$ in r and $d\check{U}_A(1|r)/d\bar{r}_{max}$ in r, there exists $\check{r} \in (0, p_0)$ such that $d\check{U}_A(1|\check{r})/d\bar{r}_{max} = 0$. Thus, given the concavity of $\check{U}_A(\bar{r}_{max}|r)$, for all $r > \check{r}$ and all $\bar{r}_{max} \in [r/p_0, 1]$, we have

$$\frac{d\check{U}_A(\bar{r}_{max}|r)}{d\bar{r}_{max}} \geq 0.$$

Therefore, it is optimal for A with ranking $r > \check{r}$ to choose $\bar{t} = \bar{t}_r^{max}$. For $r < \check{r}$, we have $d\check{U}_A(1|r)/d\bar{r}_{max} < 0$ and $d\check{U}_A(r/p_0|r)/d\bar{r}_{max} > 0$. Thus, there exists an optimal $\bar{r}_{max} \in (r/p_0, 1)$

for each r such that $d\check{U}_A(\bar{r}_{max}|r)/d\bar{r}_{max} = 0$, or equivalently,

$$\chi(\bar{r}_{max}|r) = \left[1 - \frac{(1-p_0)(v-\underline{v})}{p_0\underline{v}}\frac{(\lambda+\delta)}{\delta}\right]^{\frac{\lambda}{(\lambda+\delta)}},$$

which is *independent* of r and corresponds to deadline

$$\bar{t} = -\frac{1}{\lambda+\delta}\ln\left(1 - \frac{(1-p_0)(v-\underline{v})}{p_0\underline{v}}\frac{(\lambda+\delta)}{\delta}\right) = \bar{t}_1(\check{r}).$$

Now we show that $\check{r}$ is larger than $\hat{r}$ in Example 2. Note that $\hat{r}$ satisfies $U_A(0|\hat{r}) - U_A(\bar{t}_1(\hat{r})|\hat{r})$. Given that $\check{U}_A(\bar{t}|r)$ satisfies

$$\check{U}_A(0|r) = U_A(0|r) \quad \text{and} \quad \check{U}_A(\bar{t}_r^{max}|r) = U_A(\bar{t}_r^{max}|r),$$

we have that for $\check{U}_A(\bar{r}_{max}|\hat{r})$, $\check{U}_A(\hat{r}/p_0|\hat{r}) - \check{U}_A(1|\hat{r}) = 0$. (Remember that $\bar{r}_{max} = \hat{r}/p_0$ corresponds to $\bar{t}(0)$ and $\bar{r}_{max} = 1$ corresponds to $\bar{t} = \bar{t}_1(\hat{r})$.) Given the concavity of $\check{U}_A(\bar{r}_{max}|\hat{r})$ in $\bar{r}_{max}$, it must be that the optimal $\bar{r}_{max}$ for $\hat{r}$ is in the interior of $[\hat{r}/p_0, 1]$, which implies that the optimal deadline $\bar{t}$ must be in the open interval $(0, \bar{t}_1(\hat{r}))$. Given the definition of $\check{r}$, it is clear that $\hat{r} < \check{r}$.

Proof of Example 2

The proof for the case where $r \geq p_0$ is trivial, as employer A with such a ranking always sends an exploding offer, which allows A to recruit the top candidate with probability 1. Any non-exploding offer cannot outperform the exploding one. Therefore, for the rest of the proof, we focus on the case where $r \in (0, p_0)$. According to the expression of $U_A(\bar{t}|r)$ in (11), we have

$$U_A(0|r) = r \cdot v + (\bar{r}(0) - r) \cdot v + (1 - \bar{r}(0)) \cdot \underline{v}, \tag{30}$$

$$\begin{aligned} U_A(\bar{t}_r^{max}|r) &= r \cdot v + (1-r)[(1 - F(\bar{t}_r^{max}))p_0 + (1-p_0)]v \\ &\quad + (1-r)\int_0^{\bar{t}_r^{max}} f(t)p_0 \cdot (1 - G(t)) \cdot \underline{v}dt. \end{aligned} \tag{31}$$

The choice of the acceptance deadline has no effect on the chance of recruiting any candidate with $\bar{r} \leq r$, but will affect the chance of being matched to the candidates with $\bar{r} > r$, and even the chance of recruiting the secondary candidate. Compared with sending an exploding offer, choosing $\bar{t} = \bar{t}_r^{max}$ reduces the probability that A successfully recruit candidates with intermediate market prospects, i.e., candidates with $\bar{r} \in (r, \bar{r}(0))$, and increases its probability of recruiting candidates with good prospects, i.e., candidates with $\bar{r} > \bar{r}(0)$. The increased probability of recruiting optimistic candidates, however, is associated with another cost: if A fails to be matched with these candidates, A's probability of recruiting its secondary candidate is lower than that under an exploding offer.

To simplify notations, we define $D(r) = U_A(0|r) - U_A(\bar{t}_r^{max}|r)$, and

$$\rho = \frac{\underline{v}}{v} \quad \text{and} \quad \kappa(r) = \frac{r(1-p_0)}{(1-r)p_0}.$$

Using the result that

$$\int_0^{\bar{t}_r^{max}} f(t)p_0(1-G(t))\underline{v}dt = \frac{\lambda}{(\lambda+\delta)}\left[1-\kappa(r)^{\frac{(\lambda+\delta)}{\lambda}}\right]p_0\underline{v},$$

we can simplify $D(r)$ as

$$D(r) = \left\{\eta(1-\kappa(r)) - \frac{\lambda}{(\lambda+\delta)}\left[1-\kappa(r)^{\frac{(\lambda+\delta)}{\lambda}}\right]\right\}\underline{v}(1-r)p_0, \tag{32}$$

where $\eta = [\rho - (1-p_0)]/\rho p_0$. Since we focus on the case where $r \in (0, p_0)$, it is always true that $0 < \kappa(r) < 1$, which implies that $1-\kappa(r) < 1-\kappa(r)^{(\lambda+\delta)/\lambda}$. If $\eta \leq \lambda/(\lambda+\delta)$, i.e.,

$$\rho \leq \frac{1-p_0}{1-\frac{\lambda}{(\lambda+\delta)}p_0}, \tag{33}$$

then $D(r) < 0$, which means that it is always optimal for A to choose deadline $\bar{t}_r^{max}$, which allows A to recruit any type of candidate with a positive probability. However, the value of $\bar{t}_r^{max}$ is decreasing in r, namely A with a higher ranking will choose a shorter deadline for the offer.

We now show that if condition (33) is violated, $D(r) = 0$ has only one solution $\hat{r}$, with $D(r) > 0$ to the left of $\hat{r}$ and $D(r) < 0$ to the right of $\hat{r}$. We define the terms in the large bracket of $D(r)$ as $L(\kappa(r))$, which has the same sign as $D(r)$ for any $r \in (0, p_0)$ and has derivative

$$\frac{dL(\kappa(r))}{d\kappa(r)} = -\eta + \kappa(r)^{\frac{\delta}{\lambda}}.$$

It is clear, according to $dL(\kappa(r))/d\kappa(r)$, that for $\kappa(r) \in (0, \eta^{\lambda/\delta})$, $L(\kappa(r))$ is decreasing in $\kappa(r)$, while for $\kappa(r) \in (\eta^{\lambda/\delta}, 1)$, $L(\kappa(r))$ is increasing in $x(r)$. Since $L(\kappa(r)) > 0$ as $\kappa(r) \to 0^+$ and $L(\kappa(r)) < 0$ as $\kappa(r) \to 1^-$, there exists a unique $\hat{r}$, with $\kappa(\hat{r}) \in (0, \eta^{\lambda/\delta})$, such that $L(\kappa(\hat{r})) = 0$. To the left of $\hat{r}$, $L(\kappa(\hat{r})) > 0$, thus $D(r) > 0$, which implies that A with ranking $r < \hat{r}$ optimally chooses send an exploding offer. To the right of $\hat{r}$, $H(\kappa(\hat{r})) < 0$, thus $D(r) < 0$, which implies that A with ranking $r > \hat{r}$ optimally sets $\bar{t} = \bar{t}_r^{max}$.

Proof of Proposition 4

To prove this proposition, we show that the expected payoff of the employer in an equilibrium of the game with deadline re-negotiation is (weakly) higher than that in the benchmark model. Note that for the preferred candidate with outside option $\bar{r} \leq r$, time for considering the offer of A has no value; for the candidate with $\bar{r} > r$, more consideration time is always desirable, so these types of the candidate request a deadline extension in any equilibrium with deadline extensions. Therefore, without loss, we consider equilibria where the candidate always accepts A's offer upon reaching the initial deadline if $\bar{r} \leq r$, and always requests an extension of the deadline if $\bar{r} > r$.

Suppose that the initial deadline is $\bar{t}$. For expositional purposes, we define $\bar{r}(\Delta\bar{t}|p_{\bar{t}})$ as the highest type of the preferred candidate that accepts A's offer with a positive probability when the posterior belief of receiving an alternative offer at the end of initial deadline is $p_{\bar{t}}$ and the deadline extension is $\Delta\bar{t}$, and

$$P_N(\Delta\bar{t}|p_{\bar{t}}) = (1 - F(\Delta\bar{t}))p_{\bar{t}} + (1 - p_{\bar{t}}).$$

It is clear that $\bar{r}(\Delta\bar{t}|p_{\bar{t}}) = \bar{r}(\Delta\bar{t} + \bar{t})$, given that the candidate with $\bar{r} > r$ always asks for an extension, and $P_N(\Delta\bar{t} + \bar{t}) = P_N(\Delta\bar{t}|p_{\bar{t}})P_N(\bar{t})$. At the end of the initial deadline $\bar{t}$, when the candidate requests a deadline extension, if the secondary candidate is still available, the expected payoff of A from recruiting the types $\bar{r} > r$ of the candidate by choosing $\Delta_1\bar{t}$ is

$$\begin{aligned}\tilde{U}_A^1(\Delta_1\bar{t}|p_{\bar{t}}) &= (H(\bar{r}(\Delta_1\bar{t}|p_{\bar{t}})) - H(r))P_N(\Delta_1\bar{t}|p_{\bar{t}})v \\ &\quad + (1 - H(\bar{r}(\Delta_1\bar{t}|p_{\bar{t}})))P_N(\Delta_1\bar{t}|p_{\bar{t}})(1 - G(\Delta_1\bar{t}))\underline{v} + (1 - H(r))\int_0^{\Delta_1\bar{t}} f(t)p_{\bar{t}}(1 - G(t))\underline{v}dt,\end{aligned}$$

which is different from $U_A(\bar{t}|r)$ by the constant $H(r)v$ if we replace p_0 and $\bar{t}$ by $p_{\bar{t}}$ and $\Delta_1\bar{t}$, respectively. In equilibrium, the employer chooses $\Delta_1\bar{t}$ that maximizes $\tilde{U}_A^1(\Delta_1\bar{t}|p_{\bar{t}})$. Note that $\Delta_1\bar{t} = 0$ is equivalent to rejecting the request of the candidate for a deadline extension.

