# Consumer-Optimal Market Segmentation

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**Abstract**

Consumer surplus in a market is affected by how the market is segmented. We study the maximum consumer surplus across all possible segmentations of a given market served by a multi product monopolist. We characterize markets for which the maximum consumer surplus equals a first best benchmark (i.e., maximum total surplus minus minimum profit). The first best benchmark is achievable whenever the seller does not find it profitable to screen types by offering multiple bundles, highlighting a novel impact of screening. We also characterize markets for which consumer surplus can be increased compared to the unsegmented market, and show that these markets are generic. We construct a simple segmentation that improves consumer surplus in these markets.

## 1   Introduction

Advances in information technologies have enhanced firms' ability to personalize their offers based on consumer data. A central regulatory question regarding consumer privacy is to what extent, if at all, a firm's ability to collect consumer data should be limited. As a 2012 report by the Federal Trade Commission puts it, "The Commission recognizes the need for flexibility to permit [...] uses of data that benefit consumers. At the same time, [...] there must be some reasonable limit on the collection of consumer data."[1] We study consumer surplus when a firm uses data to segment a market and make segment-specific offers.

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[1] "Protecting Consumer Privacy in an Era of Rapid Change, Recommendations for Businesses and Policymakers", FTC report, March 2012.

A large and growing literature (Pigou, 1920, Varian, 1985, Bergemann et al., 2015) studies the welfare impacts of market segmentation. This literature typically assumes that the seller conducts third degree price discrimination, that is, sells a single product in each market (or market segment). In reality, however, sellers may offer multiple quality or quantity levels of a product or bundles of heterogeneous products. This paper investigates consumer-optimal segmentations in such environments. We show that optimal consumer surplus depends crucially on whether or not the seller finds it profitable to screen types by offering multiple bundles. In this sense, we uncover a novel connection between second and third degree price discrimination.

Consider a multi product seller, for example an online retailer such as Amazon. The seller may be able to observe certain characteristics of its buyers, perhaps noisily, such as age, sex, or location. Based on the available information, the seller may be able to segment the market and offer each market segment a potentially different menu of products and bundles of products. For instance, the seller may offer bundle discounts to consumers in certain locations, or offer products exclusively to different age groups. The resulting producer and consumer surplus depend on how the market is segmented (the "segmentation"), which in turn depends on the information available to the seller.

We study the maximum consumer surplus across all possible segmentations of a given market. The maximum consumer surplus lies between two bounds. First, it is bounded from below by the consumer surplus in the unsegmented market.[2] If the maximum consumer surplus is strictly higher than this lower bound, we say that improving consumer surplus is possible. Second, the maximum consumer surplus is bounded from above by "first best" consumer surplus, which is the total surplus from allocating products efficiently minus the seller's profit in the unsegmented market.[3] If the maximum consumer surplus equals first best consumer surplus, we say that first best consumer surplus is achievable. We investigate the possibility of improving consumer surplus and the achievability of first best consumer surplus.

To illustrate, consider the following example, based on Bergemann et al. (2015), in which the seller produces a single product at zero cost and consumers have unit demand. A third of the

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[2]This is because one possible segmentation of a market is to leave it unsegmented.

[3]The maximum consumer surplus is at most the total surplus from allocating products efficiently minus the minimum profit across all segmentations. Profit is minimized when the market is unsegmented. This is because if the market is segmented, the seller can still offer the menu offered in the unsegmented market to all market segments in the segmented market.

consumers have valuation (willingness to pay) 1 for the product, and two thirds have valuation 2. In the unsegmented market, the seller optimally sells the product at price 2 (hence only consumers with valuation 2 buy the product), profit is 4/3, and consumer surplus is 0. Because it is efficient to allocate the product to all consumers, the surplus of the efficient allocation is 5/3. The first best consumer surplus is therefore $5/3 - 4/3 = 1/3$. There exists a segmentation in which the consumer surplus is 1/3 and therefore is equal to first best. This segmentation assigns half of the consumers with valuation 2 and all the consumers with valuation 1 to one market segment, and the rest of the consumers with valuation 2 to another market segment. In the first market segment the seller optimally sells the product at price 1, which gives a consumer with valuation 2 a surplus of $2 - 1 = 1$. In the second market segment, the seller optimally sells the product at price 2. Overall, a third of all consumers (half of the consumers with valuation 2) get a surplus of 1, so the consumer surplus is 1/3. Thus, first best consumer surplus is achievable, and improving consumer surplus is possible. Bergemann et al. (2015) show that in fact for any market with a single product and unit demand consumers, first best consumer surplus is achievable.[4]

In the above example the seller sets a price for the product in each market. In our setting, the seller can produce multiple products (at zero cost), and can offer a menu of bundle-price pairs (a bundle may include a single product) in each market. There are multiple types of consumers, where each type specifies a valuation for every bundle. Consumers' valuations may be non-linear, and as a result the seller may find it profitable to screen consumers in a given market by offering multiple bundles, each targeted at a different subset of consumers. The possible profitability of screening consumers in a market is a key distinguishing feature of our setting compared to most of the related literature on third degree price discrimination.[5]

The maximum consumer surplus can in principle be identified using a concavification approach from Aumann et al. (1995) and Kamenica and Gentzkow (2011). In particular, consider the concavified consumer surplus function, which is the smallest concave function that, for each market, is at least as large as the consumer surplus for that market.[6] For a given market, the

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[4]Notice that if first best consumer surplus is strictly higher than consumer surplus in the unsegmented market, then the achievability of first best consumer surplus implies possibility of improving consumer surplus.

[5]For instance, in the example discussed before, it is not profitable to screen types by offering randomized allocations, as shown by Myerson (1981) and Riley and Zeckhauser (1983).

[6]If there are multiple menus that maximizer profit, ties are broken to maximize consumer surplus.

maximum consumer surplus is equal to the value of the concavified consumer surplus function for that market.

In our setting, however, the concavification approach is of limited use. First, to apply concavification one must identify optimal (profit-maximizing) mechanisms for each market in order to identify consumer surplus in that market. However, no general characterization of optimal mechanisms is known in settings with multiple products.[7] Second, even if consumer surplus can be identified for all markets, it is not clear how its concavification can be compared to the upper and lower bounds on the maximum consumer surplus.[8] We obtain sharp characterizations of the achievability of first best consumer surplus and the possibility of improving consumer surplus by developing new techniques that do not rely on either a characterization of optimal mechanisms or concavification.

**First best consumer surplus.** One of our of key findings relates the achievability of first best consumer surplus to the profitability of screening. For a given market, we say that screening is profitable if the optimal menu for that market includes at least two bundles.[9] We provide three results which, roughly speaking, show that profitability of screening interferes with the achievability of first best consumer surplus. Our first result states that if screening is profitable for a market, then first best consumer surplus is unachievable for that market. If screening is profitable for a market, then the optimal mechanism for that market has an inefficient allocation. Our result shows that it is impossible to restore efficiency via segmentation without appropriating some of the gains to the seller.

Our other two results characterize when first best consumer surplus is achievable for either every market or no market with a given set of types. These two results relate the achievability of first best consumer surplus for a market to the profitability of screening in other markets

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[7]More precisely, no characterization is known when valuations are non-linear in quantity. An alternative modeling approach would be to assume that valuations are linear but costs are non-linear, as in Mussa and Rosen (1978). In such a setting, ironing can be used to identify optimal mechanisms. Nevertheless, the difficulty with concavification persists since it requires identification of consumer surplus for *all* markets. A closed form expression for consumer surplus for all markets cannot be obtained since ironing needs to be performed differently for different markets. We provide a detailed analysis in Subsection 7.2.

[8]Thus, the problem remains difficult even with a single product. Bergemann et al. (2015) develop methodologies to address achievability of first best consumer surplus for a single product.

[9]More precisely, we say that screening is profitable if any menu that offers only one bundle is not optimal.

with the same set of types. Our second result characterizes when first best consumer surplus is achievable for all markets. It shows that the following three statements are equivalent for a given a set of consumer types: (1) first best consumer surplus is achievable for all markets with this set of types; (2) screening is not profitable for any market with this set of types; and (3) for any pair of types, and for any bundle of products, the ratio of the valuations of that bundle to the grand bundle of all products is larger for the type with the higher valuation for the grand bundle. The second and third statements characterize the achievability of first best consumer surplus in different ways. The second statement has a more clear economic interpretation but may be difficult to determine (since it relies on identifying optimal mechanisms), whereas the third statement is a direct condition on primitives, i.e., the set of types.

Our third result characterizes when first best consumer surplus is unachievable for any market. It states that the following three statements are equivalent for a given a set of consumer types: (1) first best consumer surplus is unachievable for any market with this set of types; (2) for any market with this set of types, there is at most one price at which selling only the grand bundle is optimal; and (3) for every pair of types, there exists a bundle of products such that the ratio of valuations of that bundle to the grand bundle of all products is smaller for the type with the higher valuation for the grand bundle.

**Improving Consumer Surplus.** We next turn to the possibility of improving consumer surplus. We first show that there are markets for which improving consumer surplus is impossible. In such markets, consumer surplus in any segmentation that is not equal to the unsegmented market is in fact strictly lower than the consumer surplus in the unsegmented market. However, we show that such markets are not generic by constructing a segmentation that improves consumer surplus for generic markets.

The segmentation consists of only two market segments. The first segment contains two consumer types, the lowest consumer type, and another type to be identified shortly. The second segment contains the remaining types. The first segment is small enough so that the second segment is almost identical to the unsegmented market. As a result, generically, the optimal mechanism for the unsegmented market is also optimal for the second segment. Thus, all the types in the second segment remain unaffected by this segmentation. We show that the segmentation improves consumer surplus by identifying a type that benefits from joining the lowest type in the first market segment. This type is one that purchases the grand bundle of

products in the unsegmented market.

**Related Work.**   Our work connects the literature on second and third degree price discrimination.

The literature that studies third degree price discrimination and its effects and producer and consumer surplus is broad. Pigou (1920) provides examples where a segmentation may decrease total and hence consumer surplus. Followup work provides conditions for a segmentation to increase or decrease total surplus or consumer surplus (Robinson, 1969; Schmalensee, 1981; Varian, 1985; Aguirre et al., 2010; Cowan, 2016). Our work differs from this literature in two significant ways. First, with third degree price discrimination, the seller offers a single product to all consumers in a market, whereas the seller in our setting may screen consumers in each market by offering a menu of multiple bundles. Second, with the exceptions we now discuss, the literature assumes that the segmentation is exogenously fixed.

