Dynamic Contracting with Flexible Monitoring*

Liang Dai † Yenan Wang ‡ Ming Yang §

February 2022

Abstract

We study a principal's joint design of optimal monitoring and compensation schemes to incentivize an agent by incorporating information design into a dynamic contracting framework. The principal can flexibly allocate her limited monitoring capacity between seeking evidence that confirms or contradicts the agent's effort, as the basis for reward or punishment. When the agent's continuation value is low, the principal seeks only confirmatory evidence. When it exceeds a threshold, the principal seeks mainly contradictory evidence. Importantly, the agent's effort is perpetuated if and only if he is sufficiently productive.

JEL: D86 D82 G39 J33 M52

*We thank Brett Green, Simon Mayer, Ming Lu, Konrad Mierendorff, Peng Sun, Felipe Varas, Vish Viswanathan, Jingshu Zhang, Lei Zhang, and seminar participants at City University London, East China University of Science and Technology, Emory University, Hong Kong Baptist University, INSEAD, ITAM, Queen Mary University London, Toulouse School of Economics, UCLA and University of Michigan for their valuable comments and discussions.

†Antai College of Economics and Management, Shanghai Jiao Tong University, 1954 Huashan Road, Shanghai, 200030, China, liangdai@sjtu.edu.cn

‡Amsterdam Business School, University of Amsterdam, 1001 NL Amsterdam, y.wang9@uva.nl

§Department of Economics and School of Management, University College London, 30 Gordon Street, London WC1H 0AX, United Kingdom, m-yang@ucl.ac.uk

1 Introduction

An employer generally faces two *interdependent* problems when designing an employment contract. First, the contract needs to specify performance indicators that correspond to different stages of the employee's career (i.e., a *monitoring scheme*). Second, the reward or punishment corresponding to each realization of the stipulated performance indicators (i.e., a *compensation scheme* based on a *given* monitoring scheme) must also be specified. This motivates the dynamic moral hazard problem studied in this paper, in which a principal ("she"), with limited monitoring capacity, *jointly* designs a monitoring scheme and a subsequent compensation scheme for the agent ("he"), to motivate him to take her desired action.

The resolution of this problem calls for a non-trivial combination of two major strands of the literature on incentive provision. One is information design, which investigates how a principal can optimally provide information about an *exogenous* state of nature to incentivize an agent, taking as given their payoffs as *exogenous* functions of the state and the agent's action. Instead, our monitoring scheme directly reveals information about the agent's hidden action, which is *endogenously* shaped by the monitoring scheme through the *jointly designed* compensation scheme.

The other strand of the literature is contract design, which studies a principal's optimal design of a compensation scheme *given an exogenous monitoring scheme*.[1] In our model, the compensation scheme is instead *jointly designed* with the monitoring scheme. Non-trivial interactions arise between them: The more monitoring capacity allocated to seeking a certain type of evidence, the more likely such evidence would arrive, and hence the more effective is the reward (punishment) associated with such evidence in incentivizing the agent. This in turn makes it more worthwhile to seek such evidence in the first place, or vice versa. Moreover, allowing for joint design leads to new implications relative to the standard contract-design literature. Specifically, with a large

[1] That is, it either assumes a *single* exogenous performance indicator, or focuses on *how much* monitoring capacity should be devoted to a *given* performance indicator.

monitoring capacity, it is possible that the principal optimally perpetuates the agent's effort.[2] We believe that this paper is among the first to explore the combination of information and dynamic contract design, which engenders non-trivial interaction with novel implications for incentive provision.

The flexibility in the principal's information design is modeled as that in the allocation of her limited monitoring capacity to searching for two polar types of evidence about whether the agent is working.[3,4] A "carrot-based search" ("C-search" hereafter) generates confirmatory evidence of the agent's effort ("C-evidence") that emerges only if the agent has worked, while a "stick-based search" ("S-search" hereafter) generates contradictory evidence of the agent's effort ("S-evidence") that emerges only if the agent has shirked. By and large, any relevant monitoring scheme in this environment can be understood as a combination of C-search and S-search. The flexibility of allocating limited monitoring capacity to searching for two types of evidence solves a technical dilemma: it renders the information design meaningful while it retains flexibility in the design and guarantees tractability.

The contracting aspects of the model are standard. The principal's project requires the agent's operation. The agent is less patient than the principal and can work or shirk at each instant. From the perspectives of both the principal and social welfare, it is optimal for the agent to work, but the agent enjoys a private benefit from shirking. Besides determining how much to reward or punish the agent when each piece of evidence arrives, the principal can also terminate the project at any time, which is socially inefficient.

[2] To the best of our knowledge, dynamic contracting models rarely yield a perpetual contractual relationship as optimum, with the exception of Sannikov (2008). Yet, even in that optimum, the agent no longer exerts effort once his continuation value reaches the payout boundary.

[3] This can be understood as specifying in employment contracts the corresponding KPIs for employees differing in seniority and past performance, which is commonly seen in practice. Che and Mierendorff (2019) adopt the same monitoring setting in a single individual's decision problem.

[4] From the modeling perspective, unlike information-design models, the principal learns about the agent's hidden actions from the monitoring scheme designed by herself. Thus, constraints on her ability to acquire information are necessary to make this problem nontrivial: otherwise, it is simply optimal for her to choose to be perfectly informed.

Our model brings to light two issues novel in the literature. First, we identify a key tradeoff between seeking confirmatory and contradictory evidence as a means to incentivize the agent. C-search generates greater variation in the agent's continuation value than S-search. This is because, given that the agent indeed works, no S-evidence exists, and thus no adjustment to the agent's continuation value is required; while C-evidence does emerge in equilibrium, which necessarily involves a reward upon its receipt ("carrots" hereafter) and the downward adjustment of the agent's continuation value in the absence of C-evidence. In this sense, C-search is less advantageous to the principal, who is effectively risk-averse due to the inefficiency of early termination. On the other hand, for S-search alone to be effective, the agent's stake in the project (continuation value) must be high, whereas the effectiveness of C-search does not depend on the agent's continuation value. Moreover, to guarantee the effectiveness of S-search, a high continuation value for the agent must be maintained, which involves high interest expenditure for the principal. This tradeoff shapes the optimal incentive scheme.

Consequently, when the agent's continuation value is low, the principal allocates all her monitoring capacity to C-search. Instead of paying the agent immediately upon receiving C-evidence, the principal adds the whole reward to the agent's continuation value to build a buffer against inefficient termination and to make S-search effective in the future. When the agent's continuation value becomes higher, the optimal incentive scheme features a "phase change." That is, instead of the carrot-only mode, the principal now relies mainly on S-search, and sets the penalty for observing S-evidence ("sticks" hereafter) to its maximum: confiscation of the entire stake promised to the agent, resulting in termination of the project. C-search is still used but is limited to the minimum, and the associated reward is decreasing in the agent's continuation value. The standard incentive versus interest tradeoff implies that payments to the agent are incurred only when the agent's continuation value grows beyond a payout boundary.

The second issue novel to the literature concerns perpetuating the agent's effort with a permanent position. Specifically, the flexibility of combining C-search with S-search offers the principal the opportunity to first build up the agent's stake (i.e., his continuation value) with C-search, and then perpetuate his effort mainly with S-search to avoid inefficient termination. Is perpetuating the agent's effort optimal? We show that the answer is affirmative if and only if the latent benefit from the agent's effort is large. Moreover, when perpetuation of the agent's effort is optimal, the value function is convex in the vicinity of the (absorbing) payout boundary when public randomization is not allowed. This is due to a new economic force in addition to the standard incentive versus interest tradeoff. That is, the higher the agent's continuation value, not only is it the less likely to reach the (inefficient) termination boundary as in existing models, it is also more likely to reach perpetuation with no inefficient termination. The latter fact makes the marginal benefit of accumulating the agent's continuation value increasing instead of decreasing in the continuation value in the vicinity of the payout boundary.

Our model offers practical suggestions for the design of incentive schemes, which turn out to be largely consistent with the real practice in academia. First, junior employees are incentivized mainly based on evidence that confirms their effort, while senior employees are incentivized mainly based on evidence that refutes theirs. This is consistent with the fact that publication matters more for junior faculty than for senior faculty. Second, permanent positions are offered only for employees with sufficiently large potential synergy. This is largely consistent with the fact that for top research universities, only research faculty can be on the tenure track, but this pattern is less clear for teaching schools. Similar pattern also applies to the viability of partnership in accounting, consulting and law firms. Third, except for those hired permanently, in the absence of evidence confirming their contribution, employees become more prone to unemployment, and the more so if they are less senior. Lastly, instead of fixed payments only, compensation for employees hired permanently should still involve carrots. This is consistent with the fact that tenured professors are still entitled to wage increase and promotions for post-tenure achievements.

1.1 Literature Review

Our paper is related to three strands of the literature: information design, dynamic attention allocation and contract design. First, our paper connects to the information design literature. Standard information design models, e.g., Kamenica and Gentzkow (2011),[5] Ely (2017), Smolin (2017), Ely and Szydlowski (2020), Ball (2019), Orlov et al. (2020) and Che et al. (2021), consider situations where the principal designs an information scheme for an *exogenous* payoff-relevant state of nature to maximize the chance that the agent takes the principal's desired actions. In contrast, in our model, the information scheme is for the agent's hidden actions, where the latter is *endogenously* shaped by the former through the *jointly designed* compensation scheme. Recent work also explores the design of monitoring schemes in moral hazard settings. Varas et al. (2020) consider a setting in which the principal designs a schedule of costly inspections of the project quality affected by the agent's hidden actions, but they abstract from the design of a compensation scheme which we explore. Hoffmann et al. (2020) and Georgiadis and Szentes (2020) consider contracting problems where the impatient agent takes only one hidden action at the very beginning, and the principal designs a deferred compensation scheme facing the tradeoff between additional information about that action before payment and costly payment deference.[6] Compared with these models, our setup is richer in three aspects. First, our model allows the agent to act and payments to occur at *every* instant, as in a standard dynamic contracting model. Second, besides *how much attention* to devote to a *single* exogenous information source, our principal can determine the *attention allocation* between *multiple* information sources. Lastly and most importantly, our main focus is exactly on the rich interaction between the first two aspects.

Our modeling approach for the principal's flexibility in information design is inspired by the literature on dynamic attention allocation. Che and

[5] Methodologically, Kamenica and Gentzkow (2011) adopt the concavification approach in Aumann et al. (1995).

[6] Li and Yang (2019) consider a static contracting problem where the cost of acquiring information is related to the accuracy of the information.

Mierendorff (2019) consider a decision maker's dynamic attention allocation between seeking confirmatory and contradictory evidence for one of the two states of nature. Mayskaya (2020) applies the modeling approach of Che and Mierendorff (2019) to study a decision maker's attention allocation between multiple dimensions of states of nature. Nikandrova and Pancs (2018) apply the same modeling approach to an investor's attention allocation between two projects to seek confirmatory evidence of their profitability. All these papers study a single individual's decision problems. Instead, we study contracting problems between strategic individuals. Kuvalekar and Ravi (2019) consider a principal's design of a compensation scheme for an agent, who allocates limited attention between seeking confirmatory and contradictory evidence of a project's quality for the principal. Similar to the aforementioned standard dynamic information design literature, the fact to be learned in these papers is *exogenous*. In our model, the fact to be learned is instead the agent's hidden actions, which is *endogenous* to the principal's attention allocation through the jointly designed compensation scheme.

Our paper adopts a standard continuous-time contract design framework, pioneered by DeMarzo and Sannikov (2006), Biais et al. (2007) and Sannikov (2008).[7] DeMarzo and Sannikov (2006) directly apply the martingale representation technique developed in Sannikov (2008) to a continuous-time setup, while Biais et al. (2007) is based on the continuous-time limit of a discrete-time model. Early work on dynamic contract design also includes Biais et al. (2010), featuring an environment where the arrival rate of huge loss can be reduced by the agent's effort. Myerson (2015) considers a similar problem in a political economy framework, where a political leader uses randomized punishment to motivate governors. In contrast to the discrete losses in Biais et al. (2010), Sun and Tian (2017) consider discrete revenue. In a setup similar to Biais et al. (2010) and Myerson (2015), Chen et al. (2020) model the principal's option of monitoring as her ability, at any instant, to enforce the agent's effort at an exogenous flow cost, regardless of incentive compatibility.

[7] Both DeMarzo and Sannikov (2006) and Biais et al. (2007) study continuous-time variants of the discrete-time dynamic security design model in DeMarzo and Fishman (2007).

As discussed in the Introduction, evidence-generating processes are exogenous in these models. That is, given the agent's hidden actions, the principal has no control over how evidence is generated. Based on DeMarzo and Sannikov (2006), Piskorski and Westerfield (2016) allow the principal to choose the arrival rate of confirmatory evidence of the agent's profit diversion. Orlov (2018) considers a situation where the agent relies on the principal's inspection to infer his past performance, and for each instant the principal can choose the percentage of projects to inspect. In our model, the principal optimally designs the evidence generating processes jointly with the compensation scheme, subject to the budget constraint of monitoring capacity. Besides *how much attention* to devote to a *single* exogenous information source, our principal can determine the *attention allocation* between *multiple* information sources. This leads to novel interaction between monitoring and compensation schemes and the possibility of optimally perpetuating the agent's effort mentioned in the Introduction.

2 The Model

2.1 Setup

Time is continuous and infinite. There is a principal ("she", designer of a bureaucratic system) and an agent ("he", a representative officer in the system). Both are risk neutral. The principal has a discount rate $r > 0$ and unlimited access to capital. The agent has a discount rate $\rho > r$ and is protected by limited liability; i.e., his cumulative payment from the principal must be non-negative and non-decreasing over time. The principal owns a project that requires the agent's operation, which involves an action $a_t \in [0, 1]$ taken by the agent. The action can be understood as the level of shirking. If action a_t is taken at instant t, in period $[t, t + dt]$, the agent enjoys a private benefit of $\lambda \cdot a_t dt$, while the principal's benefit is $z \cdot (1 - a_t)\, dt > 0$. The agent's outside option is zero. The principal can terminate the project at any time, and the project then generates a payoff of zero for both players.