If the secondary condition becomes unavailable at the end of $\bar{t}$, the expected payoff of A from recruiting the types $\bar{r} > r$ of the candidate by choosing $\Delta_0\bar{t}$ is

$$\tilde{U}_A^0(\Delta_0\bar{t}|p_{\bar{t}}) = (H(\bar{r}(\Delta_0\bar{t}|p_{\bar{t}})) - H(r))P_N(\Delta_0\bar{t}|p_{\bar{t}})v.$$

The employer chooses $\Delta_0\bar{t}$ that maximizes $\tilde{U}_A^0(\Delta_0\bar{t}|p_{\bar{t}})$ in equilibrium. The *ex ante* expected payoff of A under $(\bar{t}, \Delta_1\bar{t}, \Delta_0\bar{t})$ is

$$\begin{aligned}\tilde{U}_A(\bar{t}, \Delta_1\bar{t}, \Delta_0\bar{t}|r) = H(r)v + P_N(\bar{t})&\left[(1 - G(\bar{t}))\tilde{U}_A^1(\Delta_1\bar{t}|p_{\bar{t}}) + G(\bar{t})\tilde{U}_A^0(\Delta_0\bar{t}|p_{\bar{t}})\right] \\ &+ (1 - H(r))\int_0^{\bar{t}} f(t)p_0(1 - G(t))\underline{v}dt.\end{aligned}$$

It is easy to show that the payoff $\tilde{U}_A(\bar{t}, \Delta_1\bar{t}, \Delta_0\bar{t}|r)$ reduces to $U_A(\bar{t}|r)$, the expected payoff of A when it has full commitment power and chooses deadline $\bar{t}$, when we restrict $\Delta_1\bar{t} = \Delta_0\bar{t} = 0$. It is therefore clear that

$$\max_{\bar{t}, \Delta_1\bar{t}, \Delta_0\bar{t}} \tilde{U}_A(\bar{t}, \Delta_1\bar{t}, \Delta_0\bar{t}|r) \geq \max_{\bar{t}} \tilde{U}_A(\bar{t}, 0, 0|r) = \max_{\bar{t}} U_A(\bar{t}|r). \tag{34}$$

The inequality in (35) holds as a strict inequality, i.e., the employer gets strictly better offer when allowing for deadline re-negotiation, if the optimal deadline in the full-commitment case is interior. Suppose that the optimal deadline $\bar{t}^*$ in the full-commitment case is in $(0, \bar{t}_r^{max})$. Then from (12), we can obtain that $dP(\bar{t}^*|r)/d\bar{t} > 0$, which implies that $dP(\Delta\bar{t} + \bar{t}^*|r)/d\Delta\bar{t} > 0$ at $\Delta\bar{t} = 0$. Given that $P_N(\Delta\bar{t} + \bar{t}) = P_N(\Delta\bar{t}|p_{\bar{t}})P_N(\bar{t})$, or equivalently $P_N(\Delta\bar{t}|p_{\bar{t}}) = P_N(\Delta\bar{t} + \bar{t})/P_N(\bar{t})$ for all $\bar{t}$ and $\Delta\bar{t}$, we have at $\bar{t} = \bar{t}^*$ and $\Delta_0\bar{t} = 0$,

$$\frac{d\tilde{U}_A^0(\Delta_0\bar{t}|p_{\bar{t}^*})}{d\Delta_0\bar{t}} = \frac{dP(\Delta_0\bar{t} + \bar{t}^*|r)}{d\Delta_0\bar{t}} \frac{v}{P_N(\bar{t}^*)} > 0.$$

That is, the employer has incentive to extend its deadline $\bar{t}^*$ if the secondary candidate becomes unavailable at the end of the deadline. This implies that (35) holds as a strict inequality, because

$$\max_{\bar{t}, \Delta_1\bar{t}, \Delta_0\bar{t}} \tilde{U}_A(\bar{t}, \Delta_1\bar{t}, \Delta_0\bar{t}|r) \geq \max_{\Delta_0\bar{t}} \tilde{U}_A(\bar{t}^*, 0, \Delta_0\bar{t}|r) > U_A(\bar{t}^*|r). \tag{35}$$

Proof of Lemma 1

To begin, we simplify the (IC) constraints. Define $\hat{U}_C(\bar{r}, \bar{r}'; \bar{t}, \tau)$ as the expected payoff of the candidate if his true type is $\bar{r}$ but he reports $\bar{r}'$ and *commits* himself to accept the offer $(\bar{t}(\bar{r}'), \tau(\bar{r}'))$ provided by A when he receives no alternative offer before the acceptance deadline, so

$$\hat{U}_C(\bar{r}, \bar{r}'; \bar{t}, \tau) = (r + \tau(\bar{r}'))P_N(\bar{t}(\bar{r}')) + \max\{r + \tau(\bar{r}'), \bar{r}\}P_R(\bar{t}(\bar{r}')). \tag{36}$$

It is clear that $U_C(\bar{r}, \bar{r}'; \bar{t}, \tau) = \max\{\bar{r}p_0, \hat{U}_C(\bar{r}, \bar{r}'; \bar{t}, \tau)\}$, so the (IC) constraints can be written as

$$U_C(\bar{r}; \bar{t}, \tau) \geq \max\{\bar{r}p_0, \hat{U}_C(\bar{r}, \bar{r}'; \bar{t}, \tau)\},$$

for any $\bar{r}$, $\bar{r}' \in [0, 1]$. Since $U_C(\bar{r}; \bar{t}, \tau) \geq \bar{r}p_0$ always holds (condition (15)), the (IC) constraints can be reduced to

$$\text{(IC}'\text{)} \qquad U_C(\bar{r}; \bar{t}, \tau) \geq \hat{U}_C(\bar{r}, \bar{r}'; \bar{t}, \tau), \quad \text{for any } \bar{r}, \bar{r}' \in [0, 1].$$

It is clear that $s_1(\bar{t}, \tau)$ is always non-empty: any $\bar{r} \leq r$ is included in $s_1(\bar{t}, \tau)$, given $\tau(\bar{r}) \geq 0$. This implies that for *any* incentive feasible mechanism $(\bar{t}, \tau)$, there exists a $\bar{r}$ such that

$$r + \tau(\bar{r}) > \bar{r}p_{\bar{t}(\bar{r})}, \tag{37}$$

given that $p_{\bar{t}(\bar{r})} \leq p_0$. We now show that for any incentive compatibility of $(\bar{t}, \tau)$, if $\bar{r}$ satisfies (37), then any $\bar{r}' < \bar{r}$ also satisfies (37). From the (IC) constraints for $\bar{r}'$, we have

$$U_C(\bar{r}'; \bar{t}, \tau) - \bar{r}'p_0 \geq \hat{U}_C(\bar{r}', \bar{r}; \bar{t}, \tau) - \bar{r}'p_0, \tag{38}$$

which is

$$\max\{r+\tau(\bar{r}')-\bar{r}'p_{\bar{t}(\bar{r}')},0\}P_N(\bar{t}(\bar{r}'))+\max\{r+\tau(\bar{r}')-\bar{r}',0\}P_R(\bar{t}(\bar{r}'))$$
$$\geq (r+\tau(\bar{r})-\bar{r}'p_{\bar{t}(\bar{r})})P_N(\bar{t}(\bar{r}))+\max\{r+\tau(\bar{r})-\bar{r}',0\}P_R(\bar{t}(\bar{r})). \quad (39)$$

Since $\bar{r}$ satisfies (37) and $\bar{r}'<\bar{r}$, we have $r+\tau(\bar{r})-\bar{r}'p_{\bar{t}(\bar{r})}>0$ on the RHS of (39). Thus, the RHS of (39) is positive. To have (39) hold, there must be

$$r+\tau(\bar{r}')-\bar{r}'p_{\bar{t}(\bar{r}')}>0, \quad (40)$$

namely $\bar{r}'$ satisfies (37).

Define $\bar{r}^{**}_{sup}$ as the supremum of all the types satisfying (37), i.e.,

$$\bar{r}^{**}_{sup}\equiv\sup\{\bar{r}\in[0,1]:r+\tau(\bar{r})>\bar{r}p_{\bar{t}(\bar{r})}\}. \quad (41)$$

The arguments above imply that (37) holds for any $\bar{r}<\bar{r}^{**}_{sup}$. The value of $\bar{r}^{**}_{sup}$ depends on the mechanism. We suppress its dependence on $(\bar{t},\tau)$ to simplify notations.

We need only to prove the second and third scenarios in the lemma, given that the first scenario is trivial. For the convenience of analysis, we start by proving the third scenario of the lemma. In the proof of this scenario, we show that the non-emptiness of $s_3(\bar{t},\tau)$ implies the non-emptiness of $s_2(\bar{t},\tau)$.

Scenario 3:

Suppose that $s_3(\bar{t},\tau)$ is non-empty. First, we show that there exists $\bar{r}_2\leq 1$ such that $s_3(\bar{t},\tau)\subset[\bar{r}_2,1]$ and $U_C(\bar{r}_2;\bar{t},\tau)=\bar{r}_2p_0$. The $\bar{r}^{**}_{sup}$ characterized in (41) is a natural candidate for $\bar{r}_2$. We have proved above that (37) holds for any $\bar{r}<\bar{r}^{**}_{sup}$. Given that $s_3(\bar{t},\tau)$ is non-empty and $s_3(\bar{t},\tau)\cap[0,\bar{r}^{**}_{sup})=\emptyset$, so there must be $s_3(\bar{t},\tau)\subset[\bar{r}^{**}_{sup},1]$.

Next we prove that $r+\tau(\bar{r})\geq\bar{r}p_{\bar{t}(\bar{r})}$ for $\bar{r}=\bar{r}^{**}_{sup}$, which implies that $U_C(\bar{r}^{**}_{sup};\bar{t},\tau)=\bar{r}^{**}_{sup}p_0$. This must be true if $\bar{r}^{**}_{sup}=1$, because otherwise $s_3(\bar{t},\tau)$ is empty. For the case that $\bar{r}^{**}_{sup}<1$, we show that for $\bar{r}>\bar{r}^{**}_{sup}$,

$$r+\tau(\bar{r})<\bar{r}p_{\bar{t}(\bar{r})}. \quad (42)$$

The definition of $\bar{r}^{**}_{sup}$ implies that for $\bar{r}>\bar{r}^{**}_{sup}$, there is $r+\tau(\bar{r})\geq\bar{r}p_{\bar{t}(\bar{r})}$. Suppose that there exists $\bar{r}>\bar{r}^{**}_{sup}$ such that $r+\tau(\bar{r})=\bar{r}p_{\bar{t}(\bar{r})}$. Then the IC constraint requires that for $\bar{r}'\in(\bar{r}^{**}_{sup},\bar{r})$, the inequality (38) must hold. The RHS of (39), which is derived from (38), will be positive, because $r+\tau(\bar{r})-\bar{r}'p_{\bar{t}(\bar{r})}=(\bar{r}-\bar{r}')p_{\bar{t}(\bar{r})}>0$. This implies that the LHS of (38) is positive, so $r+\tau(\bar{r}')-\bar{r}'p_{\bar{t}(\bar{r}')}>0$, which violates the definition of $\bar{r}^{**}_{sup}$. Hence, the inequality (42) must hold for all $\bar{r}>\bar{r}^{**}_{sup}$. If we have $r+\tau(\bar{r}^{**}_{sup})>\bar{r}^{**}_{sup}p_{\bar{t}(\bar{r}^{**}_{sup})}$, then for $\bar{r}=\bar{r}^{**}_{sup}+\epsilon$, with $\epsilon>0$ being sufficiently small, we will have $r+\tau(\bar{r}^{**}_{sup})>\bar{r}p_{\bar{t}(\bar{r}^{**}_{sup})}$; thus, $U_C(\bar{r};\bar{t},\tau)<\hat{U}_C(\bar{r},\bar{r}^{**}_{sup};\bar{t},\tau)$, which violates the IC constraint for $\bar{r}$. Therefore, $r+\tau(\bar{r}^{**}_{sup})\geq\bar{r}^{**}_{sup}p_{\bar{t}(\bar{r}^{**}_{sup})}$, and $U_C(\bar{r}^{**}_{sup};\bar{t},\tau)=\bar{r}^{**}_{sup}p_0$.

Now we show that there exists $\bar{r}_1 < 1$ such that $s_1(\bar{t}, \tau) = [0, \bar{r}_1]$ and $r + \tau(\bar{r}_1) = \bar{r}_1$. We already know that $s_1(\bar{t}, \tau)$ is non-empty and bounded. Define

$$\bar{r}^*_{sup} \equiv \sup\{\bar{r} \in [0, 1] : r + \tau(\bar{r}) \geq \bar{r}\}. \tag{43}$$

It is easy to show that for $\bar{r} < \bar{r}^*_{sup}$, $r + \tau(\bar{r}) \geq \bar{r}$. This is because if $r + \tau(\bar{r}) > \bar{r}$ for some $\bar{r} < \bar{r}^*_{sup}$, then incentive compatibility requires that for all $\bar{r}' \in (\bar{r}, \bar{r}^*_{sup})$, there should be $r + \bar{r}' < \bar{r} < \bar{r}'$. This is a contradiction to the definition of $\bar{r}^*_{sup}$.

