Stochastic Sequential Screening*

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Abstract

We study when and how randomization can help improve the seller's revenue in the sequential screening setting. Using a model with discrete ex ante types and a continuum of ex post valuations, we demonstrate why the standard approach based on solving a relaxed problem that keeps only local downward incentive compatibility constraints often fails and show how randomization is needed to realize the full potential of sequential screening. Under a strengthening of first-order stochastic dominance ordering on the valuation distribution functions of ex ante types, we introduce and solve a modified relaxed problem by retaining all local incentive compatibility constraints, provide necessary and sufficient conditions for optimal mechanisms to be stochastic, and characterize optimal stochastic contracts. Our analysis mostly focuses on the case of three ex ante types, but our methodology of solving the modified problem can be extended to any finite number of ex ante types.

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1 Introduction

Random allocations through rationing and lotteries are common for selling event tickets, material inputs, or consumer products (see Gilbert and Klemperer (2000) for a list of examples). For the static environments of monopoly pricing or auctions, the literature of mechanism design (Myerson (1981), Riley and Zeckhauser (1983), and Bulow and Roberts (1989), among others) has established when and how randomization can help alleviate incentive problems. Relatively little is known in dynamic environments, however, because almost all the dynamic mechanism design literature adopts the standard approach which forms a relaxed problem by dropping all local upward incentive compatibility constraints and then imposes strong conditions under which the deterministic solution to the relaxed problem also solves the original problem.

For example, consider the classic formulation of the two-period sequential screening problem by Courty and Li (2000) where a seller of an indivisible good designs a selling mechanism for a buyer who knows which distribution that the valuation of the good is drawn from in period one (his ex ante type) but his valuation is only realized in period two after agreeing to the mechanism. With discrete ex ante types and continuous ex post valuations, the standard approach adapted to the sequential screening problem works as follows. One first replaces the second-period incentive compatibility constraints (truthful reporting of the realized valuation) by their corresponding first-order conditions and forms a relaxed problem by keeping only local downward incentive compatibility constraints in the first period as well as the individual rationality constraint of the lowest ex ante type. One then argues that all these constraints must bind in the solution to the relaxed problem and hence the objective function can be written as the sum of dynamic virtual surpluses of all types. As argued in Riley and Zeckhauser (1983), the problem of maximizing the virtual surplus of each type by choosing among all non-decreasing allocations necessarily has a deterministic solution. Therefore, point-wise maximizers of the objective function of the relaxed problem are type-wise deterministic. The last step of the standard approach is to find (strong) conditions under which the deterministic solution associates with a monotone sequence of cutoffs and hence satisfies the dropped local upward and non-local incentive compatibility constraints.

Therefore, the deterministic solution to the relaxed problem, implementable by a menu of option contracts, also solves the original problem.

The standard approach fails when these cutoffs of the solution to the relaxed problem fail to be monotone, because the point-wise maximizers violate the dropped upward incentive compatibility constraints. The existing literature on dynamic mechanism design is largely silent on how to characterize optimal mechanisms in this case. The goal of this paper is to characterize optimal mechanisms when the standard approach fails and shed light on the role of randomization in alleviating incentive compatibility constraints.

Our analysis focuses mostly on the sequential screening problem with three ex ante types, although it can be generalized to any finite number of types. We need a minimum of three types for stochastic contracts to be optimal because, with binary types ranked by first-order stochastic dominance, the cutoffs associated with the deterministic solution to the relaxed problem is necessarily monotone and hence satisfies all dropped incentive compatibility constraints. Intuitively, with two ex ante types, (deterministic) option contracts – each represented by pair of advance payment and strike price – are sufficient for sequential price discrimination. With three types, however, advance payment and strike price are generally insufficient to realize the full potential of sequential screening. In particular, upward as well as downward incentive compatibility constraints might bind at an optimal mechanism. When this happens, deterministic mechanisms are forced to have the same strike price for two ex ante types. Randomization may be then needed to fine tune sequential screening.

To find when the optimal contract involves randomization and to characterize the optimal stochastic contract, we consider a modified relaxed problem. In our relaxed problem, we impose the same local downward incentive compatibility constraints and individual rationality constraint to arrive at the same objective function of total dynamic virtual surpluses as in the standard approach, but we retain the local upward incentive compatibility constraints. We further simplify the relaxed problem by replacing local upward incentive compatibility constraints by equivalent average monotonicity constraints of allocation rules. With three ex ante types, the simplified problem is to choose non-decreasing allocations of the middle and low types to maximize the sum of the dynamic virtual surpluses of the two types, subject to a weighted average of the middle type's allocation being greater or equal to the weighted average of the low type, which is equivalent to the upward incentive compatibility constraint of the low type.

We first use our simplified problem to uncover a new sufficient condition for optimal mechanism to be deterministic when the standard regularity condition fails. The standard approach would have no say about the optimality of deterministic mechanisms in this case. This sufficient condition relies on comparisons of ratios of dynamic virtual surplus to information rent of each of the two types. Next we show that the failure of this sufficient condition and the regularity condition, together with a strengthening of first order stochastic dominance, implies that optimal mechanism must be stochastic. Third, we adapt the standard ironing techniques, used for example, by Myerson (1981) to characterize optimal auctions when the virtual value function is non-monotone, and by Riley and Zeckhauser (1983) to show that monopoly pricing is optimal mechanism for selling an indivisible good, to characterize optimal stochastic mechanisms. Finally, we use a class of examples with the exponential distributions to illustrate the use of the simplified problem and the characterizations in terms of the surplus-to-rent ratios.

Our analysis of the modified relaxed problem can be extended in a straightforward manner to any number of finite ex ante types. Both our sufficient and necessary conditions for stochastic mechanisms to be optimal and our characterization of optimal stochastic mechanisms have their counterparts with more than three types. We use the same class of examples with the exponential distributions to illustrate this generalization. Unlike the model of three types, more than a single monotonicity constraints can be binding. At this point we can not state the necessary and sufficient conditions in terms of primitives of the model, or provide a complete characterization for optimal stochastic mechanisms. We leave these tasks for future work.

Bergemann, Casto and Weintraub (2020) study a sequential screening model with ex post individual rationality constraints, and provide necessary and sufficient conditions for optimal sequential screening to be stochastic. Our model differs from Bergemann, Casto and Weintraub (2020) because we impose interim rather than ex post individual rationality constraints. In their benchmark model with two ex ante types, every incentive compatibility constraint in Bergemann, Casto and Weintraub (2020) is local. In contrast, even with three ex ante types, our model has both global as well as local incentive compatibility constraints. Correspondingly, we impose a stronger condition than first order stochastic dominance on ex ante types to construct the relaxed problem with only local incentive compatibility constraints. Our surplus-to-rent ratio is inspired by the profit-to-rent ratio defined in Bergemann, Castro and Weintraub (2020), although the form of our ratio arises from the dynamic virtual surplus while theirs is static. A similar surplus-to-rent ratio is also crucial in a sequential delegation setting of Krahmer and Kovac (2016) to determine whether it is optimal to screen the agent's initial information. Their model share similar information structure as our model, but their analysis is quite different because there are no transfers.

2 The Model

A seller has one object for sale to a potential buyer. The seller and the buyer are risk-neutral, and do not discount. The buyer's value $\omega \in \Omega \equiv [\underline{\omega}, \overline{\omega}]$ for the good is initially unknown to both the buyer and the seller. The seller's reservation value is known to be c. We assume that $c \in (\underline{\omega}, \overline{\omega})$.

In period one, the buyer privately observes a signal $\theta \in \Theta$ about ω, which we refer to as his type. We assume that the buyer's type is ternary, $\Theta = \{H, M, L\}$, with probability ϕ_θ for each $\theta = H, M, L$ and $\sum_\theta \phi_\theta = 1$. For each $\theta \in \Theta$, let $F_\theta(\cdot)$ be the conditional distribution function over Ω, and we assume that $F_\theta(\cdot)$ has positive and finite density $f_\theta(\cdot)$. We assume that type H is higher than M, which is in turn higher than L in first order stochastic dominance, that is, $F_H(\omega) \leq F_M(\omega) \leq F_L(\omega)$ for all ω, with strict inequalities for a positive measure of ω. In period two, the buyer observes his value ω. The non-participation payoff of the buyer is normalized to 0 regardless of his ex ante type.

The seller chooses a direct revelation mechanism $(x_\theta(\omega), t_\theta(\omega))$, where $x_\theta(\omega)$ is the allocation rule and $t_\theta(\omega)$ is payment rule for reported type θ in period one and reported value (θ, ω) in period two. The objective function of the seller's optimization problem is

$$\max_{(x_\theta, t_\theta)} \sum_{\theta = H,M,L} \phi_\theta \int_{\underline{\omega}}^{\overline{\omega}} (t_\theta(\omega) - c x_\theta(\omega)) f_\theta(\omega) \, d\omega.$$

There are three sets of constraints.

First, we have the incentive compatibility constraints in period two: for each $\theta = H, M, L$,

$$\omega x_\theta(\omega) - t_\theta(\omega) \geq \omega x_\theta(\omega') - t_\theta(\omega'),$$

for all $\omega, \omega' \in [\underline{\omega}, \overline{\omega}]$. We refer to the above constraint as IC_θ. Regardless of the buyer's true type, the period-two incentive comparability constraint is the same once an ex ante type θ is reported in the first period.

Second, we have the individual rationality constraints in period one: for each $\theta = H, M, L$,

$$\int_{\underline{\omega}}^{\overline{\omega}} \left(\omega x_\theta\left(\omega\right) - t_\theta\left(\omega\right)\right) f_\theta\left(\omega\right) d\omega \geq 0.$$

We refer to the above constraint as IR_θ for each t. Since we do not impose individual rationality constraints in period two, the buyer's ex post payoff may fall below his non-participation payoff of 0.

Third, we have the incentive compatibility constraints in period one: for each $\theta = H, M, L$,

$$\int_{\underline{\omega}}^{\overline{\omega}} \left(\omega x_\theta\left(\omega\right) - t_\theta\left(\omega\right)\right) f_\theta\left(\omega\right) d\omega \geq \int_{\underline{\omega}}^{\overline{\omega}} \left(\omega x_{\theta'}\left(\omega\right) - t_{\theta'}\left(\omega\right)\right) f_\theta\left(\omega\right) d\omega,$$

for all $\theta' \neq \theta = H, M, L$. We refer to the above constraint as $\mathrm{IC}_{\theta\theta'}$. In this setup, we do not need to worry about double deviations of misreporting the ex ante type in period one and then misreporting the realized valuation in period two.

3 A Simplified Problem

We first state without proof a standard result, that allocation monotonicity with respect to valuation together with an envelope condition is both necessary and sufficient for incentive compatibility in period two.

Lemma 1 *For each $\theta = H, M, L$, IC_θ holds if and only if $x_\theta\left(\omega\right)$ is non-decreasing in ω, and*

$$\omega x_\theta\left(\omega\right) - t_\theta\left(\omega\right) = u_\theta\left(\underline{\omega}\right) + \int_{\underline{\omega}}^{\omega} x_\theta\left(s\right) ds \tag{1}$$

for all ω, where $u_\theta\left(\underline{\omega}\right) = \underline{\omega} x_\theta\left(\underline{\omega}\right) - t_\theta\left(\underline{\omega}\right)$.

As an immediate implication of the above result, through integration by parts, we have:

$$\int_{\underline{\omega}}^{\overline{\omega}} \left(\omega x_{\theta}(\omega) - t_{\theta}(\omega)\right) F_{\theta'}(\omega)\, d\omega = u_{\theta}(\underline{\omega}) + \int_{\underline{\omega}}^{\overline{\omega}} x_{\theta}(\omega)\left(1 - F_{\theta'}(\omega)\right) d\omega \qquad (2)$$

for all $\theta, \theta' = H, M, L$. We can now rewrite the seller's objective function as

$$\sum_{\theta=H,M,L} \phi_{\theta} \int_{\underline{\omega}}^{\overline{\omega}} \left(\omega - c - \frac{1 - F_{\theta}(\omega)}{f_{\theta}(\omega)}\right) x_{\theta}(\omega) f_{\theta}(\omega)\, d\omega - \sum_{\theta=H,M,L} \phi_{\theta} u_{\theta}(\underline{\omega}). \qquad (3)$$

Individual rationality constraint IR_{θ} for each type $\theta = H, M, L$ becomes

$$u_{\theta}(\underline{\omega}) + \int_{\underline{\omega}}^{\overline{\omega}} x_{\theta}(\omega)\left(1 - F_{\theta}(\omega)\right) d\omega \geq 0,$$

and period one incentive compatibility constraint $\mathrm{IC}_{\theta\theta'}$ for each pair $\theta \neq \theta' = H, M, L$ becomes

$$u_{\theta}(\underline{\omega}) + \int_{\underline{\omega}}^{\overline{\omega}} x_{\theta}(\omega)\left(1 - F_{\theta}(\omega)\right) d\omega \geq u_{\theta'}(\underline{\omega}) + \int_{\underline{\omega}}^{\overline{\omega}} x_{\theta'}(\omega)\left(1 - F_{\theta}(\omega)\right) d\omega.$$

From now on we will use the lowest ex post indirect utilities $u_{\theta}(\underline{\omega})$, $\theta = H, M, L$, instead of the payment rule t_{θ}, as the choice variables together with the allocation rule x_{θ}.

Now we define a "relaxed problem" by dropping the two non-local period one incentive compatibility constraints IC_{HL} and IC_{LH}, the individual rationality constraints for the two higher types IR_{H} and IR_{M}, and an upward period one incentive compatibility constraint IC_{MH}. The objective function is (3). The choice variables are $x_{\theta}(\omega)$ and $u_{\theta}(\underline{\omega})$ for each $\theta = H, M, L$. The constraints are: IR_{L}, IC_{ML}, IC_{HM}, and IC_{LM}, together with weak monotonicity of $x_{\theta}(\omega)$ for each $\theta = H, M, L$. Unlike in a standard relaxed problem (Courty and Li, 2000), we have kept one local upward period one incentive compatibility constraint, IC_{LM}. Nonetheless, as in the standard formulation, we can immediately identify the same binding constraints.

Lemma 2 *In any solution to the relaxed problem, IR_{L}, IC_{ML} and IC_{HM} bind.*

The proof of Lemma 2 is standard and hence omitted. We can use the binding constraints identified in Lemma 2 to solve for $u_\theta(\underline{\omega})$ for each $\theta = H, M, L$:

$$\begin{aligned}
u_L(\underline{\omega}) &= -\int_{\underline{\omega}}^{\overline{\omega}} x_L(\omega)(1-F_L(\omega))d\omega, \\
u_M(\underline{\omega}) &= -\int_{\underline{\omega}}^{\overline{\omega}} x_M(\omega)(1-F_M(\omega))d\omega + \int_{\underline{\omega}}^{\overline{\omega}} x_L(\omega)(F_L(\omega)-F_M(\omega))d\omega, \\
u_H(\underline{\omega}) &= -\int_{\underline{\omega}}^{\overline{\omega}} x_H(\omega)(1-F_H(\omega))d\omega + \int_{\underline{\omega}}^{\overline{\omega}} x_M(\omega)(F_M(\omega)-F_H(\omega))d\omega \\
&\quad + \int_{\underline{\omega}}^{\overline{\omega}} x_L(\omega)(F_L(\omega)-F_M(\omega))d\omega.
\end{aligned}$$

By substitution, the objective function becomes

$$\sum_{\theta=H,M,L} \int_{\underline{\omega}}^{\overline{\omega}} x_\theta(\omega)\phi_\theta\delta_\theta(\omega)f_\theta(\omega)d\omega,$$

where $\delta_\theta(\omega)$ is the dynamic virtual surplus function of type $\theta = H, M, L$, given by

$$\begin{aligned}
\delta_H(\omega) &= \omega - c, \\
\delta_M(\omega) &= \omega - c - \frac{\phi_H(F_M(\omega)-F_H(\omega))}{\phi_M f_M(\omega)}, \\
\delta_L(\omega) &= \omega - c - \frac{(\phi_M+\phi_H)(F_L(\omega)-F_M(\omega))}{\phi_L f_L(\omega)}.
\end{aligned}$$

The choice variables are now just allocation rule $x_\theta(\omega)$ for $\theta = H, M, L$. By Lemma 1, each function $x_\theta(\omega)$ is required to be weakly increasing, with $x_\theta(\omega) \in [0,1]$ for all $\omega \in [\underline{\omega}, \overline{\omega}]$. There is only one additional constraint, IC_{LM}. By Lemma 2, IC_{ML} is binding. Thus, IC_{LM} is equivalent to the following monotonicity constraint, obtained by adding up a binding IC_{ML} and IC_{LM}:

$$\int_{\underline{\omega}}^{\overline{\omega}} \left(x_M(\omega) - x_L(\omega)\right)\left(F_L(\omega) - F_M(\omega)\right) d\omega \geq 0. \tag{4}$$

This requires a weighted average of type M's allocation x_M is greater than the average of type L's allocation x_L with the same weights. We refer to the above average monotonicity constraint as MON_{ML}.

In any solution to the relaxed problem, allocation for the highest type, type H is efficient with $x_H(\omega) = \mathbb{1}_{\omega \geq c}$. The relaxed problem thus simplifies to

$$\max_{x_M(\omega), x_L(\omega)} \int_{\underline{\omega}}^{\overline{\omega}} x_M(\omega)\phi_M\delta_M(\omega)f_M(\omega)d\omega + \int_{\underline{\omega}}^{\overline{\omega}} x_L(\omega)\phi_L\delta_L(\omega)f_L(\omega)d\omega, \tag{5}$$

subject to $0 \leq x_M(\omega), x_L(\omega) \leq 1$ for all $\omega \in [\underline{\omega}, \overline{\omega}]$, $x_M(\omega), x_L(\omega)$ both weakly increasing, and MON_{ML} (4). We refer to the above problem as the "simplified problem."

In the rest of this section, we justify our approach of focusing on the simplified problem. There are two parts to the justification. First, we provide conditions under which the solution to the simplified problem satisfies all dropped constraints, and thus corresponds an optimal mechanism. Second, we explain why our approach may work when the standard relaxed program fails.

In forming the relaxed problem, we have dropped the individual rationality constraints except for the lowest type. As is true with the standard relaxed-program approach, any solution to the simplified problem is individually rational for the two higher types M and H. This is formally stated in the following lemma whose proof is omitted.

Lemma 3 *In any solution to the simplified problem, IR_H and IR_M are satisfied.*

In the standard relaxed-program approach, where MON_{ML} is not imposed, the solution is found by point-wise maximizing the two terms in the objective function (5). When the virtual surplus functions $\delta_M(\omega)$ and $\delta_L(\omega)$ both cross 0 only once, and when the crossing point of δ_M is smaller than or equal that of δ_L, the solutions to the relaxed problem are deterministic, and satisfy the dropped period one incentive compatibility constraints IC_{HL}, IC_{MH} and IC_{LH}. The conditions on the distribution functions $\{F_\theta\}_{\theta=H,M,L}$ that ensure both single-crossing of $\delta_M(\omega)$ and $\delta_L(\omega)$, and the "right" order of the crossing points are known as "regularity conditions." This approach however is silent about what happens when the regularity conditions fail.