A growing part of the literature on third degree price discrimination studies surplus across all possible segmentations of a given market for a single product. Bergemann et al. (2015) identify the set of all producer and consumer surplus pairs that can result from some segmentation of a given market. Glode et al. (2018) study optimal disclosure by an informed agent in a bilateral trade setting, and show that the optimal disclosure policy leads to socially efficient trade, even though information is revealed only partially. Ichihashi (2018) and Hidir and Vellodi (2018) consider maximum consumer surplus when a multi product seller offers a single product to each market. Ichihashi (2018) considers a finite number of products and compares two regimes, one in which the seller may offer the same product at different prices to different segments, and another one in which the seller fixes the price in advance. Hidir and Vellodi (2018) characterize optimal segmentation with a continuum of products. Braghieri (2017) studies market segmentation with a continuum of firms each producing a single differentiated product. In contrast, the seller in our setting may offer multiple products in a market in order to screen consumers.[10] The only instance of this we are aware of is Bergemann et al. (2015)'s parametric example with two types and non-linear valuations.

Our work is also related to Roesler and Szentes (2017) and Condorelli and Szentes (2018)

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[10]We reiterate that we use the term screening to mean that there are at least two bundles in the seller's menu. A mechanism that offers a single product at a high price and therefore excludes certain consumers is not a screening mechanism.

who consider a consumer who learns about her own preferences. Roesler and Szentes (2017) assume that the distribution of the consumer's valuations is fixed, but the consumer can choose to learn a noisy signal about her valuations. They show that the optimal learning structure for the buyer achieves efficient trade. Condorelli and Szentes (2018) study a bilateral trade setting in which the consumer can choose any distribution for her valuations at no cost prior to trade. They show that trade is ex post efficient.

The literature on multi product bundling goes back to Stigler (1963) and Adams and Yellen (1976), who study bundling as an instrument to engage in second degree price discrimination. Theoretical findings on welfare effects of bundling are inconclusive.[11] The main hurdles are the difficulty with identifying optimal mechanisms and their complexity. Thanassoulis (2004) and Daskalakis et al. (2017) show that optimal mechanisms may offer randomized bundles. Vincent and Manelli (2007) and Hart and Nisan (2013) show that the optimal menu may offer infinitely many bundles. Daskalakis et al. (2014) and Chen et al. (2015) show that the problem of finding optimal mechanisms is computationally intractable. A more recent literature empirically estimates the welfare effects of bundling (Ho et al., 2012; Crawford and Yurukoglu, 2012).

Parts of our analysis combines results from the literature on information design and multi dimensional screening. Haghpanah and Hartline (2018) identify conditions under which selling only the grand bundle of products is optimal. Under these conditions we can use a segmentation from Bergemann et al. (2015) to achieve first best consumer surplus. To identify markets for which first best consumer surplus is *un*achievable, however, we develop a novel approach that does not rely on a characterization of optimal mechanisms. We use concavification Aumann et al. (1995); Kamenica and Gentzkow (2011) to study the possibility of improving consumer surplus for two types. As discussed before, the concavification approach is of limited use beyond the two type case since it requires a characterization of optimal mechanisms. We develop new techniques to show that surplus can generically be improved upon.

All proofs not given in the text are in the appendix.

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[11]Adams and Yellen (1976) show that bundling may be inefficient as it leads to oversupply or undersupply of certain goods. Salinger (1995) argues that bundling may result in lower or higher prices and therefore may increase or decrease consumer surplus.
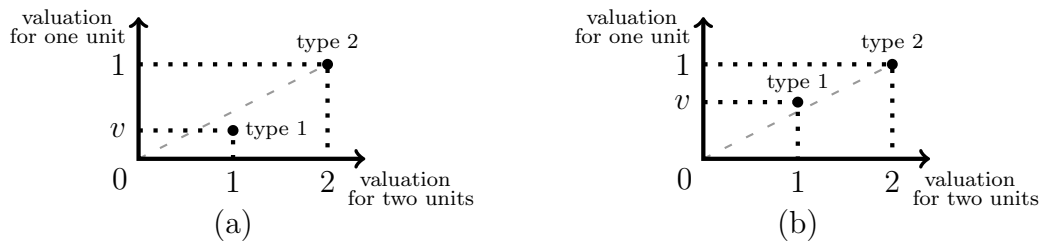
Figure 1: Type 1 has valuations $(v, 1)$ and type 2 has valuations $(1, 2)$ for one and two units. Case (a) corresponds to $v \le 0.5$, and case (b) corresponds to $v \ge 0.5$.

# 2    An Example

In this section we discuss a parametric example to highlight our results. We directly compare maximum consumer surplus with its lower and upper bounds by calculating their closed form expressions. Even though the calculations are straightforward, they are not easily extendable beyond this example. A reader who is only interested in our general treatment can skip ahead to the next section.

Consider an extension of the example discussed in the introduction. The seller produces a single product at zero cost and each consumer demands at most two units of the product. There are two types of consumers. A type 1 consumer has valuation $v \in (0, 1)$ for one unit and valuation 1 for two units. A type 2 consumer has valuation 1 for one unit and valuation 2 for two units. The two types are illustrated in Figure 1, in which case (a) corresponds to $v \le 0.5$ and case (b) corresponds to $v \ge 0.5$. A market $q$ consists of a fraction $1 - q$ of type 1 consumers and a fraction $q$ of type 2 consumers.

To identify maximum consumer surplus, it is useful to first identify optimal mechanisms. Consider the following three mechanisms and their revenue in a market $q$. Mechanism $N^1$ offers two units of the product (as a bundle) at price 1, and has revenue 1. Mechanism $N^2$ offers two units (as a bundle) at price 2, and has revenue $2q$. Mechanism $S$ screens; it offers one unit at price $v$ and two units at price $v + 1$, and has revenue $v + q$.[12] It can be shown that for any market $q$, one of the aforementioned three mechanisms is optimal, as illustrated in Figure 2.[13] If $v \le 0.5$, then mechanisms $N^1$ is optimal for markets in $[0, 0.5]$ and mechanism $N^2$ is optimal for

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[12]Type 1 purchases one unit. Type 2 is indifferent (has utility $1 - v$ for either choice) and purchases two units since we break ties to maximize revenue. Th revenue is $(1 - q)v + q(1 + v) = v + q$.

[13]We discuss how to identify optimal mechanisms with two types in Section 4. With more than two types, it is not known how to characterize optimal mechanisms, except in special cases.

Figure 2: Optimal mechanisms for the two cases (a) $v \leq 0.5$ and (b) $v \geq 0.5$.



Figure 3: (a) Consumer surplus $CS(q)$ for two the cases (a) $v \leq 0.5$ and (b) $v \geq 0.5$.

markets in $[0.5, 1]$. If $v \geq 0.5$, then mechanism $N^1$ is optimal for markets in $[0, 1-v]$, mechanism $S$ is optimal for markets in $[1 - v, v]$, and mechanism $N^2$ is optimal for markets in $[v, 1]$.

Next, we compute the (average) consumer surplus in each market $q$ generated by the optimal mechanism for that market.[14] Type 1 does not receive any information rents in any optimal mechanism. Thus, consumer surplus $CS(q)$ in a market $q$ is $q$ times the utility of type 2 in the optimal mechanism. The utility of type 2 is 1 in mechanism $N^1$, $1 - v$ in mechanism $S$, and 0 in mechanism $N^2$. Consumer surplus $CS(q)$ is illustrated in Figure 3.

A segmentation of market $q$ is a distribution $\mu$ over markets $[0, 1]$ such that $E_{q' \sim \mu}[q'] = q$. The maximum consumer surplus is $MCS(q) = \max_\mu E_{q' \sim \mu}[CS(q')]$, i.e., the maximum consumer surplus across all segmentations $\mu$. The maximum consumer surplus is obtained by concavifying $CS$. That is, $MCS(q) = \overline{CS}(q)$, where $\overline{CS}$ is the lowest concave function that is point-wise at least as high as $CS$.

The maximum consumer surplus $MCS(q)$ is at least $CS(q)$ and at most first best consumer surplus $FBCS(q)$, which is the surplus from efficient allocation (i.e., two units for each type) minus the revenue in market $q$.[15] If the two bounds are equal, i.e., $CS(q) = FBCS(q)$, then $CS(q) = MCS(q) = FBCS(q)$. This is the case for a market $q$ for which mechanism $N^1$ is optimal and for market $q = 1$.[16] We refer to such markets as trivial, because there is no scope

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[14]If there is more than one optimal mechanism we choose the one with higher consumer surplus.

[15]The surplus from efficient allocation is $1 - q + 2q = 1 + q$.

[16]If mechanism $N^1$ is optimal, then $CS(q) = q$ and $FBCS(q) = 1 + q - 1 = q$. For market $q = 1$, $CS(q) =$
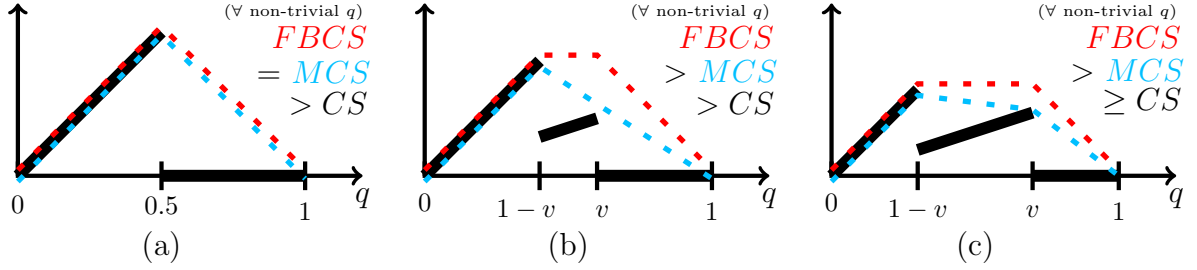
Figure 4: The relationship between $CS$, $MCS$, and $FBCS$: (a) $v \in (0, 0.5]$, (b) $v \in (0.5, \frac{\sqrt{5}-1}{2})$, and (c) $v \in [\frac{\sqrt{5}-1}{2}, 1)$.

for market segmentation to increase consumer surplus.

