Updating uncertainty-averse preferences

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March 1, 2022

Abstract

This paper characterizes axiomatically a simple consequentialist updating rule for a broad class of ambiguity-averse preferences, which nests many well-known families of ambiguity averse preferences. In particular, for the translation invariant subfamily and the positively scale invariant subfamily, explicit representations of the conditional preferences are fully characterized. These updating rules nest the familiar prior-by-prior updating of maxmin EU as a special case.

Keywords: ambiguity aversion, robustness, prior-by-prior updating, consequentialism, uncertainty-averse preferences.

JEL code: D81, D83.

1 Introduction

The relevance of ambiguity (Ellsberg, 1961) in decisions and markets has been supported by a rich theoretical and experimental literature (see Gilboa and Marinacci (2013) and Machina and Siniscalchi (2014) and references therein). For instance, a decisionmaker (DM) can perceive multiple probabilities as plausible and make decisions according to the maxmin expected utility criterion – that is, he aims to maximize the worst-case expected utility (Gilboa

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and Schmeidler, 1989). It delivers a good outcome for the DM regardless of the true probability. Numerous generalizations of the maxmin EU have been developed to account for the more general cases and less extreme ambiguity averse attitudes. A few well-known cases are the variational preferences (Maccheroni et al., 2006a), smooth ambiguity preferences (Klibanoff et al., 2005), and confidence preferences (Chateauneuf and Faro, 2009).[1] Moreover, all of these preferences are nested as special cases of the uncertainty-averse preferences (Cerreia-Vioglio et al., 2011).

Many economically relevant decisions are dynamic in nature. In a dynamic setting, the DM will receive new information and update their ambiguous beliefs upon this new information. While the standard theory of learning for subjective expected utility preferences is the Bayesian updating rule (Savage, 1954); for an agent with one of the aforementioned classes of ambiguity preferences, characterizing a good updating rule is less straightforward. Nevertheless, for the maxmin EU model, a good model of learning is the prior-by-prior Bayesian updating rule – each prior considered plausible by the DM is updated by Bayes' rule one by one. This updating rule has a few merits: (i) it generalizes the Bayesian updating rule in a simple and robust way; (ii) the behavioral property that characterizes prior-by-prior updating for the maxmin EU naturally weakens Savage's definition of conditional preferences for the SEU model; and (iii) it is shown tractable and applied wildly.[2]

Two natural questions arise from this literature. First, one would like to better understand how the prior-by-prior updating rule for maxmin EU preferences generalizes to other ambiguity-averse preference families. Second, a critique on the prior-by-prior updating rule is that it simply updates all plausible priors without discrimation. As discussed in Gilboa and Schmeidler (1993) and the Epstein and Schneider (2007), one may also wish to give priority to updating the prior that assigns a higher likelihood to the occured event, motivated by the desire to select probabilities that can better rationalize the occured information.[3]

This paper applies the familiar behavioral properties to the general case of the strongly

[1]Note that each of these three classes nests the maxmin EU preferences as a special/limit case. Moreover, the variational preferences family also nests the multiplier preferences model (Hansen and Sargent, 2001; Strzalecki, 2011); and the monotone mean-variance preferences model (Markowitz, 1952; Tobin, 1958) as special cases.

[2]See, among others, Epstein and Schneider (2007), Bose and Renou (2014), Kellner and Le Quement (2018), and Beauchêne et al. (2019).

[3]The idea is similar in spirit to the maximum likelihood method widedly used in frequentist statistics – one selects the subset of plausible priors which assign the maximum likelihood to the occurred event. For maxmin EU model, Gilboa and Schmeidler (1993) characterize a maximum likelihood updating rule; Epstein and Schneider (2007) propose a learning model which only updates plausible priors that assign likelihoods to the occurred event that are above a certain threshold level.

monotone uncertainty-averse preferences (UAP). I find that by adding a concavity assumption on the functional aggregator of the UAP representation, the familiar behavioral property that characterizes prior-by-prior updating, called conditional consistency (CC), leads to well-defined conditional certainty equivalents (Lemma 2). Under this assumption, a characterization of generalized-Bayesian updating for the UAP family is provided (Proposition 1). Moreover, in two invariant subfamilies of the UAP preferences, the variational preferences (with a translation invariant aggregator) and the confidence preferences (with a positively scaled invariant aggregator), this concavity assumption always holds and one can fully characterize explicit representations of the conditional preferences (Theorems 1, 2). A characterization of the updated conditional preferences for the subclass of smooth ambiguity preferences (Klibanoff et al., 2005) with a concave certainty equivalent function is also provided (Corollary 2). An interesting observation is that, as we investigate the two invariant subfamilies that both generalize maxmin EU, the generalized Bayesian updating rule features a normalization term (as the inversed probability of the occured event) that prioritizes selecting the probability model which assigns a higher likelihood to the occured event. It suggests that the previous critique on how prior-by-prior updating may fail to select the prior that better rationalizes the occured event could be a result of restricting to the maxmin EU functional form rather than a limitation of the updating rule.

In particular, for the variational preferences family, Theorem 1 fully characterizes a simple explicit representation of the conditional preferences in terms of the unconditional utility representation, which generalizes the Bayes' rule. Under this updating rule, as the DM learns that the true state of world belongs to some event E, he updates his preferences which still belong to the variational family. The updated cost function has two features. First, its domain contains only probabilities/posteriors are supported on the news event E. Therefore, outcomes in states that are ruled out by the news event should no longer matter in the DM's updated preferences. Second, the updated cost of choosing a posterior equals to the minimum cost (normalized by the likelihood of the news event) of choosing all priors that are updated to this posterior via the Bayes' rule. In this way, the DM displays cautiousness when incorporating the news. This updating rule nests the well-known prior-by-prior updating model of maxmin EU preferences (Pires, 2002), and a model that updates the reference prior by Bayesian updating in multiplier preferences (Strzalecki, 2011) and monotone mean-variance preferences (Corollary 1).

In the existing literature, the closest to Theorem 1 is Maccheroni et al. (2006b), which provides a recursive "no-gain" condition that connects the unconditional and the conditional cost functions within the variational preferences family. The key difference is that the updating rule studied in Theorem 1 considers all variational preferences and all information filtration but satisfies only consequentialism; while Maccheroni et al. (2006b) consider a fixed information filtration and characterize the subfamily of variational preferences that respects both assumptions. Given the above-mentioned experimental evidence, the two approaches are complementary.

For the confidence preferences family, Theorem 2 explicitly characterizes a simple generalization of the prior-by-prior updating rule. Given the news event and the unconditional confidence preferences representation, the updated conditional confidence of a posterior is equal to the maximal difference between the unconditional confidence divided by the likelihood of the news event and the odds ratio of this event, subject to the constraint that all the priors must generate this posterior via Bayesian updating and this difference must be positive.

Finally, for the smooth ambiguity preferences, due to non-separability of the utility functionals, the updated conditional utility is still an implicit solution to an equation. The analysis in section 4.2 provides the necessary and sufficient condition to guarantee concavity of the certainty equivalent function, which is needed for the conditional consistency axiom to be well-defined.

The main results (Lemma 2, Theorems 1 and 2) contribute to the literature on how to update ambiguity-averse preferences. This literature often evaluates an updating rule via two desiderata: (i) *consequentialism*—outcomes in states that are ruled out by the new information should not be relevant for decisions (Machina, 1989); (ii) *dynamic consistency*—a contingent plan that is *ex ante* optimal should remain optimal as the DM receives additional information. It is well-known that under reduction, the joint assumption of the two implies the sure-thing principle and rules out ambiguity aversion (Epstein and LeBreton, 1993). This modeling choice is consistent with documented experimental evidence on dynamic choice under ambiguity. Dominiak et al. (2012) find that, in a dynamic Ellsberg urn decision problem, subjects' behaviors often obey consequentialism while violate dynamic consistency.[4] The updating rules characterized in this paper respect consequentialism. Moreover, if one allows for preferences for temporal resolution of uncertainties, the generalized updating rule can always be embedded into a recursive preferences model that preserves dynamic consistency, albeit at the cost of relaxing reduction (Li, 2020).

1.1 Related literature

[4]See also recent experimental work by Kops and Pasichnichenko (2020).

Since incorporating ambiguity necessarily implies the failure of the sure-thing principle, under reduction, either consequentialism or dynamic consistency should be relaxed.

One direction to go is to preserve consequentialism at the cost of relaxing DC, which includes two popular updating theories—the prior-by-prior Bayesian updating and the maximum likelihood updating. The prior-by-prior Bayesian updating rule is the most straightforward generalization of Bayesian updating, which typically consists of updating all the priors the DM considers plausible without discrimination. In this direction, Pires (2002) axiomatizes full Bayesian updating for maxmin EU. Faro and Lefort (2019) provide an alternative axiomatization for the same updating rule, also for maxmin EU preferences, by imposing consistency axioms between the unconditional objectively rational preferences and the conditional subjectively rational preferences. Their work can be viewed as a new justification for this updating rule.[5]

Alternatively, the maximum likelihood updating model select the subset of plausible priors that prescribe the highest likelihood to the realized event and only update these priors. Gilboa and Schmeidler (1993) first axiomatize the maximum likelihood updating rule for the Maxmin EU preferences.[6] Ortoleva's (2012) hypothesis testing model shares a similar motivation—it considers DM who does Bayesian updating for a normal event but deviates from it when an unexpected small-probability event occurs.

Along this line, this paper further provides an axiomatic characterization of the full Bayesian updating for the more general cases of concave and strongly monotone uncertainty-averse preferences. The main representations can be viewed as an robust generalization of the prior-by-prior updating, which also incorperates the likelihood of the occured event.

Another direction is to relax consequentialism but maintain some appropriate form of dynamic consistency. Hanany and Klibanoff (2007) propose an updating rule for the maxmin EU that satisfies a weak form of dynamic consistency but not consequentialism. Gul and Pesendorfer (2018) study a proxy updating rule that satisfies consequentialism and a weak dynamic consistency property called "not all news is bad news", which is distinct from either full Bayesian updating or maximum likelihood updating. Recently, Sadowski and Sarver (2019) consider an evolutionary model and show that for the objective of maximizing long-run population growth, updating rule should be dynamically consistent for certain ambiguity-averse preference families, such as the multiplier preferences and the confidence preferences.[7]

[5]Eichberger et al. (2007) axiomatize a generalized Bayesian updating rule for the Choquet EU family.

[6]See Cheng (forthcoming) for a generalization to the relative maximum likelihood updating for the maxmin EU preferences.

[7]Note that the violation of consequentialism is interpreted differently in Sadowski and Sarver (2019),

The paper proceeds as follows. Section 2 introduces the preliminaries and characterizes updating for the general strongly monotone and concave UAP preferences. Section 3 characterize updating in two invariant subfamilies: variational preferences and confidence preference. Section 4 discusses special cases, the smooth ambiguity preferences, and the comparative ambiguity notions. Omitted proofs are relegated to the appendix.

2 The general case

2.1 Notation

Let S be a finite set of states and $\Sigma^* = 2^S \backslash \emptyset$ be the collection of non-empty events. Let X be a nonempty set of consequences, which is a convex subset of some topological vector space. An act f is a mapping from $S \mapsto X$ that assigns every state an outcome. Denote by $\mathcal{F}$ the set of acts. For each non-empty event $E \in \Sigma^*$, let $\succsim_E$ be the preferences over $\mathcal{F}$ conditional on the information that the true state belongs to event E. The primitives are the collection of conditional preferences $\{\succsim_E\}_{E \in \Sigma^*}$.

Let ΔE be the set of probabilities with support on E. Let p, q denote probabilities in ΔS. For any $p \in \Delta S$, let $p(\cdot|E)$ denote its Bayesian posterior conditional on E. Conversely, for any $p_E \in \Delta E$, let $\Delta(p_E)$ denote the subset of prior probabilities in ΔS whose Bayesian posterior is p_E:

$$\Delta(p_E) = \{p \in \Delta S : p(\cdot|E) = p_E\}.$$

Note that the subset $\Delta(p_E)$ is not closed because $p(E) > 0$.

2.2 Uncertainty-averse preferences and updating

Cerreia-Vioglio et al. (2011) characterize a broad family of ambiguity-averse preferences.

Axiom 1 (UAP axioms)**.** For all $E \in \Sigma^*$, $\succsim_E$ satisfies the following:

1. (Weak order). $\succsim_E$ is complete and transitive.

because it stems from the differences between the welfare of an individual in a population and the goal of maximizing population growth. An individual may not care about outcome in a state that has been ruled out for himself, yet outcome in this state may still be relevant for the population when other individuals in the same population faces idiosyncratic risk. This paper considers a one-step-ahead updating for a single decisiomaker who only cares about him own payoff, and hence it is natural imposing consequentialism.

2. (Continuity). For all $f \in \mathcal{F}$, $\{g \in \mathcal{F} : g \succsim_E f\}$ and $\{g \in \mathcal{F} : f \succsim_E g\}$ are closed.

3. (Monotonicity). If $f(s) \succsim_E g(s)$ for all $s \in E$, then $f \succsim_E g$.

4. (Non-degeneracy). $f \succ_E g$ for some $f, g \in \mathcal{F}$.

5. (Constant-act independence). For all $x, y, z \in X$, and $\alpha \in (0,1)$,
$$x \succsim_E y \Leftrightarrow \alpha x + (1-\alpha)z \succsim_E \alpha y + (1-\alpha)z$$

6. (Uncertainty aversion). For all $f, g \in \mathcal{F}$ such that $f \sim_E g$, $\alpha f + (1-\alpha)g \succsim_E g$ for all $\alpha \in [0,1]$.