In contrast, our simplified problem imposes MON_{ML}. This allows the solution to be either deterministic or stochastic. If the solution is deterministic, then as in the standard approach, it satisfies all dropped incentive compatibility constraints and therefore corresponds an optimal mechanism. It is important to note, however, that the solution to our simplified problem could be deterministic even though the retained constraint MON_{ML} is binding. In other words, our approach allows us to potentially identify conditions under which the regularity conditions fail — because MON_{ML} is binding — and yet the optimal mechanism remains deterministic (Proposition 1 in the next section). This is one insight we can obtain with our approach of including MON_{ML} in the simplified problem.

A solution to the simplified problem solves the seller's problem if it also satisfies the dropped period one incentive compatibility constraints IC_{HL}, IC_{MH} and IC_{LH}. Using the expressions of $u_\theta(\underline{\omega})$, $\theta = L, M, H$, from binding IR_L, IC_{ML} and IC_{HM} (Lemma 2), we find that IC_{HL} holds at any solution (x_M, x_L) to the simplified problem if and only if

$$\int_{\underline{\omega}}^{\overline{\omega}} \left(x_M(\omega) - x_L(\omega)\right)\left(F_M(\omega) - F_H(\omega)\right) d\omega \geq 0; \tag{6}$$

IC_{MH} holds if and only if

$$\int_{\underline{\omega}}^{\overline{\omega}} \left(x_H(\omega) - x_M(\omega)\right)\left(F_M(\omega) - F_H(\omega)\right) d\omega \geq 0; \tag{7}$$

and IC_{LH} holds if and only if

$$\int_{\underline{\omega}}^{\overline{\omega}} x_L(\omega)(F_L(\omega)-F_M(\omega))d\omega+\int_{\underline{\omega}}^{\overline{\omega}} x_M(\omega)(F_M(\omega)-F_H(\omega))d\omega \leq \int_{\underline{\omega}}^{\overline{\omega}} x_H(\omega)(F_L(\omega)-F_H(\omega))d\omega. \tag{8}$$

For future reference, we summarize the above observation in the following lemma:

Lemma 4 *Any solution to the simplified problem that satisfies conditions (6), (7) and (8) is an optimal mechanism.*

From now on, we will focus our analysis on the above simplified problem (5). By using information we garner from the solutions to the simplified problem, we will be able to provide conditions to ensure that a solution to the simplified problem satisfies conditions (6)-(8) and therefore corresponds an optimal mechanism.

4 Optimality of Deterministic Mechanisms

The purpose of this paper is to characterize when a stochastic mechanism, as opposed to a deterministic one, is optimal in sequential screening, and to characterize optimal randomization. To understand the necessity of randomization in maximizing the seller's profit, it is instrumental to first characterize the "optimal deterministic mechanism" which is profit-maximizing among mechanisms with deterministic allocation rules. If randomization can strictly improve the seller's profit upon the optimal deterministic mechanism, then optimal mechanism is stochastic; otherwise optimal mechanism is deterministic.

We first use the simplified problem defined in Section 3 to provide a characterization of the optimal deterministic mechanism. A deterministic mechanism is given by an allocation rule x_θ and transfer rule t_θ, $\theta = H, M, L$, such that there is a threshold k_θ for each θ with $x_\theta(\omega) = \mathbb{1}_{\omega \geq k_\theta}$. We say that $x_\theta(\omega) = \mathbb{1}_{\omega \geq k_\theta^*}$, $\theta = M, L$, is a "deterministic solution" to the simplified problem, if (k_M^*, k_L^*) maximizes

$$S_M(k_M) + S_L(k_L)$$

subject to $k_M \leq k_L$, where

$$S_\theta(k) \equiv \int_k^{\overline{\omega}} \phi_\theta \delta_\theta(\omega) f_\theta(\omega) d\omega$$

for each $\theta = M, L$.

Lemma 5 *Any deterministic solution to the simplified problem corresponds to an optimal deterministic mechanism.*

Proof. Under first order stochastic ranking of ex ante types, a deterministic mechanism satisfies local downward and upward incentive compatibility constraints if and only if $k_H \leq k_M \leq k_L$. Further, non-local incentive compatibility constraints IC_{HL} and IC_{LH} are redundant: IC_{HL} is implied by IC_{HM} and IC_{ML}, and IC_{LH} is implied by IC_{LM} and IC_{MH}. Then, we can define a deterministic relaxed problem, with choice variables k_θ and $u_\theta(\underline{\omega})$, $\theta = H, M, L$, by keeping only IR_L, IC_{ML} and IC_{HM}, together with $k_H \leq k_M \leq k_L$. As in Lemma 2, IR_L, IC_{ML} and IC_{HM} all bind at any solution to the deterministic relaxed problem. At any solution to the deterministic relaxed problem, we have $k_H = c$. Further, we have $k_M \geq c$: otherwise, since $\delta_\theta(\omega) < 0$ for each $\theta = M, L$ and for all $\omega < c$, the value of the objective function in the deterministic relaxed problem can be increased by reducing k_M and k_L by the same amount and hence leaving the constraint $k_M \leq k_L$ unaffected, a contradiction to optimality. Together with $t_\theta(\omega)$ given by (1) for each $\theta = H, M, L$, we have a solution to the original maximization problem. The lemma follows immediately. ■

We assume throughout that, for each $\theta = M, L$, there is a unique maximizer of $S_\theta(k)$, denoted as $\hat{k}_\theta$. We have $\hat{k}_\theta > c$ for each $\theta = M, L$, because a necessary condition for $\hat{k}_\theta$ is that $\delta_\theta(\hat{k}_\theta) = 0$. If $\hat{k}_M \leq \hat{k}_L$, then the constraint $k_M \leq k_L$ is not binding and $(\hat{k}_M, \hat{k}_L)$ is the deterministic solution to the simplified problem, and by Lemma 5, corresponds to an optimal deterministic mechanism. Further, the following result shows that the deterministic mechanism given by $(\hat{k}_M, \hat{k}_L)$ is overall optimal; that is, the optimal mechanism is deterministic and corresponds to $(\hat{k}_M, \hat{k}_L)$.

Lemma 6 *If $\hat{k}_M \leq \hat{k}_L$, then the optimal mechanism is deterministic.*

Proof. Consider the simplified problem without MON_{ML}. The problem is then separable in type M and type L. By Riley and Zeckhauser (1983), for each $\theta = M, L$, there is a deterministic solution to the problem of choosing $x_\theta(\omega)$ to maximize

$$\int_{\underline{\omega}}^{\overline{\omega}} x_\theta(\omega)\phi_\theta\delta_\theta(\omega)f_\theta(\omega)d\omega$$

subject to $0 \leq x_\theta(\omega) \leq 1$ for all $\omega \in [\underline{\omega}, \overline{\omega}]$ and $x_\theta(\omega)$ weakly increasing.[1] By definition, this solution is given by $\hat{x}_\theta(\omega) = \mathbb{1}_{\omega \geq \hat{k}_\theta}$. Since $\hat{k}_M \leq \hat{k}_L$, MON_{ML} is satisfied and $(\hat{k}_M, \hat{k}_L)$ is a solution to the simplified problem. Since this solution is deterministic with $c < \hat{k}_M \leq \hat{k}_L$, it satisfies the dropped constraints IC_{HL}, IC_{MH} and IC_{LH} of (6)-(8). The result follows immediately from Lemma 4. ■

A necessary condition for a stochastic mechanism to be optimal is thus $\hat{k}_M > \hat{k}_L$. Further, we impose a mild regularity condition that at least for one type θ of the two types M and L, $S_\theta(k)$ is single-peaked. This implies that the corresponding threshold $\hat{k}_\theta$ is the unique local maximizer of $S_\theta(k)$, and thus when $\hat{k}_M > \hat{k}_L$, the constraint $k_M \leq k_L$ binds at any deterministic solution to the simplified problem. The optimal deterministic mechanism thus has $k_M = k_L$. Further, the common threshold between type M and type L, denoted as $\hat{k}$, lies between $\hat{k}_L$ and $\hat{k}_M$.[2] Finally, for simplicity we assume that $\hat{k}$ is the unique deterministic solution to the simplified problem and is interior when $\hat{k}_M > \hat{k}_L$.

[1] Riley and Zeckhauser (1983) study a monopoly pricing problem, and show that there is always a deterministic solution if we restate it as an optimal mechanism design problem. See also Myerson (1981) for the same conclusion in an optimal auction problem when there is a single bidder. These conclusions are a special case of a general result that there is always a deterministic solution in maximizing a linear functional of a weakly increasing function.

[2] This does not rely on the regularity condition that $\hat{k}_\theta$ is the unique local maximizer. If $\hat{k} > \hat{k}_M$, the

Optimal mechanism may still be deterministic even when $\hat{k}_M > \hat{k}_L$. Intuitively, when the solution to the simplified problem without MON_{ML} violates MON_{ML}, the deterministic mechanism with common threshold $\hat{k}$ may be optimal because it may be better to bring the two thresholds together instead of introducing randomization for one or both types.

To formally characterize conditions under which the deterministic mechanism with common threshold $\hat{k}$ is optimal, we introduce two definitions. First, we define the average surplus-to-rent ratio for type $\theta = M, L$ over the interval $[w', w'']$ as

$$R_\theta(w', w'') = \frac{\int_{w'}^{w''} \phi_\theta \delta_\theta(\omega) f_\theta(\omega) d\omega}{\int_{w'}^{w''} (F_L(\omega) - F_M(\omega)) d\omega}.$$

The two ratios R_L and R_M have the following interpretation.[3] The numerator of $R_L(w', w'')$ is the total dynamic virtual surplus generated from type L by setting $x_L(\omega) = 1$ for all $\omega \in [w', w'']$. Correspondingly, the numerator of $R_M(w', w'')$ is the total virtual surplus generated from type M by setting $x_M(\omega) = 1$ for all $\omega \in [w', w'']$. The denominator of $R_L(w', w'')$ is the same as that of $R_M(w', w'')$, and both have the interpretation of information rent but going in opposite directions. For R_L, the denominator represents the total incentive cost of setting $x_L(\omega) = 1$ for all $\omega \in [w', w'']$, which arises because this allocation to type L makes it harder to satisfy MON_{ML}, the monotonicity constraint (4). For R_M, the denominator represents the total incentive benefit of setting $x_M(\omega) = 1$ for all $\omega \in [w', w'']$, which arises because this allocation to type M makes it easier to satisfy MON_{ML}.

Second, we define the point surplus-to-rent ratio at ω as

$$r_\theta(\omega) = \frac{\phi_\theta \delta_\theta(\omega) f_\theta(\omega)}{F_L(\omega) - F_M(\omega)}.$$

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value of the objective function could be increased by lowering the threshold for type M from $\hat{k}$ to $\hat{k}_M$ without violating MON_{ML}; if $\hat{k} < \hat{k}_L$, the value of the objective function could be increased by raising the threshold for type L from $\hat{k}$ to $\hat{k}_L$ without violating MON_{ML}. In either case we have a contradiction to the optimality of $\hat{k}$.

[3] The construction of the ratio of dynamic virtual surplus to information rent is inspired by Bergemann, Casto and Weintraub (2020). In their characterization of necessary and sufficient conditions for randomization in a two-type sequential screening model with ex post individual rationality constraint, they make use of a similar surplus-to-rent ratio.

The average surplus-to-rent ratio $R_\theta(w', w'')$ can be written in terms of $r_\theta(\omega)$ as

$$R_\theta(w', w'') = \frac{\int_{w'}^{w''} r_\theta(\omega)(F_L(\omega) - F_M(\omega))d\omega}{\int_{w'}^{w''} (F_L(\omega) - F_M(\omega))d\omega}$$

for any $[w', w''] \subseteq [\underline{\omega}, \overline{\omega}]$. Thus, $R_\theta(w', w'')$ is a weighted average of $r_\theta(\omega)$ over $\omega \in [w', w'']$. At the same time, for any $\hat{w} \in [w', w'']$,

$$r_\theta(\hat{w}) = \lim_{w' \uparrow \hat{w}} R_\theta(w', \hat{w}) = \lim_{w'' \downarrow \hat{w}} R_\theta(\hat{w}, w'').$$

That is, the point ratio $r_\theta(\hat{w})$ is the common limit of the average ratio $R_\theta(w', \hat{w})$ from the left and the average ratio $R_\theta(\hat{w}, w'')$ from the right.

Now we claim that when $\hat{k}_M > \hat{k}_L$, if for both types $\theta = M, L$,

$$\max_{\omega \leq \hat{k}} R_\theta(\omega, \hat{k}) \leq r_\theta(\hat{k}) \leq \min_{\omega \geq \hat{k}} R_\theta(\hat{k}, \omega), \tag{9}$$

then the deterministic mechanism with common threshold $\hat{k}$ is optimal. We establish the claim by the method of Lagrangian relaxation. Let $\lambda \geq 0$ be the multiplier associated with MON_{ML} in the simplified problem, and write the Lagrangian as

$$\begin{aligned}\int_{\underline{\omega}}^{\overline{\omega}} & x_M(\omega) \left(\phi_M f_M(\omega)\delta_M(\omega) + \lambda(F_L(\omega) - F_M(\omega))\right) d\omega \\ &+ \int_{\underline{\omega}}^{\overline{\omega}} x_L(\omega) \left(\phi_L f_L(\omega)\delta_L(\omega) - \lambda(F_L(\omega) - F_M(\omega))\right) d\omega.\end{aligned} \tag{10}$$

We choose a particular non-negative value $\hat{\lambda}$ for the multiplier λ, and show that with $\hat{\lambda}$ the Lagrangian (10) is maximized by $x^*_\theta(\omega) = \mathbb{1}_{\omega \geq \hat{k}}$ for each $\theta = M, L$, among all weakly increasing functions $x_\theta(\omega)$. Since $\hat{\lambda} \geq 0$, this maximum value of the Lagrangian is an upper bound of the objective function of the simplified problem (5) for any (x_M, x_L) that satisfies MON_{ML}, and since the maximizers (x^*_M, x^*_L) bind MON_{ML}, the maximized value of the Lagrangian is just the value of the objective function evaluated at (x^*_M, x^*_L). Therefore, the deterministic solution given by $\hat{k}$ solves the simplified problem. Furthermore, it satisfies all dropped constraints and hence optimal by Lemma 4.

Lemma 7 *Suppose $\hat{k}_M > \hat{k}_L$ and $\hat{k}$ is interior. If condition (9) holds, then the deterministic mechanism with common threshold $\hat{k}$ is optimal.*

Proof. Define

$$\hat{\lambda} = r_L(\hat{k}).$$

We claim that $\hat{\lambda} \geq 0$. To see this, note that since $\hat{k}$ corresponds to the deterministic solution to the simplified problem and is interior, it satisfies the first order necessary condition

$$r_L(\hat{k}) + r_M(\hat{k}) = 0.$$

If $\hat{\lambda} < 0$, then

$$r_M(\hat{k}) > 0 > r_L(\hat{k}).$$

By continuity, there exists $w' < \hat{k}$ such that $\phi_M \delta_M(\omega) f_M(\omega) > 0$ for all $\omega \in [w', \hat{k}]$, and there exists $w'' > \hat{k}$ such that $\phi_L \delta_L(\omega) f_L(\omega) < 0$ for all $\omega \in [\hat{k}, w''] > 0$. It follows that the value of the objective function of the simplified problem (5) can be improved by changing the threshold for type M from $\hat{k}$ to w' and the threshold for type L from $\hat{k}$ to w''. Such changes satisfy MON_{ML}, contradicting the optimality of $\hat{k}$ as the deterministic solution. This contradiction establishes that $\hat{\lambda} \geq 0$.

Now consider the second part of the Lagrangian (10), with λ replaced by $\hat{\lambda}$. By Riley and Zeckhauser (1983), it has a deterministic maximizer in a weakly increasing function $x_L(\omega)$. We claim that

$$\begin{aligned} &\int_{\tilde{\omega}}^{\overline{\omega}} \Big(\phi_L \delta_L(\omega) f_L(\omega) - \hat{\lambda}(F_L(\omega) - F_M(\omega))\Big)\, d\omega \\ \leq &\int_{\hat{k}}^{\overline{\omega}} \Big(\phi_L \delta_L(\omega) f_L(\omega) - \hat{\lambda}(F_L(\omega) - F_M(\omega))\Big)\, d\omega \end{aligned}$$

for all $\tilde{\omega}$. The above is the same as

$$R_L(w, \hat{k}) \leq \hat{\lambda} \leq R_L(\hat{k}, w')$$

for all $w \leq \hat{k}$ and $w' \geq \hat{k}$, which is exactly condition (9).

From the first order necessary condition for $\hat{k}$, we have

$$\hat{\lambda} = -r_M(\hat{k}).$$

A symmetric argument then establishes that for the first part of the Lagrangian (10), given the above value of $\hat{\lambda}$, the Lagrangian (10) is maximized by $x^*_M(\omega) = \mathbb{1}_{\omega \geq \hat{k}}$ among all weakly increasing functions $x_M(\omega)$. Since the maximum value of the Lagrangian is an upper bound of the objective function of the simplified problem, and since this value is achievable in the simplified problem, the deterministic solution solves the simplified problem.

Since this solution to the simplified problem is deterministic, it satisfies the dropped constraints (6)-(8) of IC_{HL}, IC_{MH} and IC_{LH}. By Lemma 4, this solution is also optimal. ■

Combining Lemma 6 and Lemma 7, we have established sufficient conditions for deterministic mechanisms to be optimal.

Proposition 1 *Optimal mechanism is deterministic if either one of the following two conditions holds: (i) $\hat{k}_M \leq \hat{k}_L$; (ii) $\hat{k}_M > \hat{k}_L$, $\hat{k}$ is interior and condition (9) holds.*

Condition (i) in Proposition 1 is well known in the literature of dynamic mechanism design, and is often referred to as the "regular case" where deterministic allocations as unconstrained maximizers of dynamic virtual surpluses are monotone in ex ante types. Condition (ii) is new. In this case, $\hat{k}_M > \hat{k}_L$, so the design problem is no longer regular and the deterministic allocation $(\hat{k}_M, \hat{k}_L)$ fails to satisfy the key monotonicity condition MON_{ML}. Proposition 1 shows that in this case, whether a deterministic mechanism is optimal depends on pairwise comparisons of average ratios of dynamic virtual surplus to information rent associated with MON_{ML}. Each pair of ratios are evaluated at an interval below and an interval above the common threshold $\hat{k}$ of types M and L when the optimal deterministic mechanism binds MON_{ML}. In particular, optimal mechanism remains deterministic and is given by $\hat{k}$ if for both types the average ratio below $\hat{k}$ is always lower than the point ratio at $\hat{k}$ which in turn always exceeds the average ratio above $\hat{k}$.