We prove by contradiction that $\bar{r}^*_{sup}$ satisfies $r + \tau(\bar{r}^*_{sup}) = \bar{r}^*_{sup}$. Based on the definition of $\bar{r}^*_{sup}$, we have $r + \tau(\bar{r}) < \bar{r}$ and $U_C(\bar{r}; \bar{t}, \tau) < \bar{r}$. Suppose $r + \tau(\bar{r}^*_{sup}) < \bar{r}^*_{sup}$, then we should have $\bar{r}^*_{sup} > U_C(\bar{r}^*_{sup}; \bar{t}, \tau)$. In this case, the type-$\bar{r}^*_{sup}$ candidate gets a higher payoff by mimicking a lower type $\bar{r} > U_C(\bar{r}^*_{sup}; \bar{t}, \tau)$, given that for $\bar{r} < \bar{r}^*_{sup}$, $r + \tau(\bar{r}) \geq \bar{r}$. Suppose $r + \tau(\bar{r}^*_{sup}) > \bar{r}^*_{sup}$, then it must be that $\bar{r}^*_{sup} < 1$, because otherwise $s_3(\bar{t}, \tau) = \emptyset$, a contradiction to our supposition of the current scenario. Given that $\bar{r}^*_{sup} < 1$, for a type $\bar{r} = \bar{r}^*_{sup} + \epsilon$ with $\epsilon > 0$ being sufficiently small, mimicking type $\bar{r}^*_{sup}$ can give this type a higher payoff $r + \tau(\bar{r}^*_{sup}) > \bar{r} > U_C(\bar{r}; \bar{t}, \tau)$. This violates the IC constraints. Therefore, there should be $r + \tau(\bar{r}^*_{sup}) = \bar{r}^*_{sup}$. The type $\bar{r}^*_{sup}$ is the $\bar{r}_1$ in the lemma.

To show that $s_2(\bar{t}, \tau)$ is non-empty, we prove that $\bar{r}_1 < \bar{r}_2$. The definitions of these two cutoff types imply that $\bar{r}_1 \leq \bar{r}_2$. For $\bar{r} > \bar{r}_2$, there should be $U_C(\bar{r}; \bar{t}, \tau) = \bar{r}p_0$. Suppose that $\bar{r}_1 = \bar{r}_2$. Then a type $\bar{r} = \bar{r}_2 + \epsilon$ with $\epsilon > 0$ being sufficiently small can be a higher payoff $\bar{r}_2 > \bar{r}p_0$ by reporting $\bar{r}_2$. Therefore, we must have $\bar{r}_1 < \bar{r}_2$, which implies that $s_2(\bar{t}, \tau)$ is non-empty and $s_2(\bar{t}, \tau) \subset (\bar{r}_1, \bar{r}_2]$.

Scenario 2:

When $s_1(\bar{t}, \tau)$ and $s_2(\bar{t}, \tau)$ are the non-empty sets, it is obvious that $\bar{r}^{**}_{sup} = 1$. The proof above regarding $\bar{r}_1$ directly applies here, so we can find a $\bar{r}_1$ satisfying $r + \tau(\bar{r}_1) = \bar{r}_1$, with $s_1(\bar{t}, \tau) = [0, \bar{r}_1]$ and $s_2(\bar{t}, \tau) = [0, 1] \setminus s_1(\bar{t}, \tau) = (\bar{r}_1, 1]$.

Proof of Proposition 9

"Only If":

All no-redundant-deadline incentive feasible mechanisms satisfy constraint (F). Thus, we only show that the incentive compatibility of such mechanisms implies conditions (Cm), (CM), (C1), (C2), (C3), and (C4).

We first consider no-redundant-deadline incentive feasible mechanisms with $s_3(\bar{t}, \tau) = \emptyset$. If $s_1(\bar{t}, \tau)$ is the only non-empty set, then we have $U_C(\bar{r}; \bar{t}, \tau) = r + \tau(\bar{r}) \geq \bar{r}$ for all $\bar{r}$, so $\bar{r}_m = \bar{r}_M = 1$, and (Cm), (CM) are satisfied. Given the definition of no-redundant-deadline mechanisms, $\bar{t}(\bar{r}) = 0$ for all $\bar{r}$, so conditions (C1) and (C2) are trivially satisfied. The (IC) constraints imply that $\tau(\bar{r}) = \tau(\bar{r}_M)$ for all $\bar{r}$, namely $U_C(\bar{r}; \bar{t}, \tau) = U_C(\bar{r}_M; \bar{t}, \tau) = r + \tau(\bar{r}_M)$. Therefore, condition (C3) is also satisfied by such mechanisms, given $P_R(\bar{t}(\bar{r})) = 0$.

If $s_1(\bar{t},\tau)$ and $s_2(\bar{t},\tau)$ are the non-empty sets, then according to Lemma 1, $\bar{r}_M = 1 \in s_2(\bar{t},\tau)$ and $\bar{r}_m$ is equal to $\bar{r}_1$ defined in (44). (Cm) and (CM) are thus satisfied. We show that conditions (C1), (C2), and (C3) are satisfied, given these two cutoff values. According to the definition of no-redundant-deadline mechanisms, $\bar{t}(\bar{r}) = 0$ for all $\bar{r} \leq \bar{r}_m$, so condition (C1) is satisfied. Consider $\bar{r} > \bar{r}' > \bar{r}_m$. Incentive compatibility requires that $U_C(\bar{r};\bar{t},\tau) \geq \hat{U}_C(\bar{r},\bar{r}';\bar{t},\tau)$, which is defined in (36), and $U_C(\bar{r}';\bar{t},\tau) \geq \hat{U}_C(\bar{r}',\bar{r};\bar{t},\tau)$, which imply

$$0 \leq \Big(U_C(\bar{r};\bar{t},\tau) - \hat{U}_C(\bar{r}',\bar{r};\bar{t},\tau)\Big) - \Big(\hat{U}_C(\bar{r},\bar{r}';\bar{t},\tau) - U_C(\bar{r}';\bar{t},\tau)\Big) = (\bar{r} - \bar{r}')\left(P_R(\bar{t}(\bar{r})) - P_R(\bar{t}(\bar{r}'))\right),$$

so $\bar{t}(\bar{r}) \geq \bar{t}(\bar{r}') \geq 0$. Thus, condition (C2) holds for the mechanism. Now we prove condition (C3). From the (IC) constraints, we have that for all $\bar{r} \leq \bar{r}_m$, $U_C(\bar{r};\bar{t},\tau) = U_C(\bar{r}_m;\bar{t},\tau) = r + \tau(\bar{r}_m)$. Consider $\bar{r} > \bar{r}' \geq 0$. It is easy to verify that $\hat{U}_C(\bar{r},\bar{r}';\bar{t},\tau) = U_C(\bar{r}';\bar{t},\tau) + (\bar{r} - \bar{r}')P_R(\bar{t}(\bar{r}'))$ and $\hat{U}_C(\bar{r}',\bar{r};\bar{t},\tau) = U_C(\bar{r};\bar{t},\tau) - (\bar{r} - \bar{r}')P_R(\bar{t}(\bar{r}))$. Thus, from the (IC) constraints, we obtain

$$(\bar{r} - \bar{r}')P_R(\bar{t}(\bar{r}')) \leq U_C(\bar{r};\bar{t},\tau) - U_C(\bar{r}';\bar{t},\tau) \leq (\bar{r} - \bar{r}')P_R(\bar{t}(\bar{r})).$$

Because $\bar{t}(\bar{r})$ was proved to be monotonically increasing, $P_R(\bar{t}(\bar{r}))$ is Riemann integrable. We have

$$U_C(\bar{r};\bar{t},\tau) - U_C(\bar{r}';\bar{t},\tau) = \int_{\bar{r}}^{\bar{r}'} P_R(\bar{t}(\tilde{r}))d\tilde{r}.$$

This completes the proof of condition (C3).

For mechanisms with $s_3(\bar{t},\tau)$ being non-empty, according to Lemma 1, there are two cutoffs, $\bar{r}_1$ and $\bar{r}_2$, such that $s_1(\bar{t},\tau) = [0,\bar{r}_1]$, $s_2(\bar{t},\tau) \subset (\bar{r}_1,\bar{r}_2]$, and $s_3(\bar{t},\tau) \subset [\bar{r}_2,1]$. Because we assume $\bar{r}_2 \in s_2(\bar{t},\tau)$, we have $\bar{r}_m = \bar{r}_1$ and $\bar{r}_M = \bar{r}_2$. Given the fact that $s_3(\bar{t},\tau)$ is non-empty, $\bar{r}_M < 1$. The proof above for the case where $s_2(\bar{t},\tau) \neq \emptyset$ directly carries over to the current case for $\bar{r} \leq \bar{r}_M$, so conditions (C1), (C2), and (C3) hold. Given that $s_3(\bar{t},\tau) = (\bar{r}_M, 1]$, the definition of no-redundant-deadline mechanisms implies that $\bar{t}(\bar{r}) = 0$ and $U_C(\bar{r};\bar{t},\tau) = \bar{r}p_0$ for any $\bar{r} \in s_3(\bar{t},\tau)$, so condition (C4) holds.

"If":

We first show that for all $\bar{r} < \bar{r}_M$, $U_C(\bar{r};\bar{t},\tau) > \bar{r}p_0$. Suppose not, i.e., $U_C(\bar{r};\bar{t},\tau) = \bar{r}p_0$ for some $\bar{r} < \bar{r}_M$. Then according to condition (C3) and the fact that $U_C(\bar{r}_M;\bar{t},\tau) \geq \bar{r}_M p_0$, we have

$$U_C(\bar{r}_M;\bar{t},\tau) - U_C(\bar{r};\bar{t},\tau) = \int_{\bar{r}}^{\bar{r}_M} P_R(\bar{t}(\tilde{r}))d\tilde{r} \geq (\bar{r}_M - \bar{r})p_0,$$

or equivalently,

$$\int_{\bar{r}}^{\bar{r}_M} \left(P_R(\bar{t}(\tilde{r})) - p_0\right) d\tilde{r} \geq 0.$$

Because $P_R(\bar{t}(\tilde{r})) \leq p_0$, with $P_R(\bar{t}(\tilde{r})) = p_0$ only if $\bar{t}(\tilde{r}) = +\infty$, the inequality above implies that $\bar{t} = +\infty$ for all the types in $(\bar{r}, \bar{r}_M]$, due to the monotonicity of $\bar{t}$ in condition (C2), and also $U_C(\bar{r}_M; \bar{t}, \tau) = \bar{r}_M p_0$, due to condition (C3). However, if $\bar{t}(\bar{r}_M) = +\infty$, there should be $U_C(\bar{r}_M; \bar{t}, \tau) > \bar{r}_M p_0$, a contradiction. Therefore, for all $\bar{r} < \bar{r}_M$, there should be $U_C(\bar{r}; \bar{t}, \tau) > \bar{r} p_0$, which implies that either $U_C(\bar{r}; \bar{t}, \tau) = r + \tau(\bar{r})$ or $U_C(\bar{r}; \bar{t}, \tau) = (r + \tau(\bar{r}))P_N(\bar{t}(\bar{r})) + \bar{r}P_R(\bar{t}(\bar{r}))$.

Case 1: $\bar{r}_M = 1$

To begin, we prove the case where $\bar{r}_M = 1$. In this case, given condition (CM) and the proof above showing $U_C(\bar{r}; \bar{t}, \tau) > \bar{r} p_0$ for all $\bar{r} < \bar{r}_M$, we have $s_3(\bar{t}, \tau) = \emptyset$. Regarding $\bar{r}_m$, because of condition (Cm), we have $s_1(\bar{t}, \tau) = [0, \bar{r}_m]$. According to condition (C1), $\bar{t}(\bar{r}) = 0$ for all $\bar{r} \in s_1(\bar{t}, \tau)$, so the mechanism is a no-redundant-deadline mechanism. We show below that the mechanism is incentive compatible.

Consider two arbitrary types $\bar{r}'' < \bar{r}' \leq \bar{r}_M$. If $\bar{r}' \leq \bar{r}_m$, then for any type $\bar{r} \in [\bar{r}'', \bar{r}']$, condition (C1) implies that $\bar{t} = 0$, and also $U_C(\bar{r}; \bar{t}, \tau) = r + \tau(\bar{r}) \geq \bar{r}$. From condition (C3), we have $U_C(\bar{r}''; \bar{t}, \tau) = U_C(\bar{r}'; \bar{t}, \tau) = r + \tau(\bar{r}')$. That is, the two types $\bar{r}'', \bar{r}'$ have no incentive to mimic each other. The (IC) constraints hold for all types of the candidate in this case given that $\bar{r}'', \bar{r}'$ are arbitrary.

If $\bar{r}' > \bar{r}_m$, then the type-$\bar{r}'$ candidate has expected payoff $U_C(\bar{r}'; \bar{t}, \tau) = (r + \tau(\bar{r}'))P_N(\bar{t}(\bar{r}')) + \bar{r}'P_R(\bar{t}(\bar{r}')) < \bar{r}'$, according to (Cm). Then we have

$$\begin{aligned} U_C(\bar{r}'; \bar{t}, \tau) - \hat{U}_C(\bar{r}', \bar{r}''; \bar{t}, \tau) &= U_C(\bar{r}'; \bar{t}, \tau) - U_C(\bar{r}''; \bar{t}, \tau) - (\bar{r}' - \bar{r}'')P_R(\bar{t}(\bar{r}'')) \\ &= \int_{\bar{r}''}^{\bar{r}'} \left(P_R(\bar{t}(\tilde{r})) - P_R(\bar{t}(\bar{r}''))\right) d\tilde{r} \geq 0, \end{aligned}$$

where the first equality is due to the definition of $\hat{U}_C(\bar{r}', \bar{r}''; \bar{t}, \tau)$ in (36), with $\bar{t}(\bar{r}'') = 0$ if $U_C(\bar{r}''; \bar{t}, \tau) = r + \tau(\bar{r}'')$; the second equality is from condition (C3); and the inequality is from condition (C2). Thus, the type-$\bar{r}'$ candidate has no incentive to report $\bar{r}''$.