Definition 1. Say $\succsim_E$ admits an *uncertainty-averse preferences representation* (u_E, G_E) if it can be represented by

$$V_E(f) = \inf_{p \in \Delta(S)} G_E\left(\int_S u_E(f)dp, p\right), \tag{1}$$

where $u_E : X \mapsto \mathbb{R}$ is an affine function and $G_E : u_E(X) \times \Delta S \mapsto (-\infty, +\infty]$ is the *uncertainty-averse index* that satisfies the following properties: (i) G_E is quasi-convex and lower semi-continuous; (ii) $G_E(\cdot, p)$ is increasing for all $p \in \Delta S$; (iii) $\inf_{p \in \Delta S} G_E(t,p) = t$ for all $t \in u_E(X)$.

Here the function $u_E(\cdot)$ corresponds to the standard vNM expected utility over lotteries. Note that in the utility representation (1), outcomes in states $s \notin E$ may still matter, so consequentialism is not yet imposed.

For all act $f \in \mathcal{F}$, E, let $\mathbf{u}_{E,f} := u_E(f) \in u_E(X)^{|S|}$ be the state-contingent expected utility vector associated with this act. Then define the UAP aggregator $I_E : u_E(X)^{|S|} \mapsto u_E(X)$ to be
$$V_E(f) = I_E(\mathbf{u}_{E,f}) = \inf_{p \in \Delta(S)} G_E\left(\int_S \mathbf{u}_{E,f}dp, p\right).$$
An *UAP aggregator* I_E is (i) continuous, (ii) monotone $\mathbf{u}_{E,f} \geq \mathbf{u}_{E,g}$ implies $I_E(\mathbf{u}_{E,f}) \geq I_E(\mathbf{u}_{E,g})$ for all $\mathbf{u}_{E,f}, \mathbf{u}_{E,g} \in u_E(X)^{|S|}$, (iii) normalized, i.e., $I_E(k\mathbf{1}_S) = k$ for all $k \in u_E(X)$, and (iv) quasi-concave, i.e., $I_E(\alpha \mathbf{u}_{E,f} + (1-\alpha)\mathbf{u}_{E,g}) \geq \min\{I_E(\mathbf{u}_{E,f}), I_E(\mathbf{u}_{E,g})\}$, for all $\alpha \in [0,1]$, $\mathbf{u}_{E,f}, \mathbf{u}_{E,g} \in u_E(X)^{|S|}$.

By Cerreia-Vioglio et al. (2011), the following statements are equivalent: (i) $\succsim_E$ satisfies Axiom 1; (ii) $\succsim_E$ admits an UAP representation (u_E, G_E); and (iii) $\succsim_E$ admits an UAP representation $V_E = I_E \circ u_E$.

To simplify exposition about updating preferences conditional on a non-empty event $E \in \Sigma^*$, I strengthen Monotonicity axiom to the following.

Axiom 2 (Strong monotonicity)**.** If $f(s) \succsim_E g(s)$ for all $s \in E$ and $f(s) \succ_E g(s)$ for some $s \in E$, then $f \succ_E g$.[8]

Strong Monotonicity emphasizes improvement in every state "count", which is used to simplify the discussions on conditional preferences elicitation.

Under UAP axioms and strong monotonicity, I_S is strictly monotone, $\mathbf{u} \geq (>)\mathbf{u}'$ implies $I(\mathbf{u}) \geq (>)I(\mathbf{u}')$ for all $\mathbf{u}, \mathbf{u}' \in u(X)^{|S|}$; and $G(\cdot, p)$ is strictly increasing for all $p \in dom(G)$.

Assumption 1. $\succsim_S$ admits some UAP representation with strongly monotone and concave $I_S : u(X)^{|S|} \mapsto \mathbb{R}$.

The next two axioms are consistency requirements connecting $\succsim_S$ and $\succsim_E$.

Axiom 3 (Stable constant-act preferences)**.** For all $E \in \Sigma^*$ and $x, y \in X$, $x \succsim_E y$ if and only if $x \succsim_S y$.

The axiom says the DM's preferences over constant acts should not change after she learns about event E.

Let $u_E \approx u$ denote that $u_E : X \mapsto \mathbb{R}$ is a positive affine transformation of $u : X \mapsto \mathbb{R}$, i.e., $u_E = au + b$ for some $a > 0$ and $b \in \mathbb{R}$.

Lemma 1. *Suppose $\succsim_S$ and $\succsim_E$ admit some UAP representations (u_S, G_S) and (u_E, G_E). Then $\succsim_S$ and $\succsim_E$ satisfy stable constant-act preferences if and only if $u_S \approx u_E$.*

Proof. Axiom 3 says that $\succsim_E$ and $\succsim_S$ agree on X, which is equivalent to u_E and u_S are ordinally equivalent. Since u_S and u_E are unique up to a positive affine transformation, $u_E = au_S + b$ for $a > 0$ and $b \in \mathbb{R}$. □

Axiom 4 (Conditional consistency, CC)**.** For all $f \in \mathcal{F}$, $x \in X$, and nonempty $E \in \Sigma^*$,

$$fEx \sim_S x \Rightarrow f \sim_E x. \tag{2}$$

The conditional consistency (CC) axiom connects the unconditional preferences $\succsim_S$ and the conditional preferences $\succsim_E$.[9] Note that the axiom only requires the $\Rightarrow$ direction. Intuitively, it specifies an updating rule from $\succcurlyeq_S$ to $\succcurlyeq_E$: one can calibrates the utility from an act f

[8]Here $f(s)$ and $g(s)$ are constant acts that give in every state outcomes $f(s)$ and $g(s)$, respectively.

[9]Pires (2002) first introduce this axiom to characterize prior-by-prior updating for Maxmin EU preferences.

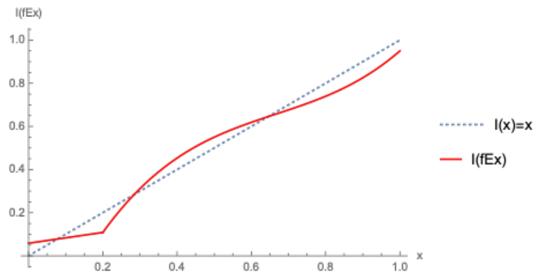

Figure 1: $I(\cdot)$ is quasi-concave

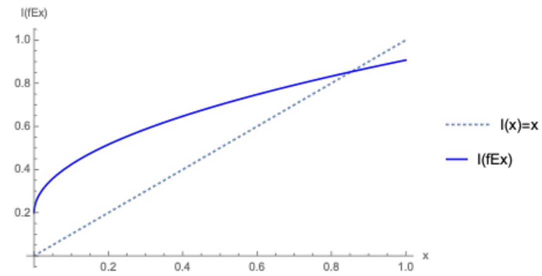

Figure 2: $I(\cdot)$ is concave

conditional on event E by finding a fixed point x to the indifference relation $fEx \sim_S x$. For an E-conditional certainty equivalent to be well defined, the fixed point above must be *essentially unique*; that is, $fEx \sim_S x$ and $fEy \sim_S y$ implies $x \sim_S y$.

Figures 1 and 2 illustrate why concavity of $I(\cdot)$ is needed for Axiom 4 to be well-defined in the UAP family. Suppose $X = [0,1]$ and $u(x) = x$. Fix some act f and event E and plot the function $I(fE\cdot) : [0,1] \mapsto [0,1]$. In Figure 1, $I(\cdot)$ is quasi-concave and strictly increasing, but there could be multiple fixed points $x \in X$ that solves the equation $I(fEx) = x$. In Figure 2, $I(\cdot)$ is additionally concave, in which case Lemma 2 below implies that there could be only a unique solution to the updating equation.

The next lemma provides a sufficient condition for the UAP preferences for the conditional certainty equivalent to be well-defined via Axiom 4.

Lemma 2. *Suppose $\succsim_S$ satisfies Assumption 1. Then, for all $f \in \mathcal{F}$ and $E \in \Sigma^*$, there exists an essentially unique $x \in X$ such that $fEx \sim_S x$.*

Remark: In general, the UAP axioms only imply $I(\cdot)$ is quasi-concave. In several special cases of UAP, including the variational preferences and the confidence preferences, concavity of $I(\cdot)$ follows from quasi-concavity. In other cases such as the smooth ambiguity preferences, additional conditions are needed to ensure that $I(\cdot)$ is concave. For this case, Lemma 7 in section 4.2 below provide an exact characterization for concavity.

Proposition 1. *Suppose $\{\succsim_E\}_{E\in\Sigma^*}$ satisfies Axioms 1 and 2, and $\succsim_S$ satisfies Assumption 1. For all $E \in \Sigma^*$, $\succsim_E$ and $\succsim_S$ satisfy stable constant-act preferences and conditional consistency if and only if $\succsim_E$ admits the induced UAP representation $V_E : \mathcal{F} \mapsto \mathbb{R}$ such that $V_E(f) = k$, and $k \in \mathbb{R}$ solves*

$$k = \inf_{\{p\in\Delta S: p(E)>0\}} G\left(\int_E u(f)dp + kp(E^c), p\right) \tag{3}$$

Proof. By Lemma 2, for UAP preferences that satisfies Assumption 1, the conditional certainty equivalent in Axiom 4 is well defined; i.e., the k that solves equation (3) is unique. $\square$

Due to the lack of separability of the general UAP functional, the conditional utility k determined by equation (3) does not always have an explicit representation. In special cases, when the uncertainty averse index $G(\cdot,\cdot)$ is additively separable or multiplicatively separable, one can separate the two terms $\int_E u(f)dp$ and $kp(E^c)$ in expression (3), and hence obtain an explicit representation of the conditional preferences $\succsim_E$ (in terms of (u, G)). This is the goal of sections 3.1 and 3.2.

2.3 Some preliminary results

First I establish a few prelimiary results for the UAP preferences.

Lemma 3. *Suppose $\succsim_S$ satisfies Assumption 1. For all $t \in u(X)$, any p_t such that $G(t, p_t) = \min_{p\in\Delta S} G(t,p) = t$, there must be $p_t(E) > 0$ for all $E \in \Sigma^*$.*

Definition 2. $\succsim_E$ satisfies *consequentialism* if $fEg \sim_E fEh$ for all $f, g, h \in \mathcal{F}$. An updating rule $(\succsim_S, E) \mapsto \succsim_E$ satisfy *consequentialism* if $\succsim_E$ satisfies *consequentialism* for all $E \in \Sigma^*$.

Clearly, for any $E \in \Sigma^*$, $\succsim_E$ and $\succsim_S$ satisfy CC implies $\succsim_E$ satisfies consequentialism.

Definition 3. Say $\succsim_E$ admits *a consequentialist conditional uncertainty-averse representation* if it is represented by

$$V_E(f) = \min_{p_E\in\Delta(E)} G_E\left(\int_E u_E(f)dp_E, p_E\right),$$

where $u_E : X \mapsto \mathbb{R}$ is affine and $G_E : u_E(X) \times \Delta E \mapsto (-\infty, +\infty]$ is a conditional UAP index that is lower semi-continuous, quasi-convex, $G_E(\cdot, p_E)$ is monotone for all $p_E \in \Delta E$, and $\inf_{p_E\in\Delta E} G_E(t, p_E) = t$ for all $t \in u_E(X)$.

Lemma 4. *$\succsim_E$ satisfy Axioms 1, 2, and consequentialism if and only if $\succsim_E$ admits a consequentialist conditional uncertainty-averse representation.*

3 Updating rules for invariant subfamilies

This section provides explicit characterization of the updating rule for two invariant subfamilies of the UAP preferences: (i) Variational preferences (VP), where $I(\cdot)$ is translation invariant; (ii) Confidence preferences (CP), where $I(\cdot)$ is positively scale invariant. Note that Assumption 1 holds in both families.

3.1 Variational preferences

The variational preferences family was characterized by Maccheroni et al. (2006a). It corresponds to the case where $u(X) = \mathbb{R}$ and the aggregator $I(\cdot)$ is *translation invariant*; i.e., $I(\mathbf{u} + b\mathbf{1}) = I(\mathbf{u}) + b$ for all $b \in \mathbb{R}$. It is equivalent to assuming the UAP index $G(\cdot, \cdot)$ is additively separable, that is, $G(t, p) = t + c(p)$. Note that for variational preferences, $I(\cdot)$ is concave and Assumption 1 always holds.

Axiom 5 (Variational preferences)**.** $\succsim_E$ satisfies the UAP Axiom 1, and

(i) (Weak Certainty Independence). For all $f, g \in \mathcal{F}$, $x, y \in X$, and $\alpha \in (0, 1)$,

$$\alpha f + (1 - \alpha)x \succsim_E \alpha g + (1 - \alpha)x \Rightarrow \alpha f + (1 - \alpha)y \succsim_E \alpha g + (1 - \alpha)y$$

(ii) (Unboundedness). For all $x \succ_E y$, there exists $z, z' \in X$ such that

$$\frac{1}{2}z + \frac{1}{2}y \succsim_E x \succ_E y \succsim_E \frac{1}{2}z' + \frac{1}{2}x$$

Say a function $g : Y \mapsto [0, +\infty]$ is *grounded* if there exists $y \in Y$ such that $g(y) = 0$.

Definition 4. For all $E \in \Sigma^*$, say the conditional preference relation $\succsim_E$ admits an *unbounded variational preferences representation* (u_E, c_E) if there exists some onto and affine vNM utility index $u_E : X \mapsto \mathbb{R}$ and some cost function $c_E : \Delta S \mapsto [0, +\infty]$ that is convex, lower semi-continuous, and grounded, such that $\succsim_E$ is represented by

$$V_E(f) = \min_{p \in \Delta S} \int_S u_E(f) dp + c_E(p). \tag{4}$$

Typically, equation (4) is interpreted as the DM is playing a zero-sum game against a malevolent player called Nature, who aims to minimize the DM's expected utility by choosing the probability model while paying a cost for her choice of probability.

For any $E \in \Sigma^*$, Maccheroni et al. (2006a) show that $\succsim_E$ satisfies Axiom 5 if and only if $\succsim_E$ admits some unbounded variational preferences representation (u_E, c_E). In addition, u_E is unique up to a positive affine transformation and c_E is unique.

