# A Cooperative Theory of Market Segmentation by Consumers

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**Abstract**

We consider a market that may be segmented and is served by a single seller. The surplus of each consumer in a segment depends on the price that the seller optimally charges, and this optimal price depends on the set of consumers in that segment. This gives rise to a novel cooperative game between consumers that determines market segmentation. We study several solution concepts. The most demanding solution concept, the core, requires that there is no objection to the segmentation by a coalition of consumers who benefit by forming a new segment. We show that the core is empty except for trivial cases. We then introduce a new solution concept, stability. A segmentation is stable if for each possible objection by a coalition, there is a counter-objection by a coalition in the original segmentation. We characterize stable segmentations as ones that are efficient and "saturated," which means that the segments are maximal in some sense. We use this characterization to constructively show that stable segmentations exist. Even though stable segmentations are efficient, they need not maximize average consumer surplus, and segmentations that maximize average consumer surplus need not be stable. Our weakest solution concept, fragmentation-proofness, rules out objections by coalitions that include consumers from more than one segment. We show that a segmentation is fragmentation-proof if and only if it is efficient.

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# 1 Introduction

An ongoing debate in academia and policy-making is about how much control consumers should have over their data, and in what forms. To answer these questions, it is important to understand the consequences of consumers' control over their data. A primary consequence of consumers' decisions about what data to disclose is that it affects market segmentation: the seller can use the available data to segment the market and to make targeted offers to different segments. Thus by controlling their data, consumers effectively choose market segmentation.

We develop a cooperative game to explore how markets are segmented by consumers. There is a seller who observes the segmentation that is chosen by consumers, and chooses a profit-maximizing price for each segment. The profit-maximizing price that the seller chooses in each segment depends on the distribution of values in that segment. Thus the surplus of each consumer in a segmentation depends on other consumers that are in the same segment with that consumer. Because consumers may have different preferences over segmentations, it is important to model how consumers interact to determine the segmentation. We model this novel interaction among consumers as a cooperative game with non-transferable utility, in which each consumer's utility from a coalition is her payoff when facing the profit-maximizing price that is offered uniformly to all consumers in that coalition.

We study three solution concepts, the core, stability, and fragmentation-proofness, each more permissive than the previous one. A segmentation is in the core of the game if no subset of consumers objects to it, in the sense that they would benefit from deviating from the segmentation and forming their own segment. We show that the core is non-empty if and only if it is profit-maximizing for the seller to serve the unsegmented market efficiently, that is, the "market is efficient," in which case the unsegmented market is essentially the unique segmentation in the core. Because many markets are inefficient, the core is often empty. So the core seems too demanding as a solution concept.

The next solution concept, stability, is new and requires that for any alternative segmentation, the stable segmentation contains a segment that objects to the alternative segmentation. This means that even though there may be objections to a stable segmentation, these objections are not credible in the sense that any resulting segmentation will have a counter-objection by a segment in the original segmentation.

So if consumers are forward-looking, such a non-credible objection will not succeed in deviating from the segmentation.

We show that stable segmentations always exist and provide a characterization. The characterization shows that a segmentation is stable if and only if it is efficient and "saturated," which means that consumers in each segment are not willing to accept additional consumers from another segment because doing so will increase the price in their own segment. We also show that stable segmentations are Pareto undominated, that is, there is no other segmentation that makes all consumers better off.

We use our characterization to show that a stable segmentation always exists. We also show that multiple stable segmentations may exist. One stable segmentation is the *maximal equal-revenue segmentation*, first discussed in Bergemann, Brooks, and Morris (2015). They show that this segmentation maximizes average consumer surplus over all segmentations. Because we show that this segmentation is stable, a natural question is whether maximizing consumer surplus implies stability or vice versa. We give negative answers to both questions.

Our notion of stability is closely related to the stable set of Morgenstern and Von Neumann (1953). In our setting, a set of segmentations is a stable set if it satisfies internal and external stability. External stability requires that for any segmentation not in the set, there is a segmentation in the set and a segment in this segmentation that objects to the segmentation that is not in the set. Internal stability requires that no segmentation in the set should contain a segment that objects to another segmentation that is also in the set. It follows from our definition of stability that in our setting any singleton set that contains only one stable segmentation is a stable set. Thus, the existence of stable segmentations implies the existence of singleton stable sets. This is notable because in general games neither stable sets nor singleton stable sets are guaranteed to exist. We characterize stable sets and show that any stable set must be a singleton set. A singleton set is a stable set if and only if the segmentation in it is Pareto undominated and satisfies a weak notion of satiation. Because the set of segmentations that are Pareto undominated and satisfy this weak notion of satiation is strictly larger than the set of stable segmentations, there exist singleton stable sets that contain one segmentation that is not stable.

Our third solution concept, fragmentation-proofness, only allows objections by coalitions of consumers that belong to the same segment. We show that a segmentation is fragmentation-proof if and only if it is efficient.

Our work sheds light on the consequences of giving complete control to consumers over how markets are segmented. This control is complete because *any* coalition of consumers can deviate from a segmentation and form their own segment. Our cooperative game takes a reduced-form view of a segmentation and does not explicitly model how segmentations arise from decisions by consumers. But to see one possibility of how disclosure decisions may allow any coalition of consumers to form a segment, suppose that each consumer has a unique identifier that distinguishes that consumer from all other consumers. If the consumer wishes, she can prove to the seller via a verifiable evidence that her identifier is in a set (so she has a verifiable evidence for every set of identifiers that contains her identifier). Now if a coalition of consumers want to form their own segment, each consumer in the coalition can verifiably disclose to the seller that they belong to this set (and this set only). The seller then learns with certainty that all these consumers belong to this coalition (and that this coalition cannot contain any other consumer), and will offer a profit-maximizing price to the coalition. For instance, for consumers with identifiers 2 and 15 to form a coalition, they can each disclose the evidence $\{2, 15\}$ to the seller, proving to the seller that each of them belongs to this set but not allowing the seller to distinguish the from each other. Because no other consumers can disclose this evidence, the seller offers to these two consumers a price that is profit-maximizing to sell to this coalition.

## 1.1 Related literature

Our paper studies how coalition formation by agents affects market outcomes. In this sense, our work is related to Peivandi and Vohra (2021) who consider stability of centralized markets against deviations by coalitions of agents. They show that fragmentation of such markets is unavoidable, despite its efficiency costs, except in special circumstances. They study a bilateral trade setting that is different than ours with a population of consumers and a seller. Further, in cases where centralized markets are fragmented, their paper does not provide a prediction of what the resulting segmentation looks like, and doing so is a main focus of our paper. Another important difference is that in Peivandi and Vohra (2021), a coalition can choose the trading mechanism, whereas in our setting the seller chooses a profit-maximizing price to sell to the coalition.

A recent literature on third-degree price discrimination studies surplus across all possible segmentations of a given market. Bergemann, Brooks, and Morris (2015) identify the set of producer and consumer surplus pairs that result from all segmentations of a given market. An implication of their results is a specification of segmentations that maximize average consumer surplus. Cummings et al. (2020) study an extension in which only certain segmentations may be chosen. Glode, Opp, and Zhang (2018) study optimal disclosure by an informed agent in a bilateral trade setting, and show that the optimal disclosure policy leads to socially efficient trade, even though information is revealed only partially. Ichihashi (2020), Hidir and Vellodi (2018), Braghieri (2017), and Haghpanah and Siegel (2022) consider maximum average consumer surplus when a multi-product seller offers different products in each market segment. These papers can be seen as identifying segmentations that are chosen *ex ante* by a consumer who does not know her own type, because such a consumer chooses the segmentation that maximizes her expected payoff. In contrast, in this paper we study market segmentation ex post by consumers who know their own type. Because agents who know their types have different preferences over segmentations, we need to model the interaction between them, which we do as a cooperative game.

The related papers that study disclosure decisions by agents who know their own types model these interactions as non-cooperative games. Ali, Lewis, and Vasserman (2020) consider voluntary disclosure of data by a single consumer, and analyze the welfare implications of various disclosure policies in both a monopolistic and a competitive environment. Sher and Vohra (2015) study a disclosure setting in which the seller can commit the mechanism that she would use after receiving information. In our setting, the seller cannot commit and will choose a profit-maximizing price for each segment. Acemoglu et al. (2019) and Bergemann, Bonatti, and Gan (2020) also study the consequences of consumers' disclosure decisions on prices and other market outcomes. The main difference between these papers and ours is that we model the interaction between consumers as a cooperative game.

The value of privacy in our game is defined endogenously in equilibrium: the payoff of each consumer depends on the set of other agents who reveal the same information to the seller and are thus pooled together with this consumer. Our work therefore relates to the literature that endogenizes privacy costs (Taylor, 2004; Calzolari and Pavan, 2006; Conitzer, Taylor, and Wagman, 2012; Cummings et al., 2016; Bonatti and Cisternas, 2020; Argenziano and Bonatti, 2021). In these papers privacy is valuable because of dynamic considerations: what an agent reveals today may be shared and used by other agents in the future. In our setting, in contrast, privacy is valuable because it allows a consumer to pool with others and by doing so she can buy the product more cheaply.