Here, we interpret z as the latent progress of a project or the reputation of an entity that is lost without the agent's due diligence and is not discernible immediately.[8] Therefore, contracts cannot be made contingent on whether z is accrued.[9] We interpret z this way for two reasons. First, it captures the reality, mentioned in the Introduction, that the agent's hidden actions are often not reflected in existing indicators, such as current output, sales or stock prices. This is because, the outcome of such actions may be realized only in the long run. For example, the daily practice of officers in charge of disease prevention can hardly be evaluated until an epidemic arrives. A manager focusing on the long-term development of her firm should not be over-responsive to the firm's current sales, output or stock prices. Second, it separates the role of output as a component of physical payoff from that as a given performance indicator; the latter having been well studied. This allows us to focus on the principal's active monitoring of the agent's action. For ease of presentation, we hereafter refer to z as the "synergy" (between principal and agent).

To model the flexibility in the principal's information design, we assume that she can freely specify in the contract for each instant how she allocates her μ units of monitoring capacity between "carrot-based search" ("C-search") and "stick-based search" ("S-search"); i.e., to seek one of two types of evidence as the basis for reward and penalty.[10] The receipt of C-evidence confirms the agent's effort, while the receipt of S-evidence contradicts it. Specifically, if the principal allocates a fraction $\alpha_t \in [0, 1]$ of her μ units of monitoring capacity to seeking S-evidence and the remaining $1 - \alpha_t$ to C-evidence, she receives S-evidence at the arrival rate $\mu \cdot \alpha_t \cdot a_t$, and C-evidence at the arrival rate $\mu \cdot (1 - \alpha_t) \cdot (1 - a_t)$. Hence, the agent's chance of being caught shirking is

[8] Alternatively, one can interpret the principal and agent (instead with discount rates 0 and $\rho - r$, respectively) as playing a repeated game that ends exogenously with arrival rate r, when the principal receives her whole payoff from the game. Accordingly, z can be understood as the instantaneous contribution to that payoff.

[9] For similar reasons, the assumption that flow payoffs are unobservable until the end of players' interaction are also made in many models of repeated games or information design, such as Aumann et al. (1995), Orlov (2018), Ball (2019) and Ely and Szydlowski (2020).

[10] The capacity μ in our model should be understood generically as resources available to the principal for monitoring the agent. In reality, this corresponds to the total budget for hiring a quality control team, installing call recorders or surveillance cameras, etc.

proportional to a_t, the level of shirking, and $\mu \cdot \alpha_t$, the capacity allocated to monitoring shirking. Intuitively, if the agent does not shirk, no evidence of shirking exists, and the principal cannot find S-evidence no matter how much capacity is allocated to seeking it; if the principal allocates no capacity to monitor shirking, she receives no S-evidence regardless of the agent's level of shirking. The arrival rate of C-evidence can be interpreted similarly. More specifically, the cumulative number of arrivals of S-evidence, Y_1, and that of C-evidence, Y_0, satisfy

$$dY_{1,t} = \begin{cases} 1, & \text{with probability } \mu\alpha_t a_t dt \\ 0, & \text{otherwise} \end{cases},$$

and

$$dY_{0,t} = \begin{cases} 1, & \text{with probability } \mu\left(1-\alpha_t\right)\left(1-a_t\right) dt \\ 0, & \text{otherwise} \end{cases},$$

respectively. To save the notation, we write $Y = (Y_0, Y_1)$.

By and large, any relevant monitoring scheme in this environment can be understood as a combination of C-search and S-search. Inspired by Che and Mierendorff (2019), the allocation of monitoring capacity between two extreme information sources maintain both flexibility and tractability in the design of the monitoring scheme.

As standard in the dynamic contracting literature, we assume that $z > \lambda > 0$; i.e., z is large enough so that shirking (action 1) is inefficient even taking into account the agent's private benefit. This assumption ensures that it is optimal for the principal to always implement $a_t = 0$, and allows us to focus on the interaction between the monitoring scheme and the agent's compensation scheme. In addition, we assume that $r < \rho < \mu$; i.e., the principal is more patient than the agent,[11] and that the principal has enough capacity to monitor the agent.

A contract $X = (a, \alpha, I, \tau)$ specifies the recommended action a taken by the

[11] As in DeMarzo and Sannikov (2006), this assumption is made for two reasons. First, it captures the fact that a firm usually has a greater risk-bearing capacity than an individual employee. Second, it rules out the possibility that the principal indefinitely postpones payments to the agent, which is neither interesting nor realistic.

agent, the monitoring scheme α,[12] the cumulative payment I to the agent and the time τ of project termination as functions of the history of past evidence. Given the contract X and an action process a, the expected discounted utility of the agent is

$$\mathbb{E}^a\left[\int_0^\tau e^{-\rho t}\left(dI_t + \lambda a_t dt\right)\right],$$

and that of the principal is

$$\mathbb{E}^a\left[\int_0^\tau e^{-rt}\left(z\left(1-a_t\right)dt - dI_t\right)\right]. \tag{1}$$

For notational convenience, we hereafter suppress all time subscripts when no confusion can be caused.

While contracts involving public randomization are of theoretical interest, they are typically not practical. Therefore, we relegate the discussion of public randomization to Section 6.1 of the Appendix, and consider only deterministic contracts for the rest of this paper unless otherwise mentioned.

2.2 Incentive Compatibility and Limited Liability

To characterize the incentive compatibility condition, we employ martingale techniques similar to those introduced by Sannikov (2008). When choosing his action at time t, the agent considers how it will affect his continuation value, defined as

$$w_t\left(X,a\right) = \mathbb{E}^a\left[\int_t^\tau e^{-\rho u}\left(dI_u + \lambda a_u du\right)\Big|\,\mathcal{F}_t\right]1_{\{t<\tau\}},$$

[12] By including the monitoring scheme α (i.e., the allocation of capacity) in the contract, we are studying the benchmark in which evaluation of the agent's performance changes focus as the contractual relationship develops, and this is explicitly stated at the outset and strictly implemented. This benchmark is realistic, especially for firms, organizations or bureaucratic systems that specify the details of their routine monitoring of employees in different positions with different seniority in contracts, charters or codes of conduct. Situations where the principal cannot commit to a monitoring scheme are also realistic in other circumstances, but are beyond the scope of this paper.

where $\{\mathcal{F}_t\}$ is the filtration generated by Y, and τ is the time when the project is terminated. Since the agent's outside option is zero, promise keeping implies that the termination time τ must be the first time when his continuation value w falls to zero. Martingale representation theorem yields the following lemma.

Lemma 2.1 *For any contract X that implements $a_t = 0$ for all $t \leq \tau$, there exist predictable processes (β_0, β_1) such that w_t evolves before termination $(t \leq \tau)$ as*

$$dw_t = \rho w_t dt - dI_t + \beta_{0,t} \left[dY_{0,t} - \mu \left(1 - \alpha_t\right) dt\right] - \beta_{1,t} dY_{1,t} \ . \tag{2}$$

The contract is incentive compatible if and only if

$$\mu \alpha_t \beta_{1,t} + \mu(1 - \alpha_t)\beta_{0,t} \geq \lambda \ . \tag{IC}$$

And the contract satisfies the limited liability constraint of the agent if and only if

$$\beta_{1,t} \leq w_t \tag{3}$$

and

$$\beta_{0,t} + w_t \geq 0 \ . \tag{4}$$

Proofs of this lemma and of all the other lemmas and propositions are relegated to the Appendix unless otherwise specified. Intuitively, β_0 refers to the agent's reward upon receipt of C-evidence, and β_1 refers to his punishment upon receipt of S-evidence. Hereafter, we refer to β_0 as "carrots," and β_1 as "sticks." (IC) highlights our model's key feature. Its left-hand side consists of the instruments, C-search and S-search, that the principal uses to incentivize the agent, which together with the associated carrots and sticks, must sum to at least λ, the agent's private benefit from shirking. The principal can choose not only the allocation of her monitoring capacity α, but also β_0 and β_1, the carrots and sticks.

Two limited liability constraints in Lemma 2.1 restrict the magnitudes of reward and punishment. (3) requires that sticks should be no more than the whole stake promised to the agent. (4) says that carrots plus the stake already promised to the agent must be non-negative, which will be shown slack.

As a preview of our results, after establishing some basic properties of the optimal contract in Section 3, Section 4.1 shows that in the optimal contract, the principal resorts only to carrots when the agent's continuation value w is low, and mainly resorts to sticks when w is high; i.e., the flexibility in capacity allocation is always valuable to her. Section 4.2 further shows that if the synergy z is large enough, the optimal contract involves the possibility of perpetuating the agent's effort (i.e., the payout boundary is an absorbing state), a feature novel in the dynamic contracting literature.

3 Basic Properties of the Optimal Contract

This section provides a heuristic derivation of some basic properties of the optimal contract. Proposition 3.1 at the end of this section verifies that this contract is indeed optimal. Notationally, superscript * hereafter denotes items in the optimal contract. First, our assumption $z > \lambda > 0$ implies that the optimal contract never induces shirking;[13] i.e.,

Property 1 $a^*(w) = 0$ *for all* w.

Let $B(w)$ denote the principal's value function. We have the Hamilton–Jacobi–Bellman (HJB) equation in the continuation region ($t < \tau$)

$$rB(w) = \max_{\alpha,\beta_0,\beta_1} z+(1-\alpha)\mu\left[B(w+\beta_0)-B(w)\right]+\left[\rho w-\beta_0\mu(1-\alpha)\right]B'(w) \ , \tag{5}$$

[13] In our formal derivation in the Appendix, we first assume that the optimal contract satisfies Property 1 when deriving its other properties, and verify this assumption at the end. This property differs from a setup featuring Brownian motions (e.g., Proposition 8 in DeMarzo and Sannikov (2006)), where it is optimal to induce the agent's effort only if the surplus generated is *significantly* greater than the agent's private benefit from shirking. Implementation of effort requires the agent to be exposed to the adverse effect of the quadratic variation of his continuation value. Such exposure has a first-order impact if his continuation value follows a Brownian motion. In our setup, the agent's continuation value follows a (generalized) Poisson process, and such exposure has no first-order impact. This is evident from the fact that V'' does not enter our HJB equation (9).

subject to

$$\mu\alpha\beta_1 + \mu(1-\alpha)\beta_0 \geq \lambda \; ; \tag{IC}$$

$$\beta_1 \leq w \; ; \tag{6}$$

$$\beta_0 + w \geq 0 \; ; \tag{7}$$

and

$$\alpha \in [0,1] \; . \tag{8}$$

The left-hand side of (5) is the principal's expected flow of value. The first term on the right-hand side, z, is the flow of synergy. The second term is due to the carrots β_0 given to the agent if C-evidence is obtained, which happens with probability $(1-\alpha)\mu dt$ conditional on $a=0$ being implemented from t to $t+dt$. The third term arises from the drift of w, where ρw is the rate at which interest accrues, and $-\beta_0\mu(1-\alpha)$ is the flip side of carrots due to promise keeping: if there is no C-evidence, the principal reduces the agent's continuation value at this rate to balance against carrots, so that the continuation value w_t net of a drift $\rho w_t dt$ is a martingale, and thus the contract does deliver w_t in expectation to the agent.

Note that no term in (5) corresponds to sticks (i.e., no term containing β_1), because S-evidence is never obtained if the agent follows the contract and takes $a=0$ at each instant. In this sense, sticks serve only as an off-equilibrium threat. Therefore, the limited liability constraint (6) must be binding: If S-evidence were obtained, the principal would maximize the penalty by terminating the project and confiscating the agent's whole stake w.

Property 2 *$\beta_1^*(w) = w$.*

Instead of $B(w)$, it is equivalent but more convenient to continue our analysis based on $V(w) = B(w) + w$, the sum of the principal's value function and the agent's continuation value, or their joint surplus. (5) then becomes

$$rV(w) = \max_{\alpha,\beta_0} z + [\rho w - \beta_0\mu(1-\alpha)]V'(w) + (1-\alpha)\mu[V(w+\beta_0) - V(w)] - (\rho - r)w \; . \tag{9}$$

Next, since $r < \rho$, we guess and later verify that there is a payout boundary $\bar{w}$ as standard in existing dynamic contracting models, e.g., DeMarzo and Sannikov (2006) and Biais et al. (2010). If $w > \bar{w}$, the principal will simply pay $dI = w - \bar{w}$ immediately and reduce the continuation value to $\bar{w}$. Otherwise, the principal will use backloading; i.e., wait for the agent's continuation value w to increase instead of paying him immediately (i.e., $dI = 0$). By construction, $V(\bar{w}+\beta_0) = V(\bar{w})$, so that when $w = \bar{w}$, the third term on the right-hand side of (9) equals zero, and $V'(\bar{w}) = 0$ if it exists. If $V'(\bar{w})$ does not exist; i.e., the left and right derivatives at $\bar{w}$ are not equal, (9) is not defined at $w = \bar{w}$, so the coefficient in front of $V'(w)$, which is the drift of the continuation value, must be zero at $\bar{w}$. As a result, the second term in (9) must also equal zero when $w = \bar{w}$, so that

$$V(\bar{w}) = \frac{z}{r} - (\rho - r)\frac{\bar{w}}{r} \tag{10}$$

and

$$B(\bar{w}) = \frac{z}{r} - \frac{\rho}{r}\bar{w} \ . \tag{11}$$

Moreover, we must have $\bar{w} \leq \frac{\lambda}{\mu}$. If not, then at any continuation value $w \in \left(\frac{\lambda}{\mu}, \bar{w}\right)$, the principal could always incentivize the agent with the following contract: paying out $w - \frac{\lambda}{\mu}$ immediately to reduce the agent's continuation value to $\frac{\lambda}{\mu}$; setting $\alpha = 1 - \frac{\rho}{\mu}$, $\beta_0 = \beta_1 = \frac{\lambda}{\mu}$, so that (IC) is binding, and that $\beta_0\mu(1-\alpha) = \rho\frac{\lambda}{\mu}$; i.e., the drift of the agent's continuation value is zero.[14] The principal's payoff from this new contract is

$$\frac{z}{r} - \frac{\rho}{r} \cdot \frac{\lambda}{\mu} - (\bar{w} - \frac{\lambda}{\mu}) > \frac{z}{r} - \frac{\rho}{r}\bar{w} = B(\bar{w}) \ ,$$

where the inequality follows $\bar{w} > \frac{\lambda}{\mu}$, contradicting the optimality of $B(\bar{w})$. As a standard result in this literature, the optimality of B implies $B'(w) > -1$ for $w < \bar{w}$ and $B'(w) = -1$ for $w > \bar{w}$. Then by definition, $V'(w) > 0$ for $w < \bar{w}$ and $V'(w) = 0$ for $w > \bar{w}$. We summarize these results in the following property.