Suppose $\succsim_E$ admits VP representation. By Lemma 4, $\succsim_E$ also satisfies consequentialism if and only if it can be represented by

$$V_E(f) = \min_{p_E \in \Delta E} \int_E u_E(f) dp_E + c_E(p_E), \tag{5}$$

where $c_E : \Delta E \mapsto [0, +\infty]$ is the conditional cost function.

The *domain of* c_E is the set $\{p \in \Delta E : c_E(p_E) < +\infty\}$ and will be denoted $dom(c_E)$. To simplify notation, I denote $u_S(\cdot)$ by $u(\cdot)$ and $c_S(\cdot)$ by $c(\cdot)$ when there is no confusion.

Definition 5. A collection of conditional variational preferences $\{\succsim_E\}_{E\in\Sigma^*}$ satisfies *generalized Bayesian updating for variational preferences* if $u_E \approx u$ and $c_E : \Delta E \mapsto [0, +\infty]$ is given by

$$c_E(p_E) = \min_{p\in\Delta(p_E)} \frac{c(p)}{p(E)}, \qquad \text{for all } p_E \in \Delta E. \tag{6}$$

Equation (6) can be interpreted as follows. In calculating the conditional cost at posterior p_E,

(i) Nature selects all prior probability models p whose Bayesian poterior is equal to p_E. [10]

(ii) Inside the minimum, the numerator $c(p)$ simply represents the unconditional cost to Nature in selecting the prior probability model p absent the information $s \in E$. It is multiplied by $\frac{1}{p(E)}$, the inverse probability of event E, which is a normalization to correct for the fact that some prior probability p does not rationalize the occurred event E well. For instance, if a probability model p induces a small penalty $c(p)$ but a small sample likelihood $p(E)$, it might still induce a high conditional cost. [11]

(iii) The minimization still reflects that the DM is playing a zero-sum game against a malevolent Nature in incorperating the information $p(E)$.

The main result (Theorem 1) says that Axioms 2, 3, 4, 5, are necessary and sufficient for the collection of conditional preferences $\{\succsim_E\}_{E\in\Sigma*}$ to admit unbounded variational preference representations that satisfy generalized Bayesian updating (equation (6)).

Theorem 1. *The following two statements are equivalent: (i) For every nonempty event* $E \in \Sigma^*$, $\succsim_E$ *satisfies Axioms 2 and 5, and* $\succsim_S$ *and* $\succsim_E$ *jointly satisfy Axioms 3 and 4; (ii) every* $\succcurlyeq_E$ *admits an unbounded variational representation* (u_E, c_E) *and the generalized Bayesian updating for variational preferences holds ; i.e.,* $u = au_E + b$ *for* $a > 0$ *and* $b \in \mathbb{R}$, *and, when* $u = u_E$, $c_E(\cdot)$ *is induced by* $c(\cdot)$ *via equation (6).*

[10] Hence, the constraint rules out model mis-specification in updating.

[11] The normalization in Eqn. (6) is similar in spirit to the inversed probability weighting (e.g., the Horvitz–Thompson estimator) commonly used in statistics, which is used to correct for unequal selection probability bias in a stratified sample.

Below is a sketch of its proof. The details are relegated to the appendix.

First, for each $E \in \Sigma^*$, the variational preferences Axiom 5 are necessary and sufficient for the preference relation $\succsim_E$ to admit an unbounded variational preferences representation (u_E, c_E) (Maccheroni et al., 2006a). Second, Strong Monotonicity of $\succsim_S$ implies that every state is non-trivial and hence a useful property of the unconditional cost function $c(\cdot)$. By Lemma 3, For all $p \in \Delta S$ such that $c(p) = 0$, $p(E) > 0$ for all $E \in \Sigma^*$. In other words, any probability p that is so plausible that $c(p) = 0$ must have full support. Third, observe that for variational preferences, the aggregator $I_S(\mathbf{u}_f) = \inf_{p \in \Delta S} \int_S \mathbf{u}_f dp + c(p)$ is the pointwise infimum of a collection of affine functions of $\mathbf{u}_f$, and hence it is concave and staisfies Assumption 1. By Lemma 2, the conditional certainty equivalent defined via Axiom 4 is well defined. Fourth, by the CC axiom, the conditional variational preference relation $\succsim_E$ satisfies consequentialism. Applying Lemma 4, in the variational representation (u_E, c_E), it is without loss to restrict the domain of c_E to ΔE. Fifth, by Lemma 1, Stable constant-act preferences implies that $\succsim$ and $\succsim_E$ agree on X, and hence $u \approx u_E$. Without loss, one can let $u = u_E$. Together, $\succsim_E$ can be represented by

$$V_E(f) = \min_{p_E \in \Delta E} \int_E u(f) dp_E + c_E(p_E),$$

where $u : X \mapsto \mathbb{R}$ is an onto and affine function and $c_E : \Delta E \mapsto [0, +\infty]$ is a convex, lower semi-continuous, and grounded conditional cost function.

Then, the key is to show that the conditional cost function $c_E(\cdot)$ must be the induced by the unconditional cost function $c(\cdot)$ exactly as equation (6). Take some variational representation (u, c) of the unconditional preference relation $\succsim_S$, where the cost function $c : \Delta S \mapsto [0, +\infty]$ is convex, lower semi-continuous, and grounded. For every non-empty event $E \in \Sigma^*$, define the following conditional cost function $\tilde{c}_E : \Delta E \mapsto [0, +\infty]$ induced by the unconditional cost function $c(\cdot)$:

$$\tilde{c}_E(p_E) = \inf_{p \in \Delta(p_E)} \frac{c(p)}{p(E)}, \qquad \forall p_E \in \Delta E. \tag{7}$$

Lemmas 9 and 10 in appendices 1.4 and 1.5 verify that $\tilde{c}_E(\cdot)$ defined by equation (7) attains its minimum within the set $\Delta(p_E)$[12] and it inherits convexity, lower semi-continuity, and groundedness from function $c(\cdot)$. Hence, $\tilde{c}_E(\cdot)$ is equivalent to the conditional cost function $c_E(\cdot)$ defined by equation (6).

Then, define the function $\tilde{V}_E : \mathcal{F} \mapsto \mathbb{R}$ that is induced by the unconditional representation

[12] Recall that $\Delta(p_E)$ requires its element to satisfy $p(E) > 0$, and hence it is not a closed set.

(u, c) as follows: for all $f \in \mathcal{F}$,

$$\begin{aligned}\tilde{V}_E(f) &= \min_{p_E \in \Delta(E)} \left(\int_E u(f) dp_E + \min_{p \in \Delta(p_E)} \frac{c(p)}{p(E)} \right) \\ &= \inf_{\{p \in \Delta S: p(E) > 0\}} \int_E u(f) dp(\cdot|E) + \frac{c(p)}{p(E)}. \end{aligned} \tag{8}$$

Lemma 5. *Suppose $\succsim_S$ satisfies Axioms 2 and 5. For all $f \in \mathcal{F}$, if $fEx \sim_S x$, then*

$$V_S(fEx) = \tilde{V}_E(f).$$

Under the CC axiom, x, the conditional certainty equivalent of f, must satisfy $V_S(fEx) = u(x)$. Lemma 5 says the utility of the conditional certainty equivalent of f is also pinned down by the induced function $\tilde{V}_E(f)$. In other words, $\succsim_E$ can be represented by the induced function $\tilde{V}_E(\cdot)$. Also, $V_E(\cdot)$ is "constructed" as variational utility function using vNM index u and the induced cost function $\tilde{c}_E(\cdot)$ defined by equation (6). Hence, it suffices to show that $\tilde{c}_E$ is the unique conditional cost function for any variational representation of $\succsim_E$ (with vNM index u), which is shown by the following lemma.

Lemma 6. *Pick any nonempty event $E \in \Sigma^*$. Suppose two unbounded variational representations (u, c_E) and (u, c'_E) both represent $\succsim_E$. Then $c_E = c'_E$.*

Lemma 6 verifies the uniqueness of the conditional cost function. For any two variational utility functions V_E and V'_E with the same vNM index u but different cost functions, these two utility functions must be strictly separated at some act. To see this, if $c_E(p_0) < c'_E(p_0)$ at some $p_0 \in \Delta E$, then one can find a hyperplane in $\mathbb{R}^{|E|+1}$ that strictly separates the point $(p_0, c_E(p_0))$ and the epigraph of $c'_E(\cdot)$.[13] Take the normal vector $\mathbf{v} \in \mathbb{R}^{|E|+1}$ of this separating hyperplane. Since $u(X) = \mathbb{R}$, there always exists some act $f \in \mathcal{F}$ whose the restriction on event E, the act f_E, satisfy that the vector $(u(f_E), 1) \in \mathbb{R}^{|E|+1}$ is proportional (by a strictly positive scalar) to this normal vector $\mathbf{v}$. For this act, $V_E(f) < V'_E(f)$, and hence the two variational functional cannot represent the same $\succsim_E$.

Remark 1. Here the unconditional preferences $\succcurlyeq_S$ and the conditional preferences $\succcurlyeq_E$ may not jointly satisfy dynamic consistency. Nevertheless, if the *ex ante* preferences allows for preferences for the temporal resolution of uncertainty, they may still be recursively generated by $\succcurlyeq_S$ (and its updates $\succcurlyeq_E$). Li (2020) proposes a general framework to generate these recursive preferences.

[13] The epigraph of c'_E is $epi(c'_E) = \{(p, r) \in \Delta E \times \mathbb{R} : c'_E(p) \leq r\}$. It is nonempty, convex, closed and bounded below.

3.2 Confidence preferences

Another special case is the confidence preferences family characterized by Chateauneuf and Faro (2009). It corresponds to the case when there exists a worst outcome $x_* \in X$ and $I(\cdot)$ is *positively scale invariant*.[14] Again, for confidence preferences $I(\cdot)$ is concave and Assumption 1 always holds.

Definition 6. Say $\succsim_E$ admits a *maximal confidence preferences representation* consisting a pair (u_E, φ_{1E}) if $\succsim_E$ is represented by

$$V_E^{CP}(f) = \inf_{p \in \Delta S} \frac{\int_S u_E(f) dp}{\varphi_{1E}(p)},$$

where $u_E : X \mapsto \mathbb{R}_+$ is an affine function with $u(x_*) = 0$, and $\varphi_{1E} : \Delta S \mapsto [0,1]$ is a *confidence function* that is quasi-concave, upper semi-continuous, and $\varphi_{1E}(p) = 1$ for some $p \in \Delta S$. Moreover, if (u_E, φ_{1E}) and (u_E, φ'_{1E}) are both confidence representations of $\succsim_E$, then $\varphi_{1E} \geq \varphi'_{1E}$.

By standard duality argument,

$$\varphi_{1E}(p) = \inf_{f \in \mathcal{F}} \left(\frac{\int_S u_E(f) dp}{u_E(c_f)} \right),$$

where $c_f \in X$ is the certainty equivalent of f such that $c_f \sim_E f$. Note that the maximal confidence function $\varphi_{1E}(\cdot)$ is the pointwise infimum of a collection of linear functions of p, and hence it is concave.

For the unconditional preference relation $\succsim_S$, it admiting a maximal confidence preferences representation is a special case of UAP representation, where the UAP index $G(\cdot, \cdot)$ is *multiplicatively separable*: for all $(t, p) \in \mathbb{R}_+ \times \Delta S$,

$$G(t,p) = \begin{cases} \frac{t}{\varphi_1(p)} & p \in C \\ +\infty & p \notin C \end{cases}$$

where $C = \{p \in \Delta : \varphi_1(p) > 0\}$ is the domain of the confidence function φ_1. It's straightforward to check that C is a non-empty, convex, and closed subset of ΔS.

Suppose X is bounded below according to $\succsim_S$; i.e., there exists some *worst outcome* $x_* \in X$ such that $x \succsim_S x_*$ for all $x \in X$.

[14]By Cerreia-Vioglio et al. (2011), it is equivalent to assuming the UAP index $G(t, p)$ is multiplicatively separable in t and p.

Axiom 6 (Confidence Preferences)**.** For $E \in \Sigma^*$, $\succsim_E$ satisfies Axiom 1 and the following axioms.

(i) (Independence to the worst outcome.) For all $f, g \in \mathcal{F}$, $\alpha, \beta \in (0,1)$,

$$\alpha f + (1-\alpha)x_* \succsim_E \alpha g + (1-\alpha)x_* \Rightarrow \beta f + (1-\beta)x_* \succsim_E \beta g + (1-\beta)x_*.$$

(ii) (Bounded attraction for certainty.) There exists $\delta \geq 1$ such that for all $f \in \mathcal{F}$ and $x, y \in X$:

$$x \sim_E f \Rightarrow \frac{1}{2}x + \frac{1}{2}y \succsim_E \frac{1}{2}f + \frac{1}{2}\left(\frac{1}{\delta}y + \left(1 - \frac{1}{\delta}x_*\right)\right)$$

That is, $\succsim_E$ satisfies independence with respect to the worst lottery outcome.

By Chateauneuf and Faro (2009) (Theorem 3, Corollary 5), $\succsim_E$ satisfies Axiom 6 if and only if $\succsim_E$ admits a unique maximal confidence preference representation (u_E, φ_{1E}). Moreover, if both (u_E, φ_{1E}) and (u'_E, φ'_{1E}) are maximal confidence representation of $\succsim_E$, then $u_E = \lambda u'_E$ for some $\lambda > 0$ and $\varphi_{1E} = \varphi'_{1E}$.

Definition 7. Say $\succsim_E$ admits a *consequentialist* maximal confidence preferences representation if

$$V_E^{CP}(f) = \inf_{p \in \Delta E} \frac{\int_E u_E(f)dp}{\varphi_{1E}(p)} \tag{9}$$

for some affine function $u_E : X \mapsto \mathbb{R}_+$ with $u_E(x_*) = 0$, and confidence function $\varphi_{1E} : \Delta E \mapsto [0,1]$.

