Persuading Communicating Voters

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Abstract

We study a multiple-receiver Bayesian persuasion model in which the sender wants to implement a proposal and commits to a communication strategy which sends correlated messages to multiple receivers who have homogeneous beliefs and vote sincerely. Receivers are connected in a network and can perfectly observe their direct neighbors' messages. After updating their beliefs accordingly, receivers vote for or against the proposal. We characterize optimal communication on various network structures and find that the limited information spillovers in the model often do not prevent the sender from attaining maximum gain from persuasion. Our results highlight the importance of the network structure when designing optimal strategies, as voters are not necessarily better off with strictly more information. Surprisingly, the creation of new links may even benefit the sender.

Keywords: Bayesian Persuasion; Networks; Spillovers; Information Design; Voting
JEL Classification: C72, D72, D82, D85

1 Introduction

Multiple-receiver Bayesian persuasion models with private communication often assume that receivers do not exchange information with each other between receiving

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signals from the sender and taking their action. In reality, however, people usually deliberate before voting or simply before buying a product, and might consult friends and acquaintances in search of additional opinions and information. We model such communication among receivers prior to making a decision with a simple setup: receivers are in a fixed network and neighbors can observe each other's private messages. An application of such communication are social networks like Facebook or Twitter, where parties can target political adverts at specific (potential) voter groups. If a person likes or shares an ad or a video on Twitter for example, it is visible to all of their followers. When parties share information via Twitter, they are aware that this will (at least to some extent) spread through the network of their followers.[1]

Similar persuasion/voting situations also occur on a smaller scale. Non-profit organizations (such as UNESCO, Red Cross, Special Olympics) usually employ a CEO (who often is not a voting member) and a board of directors, who share decision-making responsibilities.[2] It is common in such organizations for the CEO (or for another board member) to make a proposal that is put to an internal vote.[3] If the CEO wishes to pass a particular proposal, she must also consider how the board members share the information she has provided with each other.

Incorporating a communication network complicates the sender's problem of optimal persuasion significantly, as she must also take into account the intricacies of the information flow between receivers when deciding how to design her communication strategy. An immediate question that arises is whether giving more information to the receivers would always make the sender worse off. Alternatively, can the sender actually benefit from greater information sharing between the voters?

1.1 Illustrative Example

Suppose that a non-profit organization consists of a CEO and an executive board with three members, M_1, M_2, and M_3. The CEO realizes that there is a surplus in the budget and wishes to hire a new executive, who is either high quality (H) or low quality (L). Two approval votes are required for a hiring. Board members initially believe that the executive is high quality with probability $1/3$ and they approve the hire if they believe with probability at least $1/2$ that the executive is high quality. The CEO prepares three reports about the quality of the executive, two of which always favor approval while the third one presents the true findings. The CEO randomly assigns the reports among the board members.

[1] Several papers study the use of social media to spread fake news; see Allcott and Gentzkow (2017), Grinberg et al. (2019), and Zhuravskaya et al. (2020).

[2] Brickley et al. (2010) estimates that roughly half of the U.S. hospitals do not include CEOs as voting members of the board. One reason for such practices is provided by Ostrower et al. (2007), which notes that having CEOs in the board creates a conflict of interest.

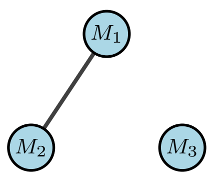
[3] The voting rule employed is usually simple majority; see UNESCO's website for an example: https://en.unesco.org/executiveboard/inbrief.

First, assume that the board members do not communicate with each other, i.e. they are in the empty network. The *communication strategy* of the CEO can be formalized by distributions $\pi(\cdot|H)$ and $\pi(\cdot|L)$ on some set of signals. Let (h, h, ℓ) denote the *signal* in which M_1 and M_2 receive *message* h (high quality) and M_3 receives *message* ℓ (low quality). While the chosen π is known by the board members, under private communication they only observe their own message. Messages h and ℓ can be interpreted as recommendations to hire and to not hire, respectively. A private communication strategy for the CEO, π, is given in the following table.

π	H	L
(h, h, h)	1	0
(h, h, ℓ)	0	$\frac{1}{4}$
(h, ℓ, h)	0	$\frac{1}{4}$
(ℓ, h, h)	0	$\frac{1}{4}$
(ℓ, ℓ, ℓ)	0	$\frac{1}{4}$

After observing h, a board member's belief that the executive is high quality is $(1/3 \cdot 1)/(1/3 \cdot 1 + 2/3 \cdot 1/2) = 1/2$. Hence, after all realizations except (ℓ, ℓ, ℓ) at least two board members approve the hire. Thus, by employing π the CEO can increase the probability of hiring the executive from the initial 0 to 5/6 (the *value* of π).

Now, assume that M_1 and M_2 communicate and *exchange* the information from the reports *before* making their decisions as shown below.



π^p	H	L
(h, h, h)	1	$\frac{1}{2}$
(ℓ, ℓ, ℓ)	0	$\frac{1}{2}$

Communication strategy π is no longer optimal: when the signal realization is (h, ℓ, h) or (ℓ, h, h), M_1 and M_2 deduce that the executive is low quality, i.e. the true state is L (since these signals only realize in state L). The executive is hired only when both M_1 and M_2 observe h, since the CEO cannot separate the beliefs of M_1 and M_2. In this case, optimal communication is public and is given by π^p above.

Note that with public communication, either *all* board members approve the hire or *none* of them do. The value of π^p is $1/3 \cdot 1 + 2/3 \cdot 1/2 = 2/3 < 5/6$. Hence, the CEO is *worse off* relative to the case where board members were not communicating. In particular, the link between M_1 and M_2 decreases the optimal probability of success due to the additional constraints imposed by the network.

The probability of implementing the sender's preferred outcome (e.g. hiring the executive) under optimal public communication turns out to be the lower bound of what the sender can achieve. It is therefore natural to ask if there are non-empty networks where this lower bound is not reached. Would the sender prefer some types of networks over others? Further, it is initially unclear whether adding a link to *any* network *always* (weakly) decreases the value of an optimal communication strategy.

1.2 Overview of Results

We consider an exogenously given network, a binary state space, and a sender who commits to a communication strategy. The sender wishes to implement a certain outcome irrespective of the true state of the world. Receivers know the joint distribution of signals (vectors of messages), but only observe their own and their direct neighbors' private messages from the signal realization. Taking all available information into account, receivers update their beliefs and vote for the alternative which they believe most likely matches the true state. If the network is empty, our model reduces to the model of Kerman, Herings, and Karos (2020), which is used as a benchmark.

We first show that the upper bound of the optimal value is achieved on an empty network, so the sender would prefer if the voters are not communicating at all. On the other hand, the lower bound of the optimal value is achieved when the network is complete: the beliefs of receivers cannot be separated via private communication and thus, optimal communication is *public*. Next, we argue that the sender's problem cannot be simplified by restricting attention to straightforward communication strategies, a result that is the information design counterpart of the revelation principle. Moreover, another common property of optimal communication in many Bayesian persuasion models, revealing the truth in the sender's preferred state, is not without loss of generality either.

Despite the challenges that the setup poses, we identify optimal communication strategies for different types of networks (e.g. line, circle, star-like) and investigate how expanding the networks by adding links changes the optimal value. While adding a link to an empty network (weakly) decreases the optimal value, this might not be the case for non-empty networks. For networks with complete components, many links can be added without decreasing the optimal value.

The upper bound of the optimal value can be achieved on certain networks with complete components, line networks, and circle networks, while it is not possible on star-like networks. Being connected to everyone, the center node in a star observes the whole signal realization, which makes it probabilistically too costly to persuade it. This is an important result, as a similar logic applies to many networks with a star-like component. Finally, in certain networks adding a link sometimes even *increases* the value. In other words, the sender can benefit from a *denser* network.[4]

The rest of the paper is organized as follows. Subsection 1.3 discusses related literature. Section 2 introduces the setup. Section 3 discusses the benchmark case and preliminary results. Section 4 focuses on optimal communication on different networks and expanding networks by adding links. Section 5 concludes.

[4]Density is the ratio of the number of actual links and the number of potential links. Hence, any network obtained by adding a link to another network is *denser*.

1.3 Related Literature

The current model comes closest to and is an extension of Kerman et al. (2020), which builds upon Kamenica and Gentzkow (2011). Kerman et al. (2020) focuses on private communication and collective decision making, where voters vote sincerely, and characterizes optimal communication under *sincere* Bayes Nash equilibrium. A crucial difference to the current setup is that in their model a receiver only has access to information revealed by the sender, whereas in our setup directly connected voters perfectly exchange information. Thus, their model is a special case of ours.

Despite it being a relatively new area of research, there are several studies that address Bayesian persuasion on networks. In Galperti and Perego (2020) the receivers play a game upon receiving information and are able to employ *mixed strategies*. In contrast, we frame the problem in a voting context and focus on *pure strategies*. Another important difference to our model is the type of information transmission. While they assume that information diffuses through *all* directed paths in the network, in our model information is *only* shared with *direct* neighbors.[5]

Liporace (2021) considers spillover effects similar to ours, however, the sender only knows the degree distribution of the agents, but not the network structure. This requires a different approach in characterizing optimal communication since individual nodes cannot be targeted. Moreover, the sender's utility is linear in the number of receivers that take the sender's preferred action. Yet, in a result that is close to ours, the paper also shows that the sender can benefit from a denser network.

In studying persuasion on networks, Babichenko, Talgam-Cohen, Xu, and Zabarnyi (2021) define the notion of information-dominating pairs (if one of two agents observes all information channels that the other one does) and show that an information structure is (weakly) better than another if and only if every such pair in the former is also information dominating in the latter. In contrast to their general top-down approach to the problem, we incorporate insights about the *specific* network structures in our analysis and outline *optimal strategies*.

In our model, receivers have to receive information directly from the sender, whereas some setups allow the receivers to avoid this. Egorov and Sonin (2020) consider a sender who communicates publicly with receivers in a fixed network, where a receiver either relies on his neighbors to learn the provided information or obtains it directly from the sender for a cost. Candogan and Drakopoulos (2020) consider a model of social network interactions, where the agents' payoffs depend on the engagement of their neighbors. A platform designs a signalling mechanism which maximizes engagement or minimizes misinformation by sending recommendations to its users. In contrast, the receivers in our model care about the collective decision and have costless access to information.

A different class of models on networks assumes that receivers' actions are strategic

[5] Another paper with similar type of spillovers to Galperti and Perego (2020) is Candogan (2020), which studies a voting game and shows that for pessimistic voters network effects do not play a role.

complements. Candogan (2019) finds that when the degrees of some nodes in the network increase, it reduces the information designer's payoff.[6] While a similar result holds in our model in some cases, we show that the converse is possible as well. Some papers also allow for the receivers to take additional actions to influence each others' beliefs (Jiménez-Martínez, 2015; Buechel and Mechtenberg, 2019).

As our paper features a sender communicating privately with receivers who make a collective decision, it contributes to the research on private communication and voting games. Some studies in this literature compare public and private communication under different settings (Wang, 2013; Mathevet, Perego, and Taneva, 2020; Titova, 2020), while others investigate voting games that focus on different voting rules (Bardhi and Guo, 2018; Chan, Gupta, Li, and Wang, 2019). Arieli and Babichenko (2019), on the other hand, do not consider collective decision making and characterize optimal communication for different utility functions of the sender, while we investigate optimal communication under various types of networks.

While our focus is on private communication, we find that in some cases public communication can also have an important role in our setup. However, we assume that neither the sender nor the receivers have additional private information about the state (as opposed to Schnakenberg (2015); Kolotilin, Mylovanov, Zapechelnyuk, and Li (2017); Alonso and Câmara (2018); Bizzotto and Vigier (2020); Hu and Weng (2020)). Unlike our setup, some models with public communication assume that receivers are heterogeneous (Alonso and Câmara, 2016; Meyer, 2017; Kosterina, 2018). Our paper also relates to models that consider information design in more general games.[7] In contrast to our model, however, the notion of straightforwardness is without loss in such models (Bergemann and Morris, 2016; Taneva, 2019).