[14] More precisely, under this contract, once $w = \frac{\lambda}{\mu}$, the continuation value never drifts away and the agent receives discrete payments of $\beta_0 = \lambda/\mu$ at the arrival rate ρ forever.

Property 3 *There exists a* $\bar{w} \in (0, \lambda/\mu]$ *such that i)* $dI^* = (w - \bar{w})^+$ *; ii)* V *is increasing in* $[0, \bar{w}]$*; iii) if* $w \geq \bar{w}$*,*

$$V(w) = z/r - (\rho - r)\bar{w}/r \ ; \tag{12}$$

and iv) either $V'(\bar{w}) = 0$*, or* $\rho\bar{w} - \mu(1 - \alpha^*(\bar{w}))\beta_0^*(\bar{w}) = 0$*, i.e., the drift at* $w = \bar{w}$ *is* 0*.*

Together with Property 2, we have $\beta_1^*(w) = w < \bar{w} \leq \lambda/\mu$ for $w < \bar{w}$. Hence, by (IC), sticks alone are not sufficient to incentivize the agent to work. Moreover, (IC) and Property 2 imply that $w\alpha^* + \beta_0^*(1 - \alpha^*) \geq \lambda/\mu$, thus $\beta_0^*(w) \geq \lambda/\mu \geq \bar{w}$ for $w < \bar{w}$. This, together with Property 3, implies

Property 4 $w + \beta_0^* \geq \bar{w}$ *for all* $w < \bar{w}$*.*

That is, a single piece of C-evidence suffices to make the continuation value w jump to the payout region $[\bar{w}, +\infty)$, so that $V(w + \beta_0^*) = V(\bar{w})$; i.e., β_0^*, carrots, raises their joint surplus only from $V(w)$ to $V(\bar{w})$, and the remaining reward, $\beta_0^* - (\bar{w} - w)$, is an immediate transfer from the principal to the agent and has no impact on their joint surplus. Also, the limited liability constraint (7) slacks as conjectured.

Property 4 simplifies our derivation of the optimal contract, given that the value function V may not always be concave.[15] To see this, note that according to Property 4, (9) becomes

$$rV(w) = \max_{\beta_0, \alpha} z + [\rho w - \beta_0 \mu(1 - \alpha)] V'(w) + (1 - \alpha)\mu[V(\bar{w}) - V(w)] - (\rho - r)w \ , \tag{13}$$

whose right-hand side is always decreasing in β_0. This has two important implications. First, it indicates the advantage of using S-search rather than C-search, regardless of the concavity of V. In equilibrium, S-evidence is never obtained, and thus S-search incentivizes the agent without causing variation in his continuation value w. But if C-search is used (i.e., $\alpha < 1$), C-evidence is

[15] This possibility is discussed in Subsection 4.2. We also concavify the value function via public randomization in Section 6.1 of the Appendix.

obtained in equilibrium and generates variation in w. Property 4 implies that effectively, the upward jump in w upon the receipt of C-evidence is always $\bar{w} - w$ (after the bonus payment), which is independent of α and β_0. But the magnitude of the downward drift of w in the absence of C-evidence, $\beta_0\mu(1-\alpha)$, is increasing in both the capacity allocated to C-search, $\mu(1-\alpha)$, and the associated carrots, β_0. Therefore, the more the principal resorts to C-search, the more adverse variation in w is generated, making it detrimental relative to sticks.

Second, the fact that the right-hand side of (13) is decreasing in β_0 implies a binding (IC) in the no-payment region $[0, \bar{w}]$; i.e.,

Property 5 $\mu\left[\alpha^* w + (1-\alpha^*)\beta_0^*\right] = \lambda.$

The incentive compatibility constraint (IC) plays a central role in our model. Property 5 establishes that the combination of C-search and S-search should be just enough to overcome the agent's private benefit from shirking. Note that the principal still has a degree of freedom to adjust the sensitivities of the agent's continuation value to evidence reflecting his actions. In the no-payment region, we have $dI = 0$ by definition and $V(w + \beta_0) = V(\bar{w})$ from Property 3. By Property 5, the HJB equation (13) becomes

$$rV(w) = \max_{\alpha} z-(\rho-r)w+(1-\alpha)\mu[V(\bar{w})-V(w)]+(\rho w-\lambda+\mu\alpha w)V'(w). \quad (14)$$

As mentioned in the literature review, this contrasts with the counterpart in models without choice among multiple performance indicators; e.g., in Sannikov (2008) and Biais et al. (2010), where there is no such degree of freedom.

Notice that α affects the right-hand side of (14) through the last two terms. As explained before, the third term reflects its impact through carrots; i.e., raising α reduces the arrival rate of C-evidence and that of the contingent increment $V(\bar{w})-V(w)$ in their joint surplus. This in turn reduces the expected instantaneous joint surplus $(1-\alpha)\mu[V(\bar{w}) - V(w)]$. The impact is linear in α, and the marginal impact is $-\mu[V(\bar{w}) - V(w)]$, whose absolute value decreases monotonically with w.

The last term on the right-hand side of (14) reflects the impact of α through the flip side of carrots; i.e., a lower arrival rate of C-evidence also reduces the downward drift of the agent's continuation value w due to promise keeping.[16] This increases the expected instantaneous joint surplus $(\rho w - \lambda + \mu\alpha w)V'(w)$. This effect is also linear in α, with a marginal impact $\mu w V'(w)$, which could be non-monotonic in w. Since the total impact of α is linear, with marginal impact

$$\mu\left[wV'(w) + V(w) - V(\bar{w})\right], \tag{15}$$

the principal would choose $\alpha^* = 0$ if $wV'(w) + V(w) - V(\bar{w}) < 0$, and $\alpha^* = 1$ if $wV'(w) + V(w) - V(\bar{w}) > 0$.

Properties 2 and 5 imply

$$\mu(1-\alpha^*)\beta_0^* = \lambda - \mu\alpha^* w. \tag{16}$$

Since $w \le \bar{w} \le \lambda/\mu$, we have $\frac{\partial\beta_0^*}{\partial\alpha^*} = \frac{\lambda-\mu w}{\mu(1-\alpha^*)^2} > 0$, and thus (16) highlights the substitution between the capacity allocated to C-search, $1-\alpha$, and the associated carrots, β_0, which is peculiar to our setup with flexibility in monitoring design. The more capacity allocated to C-search, the higher is the probability of obtaining C-evidence that confirms the agent's effort, and thus less reward is needed to incentivize the agent. Conversely, higher carrots provide a stronger incentive for the agent, and thus reduce the principal's reliance on obtaining C-evidence, enabling her to use S-search.

If $\alpha^* = 0$; i.e., incentive is completely provided through carrots, (16) yields $\beta_0^* = \lambda/\mu$. However, while the value function (14) is well-behaved when $\alpha^* = 1$, a singularity arises in the original optimization problem (13): by (16), we have $\beta_0^* \to \infty$ as $\alpha^* \to 1$. Intuitively, if $\alpha^*(w) = 1$ in (14), it is optimal for the principal to rely on sticks as much as possible when the agent's continuation value is w. Yet, since $w \le \bar{w} \le \lambda/\mu$ (by Property 3), if $\alpha = 1$, Property 5 fails; i.e., sticks do not suffice to incentivize the agent, so an additional incentive from carrots is necessary. In addition, in this no-payment region, $\alpha = 1$ further pre-

[16] Note that $\rho w - \lambda + \mu\alpha w \le 0$ since $\bar{w} \le \frac{\lambda}{\rho+\mu\bar{\alpha}}$. Raising α thus reduces the magnitude of the downward drift.

cludes carrots and thus the rise of w, rendering the principal unable to keep the promise to deliver w to the agent. But if $\alpha < 1$, the principal can always reduce the expected carrot payment, $\mu(1-\alpha)\beta_0 = \lambda - \alpha\mu w$, by making α as close to 1 as possible (i.e., reducing the probability that C-evidence arrives when the agent works, $\mu(1-\alpha)$, since the right-hand side is decreasing in α) while raising the bonus upon its arrival, β_0, creating an open-set problem for finding the optimal α and the associated β_0. That is why $\beta_0^* \to \infty$ as $\alpha^* \to 1$ in (16). Since as $\alpha \to 1$, $\mu(1-\alpha)\beta_0 \to \lambda - \mu w$, which is well defined, if we view the optimal expected carrot payment $\mu(1-\alpha^*)\beta_0^*$ as a single variable defined by (16), the resulting value function (14) is well-behaved when its optimal control $\alpha^* = 1$. Thus, this open-set problem is innocuous for our qualitative results regarding the shape of the value function and the payout boundary. But it does cause trouble for separate identification of β_0^* and α^*, and for the economic interpretation of the expected carrot payment $\mu(1-\alpha^*)\beta_0^*$. Therefore, we henceforth preclude this singularity by assuming $\alpha \leq \bar{\alpha}$ for some exogenous $\bar{\alpha}$ very close to 1. Specifically, we require $\max\left\{1/2, 1-\frac{\rho}{\mu}\right\} < \bar{\alpha} < 1$.[17] Given that, we have

Property 6 *If* $wV'(w) + V(w) < V(\bar{w})$, *then* $\alpha^* = 0$ *and* $\beta_0^* = \lambda/\mu$;
If $wV'(w) + V(w) = V(\bar{w})$, *then* $\alpha^* \in [0, \bar{\alpha}]$ *and* $\beta_0^* = \frac{\lambda - \mu\alpha^* w}{\mu(1-\alpha^*)}$;
If $wV'(w) + V(w) > V(\bar{w})$, *then* $\alpha^* = \bar{\alpha}$ *and* $\beta_0^* = \frac{\lambda - \mu\bar{\alpha} w}{\mu(1-\bar{\alpha})}$.

The following proposition verifies that our derived contract is indeed optimal.

Proposition 3.1 *The solution* V *to HJB equation* (9) *is principal and agent's joint surplus under the optimal contract. Moreover, the optimal contract is characterized by Properties 1 and 6, and is terminated at time* τ *when* w *falls to zero for the first time.*

---

[17] Note that the control that we use to establish Property 3 is still viable under the assumption that $\alpha \leq \bar{\alpha}$. So this new assumption affects none of the previous properties.

4 The Role of Flexible Monitoring

This section highlights the critical role of flexible monitoring, which is central to this paper. Section 4.1 shows that such flexibility is indeed utilized by and thus valuable to the principal. Section 4.2 articulates that such flexibility allows a long-term contractual relationship that perpetuates the agent's effort with positive probability when the synergy, z, is sufficiently large, and that the value function is convex in the vicinity of the payout boundary $\bar{w}$ if and only if such perpetuation is optimal. Section 4.3 summarizes these results with a graphic illustration using the narrative of career path and provides a few empirically plausible predictions.

4.1 Flexibility in Monitoring is Valuable

Property 6 establishes that other than in knife-edge cases, the optimal monitoring capacity allocated to S-search, α^*, is either 0 or $\bar{\alpha}$.[18] This subsection further establishes that an optimal contract necessarily involves both possibilities.

Specifically, Proposition 4.1 establishes that $\alpha^*(w) = 0$ when the agent's continuation value w is close to 0, and $\alpha^*(w) = \bar{\alpha}$ when w is close to the payout boundary $\bar{w}$. This indicates that flexibility in allocating monitoring capacity between C-search and S-search allows the principal to incentivize the agent differently at different stages of his career, and is thus valuable to the principal.

Proposition 4.1 *There exists a $\hat{w} \in (0, \bar{w})$, such that $\alpha^*(w) = 0$ and $\beta_0^*(w) = \lambda/\mu$ for $w \in (0, \hat{w})$, and that $\alpha^*(w) = \bar{\alpha}$ and $\beta_0^*(w) = \frac{\lambda - \mu\bar{\alpha}w}{\mu(1-\bar{\alpha})}$ for $w \in (\hat{w}, \bar{w}]$.*

From Property 6, the optimal contract involves only $\alpha = 0$ and $\alpha = \bar{\alpha}$ except for the knife-edge case featuring indifference. From (14) we know that for each $w \in (0, \bar{w})$, either $\alpha = 0$ and

$$rV(w) = z + [\rho w - \lambda]V'(w) + \mu[V(\bar{w}) - V(w)] - (\rho - r)w \ , \qquad (17)$$

[18] Lemma 6.3 in the Appendix shows that no interval of continuation values exists, such that the principal is indifferent between 0 and $\bar{\alpha}$. Thus the knife-edge cases are non-generic.

or $\alpha = \bar{\alpha}$ and

$$rV(w) = z + (1-\bar{\alpha})\mu[V(\bar{w}) - V(w)] + [\rho w - \lambda + \mu\bar{\alpha}w]V'(w) - (\rho - r)w \ . \quad (18)$$

Both equations can be solved in closed form (see Appendix). It can be verified that $V'(0)$ is finite. This implies $0 \cdot V'(0) + V(0) = 0 < V(\bar{w})$, and by continuity, there is a neighborhood of $w = 0$ such that $wV'(w) + V(w) < V(\bar{w})$. Thus, by Property 6, the principal relies completely on C-search when the agent's continuation value w is low. The statement for the vicinity of $\bar{w}$ can be similarly proved with closed-form solutions. We show further that the optimal ranges for using $\alpha = 0$ and $\alpha = \bar{\alpha}$ must be connected respectively, so that there is a cutoff $\hat{w}$ separating them.