Definition 8. A collection of conditional confidence preferences $\{\succsim_E\}_{E \in \Sigma^*}$ satisfies *generalized Bayesian updating for confidence preferences* if $\succsim_E$ is represented by (9) with $u_E \approx u$ and $\varphi_{1E} : \Delta E \mapsto [0,1]$ is induced by $\varphi_1 : \Delta S \mapsto [0,1]$ via the following formula:

$$\varphi_{1E}(p_E) = \max_{\{p \in \Delta(p_E) : \varphi_1(p) > p(E^c)\}} \left(1 - \frac{1 - \varphi_1(p)}{p(E)}\right) \qquad \text{for all } p_E \in \Delta E. \tag{10}$$

Equation (10) characterizes the conditional confidence at the posterior p_E. The constraint set, $\{p \in \Delta(p_E) : \varphi_1(p) > p(E^c)\}$, contains all the priors p with Bayesian posterior equal to p_E and the unconditional confidence $\varphi_1(p)$ is at least $p(E^c)$. The latter requirement makes sure that the term $1 - \frac{1-\varphi_1(p)}{p(E)} = \frac{\varphi_1(p)}{p(E)} - \frac{p(E^c)}{p(E)}$ is positive. Overall, the maximum picks the prior p with Bayesian posterior p_E that leads to the highest confidence level $\varphi_1(p)$ normalized by the maximum likelihood of event E. Again, the procedure reflects a concern for cautiousness in updating the confidence level.

Equation (10) can be rewritten as

$$\varphi_{1E}(p_E) = \max_{\{p \in \Delta(p_E): \varphi_1(p) > p(E^c)\}} \left(\frac{\varphi_1(p)}{p(E)} - \frac{p(E^c)}{p(E)} \right).$$

That is, the conditional confidence aims to maximize the difference between the normalized prior confidence level $\frac{\varphi_1(p)}{p(E)}$ and the odds ratio $\frac{p(E^c)}{p(E)}$ of the occurred event.

Theorem 2. *The following two statements are equivalent: (i) For every nonempty event $E \in \Sigma^*$, $\succsim_E$ satisfies Axioms 2 and 6 and $\succsim_S$ and $\succsim_E$ jointly satisfy Axioms 3 and 4; (ii) every $\succcurlyeq_E$ admits a maximal confidence preferences representation (u_E, φ_{1E}) and the generalized Bayesian updating for confidence preferences holds; i.e., $u = au_E$ for $a > 0$, and, when $u = u_E$, $\varphi_{1E}(\cdot)$ is induced by $\varphi_1(\cdot)$ via equation (10).*

The intuition of the proof is similar to that of Theorem 1 and hence omitted. The proof is in Appendix 1.12.

4 Discussion

4.1 Maxmin EU preferences, multiplier preferences, and monotone mean-variance preferences

In this subsection, I consider three important special cases of variational preferences—maxmin EU, multiplier preferences, and monotone mean-variance preferences. The generalized variational preferences updating formula (6) nests three well-known updating rules for the special ambiguity families: Prior-by-prior updating of the maxmin EU preferences, Bayesian updating of the multiplier preferences, and Bayesian updating of the monotone mean-variance preferences.

Example 1. Suppose the preference relation $\succsim_E$ admits the maxmin EU representation

$$V_E(f) = \min_{p_E \in P_E} \int_E u_E(f) dp_E,$$

if the conditional cost function takes the form

$$c_E(p_E) = \begin{cases} 0 & \text{if } p_E \in P_E \\ +\infty & \text{otherwise} \end{cases},$$

where $P_E \subseteq \Delta E$ is a non-empty, closed, and convex set of priors on E. Then equation (6) implies the prior-by-prior updating rule; i.e., $P_E = \{p(\cdot|E) : p \in P_S\}$.

If $P_E = \{p_E\}$ is a singleton, then $\succsim_E$ admits the SEU representation

$$V_E(f) = \int_E u_E(f) dp_E,$$

and equation (6) imply the Bayesian updating rule.

Example 2. Suppose the preference relation $\succsim_E$ admits the multiplier preferences representation

$$V_E(f) = \min_{p_E \in \Delta(E)} \int_E u_E(f) dp_E + \theta R(p_E \| q_E),$$

if the conditional cost function takes the form

$$c_E(p_E) = \theta R(p_E \| q_E) = \theta \int_E \ln \frac{p_E}{q_E}(s) dp_E(s),$$

where $q_E \in \Delta E$ is a reference prior with full support[15], $\theta > 0$ is a parameter, and $R(p_E \| q_E) = \int_E \ln \left(\frac{p_E}{q_E} \right) dp_E$ is the relative entropy distance to the reference probability q_E. Then equation (6) implies Bayesian updating of the reference prior; i.e., $q_E = q_S(\cdot | E)$.

Example 3. Suppose X is the set of all monetary lotteries and $u(z) = z$ for all $z \in \mathbb{R}$. The preference relation $\succsim_E$ admits the monotone mean-variance preferences representation (θ, q_E)

$$V_E(f) = \min_{p_E \in \Delta(E)} \int_E f dp_E + \theta G(p_E \| q_E),$$

if the conditional cost function takes the form

$$c_E(p_E) = \theta G(p_E \| q_E) = \theta \int_E \frac{1}{2} \left(\frac{p_E(s)}{q_E(s)} - 1 \right)^2 dp_E(s)$$

where $q_E \in \Delta E$ is a reference prior with full support, $\theta > 0$ is a parameter of ambiguity aversion, and $G(p_E \| q_E)$ is the Gini index. Then $\succsim_E$ also admits the classic mean-variance utility representation

$$U_E(f) = \int_E f dq_E(s) - \frac{1}{2\theta} Var_E(f)$$

for the subset of acts $\{f : f(s) - \int_E f dq_E \leq \theta, \forall s \in E\}$, where $Var_E(f)$ is the variance of f according to q_E (Maccheroni et al., 2006a, Theorem 24).

For monotone mean-variance preferences, equation (6) implies Bayesian updating of the reference prior; i.e., $q_E = q_S(\cdot | E)$.

The next corollary applies the updating equation (6) to these three special cases. The updating rule implies the well-known prior-by-prior updating rule for maxmin EU preferences, and Bayesian updating of the reference prior for multiplier preferences and monotone mean-variance preferences.

[15] The reference priors in Example 2 and 3 have full support, because $\succsim_E$ is strongly monotone.

Corollary 1. *Suppose all the assumptions in Theorem 1 hold.*

1. *If* $\{\succsim_E\}_{E\in\Sigma^*}$ *admit MEU representations* (u_E, P_E) *and* $u_E \approx u$*, then equation (6) implies the prior-by-prior updating rule; i.e.,* $P_E = \{p(\cdot|E) : p \in P_S\}$*.*

2. *If* $\{\succsim_E\}_{E\in\Sigma^*}$ *admit multiplier preferences representations* (u_E, q_E, θ) *and* $u_E \approx u$*, then equation (6) implies Bayesian updating of the reference prior; i.e.,* $q_E = q_S(\cdot|E)$*.*

3. *Suppose* X *is a set of monetary lotteries. If* $\{\succsim_E\}_{E\in\Sigma^*}$ *admit monotone mean-variance preferences representation* (u_E, q_E, θ) *where* $u_E(t) = t \in \mathbb{R}$*, then equation (6) implies Bayesian updating of the reference prior; i.e.,* $q_E = q_S(\cdot|E)$*.*

4.2 Smooth ambiguity preferences

With smooth ambiguity preferences (Klibanoff et al., 2005), we have

$$V_S(f) = \phi^{-1}\left(\int_{\Delta S} \phi\left(\int u(f)dp\right) d\mu(p)\right)$$

For $\xi \in u(X)^{|S|}$, define the aggregator

$$I_S(\xi) := \phi^{-1}\left(\int_{\Delta S} \phi\left(\int \xi dp\right) d\mu(p)\right)$$

Alternatively, for any fixed $\xi \in u(X)^{|S|}$ and $\mu \in \Delta\Delta S$, define

$$\tilde{X}_\xi(p) = \int_S \xi(s)dp(s), \qquad \forall p \in \Delta S,$$

where $\tilde{X}_\xi$ is a random variable on $\mathbb{R}$ induced by μ.

Define $C_\phi : \Delta\mathbb{R} \mapsto \mathbb{R}$ to be

$$C_\phi(\tilde{X}_\xi) := \phi^{-1}\left(\int_{\Delta S} \phi\left(\tilde{X}_\xi(p)\right) d\mu(p)\right)$$

So C_ϕ is the certainty equivalent function for the real-value random variable $\tilde{X}_\xi$.

For $\phi \in C^3$ (i.e., ϕ is three-times continuously differentiable), define the measure of absolute ambiguity aversion as $A_\phi(\cdot) := -\frac{\phi''(\cdot)}{\phi'(\cdot)}$.

Lemma 7. *For the smooth ambiguity preferences $\succsim_S$, suppose $\phi \in C^3$, $\phi' > 0$, and $\phi'' < 0$. Then the following are equivalent: (1) $I(\cdot)$ is concave; (2) The certainty equivalent function $C_\phi(\cdot)$ is concave; (3) $\frac{1}{A_\phi(\cdot)}$ is concave.*[16]

Corollary 2. *If $\{\succsim_E\}_{E\in\Sigma^*}$ admit smooth ambiguity preferences representations and $\succsim_S$ admit a smooth representation with (u, ϕ, μ). Suppose $\phi \in C^3$, $\phi' > 0$ and $\phi'' < 0$, and $\frac{1}{A_\phi(\cdot)}$ is concave.*

Then stable constant-act preferences and conditional consistency hold if and only if $\succsim_E$ admits representation $V_E : \mathcal{F} \mapsto \mathbb{R}$ with $V_E = I_E \circ u$ such that for all $f \in \mathcal{F}$, $I_E(u(f)) = k^* \in \mathbb{R}$ that uniquely solves

$$k^* = \phi^{-1}\left(\int_{\Delta S} \phi\left(\int_E u(f)dp + k^* p(E^c)\right) d\mu(p)\right) \tag{11}$$

Proof. "If" part is obvious. To see the "only if" part, by Lemma 7, $I_S(\cdot)$ satisfies Assumption 1 and hence the conditional certainty equivalent is well-defined. That is, for all $f \in \mathcal{F}$, the equivalence relation $fEx \sim_S x$ induces the equation

$$V_S(fEx) = \phi^{-1} \int_{\Delta S} \phi\left(\int_E u(f)dp + p(E^c)u(x)\right) d\mu(p) = u(x),$$

which has a unique solution $u(x) \in u(X)$. Let $k^* = u(x)$. Then equation (18) follows from conditional consistency axiom. □

Remark 2. In the smooth ambiguity representation, when $\mu \in \Delta\Delta S$ has support on Dirac probabilities on X, then μ can be identified with a $p_0 \in \Delta S$ while the DM still has second-order risk aversion captured by the concave functional $\phi(\cdot)$. In this special case, $\succsim_S$ admits a Second-order EU (SOEU) representation (u, ϕ, p_0) (Grant et al., 2009)

$$V_S(f) = \phi^{-1}\left(\int_S \phi(u(f))dp\right).$$

For this case, when ϕ satisfies the same conditions in Corollary 2, stable constant act preferences and conditional consistency are equivalent to $\succsim_E$ admits the following utility representation:

$$V_E(f) = k^* = \phi^{-1}\left(\int_E \phi(u(f))dp_0 + k^* p_0(E^c)\right) \tag{12}$$

See Appendix 1.15 for detail.

[16] I thank Todd Sarver for suggesting a reference for the proof.

4.3 Comparative ambiguity

Here I consider the notion of interpersonal comparison of ambiguity aversion introduced by Ghirardato and Marinacci (2002). The updating rule characterized for the strongly monotone and concave UAP family preserves the comparative ambiguity attitudes; i.e., if $\succsim_S^1$ is more ambiguity averse than $\succsim_S^2$, then the updated preferences $\succsim_E^1$ is still more ambiguity averse than $\succsim_E^2$.

Definition 9. For any non-empty event $E \in \Sigma^*$ and two two DMs with UAP preferences $\succsim_E^1$ and $\succsim_E^2$, say DM 1 is *more ambiguity averse* than DM 2 if (i) $\succsim_E^1 = \succsim_E^2$ on X and (ii) $f \succsim_E^1 x \Rightarrow f \succsim_E^2 x$ for all $f \in \mathcal{F}$ and all $x \in X$.

Corollary 3. *Suppose two DMs have concave and strongly monotone UAP preferences $\succsim_S^1$ and $\succsim_S^2$ satisfying Assumption 1. For all $E \in \Sigma^*$, $\succsim_S^i$ and $\succsim_E^i$ satisfy stable constant-act preferences and conditional consistency for $i = 1, 2$.*

Then, if unconditionally $\succsim_S^1$ is more ambiguity averse than $\succsim_S^2$, then the updated preferences $\succsim_E^1$ is also more ambiguity averse than $\succsim_E^2$.

Proof of Corollary 3. Take any $f \in \mathcal{F}$ and $x \in X$ such that $f \succsim_E^1 x$. Let $x_0 \in X$ be the essentially unique conditional certainty equivalent by DM 1; i.e., $fEx_0 \sim_S^1 x_0$. Then conditional consistency implies $f \sim_E^1 x_0 \succsim_E^1 x$. DM 2 is unconditionally less ambiguity averse than DM 1 thus $fEx_0 \succsim_S^2 x_0$.