For the type-$\bar{r}''$ candidate, his payoff $\hat{U}_C(\bar{r}'', \bar{r}'; \bar{t}, \tau)$ is either (1) $(r+\tau(\bar{r}'))P_N(\bar{t}(\bar{r}'))+\bar{r}''P_R(\bar{t}(\bar{r}'))$ if $r + \tau(\bar{r}') \leq \bar{r}''$ or (2) $r + \tau(\bar{r}')$ if $r + \tau(\bar{r}') > \bar{r}''$. If $r + \tau(\bar{r}') \leq \bar{r}''$, then from conditions (C3) and (C2), we have

$$U_C(\bar{r}''; \bar{t}, \tau) - \hat{U}_C(\bar{r}'', \bar{r}'; \bar{t}, \tau) = \int_{\bar{r}''}^{\bar{r}'} \left(P_R(\bar{t}(\bar{r}')) - P_R(\bar{t}(\tilde{r}))\right) d\tilde{r} \geq 0,$$

so the type-$\bar{r}''$ candidate clearly has no incentive to report $\bar{r}'$.

Suppose $r + \tau(\bar{r}') > \bar{r}''$. Given that $r + \tau(\bar{r}') < \bar{r}'$, which is proved above, there exists $\check{r} \in (\bar{r}'', \bar{r}')$ such that $\check{r} = r + \tau(\bar{r}')$. We show that there must be $U_C(\check{r}; \bar{t}, \tau) \geq \check{r}$. Suppose not, i.e.,

$U_C(\check{r};\bar{t},\tau) < \check{r}$, then from conditions (C3) and (C2), we have

$$\begin{aligned}\int_{\check{r}}^{\bar{r}'} P_R(\bar{t}(\tilde{r}))d\tilde{r} &= U_C(\bar{r}';\bar{t},\tau) - U_C(\check{r};\bar{t},\tau)\\ &= \left[(r+\tau(\bar{r}'))P_N(\bar{t}(\bar{r}')) + \bar{r}'P_R(\bar{t}(\bar{r}'))\right] - U_C(\check{r};\bar{t},\tau)\\ &= \check{r} + (\bar{r}'-\check{r})P_R(\bar{t}(\bar{r}')) - U_C(\check{r};\bar{t},\tau)\\ &> (\bar{r}'-\check{r})P_R(\bar{t}(\bar{r}')),\end{aligned}$$

where the first equality is directly from condition (C3), and the second equality is from the supposition of the current case that $U_C(\bar{r}';\bar{t},\tau) = (r+\tau(\bar{r}'))P_N(\bar{t}(\bar{r}')) + \bar{r}'P_R(\bar{t}(\bar{r}'))$. The third equality is due to that $\check{r} = r+\tau(\bar{r}')$. The inequality, which is obtained using $U_C(\check{r};\bar{t},\tau) < \check{r}$, contradicts condition (C2). Therefore, there must be $U_C(\check{r};\bar{t},\tau) \geq \check{r}$, and $\bar{r}'' < \check{r} < \bar{r}_m$. In this case, conditions (C1) and (C3) jointly imply that $r+\tau(\bar{r}'') = r+\tau(\check{r}) = r+\tau(\bar{r}')$, so the type-$\bar{r}''$ candidate clearly has no incentive to report $\bar{r}'$. Thus, the (IC) constraints are satisfied by the mechanism.

Case 2: $\bar{r}_M < 1$

In this case, given that $\bar{r}_{M;\bar{t},\tau)}$ satisfies (CM), and $U_C(\bar{r};\bar{t},\tau) > \bar{r}p_0$ for all $\bar{r} < \bar{r}_M$, which is proved at the beginning of the proof of this proposition, we have $s_3(\bar{t},\tau) \subset (\bar{r}_M, 1]$. Condition (C4) ensures that all the types in $s_3(\bar{t},\tau)$ have 0 deadline. At the beginning of Case 1, we have shown that $\bar{t}(\bar{r}) = 0$ for $\bar{r} \in s_1(\bar{t},\tau) = [0,\bar{r}_m]$, according to condition (C1). Thus, the mechanism is a no-redundant-deadline mechanism. We show below that the mechanism is incentive compatible.

Using the proof above for the case $\bar{r}_M = 1$, we can show that any two types $\bar{r}'' < \bar{r}' \leq \bar{r}_M$ have no incentive to mimic each other. Hence, we only need to show that (1) any type $\bar{r}' \leq \bar{r}_M$ has no incentive to mimic a type $\bar{r} > \bar{r}_M$, and (2) any type $\bar{r} > \bar{r}_M$ has no incentive to misreport.

For any type $\bar{r}' \leq \bar{r}_M$, if he reports to be a type strictly higher than $\bar{r}_M$, he gets payoff $\bar{r}'p_0$. We have shown above that $U_C(\bar{r}';\bar{t},\tau) \geq \bar{r}'p_0$. Thus, he has no incentive to mimic any type $\bar{r} > \bar{r}_M$.

Now suppose that the type of the candidate is $\bar{r} > \bar{r}_M$. Deviating to any other type in $(\bar{r}_M, 1]$ will given him the same expected payoff $\bar{r}p_0$, thus we only need to show that he has no incentive to mimic any type $\bar{r}' \leq \bar{r}_M$, that is,

$$U_C(\bar{r};\bar{t},\tau) \geq \hat{U}_C(\bar{r},\bar{r}';\bar{t},\tau) = (r+\tau(\bar{r}'))P_N(\bar{t}(\bar{r}')) + \max\{r+\tau(\bar{r}'),\bar{r}\}P_R(\bar{t}(\bar{r}')).$$

If $r+\tau(\bar{r}') \geq \bar{r}$, which implies $r+\tau(\bar{r}') > \bar{r}_M$, then the type $\bar{r}_M$ has incentive to report $\bar{r}'$, a contradiction to our proof in the Case 1 above. Thus, there must be $r+\tau(\bar{r}') < \bar{r}$, i.e.,

$$\hat{U}_C(\bar{r},\bar{r}';\bar{t},\tau) = (r+\tau(\bar{r}'))P_N(\bar{t}(\bar{r}')) + \bar{r}P_R(\bar{t}(\bar{r}')) = U_C(\bar{r}';\bar{t},\tau) + (\bar{r}-\bar{r}')P_R(\bar{t}(\bar{r}')).$$

Using condition 3 and the fact that $U_C(\bar{r};\bar{t},\tau) - \bar{r}p_0$,

$$U_C(\bar{r};\bar{t},\tau) - \hat{U}_C(\bar{r},\bar{r}';\bar{t},\tau) = \bar{r}p_0 - \left[(r+\tau(\bar{r}'))P_N(\bar{t}(\bar{r}')) + \bar{r}P_R(\bar{t}(\bar{r}'))\right]$$
$$= (\bar{r}-\bar{r}_M)\left(p_0 - P_R(\bar{t}(\bar{r}'))\right) + \int_{\bar{r}'}^{\bar{r}_M} \left(P_R(\bar{t}(\tilde{r})) - P_R(\bar{t}(\bar{r}'))\right) d\tilde{r} \geq 0,$$

where the second equality is obtained using condition (C3), and the inequality is due to condition (C2) and $p_0 - P_R(\bar{t}(\bar{r}')) \geq 0$. Thus, the type $\bar{r}$ has no incentive to report $\bar{r}'$.

Combining with constraint (F), the mechanism is thus no-redundant-deadline incentive feasible mechanism.

Proof of Proposition 5

****

Lemma 1. *For any incentive feasible mechanism $(\bar{t},\tau)$, $s_1(\bar{t},\tau)$ is always non-empty, and the non-emptiness of $s_3(\bar{t},\tau)$ implies the non-emptiness of $s_2(\bar{t},\tau)$. The non-empty elements of $\{s_1(\bar{t},\tau), s_2(\bar{t},\tau), s_3(\bar{t},\tau)\}$ form a monotonic partition of the candidate's type space. Specifically,*

1. *if $s_1(\bar{t},\tau)$ is the only non-empty set, then $s_1(\bar{t},\tau) = [0,1]$, i.e., all types of the candidate accept the offer provided by A with probability 1;*
2. *if $s_1(\bar{t},\tau)$ and $s_2(\bar{t},\tau)$ are the non-empty sets, there exists a type $\bar{r}_1$ satisfying*
$$r + \tau(\bar{r}_1) = \bar{r}_1 \tag{44}$$
such that $s_1(\bar{t},\tau) = [0,\bar{r}_1]$, $s_2(\bar{t},\tau) = (\bar{r}_1, 1]$;
3. *if $s_1(\bar{t},\tau)$, $s_2(\bar{t},\tau)$, and $s_3(\bar{t},\tau)$ are all non-empty, then except $\bar{r}_1$ defined above, there exists another type $\bar{r}_2$ satisfying*
$$U_C(\bar{r}_2;\bar{t},\tau) = \bar{r}_2 p_0 \tag{45}$$
such that $s_2(\bar{t},\tau) \subset (\bar{r}_1,\bar{r}_2]$, $s_3(\bar{t},\tau) \subset [\bar{r}_2, 1]$, and $s_2(\bar{t},\tau) \cup s_3(\bar{t},\tau) = (\bar{r}_1, 1]$.[9]

The non-emptiness of $s_1(\bar{t},\tau)$ is obvious, as the candidate with outside option $\bar{r} < r$ would always accept A's offer with probability 1. The monotonic structure of $\{s_1(\bar{t},\tau), s_2(\bar{t},\tau), s_3(\bar{t},\tau)\}$ reflects the monotonic relationship between the type of the candidate and the response of the candidate to the offer provided by A. For any two different types of the candidate, if they are provided the same offer, the lower type will accept the offer with a higher probability, as his outside option is worse than that of the higher type. In an incentive feasible mechanism, A may give the

[9] It is not necessarily true that $r + \tau(\bar{r}_2) = \bar{r}_2 p_0$. If $r + \tau(\bar{r}_2) = \bar{r}_2 p_0$, then result indicates that $\bar{r}_2$ is the highest type of the candidate that accepts A's offer with positive probability. If $r + \tau(\bar{r}_2) \neq \bar{r}_2 p_0$, then $\bar{r}_2$ is the lowest type of the candidate that rejects A's offer for sure.

lower-type candidate an offer that is more attractive, from the lower-type candidate's perspective, than the higher-type candidate's offer to deter the lower-type candidate from mimicking the higher-type candidate, and induce the lower-type to accept the offer. Therefore, the lower-type candidate never accepts A's offer with a lower probability than does the higher type.

To illustrate the intuition more concretely, consider two types $\bar{r}$ and $\bar{r}'$ of the candidate, with $\bar{r} < \bar{r}'$. Under an incentive feasible mechanism $(\bar{t}, \tau)$, if the type-$\bar{r}'$ candidate always accepts the offer $(\bar{t}(\bar{r}'), \tau(\bar{r}'))$ designed for his type and receives payoff $r + \tau(\bar{r}')$, namely $\bar{r}' \in s_1(\bar{t}, \tau)$, then the type-$\bar{r}$ candidate must also belong to $s_1(\bar{t}, \tau)$, i.e., he would also always accept the offer $(\bar{t}(\bar{r}), \tau(\bar{r}))$ designed for his type and get payoff $r + \tau(\bar{r})$, because otherwise, it must be that $r + \tau(\bar{r}) < \bar{r}$, and the type-$\bar{r}$ candidate can receive a higher payoff $r + \tau(\bar{r}')$ by mimicking type-$\bar{r}'$ candidate, given $r + \tau(\bar{r}') \geq \bar{r}' > \bar{r}$. Similar arguments can be extended to other cases.

Note that in the third scenario of Lemma 1, the cutoff type $\bar{r}_2$ can be included in $s_2(\bar{t}, \tau)$ or $s_3(\bar{t}, \tau)$. It is easy to verify that two incentive feasible mechanisms that differ only in the response of the type-$\bar{r}_2$ candidate give employer A and different types of the candidate the same expected payoffs. To simplify discussion in the rest of the paper, we assume without loss that $\bar{r}_2$, if it exists, belongs to $s_2(\bar{t}, \tau)$.