Intuitively, when the agent's continuation value w is low, the principal should not rely on S-search, because the agent has little to lose even if he is known to have shirked. Relying on C-search also maximizes the chance of obtaining C-evidence. This helps the principal quickly raise the agent's "skin in the game," which makes S-search (which is costless to the principal) more effective in the future, and pushes the project away from termination (which is socially inefficient). When the agent's continuation value w is higher, the principal can impose a large penalty for S-evidence. Since such a penalty is just an off-equilibrium threat, making S-search less costly than C-search, the principal should rely on S-search as much as possible.

The flexibility of combining C-search and S-search allows the principal to exploit their respective advantages. On one hand, C-search generates greater variation than S-search in the agent's continuation value, and is thus less advantageous to the principal. Given that the agent does work, no S-evidence would arrive, and thus, no adjustment of the agent's continuation value would be required. However, in equilibrium, C-evidence would be obtained, which would necessarily involve a reward and the downward adjustment of the agent's continuation value in the absence of C-evidence. On the other hand, a sufficiently high continuation value is required as the agent's skin in the game for sticks alone to be an effective incentive, whereas the effectiveness of C-search does not depend on the agent's continuation value. Moreover, even if S-search could work alone, a high continuation value for the agent has to be maintained, which involves interest expenditure for the principal, making S-search less advantageous than C-search. This tradeoff between C-search and S-search induces the principal to rely only on C-search when w is low, and on S-search, as much as possible, when w is high.

Concerning carrots, β_0, recall that the right-hand side of (13) is decreasing in β_0, since an increase in β_0 makes the drift of the agent's continuation value, $\rho w - \beta_0 \mu(1-\alpha)$, more negative due to promise keeping, and thus makes the project more prone to termination. Hence, given the optimal capacity allocation α^*, β_0^* should be set as low as possible — such that (IC) is binding. Thus, for agents facing $\alpha^* = 0$, including those with $w \in (0, \hat{w})$, we have $\beta_0^*(w) = \lambda/\mu$, and the resulting drift of w is $\rho w - \lambda < 0$. For agents facing $\alpha^* = \bar{\alpha}$, including those with $w \in (\hat{w}, \bar{w}]$, we have $\beta_0^*(w) = \frac{\lambda - \mu\bar{\alpha}w}{\mu(1-\bar{\alpha})}$, and the resulting drift of w is $\rho w - \lambda + \mu\bar{\alpha}w \leq 0$.[19]

Note first that $\beta_0^*(w)$ is constant in the region of $\alpha^*(w) = 0$, but is decreasing in the region of $\alpha^* = \bar{\alpha}$. This is because in the latter case, sticks increase with w, partially substituting carrots that are required by (IC). Second, $\beta_0^*(w)$ features an upward jump when α^* switches from 0 to $\bar{\alpha}$. To see this, notice the fact that any switching point $w < \frac{\lambda}{\mu}$ implies that the size of the jump is $\frac{\lambda - \mu\bar{\alpha}w}{\mu(1-\bar{\alpha})} - \frac{\lambda}{\mu} > \frac{\lambda - \mu\bar{\alpha}\cdot\frac{\lambda}{\mu}}{\mu(1-\bar{\alpha})} - \frac{\lambda}{\mu} = 0$. Third, the drift of w increases (i.e., becomes less negative) with w, due to the interest accrued (i.e., due to the term ρw) and the increasing reliance on S-search in lieu of C-search (i.e., due to the term $\mu\bar{\alpha}w$). Lastly, the drift of w is negative, which moves w towards 0, the termination boundary, unless w reaches the payout boundary $\bar{w}$ and $\bar{w} = \frac{\lambda}{\rho + \mu\bar{\alpha}}$, where the drift is zero; i.e., the project and the agent's effort are perpetuated. Section 4.2 characterizes when such perpetuation is optimal.

[19] This is because $w \leq \bar{w} \leq \frac{\lambda}{\rho + \mu\bar{\alpha}}$.

4.2 Possibility of Perpetuating the Agent's Effort

The assumption that $z > \lambda > 0$ implies that at *each* instant, from the planner's perspective, having the agent working *always* dominates letting him shirk, which further dominates terminating the project. Perpetuation of the agent's effort is always a feasible option for the principal as well.[20] But is this optimal for her? This subsection shows that, if and only if the synergy z is large enough relative to the agent's private benefit from shirking, λ, the payout boundary $\bar{w}$ of the optimal contract is another absorbing state in addition to 0, so that the *optimal* contract involves the perpetuation of the agent's effort with positive probability for *any* continuation value w. And the value function V is convex in $(\hat{w}, \bar{w})$, where the optimal control $\alpha^* = \bar{\alpha}$.[21] This feature is novel in the dynamic contracting literature. Otherwise, as in standard in the literature, $\bar{w}$ is reflective, so that the optimal contract results in termination with probability one and V is universally concave.

Before formally presenting the result in Proposition 4.2 and explaining the underlying mechanisim, it is worthwhile to highlight its value with the fact that, without the flexibility in capacity allocation, the possibility of optimal perpetuation for all w is out of the question regardless of the synergy z. Consider first the extreme situation where only C-search is viable (i.e., α is fixed to 0) as in Sun and Tian (2017). From Property 5, the cheapest way to satisfy (IC) is to set the carrot $\beta_0(w) = \lambda/\mu$ for all w. But promise keeping inevitably requires the reduction of continuation value w when C-evidence does not arrive, jeopardizing perpetuation. The only way to counteract is to defer bonus payments, and the cheapest option is to set an absorbing payout boundary $\bar{w} = \lambda/\rho$.[22] Despite its *feasibility*, the interest accrual required for maintaining such a high continuation value makes this option so costly that it is *never optimal* for the principal *regardless of* the synergy z. Indeed, our closed-form solution to (17) implies that $V'(\lambda/\rho) < 0$ for all z.

[20] As discussed below, one trivial way is to incentivize the agent with only carrots, and to defer payments whenever the agent's continuation value w approaches zero.

[21] Recall from Section 2 that we discuss public randomization in Section 6.1 and preclude it in the rest of the paper unless otherwise mentioned.

[22] The drift of w there is $\rho\bar{w} - (1-\alpha)\mu\beta_0 = \rho \cdot \lambda/\rho - 1 \cdot \mu \cdot \lambda/\mu = 0$.

In the opposite extreme situation, where only S-search is viable (i.e., α is fixed to 1), the cheapest way to satisfy (IC) is to maintain $w = \lambda/\mu$ with a constant flow payment $\rho\lambda/\mu$ and an immediate lump-sum payment $w - \lambda/\mu$ if $w \geq \lambda/\mu$; i.e., to have an absorbing payout boundary $\bar{w} = \lambda/\mu$. Since $z > \lambda > \rho\lambda/\mu$, this option is always optimal for the principal. But if $w < \lambda/\mu$, it is impossible to satisfy (IC), so the principal has to immediately raise w to λ/μ unconditionally. But by Proposition 4.1, the principal can instead base the rise of w on carrots if she is entitled to the flexibility of combining them with sticks. This makes it possible to have an absorbing payout boundary while satisfying (IC) *for all* w in the *optimal contract*.

Proposition 4.2 *1.* $\bar{w} \leq \frac{\lambda}{\rho+\mu\bar{\alpha}}$*, and* $\bar{w}$ *is absorbing if and only if* $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$*;*

2. $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$ *if and only if* V *is convex in* $(\hat{w}, \bar{w})$*, which holds if and only if* z/λ *is greater than a threshold* $\theta(r, \rho, \mu, \bar{\alpha})$*:*

3. Perpetuation may not be optimal while feasible: if there exist constants $c_\rho > 1$ *and* $c_\mu > c_\rho$ *such that* $\rho = c_\rho r$ *and* $\mu = c_\mu r$*, then* $\lim_{r\to+\infty} \theta(r, \rho, \mu, \bar{\alpha}) = +\infty$*;*

4. Perpetuation is optimal with large monitoring capacity: for all $r, \rho, \bar{\alpha}$*,* $\lim_{\mu\to+\infty} \theta(r, \rho, \mu, \bar{\alpha}) = 0$*.*

Since $\frac{\lambda}{\rho+\mu\bar{\alpha}} < \lambda/\mu$, Statement 1 indicates that the flexibility in combining C-search with S-search reduces the cost of perpetuating the agent's effort. As aforementioned, the perpetuation requires a minimum $w = \lambda/\mu$ if the principal can resort to only S-search, and the agent's continuation value is fixed there with a perpetuity $\rho\lambda/\mu$ from the principal. The flexibility in combining C-search with S-search allows the principal to replace this perpetuity with carrots, which in turn replaces part of S-search's role in (IC). This reduces the required continuation value and saves on accrued interest. The minimum w required for perpetuation with this flexibility is such that the flow interest ρw exactly equals the expected carrots required by (IC), $\lambda - \bar{\alpha}\mu w$, as in Property 5, which yields the absorbing $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$ in Statement 1.

Statement 2 formally establishes that it is optimal for the principal to set an absorbing payout boundary if and only if the synergy z is large enough relative to the agent's private benefit from shirking, λ. We give a precise formula for the threshold $\theta(r, \rho, \mu, \bar{\alpha})$ in the Appendix, which is derived from our closed-form solution to (18).

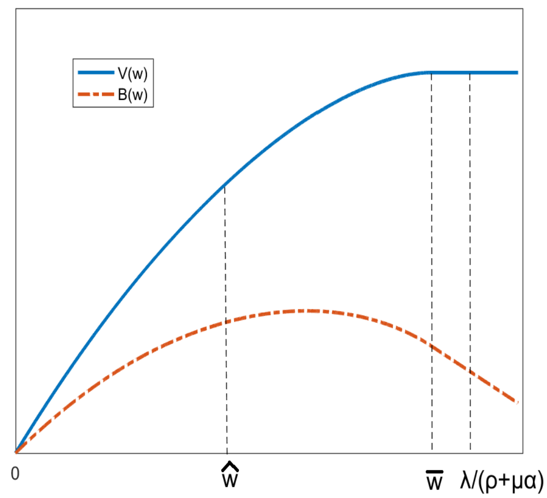

Figure 1: Reflective Payout Boundary $\bar{w}$

In addition, statement 2 points out a novel feature of our model: the value function V is convex in $(\hat{w}, \bar{w})$, where $\alpha^* = \bar{\alpha}$, if and only if $\bar{w}$ is absorbing.[23] V is always concave in $(0, \hat{w})$, where $\alpha^* = 0$. This reflects the standard incentive-versus-interest tradeoff, as illustrated in Figure 1. That is, an increase in w pushes the continuation value away from the termination boundary 0, whose marginal benefit decreases with w, but whose marginal cost, due to an increase in accrued interest, is constant.[24] But new economic forces come into play in $(\hat{w}, \bar{w})$, where $\alpha^* = \bar{\alpha}$. There, the reliance on S-search reduces the downward drift of the continuation value, $\mu(1-\alpha)\beta_0$, that balances carrots. This raises the marginal benefit of increasing w without affecting the marginal cost, and thus makes V less concave. Moreover, a fundamental change occurs when

[23] Recall from Section 2 that we discuss public randomization in Section 6.1 of the Appendix and preclude it in the text unless otherwise mentioned.

[24] Besides Sun and Tian (2017), such a tradeoff is also featured in Biais et al. (2010) and DeMarzo and Sannikov (2006).

$\bar{w}$ becomes absorbing, as illustrated in Figure 2. In that case, the marginal benefit of increasing w results not only from the fact that w is further away from the inefficient absorbing state 0, but also from the fact that w is closer to the efficient absorbing state $\bar{w}$. The latter fact, together with the constant marginal cost due to accrued interest, makes the marginal benefit increasing instead of decreasing in w and thus the value function V convex in $(\hat{w}, \bar{w})$.

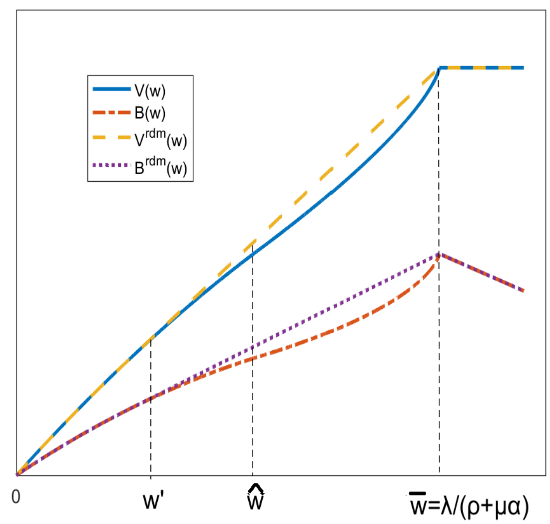

Figure 2: Absorbing Payout Boundary $\bar{w}$

Statements 3 and 4 illustrate that the flexibility in combining C-search with S-search offers the principal more flexibility to decide whether to perpetuate the agent's effort. Recall that with only C-search, such perpetuation is never optimal for the principal even if $z/\lambda \to \infty$, but that as long as $z/\lambda > 1$, with only S-search, such perpetuation is always optimal for her, provided that $w \geq \lambda/\mu$. Statement 3 shows that, with the flexibility in combining C-search with S-search, such perpetuation is no longer optimal when $w \geq \lambda/\mu$ if ρ/r and μ/r are constant and r is sufficiently large. This is because, the large difference in $\rho - r$ increases the interest expenditure for the principal given the same w. Statement 4 instead shows that it is always optimal to have an absorbing payout boundary $\bar{w}$ if the principal has large monitoring capacity μ. This is because, large monitoring capacity reduces the penalty needed when S-evidence arrives (i.e., $\bar{w}$), and in turn the interest expenditure from maintaining it.

4.3 Practical Implications

Our model offers practical suggestions for the design incentive schemes in reality, and they turn out to be largely consistent with the real practice in academia, where the majority of our readers are. First, Proposition 4.1 suggests that incentives for junior employees (i.e., agents with continuation value $w \in (0, \hat{w})$ in the model) are mainly based on carrots, since they need to accumulate a cushion against unemployment (i.e., termination) and have little to lose even if caught shirking. Senior employees are instead incentivized in stick-dominant mode (i.e., $\alpha = \bar{\alpha}$), since they have enough skin in the game, and sticks are off-equilibrium penalties, which are less costly than on-equilibrium carrots. This is largely consistent with the fact that publication matters more for junior faculty than for senior faculty.