Lemma 8. *Suppose $\succsim_S$ satisfies Assumption 1. For all $E \in \Sigma^*$, under stable constant-act preferences, the following statements are equivalent: (i) $\succsim_S$ and $\succsim_E$ satisfy conditional consistency; (ii) $fEx \succsim_S x \Rightarrow f \succsim_E x$ and $fEy \precsim_S y \Rightarrow f \precsim_E y$ for all $f \in \mathcal{F}, x, y \in X$.*

The proof of Lemma 8 is in appendix 1.14. It implies $f \succsim_E^2 x_0$. Since $\succsim_E^2 = \succsim_E^1$ on X, $f \succsim_E^2 x$ for all x that satisfies $f \succsim_E^1 x$.

□

A Appendix: Proofs

1.1 Proof of Lemma 2

Proof. Fix an act $f \in \mathcal{F}$ and event $E \in \Sigma^*$. There exists x_*, x^*, $x_* \precsim f(s)$ and $x^* \succsim f(s)$ for all $s \in E$. Let $a = u(x_*)$ and $b = u(x^*)$. Define function $g : [a, b] \mapsto [a, b]$ where

$g(z) := I(u(f)Ez)$. Then g is strictly increasing, continuous, and concave. Since $I(\cdot)$ is normalized, $g(a) \geq I(a\mathbf{1}) = a$, and $g(b) \leq I(b\mathbf{1}) = b$.

Define $h(z) = g(z) - z$. Then the function $h : [a, b] \mapsto [a, b]$ is continuous, and $h(a) \geq 0$ and $h(b) \leq 0$. By intermediate value theorem, there exists $z^* \in [a, b]$, $h(z^*) = 0$.

Next I show the solution z^* is unique. Note that $g(z) = I(u(f)Ez)$ is concave as $I(\cdot)$ is concave. So $h(z) = g(z) - z$ is concave since it is the sum of two concave functions.

First, suppose for a fixed f one can find some $x_* \in X$ such that $g(a) > a$ for $a = u(x_*)$; i.e., $f(s) \succ x_*$ for some $s \in E$. Suppose there are more than one solution to $h(z) = 0$ on $[a, b]$. Let z_1^* be the smallest solution and $z_2^* > z_1^*$ be another solution. Then there are two cases: (i) there exists $z' \in (z_1^*, z_2^*)$ such that $h(z') < 0$; (ii) $h(z') \geq 0$ for all $z' \in (z_1^*, z_2^*)$.

For case (i), there exists some $\lambda \in (0, 1)$ such that $z' = \lambda z_1^* + (1 - \lambda)z_2^*$. Yet

$$0 > h(z') = h(\lambda z_1^* + (1 - \lambda)z_2^*) \geq \min\{h(z_1^*), h(z_2^*)\} = 0$$

where the $\geq$ step follows from (quasi-)concavity of the $h(\cdot)$ function. This is a contradiction.

For case (ii), note that z_1^* is the smallest solution to $h(z) = 0$ on $[a, b]$ and $h(a) > 0$. Since h is a continuous function, there exists some $z'' \in [a, z_1^*)$ such that $h(z'') > 0$. Pick any $z' \in (z_1^*, z_2^*)$, then $h(z') \geq 0$. Moreover, there exists $\alpha \in (0, 1)$ such that $z_1^* = \alpha z'' + (1-\alpha)z'$. Then

$$0 = h(z_1^*) = h(\alpha z'' + (1 - \alpha)z') \geq \alpha h(z'') + (1 - \alpha)h(z') > 0,$$

where the $\geq$ step follows from the concavity of $h(\cdot)$. Again this is a contradiction.

Finally, I rule out the corner case where for the fixed f, one cannot find $x_* \in X$ such that $g(a) > a$ for $a = u(x_*)$. This is the case when $u(X)$ is bounded below at a and $u(f(s)) = a$ for all $s \in E$. Then there exists some $x_* \in X$ such that $x \succsim x_*$ for all $x \in X$, and for this $a = u(x_*)$ we have $g(a) = a$; i.e., $I(u(f)Ea) = a = u(x_*)$. In this case, $fEx_* \sim x_*$. Since x_* is the least preferred element in X, $f(s) \succsim x_*$ for all $s \in E$. By Strong Monotonicity of $\succsim$, $f(s) \sim x_*$ for all $s \in E$, because if not there must be $fEx_* \succ x_*$. Suppose there is another $y \in X$ that satisfies $fEy \sim y$. Then by Strong Monotonicity $f(s) \sim x_*$ for all $s \in E$ implies $x_*Ey \sim fEy \sim y$, which implies $x_* \sim y$. So the conditional certainty equivalent is essentially unique at the corner case. □

1.2 Proof of Lemma 3

Proof of Lemma 3. Take any $t \in u(X)$, it suffices to show $p_t(E) > 0$ for all non-empty $E \in \Sigma^*$.

Construct vectors ξ, $\xi' \in u(X)^{|S|}$,

$$\xi(s) = t, \ \forall s \in S, \qquad \xi'(s) = \begin{cases} t + \epsilon, \ \forall s \in E \\ t, \ \forall s \in E^c \end{cases},$$

where $\epsilon > 0$ is arbitrarily small.

Suppose $p_t(E) = 0$ for some E. Then $\int \xi' dp_t = \int \xi dp_t = t$. Hence,

$$I_S(\xi) = \min_{p \in \Delta S} G(\int \xi dp, p) = G(\int \xi dp_t, p_t) = G(t, p_t) = t$$

(by definition of p_t). And

$$I_S(\xi') = \min_{p \in \Delta S} G(\int \xi' dp, p) \leq G(\int \xi' dp_t, p_t) = G(t, p_t) = t.$$

But $\xi' > \xi$, by strong monotonicity of I_S, there must be $I_S(\xi') > I_S(\xi)$. A contradiction. □

1.3 Proof of Lemma 4

Proof. Note that $\succsim_E$ satisfies Axiom 1 and strong monotonicity if and only if $\succsim_E$ admits a strongly monotone UAP representation $(u_E, \tilde{G}_E)$

$$V_E(f) = \min_{p \in \Delta S} \tilde{G}_E \left(\int_S u_E(f) dp, p \right), \tag{13}$$

where $u_E : X \mapsto \mathbb{R}$ is an affine vNM utility and $\tilde{G}_E : u_E(X) \times \Delta S \mapsto (-\infty, +\infty]$ is a conditional UAP index that is lower semi-continuous, quasi-convex, $\tilde{G}_E(\cdot, p)$ is strongly monotone for all $p \in \Delta S$, and $\inf_{p \in \Delta S} \tilde{G}_E(t, p) = t$ for all $t \in u_E(X)$.

"If" direction is obvious. When the minimizer of equation (13) always belong to ΔE, $\succsim_E$ satisfies consequentialism.

For the "only if" direction, fix any $E \in \Sigma^*$. For an arbitrary $f \in \mathcal{F}$, because S is finite, there is some $x_* \in X$ such that $f(s) \succ_E x_*$ for all s. By consequentialism of $\succsim_E$, $fEx_* \sim_E f$. Let $p^* \in \Delta S$ be an arbitrary minimizer in equation (13). Then

$$V_E(f) = \tilde{G}_E \left(\int_S u_E(f) dp^*, p^* \right) = V_E(fEx_*) \leq \tilde{G}_E \left(\int_E u_E(f) dp^* + p^*(E^c) u_E(x_*), p^* \right).$$

Strict monotonicity of $\tilde{G}_E(\cdot, p^*)$ implies $\int_{E^c} (u_E(f) - u_E(x_*)) dp^* \leq 0$. Since $u_E(f)(s) - u_E(x_*) > 0$ for every state $s \in E^c$, the implication holds if and only if $p^*(E^c) = 0$.

Therefore, for every $f \in \mathcal{F}$, any minimizing probability p^* in $V_E(f)$ always has full support on E. Therefore, without loss one can express the utility representation of $\succsim_E$ as

$$V_E(f) = \min_{p \in \Delta S} \tilde{G}_E\left(\int_S u_E(f)dp, p\right) = \min_{p_E \in \Delta E} G_E\left(\int_E u_E(f)dp_E, p_E\right),$$

where $G_E : u_E(X) \times \Delta E \mapsto (-\infty, +\infty]$ is the restriction of $\tilde{G}_E$ on $u_E(X) \times \Delta E$. □

1.4 Lemma 9

Lemma 9. *If $\succsim_S$ satisfies Axioms 2 and 5, the conditional cost function $\tilde{c}_E$ defined by equation (7) attains its minimum.*

Proof. Given c, the function $\tilde{c}_E : \Delta E \mapsto [0, +\infty]$ is

$$\tilde{c}_E(p_E) = \inf_{p \in \Delta(p_E)} \frac{c(p)}{p(E)}.$$

Recall $\Delta(p_E) = \{p \in \Delta S : p(\cdot|E) = p_E\}$. If $c(p) = +\infty$ for all $p \in \Delta(p_E)$, then $c_E(p_E) = +\infty$ and the minimum value attains at any $p \in \Delta(p_E)$. Otherwise, $c(p) < +\infty$ for some $p \in \Delta(p_E)$ and thus $c_E(p_E) < +\infty$. Define function $\phi : \Delta S \mapsto [0, +\infty]$ where $\phi(p) = \frac{c(p)}{p(E)}$.

First, $\phi(\cdot)$ is a lower semi-continuous function on ΔS. To see this, for all $r \in [0, +\infty]$, let $L_r = \{q \in \Delta S : \frac{c(q)}{q(E)} \leq r\}$. Take any sequence q^n in L_r, which satisfies $c(q^n) \leq rq^n(E)$ for all n. This implies

$$c(q) \leq \lim_m \inf_{n \geq m} c(q^n) \leq r \lim_n q^n(E) = rq(E),$$

where the first inequality follows from lower semi-continuity of function c, the second inequality follows from that $q^n \in L_r$, and the last equality from $q^n \to q$.

Second, $\overline{\Delta(p_E)} = \Delta(p_E) \cup \Delta(E^c)$ is a compact subset $\Delta(S)$, and hence $\inf_{p \in \overline{\Delta(p_E)}} \phi(p)$ attains its minimum on $\overline{\Delta(p_E)}$. To see this, $\overline{\Delta(p_E)}$ is obviously bounded. And for any sequence $\{q^n\} \subseteq \overline{\Delta(p_E)}$, there exists a convergent subsequence $\{q^l\}$ such that $q^l \to q^* \in \Delta S$ since ΔS is compact. It remains to show that q^* belongs to $\overline{\Delta(p_E)}$, which is closed. If $q^*(E) = 0$, then $q^* \in \Delta E^c \subseteq \overline{\Delta(p_E)}$. Suppose $q^*(E) > 0$. Then, for sufficiently large l, $q^l(E) > 0$ and $q^l \in Q(p_E)$. Therefore,

$$\frac{q^*(B \cap E)}{q^*(E)} = \lim_l \frac{q^l(B \cap E)}{q^l(E)} = p_E(B) \qquad \text{for all } B \in \Sigma,$$

and thus $q^* \in \Delta(p_E) \subseteq \overline{\Delta(p_E)}$.

Hence, $\tilde{c}_E(p_E) = \inf_{p \in \overline{\Delta(p_E)}} \frac{c(p)}{p(E)}$, and there exists some $p^* \in \overline{\Delta(p_E)}$ such that $\tilde{c}_E(p_E) = \frac{c(p^*)}{p^*(E)}$. First suppose $p^* \in \Delta(E^c)$. Then $p^*(E) = 0$, and by Lemma 3, $c(p^*) > 0$. For all $p \in \Delta(p_E)$,

$$+\infty = \frac{c(p^*)}{p^*(E)} = \min_{p \in \overline{\Delta(p_E)}} \frac{c(p)}{p(E)} \leq \frac{c(p)}{p(E)} < +\infty,$$

which leads to a contradiction. Hence, $p^* \in \Delta(p_E)$ and $\tilde{c}_E(p_E) = \min_{p \in \Delta(p_E)} \frac{c(p)}{p(E)}$.

□

1.5 Lemma 10

Lemma 10. *If $\succsim_S$ satisfies Axioms 2 and 5, the function $\tilde{c}_E$ defined by equation (6) is convex, lower semi-continuous, and grounded.*

Proof. Convexity. By Lemma 9, for all $p_E, q_E \in \Delta(E)$, there exists $p^*, q^* \in \Delta S$ such that

$$p^*(\cdot|E) = p_E, q^*(\cdot|E) = q_E, \quad \text{and} \quad c_E(p_E) = \frac{c(p^*)}{p^*(E)}, c_E(q_E) = \frac{c(q^*)}{q^*(E)}.$$

Fix any $\alpha \in [0,1]$. Pick $\gamma \in [0,1]$ that satisfies $\frac{\gamma p^*(E)}{\gamma p^*(E) + (1-\gamma) q^*(E)} = \alpha$. Set $p' := \gamma p^* + (1-\gamma) q^*$. Then

$$\begin{aligned} p'(\cdot|E) &= \frac{\gamma p^*(\cdot) + (1-\gamma) q^*(\cdot)}{\gamma p^*(E) + (1-\gamma) q^*(E)} = \alpha \frac{p^*(\cdot)}{p^*(E)} + (1-\alpha) \frac{q^*(\cdot)}{q^*(E)} \\ &= \alpha p^*(\cdot|E) + (1-\alpha) q^*(\cdot|E) = \alpha p_E + (1-\alpha) q_E \end{aligned}$$

And

$$\begin{aligned} c_E(\alpha p_E + (1-\alpha) q_E) &\leq \frac{c(p')}{p'(E)} \leq \frac{\gamma c(p^*) + (1-\gamma) c(q^*)}{\gamma p^*(E) + (1-\gamma) q^*(E)} \\ &= \frac{\gamma p^*(E)}{\gamma p^*(E) + (1-\gamma) q^*(E)} c_E(p_E) + \frac{(1-\gamma) q^*(E)}{\gamma p^*(E) + (1-\gamma) q^*(E)} c_E(q_E) \\ &= \alpha c_E(p_E) + (1-\alpha) c_E(q_E). \end{aligned}$$

The first inequality follows from definition and that $p'(\cdot|E) = \alpha p_E + (1-\alpha) q_E$. The second inequality follows from convexity of c.