2 Setup

2.1 Communication Strategy

Let $N = \{1, \dots, n\}$ be the set of receivers and $\Omega = \{X, Y\}$ the set of states of the world. For any set S denote by $\Delta(S)$ the set of probability distributions over S with finite support. The receivers share a common prior belief $\lambda^0 \in \Delta^\circ(\Omega)$ about the true state of the world, where $\Delta^\circ(\Omega)$ denotes the set of strictly positive probability distributions on Ω.

Let S_i be a finite set of *messages* the sender can send to receiver i, and let $S = \prod_{i \in N} S_i$, where the elements of S are called *signals*. A *communication strategy* is a function $\pi : \Omega \to \Delta(S)$ which maps each state of the world to a joint probability distribution over signal realizations. Let Π be the set of all communication strategies.

[6] Mathevet and Taneva (2020) study how information is transmitted among agents and characterizes the outcomes different families of information structures implement.

[7] Bergemann and Morris (2019) unify information design with other strands of literature.

For each signal $s \in S$, let $s_i \in S_i$ denote the message for receiver i. For each $s_i \in S_i$ and $\omega \in \Omega$, let $\pi_i(s_i|\omega) = \sum_{t \in S: t_i = s_i} \pi(t|\omega)$, which is the probability that receiver i observes s_i given ω.

For each $\pi \in \Pi$, define $S^\pi = \{s \in S | \exists \omega \in \Omega : \pi(s|\omega) > 0\}$. That is, S^π consists of signals in S which are sent with positive probability by π. Similarly, for each $i \in N$, define $S_i^\pi = \{s_i \in S_i | \exists \omega \in \Omega : \pi_i(s_i|\omega) > 0\}$, which is the set of messages receiver i observes with positive probability under π.

2.2 Networks

An *undirected network* is a map $g : N \times N \to \{0, 1\}$ with $g_{ij} = g(i, j)$ and $g_{ij} = g_{ji}$. Given a set of receivers N, let $G(N)$ be the set of all such networks. We assume that receivers are in a fixed network and each receiver in the network observes his neighbors' message realizations. Thus, in a non-empty network, a receiver gathers more information about the true state than he would from the *same* communication strategy under the *empty* network.

A network $g \in G(N)$ is *complete* if for all $i, j \in N$ with $i \neq j$ it holds that $g_{ij} = 1$. In this case each receiver knows the signal realization, so communication is effectively public on the complete network. For any network $g \in G(N)$, we denote the empty network with the *same number* of receivers by g_0.

Let $N_i(g) = \{j \in N | g_{ij} = 1\}$ be the *neighborhood* of receiver i in g, let $\delta_i^g = |N_i(g)|$ be the *degree* of i in g, and let $\bar{N}_i(g) = N_i(g) \cup \{i\}$. For any $\pi \in \Pi$, $s \in S^\pi$, $i \in N$, and $j \in N_i(g)$, let s_{ij} be the message i observes from j in s, that is $s_{ij} = s_j$. Let $s_i(g) = (s_{ij})_{j \in \bar{N}_i(g)}$ be the *information neighborhood* of receiver i in s, that is, $s_i(g)$ is the vector of messages (with length $\delta_i^g + 1$) receiver i observes upon signal realization s. Let $A_i^\pi(g, s) = \{t \in S^\pi | t_i(g) = s_i(g)\}$ be the set of signals i *associates* with s, i.e. the set of signals i considers possible upon signal realization s. Given $s, t \in S^\pi$, we say that t *is associated with* s if there exists an agent $i \in N$ such that $t \in A_i^\pi(g, s)$.

For any $g \in G(N)$, $\pi \in \Pi$, and $s \in S^\pi$, the posterior belief vector $\lambda^{s,g} \in \Delta(\Omega)^n$ is defined by

$$\lambda_i^{s,g}(\omega) = \frac{\sum_{t \in A_i^\pi(g,s)} \pi(t|\omega)\lambda^0(\omega)}{\sum_{\omega' \in \Omega} \sum_{t \in A_i^\pi(g,s)} \pi(t|\omega')\lambda^0(\omega')}, \quad i \in N, \omega \in \Omega.$$

That is, $\lambda_i^{s,g}(\omega)$ is receiver i's posterior belief that the state is ω upon observing $s_i(g)$.

2.3 Voting

For each $i \in N$, let $B_i = \{x, y\}$ be the set of *actions* of receiver i. Let $B = \prod_{i \in N} B_i$ denote the space of action profiles and $Z = \{x, y\}$ be the set of *voting outcomes*. Upon a signal realization, a receiver chooses an action according to his posterior belief.

Let $z^k : B \to Z$ be a map, where $z^k(a)$ is the *outcome* of the vote when the action profile is a and is defined by

$$z^k(a) = \begin{cases} x & \text{if } |\{i \in N : a_i = x\}| \geq k, \\ y & \text{otherwise.} \end{cases}$$

We assume that the sender's utility function $v : Z \to \{0, 1\}$ has value 1 if x is implemented and 0 otherwise. For each $i \in N$, let $u_i : Z \times \Omega \to \{0, 1\}$ be the utility function of receiver i such that $u_i(x, X) = u_i(y, Y) = 1$ and $u_i(x, Y) = u_i(y, X) = 0$.

To keep the model simple and to focus more on the effects of information transmission on persuasion, we assume that the receivers vote sincerely.[8] In particular, for any $g \in G(N)$, $\pi \in \Pi$, and $i \in N$, let $S_i^\pi(g) = \prod_{j \in \bar{N}_i(g)} S_j^\pi$ be the space of vectors of length $\delta_i^g + 1$ that i can observe under π and on g. Let $\alpha_i^{\pi,g} : S_i^\pi(g) \to B_i$ be agent i's *sincere action function*, such that for any realization $s \in S^\pi$ it holds that

$$\alpha_i^{\pi,g}(s_i(g)) = \begin{cases} x & \text{if } \lambda_i^{s,g}(X) \geq \dfrac{1}{2}, \\ y & \text{otherwise.} \end{cases}$$

That is, a receiver chooses *action* x if he believes the *true state* is X with a probability of at least $1/2$. Throughout the paper we assume that $\lambda^0(X) < \lambda^0(Y)$, since otherwise receivers already take the sender's preferred action. Define the set of signals which implement *outcome* x on g under π as $Z_x^g(\pi) = \left\{s \in S^\pi | z^k(\alpha^{\pi,g}(s)) = x\right\}$.

Receiver i is *pivotal in* $s \in S^\pi$ if for any $a_i \in B_i$, $z^k(a_i, \alpha_{-i}^{\pi,g}(s_{-i}(g))) = a_i$. That is, i is pivotal following realization s if i's vote determines the voting outcome given that all $j \neq i$ vote sincerely.

Let $a \in B$ be an action profile and $z = z^k(a)$ be a voting outcome. The *value* of a communication strategy $\pi \in \Pi$ for quota k is defined as the sender's expected utility under π on network g. As we fix λ^0 and $\alpha_i^{\pi,g}$ throughout the paper, we write $V_k^\pi(g) = V_k^\pi(\lambda^0, g, \alpha^{\pi,g})$, where

$$V_k^\pi(g) = \mathbb{E}_{\lambda^0}\left[\mathbb{E}_\pi\left[v(z^k(\alpha^{\pi,g}(s))\right]\right] = \lambda^0(X) \sum_{s \in Z_x^g(\pi)} \pi(s|X) + \lambda^0(Y) \sum_{s \in Z_x^g(\pi)} \pi(s|Y).$$

That is, given n, k, and g, the value of a communication strategy is equal to the probability of implementing x. A communication strategy $\pi^* \in \Pi$ is *optimal on* g for *quota* k if $V_k^{\pi^*}(g) = \sup_{\pi \in \Pi} V_k^\pi(g)$.

3 Preliminaries

In this section, we discuss how the information-sharing feature of our model produces a non-trivial change in the setup of multiple-receiver Bayesian persuasion. We start by arguing that the sender performs best on the empty network, by first introducing an optimal communication strategy on the empty network, as provided in Kerman

[8] Felsenthal and Brichta (1985); Degan and Merlo (2007); Groseclose and Milyo (2010) show that voters vote sincerely under certain conditions.

et al. (2020). In their setup, it is without loss of generality to restrict attention to straightforward (à la Kamenica and Gentzkow (2011)) and anonymous communication strategies.[9] This allows one to represent a communication strategy with probability weights q_ℓ and r_ℓ, where q_ℓ (r_ℓ) is the probability that ℓ agents observe x in state X (state Y), and *each* signal in which the same number of receivers observe x has the *same* probability. An optimal communication strategy on the *empty* network is characterized below.[10]

Theorem 3.1. *(Kerman et al., 2020) Let $\pi^* \in \Pi$ with (q^*, r^*) be given by*

$$(q_n^*; r_0^*, r_k^*) = \begin{cases} (1; 0, 1) & \text{if } \lambda^0(X) \geq \frac{k}{n+k}, \\ \left(1; 1 - \frac{\lambda^0(X)}{\lambda^0(Y)}\frac{n}{k}, \frac{\lambda^0(X)}{\lambda^0(Y)}\frac{n}{k}\right) & \text{if } \lambda^0(X) < \frac{k}{n+k}. \end{cases}$$

Then π^* is optimal on the empty network with n nodes and quota k, and the value is given by $V_k^n = \min\left\{\frac{n+k}{k}\lambda^0(X), 1\right\}$.

The optimal communication strategy in Theorem 3.1 sends x to all receivers with probability 1 if the state is X ($q_n^* = 1$) and targets minimal winning coalitions (r_k^*) if the state is Y.[11] This is no longer optimal when we consider a non-empty network, since neither straightforwardness nor anonymity survive incorporating a network structure into the model.

Theorem 3.1 provides the upper bound of what the sender can achieve. The simple logic behind this observation is that while the information a receiver gathers from a communication strategy on a non-empty network can be replicated on an empty network of the same size, the converse is not necessarily true.

Proposition 3.2. *Let $g \in G(N)$. For any $\pi \in \Pi$ it holds that $V_k^\pi(g) \leq V_k^n$.*

The proofs to all statements can be found in Appendix A.

In contrast to empty networks, an observation that is particularly relevant for complete networks (and networks with complete components) is that if two nodes have *exactly* the same neighborhood, they can be treated identically.

Lemma 3.3. *Let $\pi \in \Pi$ and let $g \in G(N)$ and $i, j \in N$ be such that $\bar{N}_i(g) = \bar{N}_j(g)$. Then there exists $\pi' \in \Pi$ such that for any $s \in S^{\pi'}$, $s_i = s_j$ and $V_k^{\pi'}(g) = V_k^\pi(g)$.*

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[9] A communication strategy is *straightforward* if for all $i \in N$ it holds that (i) $S_i^\pi \subseteq B_i$ and (ii) for all $g \in G(N)$ and $s \in S^\pi$ with $s_i = a_i$, $\alpha_i^{\pi,g}(s_i(g)) = a_i$.

[10] Note that Theorem 3.1 also follows from Corollary 2 of Arieli and Babichenko (2019). In their model agents want their action to match the true state, whereas in Kerman et al. (2020) agents want the outcome of the vote to match the true state. However, the optimization problems in both are equivalent since the conditions for a sincere agent to vote in favor of the sender's preferred outcome are identical in both cases.

[11] In our motivating example, π is precisely the communication strategy provided by Theorem 3.1, where $q_3 = 1$, $r_2 = 3/4$, and $r_0 = 1/4$.