We are now ready to address the possibility of improving consumer surplus and the achievability of first best consumer surplus for all markets $q \in [0, 1]$. The relationship between consumer surplus, $CS$, maximum consumer surplus, $MCS$, and first best consumer surplus, $FBCS$, is illustrated in Figure 4, and depends on the value of $v$. If $v \in (0, 0.5]$ as in Figure 4 (a), then improving consumer surplus is possible and first best consumer surplus is achievable for all non-trivial markets.[17] If $v \in (0.5, \frac{\sqrt{5}-1}{2} \simeq 0.61)$ as in Figure 4 (b), then improving consumer surplus is possible and first best consumer surplus is unachievable for all non-trivial markets.[18] If $v \in [\frac{\sqrt{5}-1}{2}, 1)$ as in Figure 4 (c), then improving consumer surplus is possible for all non-trivial markets except $q = v$, and first best consumer surplus is unachievable for all non-trivial markets.[19]

Let us summarize. First best consumer surplus is achievable for either every non-trivial market (if $v \leq 0.5$) or no non-trivial market (if $v > 0.5$). Further, first best consumer surplus is achievable if and only if either mechanism $N^1$ or $N^2$ is optimal for every market, that is, the seller does not find it profitable to screen. We show that both these results extend to any two types

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$0 = FBCS(q) = 1 + q - 2q$.

[17] Markets in $[0, 0.5] \cup \{1\}$ are trivial. For a market $q \in (0.5, 1)$, $CS(q) = 0$, $MCS(q) = 1 - q$ and $FBCS(q) = 1 + q - 2q = 1 - q$.

[18] Markets in $[0, 1 - v] \cup \{1\}$ are trivial. The point $(v, CS(v))$ is below the line that connects the points $(1 - v, CS(1 - v))$ to $(1, 0)$, that is, $(1 - v)v < \frac{(1-v)^2}{v}$, since $v < \frac{\sqrt{5}-1}{2}$. As a result, $CS(q) < MCS(q)$ for all non-trivial $q$. To see that $MCS(q) < FBCS(q)$ for all non-trivial $q$, it is sufficient to verify that $MCS(v) < FBCS(v)$, which holds because $MCS(v) = \frac{(1-v)^2}{v} < FBCS(v) = 1 + v - 2v = 1 - v$.

[19] Markets in $[0, 1 - v] \cup \{1\}$ are trivial. $CS(v) = MCS(v)$ because $(1 - v)v \geq \frac{(1-v)^2}{v}$. To see that $MCS(q) < FBCS(q)$, it is sufficient to verify that $MCS(v) < FBCS(v)$, which holds because $MCS(v) = (1 - v)v < FBCS(v) = 1 + v - 2v = 1 - v$.

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(even with multi unit demand), although the proofs require developing new techniques. With more than two types, we characterize when consumer surplus is achievable for every non-trivial market, no non-trivial market, or some but not every non-trivial market. Further, improving consumer surplus is possible for a generic set of non-trivial markets. We show that this statement extends to any number of types (even with multi unit demand).

Notice that an equivalent interpretation of our example is of a seller who can produce a product at two quality levels (e.g., a smartphone with low or high memory) for which consumers have unit demand. Our model captures settings with multi unit demand, configurable products, and bundling with non-identical products by allowing for a finite set of alternatives, where each alternative represents a subset of products.

# 3   Model

There is a continuum of consumers with a finite set $T$ of types 1 to $n$.[20] There is a finite set $A$ of alternatives 0 to $k \geq 1$, where alternative 0 is the outside option. An alternative can correspond to a product or a bundle of products. For example, if there are two products that can be bundled, then there are four alternatives: the outside option, product 1, product 2, and a bundle that includes both products.[21] The cost of production is zero. The valuation of a type $i \in T$ for an alternative $a \in A$ is $v_a^i \geq 0$, with $v_0^i = 0$.[22] We assume that some alternative $\bar{a} \in A$ is all types' most preferred alternative, that is, $v_{\bar{a}}^i > v_a^i$ for all types $i$ and alternatives $a \neq \bar{a}$.[23] In a bundling application, it is natural to think of $\bar{a}$ as the grand bundle of all products. We do not assume that alternatives are otherwise ranked. We assume that a higher type has a higher

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[20]An equivalent interpretation is of a single consumers with unknown taste.

[21]Compare this with the case where each of the two products is a different configuration or quality level of a single good (e.g., a smartphone) for which the consumer has unit demand. In this case, the products cannot be bundled, and the alternatives are 0, product 1, and product 2. Notice further that our setting also allows for consumers with multi unit demands (with finite demands). For example, if there are two products and consumers need at most two units of product 1 and at most one unit of product 2, then there are six alternatives, where each alternative is a bundle that includes at most two units of product 1 and one unit of product 2.

[22]Assume without loss of generality that there are no redundant alternatives. That is, for each pair of alternatives $a, a'$, there exists a type $i$ such that $v_a^i \neq v_{a'}^i$.

[23]Our results hold with a weaker assumption that $v_{\bar{a}}^i \geq v_a^i$. We maintain the stronger assumption to simplify the analysis.

valuation for any alternative, that is, $v_a^1 < v_a^2 < \ldots < v_a^n$ for any alternative $a \neq 0$.[24]

An allocation $x \in X = \Delta(A)$ is a distribution over alternatives, where $x_a$ denotes the probability of alternative $a$. The (expected) utility of a type $i$ from an allocation $x$ and a payment $p$ is $v^i \cdot x - p = (\sum_a v_a^i x_a) - p$. We say that an allocation $x$ is empty if $x_0 = 1$, and is non-empty otherwise. Note that the efficient allocation $x$ for each type satisfies $x_{\bar{a}} = 1$.

A mechanism consists of an allocation function $x : T \to X$ and a payment function $p : T \to R$. Mechanism $M = (x, p)$ is incentive compatible (IC) if for all types $i$ and $j$,

$$v^i \cdot x(i) - p(i) \geq v^i \cdot x(j) - p(j). \tag{1}$$

That is, type $i$ does not benefit from "mimicking" type $j$. Mechanism $M$ is individually rational (IR) if for all types $i$,

$$v^i \cdot x(i) - p(i) \geq 0. \tag{2}$$

A *market* $f \in \Delta(T)$ is a distribution over types, where $f_i$ denotes the fraction of consumers with type $i$. The expected utility of consumers in a market $f$ and mechanism $M = (x, p)$ is $EU(f, M) = E_{i \sim f}[v^i \cdot x(i) - p(i)]$. A mechanism $(x, p)$ is optimal for market $f$ if it is IC and IR and maximizes revenue

$$E_{i \sim f}[p(i)]$$

across all IC and IR mechanisms.[25] For a market $f$, let $ER(f)$ be the maximum expected revenue, $\mathcal{M}(f)$ be the set of optimal mechanisms, and $CS(f)$ be the maximum consumer surplus (expected utility) across all optimal mechanisms,

$$CS(f) = \max_{M \in \mathcal{M}(f)} EU(f, M).$$

A *segmentation* of a market $f$ is a distribution $\mu \in \Delta(\Delta(T))$ over markets that average to $f$, that is, $E_{f' \sim \mu}[f'] = f$. We refer to a market $f'$ in the support of the segmentation $\mu$ as a market segment (or simply a segment). Let $SEG(f)$ denote the set of segmentations of $f$. Abusing notation, let $CS(\mu)$ be the consumer surplus in the segmentation $\mu$,

$$CS(\mu) = E_{f \sim \mu}[CS(f)].$$

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[24]Importantly, we do not require single-crossing. Single-crossing pis down binding incentive constraints, and simplifies the characterization of optimal mechanisms.

[25]A optimal mechanism exists. IR implies that the revenue of any mechanism is at most $E_{i \sim f}[v_{\bar{a}}^i]$, and the set of IC and IR mechanisms is closed.

When discussing segmentations of a given market $f$, we refer to $f$ as the unsegmented market.

We often represent a mechanism indirectly by a menu of allocation-price pairs, where each type chooses a pair that maximizes its utility. If a type is indifferent between two allocation-price pairs, it chooses the one with a higher price (e.g., a type $i$ chooses alternative $a$ at price $v_a^i$ over alternative 0 at price zero). If, further, the prices are identical, then the tie breaking can be arbitrary since it does not affect neither consumer surplus nor revenue. Assume without loss of generality that for each allocation-price pair, there exists a type that chooses that pair. Unless stated otherwise, every menu includes the outside option at price zero.

We distinguish between screening and non-screening mechanisms. A mechanism $(x, p)$ is a non-screening mechanism if it can be represented by a menu with a single allocation-price pair. Of particular interest are non-screening mechanisms $\{N^i\}_{i \in T}$, where $N^i$ offers the most preferred alternative $\bar{a}$ at price $v_{\bar{a}}^i$. Indeed, $N^i$ is optimal among all non-screening mechanisms for some $i$.[26] A mechanism is a screening mechanism if it can be represented by a menu with at least two allocation-price pairs. In such a mechanism, there exist two types with different and non-empty allocations. We say that non-screening is optimal if $N^i$ is optimal for some $i$. Otherwise, that is if $N^i$ is not optimal for any $i$, we say that screening is optimal. In this case, any optimal mechanism is a screening mechanism.

## 3.1 Bounds On The Maximum Consumer Surplus

For a given market $f$, we denote by $MCS(f) = \max_{\mu \in SEG(f)} CS(\mu)$ the maximum consumer surplus across all segmentations of $f$, and refer to the segmentation that achieves the maximum as the consumer-optimal segmentation. The maximum consumer surplus is bounded between two values. First, $MCS(f)$ is at least $CS(f)$, since the distribution $\mu$ over markets that puts probability 1 on market $f$ is a segmentation of $f$. Second, the maximum consumer surplus is at most the expected surplus of an efficient allocation, $E_{i \sim f}[v_{\bar{a}}^i]$, minus the maximum expected revenue in market $f$, $ER(f)$. The reason is that in any segmentation, the sum of the expected revenue and the consumer surplus is at most $E_{i \sim f}[v_{\bar{a}}^i]$. In addition, the expected revenue in any

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[26]Consider any non-screening mechanism $M$ that offers a single allocation $x$ for price $p$. The mechanism that offers $\bar{a}$ at price $p$ obtains at least as much revenue. Further, a price $p$ such that $v_{\bar{a}}^{i-1} < p < v_{\bar{a}}^i$, cannot be optimal, since then offering $\bar{a}$ at price $v_{\bar{a}}^i$ obtains a strictly higher revenue. Thus, it is optimal to offer $\bar{a}$ at price $v_{\bar{a}}^i$ for some $i$.

segmentation is at least $ER(f)$, since the seller can choose a mechanism in $\mathcal{M}(f)$ for all markets segments. Thus, consumer surplus is at most $E_{i \sim f}[v_{\bar{a}}^i] - ER(f)$. The following lemma formalizes the discussion.

**Lemma 1** *For every market $f$, $CS(f) \leq MCS(f) \leq E_{i \sim f}[v_{\bar{a}}^i] - ER(f)$.*

We study when either of the above two bounds is tight. We use the following terminology to refer to the tightness of each bound.