Lower semi-continuity. It suffices to show that for all $r \in [0, +\infty]$, $L_r = \{p_E \in \Delta E : c_E(p_E) \leq r\}$ is closed in ΔE. To that end, take any sequence $\{p_E^n\}$ in L_r and $p_E^n \to p_E \in \Delta(E)$. Then

$c_E(p_E^n) \leq r$ for all n. By Lemma 9, there exists a corresponding sequence $p^n \in \Delta(S)$ such that $p^n(\cdot|E) = p_E^n$ and $\frac{c(p^n)}{p^n(E)} = c_E(p_E^n)$. This implies $c(p^n) \leq rp^n(E)$ for all n. Since $\Delta(S)$ is compact, there exists a convergent subsequence $\{p^k\}$ such that $p^k \to p^* \in \Delta(S)$. Moreover,

$$c(p^*) \leq \liminf c(p^k) \leq r \lim p^k(E) = rp^*(E).$$

If $p^*(E) > 0$, then its Bayesian posterior exists and equals to $\lim_k p_E^k = p_E$. So $c_E(p_E) \leq \frac{c(p^*)}{p^*(E)} \leq r$ and $p_E \in L_r$. If $p^*(E) = 0$, then the above imply $c(p^*) = 0$, which contradicts Lemma 3.

Groundedness. Since c is grounded, so there exists p^* such that $c(p^*) = 0$. By Lemma 3, $p^*(E) > 0$, so $c_E(p^*(\cdot|E)) = 0$. □

1.6 Proof of Lemma 5

Proof. Suppose $fEx \sim_S x$.

First, I show $V_S(fEx) \geq \tilde{V}_E(f)$.

$$\begin{aligned} V_S(fEx) &= \min_{p\in\Delta S} \int_E u(f)dp + u(x)p(E^c) + c(p) \\ &= \int_E u(f)dp^* + u(x)p^*(E^c) + c(p^*) \end{aligned} \tag{14}$$

where p^* is the minimizing probability in equation (14). If $p^*(E) = 0$, then the above equation implies $c(p^*) = 0$, which contradicts Lemma 3. Therefore, $p^*(E) > 0$, and equation (14) becomes

$$p^*(E)u(x) = \int_E u(f)dp^* + c(p^*).$$

Thus,

$$\begin{aligned} V_S(fEx) = u(x) &= \int_E u(f)dp^*(\cdot|E) + \frac{c(p^*)}{p^*(E)} \\ &\geq \min_{p_E\in\Delta(E)} \left(\int_E u(f)dp_E + \min_{p\in\Delta(p_E)} \frac{c(p)}{p(E)} \right) \\ &= \inf_{\{p\in\Delta S: p(E)>0\}} \int_E u(f)dp(\cdot|E) + \frac{c(p)}{p(E)} = \tilde{V}_E(f). \end{aligned}$$

For the other direction $V_S(fEx) \leq \tilde{V}_E(f)$. Since $c(\cdot)$ is lower semi-continuous, there exists some $q^* \in \Delta S$ such that

$$\int_E u(f)dq^*(\cdot|E) + \frac{c(q^*)}{q^*(E)} = \inf_{\{p\in\Delta S: p(E)>0\}} \int_E u(f)dp(\cdot|E) + \frac{c(p)}{p(E)} = \tilde{V}_E(f)$$

Also, there must be $q^*(E) > 0$. Suppose not and $q^*(E) = 0$, then the fact that the expression attains its infimum at q^* rules out the case $c(q^*) > 0$. Then $c(q^*) = 0$, which contradicts Lemma 3.

Then,

$$\begin{aligned} & q^*(E) \cdot \tilde{V}_E(f) + q^*(E^c) \cdot u(x) \\ = \; & q^*(E) \left(\int_E u(f) dq^*(\cdot|E) + \frac{c(q^*)}{q^*(E)} \right) + q^*(E^c) u(x) \\ \geq \; & V_S(fEx) \end{aligned}$$

Since $V_S(fEx) = u(x)$, there must be $\tilde{V}_E(f) \geq V_S(fEx)$. □

1.7 Proof of Lemma 6

Proof. For notational simplicity, I prove the lemma for the case $E = S$. The proof for the case with an arbitrary non-empty event $E \in \Sigma^*$ is analogous.

By constant-act independence, $u \approx u'$. Without loss, let $u = u'$. Denote $\phi = u(f)$.

It suffices to show the following statement. For any two variational functionals $I(\phi) = \min_{p \in \Delta} \int_S \phi dp + c(p)$ and $I'(\phi) = \min_{p \in \Delta} \int_S \phi dp + c'(p)$, if $c(p_0) < c'(p_0)$ for some p_0, then there exists $\xi \in \mathbb{R}^{|S|}$ such that $I(\xi) < I'(\xi)$.

Consider the epigraph of c':

$$epi(c') = \{(p, r) \in \Delta S \times \mathbb{R} | r \geq c'(p)\} \subseteq \Delta S \times [0, +\infty]$$

Since c' is nonnegative, convex, lower semi-continuous, and grounded, $epi(c')$ is nonempty, closed and convex. Let $r_0 = c(p_0)$. Since $c(p_0) < c'(p_0)$, $(p_0, r_0) \notin epi(c')$. By the strict separating hyperplane theorem there exists $(\xi_0, r^*) \in \mathbb{R}^{|S|+1}$, $(\xi_0, r^*) \neq 0$, that strictly separates (p_0, r_0) from the set $epi(c')$; i.e.,

$$\int_S \xi_0 dp_0 + r_0 \cdot r^* < \min_{r' \geq c'(p')} \int_S \xi_0 dp' + r' \cdot r^*. \tag{15}$$

[17] Note that $r^* > 0$: If $r^* < 0$, take $r' = +\infty$ in inequality (15) above and the right-hand side becomes $-\infty$; or if $r^* = 0$, then for $(p_0, r') \in epi(c')$ inequality (15) becomes $\int_S \xi_0 dp_0 < \int_S \xi_0 dp_0$. Hence, multiplying both sides of (15) by $\frac{1}{r^*}$ (take $\xi = \frac{1}{r^*} \xi_0$) yields

$$\int_S \xi dp_0 + r_0 < \int_S \xi dp' + r', \qquad \forall (p', r') \in epi(c').$$

[17] Since $\inf_{r' \geq c'(p')} \int_S \xi dp' + r' = \min_{p' \in \Delta} \int_S \xi dp' + c'(p')$, the right hand side always attains its minimum.

By $r_0 = c(p_0)$,

$$\int_S \xi dp_0 + r_0 = \int_S \xi dp_0 + c(p_0) \geq \min_{p\in\Delta} \int_S \xi dp + c(p) = I(\xi).$$

By definition,

$$\min_{r'\geq c'(p')} \int_S \xi dp' + r' = \min_{p'\in\Delta} \int_S \xi dp' + c'(p') = I'(\xi).$$

Thus, $I(\xi) \leq \int_S \xi dp_0 + r_0 < \min_{r'\geq c'(p')} \int_S \xi dp' + r' = I'(\xi)$. □

1.8 Proof of Theorem 1

Proof. (i) implies (ii). Suppose Axioms 1-9 hold.

By Stable Constant-act Preferences, $\succsim_0$ and $\succsim_E$ agree on the set of constant acts X. Without loss of generality, assume $u_E = u$.

It remains to show that given u and c functions, the conditional cost function c_E must coincide with the induced cost function $\tilde{c}_E(p_E) := \min_{\{p\in\Delta(p_E)\}} \frac{c(p)}{p(E)}$. Suppose instead $c_E \neq \tilde{c}_E$. Thus there exists p^*_E such that $c_E(p^*_E) \neq \tilde{c}_E(p^*_E)$.

Suppose $c_E(p^*_E) > \tilde{c}_E(p^*_E)$. By Lemma 6, there exists some $\xi_E \in \mathbb{R}^{|E|}$ such that

$$\min_{p_E\in\Delta E} \int_E \xi_E dp_E + \tilde{c}_E(p_E) < \min_{p_E\in\Delta E} \int_E \xi_E dp_E + c_E(p_E).$$

Since $u(X) = \mathbb{R}$, there exists an act $f \in \mathcal{F}$ such that $u(f)(s) = \xi_E(s)$ for all $s \in E$.

$$\min_{p_E\in\Delta E} \int_E u(f) dp_E + \tilde{c}_E(p_E) < \min_{p_E\in\Delta E} \int_E u(f) dp_E + c_E(p_E).$$

By Continuity, there exists $x \in X$ such that $x \sim_E f$. Then,

$$u(x) = V_E(f) = \min_{p_E\in\Delta E} \int_E u(f) dp_E + c_E(p_E) > \min_{p_E\in\Delta E} \int_E u(f) dp_E + \tilde{c}_E(p_E) = \tilde{V}_E(f)$$

By Lemma 5, for all $y \in X$ such that $y \sim_S fEy$, $u(y) = V_S(fEy) = \tilde{V}_E(f) < u(x)$. By conditional consistency, $f \sim_E y$. Hence, $x \sim_E y$. By Stable Risk Preferences, $x \sim_S y$, which contradicts $u(y) < u(x)$.

The case $c_E(p^*_E) < \tilde{c}_E(p^*_E)$ at some $p^*_E \in \Delta E$ can also be ruled out by an analogous argument.

Hence, $c_E(\cdot) = \tilde{c}_E(\cdot)$ on ΔE.

(ii) implies (i). Take any non-empty event $E \in \Sigma^*$. Let (u, c) and (u_E, c_E) be the variational representations for $\succsim_S$ and $\succsim_E$, respectively.

Since $u \approx u_E$, for all $x, y \in X$, $x \succsim_S y \Leftrightarrow u(x) \geq u(y) \Leftrightarrow u_E(x) = au(x) + b \geq au(y) + b = u_E(y) \Leftrightarrow x \succsim_E y$, for some $a > 0, b \in \mathbb{R}$. Hence, Stable Constant-act Preferences holds.

Without loss, let $u(\cdot) = u_E(\cdot)$ on X. Suppose the cost functions $c_E(\cdot)$ and $c(\cdot)$ satisfies equation (6). Then, for all f,

$$V_E(f) = \min_{p_E \in \Delta E} \int_E u(f) dp_E + c_E(p_E) = \tilde{V}_E(f)$$

for $\tilde{V}_E$ defined in equation (8).

It remains to check that conditional consistency also holds. For any $f \in \mathcal{F}$ and $x \in X$, suppose $fEx \sim_S x$. By Lemma 5,

$$u(x) = V_S(fEx) = \tilde{V}_E(f) = V_E(f)$$

Since $u(x) = u_E(x)$ and $V_E(\cdot)$ represents $\succsim_E$, there must be $f \sim_E x$.

□

1.9 Proof of Corollary 1

Proof. 1. Suppose $\succcurlyeq_S$ has a MEU representation (u, P_S). Then it corresponds to the indicator cost function

$$c(p) = \begin{cases} 0 & \text{if } p \in P_S, \\ +\infty & \text{otherwise.} \end{cases}$$

For all $E \in \Sigma^*$, Strong Monotonicity of $\succcurlyeq_S$ ensures that $p(E) > 0$ for all $p \in P_S$.

Applying updating equation (6),

$$c_E(p_E) = \begin{cases} 0 & \text{if } p_E \in \{p(\cdot|E) : p \in P_S\} \\ +\infty & \text{otherwise} \end{cases}$$

2. Suppose $\succcurlyeq_S$ has a multiplier preference representation (u, q, θ). Then $\succsim_S$ corresponds to the cost function that is proportional to the relative entropy distance:

$$c(p) = \theta \int_S \ln\left(\frac{p}{q}(s)\right) dp(s)$$

For any nonempty event E, Strong Monotonicity of $\succcurlyeq_S$ ensures that $q_S(E) > 0$ for all $E \in \Sigma^*$. Applying updating equation (6),

$$\begin{aligned}
c_E(p_E) &= \min_{p\in\Delta(p_E)} \frac{\theta}{p(E)} \int_S \ln\left(\frac{p}{q}(s)\right) dp(s) \\
&= \min_{p\in\Delta(p_E)} \frac{\theta}{p(E)} \left(\int_E \ln \frac{p_E}{q_E}(s) dp_E(s)\right) p(E) + \frac{\theta}{p(E)} \left(\int_{E^c} \ln \frac{p_{E^c}}{q_{E^c}}(s) dp_{E^c}(s)\right) p(E^c) \\
&\qquad + \frac{\theta}{p(E)} \left(p(E) \ln \frac{p(E)}{q(E)} + p(E^c) \ln \frac{p(E^c)}{q(E^c)}\right) \\
&= \theta \int_E \ln \frac{p_E}{q_E}(s) dp_E(s)
\end{aligned}$$

In the last equality, the minimizer p satisfies $p(E) = q(E)$ and $p(\cdot|E^c) = q_{E^c}$.

3. Suppose X is the set of monetary lotteries and $\succcurlyeq_S$ admits a monotone mean-variance preferences representation (q_S, θ) with $u_S(t) = t \in \mathbb{R}$. Denote q_S by q. Strong Monotonicity of $\succcurlyeq_S$ ensures that $q(E) > 0$ for all $E \in \Sigma^*$.