5 Optimality of Stochastic Mechanisms

In this section, we will establish sufficient conditions for an optimal mechanism to be stochastic. Proposition 1 suggests that a necessary condition for randomization is $\hat{k}_M > \hat{k}_L$ and a failure of condition (9) for either type. It turns out that this necessary condition is also sufficient for randomization. We establish this result by first showing that, if condition (9) fails for type θ, we can perturb the allocation rule for type θ around $\hat{k}$ to form a stochastic one that does strictly better than the optimal deterministic mechanism given by $\hat{k}$ in the simplified problem. Then we provide sufficient conditions for the stochastic allocation resulted from perturbation to satisfy the dropped IC constraints (6)-(8) and hence be feasible in the seller's original problem.

Lemma 8 *Suppose $\hat{k}_M > \hat{k}_L$ and $\hat{k}$ is interior. If $\hat{k}$ satisfies*

$$\max_{\omega \le \hat{k}} R_\theta(\omega, \hat{k}) > \min_{\omega \ge \hat{k}} R_\theta(\hat{k}, \omega), \tag{11}$$

for $\theta = L$ or $\theta = M$, then the solution to the simplified problem is stochastic.

Proof. Suppose (11) holds for $\theta = L$. Since $R_L(\omega, \hat{k})$ and $R_L(\hat{k}, \omega)$ are continuous in ω, the maximum and the minimum in condition (11) are attained. Let w' and w'' attain the maximum and the minimum respectively. Then, $w' \le \hat{k} \le w''$, with at least one strict inequality. By continuity of $R_L(\omega, \hat{k})$ and $R_L(\hat{k}, \omega)$ in ω, there exist w_L^- and w_L^+ satisfying $w_L^- < \hat{k} < w_L^+$, such that

$$R_L(w_L^-, \hat{k}) > R_L(\hat{k}, w_L^+).$$

Now, starting with the deterministic allocation $\hat{x}_M(\omega) = \hat{x}_L(\omega) = \mathbb{1}_{\omega \ge \hat{k}}$, we keep $\hat{x}_M(\omega)$ for type M but change allocation for type L to $x_L(\omega)$, given by

$$x_L(\omega) = \begin{cases} 0 & \text{if} \quad \omega < w_L^- \\ \chi_L & \text{if} \quad \omega \in [w_L^-, w_L^+] \\ 1 & \text{if} \quad \omega > w_L^+ \end{cases} \tag{12}$$

where

$$\chi_L \equiv \frac{\int_{\hat{k}}^{w_L^+} (F_L(\omega) - F_M(\omega)) d\omega}{\int_{w_L^-}^{w_L^+} (F_L(\omega) - F_M(\omega)) d\omega}.$$

Since $w_L^- < \hat{k} < w_L^+$, we have $\chi_L \in (0, 1)$. Further,

$$\chi_L \int_{w_L^-}^{\hat{k}} (F_L(\omega) - F_M(\omega)) d\omega = (1 - \chi_L) \int_{\hat{k}}^{w_L^+} (F_L(\omega) - F_M(\omega)) d\omega,$$

and thus MON_{ML} remains binding. The change in the value of the objective function in the simplified problem (5) is

$$\chi_L \int_{w_L^-}^{\hat{k}} \phi_L \delta_L(\omega) f_L(\omega) d\omega - (1 - \chi_L) \int_{\hat{k}}^{w_L^+} \phi_L \delta_L(\omega) f_L(\omega) d\omega.$$

With the expression of χ_L, the above has the same sign as

$$R_L(w_L^-, \hat{k}) - R_L(\hat{k}, w_L^+),$$

which is positive.

A symmetric argument applies when (11) holds for $\theta = M$. Therefore, when (11) holds for either type M or L, there is a stochastic allocation that gives a greater value for the objective function of the simplified problem (5) than the deterministic solution $\hat{x}_M(\omega) = \hat{x}_L(\omega) = \mathbb{1}_{\omega \geq \hat{k}}$. It follows that any solution to the simplified problem is stochastic. ■

The proof of Lemma 8 can be understood as constructing a particular class of perturbations to the deterministic solution to the simplified problem represented by the common threshold $\hat{k}$ for types M and L. These perturbations are piece-wise constant allocations for each type $\theta = M, L$ separately, with the support for a random allocation spanning across $\hat{k}$ in such a way to bind the monotonicity constraint MON_{ML}. The profitability of any such perturbation over the deterministic solution $\hat{k}$, represented by inequality (11), is then sufficient for randomization to be optimal.[4]

Next, we identify conditions on the distribution functions $\{F_\theta\}_{\theta=H,M,L}$ under which the particular perturbations used to derive the sufficient condition for randomization in Lemma 8, together with $x_H(\omega) = \mathbb{1}_{\omega \geq c}$, satisfy the dropped IC constraints (6), (7) and (8). These conditions, together with the sufficient condition in Lemma 8, would allow us to conclude that optimal mechanism is stochastic. Since the distribution functions $\{F_\theta\}_{\theta=H,M,L}$ are ranked by first order stochastic dominance, we can uniquely define a function $\alpha(\omega)$ that maps $[\underline{\omega}, \overline{\omega}]$ to $[0, 1]$ such that

$$F_M(\omega) = (1 - \alpha(\omega))F_L(\omega) + \alpha(\omega)F_H(\omega). \tag{13}$$

Proposition 2 *Suppose $\hat{k}_M > \hat{k}_L$ and $\hat{k}$ is interior. Any optimal mechanism is stochastic if either of the following two conditions holds: (i) condition (11) holds for $\theta = L$ and $\alpha(\omega)$ is non-increasing; (ii) condition (11) holds for $\theta = M$ and $\alpha(\omega)$ is non-decreasing.*

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[4] Consistent with Lemma 6, condition (11) never holds if we replace $\hat{k}$ with $\hat{k}_\theta$ for each $\theta = M, L$. This is because, by Reily and Zeckhauser (1983), $\mathbb{1}_{\omega \geq \hat{k}_\theta}$ maximizes the objective function (5) without MON_{ML} among all weakly increasing allocations with range in $[0, 1]$. There is no perturbation to $\mathbb{1}_{\omega \geq \hat{k}_\theta}$ that achieves a strictly greater value for (5).

Proof. Since $\hat{k}_M > \hat{k}_L$, the deterministic solution to the simplified problem is given by $\hat{x}_M(\omega) = \hat{x}_L(\omega) = \mathbb{1}_{\omega \geq \hat{k}}$. By Lemma 4, this deterministic solution is an optimal deterministic mechanism.

First, suppose that (11) is satisfied for $\theta = L$ and $\alpha(\omega)$ is non-increasing. By Lemma 8, there exist w_L^- and w_L^+ satisfying $w_L^- < \hat{k} < w_L^+$, and $\chi_L \in (0,1)$, such that a stochastic allocation $x_L(\omega)$ given by (12), together with $\hat{x}_M$, binds MON_{ML}, and yields a greater value than $(\hat{x}_M, \hat{x}_L)$ for the objective of the simplified problem. Note that we can use the binding MON_{ML} to rewrite the dropped IC_{LH} constraint (8) as

$$\int_{\underline{\omega}}^{\overline{\omega}} \left(x_H\left(\omega\right) - x_M\left(\omega\right)\right)\left(F_L\left(\omega\right) - F_H\left(\omega\right)\right) d\omega \geq 0. \tag{14}$$

Since $x_H(\omega) = \mathbb{1}_{\omega \geq c}$ and $\hat{x}_M(\omega) = \mathbb{1}_{\omega \geq \hat{k}}$ with $\hat{k} > \hat{k}_L > c$, it is straightforward to verify that $(\hat{x}_M, x_L)$ satisfies the dropped IC_{LM} constraint of (7) and IC_{LH} constraint of (14).

For the dropped IC_{HL} constraint of (6), we rewrite MON_{ML} as

$$\chi_L \int_{w_L^-}^{\hat{k}} (F_L(\omega) - F_M(\omega)) d\omega = (1 - \chi_L) \int_{\hat{k}}^{w_L^+} (F_L(\omega) - F_M(\omega)) d\omega. \tag{15}$$

Since $\alpha(\omega)$ is non-increasing,

$$\frac{F_M(\omega) - F_H(\omega)}{F_L(\omega) - F_M(\omega)} \leq \frac{F_M(\hat{k}) - F_H(\hat{k})}{F_L(\hat{k}) - F_M(\hat{k})} \leq \frac{F_M(\omega') - F_H(\omega')}{F_L(\omega') - F_M(\omega')}$$

for all $\omega \leq \hat{k} \leq \omega'$. Integrating the left ratio over $\omega \in [w_L^-, \hat{k}]$ and the right ratio over $\omega' \in [\hat{k}, w_L^+]$ separately, we have

$$\frac{\int_{w_L^-}^{\hat{k}} (F_M(\omega) - F_H(\omega)) d\omega}{\int_{w_L^-}^{\hat{k}} (F_L(\omega) - F_M(\omega)) d\omega} \leq \frac{\int_{\hat{k}}^{w_L^+} (F_M(\omega) - F_H(\omega)) d\omega}{\int_{\hat{k}}^{w_L^+} (F_L(\omega) - F_M(\omega)) d\omega}.$$

Together with MON_{ML} in the form of (15), the above implies

$$\chi_L \int_{w_L^-}^{\hat{k}} (F_M(\omega) - F_H(\omega)) d\omega \leq (1 - \chi_L) \int_{\hat{k}}^{w_L^+} (F_M(\omega) - F_H(\omega)) d\omega,$$

which is (6). Therefore, $(\hat{x}_M, x_L)$ satisfies the dropped IC constraints (6)-(8). It follows that $(\hat{x}_M, x_L)$ is feasible in the seller's problem and generates a strictly higher revenue than the optimal deterministic mechanism $(\hat{x}_M, \hat{x}_L)$. Hence, optimal mechanism must be stochastic.

Next, suppose that (11) is satisfied for $\theta = M$ and $\alpha(\omega)$ is non-decreasing. By Lemma 8, there exist w_M^- and w_M^+ satisfying $w_M^- < \hat{k} < w_M^+$, and $\chi_M \in (0,1)$, such that a stochastic allocation $x_M(\omega)$ given by

$$x_M(\omega) = \begin{cases} 0 & \text{if} \quad \omega < w_M^- \\ \chi_M & \text{if} \quad \omega \in [w_M^-, w_M^+] \\ 1 & \text{if} \quad \omega > w_M^+ \end{cases},$$

together with $\hat{x}_L$, binds MON_{ML}, and yields a greater value for the objective of the simplified problem than $(\hat{x}_M, \hat{x}_L)$. Let w_M^* denote the maximizer of $R_M(w, \hat{k})$ over $w \leq \hat{k}$. We claim that $w_M^* \geq c$ and hence we can always choose $w_M^- \geq c$ to satisfy condition (11) for type M. To prove the claim, we note that, for any $z < z' < \hat{k}$,

$$\frac{F_M(z) - F_H(z)}{F_L(z) - F_M(z)} \geq \frac{F_M(z') - F_H(z')}{F_L(z') - F_M(z')}.$$

Integrating the left ratio over $\omega \in [z, z']$ and the right ratio over $\omega \in [z', \hat{k}]$ separately, we have

$$\frac{\int_z^{z'} (F_M(\omega) - F_H(\omega)) d\omega}{\int_z^{z'} (F_L(\omega) - F_M(\omega)) d\omega} \geq \frac{\int_{z'}^{\hat{k}} (F_M(\omega) - F_H(\omega)) d\omega}{\int_{z'}^{\hat{k}} (F_L(\omega) - F_M(\omega)) d\omega},$$

and thus

$$\frac{\int_z^{\hat{k}} (F_M(\omega) - F_H(\omega)) d\omega}{\int_z^{\hat{k}} (F_L(\omega) - F_M(\omega)) d\omega} \geq \frac{\int_{z'}^{\hat{k}} (F_M(\omega) - F_H(\omega)) d\omega}{\int_{z'}^{\hat{k}} (F_L(\omega) - F_M(\omega)) d\omega}. \tag{16}$$

By definition of $\delta_M(\omega)$, we can rewrite $R_M(w, \hat{k})$ with $w \leq \hat{k}$ as

$$R_M(w, \hat{k}) = \frac{\int_w^{\hat{k}} \phi_M(\omega - c) f_M(\omega) d\omega}{\int_w^{\hat{k}} (F_L(\omega) - F_M(\omega)) d\omega} - \frac{\phi_H \int_w^{\hat{k}} (F_M(\omega) - F_H(\omega)) d\omega}{\int_w^{\hat{k}} (F_L(\omega) - F_M(\omega)) d\omega}.$$

Suppose by contradiction that $w_M^* < c$. Note that by (16) the second term (without the negative sign) of $R_M(w, \hat{k})$ is non-increasing in w. If the first-term is non-positive, then $R_M(w_M^*, \hat{k}) < R_M(c, \hat{k})$. If the first term is positive, then it is increasing in w for $w \leq c$, and hence $R_M(w, \hat{k})$ is increasing in w for $w \leq c$. Again, we have $R_M(w_M^*, \hat{k}) < R_M(c, \hat{k})$. This is a contradiction to the assumption that w_M^* is the maximizer of $R_M(w, \hat{k})$. Therefore, we must have $w_M^* \geq c$. As a result, we can always choose $w_M^- \geq c$ such that condition (11) is satisfied for $\theta = M$. With $w_M^- \geq c$ and $x_H(\omega) = \mathbb{1}_{\omega \geq c}$, we can immediately verify that $(x_M, \hat{x}_L)$ satisfies the dropped IC_{HM} of (7) and IC_{LH} of (8).

For the dropped IC_{HL} of (6), we can use a similar argument as the case of (11) holding for $\theta = L$ to show that if $\alpha(\omega)$ is non-decreasing, then a binding MON_{ML}, or

$$\chi_M \int_{w_M^-}^{\hat{k}} (F_L(\omega) - F_M(\omega))d\omega = (1 - \chi_M) \int_{\hat{k}}^{w_M^+} (F_L(\omega) - F_M(\omega))d\omega$$

implies

$$\chi_M \int_{w_M^-}^{\hat{k}} (F_L(\omega) - F_M(\omega))d\omega \geq (1 - \chi_M) \int_{\hat{k}}^{w_M^+} (F_L(\omega) - F_M(\omega))d\omega,$$

which is (6). It follows that $(x_M, \hat{x}_L)$ is feasible in the seller's problem and generates a strictly higher revenue than the optimal deterministic mechanism $(\hat{x}_M, \hat{x}_L)$. Hence, optimal mechanism must be stochastic. ■

We refer to the case where the relative weighting function $\alpha(\omega)$ in (13) is constant as the distributions (F_H, F_M, F_L) satisfying the "alignment" condition. The proof of Proposition 2 shows that the optimal mechanism is stochastic if (11) is satisfied for either type M or type L. In fact, it is straightforward to see that under the alignment condition, any solution to the simplified problem corresponds to an optimal mechanism.[5] For IC_{HL}, note that (6) is equivalent to MON_{ML}, while both IC_{MH} as given by (7) and IC_{LH} as given by (8) (when MON_{ML} is binding) are equivalent to

$$\int_{\underline{\omega}}^{\overline{\omega}} \left(x_H\left(\omega\right) - x_M\left(\omega\right)\right)\left(F_L\left(\omega\right) - F_M\left(\omega\right)\right) d\omega \geq 0.$$

Since $x_H(\omega) = \mathbb{1}_{\omega \geq c}$, and $x_L(\omega) = 0$ for all $\omega < c$ (otherwise the value of the objective function in the simplified problem could be increased by reducing $x_L(\omega)$ to 0 for all $\omega < c$ without affecting MON_{ML}), the above is implied by binding MON_{ML}. By Lemma 4, the alignment condition provides sufficiently strong restrictions for us on the primitives to focus entirely on the simplified problem, without needing to use any characterization about solutions to the simplified problem. In the next section, we will use this observation repeatedly.

[5] The constant alignment condition was proposed in an earlier draft of Courty and Li (2000), where they noted it is a sufficient condition for local incentive compatibility constraints to imply global ones. In the same draft there were numerical examples with discrete valuations showing that stochastic mechanisms can be optimal.

6 Optimal Randomization

In this section we characterize optimal stochastic mechanisms. Our methodology is based on the simplified problem (5). We have commented at the end of the previous section that under alignment the solution to the simplified problem corresponds to an optimal mechanism. However, the characterization itself does not rely on the alignment condition.

For the analysis of this section, we will repeatedly use Theorem 1 of Luenberger (1967, p. 217). Applied to the simplified problem, the theorem states that, if $(x_L^*(\omega), x_M^*(\omega))$ solves for simplified problem, then there exists a multiplier $\lambda \geq 0$ for MON_{ML}, such that for each $\theta = M, L$, $x_\theta^*(\omega)$ maximizes the Lagrangian (10) among all weakly increasing $x_\theta(\omega)$ satisfying $x_\theta(\omega) \in [0,1]$ for all $\omega \in [\underline{\omega}, \overline{\omega}]$.[6] We refer to it as the Luenberger Theorem.

We first show that there is always a solution to the simplified problem with at most one level of stochastic allocation for types M and L. That is, for each type $\theta = M, L$, if $x_\theta(w), x_\theta(w') \in (0,1)$ then $x_\theta(w) = x_\theta(w')$. The result is due to the fact both the objective function and the constraint MON_{ML} in the simplified problem are linear functionals of non-decreasing schedules $x_\theta(\cdot)$.