Given the monotonic structure of $\{s_1(\bar{t}, \tau), s_2(\bar{t}, \tau), s_3(\bar{t}, \tau)\}$, the problem of employer A can be formulated as the following constrained optimization problem:

$$\text{(P)} \quad \max_{(\bar{t},\tau)} \int\limits_{s_1(\bar{t},\tau)} (v - \tau(\bar{r}))dH(\bar{r}) + \int\limits_{s_2(\bar{t},\tau)} \left\{ (v - \tau(\bar{r}))P_N(\bar{t}(\bar{r})) + \int_0^{\bar{t}(\bar{r})} p_0 f(t)(1 - G(t))\underline{v}dt \right\} dH(\bar{r})$$
$$+ \int\limits_{s_3(\bar{t},\tau)} \left\{ (1 - G(\bar{t}(\bar{r})))P_N(\bar{t}(\bar{r}))\underline{v} + \int_0^{\bar{t}(\bar{r})} p_0 f(t)(1 - G(t))\underline{v}dt \right\} dH(\bar{r}),$$

subject to the (F) and (IC) constraints.

Solving the employer's problem directly is challenging, given the complexity of the (IC) constraints and the objective function (P). To simplify the problem, we first show in the following lemma that to search for the mechanisms that maximize the employer's expected payoff, we can focus on the set of incentive feasible mechanisms that specify $\bar{t}(\bar{r}) = 0$ for the offers to the types in $s_2(\bar{t}, \tau)$ and $s_3(\bar{t}, \tau)$. We call such incentive feasible mechanisms *no-redundant-deadline (NRD) incentive feasible mechanisms.*

Lemma 2. *For any incentive feasible mechanism $(\bar{t}, \tau)$, there exists an incentive feasible mechanism $(\bar{t}', \tau')$, which is the same as $(\bar{t}, \tau)$, except that $\bar{t}'(\bar{r}) = 0$ for any type $\bar{r} \in s_1(\bar{t}, \tau) \cup s_3(\bar{t}, \tau)$, and gives any type of the candidate the same expected payoffs as $(\bar{t}, \tau)$ and gives A a (weakly) higher payoff than $(\bar{t}, \tau)$.*

The proof of this result is simple. Under any incentive feasible mechanism $(\bar{t}, \tau)$, if the candidate's type is included in $s_1(\bar{t}, \tau) \cup s_3(\bar{t}, \tau)$, reducing the deadline of his offer to 0 does not change his expected payoff, according to (17) and (19), so does not change his incentive to report truthfully. At the same time, because of the reduced deadline, other types of the candidate have less incentive to mimic this type. Therefore, the (IC) constraints still hold. If the candidate's type in $s_3(\bar{t}, \tau)$, we know that he always holds A's offer until the deadline. Decreasing the deadline of such a candidate thus can improve the expected payoff of A by increasing its chance of being matched to the secondary candidate.

The set of NRD incentive feasible mechanism is characterized by (F), (IC), and the NRD constraint that $\bar{t}(\bar{r}) = 0$ for any type $\bar{r} \in s_1(\bar{t}, \tau) \cup s_3(\bar{t}, \tau)$. Based on the monotonic structure of $\{s_1(\bar{t}, \tau), s_2(\bar{t}, \tau), s_3(\bar{t}, \tau)\}$, we can use the standard techniques to reformulate the (IC) and NRD constraints.

Proposition 9. *A mechanism $(\bar{t}, \tau)$ is a no-redundant-deadline incentive feasible mechanism if and only if it satisfies (F), and there exist $\bar{r}_m \leq \bar{r}_M$ satisfying*

(Cm) $\quad r + \tau(\bar{r}) \geq \bar{r}$ *if and only if* $\bar{r} \leq \bar{r}_m$, *and*

(CM) $\quad r + \tau(\bar{r}_M) \geq \bar{r}_M p_{\bar{t}(\bar{r}_M)}$,

such that for $\bar{r} \leq \bar{r}_M$,

(C1) $\quad \bar{t}(\bar{r}) = 0$ *if* $\bar{r} \leq \bar{r}_m$,

(C2) $\quad \bar{t}(\bar{r})$ *is increasing in* $\bar{r}$,

(C3) $\quad U_C(\bar{r}; \bar{t}, \tau) = U_C(\bar{r}_M; \bar{t}, \tau) - \displaystyle\int_{\bar{r}}^{\bar{r}_M} P_R(\bar{t}(\tilde{r})) d\tilde{r}$,

and if $\bar{r}_M < 1$, $U_C(\bar{r}_M; \bar{t}, \tau) = \bar{r}_M p_0$ and for $\bar{r} > \bar{r}_M$,

(C4) $\quad \bar{t}(\bar{r}) = 0$ *and* $U_C(\bar{r}; \bar{t}, \tau) = \bar{r} p_0$.

The cutoff type $\bar{r}_m$ in the above proposition is the highest type of the candidate that accepts A's offer for sure. The cutoff type $\bar{r}_M$ is the highest type of the candidate that employer A will recruit with a positive probability. At optimum, employer A may choose a mechanism with $\bar{r}_M < 1$, i.e., A gives up the chance of recruiting the candidates with sufficiently high outside options. This is because such a candidate requires an offer with attractive terms; if A tries to recruit such candidate using an attractive offer, then A also needs to give the candidates with lower outside options more information rent. In the rest of the analysis, without loss of generality, we say that employer A provides no offer to the types of the candidate in $(\bar{r}_M, 1]$ when $\bar{r}_M < 1$.

The (C2) and (C3) conditions in Proposition 9 are similar to the monotonicity condition and envelope condition in standard mechanism design problems, respectively. In the current model, $\bar{t}(\bar{r})$ determines the probability that the type-$\bar{r}$ candidate $\bar{r}$ receives his outside option: a larger $\bar{t}(\bar{r})$ induces a higher $P_R(\bar{t}(\bar{r}))$, as the candidate has more time to wait for his outside option. The (C2) condition implies that the probability that the candidate gets his outside option is increasing in the value $\bar{r}$ of the outside option, when $\bar{r} \leq \bar{r}_M$. The (C3) condition describes how the expected payoff of the candidate depends on his outside option and the probability $P_R(\bar{t}(\bar{r}))$ for all $\bar{r} \leq \bar{r}_M$. These two conditions imply how the transfer $\tau(\bar{r})$ changes with $\bar{r}$ and relates to $\bar{t}(\bar{r})$.

Corollary 3. *For any no-redundant-deadline incentive feasible mechanism $(\bar{t}, \tau)$, $\tau(\bar{r})$ is decreasing in $\bar{r}$ for $\bar{r} \leq \bar{r}_M$. Therefore, $\tau(\bar{r})$ and $\bar{t}(\bar{r})$ move in the opposite directions when $\bar{r}$ increases over the region $[0, \bar{r}_M]$.*

The reason for this corollary is intuitive. For any two different types $\bar{r}, \bar{r}' \in [0, \bar{r}_M]$, with $\bar{r} < \bar{r}'$ Proposition 9 indicates that $\bar{t}(\bar{r}) \leq \bar{t}(\bar{r}')$. If there is $\tau(\bar{r}) < \tau(\bar{r}')$, then it is obvious that the type-$\bar{r}$ candidate can get strictly better off by reporting that his outside option is $\bar{r}'$.

Why the candidate with a better outside option is provided an offer with a longer deadline and a lower transfer, instead of an offer with a shorter deadline and a higher transfer? This is resulted from a single-crossing property of the candidate's expected payoff: the candidate with a better outside option values time more than does the candidate with a lower outside option. It is easy to show that if A provides a lower-type candidate with an offer with a longer deadline and a lower transfer so that he has no incentive to mimic a higher-type candidate, then a higher-type candidate has incentive to pretend to be a lower-type candidate.

By replacing the (IC) and NRD constraints with the constraints from (Cm) to (C4), we can simplify the optimization problem of the employer. **

For convenience, we define $\Delta U_C(\bar{r}; \bar{t}, \tau) = U_C(\bar{r}; \bar{t}, \tau) - \bar{r}p_0$ as the surplus of the type-$\bar{r}$ candidate when he reports truthfully under mechanism $(\bar{t}, \tau)$. The (C3) condition in Proposition 9 thus can be reformulated as

$$\Delta U_C(\bar{r}; \bar{t}, \tau) = \Delta U_C(\bar{r}_M; \bar{t}, \tau) + \int_{\bar{r}}^{\bar{r}_M} (1 - F(\bar{t}(\tilde{r})))p_0 d\tilde{r}. \tag{46}$$

From (46), we solve out $\tau(\bar{r})$ for all $\bar{r} \leq \bar{r}_M$. Specifically, for $\bar{r} \in (\bar{r}_m, \bar{r}_M]$, we obain

$$\tau(\bar{r})P_N(\bar{t}(\bar{r})) = \Delta U_C(\bar{r}_M; \bar{t}, \tau) + \int_{\bar{r}}^{\bar{r}_M} (1 - F(\bar{t}(\tilde{r})))p_0 d\tilde{r} + \bar{r}p_0 - \bar{r}P_R(\bar{t}(\bar{r})) - rP_N(\bar{t}(\bar{r}));$$

for $\bar{r} \leq \bar{r}_m$, we have

$$\tau(\bar{r}) = \Delta U_C(\bar{r}_M; \bar{t}, \tau) + \int_{\bar{r}_m}^{\bar{r}_M} (1 - F(\bar{t}(\tilde{r})))p_0 d\tilde{r} + \bar{r}_m p_0 - r.$$

Using constraints (CM), (C1), (C4), and $\tau(\bar{r})$ obtained above using (C3), we can transform the employer's problem, using integration by parts, to the following simpler constrained maximization problem:

$$
\text{(P}'\text{)}\qquad \int_0^{\bar r_m} [(v+r)-J(\bar r)p_0]\,dH(\bar r) + (1-H(\bar r_M))\underline{v} - H(\bar r_M)\Delta U_C(\bar r_M;\bar t,\tau)
$$
$$
+\int_{\bar r_m}^{\bar r_M}\left\{\left[(v+r)-J(\bar r)p_{\bar t(\bar r)}\right]P_N(\bar t(\bar r)) + \int_0^{\bar t(\bar r)} p_0 f(t)(1-G(t))\underline{v}dt\right\}dH(\bar r),
$$

subject to that $(\bar t,\tau)$ satisfies constraints (F), (Cm), and (C2).

The following lemma shows that at optimum, A will never give any surplus to the type-$\bar r_M$ candidate, which is the most optimistic candidate that it would like to recruit with a positive probability, i.e., $\Delta U_C(\bar r_M;\bar t,\tau)=0$.

Lemma 3. *The no-redundant-deadline incentive feasible mechanism $(\bar t,\tau)$ that solves (P') satisfies*

1. $\Delta U_C(\bar r_M;\bar t,\tau)=0$, *or equivalently,* $r+\tau(\bar r_M)=\bar r_M p_{\bar t(\bar r_M)}$*;*
2. $r+\tau(\bar r_m)=\bar r_m$.

Proof. We show that if $\Delta U_C(\bar r_M;\bar t,\tau)>0$, there exists another no-redundant-deadline incentive feasible mechanism $(\bar t',\tau')$ that gives A a higher expected payoff.

Suppose that under mechanism $(\bar t,\tau)$, we have $\Delta U_C(\bar r_M;\bar t,\tau)>0$. This implies that $\bar r_M=1$ and

$$
r+\tau(\bar r_M)>\bar r_M p_{\bar t(\bar r_M)}.
$$

For such a mechanism, if $r\le \bar r_M p_{\bar t(\bar r_M)}$, we construct mechanism $(\bar t',\tau')$:

$$
\begin{aligned}
&\bar t'(\bar r_M)=\bar t(\bar r_M),\quad r+\tau'(\bar r_M)=\bar r_M p_{\bar t'(\bar r_M)},\\
&\bar t'(\bar r)=\bar t(\bar r),\quad (\tau(\bar r)-\tau'(\bar r))P_N(\bar t'(\bar r))=(\tau(\bar r_M)-\tau'(\bar r_M))P_N(\bar t'(\bar r_M)) \text{ for } \bar r<\bar r_M.
\end{aligned}\tag{47}
$$

That is, compared with mechanism $(\bar t,\tau)$, the new mechanism $(\bar t',\tau')$ gives each type of the candidate the same acceptance deadline, but a lower transfer. In the new mechanism, $\bar r'_m=r+\tau'(\bar r'_m)<\bar r_m$.