An immediate corollary to Lemma 3.3 is that public communication is optimal on the complete network. In general, the optimal public communication strategy (denoted by π^p) always yields the same value $V^p = 2\lambda^0(X)$ for any k, as either all agents are persuaded or none of them are.[12] Hence, V^p is independent of the network structure.[13]

Corollary 3.4. *Let $g \in G(N)$ be complete. Then π^p is optimal on g.*

An important feature of our set up is that, interestingly, straightforwardness is not without loss of generality, while it might not be the case with a different type of information spillovers, as in Galperti and Perego (2020). The main reason for this is that they allow receivers to have mixed strategies and for the sender to send mixed strategy recommendations, while we assume that receivers choose a pure strategy according to their posterior beliefs (i.e. they vote sincerely). The type of information spillovers in our model further hinders the ability to restrict attention to straightforward communication strategies.[14]

The difficulty our set up presents is not only due to straightforwardness not being without loss, but also due to truth-telling in state X not being optimal in general. In particular, the sender might find it beneficial to garble information in state X in some type of networks, such as the line. While this does not decrease the probability of implementing x in state X, it allows the sender to increase it in state Y.[15]

It is important to note that under π^* in Theorem 3.1, sincere voting does not constitute a Bayes Nash equilibrium (BNE) when $k < n$. This stems from the structure of π^*: receivers are pivotal upon observing x in state Y, but not in state X. Thus, upon having a posterior belief of at least $1/2$ that the true state is X, it is in a receiver's best interest to vote against his belief, since receivers are only pivotal in state Y.[16] It follows that whenever we are able to achieve V_k^n on a network and the optimal communication exhibits the same structure as π^*, sincere voting is not a BNE.[17] One remedy to the swing voter's curse provided by Kerman et al. (2020) is the following: instead of targeting minimal winning coalitions in state Y (r_k), the

[12]Since receivers share a common prior, the situation is equivalent to persuading a single receiver.

[13]Setting $k = n$ in Theorem 3.1 yields the value of the optimal *public* communication strategy.

[14]In particular, it is possible that the set of associated signals are not subsets of each other ($A_i^\pi(g,s) \subseteq A_j^\pi(g,s)$) but have a non-empty intersection ($A_i^\pi(g,s) \cap A_j^\pi(g,s) \neq \emptyset$), which increases the difficulty of devising optimal communication. In a sense, the two models can be seen as two possible extremes of information sharing: in Galperti and Perego (2020), if a path exists from player i to player j, then j learns i's signal irrespective of the length of the path (cf. Assumption 1 in Galperti and Perego (2020)). In our case, an agent only learns the signals of their *direct* neighbours. In other words, information in their model acts close to a *global* public good, while in ours it is strictly a *local* public good.

[15]Interested readers can find a detailed example in our working paper Kerman and Tenev (2021).

[16]This phenomenon is known as the *swing voter's curse* (Feddersen and Pesendorfer, 1996).

[17]Note that this is not true if $k = n$, since agents are pivotal in both states and have no incentive to deviate. More generally, the optimal public communication strategy always leads to a BNE under sincere voting.

sender might target slightly larger coalitions (r_{k+1}), so that *no agent* is pivotal in any state. This implies that voting according to one's belief constitutes a BNE.

An alternative presentation of our results would be to employ this equilibrium refinement (sincere BNE), so that we search for optimal strategies under which sincere voting is a BNE. This, however, would not make a crucial difference in the characterizations we provide throughout the paper; our benchmark in this case would be V_{k+1}^n instead of V_k^n. Therefore, we simply assume that agents vote sincerely, as this keeps the exposition simpler and allows us to focus more on the sender's problem.

4 Expanding Networks: Optimal Communication

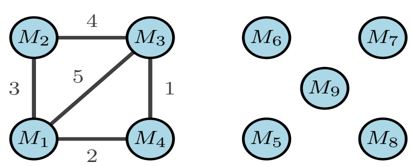
So far, we have seen that the sender achieves the upper bound of the value (V_k^n) under the empty network and the lower bound (V^p) under the complete network. It makes intuitive sense that the upper bound of the value is reached when voters are not communicating (i.e. in the least dense network), since this allows the sender to utilize private communication to its full extent. On the other hand, when each voter is communicating with every other voter (i.e. in the densest network), the effectiveness of private communication plummets.

Nevertheless, given two non-empty networks, it is unclear whether the sender would always be worse off in the denser one. By Proposition 3.2, adding a link to an empty network (weakly) decreases the optimal value. One might naively guess that this is also the case for any non-empty network since voters would have access to more information than before, making it harder for the sender to garble information in state Y. Yet, this is not the case; the optimal value stays the same in many cases and even *increases* in some. In particular, how the optimal value changes not only depends on the type of network, but also on where in the network the link is added. In this section, we provide partial characterizations of the optimal value and the change in optimal value with the addition of a link for different types of networks. To do so, we identify optimal strategies for a number of commonly investigated network structures, which can serve as a blueprint for more complicated strategies.

4.1 Networks with Many Singletons

In our motivating example, the optimal value immediately falls to its lower bound (V^p) when a link is added to the empty network. This strict decrease, however, is caused by the small size of the network. If the empty network is large, the upper bound (V_k^n) can still be achieved after adding one link (and possibly more). Hence, the CEO would not care if two board members among many are communicating with each other, as communication is almost fully private. More generally, if there are sufficiently many board members that the CEO can communicate with in private (i.e. there exist sufficiently many singleton nodes), then having more communication in the network does not harm the CEO's persuasion capabilities.

Example 4.1. Suppose there are 9 board members with $\lambda^0(X) = 1/3$ and that $k = 5$ votes are required to approve the hire. Suppose additionally that starting from the empty network g_0, links are added in the order given in the figure below. Let g_ℓ for $\ell \in \{1, \dots, 5\}$ denote the corresponding network after each addition of a link, $g_1 = g_0 + M_3M_4$, $g_2 = g_1 + M_4M_1$, and so on. It turns out that for any g_ℓ, there exists $\pi \in \Pi$ such that $V_k^\pi(g_\ell) = V_k^n$. That is, the CEO can hire the executive with the highest possible probability for up to four (fully inter-) connected board members. We present an optimal communication strategy on g_5 below, which is also optimal on any g_ℓ for $\ell \in \{1, \dots, 4\}$.



π	$\omega = X$	$\omega = Y$
$(x, x, x, x, x, x, x, x, x)$	1	0
$(y, y, y, y, x, x, x, x, x)$	0	$\frac{4}{10}$
$(x, x, x, x, x, y, y, y, y)$	0	$\frac{1}{10}$
$(x, x, x, x, y, x, y, y, y)$	0	$\frac{1}{10}$
$(x, x, x, x, y, y, x, y, y)$	0	$\frac{1}{10}$
$(x, x, x, x, y, y, y, x, y)$	0	$\frac{1}{10}$
$(x, x, x, x, y, y, y, y, x)$	0	$\frac{1}{10}$
$(y, y, y, y, y, y, y, y, y)$	0	$\frac{1}{10}$

Let $\mathbf{x}$ be such that $\mathbf{x}_i = x$ for all $i \in N$ and define $\mathbf{y}$ analogously. It holds that for any $i \in N$, $\lambda_i^{\mathbf{x}}(X) = 1/2$ and $\lambda_i^{\mathbf{y}}(X) = 0$. Thus, $V_5^\pi(g_5) = 1/3 \cdot 1 + 2/3 \cdot 9/10 = 14/15 = V_5^9$. Observe that this value can be achieved irrespective of the connections between M_1, M_2, M_3, and M_4, as they are always treated identically. △

Example 4.1's insight is generalized in the proposition below.

Proposition 4.2. *Let $g \in G(N)$ and $k \geq n/2$. If $g' = g + i'j'$ for $i', j' \in N$ and $|\{i \in N : \delta_i^{g'} = 0\}| \geq k$, then there exists $\pi \in \Pi$ such that $V_k^\pi(g') = V_k^n(g)$.*

Proposition 4.2 implies that as long as there are sufficiently many agents who are not communicating, the sender does not care about the number of links among the remaining agents. However, when the network consists only of complete components, V_k^n can be achieved under a very strict requirement. But more importantly, the addition of a link to such a network generally *does not decrease* the optimal value.

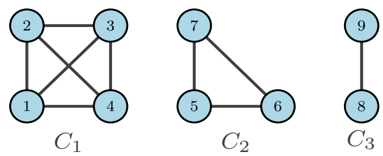
4.2 Networks with Complete Components

The implication of Lemma 3.3 that if two agents have exactly the same neighborhood, then their beliefs cannot be separated naturally extends to networks with complete components. It follows that in such networks agents in the same component have the same belief following any signal realization. Treating all nodes within a component uniformly in every signal makes this setup equivalent to an empty network with fewer *nodes* and *weighted voting*.

Suppose that a city referendum will be held and the incumbent party is organizing rallies in different regions of the city. This could (roughly) be related to the case of complete components; the party can send different messages in each rally that takes place in a different region of the city. The voters in each rally/region observe the same message and will have identical voting behaviors.[18] Thus, optimal communication in this case relies heavily on how many people live in each region of the city.

We present a simplified version of such a situation in the next example.

Example 4.3. Let $|N|= 9$, $\lambda^0(X) = 1/3$, and $k = 5$. Consider the following network g with complete components C_1, C_2, and C_3.



Consider a communication strategy $\pi \in \Pi$ which reveals the true state when it is X and targets two out of three components in each signal that implements x with probability 1/4 in Y. Thus, $V_5^\pi(g) = 1/3 \cdot 1 + 2/3 \cdot 3/4 = 5/6 = V_2^3 < V_5^9$, the value for persuading $k = 2$ out of $n = 3$ nodes in the empty network. It is straightforward to verify that no greater value can be achieved. △

Example 4.3 illustrates that the upper bound V_k^n cannot always be achieved in a network with complete components. The best strategy for the sender there is targeting two out of three components in each signal (since any two components together fulfil the quota), a situation equivalent to persuading two out of three individual agents in the empty network.

To formalize the logic of the example, denote the set of all networks with $\ell \in \{1, 2, \ldots\}$ complete components by $G_\ell^c(N)$. Given $g \in G_\ell^c(N)$, let $C(g)$ be the set of all components of g. Let $\mathfrak{C}^q$ be the set of all subsets of $C(g)$, where each subset has cardinality q. That is, $\mathfrak{C}^q = \{\mathcal{C}' \subseteq C(g) : |\mathcal{C}'|= q\}$. Let $q_* = \min\{q \in \mathbb{N} | \sum_{C\in\mathcal{C}'} |C|\geq k, \forall \mathcal{C}' \in \mathfrak{C}^q\}$, i.e., q_* is the least number of components such that whenever the elements of any q_* components are counted together, they fulfil the quota.

Proposition 4.4. *Let $g \in G_\ell^c(N)$. If there is no $q' < q_*$ such that $\sum_{C\in\mathcal{C}'} |C|\geq k$ for all $\mathcal{C}' \in \mathfrak{C}^{q'}$, then $\pi \in \Pi$ with $V_k^\pi(g) = V_{q_*}^\ell$ is optimal on g.*

In words, if a network g consists of ℓ complete components *and* combining the *same number* (q_*) of components fulfils the quota (where for any $q' < q_*$ the quota is not

[18] At the moment, we are abstracting away from the fact that a voter might obtain information about the campaign without being in the rally.

fulfilled), then a communication strategy with value equal to the optimal value of persuading q_* out of ℓ agents in the empty network is optimal on g. Note that if there exists $q' < q_*$, then it might be beneficial for the sender to target different number of components in different signals.

It immediately follows from Proposition 4.4 that if $g \in G^c_\ell(N)$, then the sender can achieve V^n_k when the ratios n/k and ℓ/q_* are equal.

Corollary 4.5. *Let $g \in G^c_\ell(N)$. If $n/k = \ell/q_*$, then there exists $\pi \in \Pi$ such that $V^\pi_k(g) = V^n_k$.*

In such networks, a link can only be added between agents in different components. Regardless of the upper bound V^n_k being achieved, in many cases adding a link to a network with complete components does not decrease the optimal value.