Lemma 9 *There is a solution $(x_L^*(\omega), x_M^*(\omega))$ to the simplified problem such that for each $\theta = L, M$, there exist w_θ^- and w_θ^+ with $\underline{\omega} \leq w_\theta^- \leq w_\theta^+ \leq \overline{\omega}$ and $\chi_\theta \in (0,1)$, such that $x_\theta^*(\omega) = 0$ for $\omega \in [\underline{\omega}, w_\theta^-)$, $x_\theta^*(\omega) = \chi_\theta$ for $\omega \in [w_\theta^-, w_\theta^+)$, and $x_\theta^*(\omega) = 1$ for $\omega \in [w_\theta^+, \overline{\omega}]$.*

Proof. We establish the lemma for type L; the proof for type M is the same. We first show that there is a solution to the simplified problem where $x_L^*(\omega)$ is piece-wise constant. Suppose instead that $x_L^*(\omega)$ is continuously strictly increasing for all $\omega \in (w', w'')$. We claim that the integrand of the Lagrangian (10)

$$\phi_L \delta_L(\omega) f_L(\omega) - \lambda(F_L(\omega) - F_M(\omega))$$

is zero for all $\omega \in (w', w'')$. To see this, note that if it is strictly positive at some $\hat{w} \in (w', w'')$, we can find a neighborhood $(\hat{w} - \epsilon, \hat{w} + \epsilon)$ of $\hat{w}$ for some $\epsilon > 0$ such that the integrand is

[6] To apply Theorem 1 of Luenberger (1967, p. 217), we need to show that that the feasible set in the simplified problem contains some (x_M, x_L) that satisfies MON_{ML} strictly. This is clearly true.

strictly positive for all $\omega \in (\hat{w}-\epsilon, \hat{w}+\epsilon)$. But then by changing $x_L^*(\omega)$ for all ω in the neighborhood to its highest value $x_L^*(\hat{w}+\epsilon)$ at $\hat{w}+\epsilon$ we can strictly increase the value of the Lagrangian, which contradicts Luenberger's Theorem. A similar argument applies if the integrand is strictly negative at any $\omega \in (w', w'')$. By the Intermediate Value Theorem, there is a $\hat{w} \in (w', w'')$ such that

$$\int_{w'}^{w''} x_L^*(\hat{w})(F_L(\omega) - F_M(\omega))d\omega = \int_{w'}^{w''} x_L^*(\omega)(F_L(\omega) - F_M(\omega))d\omega.$$

Since the integrand is zero for all $\omega \in (w', w'')$, we have

$$\int_{w'}^{w''} x_L^*(\omega)\phi_L \delta_L(\omega) f_L(\omega)d\omega = \int_{w'}^{w''} x_L^*(\hat{w})\phi_L \delta_L(\omega) f_L(\omega)d\omega.$$

Thus, the value of the part of the objective function associated with $x_L^*(\omega)$ for $\omega \in (w', w'')$ is unchanged if we replace $x_L^*(\omega)$ with $x_L^*(\hat{w})$ for all $\omega \in (w', w'')$.

Next, we show that there is a solution where there is at most one intermediate value of $x_L^*(\omega)$ strictly between 0 and 1. Suppose instead that there exist w', $\hat{w}$ and w'', such that $x_L^*(\omega) = \chi$ for $\omega \in (w', \hat{w})$ and $x_L^*(\omega) = \chi'$ for $\omega \in (\hat{w}, w'')$, with $x_L^*(w'^-) < \chi < \chi' < x_L^*(w''^+)$. Consider changing $x_L^*(\omega)$ for $\omega \in (w', \hat{w})$ by ϵ, and simultaneously changing $x_L^*(\omega)$ for $\omega \in (\hat{w}, w'')$ by ϵ', such that MON_{ML} is unchanged. This requires

$$\epsilon \int_{w'}^{\hat{w}} (F_L(\omega) - F_M(\omega))d\omega + \epsilon' \int_{\hat{w}}^{w''} (F_L(\omega) - F_M(\omega))d\omega = 0.$$

The change in the value of the objective function is given by

$$\epsilon \int_{w'}^{\hat{w}} \phi_L \delta_L(\omega) f_L(\omega)d\omega + \epsilon' \int_{\hat{w}}^{w''} \phi_L \delta_L(\omega) f_L(\omega)d\omega.$$

Since both $\epsilon > 0 > \epsilon'$ and $\epsilon < 0 < \epsilon'$ are feasible perturbations, and since $x_L^*(\omega)$ is optimal, we must have

$$R_L(w', \hat{w}) = R_L(\hat{w}, w'').$$

Then there is also a solution to the simplified problem with one fewer intermediate value strictly between 0 and 1, by setting ϵ and ϵ' such that $\chi + \epsilon = \chi' + \epsilon'$. ∎

Using Lemma 9, and slightly abusing notation, for simplicity we denote as $\chi_\theta^{[w_\theta^-, w_\theta^+]}$ the random allocation $x_\theta(\omega)$ for type θ given by $x_\theta^*(\omega) = 0$ for $\omega \in [\underline{\omega}, w_\theta^-)$, $x_\theta^*(\omega) = \chi_\theta$ for $\omega \in [w_\theta^-, w_\theta^+)$, and $x_\theta^*(\omega) = 1$ for $\omega \in [w_\theta^+, \overline{\omega}]$. This notation $x_\theta(\omega)$ includes deterministic allocations for type θ as special cases, if $w_\theta^- = w_\theta^+$, or $\chi_\theta = 0, 1$. Although Lemma 9 does not establish that all solutions to the simplified problem takes the form of $\chi_\theta^{[w_\theta^-, w_\theta^+]}$ for each $\theta = M, L$, it does say that it is without loss to restrict to this form of solution when we want to rule out certain allocations as solutions based on the value of the simplified problem. In a similar way, we now show that there is always a solution to the simplified problem where randomization occurs only for one of the two types M and L. This result is due to the fact that there is a single constraint MON_{ML} in the simplified problem for two non-decreasing functions $x_M(\cdot)$ and $x_L(\cdot)$.

Lemma 10 *There is a solution $(x_L^*(\omega), x_M^*(\omega))$ to the simplified problem such that for $\theta = L$ or $\theta = M$, or both, $x_\theta^*(\omega) = 0$ or 1 for all $\omega \in [\underline{\omega}, \overline{\omega}]$.*

Proof. By Lemma 9, there is always a solution $x_\theta^*(\omega) = \chi_\theta^{[w_\theta^-, w_\theta^+]}$, $\theta = M, L$, to the simplified problem. Suppose that $w_\theta^- < w_\theta^+$ and $\chi_\theta \in (0, 1)$ for each $\theta = M, L$. Then, by Luenberger's Theorem, since $x_\theta^*(\omega)$ maximizes (10) among all non-decreasing $x_\theta(\cdot)$, for each $\theta = M, L$, we have

$$\int_{w_M^-}^{w_M^+} (\phi_M \delta_M(\omega) f_M(\omega) + \lambda(F_L(\omega) - F_M(\omega)))\, d\omega = 0$$
$$\int_{w_L^-}^{w_L^+} (\phi_L \delta_L(\omega) f_L(\omega) - \lambda(F_L(\omega) - F_M(\omega)))\, d\omega = 0. \tag{17}$$

The objective function of the simplified problem (5) evaluated at the solution $\chi_\theta^{[w_\theta^-, w_\theta^+]}$, $\theta = M, L$, is

$$\chi_M \int_{w_M^-}^{w_M^+} \phi_M \delta_M(\omega) f_M(\omega) d\omega + \int_{w_M^+}^{\overline{\omega}} \phi_M \delta_M(\omega) f_M(\omega) d\omega$$
$$+\chi_L \int_{w_L^-}^{w_L^+} \phi_L \delta_L(\omega) f_L(\omega) d\omega + \int_{w_L^+}^{\overline{\omega}} \phi_L \delta_L(\omega) f_L(\omega) d\omega$$

If $\lambda = 0$ at the solution, then by (17) the objective function is independent of the values of χ_M and χ_L. We can change χ_M to 1, which keeps MON_{ML} satisfied, because the allocation of type M is weakly increased for all ω. Thus, there is also a solution where the allocation for type M is deterministic.

If $\lambda > 0$, then by complementary slackness, MON_{ML} is binding, and thus

$$\begin{aligned}&\chi_M \int_{w_M^-}^{w_M^+} (F_L(\omega) - F_M(\omega))d\omega + \int_{w_M^+}^{\overline{\omega}} (F_L(\omega) - F_M(\omega))d\omega \\ =&\chi_L \int_{w_L^-}^{w_L^+} (F_L(\omega) - F_M(\omega))d\omega + \int_{w_L^+}^{\overline{\omega}} (F_L(\omega) - F_M(\omega))d\omega.\end{aligned}$$

Then (17) implies that the objective function is again independent of the values of χ_M and χ_L. As a result, if we replace either χ_M or χ_L with 0 or 1, then so long as MON_{ML} holds, the resulting allocations, which have randomization for at most one type, yield the same value for the objective function of the simplified problem. Since MON_{ML} is binding, the set $[w_L^-, w_L^+] \cap [w_M^-, w_M^+]$ has a positive measure. Then, there are four cases we need to consider: (i) $w_L^- \leq w_M^- < w_M^+ \leq w_L^+$, (ii) $w_M^- \leq w_L^- < w_M^+ \leq w_L^+$, (iii) $w_M^- \leq w_L^- < w_L^+ \leq w_M^+$, and (iv) $w_L^- \leq w_M^- < w_L^+ \leq w_M^+$. For case (i), MON_{ML} is satisfied with either $\tilde{\chi}_M = 1$ and

$$\tilde{\chi}_L = \frac{\int_{w_M^-}^{w_L^+} \left(F_L\left(\omega\right) - F_M\left(\omega\right)\right) d\omega}{\int_{w_L^-}^{w_L^+} \left(F_L\left(\omega\right) - F_M\left(\omega\right)\right) d\omega} \in \left(0, 1\right],$$

or $\tilde{\chi}_M = 0$ and

$$\tilde{\chi}_L = \frac{\int_{w_M^+}^{w_L^+} \left(F_L\left(\omega\right) - F_M\left(\omega\right)\right) d\omega}{\int_{w_L^-}^{w_L^+} \left(F_L\left(\omega\right) - F_M\left(\omega\right)\right) d\omega} \in \left[0, 1\right).$$

For case (ii), MON_{ML} is satisfied with $\tilde{\chi}_M = 0$ and

$$\tilde{\chi}_L = \frac{\int_{w_M^+}^{w_L^+} \left(F_L\left(\omega\right) - F_M\left(\omega\right)\right) d\omega}{\int_{w_L^-}^{w_L^+} \left(F_L\left(\omega\right) - F_M\left(\omega\right)\right) d\omega} \in \left[0, 1\right),$$

or $\tilde{\chi}_L = 1$ and

$$\tilde{\chi}_M = \frac{\int_{w_L^-}^{w_M^+} \left(F_L\left(\omega\right) - F_M\left(\omega\right)\right) d\omega}{\int_{w_M^-}^{w_M^+} \left(F_L\left(\omega\right) - F_M\left(\omega\right)\right) d\omega} \in \left(0, 1\right].$$

Case (iii) is symmetric to case (i), and case (iv) is symmetric to case (ii), both with roles of the types switched. The lemma follows immediately. ■

Lemma 9 and Lemma 10 together imply that, if randomization occurs in a solution to the simplified problem, then there is always a solution (x_M^*, x_L^*) where for only one type $\theta = M, L$, and for only one non-degenerate interval $[w_\theta^-, w_\theta^+]$ of valuations, $x_\theta(\omega)$ is some constant χ_θ

strictly between 0 and 1. We denote such representative solution as $x_\theta(\omega) = \chi_\theta^{[w_\theta^-, w_\theta^+]}$ and $x_{\theta'}(\omega) = \mathbb{1}_{\omega \geq k_{\theta'}}$. From now on, unless explicitly mentioned, this notation presumes $w_\theta^- < w_\theta^+$ and $\chi_\theta \in (0,1)$, given by

$$\chi_\theta = \frac{\int_{k_{\theta'}}^{w_\theta^+} (F_L(\omega) - F_M(\omega)) d\omega}{\int_{w_\theta^-}^{w_\theta^+} (F_L(\omega) - F_M(\omega)) d\omega}. \tag{18}$$

Now we impose restrictions on where this interval $[w_\theta^-, w_\theta^+]$ can be located, depending on local characteristics of the point surplus-to-rent ratio function r_θ.

Lemma 11 *(i) If $r_\theta(\omega)$ is strictly increasing in $\omega \in (\omega^t, \omega^p)$ for some type $\theta = M, L$, then there is no solution $(x_L^*(\omega), x_M^*(\omega))$ to the simplified problem where $x_\theta^*(\omega) = \chi \in (0,1)$ for all $\omega \in (w, w') \subseteq (\omega^t, \omega^p)$. (ii) If $r_\theta(\omega)$ is strictly decreasing in $\omega \in (\omega^p, \omega^t)$ for some type $\theta = M, L$, then in any solution $(x_L^*(\omega), x_M^*(\omega))$ to the simplified problem $x_\theta^*(\omega)$ is constant for all $\omega \in (\omega^p, \omega^t)$.*

Proof. (i) Suppose that $r_\theta(\omega)$ is strictly increasing in $\omega \in [\omega^t, \omega^p]$ for some type $\theta = M, L$, and that $x_\theta^*(\omega) = \chi_\theta^{[w_\theta^-, w_\theta^+]}$ with $[w_\theta^-, w_\theta^+] \subseteq (\omega^t, \omega^p)$ is part of a solution to the simplified problem. Let $\theta = M$; the case of $\theta = L$ is symmetric. By Luenberger's Theorem, $x_M^*(\omega)$ maximizes type M part of the Lagrangian (10)

$$\int_{\underline{\omega}}^{\overline{\omega}} x_M(\omega) \left(\phi_M \delta_M(\omega) f_M(\omega) + \lambda(F_L(\omega) - F_M(\omega))\right) d\omega$$

among all weakly increasing $x_M(\omega)$ with the range $[\underline{\omega}, \overline{\omega}]$. Since $\chi_M \in (0,1)$, we have

$$\phi_M \delta_M(w_M^-) f_M(w_M^-) + \lambda(F_L(w_M^-) - F_M(w_M^-)) \geq 0,$$

which is equivalent to $r_M(w_M^-) \geq -\lambda$. Otherwise, an increase in w_M^- would increase the value of the Lagrangian without violating x_M being non-decreasing. Similarly, $\chi_M \in (0,1)$ implies that $r_M(w_M^+) \leq -\lambda$. Thus, $r_M(w_M^-) \geq r_M(w_M^+)$, contradicting the assumption that r_M is strictly increasing in $[\omega^t, \omega^p] \supseteq [w_M^-, w_M^+]$. The first part of the lemma follows immediately.

(ii) Suppose that $r_\theta(\omega)$ is strictly decreasing in $\omega \in [\omega^p, \omega^t]$ for some type $\theta = M, L$, and $x_\theta^*(\omega)$ is part of a solution to the simplified problem but is not constant on $\omega \in [\omega^p, \omega^t]$. Let $\theta = L$; the case of $\theta = M$ is symmetric. By Lemma 9, we can assume that x_L^* is piece-wise constant. Then, there exist w', $\hat{w}$ and w'' satisfying $\omega^p \leq w' < \hat{w} < w'' \leq \omega^t$, such that $x_L^*(\omega) = x$ for all $\omega \in (w', \hat{w})$ and $x_L^*(\omega) = x'$ for all $\omega \in (\hat{w}, w'')$, and $x < x'$. Consider replacing $x_L^*(\omega)$ with $\tilde{x}_L(\omega)$, given by $\tilde{x}_L(\omega) = x_L^*(\omega)$ for all $\omega \leq w'$ and $\omega \geq w''$, and $\tilde{x}_L(\omega) = \chi$ for all $\omega \in (w', w'')$, where χ is given by

$$\chi = \frac{x \int_{w'}^{\hat{w}} (F_L(\omega) - F_M(\omega))d\omega + x' \int_{\hat{w}}^{w''} (F_L(\omega) - F_M(\omega))d\omega}{\int_{w'}^{w''} (F_L(\omega) - F_M(\omega))d\omega}.$$

Then, $x < \chi < x'$ and MON_{ML} remains satisfied as we have not changed $x_M^*(\omega)$. The change in the value of the objective function in (5) problem is given by

$$(\chi - x) \int_{w'}^{\hat{w}} \phi_L \delta_L(\omega) f_L(\omega) d\omega - (x' - \chi) \int_{\hat{w}}^{w''} \phi_L \delta_L(\omega) f_L(\omega) d\omega,$$

which has the same sign as

$$R_L(w', \hat{w}) - R_L(\hat{w}, w'').$$

The above is strictly positive because

$$r_L(\omega) > r_L(\hat{w}) > r_L(\omega')$$

for all $\omega \in [w', \hat{w})$ and $\omega' \in (\hat{w}, w'']$, as $r_L(\omega)$ strictly decreases in $[w', w''] \subset [\omega^p, \omega^t]$. The second part of the lemma follows immediately. ■

Part (i) of Lemma 11 has an immediate implication. If for some type $\theta = M, L$, the point ratio of surplus-to-rent function $r_\theta(\omega)$ is strictly increasing for all $\omega \in [\underline{\omega}, \overline{\omega}]$, then there is no randomization for type θ in any solution to the simplified problem. This is therefore a simple sufficient condition to rule out randomization for type θ in characterizing optimal mechanisms. In contrast, part (ii) of Lemma 11 offers a way to rule in randomization for type θ. If $r_\theta(\omega)$ is strictly decreasing for all $\omega \in [\underline{\omega}, \overline{\omega}]$, then since $x_\theta^*(\omega)$ is constant for all $\omega \in [\underline{\omega}, \overline{\omega}]$, in any solution to the simplified problem the value of $x_\theta^*(\omega)$ is either 0, 1, or some $\chi_\theta \in (0, 1)$. The first two cases are deterministic and lead to immediate characterizations of the solution $(x_M^*(\omega), x_L^*(\omega))$. Only the third case, with the randomization support given by $w_\theta^- = \underline{\omega}$ and $w_\theta^+ = \overline{\omega}$, is interesting.

We now go one step further than Lemma 11 and characterize necessary conditions for solutions to the simplified problem. This is accomplished by adapting the general ironing techniques used in standard mechanism design problems (e.g., Fudenberg and Tirole, 1991).

For simplicity, we guarantee the uniqueness of the characterization by assuming that for each $\theta = M, L$, the point ratio of surplus-to-rent function r_θ is "single dipped," in that if r_θ is decreasing at both w and w' then it is decreasing at any convex combination of the two valuations. Under this assumption, r_θ has at most one interior peak, which we denote as ω_θ^p and which satisfies $dr_\theta(\omega_\theta^p)/d\omega = 0$, and at most one interior trough, which we denote as ω_θ^t and which satisfies $dr_\theta(\omega_\theta^t)/d\omega = 0$. If ω_θ^p and ω_θ^t both exist, then $\omega_\theta^p < \omega_\theta^t$. We will further adopt the convention that $r_\theta(\omega)$ is not strictly increasing for all $\omega \in [\underline{\omega}, \overline{\omega}]$ if it is single dipped.

Lemma 12 *Suppose that r_θ is single dipped for some type $\theta = M, L$. There exist unique $\omega_\theta^{*-} < \omega_\theta^{*+}$ such that*

$$r_\theta(\omega_\theta^{*-}) \geq R_\theta(\omega_\theta^{*-}, \omega_\theta^{*+}) \geq r_\theta(\omega_\theta^{*+}), \tag{19}$$

with $r_\theta(\omega_\theta^{*-}) = R_\theta(\omega_\theta^{*-}, \omega_\theta^{*+})$ if $\omega_\theta^{*-} > \underline{\omega}$ and $r_\theta(\omega_\theta^{*+}) = R_\theta(\omega_\theta^{*-}, \omega_\theta^{*+})$ if $\omega_\theta^{*+} < \overline{\omega}$. Further, if part of a solution to the simplified problem is $x_\theta^*(\omega) = \chi_\theta^{[w_\theta^-, w_\theta^+]}$ then $w_\theta^- = \omega_\theta^{*-}$ and $w_\theta^+ = \omega_\theta^{*+}$, and if it is $x_\theta^*(\omega) = \mathbb{1}_{\omega \geq k_\theta}$ then $k_\theta \notin (\omega_\theta^{*-}, \omega_\theta^{*+})$.