# 2 Model

A monopolistic seller faces a unit mass of consumers uniformly distributed on the interval $[0,1]$. Consumers have unit demand for the monopolist's product, whose production cost is normalized to zero. The value of the product for consumer $c \in [0,1]$ is $v(c) \in V = \{v_1, \ldots, v_n\} \subseteq R_+$, where $v_i$ increases in $i$. Let $\mu$ be the Borel measure on the unit interval. The measure of consumers with value $v_i$ is $f(v_i) = \mu(\{c : c \in [0,1], v(c) = v_i\})$. We assume without loss of generality that $f(v_i) > 0$ for every $i \leq n$.

A *coalition* is a measurable subset $C \subseteq [0,1]$ of consumers. Let $f^C(v_i) = \mu(\{c : c \in C, v(c) = v_i\})$ denote the measure of consumers with value $v$ in coalition $C$. A price $p \in V$ is *optimal for a coalition* $C$ if it maximizes the revenue from selling the product to consumers in $C$, that is, for any other price $p' \in V$,

$$p \sum_{i:v_i \geq p} f^C(v_i) \geq p' \sum_{i:v_i \geq p'} f^C(v_i).$$

Since for each $p \notin V$, there exists a price in $V$ with a weakly higher revenue, we do not need to consider prices that are not in $V$.

A *segment* is a pair $(C, p)$, where $C$ is a coalition and $p$ is an optimal price for that coalition. A *segmentation* $S$ is a finite set of segments $\{(C_j, p_j)\}_{j=1,\ldots,k}$ such that $C_1, \ldots, C_k$ partitions the set of consumers $[0,1]$. In words, a segmentation is a partitioning of a the set of all consumers into coalitions, and for each coalition, an optimal price. A segmentation $S$ is trivial if $|S| = 1$. Such a segmentation contains a single segment $\{([0,1], p)\}$, where $p$ is an optimal price for the set of all consumers $[0,1]$. There may be multiple trivial segmentations because multiple prices may be optimal for the set $[0,1]$ of all consumers.

Let $CS(c, p) = \max(v(c) - p, 0)$ denote the surplus of consumer $c$ if she is offered the product at price $p$. If consumer $c$ belongs to an segment $(C, p)$, then her surplus is $CS(c, p)$. Each consumer $c \in [0,1]$ belongs to a unique segment in any segmentation

$S$. Let $p_S(c)$ denote the price in that segment. Then $CS(c, S) = CS(c, p_S(c))$ specifies the surplus of consumer $c$ in segmentation $S$.

In our setting, there are many possible segmentations because there are many possible ways to partition the set of all consumers (and there may be multiple optimal prices for a given coalition). The rest of the paper develops different solution concepts that predict what segmentations would arise as the outcome of strategic choices by coalitions of agents.

# 3 The core

We start with the most demanding solution concept, the core. A segmentation is in the core if no segment $(C, p)$ "objects" to it in the sense that the consumers in the coalition would benefit from jointly deviating from the segmentation and forming their own segment, as formalized below.

**Definition 1 (Objections)** *A segment $(C, p)$ objects to a segmentation $S$ if $CS(c, p) \geq CS(c, S)$ for all consumers $c \in C$, with a strict inequality for some positive measure of consumers $c \in C$.*

Notice that if a segment $(C, p)$ objects to $S$, then $(C, p)$ is not in $S$, as otherwise the surplus of each consumer in the coalition is the same in the segment and in the segmentation. Also notice that the definition would be vacuous if we require the inequality to be strict for all (or even just almost all) consumers in $C$, because optimality of $p$ for $C$ requires that the surplus of consumers with the lowest value in segment $(C, p)$ is zero, and the surplus of any consumer in any segmentation is at least zero. So with strict inequality, there would be no objection to any segmentation.

We define the core to be the set of segmentations to which there is no objection.

**Definition 2 (Core)** *The core is a set of segmentations. A segmentation $S$ is in the core if there exists no segment that objects to $S$.*

Our first result characterizes when the core is empty, and shows that when the core is non-empty then it contains an essentially unique segmentation. We say that two segmentations $S$ and $S'$ are surplus-equivalent if the surplus of each consumer is the same in the two segmentations, that is, $CS(c, S) = CS(c, S')$ for almost all

$c \in [0,1]$. Recall that a segmentation is trivial if it has a single segment containing all consumers.

**Proposition 1** *If price $v_1$ is optimal for the set of all consumers $[0,1]$, then the core consists of all segmentations that are surplus-equivalent to the trivial segmentation $\{([0,1], v_1)\}$. Otherwise, the core is empty.*

**Proof.** Suppose that $v_1$ is optimal for the set of all consumers. Then the coalition-price pair $([0,1], v_1)$ is a segment and there is no objection to the trivial segmentation $\{([0,1], v_1)\}$ because in any segment $(C, p)$, the price is at least $v_1$. For the same reason, any segmentation that is surplus-equivalent to $\{([0,1], v_1)\}$ is also in the core.[1] Now consider a segmentation $S$ that is not surplus-equivalent to $\{([0,1], v_1)\}$, which means that the price in some segment is higher than $v_1$. So for a positive measure of consumers $c$, we have $CS(c, S) < \max(v(c) - v_1, 0)$. Also because the price in any segment is at least $v_1$, for all consumers $c$ we have $CS(c, S) \leq \max(v(c) - v_1, 0)$. Then the segment $([0,1], v_1)$, in which any consumer $c$ obtains surplus equal to $\max(v(c) - v_1, 0)$, objects to $S$ and so $S$ cannot be in the core. We conclude that the core consists of all segmentations that are surplus-equivalent to the trivial segmentation $\{([0,1], v_1)\}$.

Now suppose that $v_1$ is not optimal for the set of all consumers. We first claim that in any segmentation, there must be a segment in which the price is strictly higher than $v_1$. Suppose for contradiction that there exists a segmentation $S = \{(C_j, v_1)\}_{j=1,\ldots,k}$. Because $(C_j, v_1)$ is a segment, $v_1$ is optimal for $C_j$. That is, for any $p' \in V$ we have

$$v_1 \sum_{i: v_i \geq v_1} f^{C_j}(v_i) \geq p' \sum_{i: v_i \geq p'} f^{C_j}(v_i).$$

Because the segments $C_1, \ldots, C_k$ partition $[0,1]$, we have $\sum_{j=1}^{k} f^{C_j}(v_i) = f(v_i)$ for each $v_i$. Therefore, summing the above inequality over all $j$, we have

$$v_1 \sum_{i: v_i \geq v_1} f(v_i) \geq p' \sum_{i: v_i \geq p'} f(v_i).$$

This is a contradiction to the assumption that $v_1$ is not optimal for the set of all consumers.

[1] There are infinitely many surplus-equivalent segmentations to the trivial segmentation. For example, we can divide $[0,1]$ into two coalitions $C_1, C_2$ such that the relative measure of all the values is the same in $C_1, C_2$, and $[0,1]$, and so $\{(C_1, v_1), (C_2, v_1)\}$ is a surplus-equivalent segmentation.

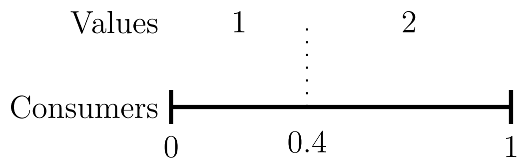


Figure 1: Example 1

So any segmentation $S$ must have a segment $(C, p)$ such that $p > v_1$. Because the price is higher than $v_1$, there are consumers with some value $v_i$ (e.g., consumers with the highest value in $C$) in $C$ whose surplus is strictly lower than $v_i - v_1$. Now consider a coalition $C'$ consisting of a positive measure of some such consumers from $C$ and some consumers with value $v_1$ (from any segment). If $f^{C'}(v_i)$ is small enough relative to $f^{C'}(v_1)$, then price $v_1$ is optimal for $C'$, and so $(C', v_1)$ is a segment. The surplus of value $v_i$ consumers strictly increases to $v_i - v_1$, and the surplus of consumers with value $v_1$ does not change because their surplus is zero in any segment. Therefore, the segment $(C', v_1)$ is an objection to $S$ and so $S$ is not in the core. Because this argument applies to any segmentation $S$, the core is empty. ■

# 4 Stable Segmentations

Given Proposition 1, the core as a solution concept seems too demanding. This is because the core is empty except for "uninteresting" cases where to sell to the set of consumers, it is optimal for the seller to offer the lowest possible price and give each consumer the highest possible surplus. We here develop a notion of stable segmentations and show that such segmentations always exist.

To gain some intuition, consider the following example. In the example, the core is empty. But its examination leads to our notion of stability.