The newly constructed mechanism automatically satisfies (CM), (C1), (C2), and (C4). The surplus of the type-$\bar r$ candidate obtained under mechanism satisfies (46), which is the reformulated (C3) in Proposition 9, because

$$
\begin{aligned}
\Delta U_C(\bar r;\bar t',\tau') &= \Delta U_C(\bar r;\bar t,\tau)-(\tau(\bar r)-\tau'(\bar r))P_N(\bar t'(\bar r))\\
&=\Delta U_C(\bar r_M;\bar t,\tau)-(\tau(\bar r)-\tau'(\bar r))P_N(\bar t'(\bar r))+\int_{\bar r}^{\bar r_M}(1-F(\bar t(\tilde r)))p_0 d\tilde r\\
&=\Delta U_C(\bar r_M;\bar t',\tau')+\int_{\bar r}^{\bar r_M}(1-F(\bar t'(\tilde r)))p_0 d\tilde r,
\end{aligned}\tag{48}
$$

where the second equality is based on (46), and the third equality is based on (47). The condition (C3) and the fact that $\bar r'_m=r+\tau'(\bar r'_m)<\bar r_m$ jointly imply that (Cm) is satisfied.

If $r > \bar{r}_M p_{\bar{t}(\bar{r}_M)}$, which means that we cannot reduce the surplus of type-$\bar{r}_M$ candidate to 0 by solely reducing the transfer included in the offer, then we construct mechanism $(\bar{t}', \tau')$ with $\tau'(\bar{r}_M) = 0$ and $\bar{t}'(\bar{r}_M)$ satisfying

$$r = \bar{r}_M p_{\bar{t}'(\bar{r}_M)}, \tag{49}$$

which makes $\Delta U_C(\bar{r}_M; \bar{t}', \tau') = 0$. Define $\Delta P_R = P_R(\bar{t}(\bar{r}_M)) - P_R(\bar{t}'(\bar{r}_M))$. We have

$$\Delta U_C(\bar{r}_M; \bar{t}, \tau) = \tau(\bar{r}_M) P_N(\bar{t}(\bar{r}_M)) + (\bar{r}_M - r)\Delta P_R. \tag{50}$$

For any $\bar{r} < \bar{r}_M$ with $P_R(\bar{t}(\bar{r})) \geq \Delta P_R$, $(\bar{t}'(\bar{r}), \tau'(\bar{r}))$ satisfies

$$P_R(\bar{t}'(\bar{r})) = P_R(\bar{t}(\bar{r})) - \Delta P_R, \quad \tau'(\bar{r}) P_N(\bar{t}'(\bar{r})) = \tau(\bar{r}) P_N(\bar{t}(\bar{r})) - \tau(\bar{r}_M) P_N(\bar{t}(\bar{r}_M)), \tag{51}$$

and for $\bar{r} < \bar{r}_M$ with $P_R(\bar{t}(\bar{r})) < \Delta P_R$, $(\bar{t}'(\bar{r}), \tau'(\bar{r}))$ satisfies $\bar{t}'(\bar{r}) = 0$ and

$$\begin{aligned}\tau'(\bar{r}) = \tau(\bar{r}) P_N(\bar{t}(\bar{r})) &- \tau(\bar{r}_M) P_N(\bar{t}(\bar{r}_M)) \\ &- (\bar{r}_M - r)\Delta P_R + (\bar{r} - r) P_R(\bar{t}(\bar{r})) + \int_{\bar{r}}^{\bar{r}_M} (P_R(\bar{t}(\tilde{r})) - P_R(\bar{t}'(\tilde{r}))) d\tilde{r}. \end{aligned} \tag{52}$$

The construction of $(\bar{t}', \tau')$ clearly indicates that $\bar{t}'(\bar{r}) \leq \bar{t}(\bar{r})$ for any $\bar{r}$. Regarding the transfer rule τ', it is obvious that $\tau'(\bar{r}_M) \leq \tau(\bar{r}_M)$, and for $\bar{r} < \bar{r}_M$ with $P_R(\bar{t}(\bar{r})) \geq \Delta P_R$, since $P_N(\bar{t}'(\bar{r}) > P_N(\bar{t}(\bar{r})$, according to (51) it is clear that $0 < \tau'(\bar{r}) < \tau(\bar{r})$. For $\bar{r} < \bar{r}_M$ with $P_R(\bar{t}(\bar{r})) < \Delta P_R$, since

$$(\bar{r} - r) P_R(\bar{t}(\bar{r})) + \int_{\bar{r}}^{\bar{r}_M} (P_R(\bar{t}(\tilde{r})) - P_R(\bar{t}'(\tilde{r}))) d\tilde{r} < (\bar{r}_M - r)\Delta P_R, \tag{53}$$

we have $\tau'(\bar{r}) < \tau(\bar{r})$ according to (52). Thus, A gets a higher expected payoff from the new mechanism.

The newly mechanism still satisfies (CM), (C1), (C2), and (C4) by construction. For $\bar{r} < \bar{r}_M$ with $P_R(\bar{t}(\bar{r})) \geq \Delta P_R$, we have

$$\begin{aligned}\Delta U_C(\bar{r}; \bar{t}', \tau') &= \Delta U_C(\bar{r}; \bar{t}, \tau) - \tau(\bar{r}_M) P_N(\bar{t}(\bar{r}_M)) - (\bar{r} - r)\Delta P_R \\ &= \Delta U_C(\bar{r}_M; \bar{t}, \tau) + \int_{\bar{r}}^{\bar{r}_M} (1 - F(\bar{t}(\tilde{r}))) p_0 d\tilde{r} - \tau(\bar{r}_M) P_N(\bar{t}(\bar{r}_M)) - (\bar{r} - r)\Delta P_R \\ &= (\bar{r}_M - r)\Delta P_R + \int_{\bar{r}}^{\bar{r}_M} (1 - F(\bar{t}(\tilde{r}))) p_0 d\tilde{r} - (\bar{r} - r)\Delta P_R \\ &= \Delta U_C(\bar{r}_M; \bar{t}', \tau') + \int_{\bar{r}}^{\bar{r}_M} (1 - F(\bar{t}'(\tilde{r}))) p_0 d\tilde{r}, \end{aligned}$$

where the second equality employs (46), the third equality is based on (50), and the fourth equality is based on the definition of $\Delta U_C(\bar{r}_M;\bar{t}',\tau')$. For $\bar{r}<\bar{r}_M$ with $P_R(\bar{t}(\bar{r}))<\Delta P_R$, we have

$$\begin{aligned}\Delta U_C(\bar{r};\bar{t}',\tau') &= (r+\tau'(\bar{r}))-\bar{r}p_0\\ &= \Delta U_C(\bar{r};\bar{t},\tau)-\tau(\bar{r}_M)P_N(\bar{t}(\bar{r}_M))-(\bar{r}_M-r)\Delta P_R\\ &\quad+\int_{\bar{r}}^{\bar{r}_M}(P_R(\bar{t}(\tilde{r}))-P_R(\bar{t}'(\tilde{r})))d\tilde{r}\\ &= \int_{\bar{r}}^{\bar{r}_M}(1-F(\bar{t}(\tilde{r})))p_0 d\tilde{r}+\int_{\bar{r}}^{\bar{r}_M}(P_R(\bar{t}(\tilde{r}))-P_R(\bar{t}'(\tilde{r})))d\tilde{r}\\ &= \Delta U_C(\bar{r}_M;\bar{t}',\tau')+\int_{\bar{r}}^{\bar{r}_M}(1-F(\bar{t}'(\tilde{r})))p_0 d\tilde{r},\end{aligned}$$

where the first and second equalities are based on (52), the third equality is based on (46) and (50), and the fourth equality is based on on the definition of $\Delta U_C(\bar{r}_M;\bar{t}',\tau')$. Therefore, the (C3) conditioon of Proposition 9 is also satisfied by the new mechanism. In the new mechanism, $\bar{r}'_m=r+\tau'(\bar{r}'_m)<\bar{r}_m$. Combining with (C3), the condition (Cm) is satisfied by $\bar{r}'_m$.

Now we show the second part of this lemma. Given that $\Delta U_C(\bar{r}_M;\bar{t},\tau)=0$ at optimum, we know that $s_2(\bar{t},\tau)\neq\emptyset$. Thus, according to Lemma 1, there must be

$$r+\tau(\bar{r}_m)=\bar{r}_m. \tag{54}$$

□

Lemma 3 introduces two new constraints on the solution to the employer's problem: (1) $\Delta U_C(\bar{r}_M;\bar{t},\tau)=0$, and (2) $r+\tau(\bar{r}_m)=\bar{r}_m$. We use these constraints to further simplify the problem of the employer. Given the definition of $x(\bar{r})$, for any $\bar{r}\in(\bar{r}_m,\bar{r}_M]$ in problem (P′), we have

$$\left[(v+r)-J(\bar{r})p_{\bar{t}(\bar{r})}\right]P_N(\bar{t}(\bar{r}))=(v+r)\left[x(\bar{r})p_0+(1-p_0)\right]-J(\bar{r})x(\bar{r})p_0, \tag{55}$$

and

$$\int_0^{\bar{t}(\bar{r})}p_0 f(t)(1-G(t))\underline{v}=\frac{\lambda}{\lambda+\delta}\left[1-x(\bar{r})^{\frac{(\lambda+\delta)}{\lambda}}\right]p_0\underline{v}. \tag{56}$$

Plugging these equations and the constraint $\Delta U_C(\bar{r}_M;\bar{t},\tau)=0$ into the objective function (P′), we can reformulate the employer's problem as follows:

$$\begin{aligned}\max_{\{\bar{r}_m,\bar{r}_M,x(\cdot)\}}&\int_0^{\bar{r}_m}[(v+r)-J(\bar{r})p_0]dH(\bar{r})+(1-H(\bar{r}_M))\underline{v}\\ &+\int_{\bar{r}_m}^{\bar{r}_M}\left\{(v+r)\left[x(\bar{r})p_0+(1-p_0)\right]-J(\bar{r})x(\bar{r})p_0+\frac{\lambda}{\lambda+\delta}\left[1-x(\bar{r})^{\frac{(\lambda+\delta)}{\lambda}}\right]p_0\underline{v}\right\}dH(\bar{r}),\end{aligned}$$

which is the objective function (P′) in Proposition 5, subject to constraints (F), (Cm), (C2), and

$r + \tau(\bar{r}_m) = \bar{r}_m$.

We prove that the constraints (F), (Cm), (C2), and $r + \tau(\bar{r}_m) = \bar{r}_m$ can be replace by (C1*) and (C2*). Given $\Delta U_C(\bar{r}_M; \bar{t}, \tau) = 0$, i.e., $r + \tau(\bar{r}_M) = \bar{r}_M p_{\bar{t}(\bar{r}_M)}$, (F) and (C2) are equivalent to requiring that $\bar{r}_M \geq r/p_0$ and $\bar{t}(\bar{r})$ is increasing in $\bar{r}$, but bounded between 0 and $\bar{t}_{\bar{r}_M}(r)$, which is exactly the constraint (C1*). When (C1) and the envelope condition (C3) hold, the condition $r + \tau(\bar{r}_m) = \bar{r}_m$ implies (Cm). Thus, we can drop (Cm) from the analysis. Using (C3), we can transform $r + \tau(\bar{r}_m) = \bar{r}_m$ into (C2*), which involves only $\bar{r}_m, \bar{r}_M$, and x.