Proof. For now we assume that r_θ has both an interior peak at ω_θ^p, and a trough at $\omega_\theta^t > \omega_\theta^p$. Since r_θ is single dipped, r_θ is strictly decreasing over $[\omega_\theta^p, \omega_\theta^t]$. Thus, for any $r \in [r_\theta(\omega_\theta^t), r_\theta(\omega_\theta^p)]$, there exists a unique value of $\hat{z}(r) \in [\omega_\theta^p, \omega_\theta^t]$ such that $r_\theta(\hat{z}(r)) = r$. For any $r \in [r_\theta(\omega_\theta^t), r_\theta(\omega_\theta^p)]$, define $z^p(r) \leq \omega_\theta^p$ as the unique value of ω such that $r_\theta(z^p(r)) = r$; let $z^p(r) = \underline{\omega}$ if $r_\theta(\omega) > r$ for all $\omega \in [\underline{\omega}, \omega_\theta^p]$. Similarly, for any $r \in [r_\theta(\omega_\theta^t), r_\theta(\omega_\theta^p)]$, define $z^t(r) \geq \omega_\theta^t$ as the unique value of ω such that $r_\theta(z^t(r)) = r$; let $z^t(r) = \overline{\omega}$ if $r_\theta(\omega) < r$ for all $\omega \in [\omega_\theta^t, \overline{\omega}]$. The three functions $z^p(r)$, $\hat{z}(r)$, $z^t(r)$ are well-defined for all $r \in [r_\theta(\omega_\theta^t), r_\theta(\omega_\theta^p)]$, and are all continuous functions.

By construction, $z^t(r_\theta(\omega_\theta^t)) = \hat{z}(r_\theta(\omega_\theta^t))$. At $r = r_\theta(\omega_\theta^t)$, since $r_\theta(\omega) > r_\theta(\omega_\theta^t)$ for all $\omega \in (z^p(r_\theta(\omega_\theta^t)), z^t(r_\theta(\omega_\theta^t)))$, we have

$$R_\theta(z^p(r_\theta(\omega_\theta^t)), z^t(r_\theta(\omega_\theta^t)) > r_\theta(\omega_\theta^t).$$

Similarly, at $r = r_\theta(\omega_\theta^p)$ we have $z^p(r_\theta(\omega_\theta^p)) = \hat{z}(r_\theta(\omega_\theta^p))$, and $r_\theta(\omega) < r_\theta(\omega_\theta^p)$ for all $\omega \in (z^p(r_\theta(\omega_\theta^p)), z^t(r_\theta(\omega_\theta^p)))$, and so

$$R_\theta(z^p(r_\theta(\omega_\theta^p)), z^t(r_\theta(\omega_\theta^p)) < r_\theta(\omega_\theta^p).$$

It follows from the Intermediate Value Theorem that there exists some $r^* \in (r_\theta(\omega_\theta^t), r_\theta(\omega_\theta^p))$ such that

$$R_\theta(z^p(r^*), z^t(r^*)) = r^*.$$

The total derivative of $R_\theta(z^p(r), z^t(r))$ with respect to r, evaluated at r^*, has the same sign as

$$\begin{aligned}&-(F_L(z^p(r^*)) - F_M(z^p(r^*)))(r_\theta(z^p(r^*)) - r^*)\frac{dz^p(r^*)}{dr}\\ &+(F_L(z^t(r^*)) - F_M(z^t(r^*)))(r_\theta(z^t(r^*)) - r^*)\frac{dz^t(r^*)}{dr}.\end{aligned}$$

The first term is the above expression is 0 because either $r_\theta(z^p(r^*)) = r^*$, or $z^p(r^*) = \underline{\omega}$ and thus $dz^p(r^*)/dr = 0$. Similarly, the second term is also zero. It follows that r^* is uniquely defined.

If r_θ has a peak at ω_θ^p but no trough, then it is strictly decreasing for all $\omega \in [\omega_\theta^p, \overline{\omega}]$. In this case, we set $z^t(r) = \overline{\omega}$ for all $r \in [r_\theta(\overline{\omega}), r_\theta(\omega_\theta^p)]$. Symmetrically, if r_θ has a trough at ω_θ^t but no peak, then we set $z^p(r) = \underline{\omega}$ for all $r \in [r_\theta(\omega_\theta^t), r_\theta(\underline{\omega})]$. Finally, if r_θ has neither a peak nor a trough, we set $z^p(r) = \underline{\omega}$ and $z^t(r) = \overline{\omega}$ for all $r \in [r_\theta(\overline{\omega}), r_\theta(\underline{\omega})]$. The rest of the proof goes through without change.

Let $\omega_\theta^{*-} = z^p(r^*)$ and $\omega_\theta^{*+} = z^t(r^*)$. These are uniquely defined because r^* is. Further, for any $\omega < \omega_\theta^p$, we have $r_\theta(\omega) \leq R_\theta(\omega, z^t(r_\theta(\omega)))$ if and only if $\omega \leq \omega_\theta^{*-}$. Symmetrically, for any $\omega > \omega_\theta^t$, we have $r_\theta(\omega) \geq R_\theta(z^p(r_\theta(\omega)), \omega)$ if and only if $\omega \geq \omega_\theta^{*+}$.

For the second part of the proposition, let $\theta = L$; the proof for the other case is symmetric. By Luenberger's Theorem, if $x_L^*(\omega)$ is part of a solution to the simplified problem, there exists $\lambda \geq 0$ such that $x_L^*(\omega)$ maximizes type L part of the Lagrangian (10)

$$\int_{\underline{\omega}}^{\overline{\omega}} x_L(\omega)\left(\phi_L \delta_L(\omega) f_L(\omega) - \lambda(F_L(\omega) - F_M(\omega))\right) d\omega$$

among all weakly increasing $x_L(\omega)$ with the range $[0, 1]$.

Suppose that $x_L^*(\omega) = \chi_L^{[w_L^-, w_L^+]}$. Since $\chi_L \in (0, 1)$, we have

$$R_L(w_L^-, w_L^+) = \lambda,$$

for otherwise we could increase the value of the Lagrangian by either increasing or decreasing χ_L. Similarly, we have $r_\theta(w_\theta^-) \geq \lambda$ and $w_\theta^- \geq \underline{\omega}$, with complementary slackness, and

$r_\theta(w_\theta^+) \le \lambda$ and $w_\theta^+ \le \overline{\omega}$, with complementary slackness. Thus, w_θ^- and w_θ^+ satisfy (19), and by the uniqueness of ω_θ^{*-} and ω_θ^{*+}, we have $w_\theta^- = \omega_\theta^{*-}$ and $w_\theta^+ = \omega_\theta^{*+}$.

Finally, suppose that $x_L^*(\omega) = \mathbb{1}_{\omega \ge k_L}$ for some k_L. By Lemma 11, we have $k_L \notin (\omega_\theta^p, \omega_\theta^t)$. Suppose that $k_L \in (\omega_\theta^{*-}, \omega_\theta^p]$. Consider replacing $x_L^*(\omega)$ with $\chi^{[w, z^t(r_L(w))]}$ for some $w < k_L$, where $\chi \in (0,1)$ satisfies

$$\chi \int_w^{k_L} (F_L(\omega) - F_H(\omega)) d\omega = (1-\chi) \int_{k_L}^{z^t(r_L(w))} (F_L(\omega) - F_H(\omega)) d\omega.$$

This does not affect MON_{ML}. The change in the value of type L part of the objective function in the simplified problem has the same sign as

$$R_L(w, k_L) - R_L(k_L, z^t(r_L(w))).$$

This is strictly positive for w sufficiently close to k_L, because $k_L > \omega_L^{*-}$ implies that $R_L(k_L, z^t(r_L(w))) < r_L(k_L)$ and $R_L(w, k_L)$ converges to $r_L(k_L)$ as w converges to k_L. We have a contradiction to the assumption that $x_L^*(\omega) = \mathbb{1}_{\omega \ge k_L}$ is part of a solution to the simplified problem. A symmetric argument leads to a similar contradiction if $k_L \in [\omega_\theta^t, \omega_L^{*+})$.

■

Under the assumption that $r_\theta(\omega)$ is single dipped for some type $\theta = M, L$, Lemma 12 claims a unique candidate randomization support for type θ in any solution to the simplified problem (x_M^*, x_L^*), given by $[\omega_\theta^{*-}, \omega_\theta^{*+}]$. The support is a superset of the interval $[\omega_\theta^p, \omega_\theta^t]$ over which $r_\theta(\omega)$ is strictly decreasing. If x_θ^* is deterministic, Lemma 12 restricts the threshold k_θ to lie not just outside of $[\omega_\theta^p, \omega_\theta^t]$, but outside of the superset $(\omega_\theta^{*-}, \omega_\theta^{*+})$. Both these two results generalize Lemma 11.

Now we are ready to present our first main characterization result on optimal stochastic mechanisms. Under assumptions that either rule out or rule in one of the two types M and L as having a random allocation in any solution to the simplified problem, we show that the allocations characterized in Lemma 12 lead to a solution to the simplified problem, if in addition they satisfy some cross-type restrictions. The additional restrictions allow us to use Lagrangian relaxation in a similar way as in the proof of Lemma 7. Once again, the role of alignment in the following proof is to reduce the optimal mechanism problem to the simplified problem.

Proposition 3 *(i) Suppose that $r_\theta(\omega)$ is single dipped and $r_{\theta'}(\omega)$ is strictly increasing, with $R_\theta(\omega_\theta^{*-},\omega_\theta^{*+}) \leq 0$ for $\theta = M$ and $R_\theta(\omega_\theta^{*-},\omega_\theta^{*+}) \geq 0$ for $\theta = L$. If there exists $k_{\theta'} \in (\omega_\theta^{*-},\omega_\theta^{*+})$ such that $r_{\theta'}(k_{\theta'}) = -R_\theta(\omega_\theta^{*-},\omega_\theta^{*+})$, then $x_\theta^*(\omega) = \chi_\theta^{[\omega_\theta^{*-},\omega_\theta^{*+}]}$ and $x_{\theta'}^*(\omega) = \mathbb{1}_{\omega \geq k_{\theta'}}$ solve the simplified problem for χ_θ given by (18). (ii) Suppose that $r_\theta(\omega)$ is single dipped and $r_{\theta'}(\omega)$ is strictly decreasing, with $R_{\theta'}(\underline{\omega},\overline{\omega}) \leq 0$ for $\theta' = M$ and $R_{\theta'}(\underline{\omega},\overline{\omega}) \geq 0$ for $\theta' = L$. If there exists $k_\theta \in (\underline{\omega},\omega_\theta^{*-}]$ or $k_\theta \in [\omega_\theta^{*+},\overline{\omega})$ such that $r_\theta(k_\theta) = -R_{\theta'}(\underline{\omega},\overline{\omega})$, then $x_\theta^*(\omega) = \mathbb{1}_{\omega \geq k_\theta}$ and $x_{\theta'}^*(\omega) = \chi_{\theta'}^{[\underline{\omega},\overline{\omega}]}$ solve the simplified problem for $\chi_{\theta'}$ given by (18). Further, under alignment these solutions each correspond to an optimal mechanism.*

Proof. (i) Let $\theta = L$ and $\theta' = M$; the proof for the other case is symmetric. By assumption, there exists k_M such that $r_M(k_M) = -R_L(\omega_L^{*-},\omega_L^{*+})$. Consider the first part of the Lagrangian (10) wth λ replaced with $\hat{\lambda} = -r_M(k_M) \geq 0$. By Riley and Zeckhouser (1983), it has a deterministic maximizer among all weakly increasing $x_M(\omega)$ with the range in $[0,1]$. Since by assumption $r_M(\omega)$ is strictly increasing, for all $k \in [\underline{\omega},\overline{\omega}]$ we have

$$\begin{aligned}
&\int_k^{\overline{\omega}} \Big(\phi_M f_M(\omega)\delta_M(\omega) + \hat{\lambda}(F_L(\omega) - F_M(\omega))\Big)\, d\omega \\
= &\int_k^{\overline{\omega}} (r_M(\omega) - r_M(k_M))\,(F_L(\omega) - F_M(\omega))d\omega \\
\leq &\int_{k_M}^{\overline{\omega}} (r_M(\omega) - r_M(k_M))\,(F_L(\omega) - F_M(\omega))d\omega.
\end{aligned}$$

Thus, $x_M^*(\omega) = \mathbb{1}_{\omega \geq k_M}$ maximizes the first part of (10) among all weakly increasing $x_M(\omega)$ with the range in $[0,1]$.

Next, consider the second part of the Lagrangian (10), with λ replaced with $\hat{\lambda} = -r_M(k_M)$. By Riley and Zeckhouser (1983), it has a deterministic maximizer in a weakly increasing function $x_L(\omega)$ with the range in $[0,1]$. Since $\hat{\lambda} = R_L(\omega_L^{*-},\omega_L^{*+})$ by equations (19),

$$\begin{aligned}
&\int_{\omega_L^{*-}}^{\overline{\omega}} \Big(\phi_L \delta_L(\omega) f_L(\omega) - \hat{\lambda}(F_L(\omega) - F_M(\omega))\Big)\, d\omega \\
= &\int_{\omega_L^{*-}}^{\overline{\omega}} \big(r_L(\omega) - R_L(\omega_L^{*-},\omega_L^{*+})\big)\,(F_L(\omega) - F_M(\omega))d\omega \\
= &\int_{\omega_L^{*+}}^{\overline{\omega}} \big(r_L(\omega) - R_L(\omega_L^{*-},\omega_L^{*+})\big)\,(F_L(\omega) - F_M(\omega))d\omega.
\end{aligned}$$

Since $r_L(\omega)$ is single dipped, $r_L(\omega) < r_L(\omega_L^{*-})$ for all $\omega < \omega_L^{*-}$ and in this case, equations

(19) require $r_L(\omega_L^{*-}) = \hat{\lambda}$. Then, for all $k < \omega_L^{*-}$,

$$\begin{aligned}
&\int_k^{\overline{\omega}} \left(\phi_L \delta_L(\omega) f_L(\omega) - \hat{\lambda}(F_L(\omega) - F_M(\omega))\right) d\omega \\
&= \int_k^{\overline{\omega}} \left(r_L(\omega) - r_L(\omega_L^{*-})\right) (F_L(\omega) - F_M(\omega)) d\omega \\
&< \int_{\omega_L^{*-}}^{\overline{\omega}} \left(r_L(\omega) - r_L(\omega_L^{*-})\right) (F_L(\omega) - F_M(\omega)) d\omega.
\end{aligned}$$

By a symmetric argument, for all $k > r_L(\omega_L^{*+})$, we have $r_L(k) > r_L(\omega_L^{*+}) = \hat{\lambda}$, and thus

$$\begin{aligned}
&\int_k^{\overline{\omega}} \left(\phi_L \delta_L(\omega) f_L(\omega) - \hat{\lambda}(F_L(\omega) - F_M(\omega))\right) d\omega \\
&< \int_{\omega_L^{*+}}^{\overline{\omega}} \left(r_L(\omega) - r_L(\omega_L^{*+}))\right) (F_L(\omega) - F_M(\omega)) d\omega.
\end{aligned}$$

Finally, consider $k \in (\omega_L^{*-}, \omega_L^{*+})$. Since $r_L(\omega)$ is single dipped, by Lemma 12, there exists a unique $\hat{w} \in (\omega_L^{*-}, \omega_L^{*+})$ such that $r_L(\hat{\omega}) = R_L(\omega_L^{*-}, \omega_L^{*+}) = \hat{\lambda}$, $r_L(\omega) \geq r_L(\hat{w})$ for any $\omega \in (\omega_L^{*-}, \hat{w})$, and $r_L(\omega) \leq r_L(\hat{w})$ for any $\omega \in (\hat{w}, \omega_L^{*+})$. Thus

$$\begin{aligned}
&\int_k^{\overline{\omega}} \left(\phi_L \delta_L(\omega) f_L(\omega) - \hat{\lambda}(F_L(\omega) - F_M(\omega))\right) d\omega \\
&= \int_k^{\overline{\omega}} \left(r_L(\omega) - r_L(\hat{w})\right) (F_L(\omega) - F_M(\omega)) d\omega
\end{aligned}$$

is decreasing for any $k \in (\omega_L^{*-}, \hat{w})$ and increasing for any $k \in (\hat{w}, \omega_L^{*+})$. Therefore, any $k \in (\omega_L^{*-}, \omega_L^{*+})$ is dominated by either ω_L^{*-} or ω_L^{*+}.

We have verified that $x_L^*(\omega) = \chi_L^{[\omega_L^{*-}, \omega^{*+}]}$ and $x_M^*(\omega) = \mathbb{1}_{\omega \geq k_M}$ maximize the Lagrangian (10) among all weakly decreasing x_L and x_M. The maximum value of the Lagrangian achieved by the allocations given by Lemma 12 is an upper bound of the objective function of the simplified problem. Since $k_M \in (\omega_L^{*-}, \omega_L^{*+})$ by assumption, χ_L given by (18) binds MON_{ML}. Thus, the maximum value of the Lagrangian is achievable in the simplified problem. It follows that (x_M^*, x_L^*) solves the simplified problem.

(ii) Let $\theta = M$ and $\theta' = L$; the proof for the other case is symmetric. Since by assumption $r_L(\omega)$ is strictly decreasing, from (19) in Lemma 12 we have $\omega_L^{*-} = \underline{\omega}$ and $\omega_L^{*+} = \overline{\omega}$. Consider the second part of the Lagrangian (10), with λ replaced with $\hat{\lambda} = R_L(\underline{\omega}, \overline{\omega})$, which is non-negative by assumption. By Riley and Zeckhouser (1983), it has a deterministic maximizer with some threshold k among all weakly increasing functions $x_L(\omega)$ with the range in $[0,1]$. We claim that the unique maximizer is $k = \underline{\omega}$. This is equivalent to

$$\int_k^{\overline{\omega}} (r_L(\omega) - \hat{\lambda})(F_L(\omega) - F_H(\omega))d\omega \leq \int_{\underline{\omega}}^{\overline{\omega}} (r_L(\omega) - \hat{\lambda})(F_L(\omega) - F_H(\omega))d\omega$$

for all k. Since $\hat{\lambda} = R_L(\underline{\omega}, \overline{\omega})$, the right hand side above is equal to 0. Replace $\hat{\lambda}$ on the left hand side with $r_L(\hat{w})$ where the unique $\hat{w}$ is chosen such that $r_L(\hat{w}) = \hat{\lambda}$. Since r_L is strictly decreasing, the left hand side is negative for any $k \geq \hat{w}$, and is strictly decreasing for any $k \in [\underline{\omega}, \hat{w})$. Thus, the left hand side is maximized at $k = \underline{\omega}$.