It is interesting to observe that the value $V^\ell_{q_*}$ from Proposition 4.4 will hold as a lower bound even if we add a significant number of new connections between the components. In Example 4.3, if we add a link between nodes 3 and 7 to form g', the same value can be achieved by slightly adjusting some signals in π. In fact, we can even add more links to g' and the sender would still be able to achieve V^3_2. In particular, the communication strategy can be adjusted similarly to π' above, as long as there is at least one node in every component that is communicating *only with* the nodes in its component. We generalize this result in the next proposition.[19]

Proposition 4.6. *Let $g \in G^c_\ell(N)$. Suppose $\pi \in \Pi$ is such that $V^\pi_k(g) = V^\ell_{q_*}$. Let $\{i_m j_{m'}\}_{m,m' \in \mathbb{N}}$ be a sequence of links such that for any $m, m' \in \mathbb{N}$, $i_m j_{m'} \notin g$. Let $g' \in G(N)$ be defined by (i) $g' = g + \{i_m j_{m'}\}_{m,m' \in \mathbb{N}}$ and (ii) for all $C \in C(g)$ there exists $i \in C$ such that $N_i(g) = N_i(g')$. Then, there exists $\pi' \in \Pi$ such that $V^{\pi'}_k(g') \geq V^\pi_k(g)$.*

It should be noted that Proposition 4.6 relies on our assumption that agents share their messages only with their immediate neighbors. If $g \in G^c_\ell(N)$, then adding a single link *never* decreases the optimal value.[20]

By Proposition 4.6, it follows that we can add up to 11 links to g in Example 4.3 without decreasing the optimal value.[21] More generally, for any $g \in G^c_\ell(N)$, as many as $\frac{1}{2}\sum_{C \in C(g)}(|C|-1)(n-|C|-|C(g)|+1)$ new links can be formed without decreasing the value, while satisfying condition (ii). To provide a large network example, let $|N| = 121$ and suppose $g \in G^c_{11}(N)$ and for all $C \in C(g)$, $|C| = 11$. Using simple majority ($k = 66$), it follows by Corollary 4.5 that there exists $\pi \in \Pi$ with $V^\pi_k(g) = V^n_k$. By Proposition 4.6 we can add up to 5,500 links to g, increasing the number of links in the network more than tenfold, while always achieving V^n_k.

[19] The proposition establishes a lower bound on the value for such networks.

[20] On the other hand, if agents were transmitting their neighbors' messages (as in Galperti and Perego (2020)), then adding even one link can decrease the optimal value.

[21] For example, after adding links between 2-6, 3-6, 4-6, 2-7, 3-7, 4-7, 2-8, 3-8, 4-8, 6-8, and 7-8.

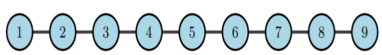
In our example of a city referendum, the party would be equally well off if some people in different regions are communicating; equivalently, we can think of this as the sender not being aware of some links between distinct components. If, however, information in one region spreads to everyone in another region, then the incumbent party can no longer effectively target different regions.

4.3 Line and Circle Networks

Connecting the end nodes of a line produces a circle and this presents a simple situation to analyze the addition of a link to a non-empty network.[22] We first provide some sufficiency conditions for achieving V_k^n in line and circle networks and then combine the two results to show that completing a line network to a circle by adding a link does not change the optimal value.

In Example 4.7, we present a situation in which V_k^n can be achieved on a line.

Example 4.7. Let $|N|= 9$, $\lambda^0(X) = 1/3$, and $k = 6$. Consider the following network g and the communication strategy $\pi \in \Pi$:



π	$\omega = X$	$\omega = Y$
$(x, x, x, x, x, x, x, x, x)$	1	0
$(x, x, x, x, x, x, x, y, x)$	0	$\frac{1}{4}$
$(x, x, x, x, y, x, x, x, x)$	0	$\frac{1}{4}$
$(x, y, x, x, x, x, x, x, x)$	0	$\frac{1}{4}$
$(y, y, y, y, y, y, y, y, y)$	0	$\frac{1}{4}$

For any $i \in N$ and $s \in S^\pi$ if $s_i = y$ and $j \in N_i(g)$, then $\lambda_i^s(X) = \lambda_j^s(X) = 0$. Thus, the sender targets a minimal winning coalition in signals that implement x in state Y. So, $V_6^\pi(g) = 1/3 \cdot 1 + 2/3 \cdot 3/4 = 5/6 = V_6^9$. △

We now generalize the example and show that line networks with a common factor 2 or 3 for n and k can achieve the optimal value by constructing optimal strategies following the same pattern.[23]

Proposition 4.8. *If $g \in G(N)$ is a line and if for $\alpha, \beta \in \mathbb{N}$: (i) $k = 3\alpha, n = 3\beta$ or (ii) $k = 2\alpha, n = 2\beta$, then there exists $\pi \in \Pi$ such that $V_k^\pi(g) = V_k^n$.*

[22] It is common to come across circle networks in the network formation literature: Bala and Goyal (2000) consider a noncooperative game of network formation and show that circle and star networks are formed in the Nash equilibrium of the game. Falk and Kosfeld (2012) show in an experimental study that this holds under certain conditions. Watts (2002) shows that circle networks might form with non-myopic agents.

[23] See also Example 3.8 in Kerman and Tenev (2021).

The main difference between a circle and line is that circle networks are *regular*, that is, for any $i, j \in N$ it holds that $\delta_i^g = \delta_j^g$. Despite this asymmetry in the degrees, if $k = 3\alpha$ and $n = 3\beta$ for $\alpha, \beta \in \mathbb{N}$, it is possible that optimal communication strategies coincide in the line and circle networks.

Example 4.9. Consider the set up of Example 4.7 and $g' = g + \{19\}$ instead of g. Communication strategy π has the same value on the circle g', i.e. $V_6^\pi(g') = V_6^9$. This is the case since the addition of the link between nodes 1 and 9 does not affect the voting behavior of any agent. △

There are a few aspects to mention about Example 4.7 and Example 4.9. First, it is important to note that π achieves the same value on g and g' due to n and k satisfying $k = 3\alpha, n = 3\beta$.[24] If $k = 2\alpha, n = 2\beta$, however, then while we can achieve V_k^n in a line with a communication strategy that employs only two messages, we might not in a circle.[25] In particular, our sufficiency condition for a circle requires the quota to not be too high, rather than having a common factor with the number of voters.

Proposition 4.10. *Let $g \in G(N)$ be a circle and let $k < n - 2$. Then there exists $\pi \in \Pi$ such that $V_k^\pi(g) = V_k^n$.*

Secondly, when we add a link between nodes 1 and 9 in g of Example 4.7 to form g', the sets of associated signals for agents 1 and 9 do not change. In other words, following any realization of π, all agents deem exactly the same signals possible and thus, vote for the same alternative as on g. More generally, if no agent's association set is affected by the addition of a link, then the optimal strategy on g and g' coincide. We present this result without proof as it is straightforward.

Lemma 4.11. *Let $\pi \in \Pi$, $g \in G(N)$, and $g' = g + ij$, for some $i, j \in N$. For any $s \in S^\pi$, if $A_i^\pi(g, s) = A_i^\pi(g', s)$ and $A_j^\pi(g, s) = A_j^\pi(g', s)$, then $V_k^\pi(g) = V_k^\pi(g')$.*

The implications of this simple result are far-reaching. For instance, the network in Example 4.7 can be expanded by additional 10 links, making it twice as dense and keeping its empty-network optimal value.

Thirdly, in both examples we have $n = 9$ and $k = 6$, which satisfy the conditions of both Proposition 4.8 and Proposition 4.10. More generally, whenever these conditions are satisfied together, expanding the line to a circle does not change the value of an optimal communication strategy.

Corollary 4.12. *Let n and k be such that (i) $k = 3\alpha, n = 3\beta$ or $k = 2\alpha, n = 2\beta$ for $\alpha, \beta \in \mathbb{N}$ and (ii) $k < n - 2$. Let $g \in G(N)$ be a line and let $\pi \in \Pi$ be optimal on g. If $g' = g + ij$ is a circle for $i, j \in N$ and $\pi' \in \Pi$ is optimal on g', then $V_k^{\pi'}(g') = V_k^\pi(g) = V_k^n$.*

[24] The same value can be achieved on a circle also with a communication strategy that employs more signals and treats agents symmetrically.

[25] According to (Babichenko et al., 2021) the optimal value on a circle should always be achievable, however, this might require "continuum-many different signals".

4.4 Star-like Networks

We first define a generalized version of the well-known star network. Given $g \in G(N)$, let $Q(g) = \{i \in N | \delta_i = n - 1\}$ and $|Q(g)| = m \in \{1, \ldots, n\}$. We say that $g \in G(N)$ is *star-like* if $Q(g) \neq \emptyset$ and for all $j, j' \in N \setminus Q(g)$ it holds that $g_{jj'} = 0$. Star-like networks have one or more agents connected to *every other* agent. If $m = 1$, g corresponds to a star network and if $m = n$, g is the complete network.

There are two main reasons for star-like networks being a point of interest. First, a star is a type of egocentric network, which is commonly observed in friendship networks on social media outlets.[26] More generally, the set $Q(g)$ in star-like networks can be interpreted as opinion leaders in society. Clearly, friendship networks on these outlets are very large. Yet, they possibly contain many subnetworks that are (or resemble) star-like networks, which political parties can take into account when communicating with voters via social media. In contrast, star networks can also represent situations of smaller scales. For instance, a board of directors similar to our motivating example could be represented by a star network, in which the chairperson is the center node.

Second, the optimal value on any star-like network is less than V_k^n for $k < n$. This is an important observation, since the reasons for the inability to achieve the upper bound also apply to other networks which include a star(-like) component. In an optimal communication strategy on a simple star network, the center node always observes the same message in state X and thus, is only persuaded after a signal realization in which he is not pivotal. This implies that it is in the sender's best interest *not* to attempt to persuade the center node. This restricts the information the peripheral nodes receive from the center, effectively transforming the star network.

Proposition 4.13. *Let $g \in G(N)$ be star-like with $|Q(g)| = m$ and let $k < n - m$. Then $\pi \in \Pi$ with $V_k^\pi(g) = V_k^{n-m}$ is optimal on g.*

In words, under an optimal strategy on a star-like network, it is as if the sender is attempting to persuade k agents out of $n-m$ in the empty network, leading to a lower optimal value. Note that as m increases, the optimal value *monotonically* decreases. In the limit, if $k \geq n - m$, the optimal value is equal to V^p.

This decrease in the optimal value can also be thought of as benefiting the voters since they want the implemented outcome to match the true state.[27] Since both V_k^n and V_k^{n-m} are achieved by implementing x in state X with probability 1, the decrease in the optimal value implies that y is implemented with a higher probability in Y.[28]

[26] An egocentric network consists of an agent who is connected to all other agents.

[27] In a completely different context, Galeotti and Goyal (2010) show that under a public good game under network formation, the star network is the unique equilibrium and that welfare is maximized in this case. While we cannot claim that welfare is maximized for the voters in a star network in our case, they are certainly better off relative to the empty network.

[28] Becker, Brackbill, and Centola (2017) build on DeGroot's formalization of local information

Recall that if $m = 1$, Proposition 4.13 refers to a star network and notice that a circle network is denser than a star network with the same number of agents. By Proposition 4.10 and Proposition 4.13, it follows that the sender can achieve a higher value on the denser of the two networks.

Using Proposition 4.13, it is easy to see that for the related network structure wheel (for $k < n-3$), V_k^{n-1} is the lower bound for the optimal value.[29]

4.5 Sender-preferred Denser Networks

So far, we have seen that while the optimal value mostly decreases when a link is added to a network, it is also possible that it remains unchanged under certain conditions. But can the addition of a link actually *benefit* the sender?

Consider a network g. When calculating the optimal value under $g' = g + ij$, it is as if we are assuming that the sender observes the new link ij and devises a new communication strategy accordingly. What would happen if the sender observed ij but did not have time or ability to adjust her strategy? For example, new relations develop in the board of directors so fast that the CEO cannot adapt her strategy or she is simply unaware of them. While this might harm the CEO if she is optimizing on the existing network, it is easy to see that if she is using a "bad" communication strategy to begin with, then the addition of a link might be beneficial. In other words, if the sender uses a suboptimal communication strategy π on a network g, then π might be more effective on $g + ij$ than g.

Let us now return to our usual case of the sender being able to adjust her strategy. One might expect that whenever a link is added, the value of an *optimal* communication strategy on the new network will be (weakly) lower than an optimal communication strategy on the initial network. This (rather crude) intuition is based on the fact that the upper bound of the optimal value is reached in the empty network and the lower bound is reached in the complete network. However, the optimal communication strategy might, in fact, have a *higher* value after adding a link, so that the sender benefits from a *denser* network.

In the following proposition, we provide a partial characterization of increasing optimal value by the addition of a single link.