**Definition 1** *A segmentation $\mu$ of $f$ improves consumer surplus if $CS(f) < CS(\mu)$, and achieves first best consumer surplus if $CS(\mu) = E_{i \sim f}[v_{\bar{a}}^i] - ER(f)$.*

When there exists a segmentation of $f$ that achieves first best consumer surplus, we say that first best consumer surplus is achievable for $f$; when there exists a segmentation of $f$ that improves consumer surplus, we say that improving consumer surplus is possible for $f$. Notice that if the two bounds are not identical for a market, then improving consumer surplus is possible if first best consumer surplus is achievable.

The two bounds can be identical, however. This is the case when a market has an optimal mechanism with an efficient allocation. Indeed, consider a market $f$ for which mechanism $N^{\underline{i}(f)}$ is optimal, where $\underline{i}(f)$ is the lowest type in the support of $f$. The utility of each type $i$ in the support of $f$ is $v_{\bar{a}}^i - v_{\bar{a}}^{\underline{i}(f)}$. Since $CS(f)$ is the maximum expected utility across all optimal mechanisms, we have $CS(f) \geq E_{i \sim f}[v_{\bar{a}}^i - v_{\bar{a}}^{\underline{i}(f)}]$. As a result,

$$CS(f) + ER(f) \geq E_{i \sim f}[v_{\bar{a}}^i - v_{\bar{a}}^{\underline{i}(f)}] + E_{i \sim f}[v_{\bar{a}}^{\underline{i}(f)}] = E_{i \sim f}[v_{\bar{a}}^i].$$

Given Lemma 1, we must have $CS(f) = E_{i \sim f}[v_{\bar{a}}^i] - EF(f)$. That is, the lower bound of Lemma 1, $CS(f)$, is identical to its upper bound, $E_{i \sim f}[v_{\bar{a}}^i] - ER(f)$. We refer to such a market $f$ as trivial. More precisely, we have the following definition.

**Definition 2** *A market $f$ is trivial if it has an optimal mechanism $N^{\underline{i}(f)}$, and is non-trivial otherwise.*

For a trivial $f$, first best consumer surplus is achievable with a segmentation that assigns probability one to $f$, and improving consumer surplus is impossible. The rest of the paper investigates the possibility of improving consumer surplus and the achievability of first best consumer surplus for non-trivial markets.

## 3.2 Preliminary Observations

We start with three useful observations.

The first observation specifies two conditions, which are together necessary and sufficient for a segmentation to achieve first best consumer surplus. First, total surplus must be maximized. That is, for every segment, a mechanism with an efficient allocation must be optimal. In other words, every segment must be trivial (Definition 2). Second, the seller should not benefit from the segmentation. The seller benefits from the segmentation if an optimal mechanism for the unsegmented market is no longer optimal for a segment.[27] Thus, every optimal mechanism for the unsegmented market must also be optimal for every segment. We have the following lemma.

**Lemma 2** *For any segmentation $\mu$ of a market $f$, the following are equivalent:*

1. *$\mu$ achieves first best consumer surplus.*

2. *for some optimal mechanism $M$ of $f$ and every segment $f'$ of $\mu$, $f'$ is trivial and has an optimal mechanism $M$.*

3. *for every optimal mechanism $M$ of $f$ and every segment $f'$ of $\mu$, $f'$ is trivial and has an optimal mechanism $M$.*

The second observation uses Lemma 2 to obtain sufficient conditions for achieving first best consumer surplus. We describe a class of segmentations used by Bergemann et al. (2015), starting with the following definition.

**Definition 3** *A market $f$ is an* equal-revenue *market if for every type $i$ in the support of $f$, the mechanism $N^i$ is optimal.*

Consider a market $f$ for which the mechanism $N^i$ is optimal for some $i$, and a segmentation $\mu$ of $f$. Assume that every segment $f'$ is an equal-revenue market and includes $i$ in its support. Notice that $f'$ is trivial since any equal-revenue market is trivial. Additionally, by definition, $N^i$ is optimal for $f'$. Thus, $\mu$ achieves first best consumer surplus by Lemma 2. We have the following corollary.

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[27]This is because the seller can offer in this segment an optimal mechanism for this segment, and in all other segments an optimal mechanism for the unsegmented market.

**Corollary 1** *Consider a market $f$ for which $N^i$ is optimal for some $i$. A segmentation $\mu$ of $f$ achieves first best consumer surplus if for every segment $f'$ of $\mu$, $f'$ is an equal-revenue market and includes $i$ in its support.*

The third observation, from Kamenica and Gentzkow (2011), is that the maximum consumer surplus can be obtained from the concavification of the function $CS$. Formally, consider the set $S = \{(f, CS(f)) \mid f \in \Delta(T)\}$ of all markets and their consumer surplus, and let $CH(S)$ be its convex hull. Define $\overline{CS}(f) = \sup_{(f, cs) \in CH(S)} cs$. The function $\overline{CS}$ is the point-wise smallest concave function that is at least as large as $CS$. We refer to $\overline{CS}$ as the concavification of $CS$. Note that a pair $(f, cs)$ is in $CH(S)$ if and only if there exists a segmentation $\mu$ of $f$ such that $CS(\mu) = cs$. Thus, the maximum consumer surplus for $f$ is equal to the largest $cs$ such that $(f, cs) \in CH(S)$, and therefore $MCS(f) = \overline{CS}(f)$. This discussion is summarized by the following lemma.

**Lemma 3 (Kamenica and Gentzkow, 2011)** *For every market $f$, $MCS(f) = \overline{CS}(f)$. Thus, improving consumer surplus is possible for a market $f$ if and only if $\overline{CS}(f) > CS(f)$.*

In spite of its power, the above lemma is of limited use in our setting when there are more than two types. The reason is that with more than two types, $CS(f)$ cannot be specified in closed form for all markets $f$ due to known issues with identifying optimal mechanisms. We apply the lemma in our analysis of two types, and develop a new approach to study the possibility of improving consumer surplus when there are more than two types.

# 4   Two Types

We begin by considering markets with only two types, 1 and 2, and any number of alternatives $k \geq 1$. We refer to a market by the probability $q \in [0, 1]$ of type 2. We first provide necessary and sufficient conditions for the achievability of first best consumer surplus. We then provide examples of markets for which improving consumer surplus is impossible. Nonetheless, we show that improving consumer surplus is possible for generic markets. Our results in the subsequent sections generalize these results to more than two types.

With two types, optimal mechanisms can be characterized in closed form. We defer the characterization to the appendix (Appendix B.1), and invoke only the properties that are needed for the analysis.

## 4.1 First Best Consumer Surplus

The following lemma shows that the set of markets $[0,1]$ (where $q \in [0,1]$ is the probability of type 2) can be qualitatively divided into at most three regions. The first region consists of markets for which mechanism $N^1$ (which sells $\bar{a}$ at price $v_{\bar{a}}^1$) is optimal. These are markets in which the probability of type 1 is high. The second region consists of markets for which mechanism $N^2$ (which sells $\bar{a}$ at price $v_{\bar{a}}^2$) is optimal. These are markets in which the probability of type 2 is high. In between, there may be markets for which neither $N^1$ nor $N^2$ is optimal. For such markets, screening is optimal, that is, $x_a(1) = 1, x_{\bar{a}}(2) = 1$ where $a \neq 0, \bar{a}$, in a revenue maximizing mechanism $(x,p)$. If there exist a single alternative $a \neq 0, \bar{a}$, then a single mechanism is optimal for all markets in $[q_1, q_2]$. But more generally, optimal mechanisms may vary across $[q_1, q_2]$. For a mechanism $M$, let $\mathcal{F}(M)$ denote the (possibly empty) set of markets for which $M$ is optimal.

**Lemma 4** *There exists thresholds $q_1$ and $q_2$, $0 < q_1 \leq q_2 < 1$, such that $\mathcal{F}(N^1) = [0, q_1], \mathcal{F}(N^2) = [q_2, 1]$, and $\mathcal{F}(M) \subseteq [q_1, q_2]$ for any mechanism $M \neq N^1, N^2$.*

Given the above lemma, we can distinguish two cases. Either $q_1 = q_2$, in which case either $N^1$ or $N^2$ is optimal for any market, or $q_1 < q_2$, in which case neither $N^1$ nor $N^2$ is optimal for all markets in the interval $(q_1, q_2)$. See Figure 5. Haghpanah and Hartline (2018) provide a characterization of when $q_1 = q_2$ or $q_1 < q_2$, generalizable to any number of types, which we discuss later.

We now relate the above two cases to the possibility of achieving first best consumer surplus. If $q_1 = q_2$ then first best consumer surplus is achievable for all markets, and if $q_1 < q_2$ then first best consumer surplus is unachievable for any non-trivial market. The set of non-trivial markets is $(q_1, 1)$.[28] Recall from Section 3 that first best consumer surplus is achievable for all trivial markets. We thus obtain a characterization of the achievability of first best consumer surplus for all markets over two types.

**Proposition 1** *Assume that there are two types. For any non-trivial market $q$, first best consumer surplus is achievable if and only if $q_1 = q_2$.*

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[28]If $q \in [0, q_1]$, then mechanism $N^1$ is optimal. If $q = 1$, then mechanism $N^2$ that has an efficient allocation for all types in the support of $q$ (type 2) is optimal.

Figure 5: (a) $q_1 = q_2$. For any market, either $N^1$ or $N^2$ is optimal. (b) $q_1 < q_2$. Neither $N^1$ nor $N^2$ is optimal for any market in the interval $(q_1, q_2)$.

The proof outline is as follows. Suppose $q_1 = q_2$. Then the market $q_1 = q_2$ is an equal-revenue market. Consider a market in $[0, q_1]$, for which mechanism $N^1$ is optimal, and its segmentation into segments 0 and $q_1$. Since both segments are equal-revenue markets and include type 1 in their support, Corollary 1 implies that the segmentation achieves first best consumer surplus. A similar argument applies for any market in $[q_1, 1]$. Now suppose $q_1 < q_2$ and consider any market $q \in (q_1, 1)$ for which first best consumer surplus is achievable. By Lemma 2, two conditions must hold. First, every segment must be trivial, that is, it must be in $[0, q_1]$ or equal to 1. Second, any optimal mechanism for $q$ must be also optimal for every segment. However, there exists no mechanism that is both optimal for market 1 and also any market in $[0, q_1]$. Thus, either all segments are equal to 1, or they are all in $[0, q_1]$. In either case, market $q$ must be trivial.