Now consider first part of the Lagrangian (10) wth λ replaced with $\hat{\lambda} = R_L(\underline{\omega}, \overline{\omega})$, which by assumption equals $-r_M(k_M)$. By Riley and Zeckhouser (1983), it has a deterministic maximizer with some threshold k among all weakly increasing functions $x_M(\omega)$ with the range in $[0,1]$. We claim that the unique maximizer is given by $k = k_M$. This is equivalent to

$$\int_k^{\overline{\omega}} (r_M(\omega) + \hat{\lambda})(F_L(\omega) - F_H(\omega))d\omega \leq \int_{k_M}^{\overline{\omega}} (r_M(\omega) + \hat{\lambda})(F_L(\omega) - F_M(\omega))d\omega$$

for all k. Suppose first that $k_M \leq \omega_M^{*-}$; by Lemma 12, $k_M < \omega_M^p$ where ω_M^p is the interior peak of r_M. Since $r_M(\omega)$ is strictly increasing for $\omega < k_M$ and since $\hat{\lambda} = -r_M(k_M)$, the above inequality holds for all $k < k_M$. For $k > k_M$, we rewrite the above inequality as

$$R_M(k_M, k) \geq r_M(k_M).$$

As in the proof of Lemma 12, define $\hat{z} \in [\omega_M^p, \omega_M^t]$ such that $r_M(\hat{z}) \geq r_M(k_M)$ and $\hat{z} \leq \omega_M^t$, with complementary slackness, and define $z^t \geq \omega_M^t$ such that $r_M(z^t) \geq r_M(k_M)$ and $z^t \leq \overline{\omega}$, with complementary slackness, where ω_M^t is the interior trough of r_M. For all $k \in [k_M, z^t]$, $r_M(k) \geq r_M(k_M)$, so the desired inequality holds. For $k \geq \hat{z}$, we have $r_M(k) \leq r_M(k_M)$ for $k \in [\hat{z}, z^t]$ and $r_M(k) \geq r_M(k_M)$ for $k > \hat{z}$. Therefore, the desired inequality holds for all $k \geq \hat{z}$ as long as $R_M(k_M, z^t) \geq r_M(k_M)$. In the proof of Lemma 12, we have shown that $k_M \leq \omega_M^{*-}$ implies that $R_M(k_M, z^t) \geq r_M(k_M)$. Thus, $R_M(k_M, k) \geq r_M(k_M)$ for all $k \geq k_M$. The argument is symmetric if $k_M \geq \omega_M^{*+}$.

We have verified that $x_L^*(\omega) = \chi_L^{[\underline{\omega}, \overline{\omega}]}$ and $x_M^*(\omega) = \mathbb{1}_{\omega \geq k_M}$ maximize the Lagrangian (10) among all weakly decreasing x_L and x_M. The maximum value of the Lagrangian achieved by the allocations given by Lemma 12 is an upper bound of the objective function of the simplified problem. Since $k_M \in (\underline{\omega}, \overline{\omega})$ by assumption, χ_L given by (18) binds MON_{ML}. Thus, the maximum value of the Lagrangian is achievable in the simplified problem. It follows that (x_M^*, x_L^*) solves the simplified problem.

Finally, under the alignment condition, any solution to the simplified problem corresponds to an optimal mechanism. The conclusion follows immediately. ■

In a similar way as Proposition 2 extends Lemma 8, we can generalize Proposition 3 to "weak alignment" where the relative weight function $\alpha(\omega)$ on the intermediate distribution F_M is monotone rather than constant. The proof of the following result is similar to that in Proposition 2 and skipped.[7]

Corollary 1 *(i) Suppose that $r_\theta(\omega)$ is single dipped and $r_{\theta'}(\omega)$ is strictly increasing, with $R_\theta(\omega_\theta^{*-}, \omega_\theta^{*+}) \leq 0$ and $\alpha(\omega)$ non-decreasing for $\theta = M$, and $R_\theta(\omega_\theta^{*-}, \omega_\theta^{*+}) \geq 0$ and $\alpha(\omega)$ non-increasing for $\theta = L$. If there exists $k_{\theta'} \in (\omega_\theta^{*-}, \omega_\theta^{*+})$ such that $r_{\theta'}(k_{\theta'}) = -R_\theta(\omega_\theta^{*-}, \omega_\theta^{*+})$, then $x_\theta^*(\omega) = \chi_\theta^{[\omega_\theta^{*-}, \omega_\theta^{*+}]}$ and $x_{\theta'}^*(\omega) = \mathbb{1}_{\omega \geq k_{\theta'}}$ correspond to an optimal mechanism for χ_θ given by (18). (ii) Suppose that $r_\theta(\omega)$ is single dipped and $r_{\theta'}(\omega)$ is strictly decreasing, with $R_{\theta'}(\underline{\omega}, \overline{\omega}) \leq 0$ and $\alpha(\omega)$ non-decreasing for $\theta' = M$, and $R_{\theta'}(\underline{\omega}, \overline{\omega}) \geq 0$ and $\alpha(\omega)$ non-increasing for $\theta' = L$. If there exists $k_\theta \in (\underline{\omega}, \omega_\theta^{*-}]$ or $k_\theta \in [\omega_\theta^{*+}, \overline{\omega})$ such that $r_\theta(k_\theta) = -R_{\theta'}(\underline{\omega}, \overline{\omega})$, then $x_\theta^*(\omega) = \mathbb{1}_{\omega \geq k_\theta}$ and $x_{\theta'}^*(\omega) = \chi_{\theta'}^{[\underline{\omega}, \overline{\omega}]}$ correspond to an optimal mechanism for $\chi_{\theta'}$ given by (18).*

To make use of Proposition 3 to examine specific examples, we strengthen the alignment condition (13) by imposing the monotone likelihood ratio property. (i) We assume that for some $\alpha \in (0, 1)$,

$$f_M(\omega) = (1 - \alpha) f_H(\omega) + \alpha f_L(\omega)$$

for all $\omega \in [\underline{\omega}, \overline{\omega}]$. This implies that $\alpha(\omega)$ given by (13) is constant and equal to α. (ii) We assume that $f_H(\omega)/f_L(\omega)$ is strictly increasing in ω. Then, so long as $f_M(\underline{\omega})/f_L(\underline{\omega}) > \alpha$,

[7] The argument that IC_{HL} is satisfied by the solution to the simplified problem depends on whether the solution has stochastic allocation for type M or for type L. In either case, it is the same as the argument in the proof of Proposition 2. The argument for IC_{MH} and IC_{LH} also depends. When x_L is stochastic, we can show that $k_M > c$; when x_M is stochastic, we can show that $\omega_M^{*-} > c$. In either case, (7) and (8) hold, and thus IC_{MH} and IC_{LH} are satisfied by the solution to simplified problem.

the implied $\{F_\theta\}_{\theta=H,M,L}$ is feasible. We refer to specifications of $\{F_\theta\}_{\theta=H,M,L}$ satisfying conditions (i) and (ii) above as "strong alignment."

Although the characterization results on optimal randomization so far all treat type M and type L symmetrically, under strong alignment we expect in any stochastic optimal mechanism, randomization is more likely to occur for type L than for type M, at least for sufficiently small c. To see this, note that under strong alignment, by condition (i) we have

$$R_M(w,w') = \frac{\int_w^{w'} \phi_M(\omega - c) f_M(\omega) d\omega}{\int_w^{w'} (F_L(\omega) - F_M(\omega)) d\omega} - \frac{\alpha \phi_H}{1-\alpha},$$
$$R_L(w,w') = \frac{\int_w^{w'} \phi_L(\omega - c) f_L(\omega) d\omega}{\int_w^{w'} (F_L(\omega) - F_M(\omega)) d\omega} - (\phi_M + \phi_H),$$

for any $w \le w'$. By condition (ii),

$$\frac{f_M(\omega)}{f_L(\omega)} < \frac{f_M(\hat{w})}{f_L(\hat{w})} < \frac{f_M(\omega')}{f_L(\omega')}$$

for any $\omega < \hat{w} < \omega'$. Thus, if $R_M(w,\hat{w}) > R_M(\hat{w},w')$ for some $c < w < \hat{w} < w'$, then $R_L(w,\hat{w}) > R_L(\hat{w},w')$. It follows that for sufficiently small c, whenever the sufficient condition (11) for randomization is satisfied for type M, it is also satisfied for type L. The following result shows that, in fact, for $c = \underline{\omega}$, whenever an optimal mechanism involves randomization for type M, it must also involve randomization for type L.

Proposition 4 *Suppose $c \le \underline{\omega}$. Under strong alignment, if no deterministic mechanism is optimal, then in any optimal mechanism randomization occurs for type L.*

Proof. Suppose that there is no deterministic mechanism that is optimal, but that there is an optimal mechanism with randomization for type M only. By strong alignment, there is no deterministic solution to the simplified problem, and there is a solution (x_M^*, x_L^*) where x_M^* is random but x_L^* is deterministic. By Lemma 9, we can assume that $x_M^*(\omega) = \chi_M^{[w_M^-, w_M^+]}$ and $x_L^*(\omega) = \mathbb{1}_{\omega \ge k_L}$. By Luenberger's Theorem, $\left(\chi_M^{[w_M^-, w_M^+]}, \mathbb{1}_{\omega \ge k_L}\right)$ maximizes (10) for some $\lambda \ge 0$ among all weakly increasing allocations for type M and type L.

First, we claim that $\lambda > 0$. Suppose instead $\lambda = 0$. Then, since $\chi_M \in (0,1)$, we have

$$\int_{w_M^-}^{w_M^+} \phi_M \delta_M(\omega) f_M(\omega) d\omega = 0.$$

It follows that replacing $x^*_M(\omega) = \chi_M^{[w_M^-, w_M^+]}$ with $\mathbb{1}_{\omega \geq w_M^-}$ does not change the value of the objective function in the simplified problem. Since the allocation for type M is weakly increased for all valuations, MON_{ML} remains satisfied. This contradicts the optimality of (x^*_M, x^*_L), and establishes that $\lambda > 0$. By complementary slackness, MON_{ML} binds. It follows that $k_L \in (w_M^-, w_M^+)$, and χ_M is given by equation (18).

Next, we claim that $R_M(w_M^-, k_L) \geq R_M(k_L, w_M^+)$. Suppose not. Then by replacing $x^*_M(\omega)$ with $\mathbb{1}_{\omega \geq k_L}$, we continue to bind MON_{ML}, and the total change in the objective function of the simplified problem (5) is given by

$$-\chi_M \int_{w_M^-}^{k_L} \phi_M \delta_M(\omega) f_M(\omega) d\omega + (1 - \chi_M) \int_{k_L}^{w_M^+} \phi_M \delta_M(\omega) f_M(\omega) d\omega,$$

which is strictly positive because $R_M(w_M^-, k_L) < R_M(k_L, w_M^+)$.

Under condition (ii) of strong alignment, since $c \leq \underline{\omega} \leq w_M^-$, we have that $R_M(w_M^-, k_L) \geq R_M(k_L, w_M^+)$ implies $R_L(w_M^-, k_L) > R_L(k_L, w_M^+)$. Then, by replacing $x^*_L(\omega)$ with $\chi_M(w_M^-, w_M^+)$, we continue to bind MON_{ML}, and the total change in the objective function of the simplified problem (5) is given by

$$\chi_M \int_{w_M^-}^{k_L} \phi_L \delta_L(\omega) f_L(\omega) d\omega - (1 - \chi_M) \int_{k_L}^{w_M^+} \phi_L \delta_L(\omega) f_L(\omega) d\omega,$$

which is strictly positive because $R_M(w_M^-, k_L) \geq R_M(k_L, w_M^+)$ implies that $R_L(w_M^-, k_L) > R_L(k_L, w_M^+)$. This contradicts the assumption that (x^*_M, x^*_L) is a solution to the simplified problem. ■

Proposition 4 does not rule out the possibility that there is an optimal mechanism with randomization for both type M and type L. By Lemma 10, in this case there is another optimal mechanism with randomization for at most one of the two types. Proposition 4 then implies that these other optimal mechanisms necessarily involve randomization for type L only.[8]

[8] In the proof of Lemma 10, we go through all four cases where a solution to the simplified problem involves randomization for both type M and type L. In cases (ii), (iii) and (iv), there is another optimal mechanism with randomization for type M only. By Proposition 4, these cases cannot happen under strong alignment and $c = \underline{\omega}$. It follows that case (i), with strict inequalities, is the only case when randomization for type M occurs.

We will use a class of examples with explicit distributions to illustrate how to apply our main characterization results Proposition 3 and Proposition 4. In the next section, we discuss how our methodology for finding optimal mechanisms can be extended to more than three types. The class of examples will be introduced in the section after the next to illustrate both the main model with three ex ante types and the extension in the next section to more than three types.

7 An Extension

In our analysis of stochastic sequential screening mechanisms, the simplified problem plays the central role. This problem is obtained from the original maximization problem by binding the lowest type's individual rationality constraint and each local downward incentive compatibility constraint to make non-decreasing allocations as choice variables, subject only to monotonicity constraints that are equivalent to local upward incentive compatibility constraints, dropping individual rationality constraints of all types higher than the lowest type and all non-local incentive compatibility constraints. As is standard under first order stochastic dominance ordering of ex ante types, by an inductive argument, individual rationality constraints of each typer higher than the lowest type is implied by the local downward incentive compatibility constraint and the lower type's individual rationality constraint. For all non-local incentive compatibility constraints, we impose the alignment condition to ensure they are satisfied by any solution to the simplified problem. With only three types, high, middle and low, alignment also allows us to drop the monotonicity constraint between the high type and the middle type, and thus dropping the high type's allocation from the simplified problem altogether.

The methodology of focusing on the simplified problem can be easily extended to more than three ex ante types. Let $\Theta = \{1, \ldots, I\}$ be the ex ante type space, with type 1 being the lowest type, ϕ_i being the fraction of type i, $f_i(\cdot)$ and $F_i(\cdot)$ being the conditional density and conditional distribution of valuations respectively, $i \in \Theta$. The counterpart of the alignment condition that $\alpha(\omega)$ given by (13) is constant, is that, for each $i = 1, \ldots, I$,

$$f_i(\omega) = (1 - \alpha_i) f_I(\omega) + \alpha_i f_1(\omega) \qquad (20)$$

for some $\alpha_i \in [0,1]$, with $1 = \alpha_1 > \alpha_2 > \ldots > \alpha_I = 0$. Proposition 4 extends immediately: under alignment, any non-local incentive compatibility constraint is implied by a chain of local ones in the same direction and a single monotonicity constraint. In particular, for all $i \geq j+2$, the downward incentive compatibility constraint $\mathrm{IC}_{i,j}$ is implied $\mathrm{IC}_{i,i-1}$, ... , $\mathrm{IC}_{j+1,j}$, and $\mathrm{IC}_{j,j+1}$, for all $i \leq j-2$, the upward incentive compatibility constraint $\mathrm{IC}_{i,j}$ is implied $\mathrm{IC}_{i,i+1}$, ... , $\mathrm{IC}_{j-1,j}$, and $\mathrm{IC}_{j,j-1}$. As is true with three ex ante types, under alignment we can focus on the following simplified problem:

$$\max_{\{x_i(\cdot)\}_{i\in\Theta}} \sum_{i\in\Theta} \int_{\underline{\omega}}^{\overline{\omega}} x_i(\omega)\phi_i\delta_i(\omega)f_i(\omega)d\omega,$$

where $\delta_i(\omega)$ is the dynamic virtual surplus function of type $i = 1,\ldots,I-1$, given by

$$\delta_i(\omega) = \omega - c - \frac{\sum_{i'=i+1}^{I} \phi_{i'}(F_i(\omega) - F_{i+1}(\omega))}{\phi_i f_i(\omega)},$$

with $\delta_I(\omega) = \omega - c$, subject to each $x_i(\cdot)$ non-decreasing with the range of $[0,1]$, and monotonicity constraint $\mathrm{MON}_{i+1,i}$

$$\int_{\underline{\omega}}^{\overline{\omega}} (x_{i+1}(\omega) - x_i(\omega))(F_i(\omega) - F_{i+1}(\omega))d\omega \geq 0,$$

for each $i = 1,\ldots,I-1$.

Under alignment, we can replace the "weighting function" $F_i(\omega) - F_{i+1}(\omega)$ for all $\mathrm{MON}_{i+1,i}$ with a single function $F_1(\omega) - F_I(\omega)$. That is, we can write $\mathrm{MON}_{i+1,i}$ as

$$\int_{\underline{\omega}}^{\overline{\omega}} (x_{i+1}(\omega) - x_i(\omega))(F_1(\omega) - F_I(\omega))d\omega \geq 0.$$

For the same reason, we define the average ratio of surplus-to-rent for type $i = 1,\ldots,I-1$ as

$$R_i(w,w') = \frac{\int_w^{w'} \phi_i\delta_i(\omega)f_i(\omega)d\omega}{\int_w^{w'} (F_1(\omega) - F_I(\omega))d\omega}$$

for all $w < w'$, the corresponding point ratio as

$$r_i(\omega) = \frac{\phi_i\delta_i(\omega)f_i(\omega)}{F_1(\omega) - F_I(\omega)}$$

for all ω.

Sufficient conditions for optimal mechanism to be stochastic are a straightforward extension of Proposition 8. Define $\{\hat{w}_i\}_{i=1,\dots,I}$ as a deterministic solution to the simplified problem:

$$\max_{\{w_i\}_{i=1,\dots,I}} \sum_{i=1}^{I} S_i(w_i),$$

where

$$S_i(w_i) = \int_{w_i}^{\overline{\omega}} \phi_i \delta_i(\omega) f_i(\omega) d\omega,$$

subject to that w_i is weakly decreasing in i. If

$$\max_{\omega \le \hat{w}_i} R_i(\omega, \hat{w}_i) > \min_{\omega \ge \hat{w}_i} R_i(\hat{w}_i, \omega),$$

for any $i = 1, \dots, I$, then the solution to the simplified problem is stochastic, and thus any optimal mechanism is stochastic. The argument extends the proof of Proposition 8. If there exist $w' < \hat{w}_i < w''$ for some i such that

$$R_i(w', \hat{w}_i) > R_i(\hat{w}_i, w''),$$

then by replacing $\mathbb{1}_{\omega \ge \hat{w}_i}$ with $\chi_{[w', w'']}$, where $\chi \in (0, 1)$ satisfies

$$\chi \int_{w'}^{\hat{w}_i} (F_1(\omega) - F_I(\omega)) d\omega = (1 - \chi) \int_{\hat{w}_i}^{w''} (F_1(\omega) - F_I(\omega)) d\omega,$$

both $\mathrm{MON}_{i+1,i}$ and $\mathrm{MON}_{i,i-1}$ are unaffected,[9] but the change in the value of type i part of the objective function is

$$\chi \int_{w'}^{\hat{w}_i} \phi_i \delta_i(\omega) f_i(\omega) d\omega - (1 - \chi) \int_{\hat{w}_i}^{w''} \phi_i \delta_i(\omega) f_i(\omega) d\omega,$$

which is strictly positive since $R_i(w', \hat{w}_i) > R_i(\hat{w}_i, w'')$.