   Then $\succsim_S$ corresponds to the unconditional cost function that is proportional to the Gini index:

$$c(p) = \theta G(p\|q) = \theta \int_S \frac{1}{2} \left(\frac{p(s)}{q(s)} - 1\right)^2 dp(s)$$

   Applying updating equation (6),

$$\begin{aligned}
c_E(p_E) &= \min_{p\in\Delta(p_E)} \frac{\theta}{p(E)} \int_S \frac{1}{2} \left(\frac{p(s)}{q(s)} - 1\right)^2 dp(s) \\
&= \min_{p\in\Delta(p_E)} \frac{\theta}{p(E)} \left[\int_E \left(\frac{p_E(s)p(E)}{q_E(s)q(E)} - 1\right)^2 dp_E(s)p(E) + \int_{E^c} \left(\frac{p(s)}{q(s)} - 1\right)^2 dp(s)\right] \\
&= \frac{\theta}{p(E)} \left[\int_E \left(\frac{p_E(s)}{q_E(s)} - 1\right)^2 dp_E(s)p(E)\right] \\
&= \theta \int_E \left(\frac{p_E(s)}{q_E(s)} - 1\right)^2 dp_E(s) = \theta G(p_E\|q_E).
\end{aligned}$$

   The second equality follows from $p(\cdot|E) = p_E$, and the third equality follows from setting the minimizing p to $p(E) = q(E)$ and $p(\cdot|E^c) = q_{E^c}$.

□

1.10 Lemma 11

Lemma 11. *For all* $p_E \in \Delta E$,

$$\tilde{\varphi}_{1E}(p_E) = \sup_{\{p\in\Delta(p_E):\varphi_1(p)>p(E^c)\}} \left(\frac{\varphi_1(p)}{p(E)} - \frac{p(E^c)}{p(E)}\right), \tag{16}$$

attains its maximum in $\{p \in \Delta(p_E) : \varphi_1(p) > p(E^c)\}$.

Proof. Define $h : \Delta S \mapsto \mathbb{R}$ to be

$$h(p) := \left(\frac{\varphi_1(p)}{p(E)} - \frac{p(E^c)}{p(E)}\right) - 1 = \frac{\varphi_1(p) - 1}{p(E)}$$

So for all element in $C(p_E) := \{p \in \Delta(p_E) : \varphi_1(p) > p(E^c)\} \subseteq \Delta S$, $h(p) \in (-1, 0]$.

Fact 1: mapping $h : \Delta S \mapsto \mathbb{R}$ is upper semi-continuous.

Recall two properties of upper semi-continuous functions: For any two functions $f : X \mapsto \mathbb{R}$ and $g : X \mapsto \mathbb{R}$, (i)if both f and g are both upper semi-continuous, then so is $f + g$; (ii) if both f, g are also positive valued, then the product function fg is upper semi-continuous (and positive valued); Since φ_1 is positive valued and upper semi-continuous, $\frac{1}{p(E)} := \frac{1}{\int_S \mathbf{1}_E dp}$ is continuous (in p) and positive valued, the claimed fact follows.

Fact 2: the closure of $C(p_E)$, denoted $\overline{C(p_E)}$, is compact.

To see this fact, observe that

$$\overline{C(p_E)} \subseteq \{p \in \Delta(p_E) : \varphi_1(p) \geq p(E^c)\} \cup \Delta E^c \subseteq \Delta S,$$

and ΔS is a compact subset of $\mathbb{R}_+^{|S|}$. Hence, $\overline{C(p_E)}$ is closed (by definition) and bounded.

By a generalized Weierstrass' Theorem (Aliprantis and Border (2007), Theorem 2.43), the real-valued upper semicontinuous function $h(\cdot) : \overline{C(p_E)} \mapsto \mathbb{R}$ attains its maximum on the compact set $\overline{C(p_E)}$.

Let $p^* \in \overline{C(p_E)}$ be a maximizer of $h(\cdot)$. Suppose $p^* \notin C(p_E)$, then either (i) $\varphi_1(p^*) = p^*(E^c)$ or (ii) $p^*(E) = 0$. Let $\{p^n\}$ be a sequence in $C(p_E)$ such that $p^n \to p^*$. By construction, $p^n(E) > 0$ and $\varphi_1(p^n) > p^n(E^c)$.

If case (i), since p^* is a maximizer,

$$-1 = \frac{p^*(E^c) - 1}{p^*(E)} = \frac{\varphi_1(p^*) - 1}{p^*(E)} \geq \frac{\varphi_1(p^n) - 1}{p^n(E)} > \frac{p^n(E^c) - 1}{p^n(E)} = -1.$$

This is a contradiction.

If case (ii), since $\varphi_1(p^*) \in (0, 1]$. Using the convention $0^-/0 = -\infty$,

$$-\infty = \frac{\varphi_1(p^*) - 1}{p^*(E)} \geq \frac{\varphi_1(p^n) - 1}{p^n(E)} > \frac{p^n(E^c) - 1}{p^n(E)} = -1.$$

Again, this is a contradiction.

Therefore, $p^* \in C(p_E)$; i.e., the function

$$h(p) + 1 = \left(\frac{\varphi_1(p)}{p(E)} - \frac{p(E^c)}{p(E)} \right)$$

attains its maximum on the constraint set $C(p_E)$.

□

1.11 Lemma 12

Lemma 12. *The conditional confidence function defined by equation (10) is a function $\tilde{\varphi}_{1E} : \Delta E \mapsto [0,1]$ that is concave, upper semi-continuous, and normal ($\tilde{\varphi}_{1E}(p_E) = 1$ for some $p_E \in \Delta E$).*

Proof. (i)$\tilde{\varphi}_{1E} : \Delta E \mapsto [0,1]$. For any $p_E \in \Delta E$, by construction, $\varphi_{1E}(p_E) > 0$ for all $p_E \in C_E$ such that $\varphi_1(p) > p(E^c)$.

For all $p \in \Delta(p_E)$ such that $\varphi_1(p) > p(E^c) \geq 0$, $1 \geq \varphi_1(p)$. Hence, $\frac{\varphi_1(p)}{p(E)} - \frac{p(E^c)}{p(E)} \leq \frac{1}{p(E)} - \frac{p(E^c)}{p(E)} = 1$. So $0 \leq \varphi_{1E}(p_E) \leq 1$ for all $p_E \in \Delta E$.

(ii) Concavity.

By Lemma 11, for all $p_E, q_E \in \Delta E$, there exists $p^*, q^* \in \Delta S$ such that $p^* \in C(p_E)$ and $q^* \in C(q_E)$; i.e.,

$$p^*(\cdot|E) = p_E, \quad \varphi(p^*) > p^*(E^c), \quad \tilde{\varphi}_{1E}(p_E) = \frac{\varphi_1(p^*) - 1}{p^*(E)} + 1;$$

$$q^*(\cdot|E) = q_E, \quad \varphi(q^*) > q^*(E^c), \quad \tilde{\varphi}_{1E}(q_E) = \frac{\varphi_1(q^*) - 1}{q^*(E)} + 1.$$

Take any $\lambda \in [0,1]$, there exists $\gamma \in [0,1]$ such that $\frac{\gamma p^*(E)}{\gamma p^*(E) + (1-\gamma) q^*(E)} := \lambda$. Set $p' := \gamma p^* + (1-\gamma) q^*$. Then, by concavity of $\varphi_1(\cdot)$,

$$\begin{aligned} p'(\cdot|E) &= \frac{\gamma p^*(\cdot) + (1-\gamma) q^*(\cdot)}{\gamma p^*(E) + (1-\gamma) q^*(E)} = \lambda \frac{p^*(\cdot)}{p^*(E)} + (1-\lambda) \frac{q^*(\cdot)}{q^*(E)} \\ &= \lambda p^*(\cdot|E) + \ (1-\lambda) q^*(\cdot|E) = \lambda p_E + \ (1-\lambda) q_E; \\ \varphi_1(p') &= \varphi_1(\gamma p^* + (1-\gamma) q^*) \geq \gamma \varphi_1(p^*) + (1-\gamma) \varphi_1(q^*) > \gamma p^*(E^c) + (1-\gamma) q^*(E^c) = p'(E^c), \end{aligned}$$

which implies $p' \in C(\lambda p_E + (1-\lambda) q_E)$. Moreover,

$$\begin{aligned}
\tilde{\varphi}_{1E}(\lambda p_E + (1-\lambda) q_E) &\geq \frac{\varphi_1(p') - 1}{p'(E)} + 1 = \frac{\varphi_1(\gamma p^* + (1-\gamma) q^*) - 1}{\gamma p^*(E) + (1-\gamma) q^*(E)} + 1 \\
&\geq \frac{\gamma(\varphi_1(p^*) - 1) + (1-\gamma)(\varphi_1(q^*) - 1)}{\gamma p^*(E) + (1-\gamma) q^*(E)} + 1 \\
&= \lambda \left(\frac{\varphi_1(p^*) - 1}{p^*(E)} + 1 \right) + (1-\lambda) \left(\frac{\varphi_1(q^*) - 1}{q^*(E)} + 1 \right) \\
&= \lambda \tilde{\varphi}_{1E}(p_E) + (1-\lambda) \tilde{\varphi}_{1E}(q_E).
\end{aligned}$$

(iii) Upper semi-continuity. It suffices to show $U_\alpha = \{p_E \in \Delta E : \tilde{\varphi}_{1E}(p_E) \geq \alpha\}$ is a closed set for all $\alpha \in \mathbb{R}$. If $\alpha \leq 0$, then $U_\alpha = \Delta E$ and the claim holds. If $\alpha > 1$, then U_α is an empty set and the claim holds vacuously. Fix any $\alpha \in (0, 1]$, take any convergence sequence p_E^n from U_α such that $p_E^n \to p_E^* \in \Delta E$. For each p_E^n, by Lemma 11, there exists $p^n \in C(p_E^n)$ such that

$$\tilde{\varphi}_{1E}(p_E^n) = \frac{\varphi_1(p^n) - 1}{p^n(E)} + 1 \geq \alpha.$$

Therefore, for all n,

$$p^n(E) \geq p^n(E)(1-\alpha) \geq 1 - \varphi_1(p^n)$$

And since ΔS is compact, the sequence $\{p^n\} \subseteq \Delta S$ has a convergent subsequence $\{p^k\}$, $p^k \to p^*$. Then

$$p^*(E)(1-\alpha) = \lim_k p^k(E)(1-\alpha) = \liminf_k p^k(E)(1-\alpha) \geq \liminf_k (1 - \varphi_1(p^k)) \geq 1 - \varphi_1(p^*),$$

where the last $\geq$ uses $1 - \varphi_1(\cdot)$ is lower semi-continuous.

If $p^*(E) = 0$, then the above implies $\varphi_1(p^*) = 1$. By Lemma 3, p^* must have full support on S, which is a contradiction. Hence, $p^*(E) > 0$.

Then $p^*(\cdot|E)$ is well defined and equals to $\lim_k p_E^k = p_E^*$.[18] Moreover, since $0 < \alpha \leq 1$, $p^*(E) > (1-\alpha) p^*(E) \geq 1 - \varphi_1(p^*)$ and hence $\varphi_1(p^*) > p^*(E^c)$. Therefore, $p^* \in C(p_E^*)$, and

$$\tilde{\varphi}_{1E}(p_E^*) \geq \frac{1 - \varphi_1(p^*)}{p^*(E)} + 1 \geq \alpha \quad \Rightarrow \quad p_E^* \in U_\alpha.$$

(iv) Normality. Since φ_1 is normalized, there exists $p^* \in C$, $\varphi_1(p^*) = 1 = \max_{\Delta S} \varphi_1(p)$. For this p^*, $\varphi_1(p^*) \geq p^*(E^c)$ and $\frac{\varphi_1(p^*)}{p^*(E)} - \frac{p^*(E^c)}{p^*(E)} = 1$. So $\max_{\Delta E} \varphi_{1E}(p_E) = 1$.

□

[18] It is straightforward to verify that if $p(E) > 0$ the Bayesian updating mapping $p(\cdot) \mapsto p(\cdot|E)$ is continuous at every event $B \subseteq S$.

1.12 Proof of Theorem 2

Proof. The (ii) $\Rightarrow$ (i) direction follows from standard argument. It suffices to show the (i) $\Rightarrow$ (ii) direction.

Take any nonempty $E \in \Sigma^*$. Suppose Confidence preferences $\succsim_S$ and $\succsim_E$ satisfy stable-constant act preferences and conditional consistency. Then there exists unique maximal confidence preferences representation (u, φ_1) and (u_E, φ_{1E}) that represents $\succsim_S$ and $\succsim_E$, and $u \approx u_E$. Without loss, normalize and let $u = u_E$. It remains to show that φ_{1E} can be derived by φ_1 via the updating equation (10).

For confidence preferences, first note that Lemma 3 suggests $\varphi_1(p) = 1$ implies $p(E) > 0$ for all $E \in \Sigma^*$. Second, The aggregator function $I_S^{CP} : u(X)^{|S|} \mapsto \mathbb{R}$ is $I_S^{CP}(\xi) = \inf_{p \in \Delta S} \frac{\int_S \xi dp}{\varphi_1(p)}$ for all $\xi \in u(X)^{|S|}$. Observe that $I_S^{CP}(\cdot)$ is the pointwise infimum of a collection of linear functions (of ξ), and hence it is concave (Aliprantis and Border (2007), Lemma 5.40). Therefore, Assumption 1 holds. By Lemma 2, the conditional certainty equivalent given by the conditional consistency axiom $fEx \sim_S x$ is essentially unique. Third, by Lemma 4, it is without loss to focus on confidence preferences representation of $\succsim_E$ with the conditional cost function φ_{1E} defined on ΔE.

Then, consider the candidate conditional confidence function $\tilde{\varphi}_{1E} : \Delta E \mapsto [0, 1]$ induced by $\varphi_1(\cdot)$:

$$\tilde{\varphi}_{1E}(p_E) = \sup_{\{p \in \Delta(p_E) : \varphi_1(p) > p(E^c)\}} \left(\frac{\varphi_1(p)}{p(E)} - \frac{p(E^c)}{p(E)} \right), \tag{17}$$

By Lemma 11, equation (17) attains its maximum in the set $\{p \in \Delta(p_E) : \varphi_1(p) > p(E^c)\}$. And by Lemma 12, the induced function $\tilde{\varphi}_{1E}$ satisfies the properties of a confidence function.