Proposition 4.14. *Let $n = 2k$ and $g \in G(N)$. Let $C \in C(g)$ be star-like with $|C| = k+1$, $|N \setminus C| = k-1$, and $|Q(g)| = m$. If $g' = g + ih$ for $i \in N \setminus C$,*

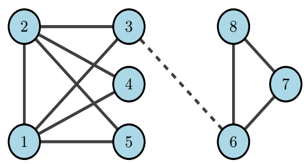
aggregation and considers the "wisdom of crowds"; in particular they show via an experiment that collective accuracy is higher in decentralized networks (e.g. the circle) relative to the empty network. This is in contrast to Proposition 4.10: if n is sufficiently high and k is simple majority, then the voters' accuracy is not improved as V_k^n can be achieved on a circle.

[29] Notice that the results in (Babichenko et al., 2021) imply that this lower bound is also the actual value for the optimal strategy on a wheel. Even though Babichenko et al. (2021) does not guarantee that this can be achieved with only two messages, an optimal communication strategy on a wheel that employs two messages can be easily derived from our Proposition 4.10.

$h \in C \setminus Q(g)$, then (i) for any $\pi \in \Pi$, $V_k^\pi(g) \leq V_k^{n-m}$ and (ii) there exists $\hat{\pi} \in \Pi$ such that $V_k^{n-m} < V_k^{\hat{\pi}}(g') = V_1^2$.

In Example 4.15, we illustrate the interesting features of Proposition 4.14.

Example 4.15. Let $|N|= 8$, $\lambda^0(X) = 1/3$ and $k = 4$. Consider network g below, which features a star-like component (left) with two "center" nodes (1 and 2). The star-like component has $k+1$ nodes. Hence, g satisfies the conditions of Proposition 4.14. After a link is added between nodes 3 and 6 to obtain network g', the optimal value increases to $V_4^\pi(g) = V_1^2$, by using π below.



π	X	Y
(x, x, x, x, x, x, x, x)	1	0
(x, x, x, x, x, y, y, y)	0	$\frac{1}{2}$
(x, x, x, y, y, x, x, x)	0	$\frac{1}{2}$

Proposition 4.14 argues that initially such a network has value of *at most* $V_4^{n-2} = V_4^6$ because the star-like component's most informed agents (1 and 2) are too costly to persuade. In the language of (Babichenko et al., 2021) they both information-dominate $3, 4$ and 5 and as such whenever the sender manages to persuade them this is as good as persuading all nodes $1-5$. However, such a strategy consumes too much probability and is inefficient for the sender. As a consequence, the best she can do is not to try influencing them at all, which inadvertently lowers the value below its empty-network optimum.

Strikingly, starting from g and adding a single link between the peripheral nodes of the star and a node from the rest of the network (nodes 3 and 6 in the figure) *increases* the value to the empty network optimal value $V_k^{2k} = V_1^2$, because with a single action it decreases the information dominance of all "centers" of the star-like component (these are all nodes in $Q(g)$ in the general case). △

Suppose the board of directors in our leading example currently has five members that are in a star-like network and three new members are to be appointed. In this case, the CEO would prefer one of the new members to have an *existing relationship* with one of the current members. Importantly, this is irrespective of the existing relationships *between* the new board members.

Combining the implications from Proposition 4.2 and Proposition 4.14 presents a surprising tradeoff for small scale voting situations. Suppose that the CEO is a voting member of the board, the remaining board members are in a network as presented in Proposition 4.14 (on which the optimal value is V_k^{n-2}), and that the voting rule is simple majority. As the CEO already votes in favor of the proposal, it suffices to persuade $n/2$ out of n remaining board members to hire the new executive. Suppose further that the sender wishes to increase the probability of implementing her preferred outcome to V_k^n, but has limited resources (time or money) to influence the outcome of the vote, in addition to her ability to manipulate information. One way to achieve this goal would be to invest her limited resources in lobbying to increase the number of board members to $2n$ (where it suffices to persuade n). Alternatively, she can simply foster the creation of a *single* link between the existing board members. This provides an insight to moral hazard problems that arise from conflict of interest by having the CEO as a voting member of the board.

Proposition 4.14 and Example 4.15 highlight the importance of the network structure for the success probability of the sender. While nodes with many sources of information can be difficult to persuade, a possible solution to this problem from the perspective of the sender is to consider a rougher partition of the network if the structure allows it. In this case, after the link is created the sender is better off refraining from trying to take advantage of the intricate connections between nodes $1-5$. Instead, it treats them almost uniformly.[30]

While it might be challenging to directly apply Proposition 4.14 to very large networks, it is common to observe clustering around certain nodes on such networks (e.g. around opinion leaders). By the use of social media, a sender can treat these clusters separately and it would be possible for her to benefit from the additional connections the peripheral nodes have.

Proposition 4.14 provides a partial characterization for situations where the optimal value increases with the addition of a link. However, even if the network does not have a star-like component, the optimal value can increase when the network becomes denser. Interestingly, this might be the case also when multiple links are added to the network. In Example 4.17, the optimal value increases despite making it twice as dense. To show this, we first introduce a technical lemma.[31]

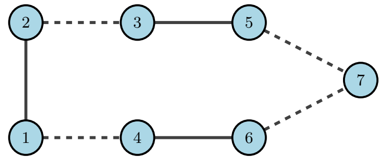
Lemma 4.16. *Let $g \in G(N)$ and $\pi \in \Pi$ be such that $V_k^\pi(g) < 1$. If there exists $s \in S^\pi$ with $s \in Z_x^g(\pi)$ and $|\{i \in N : \alpha_i^{\pi,g}(s_i(g)) = x\}| > k$, then there exists $\pi' \in \Pi$ such that $V_k^n > V_k^\pi(g)$.*

In words, if a communication strategy assigns positive probability to a signal in which more than k agents vote for x, then this communication strategy does not achieve the empty network optimal value, if the optimal value is less than 1 (which are the most interesting cases).

Example 4.17. Let $|N| = 7$, $k = 4$, and let $g \in g(N)$ (w/o dashed lines) be given as follows.

[30]The approach of dividing the network in sectors which are treated uniformly in strategies has been implicitly featured in many proofs in this paper. For a more formal version, see Proposition 5.10 in Kerman and Tenev (2021). Finally, such an approach offers rich possibilities for practical applications and leveraging the natural properties of real-life social networks, which usually exhibit high degrees of clustering (Jackson and Rogers, 2007).

[31]This result is provided without proof as it readily follows.



By Lemma 3.3, the beliefs of agents 1 and 2 cannot be separated (the same holds for agents 3 and 5, and 4 and 6). Moreover, note that if a communication strategy $\pi \in \Pi$ never persuades agent 7, then $V_4^{\pi}(g) < V_k^n$. However, whenever agent 7 is persuaded in a signal that implements x, the signal must feature at least 5 agents voting for x. Thus, by Lemma 4.16, it holds that for any $\pi \in \Pi$, $V_4^{\pi}(g) < V_k^n$.

Now, let $g' \in G(N)$ be a circle obtained by adding links to g (the dashed lines). By Proposition 4.10, it follows that there exists $\pi' \in \Pi$ with $V_4^{\pi'}(g') = V_k^n$. △

Discussion: The Voters' Perspective

We mentioned after Proposition 4.13 that a decrease in the optimal value might be interpreted as benefiting the voters. Similarly, we can interpret Proposition 4.14 also from the perspective of the voters, in terms of voting for the "correct" outcome. One way to measure whether voters vote correctly, as proposed by Lau and Redlawsk (1997), is to check if they are making the same choices they would have made under perfect information. In our case, a similar logic would lead us to examine whether a voter is voting for the same alternative he would have voted for if he knew the true state of the world (or equivalently, if the communication strategy is fully informative).

The optimal communication strategies on g and g' in Example 4.15 both put probability 1 on the signal that sends x to all agents, i.e. x is implemented with probability 1 in state X. This implies that under the denser network g', the probability of implementing the correct outcome (the one matching the true state) is lower. In particular, the probability of each individual voter voting correctly is lower on g'.[32]

It is intuitive that communication networks might help voters make correct decisions with higher accuracy (Ryan, 2011; Sokhey and McClurg, 2012), which is, in a broader sense, the case in our model as well; as Proposition 3.2 implies, adding a link to an empty network (weakly) decreases the optimal value, increasing the probability of a voter making the correct decision. In situations that involve heterogeneity in voters, however, higher levels of incorrect voting (for some types of voters) might be observed (Ryan, 2011; Watts, 2014). In our case, despite voters' prior and preferences being homogeneous, the probability of voting correctly might decrease when the network is denser, depending on the specific positioning of new links.

Given two networks, while the optimal value might be higher in the denser one, the lower bound of optimal values is achieved under the complete network (via the optimal public communication strategy). This provides the highest probability of

[32] Hahn, von Sydow, and Merdes (2019) show that while voters might individually have the same accuracy as before, the probability of the correct collective decision might be worse on a network.

the correct outcome being implemented among all *optimal* communication strategies (for any possible network). Thus, even though the receivers in our model are not strategic, since they want the implemented outcome to match the true state, the complete network would provide the highest welfare for the receivers. Hence, if the receivers had the option to disclose their information to their neighbors (without a cost), then they would be willing to create links up to a complete network. In other words, while a denser network is not always beneficial to the receivers, the *densest* network is the most beneficial.

5 Conclusion

This paper investigates the optimal persuasion of voters who exchange private information. This is modeled as a fixed network, where neighbors can perfectly observe each other's private messages sent by a centralized body. The sender wants to implement a certain proposal and commits in advance to a communication strategy which sends correlated messages to all receivers. This presents several difficulties as the sender's problem cannot be readily simplified. Crucially, while there are parallels to the empty network case, straightforward strategies or strategies which reveal the truth in state X are not optimal in general.

The paper tests the naive intuition that more information provided to the receivers through the network would make them less manipulable. This is true in some cases (e.g. on a star or when adding links to an empty network). However, the presence of a network structure does not *always* impede the persuasion abilities of the sender. In fact, there exist many non-empty networks on which the sender can do as well as on the empty network (line, circle, networks with complete components).

Interestingly, it is possible that given two networks, the sender achieves a higher value on the denser one (e.g. star and circle). More importantly, the value of an optimal communication strategy does not monotonically decrease when we add links to a network. While in many cases the sender's persuasion capabilities are not affected when a link is added and the optimal value stays the same, in others it is even possible that the sender benefits from more communication among the receivers. In particular, the optimal value can experience significant fluctuations, reaching its upper bound by the addition of a single link. This is due to the fact that in some network structures, additional connections enable the sender to fully exploit the channels of information transmission among agents to her benefit. Moreover, this can be achieved by relatively simple communication strategies, which use very few signals and in which the cardinality of the message space is two.

Our results imply that simply encouraging more communication among voters is not necessarily a good solution to collective decision making problems. In fact, increased communication might make the implementation of the "correct" outcome less likely, which harms welfare. Thus, a policy intervention that encourages the creation of more social ties requires a specific analysis of the network structure to ensure maximum efficacy, lest it yield counterproductive results.

An interesting direction for future research would be to test to what extent optimal communication under different networks exhibits a simple structure given that agents use pure actions, or if certain networks would require optimal communication strategies to employ many more messages and incorporate more information garbling. Additionally, to bridge the gap between the current paper and settings where information flows freely through all directed paths, one might investigate the intermediate cases of limited information transmission, where private messages are shared beyond direct neighbors but not to all agents on a directed path.

A Proofs

Proof of Proposition 3.2 Let $\pi \in \Pi$. For each $i \in N$, assume that $|S_i^\pi(g)| = c(i)$. Let $R(i) = \left\{m_i^1, \ldots, m_i^{c(i)}\right\} \subseteq S_i$ be a set of distinct messages for i. Moreover for any $j \in N$, $q \in \{1, \ldots, c(i)\}$, and $q' \in \{1, \ldots, c(j)\}$ let $m_i^q \neq m_j^{q'}$.

For each $i \in N$, let $\phi_i : S_i^\pi(g) \to R(i)$ be a bijection, so each *information neighborhood* of i is mapped to a *unique message* in $R(i)$. For each $\omega \in \Omega$ and $s' \in S$, define $\pi' \in \Pi$:

$$\pi'(s'|\omega) = \begin{cases} \pi(s|\omega) & \text{if } \phi_i(s_i(g)) = s'_i, \quad \forall i \in N, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the definition of π' implies that there is a bijection $\phi : S^\pi \to S^{\pi'}$ such that for each $i \in N$, $\phi(s) = s'$ if and only if $\phi_i(s_i(g)) = s'_i$. Hence, π' is a communication strategy.