Proof of Proposition 6

Ignoring constraint (C2*) and the cutoff $\bar{r}_m$, the objective function (P*) of the employer reduces to

$$\max_{\{\bar{r}_M, x(\cdot)\}} \int_0^{\bar{r}_M} \Pi_A(x(\bar{r}); \bar{r}) dH(\bar{r}) + (1 - H(\bar{r}_M))\underline{v}. \tag{57}$$

Fixing the value of $\bar{r}_M \geq r/p_0$, the derivative of $\Pi_A(x(\bar{r}); \bar{r})$ with respect to $x(\bar{r})$ is

$$\frac{d\Pi_A(x(\bar{r}); \bar{r})}{dx(\bar{r})} = \left[(v + r) - J(\bar{r}) - \underline{v}\right] p_0.$$

Case 1: $(v + r) - \underline{v} \geq J(1)$

In this case, given Assumption 2, we have $d\Pi_A(x(\bar{r}); \bar{r})/dx(\bar{r}) \geq 0$ for all $\bar{r}$. Because $x(\bar{r})$ is bounded from above by 1 due to constraint (C1*), it is optimal for A to choose $x(\bar{r}) = 1$ for all $\bar{r} \leq \bar{r}_M$. Therefore, the optimal payoff of the employer for any fixed $\bar{r}_M \geq r/p_0$ is

$$\int_0^{\bar{r}_M} \Pi_A(1; \bar{r}) dH(\bar{r}) + (1 - H(\bar{r}_M))\underline{v} = \int_0^{\bar{r}_M} \left[(v + r) - J(\bar{r}) p_0\right] dH(\bar{r}) + (1 - H(\bar{r}_M))\underline{v}.$$

Taking the derivative of this optimal payoff with respect to $\bar{r}_M$, we obtain

$$\left[(v + r) - J(\bar{r}_M) p_0 - \underline{v}\right] h(\bar{r}_M) > 0,$$

for all $\bar{r}_M \leq 1$, given that $(v + r) - \underline{v} \geq J(1)$ in this case. Therefore, the objective function (57) is maximized when $\bar{r}_M = 1$ and $x(\bar{r}) = 1$ for all $\bar{r} \in [0, 1]$. The constraint (C1*) is automatically satisfied. If we choose $\bar{r}_m = p_0$, then the constraint (C2*) is satisfied, and the payoff of the employer is unchanged. This optimal mechanism can be represented by (x, τ) with $x(\bar{r}) = 1$ and $\tau(\bar{r}) = p_0 - r$ for all $\bar{r}$.

Case 2: $\exists \hat{r}^* \in (r, 1)$ such that $(v + r) - \underline{v} = J(\hat{r}^*)$

We still consider the relaxed maximization problem (57). Because there exists $\hat{r}^* \in (r, 1)$ such that $(v + r) - \underline{v} = J(\hat{r}^*)$, it is optimal to choose $x(\bar{r}) = 1$ for $\bar{r} \leq \hat{r}^*$ and $x(\bar{r}) = \underline{\chi}(\bar{r}_M)$ for $\bar{r} \in (\hat{r}^*, \bar{r}_M]$, given Assumption 2. Thus, the optimal payoff of the employer for any fixed $\bar{r}_M \geq r/p_0$ is

$$\int_0^{\hat{r}^*} \Pi_A(1;\bar{r})dH(\bar{r}) + \int_{\hat{r}^*}^{\bar{r}_M} \Pi_A(\underline{\chi}(\bar{r}_M);\bar{r})dH(\bar{r}) + (1-H(\bar{r}_M))\underline{v}.$$

The employer chooses $\bar{r}_M \geq \max\{r/p_0, \hat{r}^*\}$ to maximize the payoff above, i.e.,

$$\bar{r}_M^* \in \arg \max_{\bar{r}_M \geq \max\{r/p_0,\hat{r}^*\}} \int_{\hat{r}^*}^{\bar{r}_M} \Pi_A(\underline{\chi}(\bar{r}_M);\bar{r})dH(\bar{r}) + (1-H(\bar{r}_M))\underline{v},$$

given that $\hat{r}^*$ is independent of $\bar{r}_M$. The solution of the relaxed problem, $x(\bar{r}) = 1$ for $\bar{r} \leq \hat{r}^*$ and $x(\bar{r}) = \underline{\chi}(\bar{r}_M^*)$ for $\bar{r} \in (\hat{r}^*, \bar{r}_M^*]$, satisfies (C1*). Constraint (C2*) can be simplified to

$$\bar{r}_m(1-p_0) = (\hat{r}^* - \bar{r}_m)p_0 + (\bar{r}_M^* - \hat{r}^*)\underline{\chi}(\bar{r}_M^*)p_0.$$

There is always a unique $\bar{r}_m^* \in [r, \hat{r}^*)$ satisfying the constraint. Because $\bar{r}_m^* < \hat{r}^*$, the solution to the relaxed problem also maximizes the problem with constraint (C2*). The optimal mechanism of A's original problem thus can be represented by (x,τ) with $x^*(\bar{r}) = 1$ and $\tau^*(\bar{r}) = (\hat{r}^* - r)(1-\underline{\chi}(\bar{r}_M^*))p_0$ for $\bar{r} \leq \hat{r}^*$; $x^*(\bar{r}) = \underline{\chi}(\bar{r}_M^*)$ and $\tau^*(\bar{r}) = 0$ for $\bar{r} \in (\hat{r}^*, \bar{r}_M^*]$; $x^*(\bar{r}) = 1$ and $\tau^*(\bar{r}) = 0$ for $\bar{r} > \bar{r}_M^*$.

It is worth mentioning that if $\hat{r}^* \geq r/p_0$, we have have $\bar{r}_M^* \geq \hat{r}^*$.

Case 3: $(v+r) - \underline{v} \leq J(r)$

Constraints (C1*) and (C2*) imply that $\bar{r}_m \geq r$. Due to constraint (C1*), we require $x(\bar{r}) \geq \underline{\chi}(\bar{r}_M)$. Thus, to have (C2*) satisfied, there should be

$$\bar{r}_m(1-p_0) \geq \int_{\bar{r}_m}^{\bar{r}_M} \underline{\chi}(\bar{r}_M)p_0 = (\bar{r}_M - \bar{r}_m)\underline{\chi}(\bar{r}_M)p_0.$$

By definition, $\underline{\chi}(\bar{r}_M) = r(1-p_0)/(\bar{r}_M - r)p_0$. Plugging this expression to the inequality above, we can derive $\bar{r}_m \geq r$. This implication means that for any $\bar{r} \geq r$, there should always be $x(\bar{r}) = 1$ in any incentive feasible mechanism. We consider following payoff function

$$\max_{\{\bar{r}_M, x(\cdot)\}} \int_0^r \Pi_A(1;\bar{r})dH(\bar{r}) + \int_r^{\bar{r}_M} \Pi_A(x(\bar{r});\bar{r})dH(\bar{r}) + (1-H(\bar{r}_M))\underline{v}.$$

In the current case, because of Assumption 2, we have $d\Pi_A(x(\bar{r});\bar{r})/dx(\bar{r}) < 0$ for all $\bar{r} > r$. Thus, it is optimal for the employer to choose $x(\bar{r}) = \underline{\chi}(\bar{r}_M)$ for all $\bar{r}$, to maximize the relaxed object function above. Therefore, for any fixed $\bar{r}_M \geq r/p_0$, the optimal payoff of the employer is

$$\int_0^r \Pi_A(1;\bar{r})dH(\bar{r}) + \int_r^{\bar{r}_M} \Pi_A(\underline{\chi}(\bar{r}_M);\bar{r})dH(\bar{r}) + (1-H(\bar{r}_M))\underline{v}.$$

The employer chooses $\bar{r}_M$ to maximize the above payoff, i.e.,

$$\bar{r}_M^* \in \arg\max_{\bar{r}_M \geq r/p_0} \int_r^{\bar{r}_M} \Pi_A(\underline{\chi}(\bar{r}_M); \bar{r}) dH(\bar{r}) + (1 - H(\bar{r}_M))\underline{v}.$$

The constraint (C1*) is automatically satisfied. If we choose $\bar{r}_m = r$, then the constraint (C2*) is also satisfied, and the payoff of the employer is unchanged. This mechanism can be represented by (x, τ) with $x(\bar{r}) = 1$ for $\bar{r} \in [0, r] \cup (\bar{r}_M^*, 1]$, $x(\bar{r}) = \underline{\chi}(\bar{r}_M^*)$ for $\bar{r} \in (r, \bar{r}_M^*]$, and $\tau(\bar{r}) = 0$ for all $\bar{r}$.

Proof of Proposition 7

Similar to the proof of Proposition 6, we first ignore (C2*) and $\bar{r}_m$, and consider the following relaxed maximization problem:

$$\max_{\{\bar{r}_M, x(\cdot)\}} \int_0^{\bar{r}_M} \Pi_A(x(\bar{r}); \bar{r}) dH(\bar{r}) + (1 - H(\bar{r}_M))\underline{v}, \tag{58}$$

subject to constraint (C1*).

Remember that without (C1*), it is optimal to choose $x(\bar{r}) = \hat{x}(\bar{r})$ defined in (25). Because there exists $\hat{r}^* \in (r, 1)$ such that $(v + r) - \underline{v} = J(\hat{r}^*)$, we have $\hat{x}(\bar{r}) \geq 1$ for all $\bar{r} \leq \hat{r}^*$. Thus, with the constraint (C1*), it is optimal to choose $x(\bar{r}) = 1$ for all $\bar{r} \leq \hat{r}^*$, given that $\Pi_A(x(\bar{r}); \bar{r})$ is concave in $x(\bar{r})$. It is easy to verify that $\Pi_A(1; \bar{r}) = (v + r) - J(\bar{r})p_0 > \underline{v}$ for $\bar{r} \leq \hat{r}^*$. That is, recruiting these types of the candidate using $x(\bar{r}) = 1$ is better than giving them up, which gives the employer payoff $\underline{v}$. Thus, the optimal $\bar{r}_M$ must be larger than $\hat{r}^*$.

For the types of the candidate in $(\hat{r}^*, \bar{r}_M]$ when $\hat{r}^* < \bar{r}_M$, it is optimal for the employer to choose $x(\bar{r}) = \max\{\hat{x}(\bar{r}), \underline{\chi}(\bar{r}_M)\}$, because of constraint (C1*). Thus, the optimal payoff of the employer for any fixed $\bar{r}_M \geq \max\{r/p_0, \hat{r}^*\}$ is

$$\int_0^{\hat{r}^*} \Pi_A(1; \bar{r}) dH(\bar{r}) + \int_{\hat{r}^*}^{\bar{r}_M} \Pi_A(x(\bar{r}); \bar{r}) dH(\bar{r}) + (1 - H(\bar{r}_M))\underline{v},$$

with $x(\bar{r}) = \max\{\hat{x}(\bar{r}), \underline{\chi}(\bar{r}_M)\}$. The employer chooses $\hat{r}_M^*$ that maximizes the payoff above, i.e.,

$$\hat{r}_M^* \in \arg\max_{\bar{r}_M \geq \max\{r/p_0, \hat{r}^*\}} \int_{\hat{r}^*}^{\bar{r}_M} \Pi_A(x(\bar{r}); \bar{r}) dH(\bar{r}) + (1 - H(\bar{r}_M))\underline{v},$$

subject to $x(\bar{r}) = \max\{\hat{x}(\bar{r}), \underline{\chi}(\bar{r}_M)\}$, given that $\hat{r}^*$ is independent of $\bar{r}_M$. The solution of the relaxed problem satisfies (C1*), including the monotonicity condition.

Now we show that there exists a unique $\bar{r}_m^* \in (r, \hat{r}^*]$ making the solution to the relaxed problem satisfy the constraint (C2*). Suppose $\bar{r}_m \leq \hat{r}^*$. We plug the solution of the relaxed problem into (C2*), and obtain

$$\bar{r}_m(1 - p_0) = (\hat{r}^* - \bar{r}_m)p_0 + \int_{\hat{r}^*}^{\hat{r}_M^*} \max\{\hat{x}(\bar{r}), \underline{\chi}(\hat{r}_M^*)\} p_0 d\bar{r}.$$

By reformulating the equation, we obtain $\bar{r}_m^* = \hat{r}^* - Q(\hat{r}^*, \hat{r}_M^*)$, which is the value of $\bar{r}_m$ that makes (C2*) hold. It is obvious that $\bar{r}_m^* \le \hat{r}^*$, as $Q(\hat{r}^*, \hat{r}_M^*) \ge 0$. Because $\hat{r}^* > r$ and $x(\bar{r}) = \max\{\hat{x}(\bar{r}), \underline{\chi}(\bar{r}_M)\} \ge \underline{\chi}(\bar{r}_M)$, we have $\bar{r}_m^* > r$. Therefore, the optimal mechanism solving the relaxed problem is also optimal to the original problem of the employer.

Proof of Proposition 8

Below we first consider the maximization problem in Proposition 5 with *fixed* $\bar{r}_m$ and $\bar{r}_M$, that is,

$$\max_{x(\cdot)} \int_0^{\bar{r}_m} \Pi_A(1;\bar{r})dH(\bar{r}) + \int_{\bar{r}_m}^{\bar{r}_M} \Pi_A(x(\bar{r});\bar{r})dH(\bar{r}) + (1 - H(\bar{r}_M))\underline{v},$$

subject to constraints

(C1*) $\quad x(\bar{r})$ is decreasing in $\bar{r}$ and bounded by 1 and $\underline{\chi}(\bar{r}_M)$, with $\bar{r}_M \ge r/p_0$;

(C2*) $\quad \bar{r}_m(1-p_0) = \displaystyle\int_{\bar{r}_m}^{\bar{r}_M} x(\bar{r})p_0 d\bar{r}$.