Consider the following conditional confidence preferences representation $(u, \tilde{\varphi}_{1E})$ of $\succsim_E$:

$$\begin{aligned} \tilde{V}_E^{CP}(f) &= \inf_{p_E \in \Delta E} \frac{\int_E \xi dp_E}{\tilde{\varphi}_{E1}(p_E)} = \inf_{p_E \in \Delta E} \left(\int_E \xi dp_E \cdot \inf_{\{p \in \Delta(p_E) : \varphi_1(p) > p(E^c)\}} \frac{p(E)}{\varphi_1(p) - p(E^c)} \right) \\ &= \inf_{\{p \in \Delta S : p(E) > 0, \varphi_1(p) > p(E^c)\}} \frac{\int_E \xi dp(\cdot|E)}{\frac{\varphi_1(p) - p(E^c)}{p(E)}} \end{aligned}$$

Lemma 13. *Suppose $\succsim_S$ satisfies Axiom 6. For all $f \in \mathcal{F}$, if $fEx \sim_S x$, then*

$$V_S(fEx) = \tilde{V}_E^{CP}(f).$$

Proof of Lemma 13. For any $f \in \mathcal{F}$. Let $\xi = u(f) \in u(X)^{|S|}$ and $I_S(\xi) = V_S(f)$ for $f \in u^{-1}(\xi)$. Note that $I_S(\cdot)$ is concave. By Lemma 2, there exists some $x \in X$ such that

$fEx \sim_S x$ and $u(x) := k \in \mathbb{R}_+$ is unique. Let $V_E^{CP}(\cdot)$ given by (u, φ_{1E}) be a consequentialist confidence preferences representation of $\succsim_E$. By conditional consistency, $f \sim_E x$ and hence $V_E^{CP}(f) = u(x) = k$.

So

$$\begin{aligned} k = I_S(\xi E k) &= \inf_{p \in \Delta S} \frac{\int_E \xi dp + kp(E^c)}{\varphi_1(p)} \\ &= \frac{\int_E \xi dp^* + kp^*(E^c)}{\varphi_1(p^*)}, \end{aligned}$$

where p^* is from $\arg\min_{p \in \Delta S} \frac{\int_E \xi dp + kp(E^c)}{\varphi_1(p)}$. Note that the infimum attains because $\varphi_1(\cdot)$ is positive valued and upper semi-continuous on ΔS, so $\frac{1}{\varphi_1(\cdot)}$ is positive valued and lower semi-continuous. The function $p \mapsto \int_E \xi dp + kp(E^c)$ is positive valued and linear. So the function $p \mapsto \frac{\int_E \xi dp + kp(E^c)}{\varphi_1(p)}$ is lower semi-continuous on the compact set ΔS.

Then

$$\begin{aligned} k &= \frac{\int_E \xi dp^*}{\varphi_1(p^*) - p^*(E^c)} = \frac{\int_E \xi dp^*(\cdot|E)}{\frac{\varphi_1(p^*) - p^*(E^c)}{p^*(E)}} \\ &\geq \inf_{\{p \in \Delta S : p(E) > 0, \varphi_1(p) > p(E^c)\}} \frac{\int_E \xi dp(\cdot|E)}{\frac{\varphi_1(p) - p(E^c)}{p(E)}} \\ &= \inf_{p_E \in \Delta E} \left(\int_E \xi dp_E \cdot \inf_{p \in \Delta(p_E) : \varphi_1(p) > p(E^c)} \frac{p(E)}{\varphi_1(p) - p(E^c)} \right) = \inf_{p_E \in \Delta E} \frac{\int_E \xi dp_E}{\tilde{\varphi}_{E1}(p_E)} := \tilde{V}_E^{CP}(f) \end{aligned}$$

Hence, $k \geq \tilde{V}_E(f)$.

Now suppose $k > \tilde{V}_E(f)$.

Since $\varphi_1(\cdot)$ is upper semi-continuous, the function $p \mapsto \phi(p) - \int_S \mathbf{1}_{E^c} dp$ is upper semi-continuous, so $\{\varphi_1(p) \geq p(E^c)\}$ is a compact subset of ΔS. On $\{\varphi_1(p) \geq p(E^c)\}$, the function $p \mapsto \phi(p) - p(E^c)$ is also positive valued. So $p \mapsto \frac{1}{\phi(p) - p(E^c)}$ is lower semi-continuous. Given $\xi \in \mathbb{R}_+^{|S|}$, $\xi \neq \mathbf{0}$, $p \mapsto \int_E \xi dp$ is positive valued and continuous. Hence the function $p \mapsto \frac{\int_E \xi dp}{\phi(p) - p(E^c)}$ is positive valued and lower semi-continuous, and must attain its minimum on a compact set. Hence, there exists $\bar{p} \in \{\varphi_1(p) \geq p(E^c)\} \subseteq \Delta S$ such that

$$\frac{\int_E \xi d\bar{p}}{\varphi_1(\bar{p}) - \bar{p}(E^c)} = \inf_{\{p \in \Delta S : p(E) > 0, \varphi_1(p) > p(E^c)\}} \frac{\int_E \xi dp}{\varphi_1(p) - p(E^c)}.$$

Note that there must be (i) $\bar{p}(E) > 0$ and (ii) $\varphi(\bar{p}) > \bar{p}(E^c)$. To see this, if (i) does not hold and $\bar{p}(E) = 0$. Then $\bar{p}(E^c) = 1$ and $1 \geq \varphi_1(\bar{p}) \geq \bar{p}(E^c) = 1$. Then $\varphi_1(\bar{p}) = 1$ but this contradicts the conclusion of Lemma 3. If (ii) does not hold and $\varphi(\bar{p}) = \bar{p}(E^c)$, then for any $\xi \neq \mathbf{0}$, the above expression implies

$$+\infty = \frac{\int_E \xi d\bar{p}}{\varphi_1(\bar{p}) - \bar{p}(E^c)} \leq \frac{\int_E \xi dp}{\varphi_1(p) - p(E^c)} < +\infty$$

for all $p(E) > 0$ and $\varphi_1(p) > p(E^c)$, which is a contradiction.

Hence, there exists some $\bar{p} \in \Delta S$, $\bar{p}(E) > 0, \varphi_1(\bar{p}) > \bar{p}(E^c)$ such that

$$k > \tilde{V}_E^{CP}(f) = \frac{\int_E \xi d\bar{p}}{\varphi_1(\bar{p}) - \bar{p}(E^c)}.$$

This implies

$$k > \frac{\int_E \xi E k d\bar{p}}{\varphi_1(\bar{p})} \quad \geq \quad \min_{p \in \Delta S} \frac{\int_S \xi E k dp}{\varphi_1(p)} = I_S(\xi E k) = k.$$

However, $k > k$ is a contradiction.

So $k = V_E^{CP}(f) = \tilde{V}_E^{CP}(f)$.

□

Hence, $\succsim_E$ can be represented by $\tilde{V}_E^{CP}(\cdot)$, which is a consequentialist confidence preference representation consisting $(u, \tilde{\varphi}_{1E})$. By Chateauneuf and Faro (2009), given u, the maximal confidence representation $\tilde{\varphi}_{1E}$ is unique. And this finishes the proof.

□

1.13 Proof of Lemma 7

Proof. (2) ⇔ (3). $\tilde{X}_\xi$ is a real value random variable. For $\phi \in C^3$, $\phi' > 0$ and $\phi'' < 0$, by construction $C_\phi(\cdot)$ is

$$C_\phi(\tilde{X}_\xi) := \phi^{-1} \int_{\Delta S} \phi\left(\tilde{X}_\xi(p)\right) d\mu(p),$$

which is the same as a certainty equivalent function for a EU DM with utility index $\phi(\cdot)$ on $u(X) \subseteq \mathbb{R}$.

By Corollary 5.1 from Ben-Tal and Teboulle (2007), the certainty equivalent function $C_\phi(\cdot)$ is concave if and only if $\frac{1}{A_\phi(\cdot)}$ is concave.

(1) $\Leftrightarrow$ (2). For all $\xi \in u(X)^{|S|}$, by definition

$$C_\phi(\tilde{X}_\xi) = I(\xi) = \phi^{-1} \int_{\Delta S} \phi\left(\int_S \xi(s) dp(s)\right) d\mu(p).$$

For any $\xi_1, \xi_2 \in u(X)^{|S|}$ and any $\alpha \in [0,1]$, since $\tilde{X}_\xi = \int_S \xi dp$,

$$\begin{aligned} I(\alpha\xi_1 + (1-\alpha)\xi_2) &= C_\phi\left(\tilde{X}_{\alpha\xi_1+(1-\alpha)\xi_2}\right) \\ &= C_\phi\left(\int_S (\alpha\xi_1 + (1-\alpha)\xi_2) dp\right) = C_\phi\left(\alpha\tilde{X}_{\xi_1} + (1-\alpha)\tilde{X}_{\xi_2}\right). \end{aligned}$$

Since $I(\xi_1) = C_\phi(\tilde{X}_{\xi_1})$ and $I(\xi_2) = C_\phi(\tilde{X}_{\xi_2})$,

$$I(\alpha\xi_1 + (1-\alpha)\xi_2) \;\geq\; \alpha I(\xi_1) + (1-\alpha) I(\xi_2).$$

if and only if

$$C_\phi\left(\alpha\tilde{X}_{\xi_1} + (1-\alpha)\tilde{X}_{\xi_2}\right) \;\geq\; \alpha C_\phi(\tilde{X}_{\xi_1}) + (1-\alpha) C_\phi\left(\tilde{X}_{\xi_2}\right).$$

□

1.14 Proof of Lemma 8

Proof. (ii) $\Rightarrow$ (i) is obvious. For (i) $\Rightarrow$ (ii), start with the first half. Suppose $fEx \succsim_S x$ for some f and x. Then by Lemma 2, there exists an essentially unique $x_0 \in X$ such that $fEx_0 \sim_S x_0$. Axiom 4 implies $f \sim_E x_0$. It suffices to show $x_0 \succsim_S x$. Note that $\succsim_S$ admits some concave and strongly monotone unconditional UAP representation (u, I). And recall from the proof of Lemma 2, fix $\xi = u(f) \in \mathbb{R}^{|S|}$ and denote $z_0 := u(x_0)$ and $z = u(x)$, then recall $h(z') = I(\xi E \cdot z') - z'$. Pick real numbers $a \leq u(f(s))$ and $b \geq u(f(s))$ for all $s \in S$. Note function $h : [a, b] \mapsto \mathbb{R}$ crosses zero exactly once from above. Since $fEx_0 \sim_S x_0$ implies $h(z_0) = 0$. Then $fEx \succsim_S x$ implies $h(z) \geq 0$, which can only happen when $z \leq z_0$; i.e., $u(x) \leq u(x_0)$. Hence, $f \succsim_E x$. The proof of the second part goes by analogy. (It suffices to pick $w = u(y)$ and use the fact that $h(w) \leq 0$ implies $w \geq z_0$; i.e., $u(y) \geq u(x_0)$.) □

1.15 The SOEU case

When the second-order belief μ has support on Dirac probabilities, then μ can be identified with a probability on S and the smooth preferences become the special case of SOEU. Let $p_0 \in \Delta S$ be the second-order belief in this case.

Then $\succsim_S$ admits a SOEU representation (u, ϕ, p_0) (Grant et al., 2009)

$$V_S(f) = \phi^{-1}\left(\int_S \phi(u(f))dp\right)$$

Or $I_S : (u(X))^{|S|} \mapsto \mathbb{R}$ is

$$I_S(\xi) = \phi^{-1}\left(\int_S \phi(\xi)dp\right)$$

Let $C_\phi(\cdot)$ be the certainty equivalent function. Then by the same argument as Lemma 7, the following statements are equivalent: (1) $I(\cdot)$ is concave (2) The certainty equivalent function $C_\phi(\cdot)$ is concave; (3) $\frac{1}{A_\phi(\cdot)}$ is concave.

Lemma 14. *For SOEU preferences $\succsim_S$. Suppose $\phi \in C^3$, $\phi' > 0$, and $\phi'' < 0$. Then the following are equivalent: (1) $I(\cdot)$ is concave; (2) The certainty equivalent function $C_\phi(\cdot)$ is concave; (3) $\frac{1}{A_\phi(\cdot)}$ is concave.*

Proof. The proof is analogous to that of Lemma 7 and hence omitted. ☐

Corollary 4. *If $\{\succsim_E\}_{E\in\Sigma^*}$ admit SOEU representation and $\succsim_S$ admit a smooth representation with (u, ϕ, p_0). Suppose $\phi \in C^3$, $\phi' > 0$ and $\phi'' < 0$, and $\frac{1}{A_\phi(\cdot)}$ is concave.*

Then stable constant-act preferences and conditional consistency hold if and only if $\succsim_E$ admits representation $V_E : \mathcal{F} \mapsto \mathbb{R}$ with $V_E = I_E \circ u$ such that for all $f \in \mathcal{F}$, $I_E(u(f)) = k^* \in \mathbb{R}$ that uniquely solves

$$k^* = \phi^{-1}\left(\int_E \phi(u(f))dp_0 + k^* p_0(E^c)\right) \tag{18}$$

Proof. "If" direction is obvious.

"Only if" direction. By Lemma 14, $I_S(\cdot)$ satisfies Assumption 1 and hence the conditional certainty equivalent is well-defined. That is, for all $f \in \mathcal{F}$, the equivalence relation $fEx \sim_S x$ induces the equation

$$V_S(fEx) = \phi^{-1}\left(\int_E \phi(u(f))dp + p(E^c)u(x)\right) = u(x),$$

which has a unique solution $u(x) \in u(X)$. Let $k^* = u(x)$. Then equation (18) follows from conditional consistency. ☐

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