We want to show that the value of π' under the empty network is equal to the value of π under g, i.e., $V_k^{\pi'}(g_0) = V_k^\pi(g)$. What remains to be shown is that each receiver i has the same posterior belief upon observing $s_i(g)$ under π and upon observing $\phi_i(s_i(g))$ under π'. Let $s' \in S^{\pi'}$ be such that $s'_i \in \left\{m_i^1, \ldots, m_i^{c(i)}\right\}$. For any $\omega \in \Omega$, we have

$$\begin{aligned}\lambda_i^{s'}(\omega) &= \frac{\sum_{s \in S^{\pi'}: s_i = s'_i} \pi'(s|\omega)\lambda^0(\omega)}{\sum_{\omega' \in \Omega} \sum_{s \in S^{\pi'}: s_i = s'_i} \pi'(s|\omega')\lambda^0(\omega')} = \frac{\sum_{s \in S^\pi: s_i(g) = \phi^{-1}\left(s'_i\right)} \pi(s|\omega)\lambda^0(\omega)}{\sum_{\omega' \in \Omega} \sum_{s \in S^\pi: s_i(g) = \phi^{-1}\left(s'_i\right)} \pi'(s|\omega')\lambda^0(\omega')} \\ &= \frac{\sum_{s \in A_i^\pi(g, \phi^{-1}(s'))} \pi(s|\omega)\lambda^0(\omega)}{\sum_{\omega' \in \Omega} \sum_{s \in A_i^\pi(g, \phi^{-1}(s'))} \pi(s|\omega')\lambda^0(\omega')} = \lambda_i^{\phi^{-1}(s'), g}(\omega).\end{aligned}$$

Thus, for each $s \in S^\pi$ it holds that $\alpha^{\pi, g}(s) = \alpha^{\pi', g_0}(\phi(s))$. Hence, $V_k^{\pi'}(g_0) = V_k^\pi(g)$. Since any $\pi \in \Pi$ on some network g can be replicated on the empty network, $V_k^n \geq V_k^\pi(g)$. $\square$

Proof of Lemma 3.3 First, note that since $\bar{N}_i(g) = \bar{N}_j(g)$, we have $A_i^\pi(g,s) = A_j^\pi(g,s)$. Hence, i and j have the same posterior belief, i.e. for any $\omega \in \Omega$ and any $s \in S^\pi$ it holds that $\lambda_i^{s,g}(\omega) = \lambda_j^{s,g}(\omega)$.

Let $|S_i^\pi \times S_j^\pi| = c$. Let $R = \{m^1, \ldots, m^c\}$ be a set of distinct messages. Define a bijection $\phi : S_i^\pi \times S_j^\pi \to R$. That is, for any *tuple* $(s_i, s_j), (t_i, t_j) \in S_i^\pi \times S_j^\pi$ it holds that $\phi(s_i, s_j) = \phi(t_i, t_j)$ if and only if $(s_i, s_j) = (t_i, t_j)$, so that each distinct *combination of messages* of i and j (and not every distinct neighborhood) is mapped to a distinct message in R.

Define $S' = \left\{s' \in S |\, s \in S^\pi, s'_{-ij} = s_{-ij} \text{ and } \phi(s_i, s_j) = s'_i = s'_j \in R\right\}$. In words, S' consists of signals obtained by replacing the messages of i and j with distinct messages in R (for each distinct message combination) and leaving the other receivers' messages unchanged, in each signal in S^π. Let $\tau : S^\pi \to S'$ be a bijection such that for any $s \in S^\pi$ we have $\tau(s) = s'$ if $\tau(s_i, s_j) = s'_i = s'_j$ and $s'_{-ij} = s_{-ij}$.

For every $s \in S^\pi$ and $\omega \in \Omega$, define $\pi'(\tau(s)|\omega) = \pi(s|\omega)$. It is clear that π' is a communication strategy. Note that since the probability weights are the same under π and π', receivers i and j still have the same posterior belief under π', i.e. for any $\omega \in \Omega$ and $s \in S^{\pi'}$ it holds that $\lambda_i^{s,g}(\omega) = \lambda_j^{s,g}(\omega)$.

Next, we show that for any $r \in \bar{N}_i(g)$, $\omega \in \Omega$, and $s \in S^\pi$ we have $\lambda_r^{s,g}(\omega) = \lambda_r^{\tau(s),g}(\omega)$. That is,

$$\lambda_r^{s,g}(\omega) = \frac{\sum_{t \in A_r^\pi(g,s)} \pi(t|\omega)\lambda^0(\omega)}{\sum_{\omega' \in \Omega} \sum_{t \in A_r^\pi(g,s)} \pi(t|\omega')\lambda^0(\omega')} = \frac{\sum_{t \in A_r^\pi(g,s)} \pi'(\tau(t)|\omega)\lambda^0(\omega)}{\sum_{\omega' \in \Omega} \sum_{t \in A_r^\pi(g,s)} \pi'(\tau(t)|\omega')\lambda^0(\omega')}$$

$$= \frac{\sum_{t' \in A_r^{\pi'}(g,\tau(s))} \pi'(t'|\omega)\lambda^0(\omega)}{\sum_{\omega' \in \Omega} \sum_{t' \in A_r^{\pi'}(g,\tau(s))} \pi'(t'|\omega')\lambda^0(\omega')} = \lambda_r^{\tau(s),g}(\omega).$$

Finally, any $r \notin \bar{N}_i(g)$ has the same posterior belief under π and π', as it is not affected by the transformation. Hence, $V_k^{\pi'}(g) = V_k^\pi(g)$. □

Proof of Proposition 4.2 We will provide an optimal communication strategy for the case of $|\{i \in N : \delta_i^{g'} = 0\}| = k$, which also yields value V_k^n for each network obtained by adding a link to the empty network up to that point, that is for all networks g' with $|\{i \in N : \delta_i^{g'} = 0\}| > k$. Assume that $\lambda^0(Y)/\lambda^0(X) = \ell$. Moreover, let $|\{i \in N : \delta_i^g = 0\}| = q \geq k$ and $2k \geq n$. So, there are q singleton receivers and $n - q$ connected receivers. Denote the set of singleton receivers by N^q and the set of connected receivers by N^c. Let $S' = \{x, y\}^n$. Define:

$$R = \{s \in S' : \forall i \in N^c, s_i = x \text{ and } |\{j \in N^q : s_j = x\}| = k - (n - q)\}.$$

In words, R is the set of signals in which all connected receivers and $k - n + q$ of the singleton receivers observe x. Note that $k - (n - q)$ is the required amount of x votes to fulfil the quota given that all connected receivers vote for x. Moreover, $|R| = \binom{q}{k-n+q}$. Finally, define:

$$T = \{t \in S' : \forall i \in N^c, t_i = y \text{ and } |\{j \in N^q : t_j = x\}| = k\}.$$

So, T is the set of signals in which all $n-q$ connected receivers and $q-k$ singletons observe y, while k singleton receivers observe x. Here $|T|=\binom{q}{k}$.

Recall that $\mathbf{x}$ is such that $\mathbf{x}_i = x$ for all $i \in N$. Define π as follows:

$$\pi\left(s|\omega\right) = \begin{cases} 1 & \text{if } s = \mathbf{x} \text{ and } \omega = X, \\ 1 - \frac{n}{k\ell} & \text{if } s = \bar{y} \text{ and } \omega = Y, \\ \frac{n-k}{q\ell}\binom{q-1}{k-1}^{-1} & \text{if } s \in T \text{ and } \omega = Y, \\ \frac{1}{\binom{q}{k-n+q}\ell} & \text{if } s \in R \text{ and } \omega = Y. \end{cases}$$

It can be easily checked that π is a communication strategy:

$$\begin{aligned} \sum_{s \in S^\pi} \pi(s|Y) &= 1 - \frac{n}{k\ell} + \binom{q}{k}\frac{n-k}{q\ell}\binom{q-1}{k-1}^{-1} + \binom{q}{k-n+q}\frac{1}{\binom{q}{k-n+q}\ell} \\ &= 1 - \frac{n}{k\ell} + \frac{n-k}{q\ell}\frac{q}{k} + \frac{1}{\ell} = 1 - \frac{n}{k\ell} + \frac{n}{k\ell} = 1. \end{aligned}$$

We will show that $V_k^\pi(g) = V_k^n = \lambda^0(X)(n+k)/k$. Under π, the connected agents always observe the same message. For any $s \in S^\pi$ with $s_i = x$ for all $i \in N^c$, we denote the information neighborhood $s_i(g)$ of a connected receiver by $\tilde{x}(i)$. Note that for any $i \in N^c$, we have $\pi_i(\tilde{x}(i)|Y) = \frac{\binom{q}{k-n+q}}{\binom{q}{k-n+q}\ell} = 1/\ell$. Hence, for any $i \in N^c$ and $s \in S^\pi$ with $s_i(g) = \tilde{x}(i)$ it holds that:

$$\lambda_i^{s,g}(X) = \frac{\pi_i(\tilde{x}(i)|X)\lambda^0(X)}{\pi_i(\tilde{x}(i)|X)\lambda^0(X) + \pi_i(\tilde{x}(i)|Y)\lambda^0(Y)} = \frac{\lambda^0(X)}{\lambda^0(X) + \frac{1}{\ell}\lambda^0(Y)} = \frac{1}{2}.$$

Thus, a connected receiver i votes in favor of x upon observing $\tilde{x}(i)$.

Now, let $i \in N^q$. The probability of i observing x in state Y is:

$$\begin{aligned} \pi_i(x|Y) &= \sum_{s \in S^\pi : s_i = x} \pi(s|Y) = \sum_{s \in R : s_i = x} \pi(s|Y) + \sum_{t \in T : t_i = x} \pi(t|Y) \\ &= \frac{\binom{q-1}{k-n+q-1}}{\binom{q}{k-n+q}\ell} + \binom{q-1}{k-1}\frac{n-k}{q\ell}\binom{q-1}{k-1}^{-1} = \frac{k-n+q}{q\ell} + \frac{n-k}{q\ell} = \frac{1}{\ell}. \end{aligned}$$

Similar calculations as in the connected receiver case follow and thus, each singleton receiver has posterior $1/2$ that the state is X upon observing x. The value of π is then:

$$\begin{aligned} V_k^\pi(g) &= \lambda^0(X) \cdot 1 + \lambda^0(Y)\left(\frac{n-k}{k\ell} + \frac{1}{\ell}\right) = \lambda^0(X) + \lambda^0(Y)\frac{n}{k\ell} \\ &= \lambda^0(X) + \lambda^0(Y)\frac{n}{k}\frac{\lambda^0(X)}{\lambda^0(Y)} = \frac{n+k}{k}\lambda^0(X) = V_k^n. \quad \square \end{aligned}$$

Proof of Proposition 4.4 As all components are complete, all of their elements can be sent the same private message within every signal by Lemma 3.3. Let $q_* \in \mathbb{N}$

be such that for each $\mathcal{C}' \in \mathfrak{C}^{q_*}$ it holds that $\sum_{C\in\mathcal{C}'}|C| \geq k$ and assume that there is no $q' < q_*$ with the same property. Note that there are $\binom{\ell}{q_*}$ many ways to choose q_* components such that the total number of receivers in the components is at least k. Then, by Theorem 3.1 it follows that there exists $\pi \in \Pi$ such that $V_k^{\pi}(g) = V_{q_*}^{\ell}$. ☐

Proof of Proposition 4.6 It follows readily that the value of the strategy outlined in Proposition 4.4 can be achieved with an alternative strategy which sends only a single y message per signal to every complete component in the network and uses an all-x signal with probability 1 in state X. ☐

Proof of Proposition 4.8 In the empty network the value corresponding to $k = q\alpha, n = q\beta$ (for $q = 2, 3$) is the same as the value for $k' = \alpha, n' = \beta$, since $V_k^n(\lambda^0) = \min\left\{\frac{n+k}{k}\lambda^0(X), 1\right\}$ (Theorem 3.1) and $\frac{n+k}{k}\lambda^0(X) = \frac{q\alpha+q\beta}{q\beta}\lambda^0(X) = \frac{\alpha+\beta}{\beta}\lambda^0(X)$. Therefore, if the network allows uniform treatment of parts with the minimal necessary size (q) so that an equal number of nodes in every part has a neighborhood with at least one y message in it, the setup becomes equivalent to the empty network and allows obtaining the optimal value with private communication, so that if $V_k^n = V_{q\alpha}^{q\beta}$, then $V_k^n = V_{\alpha}^{\beta}$. ☐

Proof of Proposition 4.10. Consider communication strategy π given in the table, where $w_1 = r_k^*$, $w_2 = r_0^*$ from Proposition 3.1 and $a = n - 2 - k$. This makes a total of $n + 2$ signals. Every node observes a message y in their information neighborhood in exactly $a + 3$ signals. This leaves $n - 1 - a$ signals in which i and all neighbors of i observe x. Given $s' \in S^{\pi}$, denote the information neighborhood $s_i'(g)$ of $i \in N$ by $\tilde{x}(i)$ if for all $j \in \bar{N}_i(g)$ it holds that $s_j' = x$. Let $i \in N$ and $s \in S^{\pi}$ be such that $s_i(g) = \tilde{x}(i)$.