Unlike in the three-type case, the sufficient conditions above are no longer necessary. This is because, with more than a single monotonicity constraint in the simplified problem, it is generally difficult to know which ones of them are binding at the deterministic solution to the simplified problem. Following the same steps of the proof of Proposition 7, let $\lambda_{i+1,i} \ge 0$ be the multiplier associated with $\mathrm{MON}_{i+1,i}$ in the simplified problem for each $i = 1, \dots, I-1$, and write the Lagrangian as

$$\sum_{i \in \Theta} \int_{\underline{\omega}}^{\overline{\omega}} x_i(\omega) \left(\phi_i \delta_i(\omega) f_i(\omega) + (\lambda_{i,i-1} - \lambda_{i+1,i})(F_1(\omega) - F_I(\omega)) \right) d\omega,$$

[9] Only $\mathrm{MON}_{2,1}$ is present if $i = 1$, and only $\mathrm{MON}_{I,I-1}$ is present if $i = I$.

with the convention of $\lambda_{1,0} = \lambda_{I+1,I} = 0$. With three types, we are able to show that MON_{HM} is equivalent to IC_{MH} and holds at any solution to the simplified problem (Proposition 4), and so we have a single multiplier λ_{ML}. In the proof of Proposition 7, this allows us to guess that $\lambda_{ML} = r_L(\hat{k})$ if the solution to the simplified problem is deterministic with common threshold $\hat{k}$ for types M and L, and establish an upper bound on the value of the simplified problem when the reverse of condition (11) holds. By Lagrangian relaxation, $\hat{k}$ indeed represents the solution to the simplified problem. With more than three types and more than one monotonicity constraint possibly binding if the solution to the simplified problem is deterministic, we can no longer guess the values of the multipliers at the solution. Nonetheless, conditional on these values of the multipliers $\hat{\lambda}_{i+1,i}$, $i = 1, \ldots, I-1$, using the same logic as in the proof of Proposition 7 we can write the sufficient conditions for the solution to the simplified problem to be deterministic, or equivalently, the necessary conditions for randomization, as

$$\max_{\omega \le \hat{w}_i} R_i(\omega, \hat{w}_i) \le \hat{\lambda}_{i+1,i} - \hat{\lambda}_{i,i-1} \le \min_{\omega \ge \hat{w}_i} R_i(\hat{w}_i, \omega),$$

for each $i = 1, \ldots, I$. Thus, to the extent that finding the deterministic solution to the simplified problem and corresponding multipliers is straightforward, the above conditions are not much harder to verify than in the three-type case.

With more than three types and more than a single monotonicity constraint, characterizing optimal stochastic mechanisms becomes more involved, but most of our characterization results generalize, at least partially, to provide restrictions we can use to construct optimal stochastic mechanisms. Lemma 9 continues to hold: randomization occurs at more than one level strictly between 0 and 1 for each type $i \in \Theta$, and so without loss we can write the solution to the simplified problem as $\left\{\chi_i^{[w_i^-, w_i^+]}\right\}_{i\in\Theta}$.[10] Of course, we do not expect Lemma 10 to hold, as generally randomization occurs for more than one type in any solution to the simplified problem; indeed we will construct such an example in the next section.[11]

Since they deal with necessary conditions for allocations of individual types, both Lemma

[10] The argument is simply noting that we can treat the difference in multipliers $\lambda_{i,i-1} - \lambda_{i+1,i}$ as a single multiplier in the proof of Lemma 9.

[11] The first part of the proof of Lemma 10 can be generalized: if at some solution $\{\chi_i^{[w_i^-, w_i^+]}\}_{i\in\Theta}$ there exist some $i_1, i_2 \in \Theta$ with $i_2 \ge i_1 + 1$ such that MON_{i_1,i_1-1} and MON_{i_2+1,i_2} are both slack, and $\lambda_{i+1,i} > 0$ for all

11 and Lemma 12 completely generalize. In particular, a generalization of Lemma 11 states that no solution to the simplified problem can have randomization for some type $i \in \Theta$ with a support a subset of an interval (w, w') over which type i's point ratio r_i of surplus-to-rent is strictly increasing; and in any solution type i's allocation is constant on any interval (w, w') over which r_i is strictly decreasing.[12] For Lemma 12, if a type $i = 1, \dots, I-1$ has a point ratio of surplus-to-rent function r_i that is single dipped, then there exist unique $\omega_i^{*-} < \omega_i^{*+}$ satisfying

$$r_i(\omega_i^{*-}) \geq R_i(\omega_i^{*-}, \omega_i^{*+}) \geq r_i(\omega_i^{*+}),$$

and $\omega_i^{*-} \geq \underline{\omega}$ and $\omega_\theta^{*+} \leq \overline{\omega}$, both with corresponding complementary slackness, such that, in any solution to the simplified problem, the support for randomization in the allocation for type i is $[\omega_i^{*-}, \omega_i^{*+}]$ if it is random, and the threshold k_i lies outside of $[\omega_i^{*-}, \omega_i^{*+}]$ if it is deterministic.

Extending Proposition 3 is difficult without additional information about the shape of each point ratio of surplus-to-rent r_i and the structure of binding monotonicity constraints, although the general idea of using Lagrangian relaxation to construct a solution to the simplified problem is applicable in specific examples. We can learn more about the structure of randomization in an optimal stochastic mechanism if we assume strong alignment. Suppose that (i) condition (20) holds, and (ii) $f_I(\omega)/f_1(\omega)$ is strictly increasing for all $\omega \in [\underline{\omega}, \overline{\omega}]$. We have

$$F_i(\omega) - F_{i+1}(\omega) = (\alpha_i - \alpha_{i+1})(F_1(\omega) - F_I(\omega))$$

for each $i = 1, \dots, I-1$, and thus

$$R_i(w_1, w_2) = \frac{\int_{w_1}^{w_2} \phi_i(\omega - c) f_i(\omega) d\omega}{\int_{w_1}^{w_2} (F_1(\omega) - F_I(\omega)) d\omega} - (\alpha_i - \alpha_{i+1}) \sum_{i'=i+1}^{I} \phi_{i'},$$

$i = i_1, \dots, i_2 - 1$, then the value of the simplified problem does not depend on χ_i, $i = i_1, \dots, i_2$. However, in general we no longer have the freedom to change the values of χ_i to reduce the number of random allocations between i_1 and i_2, because changing χ_i for any $i = i_1, \dots, i_2$ can violate $\text{MON}_{i+1,i}$ and/or $\text{MON}_{i,i-1}$.

[12] Even though the allocation of type i affects two monotonicity constraints $\text{MON}_{i+1,i}$ and $\text{MON}_{i,i-1}$ (if $i \geq 2$ and $i \leq I-1$), under alignment the weighting function $F_1(\omega) - F_I(\omega)$ is the same for all i. This implies that whenever we switch type i's allocation $x_i(\omega)$ from a random one to a deterministic one, or vice versa, so long as we keep as fixed the weight average of $x_i(\omega)$, neither of the two relevant monotonicity constraints is unaffected. The proof of Lemma 11 goes through without change.

for all $w_1 < w_2$, and

$$r_i(\omega) = \frac{\phi_i(\omega - c) f_i(\omega)}{F_1(\omega) - F_I(\omega)} - (\alpha_i - \alpha_{i+1}) \sum_{i'=i+1}^{I} \phi_{i'},$$

for all ω. Then, $r_i(\omega) \geq r_i(\omega')$ for any $\omega > \omega' > c$ implies that $r_{i'}(\omega) > r_{i'}(\omega')$ for all $i, i' \in \Theta$ with $i' \geq i + 1$. Under the assumption that $r_i(\omega)$ is single dipped for each $i = 1, \ldots, I$, with (ω_i^p, ω_i^t) being the largest interval over which $r_i(\omega)$ is strictly decreasing, if $c = \underline{\omega}$, then the intervals are all ordered by type, so that $\omega_{i'}^p \geq \omega_i^p$ with strict inequality if $\omega_i^p > \underline{\omega}$, and $\omega_{i'}^t \leq \omega_i^t$ with strict inequality if $\omega_i^t < \overline{\omega}$. Further, if $c = \underline{\omega}$, then $i' \geq i+1$ implies that either $\omega_{i'}^{*-} \geq \omega_i^{*-}$ or $\omega_{i'}^{*+} \leq \omega_i^{*+}$, with at least one holding as a strict inequality.[13] These results can help us make the correct guesses about the values of the multipliers in order to apply the argument of Proposition 3. This will be illustrated in the next section with the class of examples with explicit distribution functions.

Under strong alignment with $c = \underline{\omega}$, the argument in Proposition 4 can be extended to more than three types. We can show that randomization for any type $i = 2, \ldots, I$ at an optimal mechanism implies we cannot have both a deterministic allocation for type $i - 1$ and a binding $\text{MON}_{i,i-1}$. This suggests that in optimal stochastic stochastic mechanisms randomization occurs in "clusters," where each cluster of adjacent types has binding monotonicity constraints among them, and clusters are separated from each other. In the next section, we will use a class of examples with explicit distributions to illustrate this idea.

[13]To see this, assume for simplicity $r_i(\omega_i^{*-}) = R_i(\omega_i^{*-}, \omega_i^{*+}) = r_i(\omega_i^{*+})$. This implies that $\omega_{i'}^p > \omega_i^p$ and $\omega_{i'}^t < \omega_i^t$ are all interior. By Lemma 12, there exists $\hat{w} \in (\omega_i^p, \omega_i^t) \subset (\omega_i^{*-}, \omega_i^{*+})$ such that $r_i(\hat{w}) = R_i(\omega_i^{*-}, \omega_i^{*+})$. If $r_{i'}(\hat{w}) < r_{i'}(\omega_{i'}^t)$ then $\omega_{i'}^{*-} > \hat{w} > \omega_i^p > \omega_i^{*-}$, and if $r_{i'}(\hat{w}) > r_{i'}(\omega_{i'}^p)$ then $\omega_{i'}^{*+} < \hat{w} < \omega_i^t < \omega_i^{*+}$. Suppose $r_{i'}(\omega_{i'}^t) \leq r_{i'}(\hat{w}) \leq r_{i'}(\omega_{i'}^p)$. By condition (ii) of strong alignment with $c = \underline{\omega}$, $r_i(\omega_i^{*-}) < r_i(\hat{w}) < r_i(\omega_i^{*+})$ implies $r_{i'}(\omega_i^{*-}) < r_{i'}(\hat{w}) < r_{i'}(\omega_i^{*+})$. Then there exist $z^p \in (\omega_i^{*-}, \omega_{i'}^p]$, and $\check{w} \in [\omega_{i'}^p, \omega_{i'}^t]$ and $z^t \in [\omega_{i'}^t, \omega_i^{*+})$ such that $r_{i'}(z^p) = r_{i'}(\check{w}) = r_{i'}(z^t)$, with either $R_{i'}(z^p, z^t) \geq r_{i'}(z^p)$, or $R_{i'}(z^p, z^t) \leq r_{i'}(z^t)$, or both. By Lemma 12, in the first case $\omega_{i'}^{*-} \geq z^p > \omega_i^{*-}$; in the second case $\omega_{i'}^{*+} \leq z^t < \omega_i^{*+}$.

8 Examples

In this section, we explicitly solve for optimal mechanisms for a class of sequential screening problems. This class of problems satisfies conditions (i) and (ii) of strong alignment. We use them illustrate the results from both the main model with three ex ante types in Section 6 and the extension with more than three types in Section 7. Since the model in Section 6 is a special case of the model in Section 7, we use the latter, and specialize to three types when necessary. In addition to illustrating how to characterize optimal mechanisms, the examples solved in this section also demonstrate an often more convenient alternative to verifying the sufficient and necessary conditions (11) for randomization in Propositions 8 and 7.

For all $\omega \in [0, \infty)$, let

$$f_i(\omega) = \gamma_i e^{-\gamma_i \omega},$$

for $i = 1$ and $i = I \geq 3$, with $\gamma_1 > \gamma_I > 0$, and for each $i = 1, \ldots, I$ let

$$f_i(\omega) = (1 - \alpha_i)\gamma_I e^{-\gamma_I \omega} + \alpha_i \gamma_1 e^{-\gamma_1 \omega}$$

for some $\alpha_i \in [0, 1]$, with $1 = \alpha_1 > \alpha_2 > \ldots > \alpha_I = 0$. The resulting class of distributions $\{F_i(\omega)\}_{i=1,\ldots,I}$ satisfies conditions (i) and (ii) of strong alignment. We have $\delta_I(\omega) = \omega - c$, and for each $i = 1, \ldots, I - 1$,

$$\delta_i(\omega) = \omega - c - \frac{(\alpha_i - \alpha_{i+1}) \left(e^{-\gamma_I \omega} - e^{-\gamma_1 \omega}\right) \sum_{i'=i+1}^{I} \phi_{i'}}{\left((1 - \alpha_i)\gamma_I e^{-\gamma_I \omega} + \alpha_i \gamma_1 e^{-\gamma_1 \omega}\right) \phi_i},$$

and

$$r_i(\omega) = \frac{\phi_i(\omega - c) \left((1 - \alpha_i)\gamma_I e^{-\gamma_I \omega} + \alpha_i \gamma_1 e^{-\gamma_1 \omega}\right)}{e^{-\gamma_I \omega} - e^{-\gamma_1 \omega}} - (\alpha_i - \alpha_{i+1}) \sum_{i'=i+1}^{I} \phi_{i'}.$$

It is straightforward to verify that $r_i(0) = -\infty$ and $dr_i(0)/d\omega = \infty$ if $c > 0$, and if $c = 0$,

$$r_i(0) = \frac{\phi_i((1 - \alpha_i)\gamma_I + \alpha_i \gamma_1)}{\gamma_1 - \gamma_I} - (\alpha_i - \alpha_{i+1}) \sum_{i'=i+1}^{I} \phi_{i'}$$

and

$$\frac{dr_i(0)}{d\omega} = \frac{\phi_i((1 - \alpha_i)\gamma_I - \alpha_i \gamma_1)}{2}.$$

Also, $r_1(\infty) = -(1 - \alpha_2)(1 - \phi_1)$, and $r_i(\infty) = \infty$ for $i = 2, \ldots, I - 1$. We first derive two claims we need for explicit characterizations of optimal mechanisms. The proofs are in the appendix.

Claim 2 *For each $i = 1, \ldots, I-1$, $r_i(\omega)$ is single dipped. Further, if $c = 0$, then $r_1(\omega)$ is strictly decreasing, and for any $i = 2, \ldots, I-1$ there exists a strictly positive and finite ω_i^t such that $r_i(\omega)$ is strictly decreasing for any $\omega < \omega_i^t$ and strictly increasing for any $\omega > \omega_i^t$. If $c > 0$, then there exists a strictly positive and finite ω_1^p such that $r_1(\omega)$ is strictly increasing for any $\omega < \omega_1^p$ and strictly decreasing for any $\omega > \omega_1^p$, and $r_i(\omega)$ is strictly increasing in ω if $(1-\alpha_i)\gamma_I \geq \alpha_i \gamma_1$.*

For each $i = 1, \ldots, I-1$, the total dynamic virtual surplus of type i under a threshold allocation rule $\mathbb{1}_{\omega \geq k}$ is given by

$$S_i(k) = \int_k^\infty r_i(\omega) \left(e^{-\gamma_I \omega} - e^{-\gamma_1 \omega} \right) d\omega.$$

The following claim provides a characterization of $S_i(k)$. Let $\hat{k}_i$ be the smallest maximizer of $S_i(k)$, $i = 1, \ldots, I-1$.

Claim 3 *If $c = 0$, then $\hat{k}_1 = 0$ when $S_1(0) \geq 0$ and $\hat{k}_1 = \infty$ otherwise, and for each $i = 2, \ldots, I-1$, $\hat{k}_i$ is uniquely defined by $r_i(\hat{k}_i) = 0$ and $dr_i(\hat{k}_i)/d\omega > 0$ when $S_i(\hat{k}_i) \geq S_i(0)$, and $\hat{k}_i = 0$ otherwise. If $c > 0$, then $\hat{k}_1 = \infty$ when $r_1(\omega_1^p) \leq 0$, and is otherwise uniquely defined by $r_1(\hat{k}_1) = 0$ and $dr_1(\hat{k}_1)/d\omega > 0$, and $\hat{k}_i$ is uniquely defined by $r_i(\hat{k}_i) = 0$ for any i such that $(1-\alpha_i)\gamma_I \geq \alpha_i \gamma_1$.*

Now we are ready to illustrate explicitly constructed optimal mechanisms through a series of examples. For the first two examples, we have $I = 3$. We revert back to the notation of H, M and L. So type I becomes type H, and type 1 becomes type L, with $\alpha \in (0,1)$ representing the weight on f_L in f_M. The first example provides a straightforward application of part (i) of Proposition 3.

Example 1: $I = 3$ and $c > 0$. We assume $(1-\alpha)\gamma_H \geq \alpha\gamma_L$. By Claim 3, $\hat{k}_M$ is interior and given by $r_M(\hat{k}_M) = 0$, and $\hat{k}_L$ is given by $r_L(\hat{k}_L) \leq 0$ and $\hat{k}_L \leq \infty$, with complementary slackness.

First, suppose that $\hat{k}_M \leq \hat{k}_L$. By Lemma 6, the optimal mechanism is deterministic, with threshold allocation for all three types: the threshold for type H is c, the threshold for type M is $\hat{k}_M$, and the threshold for type L is $\hat{k}_L$. This corresponds to the regular case that the existing literature focuses on.