Second, Proposition 4.2 suggests that permanent positions are offered only for employees with sufficiently large potential synergy. This is largely consistent with the fact that for top research universities, only research faculty can be on the tenure track, but this pattern is less clear for teaching schools. Similar pattern also applies to the viability of partnership in accounting, consulting and law firms. Third, except for those hired permanently, in the absence of evidence confirming their contribution, employees become more prone to unemployment, and the more so if they are less senior. Lastly, instead of fixed payments only, compensation for employees hired permanently should still involve carrots.[25] This is consistent with the fact that tenured professors are still entitled to wage increase and promotions for post-tenure achievements.

[25] Recall from Section 3 that this is true even if $\bar{\alpha} \to 1$.

5 Conclusion

This paper studies the joint design of monitoring and compensation schemes in a continuous-time moral hazard model. In the model, a principal (i.e., the designer of the scheme) can flexibly combine C-search with S-search to incentivize an agent. That is, the principal can flexibly allocate her limited monitoring capacity between confirmatory and contradictory evidence concerning the agent's effort as the basis for reward or punishment. We find that such flexibility generates rich dynamics, which differ qualitatively from the situation where only one of the two methods is feasible. When the agent has little skin in the game, the principal resorts only to C-search; when the agent has sufficient skin in the game, the principal instead assigns the highest possible weight to S-search. Moreover, only with such flexibility can the agent's effort be perpetuated with positive probability when the agent is less patient than the principal.

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6 Appendix

6.1 Public Randomization

In the text, we have been focusing on deterministic contracts, on the basis that random contracts are of little practical relevance in reality. This is also theoretically without loss of generality if the resulting value function is globally concave as in the case illustrated in Figure 1 and as in most models in the literature. However, as established in Proposition 4.2, our value function is convex in the vicinity of the payout boundary $\bar{w}$ if it is absorbing (Figure

2). For this situation, this section discusses the extension in which public randomization of the following form is allowed. At time 0, in addition to starting the contractual relationship with a deterministic continuation value w_0, the principal can choose a mean-preserving spread of w_0 as the basis for random contracts, but no further randomization is allowed for $t > 0$. Since $B = V - w$, and the linear term has no effect on the concavification operation, we can work with the joint surplus function V without loss of generality.

Proposition 6.1 *With public randomization, the principal's value function is $B^{rdm} = V^{rdm} - w$, where V^{rdm} is the concavification of V.*

Proof. Proposition 4.2 establishes that when V is not globally concave, we must have $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$, and $V(\bar{w})$ is uniquely determined by Property 3. In addition, V is concave in $(0, \hat{w})$ and convex in $(\hat{w}, \bar{w})$. Therefore, the concavification of V must be over $\bar{w}$ and some $w' \in (0, \hat{w})$ as shown with the yellow broken line in Figure 2.[26]

We check that the values of non-randomized states are not changed. First, $V(\bar{w})$ does not change because $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$ is absorbing and its value does not depend on the values of other states. For $w \in (0, w')$, notice that the continuation value may only drift downward or jump upward over $\bar{w}$. Since $V(\bar{w})$ remains the same and $V^{rdm} = V$ for $w \in (0, w')$, the values of these states satisfy the same HJB equation and thus remain the same. ■

6.2 Proofs in Section 2

6.2.1 Proof of Lemma 2.1

Proof. The proof is a standard application of the martingale representation theorem. For any given contract $X = (\alpha, I, \tau)$ and effort process a, define

$$M_t^{1,a} = Y_t^1 - \mu \int_0^t \alpha_s a_s ds$$

[26] The purple dotted line in Figure 2 illustrates the corresponding concavification B^* of the principal's value function B.

and

$$M_t^{0,a} = Y_t^0 - \mu \int_0^t (1-\alpha_s)(1-a_s)ds \ .$$

If the agent follows the effort process a, his lifetime expected payoff, conditional on information at time t, is

$$U_t = \int_0^{t\wedge\tau} e^{-\rho s}(dI_s + \lambda a_s ds) + e^{-\rho t} W_t \ .$$

Let $\tilde{a}$ be an arbitrary effort process. Let $\tilde{U}_t$ denote the agent's lifetime expected payoff conditional on information at time t if he follows $\tilde{a}$ until time t and then reverts to a. Then by the martingale representation theorem, U_t can be written as

$$U_t = U_0 - \int_0^{t\wedge\tau} e^{-\rho s}\beta_{1,s} dM_s^{1,a} + \int_0^{t\wedge\tau} e^{-\rho s}\beta_{0,s} dM_s^{0,a}$$

For each $t \geq 0$,

$$\begin{aligned}
\tilde{U}_t =& U_t + \int_0^{t\wedge\tau} e^{-\rho s}\lambda(\tilde{a}_s - a_s)ds \\
=& U_0 - \int_0^{t\wedge\tau} e^{-\rho s}\beta_{1,s} dM_s^{1,a} + \int_0^{t\wedge\tau} e^{-\rho s}\beta_{0,s} dM_s^{0,a} + \int_0^{t\wedge\tau} e^{-\rho s}\lambda(\tilde{a}_s - a_s)ds \\
=& U_0 - \int_0^{t\wedge\tau} e^{-\rho s}\beta_{1,s} dM_s^{1,\tilde{a}} + \int_0^{t\wedge\tau} e^{-\rho s}\beta_{0,s} dM_s^{0,\tilde{a}} + \int_0^{t\wedge\tau} e^{-\rho s}\lambda(\tilde{a}_s - a_s)ds \\
&- \int_0^{t\wedge\tau} e^{-\rho s}\mu\alpha_s\beta_{1,s}(\tilde{a}_s - a_s)ds - \int_0^{t\wedge\tau} e^{-\rho s}\mu(1-\alpha_s)\beta_{0,s}(\tilde{a}_s - a_s)ds
\end{aligned}$$

Hence, $a_t = 0$ for all t is incentive compatible if and only if the drift term of the above expression is non-positive for any effort process $\tilde{a} \neq 0$; i.e.,

$$\lambda \leq \mu\alpha_t\beta_{1,t} + \mu(1-\alpha_t)\beta_{0,t}$$

for all t before termination. ■

6.3 Proofs in Section 3

We first prove statement 1) of Proposition 4.2, which establishes a tighter upper bound for the payout boundary $\bar{w}$ than that in Property 3. Then we prove Property 3 and Proposition 3.1, **assuming that Property 1 holds.** We verify Property 1 in Section 6.5. Proofs of all the other properties are straightforward from the text and are therefore omitted.

Lemma 6.1 $\bar{w} \le \frac{\lambda}{\rho+\mu\bar{\alpha}}$*, and* $\bar{w}$ *is absorbing if and only if* $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$.

Proof. If $\bar{w} > \frac{\lambda}{\rho+\mu\bar{\alpha}}$, then at any continuation value $w \in \left(\frac{\lambda}{\rho+\mu\bar{\alpha}}, \bar{w}\right)$, the principal could always incentivize the agent with the following contract: paying out $w - \frac{\lambda}{\rho+\mu\bar{\alpha}}$ immediately to reduce the agent's continuation value to $\frac{\lambda}{\rho+\mu\bar{\alpha}}$; setting $\alpha = \bar{\alpha}$, $\beta_1 = \frac{\lambda}{\rho+\mu\bar{\alpha}}$ and $\beta_0 = \frac{\rho\lambda}{\mu(1-\bar{\alpha})(\rho+\mu\bar{\alpha})}$, so that (IC) is binding, and that $\beta_0\mu(1-\bar{\alpha}) = \rho\frac{\lambda}{\rho+\mu\bar{\alpha}}$; i.e., the drift of the agent's continuation value is zero, and thus $w = \frac{\lambda}{\rho+\mu\bar{\alpha}}$ is an absorbing state. The principal's payoff from this new contract is

$$\frac{z}{r} - \frac{\rho}{r} \cdot \frac{\lambda}{\rho+\mu\bar{\alpha}} - (\bar{w} - \frac{\lambda}{\rho+\mu\bar{\alpha}}) > \frac{z}{r} - \frac{\rho}{r}\bar{w} = B(\bar{w}) \ ,$$

where the inequality follows $\bar{w} > \frac{\lambda}{\rho+\mu\bar{\alpha}}$, contradicting the optimality of $B(\bar{w})$.

For the second conclusion, first consider the "if" statement. If $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$, we show that the following strategy is feasible and optimal, and makes $\bar{w}$ absorbing: $\alpha = \bar{\alpha}$, $\beta_0 = \frac{\rho\lambda}{\mu(1-\bar{\alpha})(\rho+\mu\bar{\alpha})}$ and $\beta_1 = \bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$. Feasibility results from the binding (IC) constraint. To see why $\bar{w}$ is absorbing, note that when $w = \bar{w}$, the positive component of the drift of the agent's continuation value due to accrued interest is $\rho\bar{w}dt = \frac{\rho\lambda}{\rho+\mu\bar{\alpha}}dt$, and the negative component as the flip side of carrots is $\mu(1-\bar{\alpha})\beta_0 dt$, which also equals $\frac{\rho\lambda}{\rho+\mu\bar{\alpha}}dt$, so that w remains constant when there is no C-evidence, and when it is obtained, the whole reward β_0 is paid out immediately so that w remains at $\frac{\lambda}{\rho+\mu\bar{\alpha}}$.

To see the optimality of this strategy, observe that the principal's expected payoff at $w = \frac{\lambda}{\rho+\mu\bar{\alpha}}$ is $\mathbb{E}(\int_0^{+\infty} ze^{-rt}dt - \beta_0 \int_0^{+\infty} e^{-rt}dY_{0,t})$. Since $Y_{0,t} - \mu(1-\bar{\alpha})t$

is a martingale,

$$\mathbb{E}(\int_0^{+\infty} ze^{-rt}dt - \beta_0 \int_0^{+\infty} e^{-rt}dY_{0,t}) = \frac{z}{r} - \frac{\beta_0\mu(1-\bar{\alpha})}{r} = \frac{z}{r} - \frac{\rho}{r} \cdot \frac{\lambda}{\rho + \mu\bar{\alpha}} \ .$$

Thus, the expected joint surplus is

$$\frac{z}{r} - \frac{\rho}{r} \cdot \frac{\lambda}{\rho + \mu\bar{\alpha}} + \bar{w} = \frac{z}{r} - \frac{\rho - r}{r} \cdot \bar{w} \ .$$

From (10), this strategy achieves the optimal joint surplus at the payout boundary.

Now consider the "only if" statement. From Property 3, it suffices to show that any $\bar{w} < \frac{\lambda}{\rho+\mu\bar{\alpha}}$ cannot be absorbing. Any contract respecting (IC) satisfies

$$\beta_0\mu(1-\bar{\alpha}) \geq \lambda - \bar{w}\mu\bar{\alpha} > \rho \cdot \frac{\lambda}{\rho + \mu\bar{\alpha}} > \rho\bar{w} \ .$$

Thus, when there is no C-evidence, the agent's continuation value always has a downward drift term $\rho\bar{w} - \beta_0\mu(1-\bar{\alpha}) < 0$. This implies that the payout boundary $\bar{w} < \frac{\lambda}{\rho+\mu\bar{\alpha}}$ is reflective. ■

6.3.1 Proof of Property 3

Proof. Note that the joint value function V must be nondecreasing in continuation value w. This is because in any region where V is strictly decreasing in w, the principal can benefit from paying out to the agent, contradicting the optimality of V. Let $A \subset \mathbb{R}_+$ denote the region of continuation values in which V is strictly increasing. Then the principal does not make any payment when $w \in A$ and $\mathbb{R}_+ \backslash A$ is the payout region. Since $\rho > r$, deferring payment becomes infinitely costly as $w \to +\infty$. Thus the payout region $\mathbb{R}_+ \backslash A$ is nonempty and there exists a $\bar{w} = \inf(\mathbb{R}_+ \backslash A)$.

By construction, V is strictly increasing for $w \in [0, \bar{w}]$ and is constant in a right neighborhood of $\bar{w}$, $(\bar{w}, \bar{w} + \Delta)$. Then, if $V'(\bar{w})$ exists, it must be zero. If $V'(\bar{w})$ does not exist, i.e., the left and the right derivatives are not equal, (9) is not defined at $w = \bar{w}$ and the coefficient in front of $V'(w)$ must be zero at $\bar{w}$. Notice that this coefficient is the drift of the continuation value. Hence, when $V'(\bar{w})$ does not exist, $\bar{w}$ is an absorbing payout boundary. As a result, no matter whether $V'(\bar{w})$ exists or not, the second and the third terms on the right-hand side of (9) must be zero when $w = \bar{w}$, leading to $V(\bar{w}) = \frac{z}{r} - \frac{\rho - r}{r}\bar{w}$.

By definition, the payout region is a subset of $(\bar{w}, +\infty)$. Actually, the payout region is $(\bar{w}, +\infty)$. Otherwise, there exists an interval $(\bar{w}' - \Delta, \bar{w}') \subset (\bar{w}, +\infty)$ such that V is strictly increasing on $[\bar{w}' - \Delta, \bar{w}']$ and is constant in a right neighborhood of $\bar{w}'$. It must be the case that $\bar{w}' < \infty$, since $\rho > r$ and deferring payment is infinitely costly as $w \to +\infty$. Then a similar argument regarding the existence of $V'(\bar{w})$ also applies here: no matter whether $V'(\bar{w}')$ exists or not, the second and the third terms on the right-hand side of (9) must be zero when $w = \bar{w}'$, and thus $V(\bar{w}') = \frac{z}{r} - \frac{\rho - r}{r}\bar{w}' < \frac{z}{r} - \frac{\rho - r}{r}\bar{w} = V(\bar{w})$, a contradiction to the non-decreasing property of V. Hence, the above defined $\bar{w}$ is the payout boundary and the payout region is $(\bar{w}, +\infty)$. As an immediate implication, the optimal payment is $dI^* = (w - \bar{w})^+$ and for $w \in [\bar{w}, +\infty)$, $V(w) = V(\bar{w})$.