π	$\omega = X$	$\omega = Y$
$(x, x, x, x, \ldots, x, x, x, x)$	1	0
$(\underbrace{y, \ldots, y}_{a}, x, x, x, \ldots, x, x)$	0	$\frac{w_1}{n}$
$(x, \underbrace{y, \ldots, y}_{a}, x, x, \ldots, x, x)$	0	$\frac{w_1}{n}$
$\ldots$	$\ldots$	$\ldots$
$(x, x, \ldots, x, x, \underbrace{y, \ldots, y}_{a}, x)$	0	$\frac{w_1}{n}$
$(x, x, \ldots, x, x, x, \underbrace{y, \ldots, y}_{a})$	0	$\frac{w_1}{n}$
$\ldots$	$\ldots$	$\ldots$
$(\underbrace{y, \ldots, y}_{a-1}, x, x, \ldots, x, x, y)$	0	$\frac{w_1}{n}$
$(y, y, y, y, \ldots, y, y, y, y)$	0	w_2

It holds that:

$$\lambda_i^{s,g}(X) = \frac{\sum_{t\in A_i^\pi(g,s)} \pi(t|X)\lambda^0(X)}{\sum\limits_{t\in A_i^\pi(g,s)} \pi(t|X)\lambda^0(X) + \sum\limits_{t\in A_i^\pi(g,s)} \pi(t|Y)\lambda^0(Y)} = \frac{\lambda^0(X)}{\lambda^0(X) + \frac{(n-2-a)w_1}{n}\lambda^0(Y)}$$

Therefore,

$$\lambda_i^{s,g}(X) = \begin{cases} \frac{\lambda^0(X)}{1\lambda^0(X)+\frac{(n-2-a)}{n}\frac{\lambda^0(X)n}{\lambda^0(Y)k}\lambda^0(Y)} = \frac{1}{1+\frac{n-2-a}{k}} = 1/2 & \text{if } \lambda^0(X) < \frac{k}{k+n}, \\ \frac{\lambda^0(X)}{\lambda^0(X)+(n-2-a)\lambda^0(Y)/n} \geq 1/2 & \text{if } \lambda^0(X) \geq \frac{k}{k+n}, \end{cases}$$

as the second condition always holds for $\lambda^0(X) \geq \frac{k}{k+n}$.
In each signal $s \in S^\pi$ such that there exists $i \in N$ with $s_i = y$, there are $n-2-a$ many receivers $j \in N$ such that $s_j(g) = \tilde{x}(j)$. Hence, in each such signal at least k receivers are persuaded. The value is equal to the empty network one, i.e. $V_k^\pi(g) = \lambda^0(X)\cdot 1 + \lambda^0(Y)w_1 = \min\left\{\frac{n+k}{k}, 1\right\} = V_k^n$. □

Proof of Proposition 4.13. First, we introduce the notion of an "anchor", as it will be central in proving this and other results.

Definition A.1. For any $\pi \in \Pi$ and $s \in S^\pi$, the signal s is an *anchor* if $\pi(s|X)\lambda^0(X) \geq \pi(s|Y)\lambda^0(Y)$. The set of all anchors is denoted by $An(\pi)$.

It is easy to see that if x is implemented with positive probability under some $\pi \in \Pi$, then π must have an anchor, and that every x-vote under a communication strategy with positive value is associated with at least one anchor. Moreover, if a receiver i can uniquely identify the signal realization as an anchor, he votes for x.

Second, Lemma A.2 shows that without loss of generality any $c \in Q(g)$ is *not pivotal* whenever it votes for x.

Lemma A.2. *Let $g \in G(N)$ be star-like, $k < n$, and let $\pi \in \Pi$ be a communication strategy such that for any $c \in Q(g)$, there exists $s \in S^\pi$ with $\alpha_c^{\pi,g}(s) = x$ and c is pivotal in s. Then there exists $\pi' \in \Pi$ such that for any $s' \in S^{\pi'}$ with $\alpha_c^{\pi',g}(s') = x$, node c is not pivotal in s' and $V_k^{\pi'}(g) = V_k^\pi(g)$.*

Proof. First observe that for any $c, c' \in Q(g)$, $N_c(g) = N_{c'}(g)$. Thus, by Lemma 3.3, either all $c \in Q(g)$ are pivotal or none of them is. Take any $c \in Q(g)$ and note that $t_c(g) = t$ for all $t \in S^\pi$ (the information neighbourhood of c is t). Therefore, $\lambda_c^{s,g}(X) \geq 1/2$ if and only if s is an anchor, so if $\alpha_c^{\pi,g}(s) = x$ for some $s \in S^\pi$, it follows that $s \in An(\pi)$. Moreover, if c is pivotal in s and $k < n$, there exists $i \in N \setminus Q(g)$ such that $\alpha_i^{\pi,g}(s_i(g)) = y$. Furthermore, i votes for y in any other signal he associates with s, i.e. for any $t \in A_i^\pi(g,s)$ it holds that $\alpha_i^{\pi,g}(t_i(g)) = y$. Thus, replacing i's message in the anchor s with a *unique* message would enable i to uniquely identify the anchor and hence *reverse* i's vote from y to x in s (see the remark after Definition A.1). Since c votes for x if and only if the signal is an anchor, c's vote would *not* *change* if the probabilities of the communication strategy do not change. It would also keep everyone else's vote the same, as i is observed only by $c' \in Q(g)$.

To this end, let $S' = \{s \in S^{\pi} | \ \alpha_c^{\pi,g}(s) = x \text{ and } c \text{ is pivotal in } s\}$. In particular, let $S' = \{s^1, \ldots, s^r\}$. Let $t \in S'$ be such that there is $i \in N$ with $\alpha_i^{\pi,g}(t) = y$. Note that such i exists as per the discussion above. Let $R = \{m^1, \ldots, m^r\}$ be a set of distinct messages such that for any $j \in \{1, \ldots, r\}$, $m^j \notin S_i^{\pi}$. Let $S'' \subseteq S$ and define a bijection $\phi : S' \to S''$ such that for every $j \in \{1, \ldots, r\}$ and $s^j \in S'$ it holds that $\phi_i\left(s_i^j\right) = m^j$ and $\phi_{-i}\left(s_{-i}^j\right) = s_{-i}^j$.

Now, for any $\omega \in \Omega$ and any $s' \in (S^{\pi} \setminus S') \cup S''$, let $\pi' \in \Pi$ be defined as

$$\pi'(s'|\omega) = \begin{cases} \pi(\phi^{-1}(s')|\omega) & \text{if } s' \in S'', \\ \pi(s'|\omega) & \text{if } s' \in S^{\pi} \setminus S'. \end{cases}$$

That is, $S^{\pi'} = (S^{\pi} \setminus S') \cup S''$. By the definition of an anchor, for any $t \in S''$, we have $\alpha_i^{\pi',g}(t_i(g)) = x$, since *by construction* i observes the unique message only in this anchor t. Therefore, there are $k+1$ receivers voting for x in t, which implies that node c is *no longer pivotal*. Since π' preserves all probability weights it is true that if $s \in S''$, then $s \in An(\pi')$.

Moreover, i's votes in signals that are not in S'' are unchanged, i.e. for any $t \in S^{\pi'} \setminus S''$, $\alpha_i^{\pi,g}(t_i(g)) = \alpha_i^{\pi',g}(t_i(g))$. This holds because if $s \in S'$ and $t \in A_i^{\pi}(g, s)$, then it holds that $\alpha_i^{\pi,g}(t_i(g)) = y$ by the definition of S'.The transformation removes the anchors in S'' from the association set of every signal $t \in S^{\pi'} \setminus S''$, so if $s \in S''$ then for every $t \in S^{\pi'} \setminus S''$ it is true that $t \notin A_i^{\pi'}(g, s)$. This makes it even less likely that i would vote for x in such signals, preserving its y votes between π and π'. The transformation does not affect any other receivers' votes, hence $V_k^{\pi'}(g) = V_k^{\pi}(g)$. □

By Lemma A.2, assume without loss of generality that under π, node c is not pivotal in signals in which he votes for x. In other words, if node c votes for x, then so do at least k other nodes.

For all nodes $i \in N$ and all $t \notin An(\pi)$, if $t_c \neq s_c$ for all $s \in An(\pi)$ then $\lambda_i^{t,g}(X) < 1/2$. So, if in a certain signal $c' \in Q(g)$ receives a message different from all anchors, *all receivers* would vote y in this signal.

Note that for two anchors $s, t \in An(\pi)$ with $s_c \neq t_c$, it holds that $A^{\pi}(g, s) \cap A^{\pi}(g, t) = \emptyset$. Define a bijection $\phi : S^{\pi} \to S'$ such that $\phi(s) = s'$ if $s'_c = x$ and for every $j \in N \setminus Q(g)$, $s'_j = (s_j, s_c)$. That is, in signals in S' node c always observes x and messages of all $j \in N \setminus Q(g)$ are modified so that they contain the information previously provided by node c in signal s. In other words, the information c reveals to nodes in $N \setminus Q(g)$ is shifted to them while c observes the same message x in every signal.

For every $s' \in S'$ such that $\phi(s) = s'$ and $\omega \in \Omega$, let $\pi' \in \Pi$ be defined by $\pi'(s'|\omega) = \pi(\phi^{-1}(s')|\omega)$. As the probabilities of corresponding signals are the same under π' as under π and c's information under π is shifted to nodes in $N \setminus Q(g)$

under π' (which are observed by c), node c's vote does not change. Moreover, the votes of the nodes in $N \setminus Q(g)$ do not change either. To see this, note that for any $t' \in A_i^{\pi'}(g, s')$ there exists $t \in A_i^{\pi}(g, s)$ such that $\phi(t) = t'$. This, together with the definition of ϕ implies that $\sum_{t' \in A_i^{\pi'}(g,s')} \pi'(t'|\omega) = \sum_{t \in A_i^{\pi}(g,s)} \pi(t|\omega)$. Thus, each $j \in N \setminus Q(g)$ has the same posterior belief upon observing $s \in S^{\pi}$ and $\phi(s) \in S^{\pi'}$. Hence, $V_k^{\pi'}(g) = V_k^{\pi}(g)$.

As node c always observes the same message under π', it has no effect on the voting decisions of the other receivers. Moreover, node c is never pivotal in signals in which he votes for x. Observe that under π', it is *as if* c is always voting for y, since *all* of his y votes are preserved in π' and *none* of his x votes have an impact on whether a signal implements x or not.

We can consecutively repeat the above procedure for node c for all $c' \in Q(g)$. Thus, the setup is equivalent to having an empty network with $n - m$ nodes. Hence, we can assume without loss of generality that there exists a communication strategy $\pi'' \in \Pi$ with $|S_i^{\pi''}|= 2$ for any $i \in N$ such that $V_k^{n-m} = V_k^{\pi''}(g) \geq V_k^{\pi'}(g)$. □

Proof of Proposition 4.14 Take $c \in Q(g)$. Note that any observation that holds for c, will also hold for any $c' \in Q(g)$ by Lemma 3.3.