**Example 1** *There are two values* 1 *and* 2 *with measures* 0.4 *and* 0.6*, respectively, as shown in Figure 1.*

Because there is a strictly higher measure of value 2 consumers than value 1 consumers, price 1 is not optimal for the set of all consumers. So by Proposition 1, the core is empty.

To understand this emptiness better, consider segmentation $S = \{[0, 0.6], 1), ((0.6, 1], 2)\}$. There is an "obvious" objection to this segmentation of the form $([0, 0.8], 1)$. The coalition $[0, 0.8]$ in this segment consists of some consumers from the second coalition in $S$ who join all the consumers from the first coalition in $S$. Because the price offered to the consumers in the first coalition does not go up, they are willing to form this objection.

What about the segmentation $S = \{([0, 0.8], 1), ((0.8, 1], 2)\}$? There are no objections of the form discussed above because if some consumers leave the second segment and join the first segment, then the price in the first segment goes up, and so the surplus of consumers $[0.4, 0.8]$ strictly goes down. But there are other objections to this segmentation. Instead of joining the first segment, consumers in the second segment can "steal" the consumers with value 1 from the first segment. For instance, the segment $([0, 0.4] \cup [0.8, 1], 1)$ is an objection to $S$.

However, one could argue this objection may not be credible because there is a counter-objection to it. To see this, consider what happens after $([0, 0.4] \cup [0.8, 1], 1)$ objects to $S$. A possibility for the resulting segmentation is $S' = \{([0, 0.4] \cup [0.8, 1], 1), ((0.4, 0.8), 2)\}$. But notice that $S'$ itself has an objection by segment $([0, 0.8], 1)$ in $S$. So if consumers are forward looking and it is expected that a transition from $S$ to $S'$ will face a counter-objection that takes $S'$ back to $S$, then the consumers in coalition $[0, 0.4] \cup [0.8, 1]$ do not expect any benefit in deviating from $S$, and so this deviation will not occur. The segmentation $S$ is therefore in a sense immune to such an objection. If a segmentation is in the same sense immune to all possible objections, then we call it stable. In other words, suppose that whenever two segmentations $S$ and $S'$ have segments that object to each other, we give "priority" to the objection from segment $S$ and remove the objection in $S'$. Then $S$ is stable if it has no objection among the remaining objections.

To define stability formally, we first define the notion of a blocking segmentation.

**Definition 3 (Blocking)** *A segmentation $S'$ blocks a segmentation $S$ if there exists a segment $(C', M')$ in $S'$ that objects to $S$.*

Like an objection by a segment, blocking by a segmentation specifies a deviation by a group of consumers. But unlike an objection, a blocking segmentation specifies not only the objecting segment, but also how the other consumers re-group into segments

**Definition 4 (Stability)** *A segmentation is stable if it blocks any segmentation that is not surplus-equivalent to it.*

It is not immediately obvious that stability is a less demanding notion than being in the core. It is in principle possible that a segmentation $S$ is in the core and there is another segmentation $S'$ that contains no objecting segment to $S$ but $S$ also contains no objecting segment to $S'$, so $S$ is not stable. We show that stability is indeed less demanding by first showing that the two solution concepts coincide when the core is non-empty, and later that stable segmentations exist even when the core is empty.

**Proposition 2** *If the core is non-empty, then it is equal to the set of all stable segmentations.*

**Proof.** Recall from Proposition 1 that if the core is non-empty, then it consists of the trivial segmentation $\{([0,1], v_1)\}$ and all its surplus-equivalent segmentations. These segmentations are clearly stable because any such segmentation $S$ only contain segments of the form $(C, v_1)$, so in any non-surplus-equivalent segmentation some consumers are offered a price higher than $v_1$, and then there is a segment in $S$ that objects to the other segmentation. If a segmentation is not surplus-equivalent to the trivial segmentation, then some consumers must be offered a price higher than $v_1$, so that segmentation does not block $\{([0,1], v_1)\}$ and is therefore not stable. ■

For the rest of the paper we study stable segmentations in the general case where the core may be empty. We show that stable segmentations exist and characterize them. Because we use our characterization to prove existence, we start with the characterization.

## 4.1 Characterization

To present our characterization, we start by formulating a *canonical* representation of a segmentation so that surplus-equivalent segmentations have the same canonical representation. For this, we say a segmentation is canonical if no two segments have the same price (so any two different segments in it have different prices). Each segmentation $S$ is surplus-equivalent to a unique canonical segmentation, which we call the *induced canonical segmentation* of $S$. To obtain the induced canonical segmentation we can marge all segments that have the same price into a single segment with that price and a coalition that is the union of all the coalitions that face that price.

Our characterization says that a segmentation is stable if and only if its induced canonical segmentation satisfies two properties. The first properties is efficiency. A segmentation is efficient if for any segment $(C, p)$ in the segmentation, the price $p$ is equal to the lowest value $\underline{v}(C) := \min\{v : f^C(v) > 0\}$ in $C$. So all consumers efficiently buy the product. The second properties is saturation. A segmentation is saturated if for any segment $(C, p)$ in the segmentation, whenever we add consumers from segments with higher prices than $p$ to $C$, price $p$ is no longer optimal for this larger coalition. That is, for any two segments $(C, p)$ and $(C', p')$ where $p < p'$, any optimal price for any coalition $C''$ that contains all consumers in $C$ and some consumers in $C'$ is strictly larger than $p$. This property can be expressed more succinctly by looking at the set of prices that are optimal for $C$. A segmentation is saturated if for any two segments $(C, p)$ and $(C', p')$ in the segmentation where $p < p'$, there exists $\hat{p}$ that is optimal for $C$ such that $p < \hat{p} \leq \underline{v}(C)$. If such a price $\hat{p}$ exists, then by adding consumers from $C'$ to $C$, all of whom have value at least $\underline{v}(C')$, the revenue of price $\hat{p}$ increases more than the revenue of price $p$. And because both $p$ and $\hat{p}$ are optimal for $C$, $p$ is no longer is optimal when we add these consumers. Conversely, if no such $\hat{p}$ exists, then we can add a small amount of consumers with value $\underline{v}(C')$ from $C'$ to $C$ without changing the optimal price in $C$. This alternative representation of saturation shows that if we add some consumers to $C$ from possibly *more than one other segments* where the price is higher than $C$, then the price also has to go up.

To prove the result, we use another property which we call Pareto dominance. We say that a segmentation $S$ Pareto dominates another segmentation $S'$ if $CS(c, S) \geq CS(c, S')$ for all consumers $c \in [0, 1]$, with strictly inequality for a positive measure of consumers. And a segmentation $S$ is Pareto undominated if there exists no segmentation $S'$ that Pareto dominates $S$.

**Lemma 1** *If a segmentation is Pareto dominated, then it is not stable.*

**Proof.** If $S$ is Pareto dominated by $S'$, then no segment in $S$ objects to $S'$, and $S'$ is not surplus-equivalent to $S$. Therefore, $S$ is not stable. ■

We now state and prove our main result.

**Theorem 1** *A segmentation is stable if and only if its induced canonical segmentation is efficient and saturated.*

**Proof.** To see the necessity of efficiency, consider any segmentation $S'$ with an induced canonical segmentation $S$. Suppose that $S$ is inefficient, so there is a segment

$(C,p)$ in $S$ such that $p > \underline{v}(C)$. Consider a subset $\bar{C}$ of $C$ that contains all the consumers with value lower than $p$ from $C$ and also a small enough measure of highest value consumers from $C$. Then any optimal price $p'$ for $\bar{C}$ is strictly lower than $p$, so $(\bar{C}, p')$, $p' < p$, is a segment. Notice that $p$ remains optimal for $C \backslash \bar{C}$. To see this, first notice that removing the consumers who are excluded in $C$ does not change the optimal price. In addition, if we remove consumers of the highest value from a coalition, then the optimal price may only go down, but because $p$ is already the lowest possible value after removing all consumers with value lower than $p$, then $p$ remains optimal. Now consider a segmentation $\bar{S}$ that is identical to $S$, except segment $(C,p)$ is replaced with two segments $(C \backslash \bar{C}, p)$ and $(\bar{C}, p')$. The consumers in $\bar{C}$ who have value higher than $\underline{v}(\bar{C})$ have a strictly higher surplus in $\bar{S}$ compared to $S$ (such consumers exist because $\bar{C}$ contains consumers of at least two different values), and all the other consumers have the same surplus in the two segmentations, so $\bar{S}$ Pareto dominates $S$ and therefore $S'$. So by Lemma 1, $S'$ is not stable.