**Proof of Proposition 1.** Assume first that $q_1 = q_2$. We show that for any market $q$, the exists a segmentation that achieves first best consumer surplus. The segmentation is identical to that of Bergemann et al. (2015). First consider $q \in [0, q_1]$, which implies that the mechanism $N^1$ is optimal for $q$. Consider a segmentation of $q$ into two segments $q' = 0$ and $q'' = q_1 = q_2$.[29] Notice that both $q'$ and $q''$ are equal-revenue markets (Definition 3) and include type 1 in their support. Thus the segmentation achieves first best consumer surplus by Corollary 1. Now consider $q \in [q_2, 1]$, which implies that the mechanism $N^2$ is optimal for $q$. Consider a segmentation of $q$ into two segments $q' = 1$ and $q'' = q_1 = q_2$.[30] Notice that both $q'$ and $q''$ are equal-revenue markets and include type 2 in their support. Thus the segmentation achieves first best consumer surplus by Corollary 1.

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[29]Segmentation $\mu$ assigns probability $\alpha$ to segment $q''$ and probability $1 - \alpha$ to segment $q'$, where $\alpha = \frac{q}{q_2}$. Since $q = (1 - \alpha) \cdot 0 + \alpha \cdot q_2$, $\mu$ is a segmentation of $q$.

[30]Segmentation $\mu$ assigns probability $\alpha$ to segment $q'$, and probability $1 - \alpha$ to segment $q''$, where $\alpha = \frac{q - q_2}{1 - q_2}$. Since $q = \alpha \cdot 1 + (1 - \alpha) \cdot q_2$, $\mu$ is a segmentation of $q$.

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Now assume $q_1 < q_2$, and consider some segmentation $\mu$ of a market $q$ that achieves first best consumer surplus. We show that $q$ must be trivial. By Lemma 2, two conditions must hold. First, every segment must be trivial, that is, it must be in $[0, q_1] \cup \{1\}$. Second, any optimal mechanism for $q$ must be also optimal for every segment. But there exists no mechanism that is optimal for market 1 and also for some market in $[0, q_1]$. To see this, note that by Lemma 4, the only optimal mechanism for market 1 is mechanism $N^1$. However, since $q_1 < q_2$ and $\mathcal{F}(N^1) = [q_2, 1]$, $N^1$ is not optimal for any market in $[0, q_1]$. Therefore, either every segment must be equal to 1, in which case $q = 1$, or every segment must be in $[0, q_1]$, in which case $q \in [0, q_1]$. Therefore, $q$ is trivial. $\blacksquare$

The two cases $q_1 = q_2$ or $q_1 < q_2$ can be characterized given the valuations of the two types. The two thresholds $q_1$ and $q_2$ are equal if and only if for any alternative $a$, type 2 has a higher ratio of valuations of alternative $a$ to $\bar{a}$, that is, $r_a^1 \leq r_a^2$ where $r_a^i = v_a^i/v_{\bar{a}}^i$. Indeed, if $r_a^1 \leq r_a^2$ for all $a$, then the thresholds $q_1 = q_2$ are equal to the market $q$ at which mechanisms $N^1$ and $N^2$ have the same revenue, $v_{\bar{a}}^1 = qv_{\bar{a}}^2$. If $r_a^1 > r_a^2$ for some $a$, then $q_1 < q_2$. We provide a closed-form characterization in Appendix B.1.

## 4.2   Improving Consumer Surplus

We now turn to the possibility of improving consumer surplus for two types. We combine the concavification approach (Lemma 3) with a characterization of optimal mechanisms to show that while there are markets for which improving consumer surplus is impossible, such markets are not generic.

By Lemma 3, for every market $q$, the maximum consumer surplus is $MCS(q) = \overline{CS}(q)$, where $\overline{CS}$ is the concavification of $CS$. Improving consumer surplus is possible if and only if $\overline{CS}(q) > CS(q)$. As a result, we need to identify $CS$. The lemma below shows that $CS$ consists of at most $k + 1$ linear pieces with decreasing slopes ($k$ is the number of alternatives). Further, each linear piece is a ray to the origin. See Figure 6 (a).

**Lemma 5** *For some $m \leq k$, there exist thresholds $\tau_0(= 0) < \ldots < \tau_{m+1}(= 1)$, and slopes $\alpha_0(= v_{\bar{a}}^2 - v_{\bar{a}}^1) > \ldots > \alpha_m(= 0)$, such that if $q \in (\tau_j, \tau_{j+1}]$ then $CS(q) = q\alpha_j$.*

The reasoning behind the structure of $CS$ is as follows. In any optimal mechanism, type 1 gets zero utility, whereas type 2 may get a positive utility (information rents) based on the
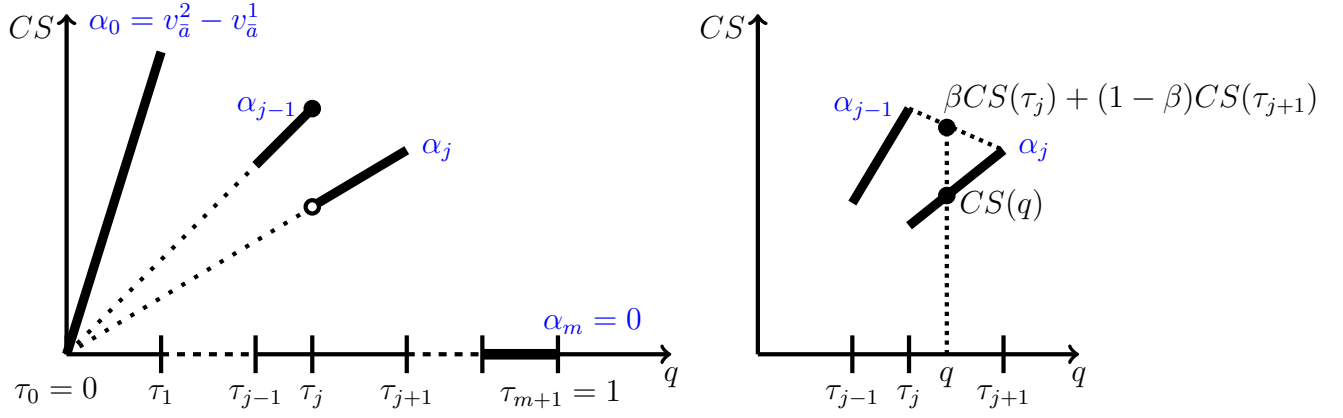
Figure 6: (a) The function $CS$ consists of $m$ linear pieces, each a ray to the origin, of decreasing slopes. (b) If $q = \beta\tau_j + (1-\beta)\tau_{j+1}$, then $CS(q) < \beta CS(\tau_j) + (1-\beta)CS(\tau_{j+1}) \leq \overline{CS}(q)$.

allocation of type 1. Thus, $CS(q)$ is equal to the probability $q$ of type 2 times the utility $u(2)$ of type 2 in the optimal mechanism. As $q$ increases, optimal mechanisms stay the same within an interval specified by thresholds $\tau$, and changes at a threshold. Therefore, within an interval, $CS(q) = qu(2)$ for a fixed $u(2)$. As $q$ crosses a threshold $\tau$, the allocation of type 1 changes in a way that it decreases the information rents of type 2. Notice that such an ordering of information rents holds even though there is no a priori ranking on alternatives. For $q \in [0, \tau_1]$, mechanism $N^1$ is optimal and thus $u(2) = v_{\bar{a}}^2 - v_{\bar{a}}^1$ (i.e., $\tau_1$ is $q_1$ from Proposition 1). For $q \in [\tau_m, 1]$, mechanism $N^2$ is revenue maximizing and thus $u(2) = 0$ (i.e., $\tau_m$ is $q_2$ from Proposition 1 ).

Given Lemma 5, improving consumer surplus is possible for any market $q \in (\tau_j, \tau_{j+1})$ where $j \geq 1$ (markets in $[\tau_0, \tau_1]$ are trivial). This can be seen in Figure 6 (b). If $q \in (\tau_j, \tau_{j+1})$, we have $q = \beta\tau_j + (1-\beta)\tau_{j+1}$ for some $\beta$, $0 < \beta < 1$. Since the slopes of the linear pieces are strictly decreasing, $CS(q)$ is strictly less than

$$\beta CS(\tau_j) + (1-\beta)CS(\tau_{j+1}) \leq \beta\overline{CS}(\tau_j) + (1-\beta)\overline{CS}(\tau_{j+1}) \leq \overline{CS}(q),$$

where the first inequality followed from $CS \leq \overline{CS}$, and the second inequality followed from concavity of $\overline{CS}$. Thus, $CS(q) < \overline{CS}(q)$, that is, improving consumer surplus is possible for $q$. We thus have the following proposition.

**Proposition 2** *Assume that there are two types. Improving consumer surplus is possible for a market $q$ if $q \in (\tau_j, \tau_{j+1})$ for some $j \geq 1$. Improving consumer surplus is impossible for at most the $k-1$ non-trivial markets $\tau_2, \ldots, \tau_m$.*
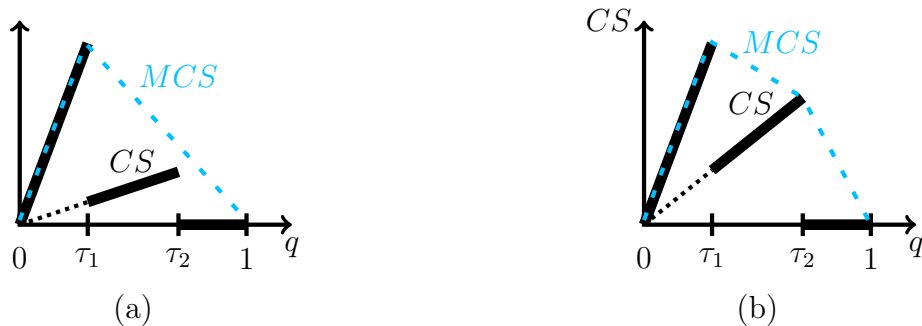
Figure 7: (a) $CS(q)$ is strictly lower than its concavification for all non-trivial markets. Therefore improving consumer surplus is possible for every non-trivial market. (b) $CS$ is equal to its concavification at $\tau_2$. Therefore improving consumer surplus is impossible for market $q = \tau_2$.

For a threshold market $q = \tau_j$, improving consumer surplus may or may not be possible. To see this, it is enough to consider a market with $k = 2$ (i.e., two alternatives other than the outside option). By Lemma 5, $CS$ consists of at most three linear pieces, identified by thresholds $\tau_1$ and $\tau_2$. Market $q = \tau_2$ is non-trivial. There are two possibilities. In Figure 7 (a), improving consumer surplus is possible for market $q = \tau_2$. In Figure 7 (b), improving consumer surplus is impossible for market $q = \tau_2$. See Section 2 for examples with two alternatives, and Appendix B.4 for extensions to any number of alternatives.