Observe that *in an optimal strategy*, if node c is *never* pivotal in signals which implement x, then any transformation of the communication strategy which preserves the other nodes' votes will *not* change the value of the strategy.

Lemma A.3. *For any optimal $\hat{\pi} \in \Pi$ there exists an optimal $\hat{\pi}' \in \Pi$ such that: (i) c is never pivotal in signals which implement x, (ii) $|S_c^{\hat{\pi}'}|= 1$, and (iii) $V_k^{\hat{\pi}'}(g) = V_k^{\hat{\pi}}(g)$.*

If the lemma holds, then it follows that node c never reveals or receives any consequential information (in terms of value). *Observe that it is irrelevant if c is pivotal in signals in which it votes for y. If such a signal implements x, changing the vote of c does not make a difference. If the signal does not implement x and c is pivotal in it, changing the vote of c from y to x while keeping all other votes constant will strictly increase the value of the strategy, contradicting the assumption that it is optimal.*

Therefore, if the lemma holds, in the best case-scenario for the sender, the situation would be equivalent to an empty network with $n - m$ nodes and quota k where $V_k^{\hat{\pi}'}(g) \leq V_k^{n-m}$.

Proof. ***(i)*** Suppose that there is a signal $t \in S^{\hat{\pi}}$ in which node c votes for x. Hence, there is at least one anchor $s \in An(\hat{\pi})$ with $s_i = t_i$ for all $i \in C$ and for every $r \in S^{\hat{\pi}}$ such that $r_i = s_i$ for all $i \in C$, node c also votes for x. The possible voting patterns of nodes in C in such signals are: *(a)* all x; *(b)* c votes x and zero or more nodes in $C \setminus Q(g)$ vote for x.

In case (a), c is not pivotal. Consider case (b). It must be true that if some node $\ell \in C \setminus Q(g)$ votes for y this is because it associates t with *more* signals than c. In other words, in all signals $r \in S^{\hat{\pi}}$ where $(r_\ell, r_c) = (t_\ell, t_c)$ node ℓ votes for y and this *includes* the signals in which c *does not* vote for x. (This also includes the associated anchors.) Thus, $A_c^{\hat{\pi}}(g,t) \subsetneq A_\ell^{\hat{\pi}}(g,t)$.

Notice the trivial fact that for every $s,t \in S^{\hat{\pi}}$ with $s_c \neq t_c$ and $i \in C$, it holds that $A_i^{\hat{\pi}}(g,s) \cap A_i^{\hat{\pi}}(g,t) = \emptyset$. So, whenever node c receives a different message in different signals, these signals belong to *disjoint* association sets and the same observation for every $i \in C$.

Let $S_c^{\hat{\pi}} = \left\{m^1, \ldots, m^\ell\right\}$ and let T be the set of signals in which receiver ℓ votes for y and receiver c votes for x. That is,

$$T = \left\{t \in S^{\hat{\pi}} \mid \alpha_\ell^{\hat{\pi},g}(t_\ell(g)) = y \text{ and } \alpha_c^{\hat{\pi},g}(t_c(g)) = x\right\}.$$

Define a bijection such that in signals in $\hat{\pi}$ in which ℓ votes for y and c votes for x, the message of ℓ is changed to a *unique* message that is specific to each distinct message of c and keep all other messages the same. Formally, let $T' \subsetneq S$ and define $\phi : T \to T'$ such that for any $t \in T$ it holds that $\phi(t) = t'$ if $t'_\ell = (t_\ell, t_c) \in S'_\ell \setminus S_\ell^{\hat{\pi}}$ and $t'_{-\ell} = t_{-\ell}$.

Now for any $\omega \in \Omega$ define a new strategy $\hat{\pi}' \in \Pi$, which transforms the signals in T according to ϕ and keeps all other signals the same while preserving the probability weights:

$$\hat{\pi}'\left(s'|\omega\right) = \begin{cases} \hat{\pi}(s'|\omega) & \text{if } s' \in S^{\hat{\pi}} \setminus T, \\ \hat{\pi}(\phi^{-1}(s')|\omega) & \text{if } s' \in T. \end{cases}$$

Let $s' \in S^{\hat{\pi}'}$ be such that $\phi(s) = s'$ for some $s \in T$. Then,

$$\lambda_\ell^{s',g}(X) = \frac{\sum_{t' \in A_\ell^{\hat{\pi}'}(g,s')} \hat{\pi}'(t'|X)\lambda^0(X)}{\sum_{\omega\in\Omega}\sum_{t' \in A_\ell^{\hat{\pi}'}(g,s')} \hat{\pi}'(t'|\omega)\lambda^0(\omega)} = \frac{\sum_{t' \in A_\ell^{\hat{\pi}'}(g,s')} \hat{\pi}(\phi^{-1}(t')|X)\lambda^0(X)}{\sum_{\omega\in\Omega}\sum_{t' \in A_\ell^{\hat{\pi}'}(g,s')} \hat{\pi}(\phi^{-1}(t')|\omega)\lambda^0(\omega)}$$

$$= \frac{\sum_{t \in A_\ell^{\hat{\pi}}(g,s)\cap A_\ell^{\hat{\pi}}(g,s)} \hat{\pi}(t|X)\lambda^0(X)}{\sum_{\omega\in\Omega}\sum_{t \in A_\ell^{\hat{\pi}}(g,s)\cap A_\ell^{\hat{\pi}}(g,s)} \hat{\pi}(t|\omega)\lambda^0(\omega)} = \frac{\sum_{t \in A_\ell^{\hat{\pi}}(g,s)\subseteq T} \hat{\pi}(t|X)\lambda^0(X)}{\sum_{\omega\in\Omega}\sum_{t \in A_\ell^{\hat{\pi}}(g,s)\subseteq T} \hat{\pi}(t|\omega)\lambda^0(\omega)} \geq \frac{1}{2},$$

where $\phi(t) = t'$ and the third equality follows from the definition of ϕ; $A_\ell^{\hat{\pi}}(g,s) \cap A_c^{\hat{\pi}}(g,s) = A_c^{\hat{\pi}}(g,s) \subseteq T$ follows from $A_c^{\pi}(g,t) \subsetneq A_\ell^{\pi}(g,t)$ and the inequality follows from the definition of case (b). Similarly, it holds that $\lambda_c^{s',g}(X) \geq 1/2$. This implies that in $\hat{\pi}'$ node ℓ will vote for x whenever c votes for x in $\hat{\pi}'$. Additionally, node c will keep its vote for x in the corresponding signals in $\hat{\pi}$ and $\hat{\pi}'$. Thus, the transformation does not change the vote of c in any signals. It only *increases* the number of x votes. Observe that for $s \in A_\ell^{\pi}(g,t) \setminus A_c^{\pi}(g,t)$ such that $t \in T$, it holds that $\alpha_\ell^{\hat{\pi},g}(t_\ell(g)) = y$ and the transformation will not decrease the value, as in such s nodes ℓ and c must already be voting for y. Hence, $V_k^{\hat{\pi}'}(g) = V_k^{\hat{\pi}}(g)$.

Such a transformation produces $k+1$ x votes every time c votes for x in the original strategy, making c not pivotal in such signals. Moreover, signals which do not implement x under $\hat{\pi}$ but have c vote for x will implement x under π' after the transformation.

Therefore, for any optimal $\hat{\pi} \in \Pi$, there exists $\hat{\pi}' \in \Pi$ such that in every signal in which c votes for x in $\hat{\pi}$, nodes in C vote for x in $\hat{\pi}'$ such that $V_k^{\hat{\pi}'}(g) = V_k^{\hat{\pi}}(g)$. Thus, c is never pivotal in $\hat{\pi}'$ in signals which implement x.

(ii) Keeping the messages of nodes in $N \setminus C$ the same as in $\hat{\pi}$ (and in $\hat{\pi}'$), from here onward, the transformation is the same as in the star network (see proof of Proposition 4.13). Note that for two anchors $s, t \in S^{\hat{\pi}}$ with $s_c \neq t_c$, it holds that $A_c^{\pi}(g, s) \cap A_c^{\pi}(g, t) = \emptyset$. Let $S' \subseteq S$. Define a bijection $\tau : S^{\hat{\pi}'} \to S'$ such that $\tau(s) = s'$ if $s'_c = x$, for $j \in C \setminus Q(g)$, $s'_j = (s_j, s_c)$, and for $\ell \in N \setminus C$, $s'_\ell = s_\ell$. That is, in signals in S' node c always observes x and the messages of nodes $C \setminus Q(g)$ are modified so that they contain the information previously provided by node c in signal s. So, the information that node c reveals to nodes in $C \setminus Q(g)$ is shifted to them while node c observes the same message in every signal.

For any $s' \in S'$ such that $\tau(s) = s'$ and $\omega \in \Omega$, let $\hat{\pi}'' \in \Pi$ be defined by $\hat{\pi}''(s'|\omega) = \hat{\pi}'(\tau^{-1}(s')|\omega)$. As the probabilities of corresponding signals are the same under $\hat{\pi}''$ as under $\hat{\pi}'$ and node c's information under $\hat{\pi}'$ is shifted to nodes in $C \setminus Q(g)$ under $\hat{\pi}''$ (which are observed by node c), node c's vote does not change. Moreover, the votes of nodes in $C \setminus Q(g)$ and in $N \setminus C$ do not change either. To see this, note that for any $i \in N$ and $t' \in A_i^{\hat{\pi}''}(g, s')$ there exists $t \in A_i^{\hat{\pi}'}(g, s)$ such that $\tau(t) = t'$. This, together with the definition of τ implies that $\sum_{t' \in A_i^{\hat{\pi}''}(g,s')} \hat{\pi}''(t'|\omega) = \sum_{t \in A_i^{\hat{\pi}'}(g,s)} \hat{\pi}'(t|\omega)$. Thus, every node has the same posterior belief upon observing $s \in S^{\hat{\pi}'}$ and $\tau(s) \in S^{\hat{\pi}''}$.

(iii) Parts (i) and (ii) imply that $V_k^{\hat{\pi}''}(g) = V_k^{\hat{\pi}'}(g) = V_k^{\hat{\pi}}(g)$.

Hence, a communication strategy $\hat{\pi}$ with $V_k^{\hat{\pi}}(g)$ can be transformed into a strategy such that: node c is never pivotal in signals which implement x, it always receives the same message and the strategy preserves the value of the initial strategy. $\square$

Node c is thus a dummy node, whose x votes are inconsequential in the optimal strategy. Its y votes were irrelevant for the value to begin with, which also hold for all $c' \in Q(g)$. The maximum value of such an optimal strategy is therefore V_k^{n-m}.

Finally, we show that there exists $\hat{\pi} \in \Pi$ such that $V_k^{\hat{\pi}}(g') = V_1^2$. Let $S' = \{x, y\}^n$. Define $R = \{s \in S' : \forall i \in (N \setminus C) \cup Q(g) \cup \{h\}, s_i = x, \text{ and } \forall j \in C \setminus (Q(g) \cup \{h\}), s_j = y\}$ and let $t \in S'$ be such that for all $i \in N \setminus C, t_i = y$ and for all $j \in C, t_j = x$. Let $\hat{\pi} \in \Pi$ be defined as

$$\hat{\pi}\,(s|\omega) = \begin{cases} 1 & \text{if } s = \bar{x} \text{ and } \omega = X, \\ min\{\frac{\lambda^0(X)}{\lambda^0(Y)}, \frac{1}{2}\} & \text{if } s = t \text{ and } \omega = Y, \\ min\{\frac{\lambda^0(X)}{\lambda^0(Y)}, \frac{1}{2}\} & \text{if } s \in R \text{ and } \omega = Y. \\ max\{1 - 2\frac{\lambda^0(X)}{\lambda^0(Y)}, 0\} & \text{if } s = \bar{y} \text{ and } \omega = Y. \end{cases}$$

Hence, we have $V_k^{\hat{\pi}}(g') = min\{3\lambda^0(X), 1\}$. $\square$

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