To see the necessity of saturation, consider any segmentation $S'$ with an induced canonical segmentation $S$. Suppose that $S$ is not saturated. If $S$ is inefficient, then the argument above implies that $S'$ is not stable. So suppose that $S$ is efficient, which together with non-saturation implies that there are two segments $(C, \underline{v}(C))$ and $(C', \underline{v}(C'))$ in $S$ such that $\underline{v}(C) < \underline{v}(C')$ and exists no $\hat{p}$ that is optimal for $C$ such that $\underline{v}(C) < \hat{p} \leq \underline{v}(C')$. So if we add some consumers of value $\underline{v}(C')$ to $C$, price $\underline{v}(C)$ remains optimal. That is, price $\underline{v}(C)$ is optimal for any coalition $\bar{C} = C \cup C''$ where $C''$ is any set of measure small enough $\delta > 0$ of consumers with value $\underline{v}(C')$ from $C'$. Let $C''$ be such that it contains a positive measure of consumers with value $\underline{v}(C')$ from *every* segment in $S'$ where the price is $\underline{v}(C)$ (recall that $S'$ need not be canonical). Now consider a segmentation $\bar{S}$ that is equivalent to $S$ except $(C, \underline{v}(C))$ and $(C', \underline{v}(C'))$ are replaced with $(\bar{C}, \underline{v}(C))$ and $(C' \backslash \bar{C}, p)$, where $p$ is any optimal price for $C' \backslash \bar{C}$. Segmentations $S'$ and $\bar{S}$ are not surplus-equivalent, because consumers in $C''$ have zero surplus in $S'$ but a positive surplus in $\bar{S}$.

To show that $S'$ is not stable, we next argue that $S'$ does not block $\bar{S}$. First, notice that the surplus of consumers from any segment in $S'$ where the price is neither $\underline{v}(C)$ nor $\underline{v}(C')$ does not change in $\bar{S}$, so these segments do not object to $\bar{S}$. Also all consumers that are in a segment in $S'$ with price $\underline{v}(C)$ are in segment $(\bar{C}, \underline{v}(C))$ in $\bar{S}$, so these segments do not object to $\bar{S}$ either. Finally, notice that for any segment $(C''', \underline{v}(C'))$ in $S'$, some of the value $\underline{v}(C')$ consumers who get a surplus of 0 in $S'$ are put in segment $(\bar{C}, \underline{v}(C))$ where their surplus is strictly positive, so $(C''', \underline{v}(C'))$ does not object to $\bar{S}$. We conclude that $S'$ does not block $\bar{S}$ so $S'$ is not stable.

We now show sufficiency of efficiency and saturation. So consider a segmentation $S'$ with a canonical segmentation $S$ that is efficient and saturated. Consider any segmentation $\bar{S}$ that is not blocked by $S'$. Then the canonical representation of $\bar{S}$ is also not blocked by $S'$, so suppose without loss of generality that $\bar{S}$ is canonical. Write the two segmentations as $S = \{(C_1, v_1), \ldots, (C_n, v_n)\}$ and $\bar{S} = \{(\bar{C}_1, v_1), \ldots, (\bar{C}_n, v_n\}$ where each $C_i$ or $\bar{C}_i$ may be empty. We show using induction that $\bar{S}$ must be surplus-equivalent to $S'$ by showing that $C_i = \bar{C}_i$ for all $i$ (for the rest of this proof, $C_i = \bar{C}_i$ is in the "almost all" sense, that is the measure of consumers in $C_i$ but not in $\bar{C}_i$ is zero, and the same holds for consumers in $\bar{C}_i$ but not in $C_i$).

Suppose that $C_j = \bar{C}_j$ for all $j < i$ (the basis of the induction is when $i = 1$). This means that a consumer is in a segment where the price is $v_i$ or higher in $S$ if an only that consumer is in such a segment in $\bar{S}$. This means in particular that consumers in $C_i$ are offered a price of $v_i$ or higher in $\bar{S}$. We show that $C_i = \bar{C}_i$. If $i = n$, then we are done because coalitions in $S$ and $\bar{S}$ partition the same set of consumers. So suppose that $i < n$. First notice that consumers in $C_i$ with value higher than $v_i$ must be in $\bar{C}_i$, otherwise these consumers are offered a price strictly higher than $v_i$ in $\bar{S}$ and so their surplus decreases, and therefore any segment in $S'$ that contains some of these consumers objects to $\bar{S}$. So $C_i$ and $\bar{C}_i$ are identical, except $\bar{C}_i$ may not contain some consumers of value $v_i$ from $C_i$ and may contain some consumers from coalitions $C_{i+1}, \ldots, C_n$, all of whom have value higher than $v_i$. In either case, if $C_i$ and $\bar{C}_i$ are not identical, because $v_i$ is optimal for $\bar{C}_i$, $v_i$ must have a strictly higher revenue for $C_i$ than any other price. But this contradicts saturation because we can add consumers to $C_i$ without increasing the price. ■

To verify the stability of a segmentation, it is important that we establish the efficiency and saturation of its *canonical* representation. Below is an example with a non-canonical segmentation that is efficient and saturated, but is not stable.

**Example 2** *There are four values* $1$ *to* $4$ *with measures* $0.5, 0.25, 0.125, 0.125$*, respectively, as shown in Figure 2.*

Consider the segmentation $\{(C_1, 1), (C_2, 1), (C_3, 4)\}$ with coalitions $C_1, C_2, C_3$ shown in the figure.

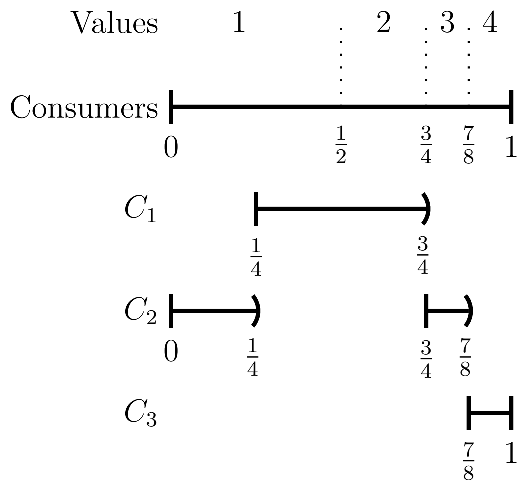


Figure 2: Example 2.

For coalition $C_1$, prices 1 and 2 are both optimal. For coalition $C_2$, prices 1 and 3 are both optimal. So it is not possible to take consumers from segment $(C_3, 4)$ where the price is high and add them to either segment $(C_1, 1)$ or $(C_2, 1)$ where the price is low without increasing the price. So this segmentation is saturated. This segmentation is also efficient.

But notice that for coalition $C_1 \cup C_2$, price 1 is the unique optimal price. So we can combine these two coalitions and add some consumers with value 4 without changing the price. In particular, consider segmentation $S' = \{([0, \frac{7}{8} + \epsilon), 1), ([\frac{7}{8} + \epsilon, 1], 4)\}$ for some small $\epsilon > 0$. The segmentation $S'$ Pareto dominates $S$, and so $S$ is not stable.

## 4.2 Existence

We now use our characterization to establish existence of stable segmentations. We start with an example which leads to an algorithm to construct stable segmentations.

**Example 3 (Maximal Equal-revenue Segmentation)** *There are three values* $1, 2, 3$ *with measures* $\frac{1}{3}, \frac{1}{6}, \frac{1}{2}$*, respectively, as shown in Figure 3.*

Consider the segmentation $S = \{(C_1, 1), (C_2, 2), (C_3, 3)\}$ as shown the figure. Coalition $C_1$ is the largest "equal-revenue" coalition that includes all values. That is, the measure of the three values $\frac{3}{9}, \frac{1}{9}, \frac{2}{9}$ in coalition $C_1$ are such that all prices 1, 2, and 3 are optimal, and further $C_1$ is the largest such coalition because it contains all

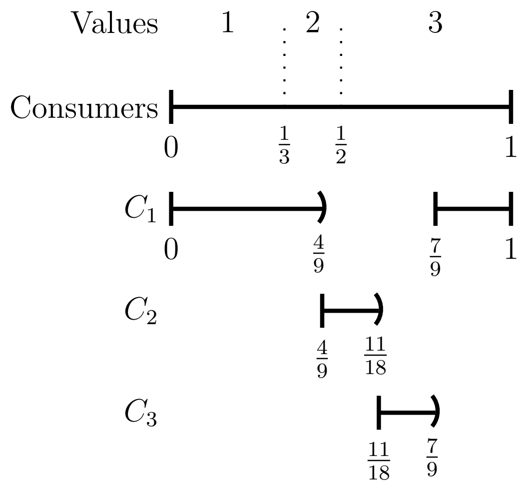


Figure 3: Example 3.

the value 1 consumers (any two coalitions $C, C'$ for which all three prices are optimal are proportional, i.e., $f^C(v) = \alpha f^{C'}(v)$ for some $\alpha$, so the largest such coalition is well-defined). Segment $(C_1, 1)$ is efficient. Also notice that because $C_1$ contains all value 1 consumers, consumers not in $C_1$ have value either 2 or 3, so adding them to $C_1$ means that price 1 is no longer optimal.