Notice two features in Figure 7. First, in Figure 7 (b), the consumer-optimal segmentation of any non-trivial market has a segment in which allocation is inefficient. In particular, the consumer-optimal segmentation of a market in $(\tau_1, \tau_2)$ has segments $\tau_1$ and $\tau_2$, and the consumer-optimal segmentation of a market in $(\tau_2, 1)$ has segments $\tau_2$ and 1. In market $\tau_2$, the allocation of any optimal mechanism is inefficient. Notice that this is not simply a consequence of the unachievability of first best consumer surplus. In Figure 7 (a), even though first best consumer surplus is unachievable for any non-trivial market, the consumer-optimal segmentation of any non-trivial market has two segments $\tau_1$ and 1. Both $\tau_1$ and 1 are markets in which an optimal mechanism has efficient allocation. Second, in Figure 7, any segmentation of market $\tau_2$ that does not assign probability one to segment $\tau_2$ has a strictly lower consumer surplus than $CS(\tau_2)$.

# 5    More Than Two Types: First Best Consumer Surplus

In this section we generalize Proposition 1 to any number of types (and any number of alternatives $k \geq 1$). We start by the following proposition, which states that if first best consumer

surplus is possible for a given market, then selling only alternative $\bar{a}$ must be optimal for that market. In other words, if screening is optimal for a market, then achieving first best consumer surplus is impossible.

**Proposition 3** *If first best consumer surplus is possible for $f$, then there exists a type $i$ such that the mechanism $N^i$ is optimal for $f$.*

The proof outline is as follows. Consider a market $f$, with an optimal mechanism $M$, for which improving consumer surplus is possible. We show that mechanism $N^j$ must also be optimal for $f$, where $j$ is the lowest type that is not excluded in $M$. Indeed, consider any segment $f'$. By Lemma 2, $M$ must be optimal for $f'$. Consider the lowest type $i$ in the support of $f'$. We must have $i \leq j$, as otherwise every type in the support of $f'$ gets strictly positive utility in $M$, which is a contradiction to the optimality of $M$ for market $f'$. Now by Lemma 2, both mechanisms $M$ and $N^i$ must be optimal for $f'$, where $i \leq j$. The following lemma shows that in this case, $N^j$ must also be optimal for $f'$. If $N^j$ is optimal for every segment $f'$, it is also optimal for $f$.

**Lemma 6** *Consider a market $f$ and an optimal mechanism $M = (x, p)$, and let $j$ be the lowest type that gets a non-empty allocation, i.e., $\{j\} = \arg\min_{j'} x_0(j') < 1$. Suppose that for some $i \leq j$, $N^i$ is also optimal for $f$. Then, $N^j$ is also optimal for $f$.*

**Proof.** Assume without loss of generality that $f$ has full support on types 1 to $n$.[31] Assume for contradiction that mechanisms $M$ and $N^i$ are optimal for $f$ but mechanism $N^j$ is not optimal. We show that mechanism $M$ can be improved upon. In particular, consider a mechanism $M'$ as follows. Types below $i$ get an empty allocation (as they do in $M$). Types $i$ to $j-1$ get alternative $\bar{a}$ with probability $\epsilon$ (and the outside option with probability $1 - \epsilon$) and pay $\epsilon v_{\bar{a}}^i$. Types $j$ to $n$ have the same allocation as in $M$, but their payment is decreased by $\epsilon(v_{\bar{a}}^j - v_{\bar{a}}^i)$ relative to their payment in $M$. See Figure 8.

Mechanism $M'$ has a higher revenue than $M$ for $\epsilon > 0$. Compared to $M$, $M'$ generates an additional revenue of $\epsilon v_{\bar{a}}^i$ from all types $i' \geq i$, and loses a revenue of $\epsilon v_{\bar{a}}^j$ from all types $i' \geq j$. The difference in revenue is $\epsilon v_{\bar{a}}^i \Pr[i' \geq i] - \epsilon v_{\bar{a}}^j \Pr[i' \geq j]$, which is $\epsilon$ times the difference in the revenue of mechanism $N^i$ compared to the revenue of mechanism $N^j$. This difference is strictly

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[31] Consider the support $\{i_1, \ldots, i_\ell\}$ of $f$. Relabel the types such that type 1 refers to $i_1$, type 2 refers to $i_2$, and so on.

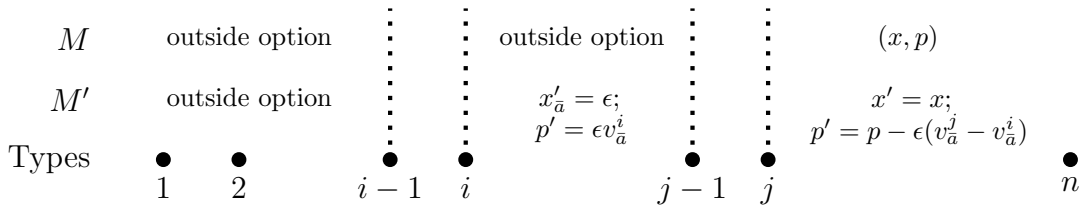| $M$ | outside option | | outside option | | | $(x, p)$ |
|---|---|---|---|---|---|---|
| $M'$ | outside option | | $x'_{\bar{a}} = \epsilon;$ $p' = \epsilon v^i_{\bar{a}}$ | | | $x' = x;$ $p' = p - \epsilon(v^j_{\bar{a}} - v^i_{\bar{a}})$ |
| Types | • 1  • 2 | | • $i-1$  • $i$ | | | • $j-1$  • $j$ • $n$ |

Figure 8: Construction of the mechanism $M'$ from the mechanism $M$ in the proof of Lemma 6.

positive by the assumption that $N^i$ is optimal but $N^j$ is not. It remains to show that $M'$ is IR and IC for small enough $\epsilon > 0$.

IR holds for types 1 to $i-1$, because they are excluded in $M'$. A type $i' = i, \ldots, j-1$ has utility $\epsilon v^{i'}_{\bar{a}} - \epsilon v^i_{\bar{a}} \geq 0$, and a type $i' \geq j$ has a higher utility in $M'$ than in $M$. Thus, IR holds for any $\epsilon > 0$.

For IC, observe that $M'$ coincides with $M$ in the limit as $\epsilon$ goes to zero. Thus, if an IC constraint is slack (holds with a strict inequality) in $M$, then it is satisfied in $M'$ for small enough $\epsilon$. Now, in mechanism $M$ a type $i'$ strictly prefers not to mimic another type $i''$ in two cases: (1) if $i' > j$ and $i'' < j$; (2) if $i' < j$ and $i'' \geq j$. In case (1), type $i'$ has a strictly positive utility in $M$. This is because type $j$ must get utility of zero in $M$ (otherwise $M$ is not optimal as prices can increase), and type $i'$ can mimic type $j$ and get $j$'s allocation and price but has a strictly higher valuation for any alternative $a \neq 0$ by assumption. Thus $i'$ strictly prefers not mimic type $i''$ (and get utility zero) in $M$. In case (2), type $i'$ gets a strictly negative utility from mimicking $i''$ since $v^i \cdot x(i'') - p(i'') < v^j \cdot x(i'') - p(i'') \leq 0$, where the last inequality followed since the utility of type $j$ is zero.

We next verify the remaining IC constraints in mechanism $M'$. Consider a type $i' < j$. As discussed in case (2) above, such a type $i'$ does not benefit from mimicking types $i'' \geq j$. Type $i'$ prefers the allocation of types $1, \ldots, i-1$ (the outside option) to the allocation of types $i, \ldots, j-1$ if and only if $\epsilon(v^{i'}_{\bar{a}} - v^i_{\bar{a}}) \leq 0$, i.e., $i' \leq i$. Thus truthtelling maximizes the utility of a type $i' < j$. For a type $i' \geq j$, note that mimicking a type $j, \ldots, n$ is not beneficial since $M$ is IC and all such types get the same additional payment in $M'$. From case (1) above, a type $i' > j$ does not benefit from mimicking types $1, \ldots, j-1$. Finally, the utility of type $j$ in $M'$ is at least $\epsilon(v^j_{\bar{a}} - v^i_{\bar{a}}) > 0$, which is the utility it would get by mimicking types $i, \ldots, j-1$, and is no lower than the utility of zero it would get by mimicking types $1, ..., i-1$. ∎

A subtlety in the proof of Lemma 6 is that if $\epsilon$ is large, then mechanism $M'$ may not be IC.
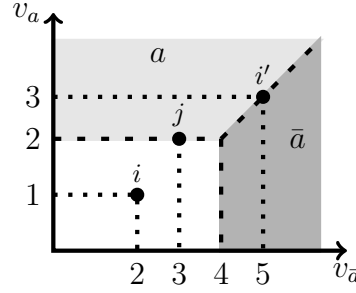
Figure 9: Mechanism $M$ offers $a$ at price 2 and $\bar{a}$ at price 4. Types in the dark shaded region choose $\bar{a}$, types in the light shaded region choose $a$, and other types are excluded. In mechanism $M'$, type $i'$ benefits from mimicking $i$ and getting utility $4\epsilon$ if $\epsilon$ is large.

This is because for large $\epsilon$, a type $i' > j$ may prefer to mimic $i, \ldots, j-1$ and get utility $\epsilon(v_{\bar{a}}^{i'} - v_{\bar{a}}^{i})$ instead of getting a utility of $\epsilon(v_{\bar{a}}^{j} - v_{\bar{a}}^{i})$ in addition to its utility in $M$. For instance, consider three types $i, j, i'$ and two alternatives $a, \bar{a}$ with valuations $(v_a^i, v_{\bar{a}}^i) = (1, 2)$, $(v_a^j, v_{\bar{a}}^j) = (2, 3)$, and $(v_a^{i'}, v_{\bar{a}}^{i'}) = (3, 5)$. Consider a mechanism $M$ represented by a menu that offers alternative $a$ at price 2 and alternative $\bar{a}$ at price 4. Type $i'$ gets $\bar{a}$, pays 4, and gets utility 1, as illustrated in Figure 9. In mechanism $M'$, type $i'$ gets $\bar{a}$ and pays $4 - \epsilon$, and type $i$ gets $\bar{a}$ with probability $\epsilon$ and pays $2\epsilon$. Utility of type $i'$ is $1 + \epsilon$, and its utility from mimicking type $i$ is $3\epsilon$. If $\epsilon > 1/2$, then mechanism $M'$ is not IC.

Before completing the proof of Proposition 3, we observe that the set of markets for which a given mechanism is optimal is convex. Convexity implies that if a mechanism is optimal for every segment of a segmentation, then it must also be optimal for the unsegmented market. To see this, consider a segmentation $\mu$ of market $f$ such that $M$ is optimal for every segment $f'$ of $\mu$. If $M$ is not optimal for $f$, then a mechanism $M'$ must give higher revenue than $M$ in at least one segment $f'$ of $\mu$, contradicting the assumption that $M$ is optimal for $f'$. Therefore, $M$ must be optimal for $f$. We therefore have the following result (recall that $\mathcal{F}(M)$ is the set of markets for which $M$ is optimal).