The above proof has already shown that either $V'(\bar{w}) = 0$, or $V'(\bar{w})$ does not exist and $\rho\bar{w} - \mu(1 - \alpha^*(\bar{w}))\beta_0^*(\bar{w})$, the drift at $w = \bar{w}$, is 0.

The proof for $\bar{w} \leq \lambda/\mu$ is straightforward from the text. ■

6.3.2 Proof of Proposition 3.1

Lemma 6.2 *For any $\bar{w} \in (0, \frac{\lambda}{\rho + \mu\bar{\alpha}}]$, let $\bar{V} \equiv \frac{z}{r} - \frac{\rho - r}{r}\bar{w}$. Then the ODE*

$$rV(w) = \max_{\alpha \in [0, \bar{\alpha}]} z - (\rho - r)w + \rho w V'(w) + (1 - \alpha)\mu[\bar{V} - V(w)] - (\lambda - \mu\alpha w)V'(w) \tag{19}$$

with boundary condition $V(0) = 0$ has a unique solution on $[0, \bar{w}]$.

Proof. For any $w < \frac{\lambda}{\rho + \bar{\alpha}\mu}$, since $\lambda - \mu\alpha w - \rho w > 0$, we can rearrange (19) to obtain

$$V' = \max_{\alpha \in [0, \bar{\alpha}]} \frac{z - (\rho - r)w + (1 - \alpha)\mu[\bar{V} - V] - rV}{\lambda - \mu\alpha w - \rho w} .$$

Let

$$F(w,V)=\max_{\alpha\in[0,\bar{\alpha}]}\frac{z-(\rho-r)w+(1-\alpha)\mu[\bar{V}-V]-rV}{\lambda-\mu\alpha w-\rho w}\ .$$

For any fixed $\epsilon>0$, for any $(w_1,V_1),(w_2,V_2)\in[0,\frac{\lambda}{\rho+\bar{\alpha}\mu}-\epsilon]\times[0,\bar{V}]$, there exists an M such that $|F(w_1,V_1)-F(w_2,V_2)|\leq M|V_1-V_2|$. Then, by the Cauchy-Lipschitz theorem, the initial value problem has a unique solution over $[0,\frac{\lambda}{\rho+\bar{\alpha}\mu}-\epsilon]$. Further, notice that V is increasing and upper bounded, and therefore V does not explode as $w\to\bar{w}$. Then the maximum interval of existence reaches the boundary $\bar{w}$ for all $\bar{w}\leq\frac{\lambda}{\rho+\bar{\alpha}\mu}$. When $\bar{w}=\frac{\lambda}{\rho+\bar{\alpha}\mu}$, taking $\epsilon\to 0$, we can extend the solution over $\left[0,\frac{\lambda}{\rho+\bar{\alpha}\mu}\right]$. ■

Proposition 6.2 *Consider two ODEs*

$$rV_1=max_{\alpha\in[0,\bar{\alpha}]}z-(\rho-r)w+\rho wV_1^{'}+(1-\alpha)\mu[\bar{V}_1-V_1]-(\lambda-\mu\alpha w)V_1^{'}$$

and

$$rV_2=max_{\alpha\in[0,\bar{\alpha}]}z-(\rho-r)w+\rho wV_2^{'}+(1-\alpha)\mu[\bar{V}_2-V_2]-(\lambda-\mu\alpha w)V_2^{'}\ ,$$

where $\bar{V}_1=\frac{z}{r}-\frac{\rho-r}{r}\bar{w}_1$, $\bar{V}_2=\frac{z}{r}-\frac{\rho-r}{r}\bar{w}_2$, $\bar{w}_1<\bar{w}_2\leq\frac{\lambda}{\rho+\bar{\alpha}\mu}$; *and* $V_1(0)=V_2(0)=0$. *Then* $V_1>V_2$ *for* $w\in(0,\bar{w}_1)$.

Proof. Suppose the opposite holds. Note that $V_1^{'}(0)>V_2^{'}(0)$. Then, there exists a $w\in(0,\bar{w}_1)$ such that $V_1(w)=V_2(w)$. Define $\tilde{w}=\inf\{w\in(0,\bar{w}_1):V_1(w)=V_2(w)\}$. By the continuity of V_1 and V_2, we have $V_1(\tilde{w})=V_2(\tilde{w})$. Let α_2 be the α that solves the maximization problem for V_2 at $\tilde{w}$. Taking the difference between the two ODEs at $w=\tilde{w}$, we obtain

$$(\rho\tilde{w}+\mu\alpha_2\tilde{w}-\lambda)\cdot(V_1-V_2)^{'}+(1-\alpha_2)\mu(\bar{V}_1-\bar{V}_2)\leq 0\ .$$

Since $\alpha_2<1$ and $\bar{V}_1-\bar{V}_2>0$,

$$(\rho\tilde{w}+\mu\alpha_2\tilde{w}-\lambda)\cdot(V_1-V_2)^{'}<0\ .$$

Since $\bar{w}_1 < \frac{\lambda}{\rho+\bar{\alpha}\mu}$, $\rho\tilde{w} + \mu\alpha_2\tilde{w} - \lambda < 0$. Thus, $V_1^{'}(\tilde{w}) - V_2^{'}(\tilde{w}) > 0$. Note that this inequality holds whenever $V_1 = V_2$. Since $V_1(w) - V_2(w)$ is continuous and the inequality is strict, it also holds for w close to $\tilde{w}$; i.e., $V_1^{'}(w) - V_2^{'}(w) > 0$ in $(\tilde{w} - \delta, \tilde{w})$ for some $\delta > 0$. By the definition of $\tilde{w}$, $V_1(w) - V_2(w) > 0$ for $w \in (\tilde{w} - \delta, \tilde{w})$. Then, it is impossible to have $V_1(\tilde{w}) = V_2(\tilde{w})$, a contradiction. ■

According to the above results, the candidate for the optimal payout boundary is the smallest $\bar{w} \in (0, \frac{\lambda}{\rho+\bar{\alpha}\mu}]$ such that the solution of ODE (14) satisfies $V(\bar{w}) = \frac{z}{r} - \frac{\rho-r}{r}\bar{w}$. The existence of such $\bar{w}$ is guaranteed by the continuity of V. Now we are ready to prove Proposition 3.1, assuming Property 1 holds (verified in Section 6.5).

Proof. Let τ denote the first time that w_t hits zero. We first verify that the principal's value function can be induced by the proposed control processes in Property 6 and the proposed payment process $dI_t = (\beta_0 + w - \bar{w})^+ dY_t^0$. Note that by Property 4, $\beta_0 + w > \bar{w}$, so that $dI_t = (\beta_0 + w - \bar{w})dY_t^0$. By Ito's Formula for jump processes,

$$\begin{aligned} e^{-r(t\wedge\tau)}B(w_{t\wedge\tau}) =& B(w_0) + \int_0^{t\wedge\tau} e^{-rs}[(\rho w_s - \beta_{0,s}\mu(1-\alpha_s))B^{'}(w_s) - rB(w_s)]ds \\ &+ \int_0^{t\wedge\tau} e^{-rs}[B(\bar{w}) - B(w_s)]dY_s^0 \ . \end{aligned}$$

Under the optimal control processes, the HJB equation becomes

$$rB(w) = z + (\rho w - \beta_0\mu(1-\alpha))B^{'}(w) + (1-\alpha)\mu[B(\bar{w}) - B(w) - (w + \beta_0 - \bar{w})] \ .$$

Thus,

$$\begin{aligned} B(w_0) =& \int_0^{t\wedge\tau} e^{-rs}[z + (1-\alpha_s)\mu(B(\bar{w}) - B(w_s) - (w_s + \beta_{0,s} - \bar{w}))]ds \\ &- \int_0^{t\wedge\tau} e^{-rs}[B(\bar{w}) - B(w_s)]dY_s^0 - e^{-r(t\wedge\tau)}B(w_{t\wedge\tau}) \ . \end{aligned}$$

Due to the fact that $Y_s^0 - (1-\alpha_s)\mu s$ is a martingale and $w_\tau = 0$, letting $t \to \infty$ and taking expectation on the right hand side of the above equation, we obtain

$$B(w_0) = \mathbb{E}(\int_0^\tau e^{-rs}[zds - (w_s + \beta_{0,s} - \bar{w})dY_s^0]) \ ,$$

which verifies that the principal's expected payoff given by (1) is indeed achieved with the proposed control and payment processes.

We then verify that the proposed contract is optimal. Since the cumulative payment process is increasing in time, without loss of generality, we write a general payment process as

$$I_t = I_t^c + I_t^d \ ,$$

where I_t^c is a continuous increasing process and I_t^d includes discrete upward jumps. By Ito's Formula for jump processes,

$$\begin{aligned} e^{-r(t\wedge\tau)}B(w_{t\wedge\tau}) =& B(w_0) + \int_0^{t\wedge\tau} e^{-rs}[(\rho w_s - \beta_{0,s}\mu(1-\alpha_s))B^{'}(w_s) - rB(w_s)]ds \\ & - \int_0^{t\wedge\tau} e^{-rs}B^{'}(w_s)dI_s^c + \int_0^{t\wedge\tau} e^{-rs}[B(w_s + \beta_{0,s}) - B(w_s)]dY_s^0 \\ & + \sum_{s\in[0,t\wedge\tau]} e^{-rs}[B(w_s + \beta_{0,s}\Delta Y_s^0 - \Delta I_s^d) - B(w_s + \beta_{0,s}\Delta Y_s^0)] \ , \end{aligned}$$

where $\Delta Y_s^0 \equiv Y_s^0 - Y_{s^-}^0$. We then rearrange the terms to get

$$\begin{aligned} B(w_0) =& e^{-r(t\wedge\tau)}B(w_{t\wedge\tau}) \\ & + \int_0^{t\wedge\tau} e^{-rs}\{rB(w_s) - (\rho w_s - \beta_{0,s}\mu(1-\alpha_s))B^{'}(w_s) - (1-\alpha_s)\mu[B(w+\beta_{0,s}) - B(w)]\}ds \\ & + \int_0^{t\wedge\tau} B^{'}(w_s)e^{-rs}dI_s^c + \int_0^{t\wedge\tau} [B(w+\beta_{0,s}) - B(w)][(1-\alpha_s)\mu ds - dY_s^0] \\ & - \sum_{s\in[0,t\wedge\tau]} e^{-rs}[B(w_s + \beta_{0,s}\Delta Y_s^0 - \Delta I_t^d) - B(w_s + \beta_{0,s}\Delta Y_s^0)] \ . \end{aligned}$$

Taking expectation on both sides and using the fact that $Y_t^0 - \int_0^s (1-\alpha_s)\mu ds$ is a martingale, we obtain

$$B(w_0) = \mathbb{E}(e^{-r(t\wedge\tau)}B(w_{t\wedge\tau})) + \mathbb{E}\left(\begin{array}{c} \int_0^{t\wedge\tau} e^{-rs}\{rB(w_s) - (\rho w_s - \beta_{0,s}\mu(1-\alpha_s))B'(w_s) \\ -(1-\alpha_s)\mu[B(w+\beta_{0,s}) - B(w)]\}ds \end{array}\right)$$
$$+ \mathbb{E}(\int_0^{t\wedge\tau} B'(w_s)e^{-rs}dI_s^c) - \mathbb{E}(\sum_{s\in[0,t\wedge\tau]} e^{-rs}[B(w_s + \beta_{0,s}\Delta Y_s^0 - \Delta I_t^d) - B(w_s + \beta_{0,s}\Delta Y_s^0)]) \ .$$

Notice that

$$rB(w) \geq z + (\rho w - \beta_0\mu(1-\alpha))B'(w) + (1-\alpha)\mu[B(\bar{w}) - B(w) - (w + \beta_0 - \bar{w})]$$

and for any incentive compatible contract,

$$B(w + \beta_{0,s}) = B(\bar{w}) - (w + \beta_{0,s} - \bar{w}) \ .$$

Moreover, since $B'(w) \geq -1$,

$$B(w_0) \geq \mathbb{E}(e^{-r(t\wedge\tau)}B(w_{t\wedge\tau})) + \mathbb{E}(\int_0^{t\wedge\tau} ze^{-rs}ds - \int_0^{t\wedge\tau} e^{-rs}dI_s^c) - \mathbb{E}(\sum_{s\in[0,t\wedge\tau]} e^{-rs}\Delta I_t^d) \ .$$

Letting $t \to \infty$ and using the fact that $B(w)$ is bounded, we obtain

$$B(w_0) \geq \mathbb{E}(\int_0^{\tau} e^{-rs}(zds - dI_s)) \ .$$

Therefore, any function satisfying all these conjectured properties is indeed the value function for the principal. ■

6.4 Proofs in Section 4

6.4.1 Proof of Proposition 4.1

Proof. By Lemma 6.1, $\bar{w} \leq \frac{\lambda}{\rho+\mu\bar{\alpha}}$. If $w = \bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$, (18) is exactly (10), so $\alpha^*(w) = \bar{\alpha}$.

For $w \in \left(0, \frac{\lambda}{\rho+\mu\bar{\alpha}}\right)$, (14) is equivalent to

$$V'(w) = \max_{\alpha\in[0,\bar{\alpha}]} \frac{z-(\rho-r)w+(1-\alpha)\mu[V(\bar{w})-V(w)]-rV(w)}{\lambda-\rho w-\mu\alpha w}. \tag{20}$$

Let $G(\alpha;w) = \frac{z-(\rho-r)w+(1-\alpha)\mu[V(\bar{w})-V(w)]-rV(w)}{\lambda-\rho w-\mu\alpha w}$, which is obviously continuous in both α and w. Property 6 establishes that the maximizer of the right-hand side (RHS) of (20) must be 0 or $\bar{\alpha}$. So to figure out $\alpha^*(w)$, it suffices to compare $G(0;w)$ with $G(\bar{\alpha};w)$, taking as given $V(0)=0$ and $V(\bar{w})$.