We can now define the segmentation recursively on the remaining consumers, hence guaranteeing efficiency and saturation.

The remaining consumers have value either 2 or 3, and their measures are $\frac{1}{18}$ and $\frac{5}{18}$, respectively. Coalition $C_2$, in which values 2 and 3 have measures $\frac{1}{18}$ and $\frac{2}{18}$, is the largest coalition where both prices $2, 3$ are optimal. The segment $(C_2, 2)$ is efficient and adding any of the remaining consumers, all of whom have value 3, increases the price.

The last segment is $(C_3, 3)$, which is efficient. We have established that the segmentation $S$ is efficient and saturated. Because this segmentation is also canonical, it is stable.

We now formally define the maximal equal-revenue segmentation. Let $\bar{F}^C(v_i)$ be the cumulative measure of consumers with values $v_i$ or higher in $C$. If in a coalition $C$, the measures of consumers of values at least $v_i$ or $v_j$ satisfies $v_i\bar{F}^C(v_i) = v_j\bar{F}^C(v_j)$, then prices $v_i$ and $v_j$ are equally profitable for the seller. A coalition $C$ is an equal-revenue coalition if $v_i\bar{F}^C(v_i)$ is constant for all $v_i$ such that $f^C(v_i) > 0$, that is, revenue is constant for all values that are included in it. A maximal equal-revenue segmentation is defined recursively. The first coalition $C_1$ is the largest equal-revenue coalition that includes all values $V$. Formally, let

$$\lambda = \min_{v_i \in V} \frac{f(v_i)}{\frac{1}{v_i} - \frac{1}{v_{i+1}}}, \tag{1}$$

and let $\bar{F}^{C_1}(v_i) = \lambda / v_i$ for all $v_i$. Then $v_i \bar{F}^{C_1}(v_i) = \lambda$ for all $v_i \in V$ and all prices are optimal. Further, $f^{C_1}(v_i) = \bar{F}^{C_1}(v_i) - \bar{F}^{C_1}(v_{i+1}) = \lambda(\frac{1}{v_i} - \frac{1}{v_{i+1}}) \leq f(v_i)$ for all $v_i$, and since the value $v_i$ at which the right hand side of (1) is minimized satisfies $\lambda(\frac{1}{v_i} - \frac{1}{v_{i+1}}) = f(v_i)$, we have $f^{C_1}(v_i) = f(v_i)$. That is, coalition $C_1$ contains all consumers with value $v_i$, and adding any other positive measure of consumers would make price $v_i$ sub-optimal. Therefore segment $C_1$ cannot be any larger. The first segment in an equal-revenue segmentation is $(C_1, v_1)$, and the rest of the segmentation is defined recursively, where $C_j$ is the largest equal-revenue coalition that includes all the values that remain after having removed consumers in $C_1, \ldots, C_{j-1}$, that is, $\{v_i : f^{C_j}(v_i) > 0\} = \{v_i : f^{C \setminus \cup_{j' < j} C_{j'}}(v_i) > 0\}$, and the price in the $j$'th segment is $\min\{v_i : f^{C_j}(v_i) > 0\}$. This process ends because in each step, the number of remaining values decreases by at least 1.

The maximal equal-revenue segmentation is not necessarily canonical. For example, it could be that the first equal-revenue coalition $C_1$ exhausts some value other than $v_1$, in which case the second coalition is also going to include consumers with value $v_1$. So to establish that this segmentation is stable, we need to consider its induced canonical segmentation and show that it is efficient.

**Proposition 3** *The maximal equal-revenue segmentation is stable.*

**Proof.** Given Theorem 1, it is sufficient to show that the induced canonical segmentation of the maximal equal-revenue segmentation satisfies the two properties of Theorem 1. The maximal equal-revenue segmentation is efficient by construction, so is its canonical representation. We show that the induced canonical segmentation of the maximal equal-revenue segmentation is saturated.

Sort the segments by their optimal price, from low to high (break ties arbitrarily if multiple segments have the same price). Let $C_j$ be the lowest coalition in the equal-revenue segmentation where the price is not $v_1$, and let $v_i$ be the lowest value in segment $C_j$. Value $v_i$ must also be in any segment $C_1, \ldots, C_{j-1}$, and in fact must be an optimal price in any of those segments. In the induced canonical segmentation, segments $(C_1, v_1), \ldots, (C_{j-1}, v_1)$ are merged into a single segment $(C'_1, v_1)$, and all segments where the price is $v_i$ are merged into an segment $(C'_2, v_i)$. Since price $v_i$ is also optimal in each of those segments, it must also be optimal in segment $C'_1$. Therefore, adding any consumers from $C'_2$ to $C'_1$ necessarily increases the price in $C'_1$. A recursive argument shows that the induced canonical segmentation is saturated. ■

The maximal equal-revenue segmentation is not the unique stable segmentation. Here is an informal description of another algorithm that leads to a stable segmentation. Put all consumers of value $v_1$ in the first coalition, and continually add consumers of the lowest remaining value to the first coalition until some price $v_i$ other than $v_1$ also becomes optimal. The first segment is $(C, v_1)$. Repeat this process with the remaining consumers. This segmentation is canonical, efficient, and saturated. Saturation follows because value $v_i$ becomes optimal for $C$ only when we have already added all the lower values to $C$, so any consumer in a segment with a higher price must have value at least $v_i$, and so adding them to the first segment must necessarily increase the optimal price. This segmentation is also in general different than the maximal equal-revenue segmentation because generally the first segment does not include all values. We proved the existence of stable segmentations using the maximal equal-revenue segmentation because it is special in the sense that it maximizes average consumer surplus, as we discuss next.

## 4.3 Stability vs. maximizing average consumer surplus

The maximal equal-revenue segmentation was first introduced in Bergemann, Brooks, and Morris (2015). They show that this segmentation maximizes average consumer surplus, but is not necessarily the *only* segmentation that does so. So one may wonder if stability either implies or is implied by maximizing average consumer surplus. This is not the case, as the following two examples show.

**Example 4 (A stable segmentation that does not maximize average consumer surplus)** *There are three values* $1, 2, 3$ *with measures* $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$*, respectively, as shown in Figure 4.*

Consider a segmentation $S = \{(C_1, 1), (C_2, 3)\}$ shown in Figure 4 in which $C_1$ contains all consumers with values 1 and 2 and $C_3$ contains all consumers with value 3. This segmentation is canonical, efficient, and saturated. It is therefore stable.

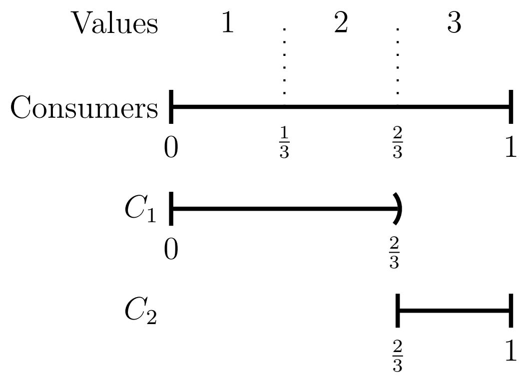


Figure 4: Example 4

But this segmentation does not maximize average consumer surplus. One way to see this is to compare the average consumer surplus of this segmentation, $\frac{1}{3}$, to the average consumer surplus of the maximal equal-revenue segmentation, $\frac{2}{3}$.[2] But a more illuminating way is to study a marginal improvement over the average consumer surplus of $S$ that is obtained from swapping the same measure of value 2 and 3 consumers. In particular, consider a coalition $C_1'$ that is $C_1$ minus an $\epsilon$ measure of value 2 consumers plus an $\epsilon$ measure of value 3 consumers, and a coalition $C_2'$ that contains the remaining consumers. If $\epsilon > 0$ is small enough, price 1 is optimal for $C_1'$ and price 3 is optimal for $C_3'$. So $S' = \{(C_1', 1), (C_3', 3)\}$ is a segmentation. To see the change in average consumer surplus, we can focus only on the value 2 and 3 consumers who are swapped. The value 2 consumers lose 1 unit of surplus each: their surplus in the original segmentation is 1, but their surplus in $S'$ is 0. The value 3 consumers gain 2 units of surplus each: their surplus in the original segmentation is 0, but their surplus in $S'$ is 2. Because the measures of these two sets are the same, the change increases average consumer surplus.

Notice that if we consider the *change in the offered price*, this change is 2 for the value 2 consumers (from 1 to 3), and is $-2$ for the value 3 consumers (from 3 to 1). So even though this change has the same absolute value, in terms of surplus it is more important to the value 3 consumers because for the value 2 consumers, some of the change has no effect on surplus: an increase in the price offered from 1 to 2 does not

[2]The maximal equal-revenue segmentation is $\{(C_1, 1), (C_2, 2)\}$ where $(f^{C_1}(1), f^{C_1}(2), f^{C_1}(3)) = (\frac{1}{3}, \frac{1}{9}, \frac{2}{9})$ and $(f^{C_2}(1), f^{C_2}(2), f^{C_2}(3)) = (0, \frac{2}{9}, \frac{1}{9})$. The surplus of value 2 and 3 consumers is 1 and 2 in the first segment, and the surplus of value 3 consumers is 1 in the second segment. So the average consumer surplus is $\frac{1}{9} \cdot 1 + \frac{2}{9} \cdot 2 + \frac{1}{9} \cdot 1 = \frac{2}{3}$.

change their surplus.