We show that if $x^* : [\bar{r}_m, \bar{r}_M] \to R_{++}$ is a solution to the problem above, there exist two cutoffs $\bar{r}^*, \bar{r}^{**} \in [\bar{r}_m, \bar{r}_M]$ such that

$$x^*(\bar{r}) = \begin{cases} 1, & \text{if } \bar{r} < \bar{r}^*; \\ \left[\frac{(v+r) - J(\bar{r}) - \mu^*/h(\bar{r})}{\underline{v}}\right]^{\frac{\lambda}{\delta}}, & \text{if } \bar{r}^* < \bar{r} < \bar{r}^{**}; \\ \underline{\chi}(\bar{r}_M), & \text{if } \bar{r}^{**} < \bar{r}. \end{cases} \tag{59}$$

From the constraints (C1*) and (C2*), we know that $\bar{r}_m \ge r$ (see Case 3 in the proof of Proposition 6) and $\bar{r}_M \ge \bar{r}_m/p_0$. If $\bar{r}_m = r$, then there must be $x^*(\bar{r}) = \underline{\chi}(\bar{r}_M)$ for all $\bar{r} \in (\bar{r}_m, \bar{r}_M]$ and all $\bar{r}_M \ge 1$. In this case, $\bar{r}^{**} = \bar{r}_m$. If $\bar{r}_M = \bar{r}_m/p_0$, then there must be $x^*(\bar{r}) = 1$ for all $\bar{r} \in (\bar{r}_m, \bar{r}_M]$. In this case, $\bar{r}^* = \bar{r}_M$.

Below we consider $\bar{r}_m > r$ and $\bar{r}_M > \bar{r}_m/p_0$. We ignore the monotonicity constraint on $x(\bar{r})$ in (C1*). According to the Lagrange theorem, if x^* solves the above maximization problem, then there exist $\mu^*(\bar{r}_m, \bar{r}_M) \in R$, $\mu_1^* : [\bar{r}_m, \bar{r}_M] \to R$, and $\mu_2^* : [\bar{r}_m, \bar{r}_M] \to R$ such that x^*, $\mu^*(\bar{r}_m, \bar{r}_M)$, μ_1^*, and μ_2^* are the solution to the following Lagrangian

$$\begin{aligned} &L(x, \mu, \mu_1, \mu_2) \\ = &\int_{\bar{r}_m}^{\bar{r}_M} \left\{ \Pi_A(x(\bar{r});\bar{r})h(\bar{r}) - \mu \left[x(\bar{r})p_0 - \frac{\bar{r}_m(1-p_0)}{\bar{r}_M - \bar{r}_m} \right] - \mu_1(\bar{r})\left(x(\bar{r}) - 1\right) - \mu_2(\bar{r})\left(\underline{\chi}(\bar{r}_M) - x(\bar{r})\right) \right\} d\bar{r}. \end{aligned}$$

The first order conditions are

$$\frac{\partial L(x,\mu,\mu_1,\mu_2)}{\partial x(\bar r)}\frac{1}{h(\bar r)p_0} = \left[(v+r)-J(\bar r)-x(\bar r)^{\frac{\delta}{\lambda}}\underline{v}-\frac{\mu}{h(\bar r)}\right]-\frac{\mu_1(\bar r)}{h(\bar r)p_0}+\frac{\mu_2(\bar r)}{h(\bar r)p_0}=0, \tag{60}$$

$$\frac{\partial L(x,\mu,\mu_1,\mu_2)}{\partial \mu} = -\int_{\bar r_m}^{\bar r_M} x(\bar r)p_0 d\bar r + \bar r_m(1-p_0)=0, \tag{61}$$

with the complementary slackness conditions: for all $\bar r \in [\bar r_m, \bar r_M]$,

$$\mu_1(\bar r)\left(x(\bar r)-1\right)=0, \text{ with } \mu_1(\bar r)\geq 0 \text{ and } x(\bar r)-1\leq 0,$$
$$\mu_2(\bar r)\left(\underline{\chi}(\bar r_M)-x(\bar r)\right)=0, \text{ with } \mu_2(\bar r)\geq 0 \text{ and } x(\bar r)-\underline{\chi}(\bar r_M)\geq 0.$$

To proceed, we neglect (61), and show that for any μ, the function $x:[\bar r_m,\bar r_M]\to R_{++}$ that satisfies (60) and the complementary slackness conditions is in the form of (59) and characterized by two cutoffs $\bar r^*_\mu$ and $\bar r^{**}_\mu$.

We define $\bar\mu$ as the value μ satisfying

$$(v+r)-J(\bar r_m)-\underline{\chi}(\bar r_M)^{\frac{\delta}{\lambda}}\underline{v}-\frac{\bar\mu}{h(\bar r_m)}=0. \tag{62}$$

Since $(v+r)\geq \underline{v}+\bar r$ for all $\bar r\in[0,1]$, there should be $H(\bar r_m)+\bar\mu>0$. Due to the concavity of $H(\bar r)$, we have that for all $\bar r>\bar r_m$ and all $\mu\geq\bar\mu$, $(v+r)-\underline{\chi}(\bar r_M)^{\frac{\delta}{\lambda}}\underline{v}<J(\bar r)+\mu/h(\bar r)$. Thus, to have (60) satisfied, $x(\bar r)=\underline{\chi}(\bar r_M)$ for all $\bar r>\bar r_m$ and $\mu\geq\bar\mu$. In this case, we have $\bar r^{**}_\mu=\bar r_m$.

We define $\underline{\mu}$ as the value μ satisfying

$$(v+r)-J(\bar r_M)-\underline{v}-\frac{\underline{\mu}}{h(\bar r_M)}=0. \tag{63}$$

It is clear that $\underline{\mu}<\bar\mu$. For all $\mu\leq\underline{\mu}$, $(v+r)-\underline{v}>J(\bar r)+\mu/h(\bar r)$ for all $\bar r<\bar r_M$. That is, when $\mu\leq\underline{\mu}$, to have (60) satisfied, $x(\bar r)=1$ for all $\bar r<\bar r_M$. We prove this by contradiction. Suppose that there exists some $\bar r'<\bar r_M$ such that $(v+r)-\underline{v}\leq J(\bar r')+\mu/h(\bar r')$ for some $\mu\leq\underline{\mu}$. Since $(v+r)\geq\underline{v}+\bar r$ for all $\bar r$, there should be $H(\bar r')+\underline{\mu}>0$. Due to the concavity of $H(\bar r)$, there must be $(v+r)-\underline{v}<J(\bar r_M)+\mu/h(\bar r_M)$. A contradiction to (63). In this case, we have $\bar r^*_\mu=\bar r_M$.

Now we consider $\mu\in(\underline{\mu},\bar\mu)$. If there exists $\bar r\in(\bar r_m,\bar r_M)$ such that

$$(v+r)-J(\bar r)-\underline{\chi}(\bar r_M)^{\frac{\delta}{\lambda}}\underline{v}-\frac{\mu}{h(\bar r)}=0, \tag{64}$$

we define this $\bar r$ as $\bar r^{**}_\mu$. Given that $(v+r)\geq\underline{v}+\bar r$ for all $\bar r$, there should be $H(\bar r^{**}_\mu)+\mu>0$. Due to the concavity of $H(\bar r)$, we have that $(v+r)-\underline{\chi}(\bar r_M)^{\frac{\delta}{\lambda}}\underline{v}<J(\bar r)+\mu/h(\bar r)$ for all $\bar r>\bar r^{**}_\mu$. That is, $x(\bar r)=1$ for all $\bar r>\bar r^{**}_\mu$. If no $\bar r$ satisfies (64), then we define $\bar r^{**}_\mu=\bar r_M$.

If there exists $\bar r$ such that

$$(v+r)-J(\bar r)-\underline{v}-\frac{\mu}{h(\bar r)}=0, \tag{65}$$

we define this $\bar{r}$ as $\bar{r}^*_\mu$. There must be that $(v+r)-\underline{v} > J(\bar{r}) + \mu/h(\bar{r})$ for all $\bar{r} < \bar{r}^*_\mu$. (This implies that $\bar{r}^*_\mu < \bar{r}^{**}_\mu$.) We prove this by contradiction. Suppose that there exists some $\bar{r}'$ such that $(v+r)-\underline{v} < J(\bar{r}') + \mu/h(\bar{r}')$. Then there should be $H(\bar{r}')+\mu > 0$. Due to the concavity of $H(\bar{r})$, there must be $(v+r)-\underline{v} < J(\bar{r}^*_\mu) + \mu/h(\bar{r}^*_\mu)$. A contradiction to that $\bar{r}^*_\mu$ satisfies (65). Therefore, to have (60) satisfied, $x(\bar{r}) = 1$ for all $\bar{r} < \bar{r}^*_\mu$. If no $\bar{r}$ satisfies (65), then we define $\bar{r}^*_\mu = \bar{r}_m$.

For $\bar{r} \in (\bar{r}^*_\mu, \bar{r}^{**}_\mu)$, we have

$$\begin{aligned}(v+r)-\underline{v} &< J(\bar{r}) + \mu/h(\bar{r}),\\ (v+r)-\underline{\chi}(\bar{r}_M)^{\frac{\delta}{\lambda}}\underline{v} &> J(\bar{r}) + \mu/h(\bar{r}).\end{aligned}$$

According to the complementary slackness condition and (60), there should be $x(\bar{r}) \in (\underline{\chi}(\bar{r}_M), 1)$, $\mu_1(\bar{r}) = \mu_2(\bar{r}) = 0$, and

$$\frac{\partial L(x,\mu,\mu_1,\mu_2)}{\partial x(\bar{r})} = (v+r) - J(\bar{r}) - x(\bar{r})^{\frac{\delta}{\lambda}}\underline{v} - \frac{\mu}{h(\bar{r})} = 0.$$

Therefore, we obtain, for $\bar{r} \in (\bar{r}^*_\mu, \bar{r}^{**}_\mu)$,

$$x(\bar{r}) = \left[\frac{(v+r) - J(\bar{r}) - \mu/h(\bar{r})}{\underline{v}}\right]^{\frac{\lambda}{\delta}}.$$

For all $\bar{r} \in (\bar{r}^*_\mu, \bar{r}^{**}_\mu)$, given that $H(\bar{r})+\mu > 0$ and $H(\bar{r})$ is concave, $x(\bar{r})$ is strictly decreasing in $\bar{r}$.

Therefore, for any μ, the function $x(\bar{r})$ that satisfies (60) and the complementary slackness conditions is in the form of (59). Moreover, $x(\bar{r})$ is decreasing in $\bar{r} \in [\bar{r}_m, \bar{r}_M]$.

We plug $x(\bar{r})$ into (61) to pin down the value of $\mu^*(\bar{r}_m, \bar{r}_M)$. That is, $\mu^*(\bar{r}_m, \bar{r}_M)$ is the solution to equation

$$(\bar{r}_m - \bar{r}^*_\mu)p_0 + \int_{\bar{r}^*_\mu}^{\bar{r}^{**}_\mu} \left[\frac{(v+r) - J(\bar{r}) - \mu/h(\bar{r})}{\underline{v}}\right]^{\frac{\lambda}{\delta}} p_0 d\bar{r} + (\bar{r}_M - \bar{r}^{**}_\mu)\underline{\chi}(\bar{r}_M)p_0 - \bar{r}_m(1-p_0) = 0.$$

There exists a unique $\mu^*(\bar{r}_m, \bar{r}_M)$ satisfying the equation, because $x(\bar{r})$ is continuously decreasing in μ, given that $\bar{r}^*_\mu$ and $\bar{r}^{**}_\mu$ are continuous in μ. Moreover, $\mu^*(\bar{r}_m, \bar{r}_M)$ is continuous in $\bar{r}_m, \bar{r}_M$. Because the function $x(\bar{r})$ for any μ is decreasing in $\bar{r}$, the solution $x^*(\bar{r})$ corresponding to $\mu^*(\bar{r}_m, \bar{r}_M)$ satisfies the monotonicity constraint in (C1*).

To find the optimal $\bar{r}_m$ and $\bar{r}_M$, we plug $x^*(\bar{r})$ into the objective function (P*) in Proposition 5. The derived objective function is continuous in $\bar{r}_m$ and $\bar{r}_M$, and the set of feasible $\bar{r}_m$ and $\bar{r}_M$ is compact. Thus, there exists a pair of $\bar{r}^*_m$ and $\bar{r}^*_M$ solving (P*).