For $w \in \left(0, \frac{\lambda}{\rho+\mu\bar{\alpha}}\right)$, $G(\bar{\alpha};w) \geq G(0;w)$ is equivalent to

$$w\left[z-(\rho-r)w-rV(w)\right] \geq \left[\lambda-(\mu+\rho)w\right](V(\bar{w})-V(w)). \tag{21}$$

Notice that when $w \to 0$, the left-hand side of (21) goes to zero while its right-hand side is positive. Thus, there exists $\hat{w}_0 > 0$ such that for any $w \in (0,\hat{w}_0)$, $\alpha^*(w) = 0$. $\beta_0^*(w)$ for $w \in (0,\hat{w}_0)$ results from (16).

Now we establish the optimality of $\alpha(w) = \bar{\alpha}$ for w in the vicinity of $\bar{w}$. Note that by (10), (21) is equivalent to

$$w(\rho-r)(\bar{w}-w) \geq \left[\lambda-(\mu+\rho+r)w\right](V(\bar{w})-V(w)), \tag{22}$$

which holds for all $w \geq \frac{\lambda}{\rho+\mu+r}$. So if $\bar{w} \in (\frac{\lambda}{\rho+\mu+r}, \frac{\lambda}{\rho+\mu\bar{\alpha}}]$, $\alpha^*(w) = \bar{\alpha}$ for all $w \in (\frac{\lambda}{\rho+\mu+r}, \bar{w}]$.

Next consider the case where $\bar{w} \leq \frac{\lambda}{\rho+\mu+r}$. Note that (22) is equivalent to $\frac{\bar{w}-w}{V(\bar{w})-V(w)} \geq \frac{\lambda-(\mu+\rho+r)w}{(\rho-r)w}$. If $\bar{w} \leq \frac{\lambda}{\rho+\mu+r} < \frac{\lambda}{\rho+\mu\bar{\alpha}}$, by Lemma 6.1 (whose proof does not require Proposition 4.1), $\bar{w}$ is reflective so that $V'(\bar{w}) = 0$. Then by L'Hôpital's rule, $\lim_{w\to\bar{w}^-} \frac{\bar{w}-w}{V(\bar{w})-V(w)} = \lim_{w\to\bar{w}^-} \frac{1}{V'(w)} = +\infty$, while $\lim_{w\to\bar{w}^-} \frac{\lambda-(\mu+\rho+r)w}{w(\rho-r)} = \frac{\lambda-(\mu+\rho+r)\bar{w}}{(\rho-r)\bar{w}} < +\infty$. Hence, there also exists a $\hat{w}_{\bar{\alpha}} < \bar{w}$, such that $\alpha(w) = \bar{\alpha}$ for all $w \in (\hat{w}_{\bar{\alpha}}, \bar{w}]$. $\beta_0^*(w)$ for $w \in (\hat{w}_{\bar{\alpha}}, \bar{w}]$ again results from (16).

We hereby establish that $\hat{w}_0 = \hat{w}_{\bar{\alpha}} = \hat{w}$ when $\bar{w} < \frac{\lambda}{\rho+\mu+r}$. $\hat{w}_0 = \hat{w}_{\bar{\alpha}} = \hat{w}$ when $\bar{w} \geq \frac{\lambda}{\rho+\mu+r}$ is delegated to Proposition 6.3. From (22), $\alpha = \bar{\alpha}$ when $V(w) + F(w) \geq \bar{V}$, where $F(w) = \frac{w(\rho-r)(\bar{w}-w)}{\lambda-(\mu+\rho+r)w}$. When $\bar{w} < \frac{\lambda}{\rho+\mu+r}$, $F(w)$ and its derivatives are well defined on $w \in [0, \bar{w}]$. Since V is concave, if $F'' < 0$, then $V + F$ is concave and thus $\{w : V(w) + F(w) \geq \bar{V}\}$ is connected and $\hat{w}_0 = \hat{w}_{\bar{\alpha}} = \hat{w}$. Tedious algebra yields $F''(w) = \frac{-2\lambda(\rho - r)}{[\lambda - (\mu+\rho+r)w]^3}[\lambda - (\mu + \rho + r)\bar{w}]$, which is negative if $\bar{w} < \frac{\lambda}{\rho+\mu+r}$. ■

Here we provide the closed-form solutions to (17) and 18). As a first-order linear ODE, (17) has general solutions

$$V(w) = \frac{\rho - r}{r + \mu - \rho}(\frac{\lambda}{\rho} - w) + \frac{\mu V(\bar{w}) + z - (\rho - r)\frac{\lambda}{\rho}}{r + \mu} + K(\frac{\lambda}{\rho} - w)^{\frac{r+\mu}{\rho}}, \quad (23)$$

which are all strictly concave in $(0, \bar{w})$. From $V(0) = 0$, we can pin down for $w \in (0, \hat{w}_0)$ that $K = -\frac{\rho(\rho-r)}{(r+\mu)(r+\mu-\rho)} \cdot \frac{\lambda}{\rho}^{-\frac{r+\mu-\rho}{\rho}} - \frac{\mu V(\bar{w})+z}{r+\mu} \cdot \frac{\lambda}{\rho}^{-\frac{r+\mu}{\rho}}$.

Also as a first-order linear ODE, (18) has general solutions

$$\begin{aligned} V(w) \;=\; & \frac{\rho - r}{r + (1-\bar{\alpha})\mu - (\rho + \bar{\alpha}\mu)}(\frac{\lambda}{\rho + \mu\bar{\alpha}} - w) + \frac{(1-\bar{\alpha})\mu V(\bar{w}) + z - (\rho - r)\frac{\lambda}{\rho+\bar{\alpha}\mu}}{r + \mu(1-\bar{\alpha})} \\ & + K(\frac{\lambda}{\rho + \mu\bar{\alpha}} - w)^{\frac{r+(1-\bar{\alpha})\mu}{\rho+\mu\bar{\alpha}}} \end{aligned} \quad (24)$$

if $r + (1 - \bar{\alpha})\mu \neq \rho + \bar{\alpha}\mu$, and

$$\begin{aligned} V(w) \;=\; & -\frac{\rho - r}{\rho + \mu\bar{\alpha}}(\frac{\lambda}{\rho + \mu\bar{\alpha}} - w)\ln(\frac{\lambda}{\rho + \mu\bar{\alpha}} - w) + \frac{(1-\bar{\alpha})\mu V(\bar{w}) + z - (\rho - r)\frac{\lambda}{\rho+\bar{\alpha}\mu}}{r + \mu(1-\bar{\alpha})} \\ & + K(\frac{\lambda}{\rho + \mu\bar{\alpha}} - w) \end{aligned} \quad (25)$$

if $r+(1-\bar{\alpha})\mu = \rho+\bar{\alpha}\mu$. The assumption that $\bar{\alpha} > 1/2$ implies that $r+(1-\bar{\alpha})\mu < \rho + \bar{\alpha}\mu$, and it is shown later in the proof of Proposition 4.2 that the solutions that are increasing in $(0, \bar{w})$ are strictly convex in $(0, \bar{w})$ if $K < 0$, linear if $K = 0$, and strictly concave in $(0, \bar{w})$ otherwise.

With the closed-form solutions and their concavity properties discussed above, we show the following proposition:

Proposition 6.3 *If $\bar{w} \geq \frac{\lambda}{\rho+\mu+r}$, then $\hat{w}_0 = \hat{w}_{\bar{\alpha}}$.*

To prove Proposition 6.3, we first prove Lemma 6.3, which articulates that the optimal α takes values in $\{0, \overline{\alpha}\}$ almost surely.

Lemma 6.3 *There does not exist an interval* (w_1, w_2) *such that* $w \cdot V'(w) = V(\bar{w}) - V(w)$ *for all* $w \in (w_1, w_2)$*.*

Proof. Suppose the contrary. Then $w \cdot V'(w) = V(\bar{w}) - V(w)$ implies

$$V(w) = \frac{c}{w} + V(\bar{w}) \tag{26}$$

in (w_1, w_2) for some constant c. Plugging $w \cdot V'(w) = V(\bar{w}) - V(w)$ into the HJB equation (9) we obtain

$$V(w) = \frac{z - (\rho - r) w + (\rho + \mu - \lambda/w) V(\bar{w})}{r + \rho + \mu - \lambda/w}. \tag{27}$$

It is straightforward to verify that (26) and (27) cannot both be satisfied in any interval. ■

Lemma 6.4 shows that the convexity of V in an interval below the payout boundary $\bar{w}$ is "contagion" up to $\bar{w}$.

Lemma 6.4 *If there exists an interval* $[w_1, w_2) \subset (0, \bar{w})$ *such that* $w_1 \cdot V'(w_1) \geq V(\bar{w}) - V(w_1)$ *and* V *is convex in* (w_1, w_2)*, then* $\alpha^*(w) = \bar{\alpha}$ *for all* $w \in (w_1, \bar{w}]$ *and* V *is convex in* $[w_1, \bar{w}]$*.*

Proof. If V is convex in (w_1, w_2), since V is continuously differentiable in $(0, \bar{w})$, $w \cdot V'(w) + V(w)$ is strictly increasing in $[w_1, w_2)$. Given that $w_1 \cdot V'(w_1) \geq V(\bar{w}) - V(w_1)$, we have $w \cdot V'(w) > V(\bar{w}) - V(w)$ for all $w \in (w_1, w_2)$. So there exists $w_3 \in (w_2, \bar{w})$ such that $w \cdot V'(w) > V(\bar{w}) - V(w)$ for all $w \in (w_1, w_3)$. Iteration of this argument yields $w \cdot V'(w) > V(\bar{w}) - V(w)$ and thus $\alpha^*(w) = \bar{\alpha}$ for all $w \in (w_1, \bar{w})$. By Proposition 4.1, $\alpha^*(\bar{w}) = \bar{\alpha}$ as well.

Given that $\alpha^*(w) = \bar{\alpha}$ for all $w \in (w_1, \bar{w}]$, the specific solution to (18) that matches the value function V in $[w_1, w_2)$ must also match V in $[w_1, \bar{w}]$. Since V is convex in $[w_1, w_2)$, that specific solution must be given by (24) with

$K \leq 0$ and $r + (1 - \bar{\alpha})\mu < \rho + \bar{\alpha}\mu$. This proves the convexity of V in $[w_1, \bar{w}]$. ■

With Lemmas 6.3 and 6.4, we can now prove Proposition 6.3.

Proof. Let $\hat{W} \equiv \left\{ w \in (0, \bar{w}) : w \cdot V'(w) = V(\bar{w}) - V(w) \right\}$. We are to show that $\hat{W}$ is a singleton if $\bar{w} \geq \frac{\lambda}{\rho+\mu+r}$. By Proposition 4.1, $\hat{W}$ is non-empty and has a maximum. Without loss of generality, assume $\hat{w}_{\bar{\alpha}} = \max \hat{W}$. Then V must be strictly concave in $(0, \hat{w}_{\bar{\alpha}}]$. To see this, Lemma 6.3 and the properties of the general solutions to (17) and (18) imply that V must be piecewise concave or convex in $(0, \hat{w}_{\bar{\alpha}}]$. If there is an interval $(w_1, w_2) \subset (0, \hat{w}_{\bar{\alpha}}]$ such that V is convex in it, then by Lemma 6.4, $\alpha^*(w) = \bar{\alpha}$ for all $w \in (w_1, \bar{w}]$, contradicting the fact that $\hat{w}_{\bar{\alpha}} = \max \hat{W}$.

Note that (27) holds for $w = \hat{w}_{\bar{\alpha}}$. Plug it into $V'(\hat{w}_{\bar{\alpha}}) = \frac{V(\bar{w}) - V(\hat{w}_{\bar{\alpha}})}{\hat{w}_{\bar{\alpha}}}$, we have $V'(\hat{w}_{\bar{\alpha}}) = \frac{\rho-r}{r+\rho+\mu}(1 - \frac{\frac{\lambda}{r+\mu+\rho} - \bar{w}}{\frac{\lambda}{r+\mu+\rho} - \hat{w}_{\bar{\alpha}}})$. Similarly, if there exists $\hat{w}' \in \hat{W}$ such that $\hat{w}' < \hat{w}_{\bar{\alpha}}$, then $V'(\hat{w}') = \frac{\rho-r}{r+\rho+\mu}(1 - \frac{\frac{\lambda}{r+\mu+\rho} - \bar{w}}{\frac{\lambda}{r+\mu+\rho} - \hat{w}'})$. If $\bar{w} \geq \frac{\lambda}{\rho+\mu+r}$, then we have $V'(\hat{w}') \leq V'(\hat{w}_{\bar{\alpha}})$, contradicting the concavity of V in $(0, \hat{w}_{\bar{\alpha}}]$. ■

6.4.2 Proof of Proposition 4.2

Proof. Statement 1 is already shown as Lemma 6.1 before. We now prove statement 2. By Proposition 4.1, $\alpha^*(w) = \bar{\alpha}$ if $w \in (\hat{w}, \bar{w}]$, so here we focus on the solutions to (18) when studying the property of the payout boundary $\bar{w}$. Let V_K be the solution with constant K in (24) or (25). We first show that $\bar{w}$ is absorbing (i.e., $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$) if and only if $K \leq 0$; i.e., if and only if V is (weakly) convex in $(\hat{w}, \bar{w})$.

$\bar{\alpha} > 1/2$ implies that $r + (1 - \bar{\alpha})\mu < \rho + \bar{\alpha}\mu$. Since $-\frac{\rho-r}{r+(1-\bar{\alpha})\mu-(\rho+\bar{\alpha}\mu)} > 0$, K in (24) can be either positive or negative. (**??**) yields

$$V_K'' = K\frac{r + (1 - \bar{\alpha})\mu}{\rho + \mu\bar{\alpha}}(\frac{r + (1 - \bar{\alpha})\mu}{\rho + \mu\bar{\alpha}} - 1)(\frac{\lambda}{\rho + \mu\bar{\alpha}} - w)^{\frac{r+(1-\bar{\alpha})\mu}{\rho+\mu\bar{\alpha}} - 2} .$$

If $K > 0$, since $\frac{r+(1-\bar{\alpha})\mu}{\rho+\mu\bar{\alpha}} - 1 < 0$, $V_K'' < 0$ so that V_K is concave. Moreover, as $w \to \frac{\lambda}{\rho+\mu\bar{\alpha}}$, $V_{\bar{\alpha}}' \to -\infty$. Again, it must be that $\bar{w} < \frac{\lambda}{\rho+\mu\bar{\alpha}}$, and $\bar{w}$ is reflective.