**Example 5 (A segmentation that maximizes average consumer surplus and is not stable)** *Consider again the example from Example 3 with three values* $1, 2, 3$ *and measures* $\frac{1}{3}, \frac{1}{6}, \frac{1}{2}$ *and a segmentation* $S = \{(C_1, 1), (C_2, 2)\}$ *with coalitions* $C_1 = [0, \frac{1}{3}] \cup [\frac{5}{6}, 1]$ *and* $C_2 = (\frac{1}{3}, \frac{5}{6})$.

Coalition $C_1$ contains all the value 1 consumers and some of the value 3 consumers in such a way that both prices are optimal. Coalition $C_2$ contains value 2 and 3 consumers in such a way that both prices are optimal. Segmentation $S$ maximizes average consumer surplus over all possible segmentations.[3]

But the segmentation is not stable. This segmentation is canonical and efficient. So to show that it is not stable, we show that it is not saturated. The reason is that there are no value 2 consumers in coalition $C_1$. So adding a small measure of value 2 consumers to $C_1$ does not change the fact that price 1 is optimal: if this measure is small enough, price 2 is not going to become optimal, and this addition increases the revenue of price 1 but does not change the revenue of price 3. This segmentation is therefore not saturated and not stable.

## 4.4 Pareto dominance and efficiency

Recall from Section 4.1 that any stable segmentation is Pareto undominated and efficient. Bergemann, Brooks, and Morris (2015) show that any segmentation that maximizes average consumer surplus is also Pareto undominated and efficient. So we here study the connection between Pareto undominance and efficiency.

**Proposition 4** *Any Pareto undominated segmentation is efficient.*

**Proof.** Consider any inefficient segmentation $S$. We show that $S$ must be Pareto dominated by some other segmentation. By definition there must be a segment $(C, p)$ in $S$ such that $p$ is higher than the lowest value $\underline{v}(C)$ in $C$. Construct a coalition $C' \subseteq C$ that contains all the consumers of lowest value $\underline{v}(C)$ from $C$, and an $\epsilon$ fraction of consumers of each other value from $C$. If $\epsilon$ is small enough, price $\underline{v}(C)$ is optimal

[3] The segmentation is efficient so it maximizes the total surplus. It also minimizes the seller's revenue over all segmentation. This is because the same price that is optimal for the set of all consumers, 3, is also optimal for each coalition.

in $C'$, so $(C', \underline{v}(C))$ is a segment. Also, price $p$ remains optimal in the remaining segment $C \backslash C'$. To see this, first notice that $\underline{v}(C)$ is lower than all values in $C \backslash C'$, and so price $\underline{v}(C)$ is not optimal for $C \backslash C'$. Second, for any $v_i > \underline{v}(C)$, we have $f^{C \backslash C'}(v_i) = (1-\epsilon) f^C(v_i)$. Therefore the revenue of any price $p' \in V \backslash \{\underline{v}(C)\}$ in $C \backslash C'$ is

$$p' \sum_{v_i \geq p'} f^{C \backslash C'}(v_i) = (1-\epsilon)(p' \sum_{v_i \geq p'} f^C(v_i)),$$

which is maximized at $p' = p$ since price $p$ is optimal in $C$.

Now consider a segmentation $S'$ that is identical to $S$, except the segment $(C, p)$ is replaced with two segments $(C', \underline{v}(C))$ and $(C \backslash C', p)$. Since the price in the segment $(C', \underline{v}(C))$ is lower than that in $(C, p)$, all consumers other than those with value $\underline{v}(C)$ are strictly better off in $C'$ relative to $C$, and the consumers with value $\underline{v}(C)$ have zero surplus in either case. Further, since the prices in segments $(C \backslash C', p)$ and $(C, p)$ are the same, the surplus of each consumer in $C \backslash C'$ remains unchanged. Therefore $S'$ Pareto dominates $S$. ■

Notice that there exist efficient segmentations that are not Pareto efficient. For instance, the segmentation that puts all the consumers of each value in a different segment is efficient but is not Pareto undominated.

## 4.5 Relationship with the stable set

Our notion of stability is related to the notion of a *stable set* from Morgenstern and Von Neumann (1953). The stable set is defined for any cooperative game, but here we present only the application of this solution concept to our game. Notice that whereas in Morgenstern and Von Neumann (1953), state below, stability is a property of a *set* of segmentations, our definition of stability is a property of a single segmentation.

**Definition 5 (Stable set, Morgenstern and Von Neumann, 1953)** *A* set *of segmentations $\mathcal{S}$ is a stable set if it satisfies the following two properties:*

1. *Internal Stability: For all $S \in \mathcal{S}$, there exists no $S' \in \mathcal{S}$ that blocks $S$.*

2. *External Stability: For all $S \notin \mathcal{S}$, there exists $S' \in \mathcal{S}$ that blocks $S$.*

If a segmentation $S$ is stable, then the set of all segmentations that are surplus-equivalent to $S$ is a stable set. This is easy to see: internal stability is trivially satisfied because a segmentation does not block a surplus-equivalent segmentation, and external stability is satisfied by definition of stability. So stable sets exist in our setting. This is noteworthy because stable sets may not exist for a general game. Further, in some games in which stable sets exists, they must necessarily contain multiple elements. In contrast, our proposition below shows that any stable set contains an essentially unique element in the sense that it consists of all segmentations that are surplus-equivalent to some segmentation $S$.

Surprisingly, however, the set of segmentations that are surplus-equivalent to $S$ may be a stable set even if $S$ is not stable. This is because stability is a strong notion: it requires that a *single* segmentation blocks any other non-surplus-equivalent segmentation; the stable set, on the other hand, requires that any non-surplus-equivalent segmentation to $S$ be blocked by *some* segmentation that is surplus-equivalent to $S$, and it could in fact be the case that $S$ does not block $S'$ but a surplus-equivalent segmentation to $S$ does. To see this, suppose that $S$ is canonical but not stable. So there exists another segmentation $S'$ that is not blocked by $S$. This does not mean, however, that there exists no segmentation $S''$ that is surplus-equivalent to $S$ that blocks $S'$. Take a segment $(C, p)$ in $S$ that does not object to $S'$. This segment may contain some consumers who prefer $S$ to $S'$, and some consumers who prefer $S'$ to $S$. Now if we replace $(C, p)$ with two segments $(C', p)$ and $(C \backslash C', p)$, it could be that all the consumers in $C'$ prefer this new segmentation to $S'$, and so this segmentation that is surplus-equivalent to $S$ blocks $S'$. The following example illustrates this.

**Example 6** *There are three values* $1, 2, 3$ *with measures* $\frac{6}{21}, \frac{4}{21}, \frac{11}{21}$*, respectively, as shown in Figure 4, and a segmentation* $S = \{(C_1, 1), (C_2, 2)\}$ *where* $C_1 = [0, \frac{6}{21}) \cup [\frac{18}{21}, 1]$ *and* $C_2 = [\frac{6}{21}, \frac{18}{21})$*.*

Segmentation $S$ is not stable because it is not saturated. This is because we can add some consumers with value 2 from $C_2$ to $C_1$ without changing the price in the first segment. For example, segmentation $S' = \{(C'_1, 1), (C'_2, 3)\}$ with coalitions $C'_1 = [0, \frac{7}{21}) \cup [\frac{18}{21}, 1]$ and $C'_2 = [\frac{7}{21}, \frac{18}{21})$ shown in the figure is not blocked by $S$. Segment $(C_1, 1)$ in $S$ does not object to $S'$ because all consumers in $C_1$ are indifferent between the two segmentations. Segment $(C_2, 2)$ does not object to $S'$ either because the value 2 consumers who join the first segment in $S'$ strictly prefer $S'$ to $S$. However, this segmentation $S'$ is blocked by the segmentation $S'' = \{(C''_1, 1), (C''_2, 2), (C'''_2, 2)\}$, with coalitions $C''_1 = [0, \frac{6}{21}) \cup [\frac{18}{21}, 1]$, $C''_2 = [\frac{6}{21}, \frac{7}{21}) \cup [\frac{16}{21}, \frac{18}{21})$, and $C'''_2 = [\frac{7}{21}, \frac{16}{21})$,

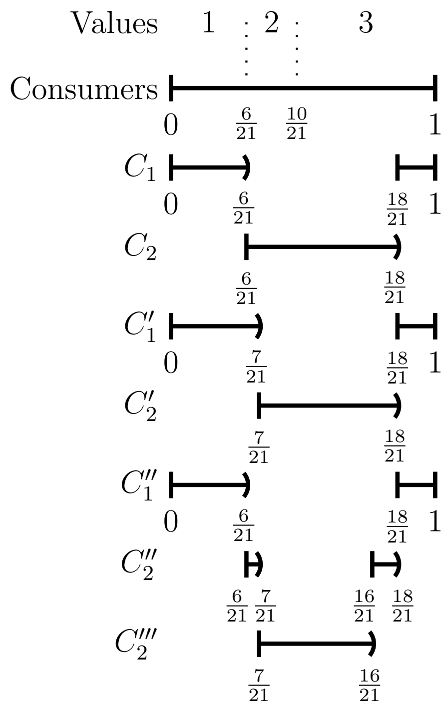


Figure 5: Example 6

which is surplus-equivalent to $S$. In particular, segment $(C_2''', 2)$ objects to $S'$ because these consumers are offered price 2 in $S''$ but price 3 in $S'$. The proposition below characterizes stable sets and shows that the set of all segmentations that are surplus-equivalent to $S$ is in fact a stable set.