Second, suppose instead $\hat{k}_M > \hat{k}_L$. This requires $\hat{k}_L < \infty$ and thus $r_L(\omega_L^p) > 0$, where ω_L^p is the unique interior peak of r_L by Claim 2. The deterministic solution $\hat{k}$ to the simplified problem is uniquely determined by $r_L(\hat{k}) + r_M(\hat{k}) = 0$, and is strictly between $\hat{k}_L$ and $\hat{k}_M$. By Claim 2, r_M is strictly increasing because $(1-\alpha)\gamma_H \geq \alpha\gamma_L$. Lemma 11 then implies that, if there is a stochastic solution to the simplified problem then randomization occurs only for type L. By Claim 2, $r_L(\omega)$ has a unique interior peak at ω_L^p with $r_L(0) = -\infty$ and $r_L(\infty) = 0$. By Lemma 12, in any stochastic solution (x_M^*, x_L^*) to the simplified problem, equations (19) imply the support of type L's random allocation $x_L^*(\omega)$ is given by $[\omega_L^{*-}, \infty)$, with ω_L^{*-} uniquely defined by

$$R_L(\omega_L^{*-}, \infty) = r_L(\omega_L^{*-}),$$

implying that $r_L(\omega_L^{*-}) > 0$ and so $\omega_L^{*-} \in (\hat{k}_L, \omega_L^p)$. Part (i) of Proposition 3, with $\theta = L$ and $\theta' = M$, then establishes that if there exists $k_M > \omega_L^{*-}$ such that[14]

$$r_M(k_M) = -R_L(\omega_L^{*-}, \infty) = -r_L(\omega_L^{*-}),$$

then $x_L^*(\omega) = \chi_L^{[\omega_L^{*-}, \infty)}$ and $x_M^*(\omega) = \mathbb{1}_{\omega \geq k_M}$ solve the simplified problem, with

$$\chi_L = \frac{\int_{k_M}^{\infty}(F_L(\omega) - F_M(\omega))d\omega}{\int_{\omega_L^{*-}}^{\infty}(F_L(\omega) - F_M(\omega))d\omega},$$

and thus corresponds to an optimal stochastic mechanism.[15] Since r_M is strictly increasing, such k_M exists if and only if

$$r_L(\omega_L^{*-}) + r_M(\omega_L^{*-}) < 0.$$

If the above condition is violated, there is no stochastic solution to the simplified problem. The solution is deterministic with a common threshold $\hat{k}$, and the optimal mechanism is deterministic. We have an example of optimal mechanism that is deterministic even though the unconstrained solution to the simplified problem violates MON_{ML}. ■

[14]Consistent with Lemma 6, the condition below cannot be satisfied if $\hat{k}_M \leq \hat{k}_L$. To see this, note that by Claim 2, $r_L(\omega)$ crosses 0 only once at $\hat{k}_L$ from below. Since $r_L(\omega_L^{*-}) > 0$, we have $\omega_L^{*-} > \hat{k}_L$ and thus $r_M(\omega_L^{*-}) > r_M(\hat{k}_L) \geq r_M(\hat{k}_M) = 0$. As a result, $r_L(\omega_L^{*-}) + r_M(\omega_L^{*-}) > 0$.

[15]Since $\omega_L^{*-} \in (\hat{k}_L, \omega_L^p)$ and since r_M is strictly increasing, $r_L(\omega_L^{*-}) + r_M(\omega_L^{*-}) < 0$ implies that $\omega_L^{*-} < \hat{k}$, where $\hat{k}$ satisfies $r_L(\hat{k}) + r_M(\hat{k}) = 0$. The proof of Lemma 12 establishes that $r_L(\omega) \geq R_L(\omega, \infty)$ if and only $\omega \geq \omega_L^{*-}$. Thus, when $r_L(\omega_L^{*-}) + r_M(\omega_L^{*-}) < 0$, we have $r_L(\hat{k}) > R_L(\hat{k}, \infty)$, and the sufficient condition for randomization (11) in Proposition 8 is satisfied for type L.

Our second example assumes $c = 0$ and uses Proposition 4 and Lemma 12 to pin down a unique candidate solution to the simplified problem when the unconstrained solution violates MON_{ML}. We then apply part (ii) of Proposition 3 to establish a sufficient condition to validate the candidate solution and thus correspond to an optimal stochastic mechanism.

Example 2: $I = 3$ and $c = 0$. By Claim 3, we have $\hat{k}_L = 0$ if $S_L(0) \geq 0$, and otherwise $\hat{k}_L = \infty$. For type M, by Claim 2, there is a unique minimizer ω_M^t of $r_M(\omega)$. By Claim 3, a sufficient condition for $\hat{k}_M$ to be interior is $r_M(0) < 0$.

First, suppose that $\hat{k}_L = \infty$, or $\hat{k}_L = \hat{k}_M = 0$. By Lemma 6, the optimal mechanism is deterministic, with threshold allocation for all three types: the threshold for type H is 0, the threshold for type M is $\hat{k}_M$, and the threshold for type L is $\hat{k}_L$.

Second, suppose that $\hat{k}_L = 0$ and $\hat{k}_M > 0$. If $\hat{k} > 0$, then since $r_L(\omega)$ is strictly decreasing by Claim 2, Proposition 8 implies that any solution to the simplified problem is stochastic. If $\hat{k} = 0$, Proposition 8 does not apply, and the solution to the simplified problem may be stochastic, or deterministic given by $x_M^*(\omega) = x_L^*(\omega) = \mathbb{1}_{\omega \geq 0}$. By Proposition 4, if randomization occurs in any optimal mechanism, it occurs for type L and takes the form of $x_L^*(\omega) = \chi_L^{[w_L^-, w_L^+]}$ and $x_M^*(\omega) = \mathbb{1}_{\omega \geq k_M}$. By Lemma 12, since $r_L(\omega)$ is strictly decreasing, equations (19) imply that $w_L^- = \omega_L^{*-} = 0$ and $w_L^+ = \omega_L^{*+} = \infty$. Further, since r_M has a unique interior trough and $r_M(\infty) = \infty$, we have $\omega_M^{*-} = 0$ and ω_M^{*+} is uniquely defined by

$$r_M(\omega_M^{*+}) = R_M(0, \omega_M^{*+}).$$

Since $\hat{k}_L = 0$, we have $S_L(0) \geq S_L(\infty) = 0$, and thus $R_L(0, \infty) \geq 0$. By part (ii) of Proposition 3, with $\theta = M$ and $\theta' = L$, if there exists $k_M \geq \omega_M^{*+}$ such that[16]

$$r_M(k_M) = -R_L(0, \infty),$$

then $x_L^*(\omega) = \chi_L^{[0,\infty)}$ and $x_M^*(\omega) = \mathbb{1}_{\omega \geq k_M}$ solve the simplified problem, with

$$\chi_L = \frac{\int_{k_M}^{\infty} (F_L(\omega) - F_M(\omega)) d\omega}{\int_0^{\infty} (F_L(\omega) - F_M(\omega)) d\omega},$$

[16] If $\hat{k}_M = 0$, then $S_M(0) \geq S_M(\omega)$ for all ω. This implies that $R_M(0, \omega) \geq 0$ for all ω, and in particular, $r_M(\omega_M^{*+}) = R_M(0, \omega_M^{*+}) \geq 0$. If $\hat{k}_L = 0$, we also have $R_L(0, \infty) \geq 0$. Thus, consistent with Lemma 6, the condition $R_L(0, \infty) + r_M(\omega_M^{*+}) < 0$ can never be satisfied if $\hat{k}_M = \hat{k}_L = 0$.

and thus corresponds to an optimal stochastic mechanism. Since $r_M(\omega)$ is strictly increasing for $\omega > \omega_M^{*+}$, the above condition is equivalent to[17]

$$R_L(0,\infty) + r_M(\omega_M^{*+}) \leq 0.$$

If $R_L(0,\infty) + r_M(\omega_M^{*+}) > 0$, there is no stochastic solution to the simplified problem, and deterministic allocations $x_M^*(\omega) = x_L^*(\omega) = \mathbb{1}_{\omega \geq 0}$ correspond to an optimal mechanism. ■

The third example below illustrates what we call randomization clusters with $I = 4$ and $c = 0$. We construct an optimal mechanism where types 1 and 2 have random allocations while types 3 and 4 have deterministic allocations. To do so, we first use Lemma 12 to propose the unique candidate solution to the simplified problem that is consistent with this randomization cluster. We then apply the same Lagrangian relaxation method used in Proposition 3 to establish a sufficient condition for the candidate solution to solve the simplified problem and thus correspond to optimal stochastic mechanisms.

Example 3: $I = 4$ and $c = 0$. We consider a solution $\{x_i^*(\omega)\}_{i=1,2,3}$ to the simplified problem of the form $x_i^*(\omega) = \chi_i^{[w_i^-, w_i^+]}$ for $i = 1, 2$, and $x_3^*(\omega) = \mathbb{1}_{\omega \geq k_3}$. By Claim 2, $r_1(\omega)$ is strictly decreasing, and both $r_2(\omega)$ and $r_3(\omega)$ have a unique interior trough. It follows from Lemma 12 that $w_1^- = \omega_1^{*-} = 0$ and $w_1^+ = \omega_1^{*+} = \infty$, $w_2^- = \omega_2^{*-} = 0$ and $w_2^+ = \omega_2^{*+}$, and $k_3 \geq \omega_3^{*+}$ with $\omega_3^{*-} = 0$, where ω_i^{*+} is uniquely defined by

$$r_i(\omega_i^{*+}) = R_i(0, \omega_i^{*+})$$

for each $i = 2, 3$. As we have argued in Section 7, since $\omega_2^{*-} = \omega_3^{*-}$, we have $\omega_2^{*+} > \omega_3^{*+}$. We claim that if $R_1(0,\infty) \geq 0$, $R_1(0,\infty) + r_2(\omega_2^{*+}) \geq 0$, and

$$-r_3(\omega_2^{*+}) < R_1(0,\infty) + r_2(\omega_2^{*+}) \leq -r_3(\omega_3^{*+}),$$

[17] Sufficient conditions for optimal mechanisms to be stochastic are $R_L(0,\infty) > 0$ and $r_L(0) + r_M(0) < 0$. Since $R_L(0,\infty) > 0$ we have $S_L(0) > 0$ and thus $\hat{k}_L = 0$ by Claim 3. It is straightforward to verify that $R_L(0,\infty) > 0$ implies that $r_L(0) > 0$. Since $r_L(0) + r_M(0) < 0$, we have $r_M(0) < 0$, which is sufficient for $\hat{k}_M > 0$. Since $r_L(\omega)$ is strictly decreasing, and since $r_M(\omega_M^{*+}) < r_M(0)$, we have $R_L(0,\infty) + r_M(\omega_M^{*+}) < r_L(0) + r_M(0) < 0$. Indeed, $r_L(0) + r_M(0) < 0$ is sufficient for $\hat{k}$ to be interior, as it implies that $dS_L(k)/dk + dS_M(k)/dk$ is strictly positive for k arbitrarily close to 0. Since $r_L(\omega)$ is strictly decreasing, condition (11) is satisfied for type L, and by Proposition 8, any optimal mechanism is stochastic.

then there exists a unique value of k_3, together with some $\chi_1, \chi_2 \in (0,1)$, such that $x_1^*(\omega) = \chi_1^{[0,\infty)}$, $x_2^*(\omega) = \chi_2^{[0,\omega_2^{*+}]}$, and $x_3^*(\omega) = \mathbb{1}_{\omega \geq k_3}$ form a solution to the simplified problem, and thus correspond to an optimal mechanism.

The claim is established by a generalization of the Lagrangian relaxation argument in Proposition 3. Since $r_3(\omega)$ is strictly increasing for $\omega \geq \omega_3^{*+}$ with $r_3(\infty) = \infty$, under the stated conditions there exists a unique $k_3 \in [\omega_3^{*+}, \omega_2^{*+})$ such that

$$R_1(0,\infty) + r_2(\omega_2^{*+}) + r_3(k_3) = 0.$$

We choose the multipliers as follows: $\lambda_{2,1} = R_1(0,\infty)$ and $\lambda_{3,2} = R_1(0,\infty) + r_2(\omega_2^{*+})$. By assumption, $\lambda_{2,1}, \lambda_{3,2} \geq 0$. With these values of the multipliers, we argue that for each type $i = 1,2,3$, the given allocation $x_i^*(\omega)$ maximizes the part of the Lagrangian function associated with type i among all weakly increasing functions $x_i(\omega)$ with the range of $[0,1]$. For type 1, given by that $\lambda_{2,1} = R_1(0,\infty)$, the argument is the same as for type θ' in part (ii) of Proposition 3. For type 2, given that $R_2(0,\omega_2^{*+}) = r_2(\omega_2^{*+}) = \lambda_{3,2} - \lambda_{2,1}$, the argument is the same as for type θ in part (i) of Proposition 3. Finally, for type 3, given that $\lambda_{3,2} = R_1(0,\infty) + r_2(\omega_2^{*+}) = -r_3(k_3)$ and $k_3 \geq \omega_3^{*+}$, the argument is the same for type θ in part (ii) of Proposition 3. The claim is then established by noting that since $k_3 < \omega_2^{*+}$, we can find values of χ_1 and χ_2 to bind $\mathrm{MON}_{1,2}$ and $\mathrm{MON}_{3,2}$:

$$\chi_1 = \frac{\int_{k_3}^{\infty}(F_1(\omega) - F_4(\omega))d\omega}{\int_0^{\infty}(F_1(\omega) - F_4(\omega))d\omega}, \ \chi_2 = \frac{\int_{k_3}^{\omega_2^{*+}}(F_1(\omega) - F_4(\omega))d\omega}{\int_0^{\omega_2^{*+}}(F_1(\omega) - F_4(\omega))d\omega}.$$

By complementary slackness, the value of the Lagrangian function achieved by the proposed solution $x_1^*(\omega) = \chi_1^{[0,\infty)}$, $x_2^*(\omega) = \chi_2^{[0,\omega_2^{*+}]}$, and $x_3^*(\omega) = \mathbb{1}_{\omega \geq k_3}$ is feasible in the simplified problem. It follows that the proposed solution solves the simplified problem, and thus corresponds to an optimal mechanism. ■

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Appendix

Proof of Claim 2. By taking derivatives, we can show that $dr_i(\omega)/d\omega$ has the same sign as

$$(1-\alpha_i)\gamma_I\left(e^{(\gamma_1-\gamma_I)\omega}-1\right)+\alpha_i\gamma_1\left(1-e^{-(\gamma_1-\gamma_I)\omega}\right)-(\gamma_1-\gamma_I)((1-\alpha_i)\gamma_I+\alpha_i\gamma_1)(\omega-c).$$

Thus, $dr_1(\omega)/d\omega>0$ if and only if

$$1-e^{-(\gamma_1-\gamma_I)\omega}>(\gamma_1-\gamma_I)(\omega-c).$$

The left-hand side is strictly concave in ω, with a derivative equal to $\gamma_1-\gamma_I$ at $\omega=0$. It follows that if $c=0$, then $dr_1(\omega)/d\omega<0$ for all ω, and if $c>0$, there exists a strictly positive and finite ω_1^p which equates the two sides of the inequality above, such that $r_1(\omega)$ is strictly increasing for any $\omega<\omega_1^p$ and strictly decreasing for any $\omega>\omega_1^p$.

Next, fix any $i = 2, \ldots, I-1$. At any $\hat{\omega}$ such that $dr_i(\hat{\omega})/d\omega = 0$, the sign $d^2 r_i(\hat{\omega})/d\omega^2$ is the same as

$$(1-\alpha_i)\gamma_I e^{(\gamma_1-\gamma_I)\hat{\omega}} + \alpha_i\gamma_1 e^{-(\gamma_1-\gamma_I)\hat{\omega}} - ((1-\alpha_i)\gamma_I + \alpha_i\gamma_1).$$

The sign of the above is the same as

$$(1-\alpha_i)\gamma_I e^{(\gamma_1-\gamma_I)\hat{\omega}} - \alpha_i\gamma_1.$$

Thus, the sign of $d^2 r_i(\hat{\omega})/d\omega^2$ at any $\hat{\omega}$ such that $dr_i(\hat{\omega})/d\omega = 0$ can only change from negative to positive. It follows that $r_i(\omega)$ is single dipped. If $c = 0$, then since $dr_i(0)/d\omega < 0$ and $r_i(\infty) = \infty$, and since $r_i(\omega)$ is single dipped, $r_i(\omega)$ has a unique interior trough. If $c > 0$, then $(1-\alpha_i)\gamma_I \geq \alpha_i\gamma_1$ implies that $d^2 r_i(\hat{\omega})/d\omega^2 > 0$ at any $\hat{\omega}$ such that $dr_i(\hat{\omega})/d\omega = 0$. As a result, $\hat{\omega}$ is a local minimum of $r_i(\omega)$. Since $dr_i(0)/d\omega = \infty$, and since $r_i(\omega)$ is single dipped, it cannot have a local minimum without having a local maximum. This is a contradiction, and it follows there is no $\hat{\omega}$ such that $dr_i(\hat{\omega})/d\omega = 0$ when $(1-\alpha_i)\gamma_I \geq \alpha_i\gamma_1$. Thus, $r_i(\omega)$ is strictly increasing in ω. ■

Proof of Claim 3. We have that $dS_i(k)/dk$ has the same sign as $-r_i(k)$. At any $\hat{\omega}$ such that $dS_i(\hat{\omega})/d\omega = 0$, the sign of $d^2 S_i(\hat{\omega})/dk^2$ is the same as $-dr_i(\hat{\omega})/dk$.

Suppose that $c = 0$. By Claim 2, since $r_1(\omega)$ is strictly decreasing, $S_1(k)$ has no interior local maximum. It follows that $S_1(k)$ is maximized at either $\hat{k}_1 = 0$ or $\hat{k}_1 = \infty$. Since $S_1(\infty) = 0$, the maximum is either attained at $\hat{k}_1 = 0$ if $S_1(0) \geq 0$, or else at $\hat{k}_1 = \infty$. For any $i = 2, \ldots, I-1$, by Claim 2, since $r_i(\omega)$ has a unique interior trough at ω_i^t, there are three cases. If $r_i(\omega_i^t) \geq 0$, then $S_i(k)$ is strictly decreasing for all k. The maximum of $S_i(k)$ is reached at $\hat{k}_i = 0$. If $r_i(0) < 0$, then since $dr_i(0)/d\omega < 0$ and $r_i(\infty) = \infty$, there exists a unique $\hat{w}$ strictly positive and finite, satisfying $r_i(\hat{w}) = 0$ with $dr_i(\hat{w})/d\omega > 0$, such that $S_i(k)$ is strictly increasing for all $k \in (0, \hat{w})$ and strictly decreasing for all $k > \hat{w}$. The maximum of $S_i(k)$ is reached at $\hat{k}_i = \hat{w}$. If $r_i(\omega_i^t) < 0 \leq r_i(0)$, then there is a unique $\hat{w} > \omega_i^t$ such that $r_i(\hat{w}) = 0$, with $dr_i(\hat{w})/d\omega > 0$. In this case $\hat{w}$ is a local maximizer of $S_i(k)$. The maximum of $S_i(k)$ is reached at $\hat{k}_i = \hat{w}$ if $S_i(\hat{w}) \geq S_i(0)$ and otherwise at $\hat{k}_i = 0$.

Suppose that $c > 0$. By Claim 2, $r_1(\omega)$ has a unique interior peak at some ω_1^p. If $r_1(\omega_1^p) \leq 0$, then $S_1(k)$ is increasing for all k, and is therefore maximized at $\hat{k}_1 = \infty$.

Otherwise, by Claim 2 there exists a unique $\hat{w}$ such that $r_1(\hat{w}) = 0$ and $dr_1(\hat{w})/d\omega > 0$. It follows that $S_1(k)$ is maximized at $\hat{k}_1 = \hat{w}$. For any $i = 2, \ldots, I-1$, by Claim 2, $r_1(\omega)$ is strictly increasing in ω when $(1-\alpha_i)\gamma_I \geq \alpha_i \gamma_1$. Since $r_i(0) = -\infty$ and $r_i(\infty) = \infty$, there exists a unique $\hat{w}$ such that $r_i(\hat{w}) = 0$. It follows that $S_i(k)$ is maximized at $\hat{k}_i = \hat{w}$. ■