If $K = 0$, then $V_K'(w) = -\frac{\rho-r}{r+(1-\bar{\alpha})\mu-(\rho+\bar{\alpha}\mu)} > 0$ for all $w \in (\hat{w}_{\bar{\alpha}}, \bar{w}]$. Thus we must have $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$ as an absorbing state.

If $K < 0$, $V_K^{''} > 0$ so that V_K is strictly convex. Thus, the value function V satisfies $V^{'} > 0$ for all $w < \frac{\lambda}{\rho+\mu\bar{\alpha}}$. This implies that $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$, and $\bar{w}$ is absorbing by Lemma 6.1.

Now we prove that $\bar{w}$ is absorbing if and only if $z/\lambda \geq \theta(r, \rho, \mu, \bar{\alpha})$, where $\theta(r, \rho, \mu, \bar{\alpha})$ is defined by (31). From (23), we have

$$V^{'}(\hat{w}) = \frac{\rho - r}{r+\mu-\rho}(1-\frac{\rho}{\lambda}\hat{w})^{\frac{r+\mu}{\rho}-1} + \frac{\mu V(\bar{w})+z}{\lambda}(1-\frac{\rho}{\lambda}\hat{w})^{\frac{r+\mu}{\rho}-1} - \frac{\rho-r}{r+\mu-\rho}. \tag{28}$$

On the other hand, by Property 6, we have $V'(\hat{w}) = \frac{V(\bar{w})-V}{\hat{w}}$. Plugging this into (17), we have

$$V^{'}(\hat{w}) = \frac{\rho - r}{r+\rho+\mu}(1 - \frac{\frac{\lambda}{r+\mu+\rho} - \bar{w}}{\frac{\lambda}{r+\mu+\rho} - \hat{w}}). \tag{29}$$

As $\hat{w}$ increases from 0 to $\frac{\lambda}{r+\mu+\rho}$, the right-hand side of (28) is decreasing from $\frac{\mu V(\bar{w})+z}{\lambda}$, and that of (29) is increasing from $\frac{z-V(\bar{w})}{\lambda}$ to $+\infty$. Thus, there exists a unique $\hat{w} \in (0, \frac{\lambda}{r+\mu+\rho})$ such that both equations hold simultaneously.

Next, we show that V_K is convex if and only if $\hat{w} \geq \frac{\lambda}{2(\rho+\mu\bar{\alpha})}$. Observe that $V(\hat{w})$ should also satisfy (24), and thus

$$V^{'}(\hat{w}) = -\frac{\rho - r}{r+(1-\bar{\alpha})\mu-(\rho+\bar{\alpha}\mu)} - K\frac{r+(1-\bar{\alpha})\mu}{\rho+\mu\bar{\alpha}}(\frac{\lambda}{\rho+\mu\bar{\alpha}} - \hat{w})^{\frac{r+(1-\bar{\alpha})\mu}{\rho+\mu\bar{\alpha}}-1}. \tag{30}$$

We have shown that V_K is convex if and only if $K \leq 0$. From (29) and (30),

$$K \leq 0 \Leftrightarrow \frac{\rho - r}{r+\rho+\mu}(1 - \frac{\frac{\lambda}{r+\mu+\rho} - \bar{w}}{\frac{\lambda}{r+\mu+\rho} - \hat{w}}) \geq -\frac{\rho-r}{r+(1-\bar{\alpha})\mu-(\rho+\bar{\alpha}\mu)},$$

where $\bar{w} = \frac{\lambda}{\rho+\bar{\alpha}\mu}$. This reduces to $\hat{w} \geq \frac{\lambda}{2(\rho+\mu\bar{\alpha})}$. Notice that since $r+(1-\bar{\alpha})\mu < \rho+\bar{\alpha}\mu$, $\frac{\lambda}{2(\rho+\mu\bar{\alpha})} < \frac{\lambda}{r+\mu+\rho}$.

Therefore, V_K is convex if and only if the right-hand sides of (28) and (29) intersect at some $\hat{w} \in [\frac{\lambda}{2(\rho+\mu\bar{\alpha})}, \frac{\lambda}{r+\mu+\rho})$. The second condition holds if and only if

$$\left(\frac{\rho-r}{r+\mu-\rho}+\frac{\mu\bar{V}+z}{\lambda}\right)(1-\frac{\rho}{\lambda}\cdot\frac{\lambda}{2(\rho+\mu\bar{\alpha})})^{\frac{r+\mu}{\rho}-1}-\frac{\rho-r}{r+\mu-\rho}\geq\frac{\rho-r}{r+\rho+\mu}(1-\frac{\frac{\lambda}{r+\mu+\rho}-\frac{\lambda}{\rho+\mu\bar{\alpha}}}{\frac{\lambda}{r+\mu+\rho}-\frac{\lambda}{2(\rho+\mu\bar{\alpha})}})\ ,$$

which is equivalent to

$$\begin{aligned}\frac{z}{\lambda} &\geq \frac{r(\rho-r)}{\mu+r}\left\{\frac{2(\rho+\mu\bar{\alpha})}{\rho+2\mu\bar{\alpha}}\frac{2\mu\bar{\alpha}}{(r+\mu-\rho)[(\rho+\mu\bar{\alpha})-(r+\mu(1-\bar{\alpha}))]}-\frac{1}{r+\mu-\rho}+\frac{\mu}{\rho+\mu\bar{\alpha}}\right\}\\ &\equiv \theta\left(r,\rho,\mu,\bar{\alpha}\right)\ . \end{aligned} \tag{31}$$

Statements 3 and 4 are straightforward from (31). ■

6.5 Proof of Property 1

First, if $z>\lambda>0$, it is suboptimal for the principal to implement $a>0$ in the payout region, $[\bar{w},+\infty)$. To see this, consider any contract that implements $a_t>0$ for some $w_t>\bar{w}$. This implies that in $[t,t+dt]$, the agent receives a private benefit of $\lambda a_t dt$. Instead, the principal could implement $a_t=0$ (which generates additional synergy $z\cdot a_t dt$), and increases the payment to the agent by $\lambda a_t dt$ in $[t,t+dt]$, without altering the contract afterwards. This raises the principal's payoff by $(z-\lambda)\,a_t dt>0$, while leaving the dynamics of the agent continuation value unchanged. Iteration of this argument rules out profitable deviation from $a=0$ in the payout region. Thus, we only need to consider the possibility of shirking in the *no-payment region* henceforth. We will rule out the profitability of deviation of $a=1$ and $a\in(0,1)$, respectively.

If the principal implements $a=1$ for some $w_t\in(0,\bar{w})$, then the agent's incentive compatibility constraint is $\mu\alpha\beta_1+\mu(1-\alpha)\beta_0\leq\lambda$, and his continuation value follows

$$dw_t=\rho w_t dt-\lambda dt-\beta_{1,t}[dY_{1,t}-\mu\alpha_t dt]\ .$$

To rule out the profitability of such deviation, we need to show that for any $w \in (0, \bar{w})$,

$$rV(w) \geq \max_{\alpha, \beta_1} \lambda + [\rho w - \lambda + \alpha\mu\beta_1] V^{'}(w) + \alpha\mu[V(w - \beta_1) - V(w)] - (\rho - r)w \; . \quad (32)$$

If $\beta_1 \geq 0$, then

$$\begin{aligned} \text{RHS of (32)} \;\; & \leq \;\; \max_{\alpha} z + (\rho w - \lambda + \alpha\mu w) V^{'}(w) - (\rho - r)w \\ & < \;\; \max_{\alpha} z + (\rho w - \lambda + \alpha\mu w) V^{'}(w) + \mu (1 - \alpha) [V(\bar{w}) - V(w)] - (\rho - r)w \\ & \leq \;\; rV(w) \;\; , \end{aligned}$$

where the first two inequalities follow the fact that V is increasing in w, and the third inequality holds because its LHS is the flow value achieved with $\beta_1 = w$ when implementing $a = 0$ and thus the LHS is dominated by the optimal flow value rV. Hence, it suffices to show that (32) holds for $\beta_1 < 0$.

Denote the objective function of the RHS of (32) by D. If $\beta_1 \leq 0$, we show that D can achieve its maximum only when $\beta_1 = 0$ or $\beta_1 = -(\bar{w} - w)$. In particular, for $\beta_1 < -(\bar{w} - w)$, we have $V(w - \beta_1) = V(\bar{w})$ and $\partial D / \partial \beta_1 = \alpha\mu V^{'}(w) > 0$. Thus D can achieve its maximum only when $\beta_1 \in [-(\bar{w} - w), 0]$. For $\beta_1 > -(\bar{w} - w)$, $\partial D / \partial \beta_1 = \alpha\mu \left[V^{'}(w) - V'(w - \beta_1)\right]$. If $V^{'}(w) \geq V'(w - \beta_1)$ for all $\beta_1 \in [-(\bar{w} - w), 0]$, then $\partial D / \partial \beta_1 > 0$ and D is maximized with $\beta_1 = 0$. If $V^{'}(w) < V'(w - \beta_1)$ for some $\beta_1 \in [-(\bar{w} - w), 0]$, then V is not globally concave. By Proposition 4.2, V is concave in $(0, \hat{w})$ and convex in $[\hat{w}, \bar{w}]$. This implies the existence of $\hat{\beta}$ such that $w - \hat{\beta} \in [\hat{w}, \bar{w}]$ and that $\partial D / \partial \beta_1 = \alpha\mu \left[V^{'}(w) - V'(w - \beta_1)\right] < 0$ if and only if $\beta_1 < \hat{\beta}$. Therefore, if D is maximized with $\beta_1 \in [-(\bar{w} - w), \hat{\beta})$, then the maximizer is $\beta_1 = -(\bar{w} - w)$. If D is instead maximized with $\beta_1 \in \left[\hat{\beta}, 0\right]$, then the maximizer is $\beta_1 = 0$. Since we have already shown that (32) holds for $\beta_1 \geq 0$, we only needs to show that it holds for $\beta_1 = -(\bar{w} - w)$.

Note also that D is linear in α, so D is maximized with either $\alpha = 0$ or $\alpha = \bar{\alpha}$.[27] If D is maximized with $\beta_1 = -(\bar{w} - w)$ and $\alpha = 0$, then the RHS of

[27] If D is maximized at some interior value of α, the coefficient of α must be zero and thus it is equivalent to evaluating D at $\alpha = 0$.

(32) equals

$$\lambda + (\rho w - \lambda)V'(w) - (\rho - r)w \ ;$$

If D is maximized with $\beta_1 = -(\bar{w} - w)$ and $\alpha = \bar{\alpha}$, then the RHS of (32) equals

$$\lambda + (\rho w - \lambda)V'(w) + \bar{\alpha}\mu[V(\bar{w}) - V(w) - (\bar{w} - w)V'(w)] - (\rho - r)w \, .$$

For both cases, we have

$$\text{RHS of (32)} < z + (\rho w - \lambda)V'(w) + \mu[V(\bar{w}) - V(w)] - (\rho - r)w \leq rV(w) \, ,$$

where the last inequality results from (14). Therefore, deviation to $a = 1$ is never profitable.

If the principal instead implements $a \in (0, 1)$ for some $w_t \in (0, \bar{w})$, then the agent's continuation value follows

$$dw_t = \rho w_t dt - a\lambda dt + \beta_{0,t} \left[dY_{0,t} - \mu (1 - \alpha_t) (1 - a)dt\right] - \beta_{1,t}[dY_{1,t} - \mu\alpha_t a dt] \ .$$

To guarantee that the agent does not choose either $a = 0$ or $a = 1$, his incentive compatibility constraint is

$$\mu\alpha\beta_1 + \mu(1 - \alpha)\beta_0 = \lambda.$$

To rule out the profitability of such deviation, we need to show that for any $w \in (0, \bar{w})$,

$$\begin{aligned} rV(w) \geq \max_{\alpha,\beta_0,\beta_1} & z(1 - a) + a\lambda + [\rho w - a\lambda - \beta_0\mu(1 - \alpha)(1 - a) + \mu\alpha\beta_1 a]V'(w) \\ & + (1 - \alpha)\mu(1 - a)[V(\bar{w}) - V(w)] + \alpha\mu a[V(w - \beta_1) - V(w)] - (\rho - r)w \ . \end{aligned} \tag{33}$$

From the incentive compatibility condition, the RHS of (33) equals

$$\max_{\alpha,\beta_0,\beta_1} z(1-a) + a\lambda + [\rho w - \beta_0\mu(1-\alpha)]V^{'}(w)$$
$$+ (1-\alpha)\mu(1-a)[V(\bar{w}) - V(w)] + \alpha\mu a[V(w-\beta_1) - V(w)] - (\rho - r)w \ .$$

If $\beta_1 \geq 0$, then (14) implies that (33) holds. If $\beta_1 < 0$, then due to the IC constraint, the RHS of (33) equals

$$\max_{\alpha,\beta_1} z(1-a) + a\lambda + [\rho w - \lambda]V^{'}(w) + (1-\alpha)\mu(1-a)[V(\bar{w}) - V(w)]$$
$$+ \alpha\mu a[V(w-\beta_1) - V(w)] + \mu\alpha\beta_1 V^{'}(w) - (\rho - r)w$$
$$< \max_{\alpha} z + [\rho w - \lambda]V^{'}(w) + [(1-\alpha)(1-a) + a\alpha]\mu[V(\bar{w}) - V(w)] - (\rho - r)w$$
$$< z + [\rho w - \lambda]V^{'}(w) + \mu[V(\bar{w}) - V(w)] - (\rho - r)w \leq rV(w) \, ,$$

where the last inequality again results from (14). Thus, deviation to $a \in (0,1)$ is never profitable either.