To state the proposition, we first define a weak notion of saturation. We say that a segmentation $S$ is *weakly saturated* if we can add *all* of consumers of one value that is optimal for one segment from that segment to segments with lower prices without increasing the prices in any of those lower priced segments. In particular, for a segment $(C, p)$ and a value $v > p$, let $\nu(C, p, v)$ denote the largest measure of value $v$ consumers that can be added to $C$ while maintaining the optimality of price $p$. A segmentation is weakly saturated if for any segment $(C, p)$ and any price $v$ that is optimal for $C$, we have $f^C(v) > \sum_{(C',p') \in S, p' < p} \nu(C', p', v)$.

The segmentation $S$ in Example 6 is weakly saturated because adding all the consumers with value 2 from $C_2$ to $C_1$ increases the optimal price in the first segment to 2, and adding all the consumers with value 3 from $C_2$ to $C_1$ increases the optimal price in the first segment to 3. Notice that a saturated segmentation is weakly saturated because for any two segments $(C, p)$ and $(C', p')$ with $p' < p$ and any value $v$ in the support of $C$, we have $\nu(C', p', v) = 0$. Recall from Lemma 1 that a stable segmentation is also Pareto undominated. So the result below, which shows that the set of all surplus-equivalent segmentations to a canonical segmentation that is Pareto undominated and weakly saturated is a stable set, implies that the set of all surplus-equivalent segmentations to a stable segmentation is a stable set.

**Proposition 5** *A set of segmentations is a stable set if and only if it consists of all segmentations that are surplus-equivalent to some canonical, Pareto undominated, and weakly saturated segmentation.*

**Proof.** To see the necessity of these conditions, consider any stable set $\mathcal{S}$. We first show that any segmentation in $\mathcal{S}$ must be Pareto undominated. Suppose for contradiction that there exists a segmentation $S$ in $\mathcal{S}$ that is Pareto dominated by another segmentation $S'$. If $S'$ is $\mathcal{S}$, then internal stability is violated because $S'$ blocks $S$. If $S'$ is not in $\mathcal{S}$, then by external stability, there is a segmentation $S''$ in $\mathcal{S}$ that blocks $S'$. But then $S''$ also blocks $S$, and so again internal stability is violated. Pareto undominance in particular implies that any segmentation in $\mathcal{S}$ must be efficient.

We next show that any two segmentations in $\mathcal{S}$ must be surplus-equivalent. Suppose for contradiction that there are two segmentations $S_1$ and $S_2$ in $\mathcal{S}$ that are not surplus-equivalent. So their induced canonical segmentations $S_1'$ and $S_2'$ are also not surplus-equivalent, so there must be segments $(C_1, p)$ in $S_1'$ and $(C_2, p)$ in $S_2'$ such that $p = \underline{v}(C_1) = \underline{v}(C_2)$ but $C_1 \neq C_2$ (in the "almost all" sense). Suppose without loss of generality that $p$ is the lowest such value, so any consumer in $C_1$ is either in $C_2$ or in a segment with a higher price, and similarly any consumer in $C_2$ is either in $C_1$ or in a segment with a higher price. Because $C_1 \neq C_2$, we either have $C_1 \backslash C_2 \neq \emptyset$ or $C_2 \backslash C_1 \neq \emptyset$. Suppose without loss of generality that $C_1 \backslash C_2 \neq \emptyset$.

First suppose that $C_1 \backslash C_2$ contains only consumers with value $p$. So $C_2$ has a lower measure of value $p$ consumers, and in addition may have some higher value consumers. Because $p$ is optimal for $C_2$, it must be strictly optimal for $C_1$. But this means that $S_1'$ is Pareto dominated because for any segment $(C', p')$ in $S_1'$ where $p' > p$, we can add some of the highest value consumers from $C'$ to $C_1$ without increasing the price in either segment. As we argued above, a Pareto dominated segment cannot be in $\mathcal{S}$.

So suppose that $C_1 \backslash C_2$ contains some consumers with value higher than $p$. Because by assumption, these consumers are not offered a lower price in $S_2'$, they must be offered a strictly higher price in $S_2'$. Consider any segment $(C'', p)$ in $S_1$ that contains some such consumers. These consumers are offered price at least $p$ in $S_2$, and some of them have a strictly higher surplus in $S_1$ than in $S_2$. So $S_1$ blocks $S_2$, which contradicts internal stability.

So we have established that any stable set $\mathcal{S}$ may only contain segmentations that are surplus-equivalent to a Pareto undominated segmentation $S$. If some $S'$ that is surplus-equivalent to $S$ is not in $\mathcal{S}$, then no segmentation in $\mathcal{S}$ blocks $S'$ so external stability is violated. So $\mathcal{S}$ must contain *all* segmentations that are surplus-equivalent to a Pareto undominated segmentation $S$, which we can assume to be canonical without loss of generality. So to complete the necessity direction, it only remains to show that the canonical segmentation $S$ must be weakly saturated.

Suppose that $S$ is not weakly saturated. Because $S$ is Pareto undominated, it is efficient by Proposition 4. So there is a segment $(C, \underline{v}(C))$ in $S$ and a price $p$ that is optimal for $C$ such that we can add all the consumers with value $p$ from $C$ to segments with lower price without increasing the price in those segments. Let $S'$ be the resulting segmentation. We argue that no segmentation $S''$ that is surplus-equivalent to $S$ blocks $S'$, violating external stability. Indeed, any consumer that is not in $C'$ has the same surplus in both segmentations. Now consider any segment in $S''$ in which the price is $\underline{v}(C)$. Any such segment must contain some consumers of value $p$ from $C$. These consumers strictly prefer $S'$ to $S''$. So $S''$ does not block $S'$.

To establish sufficiency, consider any canonical segmentation $S = \{(C_1, v_1), \dots, (C_n, v_n)\}$ that is Pareto efficient and weakly saturated ($C_i$ may be empty for some $i$). Suppose for contradiction that there is a segmentation $S'$ that is not blocked by any segmentation that is surplus-equivalent to $S$. We can assume without loss of generality that $S'$ is canonical, so $S'$ can be written as $S' = \{(C_1', v_1), \dots, (C_n', v_n)\}$. Because $S$ is Pareto efficient, some consumers must strictly prefer $S$ to $S'$. Consider the lowest segment $(C_i, v_i)$ in $S$ that contains some consumer that strictly prefers $S$ to $S'$.

We show inductively that $C_j \subseteq C_j'$ for all $j < i$. It must be that $C_1 \subseteq C_1'$, as otherwise $(C_1, v_1)$ objects to $S'$. Suppose that $C_1 \subseteq C_1', \dots, C_{j-1} \subseteq C_{j-1}'$ for some $j < i$. Coalition $C_{j'}$ cannot have a non-empty intersection with $C_1', \dots, C_{j-1}'$ because then segment $(C_j, v_j)$ contains some consumers that strictly prefer $S$ to $S'$, contradicting the assumption that $(C_i, v_i)$ is the lowest such segment. Also $C_j$ cannot have a non-empty intersection with $C'_{j+1}, \ldots, C'_n$, because then $(C_j, v_j)$ objects to $S'$. We conclude inductively that $C_j \subseteq C'_j$ for all $j < i$.