**Lemma 7** *For every mechanism $M$, the set $\mathcal{F}(M)$ is convex.*

We now prove Proposition 3.

**Proof of Proposition 3.** Consider a market $f$ and a segmentation $\mu$ of $f$ that achieves first best consumer surplus. By Lemma 2, there exists a mechanism $M$ that is optimal for market $f$ and also all segments $f'$ of $\mu$. Let $j$ the the lowest type that gets a non-empty allocation in $M$, and notice that $\underline{i}(f') \leq j$ for any segment $f'$, where $\underline{i}(f')$ is the lowest type in the support of

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market $f'$. Indeed, if the inequality were reversed, all types in the support of $f'$ would obtain a strictly positive utility, so $M$ would not be optimal for $f'$.

By Lemma 2, $N^{\dot{\imath}(f')}$ and $M$ are both optimal for $f'$. Lemma 6 implies that $N^j$ must also be optimal for $f'$. Since $N^j$ is optimal for every segment $f'$, Lemma 7 shows that it is also be optimal for $f$. Thus we have shown that if first best consumer surplus is possible for a market $f$, then there exists a non-discriminating mechanism, $N^j$, that is optimal for $f$. ∎

## 5.1 Achievability Of First Best Consumer Surplus, Non-Screening, And Increasing Ratios

Proposition 3 implies the following result. First best consumer surplus is achievable for all markets over a given set of types, if and only if non-screening is optimal for all markets over the set of types. Notice that the second statement is equivalent to saying that the sets $\mathcal{F}(N^i)$ collectively cover the set of all possible markets, $\cup_i \mathcal{F}(N^i) = \Delta(T)$.

**Theorem 1** *For any set of types $T$, the following are equivalent:*

1. *For every market, first best consumer surplus is achievable.*

2. *For every market, $N^i$ is optimal for some $i$.*

**Proof.** $(1) \rightarrow (2)$: Immediate from Proposition 3.

$(2) \rightarrow (1)$: We apply a segmentation that uses Corollary 1, and is identical to that of Bergemann et al. (2015). When offering only $\bar{a}$ is optimal for all markets, we can think of the problem as one with a single alternative $\bar{a}$. We defer the details to Appendix C.2. ∎

Theorem 1 characterizes the achievability of first best consumer surplus in terms of whether non-screening is optimal for all markets. Whether non-screening is optimal for a given market is in general difficult to ascertain. Nevertheless, Haghpanah and Hartline (2018) provide a simple characterization of the type spaces for which non-screening is optimal for all markets. For the characterization, let $r_a^i = v_a^i / v_{\bar{a}}^i$ be the ratio of type $i$'s valuation of alternatives $a$ and $\bar{a}$.

**Proposition 4 (Haghpanah and Hartline, 2018)** *For any set of types $T$, the following are equivalent:*
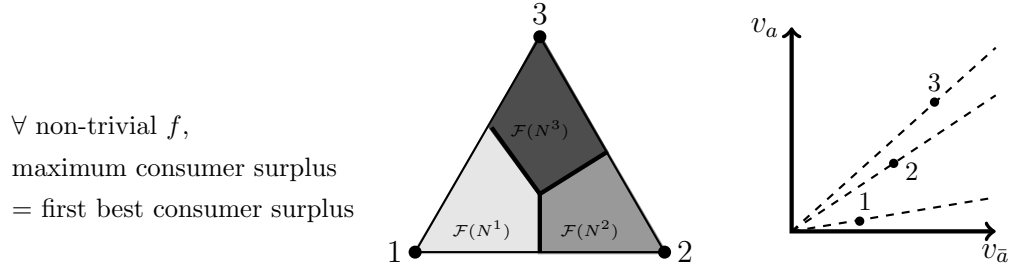
1. *For every market, $N^i$ is optimal for some $i$.*

Figure 10: The equivalence from Corollary 2. (1) First best consumer surplus is achievable (2) for every market, a non-screening mechanism is optimal. (3) The ratio of valuations increases in the valuation for the most preferred alternative.

   2. *The ratio $r_a^i$ is non-decreasing in $i$ for all $a$.*

   From Theorem 1 and Proposition 4 we have the following corollary. The corollary is illustrated in Figure 10.

**Corollary 2** *For any set of types $T$, the following are equivalent:*

   1. *For every market, first best consumer surplus is achievable.*

   2. *For every market, $N^i$ is optimal for some $i$.*

   3. *The ratio $r_a^i$ is non-decreasing in $i$ for all $a$.*

   Corollary 2 relates achievability of first best consumer surplus to the optimality of non-screening and to the ratio of valuations for different types.[32] Let us relate the corollary to our two type analysis (Section 4). Recall that with two types, the set of markets for which mechanism $N^1$ is optimal is $[0, q_1]$, and the set of markets for which mechanism $N^2$ is optimal is $[q_2, 1]$, where $q_1 \leq q_2$. For either $N^1$ or $N^2$ to be optimal for every market, we must have $q_1 = q_2$. Thus, with two types, Corollary 2 states that first best consumer surplus is achievable for every market if and only if $q_1 = q_2$, generalizing Proposition 1. However, Proposition 1 shows that if $q_1 < q_2$, then first best consumer surplus is unachievable for *all* non-trivial markets. Corollary 2

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   [32]The corollary partially generalizes Theorem 1 of Bergemann et al. (2015), which shows that first best consumer surplus is achievable for all markets if there is only one alternative other than the outside option $k = 1$. In this case, then the third property of Corollary 2 holds for any type space since $r_{\bar{a}}^i = 1$ and $r_0^i = 0$ for all types $i$. Thus, first best consumer surplus is achievable for all markets. Corollary 2 therefore generalizes Theorem 1 of Bergemann et al. (2015) regarding the achievability first best consumer surplus, although it does not address what other consumer-producer surplus pairs are achievable, which is the main focus of Bergemann et al. (2015).

does not make such a statement. The next subsection provides necessary and sufficient condition for first best consumer surplus to be unachievable for all (non-trivial) markets.

## 5.2 Unachievability Of First Best Consumer Surplus, Screening, And Decreasing Ratios

Corollary 2 characterizes when first best consumer surplus is achievable for all markets. We now characterize when first best consumer surplus is *un*achievable for all (non-trivial) markets. By Corollary 2, for first best consumer surplus to be achievable for all markets, a non-screening mechanism must be optimal for all markets. Therefore, one might guess that for first best consumer surplus to be unachievable for all markets, screening must be optimal for all markets. However, it is impossible for screening to be optimal for all markets. If a market consists of a single type, then selling the most preferred alternative to at a price equal to that type's willingness is optimal. Similarly, if a markets consists almost of a single type, then a non-screening mechanism is optimal for that market. To see the right condition, one can interpret the second statement of Corollary 2 as stating that the sets of markets for which non-screening is optimal collectively cover all markets. In contrast, we show below that first best consumer surplus is unachievable if the sets of markets for which non-screening is optimal do not intersect at all. This is illustrated in Figure 11.

To be precise, the theorem below states that, for a given set of types, the following three statements are equivalent: (1) first best consumer surplus is unachievable for all non-trivial markets; (2) for every market, $N^i$ is optimal for at most one $i$. This statements is equivalent to the following statement. For every $i \neq j$, the set of markets for which mechanism $N^i$ is optimal does not intersect with the set of markets for which mechanism $N^j$ is optimal. That is, the set of markets for which screening is optimal separates the sets of markets for which $N^i$ is optimal for some $i$; (3) for every pair of types, there exists an alternative $a$ such that the lower type has a strictly higher ratio of valuations $v_a/v_{\bar{a}}$ than does the higher type.

**Theorem 2** *For any set of types $T$, the following are equivalent:*

1. *For every non-trivial market, first best consumer surplus is unachievable.*

2. *For every market, $N^i$ is optimal for at most one $i$.*

3. *For every pair of types $i < j$, there exists an alternative $a$ such that $r_a^i > r_a^j$.*
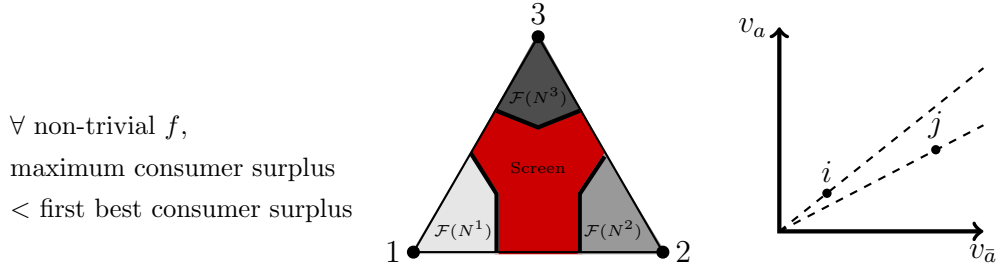
Figure 11: The equivalence from Theorem 2. (1) First best consumer surplus is unachievable (2) the sets of markets for which two different mechanisms $N^i$ and $N^j$ are optimal do not intersect. (3) The ratio of valuations decreases in the valuation for the most preferred alternative.

The proof uses the following lemma. The lemma states that if pair of types $i < j$ such that $r_a^i > r_a^j$ for some $a$, then there exists no market for which both mechanisms $N^i$ and $N^j$ are optimal. Recall from our two type analysis (Section 4) that if $r_a^i > r_a^j$, then $N^i$ and $N^j$ cannot both be optimal for any market with support $\{i, j\}$. The lemma generalizes this statement to all markets, regardless of the support. In contrast to our two type analysis, the proof of the lemma does rely on identifying optimal mechanisms. Instead, assuming both $N^i$ and $N^j$ are optimal, the lemma constructs a mechanism that outperforms both.

**Lemma 8** *Consider a pair of types $i < j$ such that $r_a^i > r_a^j$ for some $a$. For any market $f$, mechanisms $N^i$ and $N^j$ cannot both be optimal.*

**Proof.** Assume for contradiction that $r_a^i > r_a^j$ for some $i < j$ and $a$, and $N^i$ and $N^j$ are both optimal for a market $f$. Let $q_i$ be the probability of types $i$ and higher, and $q_j$ the probability of types $j$ and higher. For $N^i$ and $N^j$ to be both optimal, we must have $v_{\bar{a}}^i q_i = v_{\bar{a}}^j q_j$, that is $q_i = \frac{v_{\bar{a}}^j q_j}{v_{\bar{a}}^i}$. Thus we can write

$$v_a^i q_i = v_a^i \left(\frac{v_{\bar{a}}^j q_j}{v_{\bar{a}}^i}\right) = \left(\frac{v_{\bar{a}}^j v_a^i}{v_{\bar{a}}^i}\right) q_j > v_a^j q_j. \tag{3}$$

where the inequality followed from the assumption that $r_a^i > r_a^j$ (i.e., $v_a^i / v_{\bar{a}}^i > v_a^j / v_{\bar{a}}^j$).