Now suppose that for all values $v$ that are optimal for $C_j$, not all consumers with value $v$ from $C_j$ are in $C'_1$ to $C'_{i-1}$. So there exist some consumers with value $v$ in $C_j$ that weakly prefer $S$ to $S'$. Then we can construct a segmentation $S''$ that is surplus-equivalent to $S$ that blocks $S'$. This segmentation is the same as $S$, except $(C_j, v_j)$ is replaced with two segments $(C', v_j)$ and $(C_j \backslash C', v_j)$ such that $C'$ contains a small measure of each value $v$ that is optimal for $C_j$ that weakly prefers $S$ to $S'$, and a small measure of the consumers that strictly prefer $S$ to $S'$. It must therefore be that for some $v$ that is optimal for $C_j$, all consumers with value $v$ from $C_j$ are in $C'_1$ to $C'_{i-1}$. But this contradicts weak saturation. ■

## 4.6 Two values

We have seen examples showing that in general, stability, maximizing average consumer surplus, and Pareto undominance are different concepts. We show here that with two values, all these concepts coincide. And further, there is essentially a single segmentation that satisfies these properties, in a slightly weaker sense than surplus-equivalence. Formally, two segmentations are weakly surplus-equivalent if the induced canonical representation of each segmentation can be obtained from the induced canonical representation of the other one via a measure-preserving mapping that for each value, maps the set of consumers with that value to itself. Any two surplus-equivalent segmentations are weakly surplus-equivalent.

**Proposition 6** *Suppose that there are two values. The following three are equivalent for any segmentation $S$:*

1. *$S$ is stable.*
2. *$S$ is Pareto undominated.*
3. *$S$ maximizes average consumer surplus.*

*Segmentations that satisfy these three equivalent properties are weakly surplus-equivalent.*

**Proof.** Suppose first that $v_1$ is optimal for $[0, 1]$. Then the trivial segmentation $\{([0, 1]), v_1\}$ gives the highest possible surplus to all consumers, and so a segmentation is Pareto undominated if and only if it maximizes average consumer surplus if and only if it is surplus-equivalent (and therefore weakly surplus-equivalent) to this segmentation. And we have argued already in Proposition 2 that if $v_1$ is optimal for $[0,1]$, then all stable segmentations are surplus-equivalent to $\{([0,1]), v_1\}$.

Now suppose that $v_1$ is not optimal for $[0,1]$, that is, $v_1(f^{[0,1]}(v_1) + f^{[0,1]}(v_2)) < v_2 f^{[0,1]}(v_2)$. Consider any segmentation $S = \{(C_1, v_1), (C_2, v_2)\}$, where $C_1$ contains all the value 1 consumers and as many value 2 consumers as possible, so $v_1(f^{C_1}(v_1) + f^{C_1}(v_2)) = v_2 f^{C_1}(v_2)$, and $C_2$ contains the remaining value 2 consumers. We show that any segmentation that satisfies either of the three properties, stability, Pareto undominance, and maximizing average consumer surplus, is weakly surplus-equivalent to $S$.

Consider the induced canonical segmentation $S'' = \{(C_1'', v_1), (C_2'', v_2)\}$ of any segmentation $S'$. The surplus of value $v_2$ consumers in $C_1''$ is $v_2 - v_1$, and the surplus of all other consumers is zero. So $S'$ maximizes average consumer surplus if and only if it is Pareto undominated if and only there are as many value $v_2$ consumers in $C_1''$. That is $f^{C_2''}(v_1) = 0$ and $v_1(f^{C_1''}(v_1) + f^{C_1''}(v_2)) = v_2 f^{C_1''}(v_2)$ because otherwise in the first case we can add some consumers of value $v_1$ and $v_2$ from the second segment to the first segment, or in the second case just add some consumers with value $v_2$ from the second segment to the first segment. So $S'$ is weakly surplus-equivalent to $S$. Also, $f^{C_2''}(v_1) = 0$ and $v_1(f^{C_1''}(v_1) + f^{C_1''}(v_2)) = v_2 f^{C_1''}(v_2)$ mean that $S''$ is saturated and efficient and so $S'$ is stable. ■

# 5 Fragmentation-proofness

We now discuss our last solution concept, fragmentation-proofness. Fragmentation-proofness rules out objections by coalitions that include consumers from more than one segment. In other words, a fragmentation-proof segmentation does not have an objection by coalition that is a subset of consumers in an existing segment.

**Definition 6** *A segmentation $S$ is fragmentation-proof if there exists no objection $(C, p)$ to $S$ such that $C \subseteq C'$ for some segment $(C', p')$ in $S$.*

We show that fragmentation-proofness is equivalent to efficiency.

**Proposition 7** *A segmentation is fragmentation-proof if and only if it is efficient.*

**Proof.** Consider an efficient segmentation and any segment $(C, p)$ in it. Because the segmentation is efficient, $p$ is equal to the lowest value in $C$. Then the optimal price for any subset $C'$ of $C$ is at least $p$. So there exists no objecting segment $(C', p')$ where $C' \subseteq C$.

Now consider an inefficient segmentation $S$. So there must exist a segment $(C, p)$ in the segmentation such that $p$ is strictly higher than the lowest value in $C$. Consider a coalition $C'$ that contains all the value $\underline{v}(C)$ consumers from $C$, and a small measure of the consumers with the highest value from $C$. Then the unique optimal price for $C'$ is $\underline{v}(C)$, so $(C', \underline{v}(C))$ is a segment. This segment objects to $S$ and $C' \subseteq C$. So $S$ is not fragmentation-proof. ■

Notice that even though stable segmentations are efficient, an efficient segmentation need not be stable. For instance, a segmentation that has a segment $(C_i, v_i)$ for each $v_i$ where $C_i$ consists of all consumers with value $v_i$ is efficient. But this segmentations is not stable because it is not saturated. We can add some consumers with value $v_i > v_1$ to the first segment $(C_1, v_1)$ without changing the price in the first segment. Thus fragmentation-proofness is a less demanding notion than stability.

# 6 Conclusions

We develop a cooperative game to study what markets segmentations arise when they are controlled by consumers. The surplus of each consumer depends on the coalition of consumers that are in the same segment. We capture this novel interaction with a game in which consumers choose coalitions and then the seller offers a profit-maximizing price for each coalition.

We study several solution concepts, including the core, stability, and fragmentation-proofness. The core is empty except for uninteresting cases. When the core is empty, our results are shown in Figure 6. The sets of stable segmentations and the ones that maximize average consumers surplus have a non-empty intersection which includes the maximal equal-revenue segmentation denoted MER in Figure 6, but neither set is contained in the other. Both of these sets are contained in the set of Pareto undominated segmentations, which is itself contained in the set of efficient segmentations. The set of efficient segmentations is the set of segmentations that are fragmentation-proof. When the core is non-empty, all five notions (the core, stable segmentations, segmentations that are Pareto undominated, those that maximize average consumer

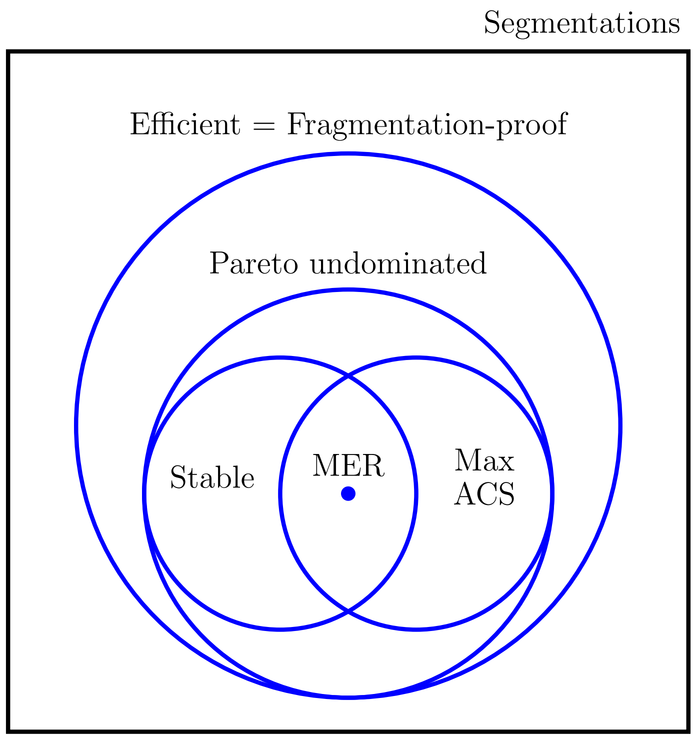


Figure 6: Summary of the relationship between different notions.

surplus, and efficient segmentations) collapse into a single segmentation, the trivial segmentation.

With two values, if the core is empty, then stable segmentations, those that are Pareto undominated, and those that maximize average consumer surplus are the same. They contain a class of segmentations that are not surplus-equivalent but are weakly surplus-equivalent.

The assumption that *any* coalition of consumers can form a segment can be seen as a richness assumption on the set of verifiable messages that each agent may send to the seller. In particular, for any consumer and any set that contains her, the consumer can prove to the seller that she belongs to the coalition. Therefore, any coalition of consumers can form a segment if each of the consumers in any coalition verifiably disclose to the seller that they belong to the coalition. In many real-world settings, the information that consumers can reveal to the seller may be limited. Our work abstracts away from such limitations in order to focus on the main economics forces that arise when consumers have control over their data, and we leave investigations of models that explicitly specify what information can be revealed to the seller as future work.

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