Construct a mechanism $M$ that improves upon $N^j$ as follows. Type $i, \ldots, j-1$ get alternative $a$ with probability $\epsilon$ and pay $\epsilon v_a^i$. Types $j, \ldots, n$ get alternative $\bar{a}$ and pay $v_{\bar{a}}^j - \epsilon(v_a^j - v_a^i)$. This is illustrated in Figure 12.

Let us compare the revenue of $M$ with the revenue of $N^j$. Types $i, \ldots, j-1$ pay $\epsilon v_a^i$ more in $M$ than in $N^j$. Types $j$ and higher pay $\epsilon(v_a^j - v_a^i)$ less in $M$ than in $N^j$. The difference in
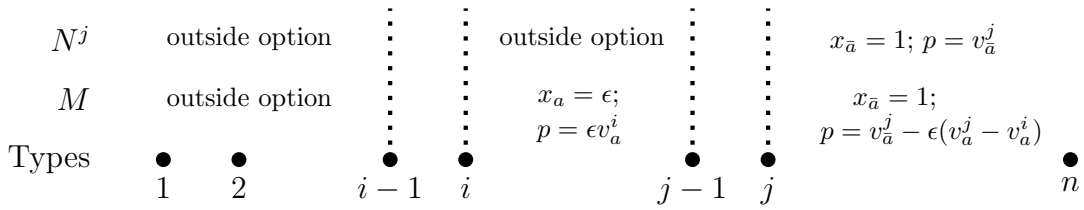
Figure 12: Construction of the mechanism $M$ in the proof of Lemma 8.

expected revenue is

$$\epsilon v_a^i (q_i - q_j) - \epsilon (v_a^j - v_a^i) q_j = \epsilon (v_a^i q_i - v_a^j q_j) > 0,$$

where the inequality followed from inequality 3. So to complete the proof, we show that $M$ is IC and IR, which contradicts the assumption that $N^j$ is optimal.

Mechanism $M$ is IR. Types lower than $i$ are excluded. A type $i'$ from $i$ to $j - 1$ has utility $\epsilon (v_a^{i'} - v_a^i) \geq 0$. Types $j$ and higher has a higher utility in $M$ than in $N^j$.

For IC, observe similarly to the proof of Lemma 6 that if an incentive constraint is slack in $N^j$, then it is satisfied in $M$ for small enough $\epsilon > 0$. In particular, (1) a type $i' > j$ does not benefit from mimicking a type $i'' < j$, (2) a type $i' < j$ does not benefit from mimicking a type $i'' \geq j$.

We now verify the remaining incentive constraints. A type $i' < j$ prefers the allocation of types $i, \ldots, j - 1$ to the outside option if and only if $\epsilon (v_a^{i'} - v_a^i) \geq 0$, i.e., $i' \geq i$. Thus the incentive constraints are satisfied for types $i' < j$. For types $i' \geq j$, note that mimicking any type $j, \ldots, n$ is not beneficial since all such types have the same allocation and payment. Finally, the utility of type $j$ in $M$ is $\epsilon (v_a^j - v_a^i)$, which is the utility it would receive by mimicking types $i, \ldots, j - 1$, and is strictly more that the utility it would receive by mimicking types $1, \ldots, i - 1$. ∎

Similar to the proof of Lemma 6, it is crucial in the proof of Lemma 8 to choose an $\epsilon$ that is small. For instance, if $\epsilon = 1$, then the mechanism $M$ may not be incentive compatible. Such a mechanism assigns alternative $a$ with payment $v_a^i$ to types $i, \ldots, j - 1$, and alternative $\bar{a}$ with payment $v_{\bar{a}}^j - (v_a^j - v_a^i)$ to types $j, \ldots, n$. The payment for alternative $\bar{a}$ is set such that type $j$ is indifferent between its own allocation and the allocation of types $i, \ldots, j - 1$. A type $i' > j$ may benefit from deviating to $i, \ldots, j - 1$.

We are now ready to complete the proof of Theorem 2. To develop a partial intuition, assume that the second statement of Theorem 2 holds, namely, $N^i$ is optimal for at most one $i$ for every market. If this is the case, then there exists no equal-revenue market (Definition 3) other than

markets that assign probability one to a single type. As a result, Corollary 1 cannot be applied since it concerns only segmentations wherein all segments are equal-revenue markets. We prove further that no segmentation achieves first best consumer surplus.

**Proof of Theorem 2.** $(1) \to (3)$: Assume for contradiction that for some $i < j$, $r_a^i \leq r_a^j$ for all $a$. By Corollary 2, first best consumer surplus is achievable for every market with support $\{i, j\}$. By Lemma 4, some markets with support $\{i, j\}$ are non-trivial.[33]

$(3) \to (2)$: Directly from Lemma 8.

$(2) \to (1)$: Assume that some segmentation $\mu$ of a non-trivial market $f$ achieves first best consumer surplus. From Proposition 3, $N^i$ must be optimal for some $i$. By Lemma 2, $N^i$ is optimal for every segment $f'$ of $\mu$. The lowest type $\underline{i}(f)$ in the support of $f'$ must satisfy $\underline{i}(f') \leq i$, as otherwise, all types in the support of $f'$ get strictly positive utility in an optimal mechanism $N^i$. Further, at least one segment $f'$ must satisfy $\underline{i}(f') < i$. Otherwise, if $\underline{i}(f') = i$ for all segments $f'$, then $\underline{i}(f) = i$. Since $N^i$ is optimal for $f$, $f$ is trivial which is a contradiction. Now consider a segment $f'$ such that $\underline{i}(f') < i$. By Lemma 2, for market $f'$, $N^{\underline{i}(f')}$ is optimal. That is, for $\underline{i}(f') \neq i$, $N^i$ and $N^{\underline{i}(f')}$ are both optimal for some market $f'$. ∎

Corollary 2 and Theorem 2 collectively generalize our two type analysis (Proposition 1). Corollary 2 states that first best consumer surplus is achievable for all non-trivial markets if and only if $q_1 = q_2$ (Figure 5). On the other hand, Theorem 2 states that first best consumer surplus is unachievable for any non-trivial market if and only if $q_1 < q_2$.

With two types, Corollary 2 and Theorem 2 collectively cover all possible cases. In particular, with two types, the second statement of Corollary 2 is the logical complement of the second statement of Theorem 2, and similarly for the third statement. That is, either non-screening is optimal for all markets, or the sets of markets for which non-screening is optimal do not intersect at all (as shown in Figure 5). With more than two types, this is no longer the case. It is possible that screening is optimal for some market, and yet, two sets of markets over which non-screening is optimal intersect, as illustrated in Figure 13. The next subsection addresses the remaining cases.

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[33]In particular, consider thresholds $q_1$ and $q_2$ from Lemma 4. Any market in which the probability $q$ of $j$ is $q_2 < q < 1$ is non-trivial.
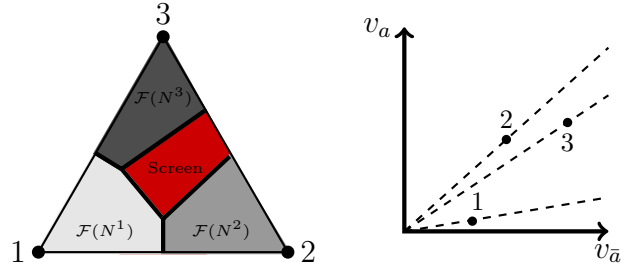
Figure 13: The equivalence from Corollary 3. (1) First best consumer surplus is achievable for some but not all non-trivial markets (2) screening is optimal for some markets, but the screening region does not separate non-screening regions (3) The ratio of valuations decreases in the valuation for the most preferred alternative for some but not all pairs of types.

## 5.3 The Remaining Cases

The following is an immediate corollary of Corollary 2 and Theorem 2. The first statement in Corollary 3 is true if and only the first statements of both Corollary 2 and Theorem 2 are false, and similarly for the second and the third statements. In particular, the following are equivalent: (1) first best consumer surplus is achievable for some but not all non-trivial markets; (2) screening is optimal for some market, but the sets of markets for which mechanism $N^i$ is optimal intersects that of $N^j$ for some $i \neq j$; (3) for some, but not all, pairs $i < j$, $r_a^i \leq r_a^j$ holds for all alternatives $a$. See Figure 13.

**Corollary 3** *For any set of types $T$, the following are equivalent:*

1. *For some but not all non-trivial markets, first best consumer surplus is achievable.*

2. *There exists a market for which $N^i$ is not optimal for any $i$, and there exists a market for which $N^i$ and $N^j$ are both optimal for some $i \neq j$.*

3. *There exist a pair of types $i < j$ such that $r_a^i \leq r_a^j$ for all $a$, and there exists a pair of types $i' < j'$ such that $r_a^{i'} > r_a^{j'}$ for some $a$.*

In Appendix C.3, we discuss with an example the markets for which first best consumer surplus is achievable when conditions (2) and (3) from Corollary 3 hold.

# 6 More Than Two Types: Improving Consumer Surplus

Recall from Subsection 4.2 that with two types, there may be markets for which no segmentation can improve consumer surplus. We now construct a simple parametric segmentation for any number of types and alternatives. We show that, parameterized appropriately, the segmentation improves consumer surplus for a generic market $f$ (for a notion of genericity to be formalized shortly). Without loss of generality, we assume that $f$ has full support.

The segmentation of $f$ consists of only two segments. The first segment contains only type 1 and some other type $i$ (that gets alternative $\bar{a}$ with probability one in some optimal mechanism for $f$). The second segment contains the remaining types. If the probability of the first segment is small enough, then the second segment is almost identical to market $f$, and thus generically an optimal mechanism for $f$ is also optimal for the second segment. As a result, types in the second segment remain unaffected by this segmentation. To show that the segmentation improves consumer surplus, we show that the utility of type $i$ is strictly higher in the first segment than in $f$ (the utility of type 1 is 0 in $f$ and in the first segment).

We now formalize the above discussion. We first define a parametric segmentation and later identify the appropriate parameters. Recall from our discussion in Section 4 that for every pair of types $i < j$, there exists $\tau < 1$ such that the mechanism $N^i$ is optimal for a market in which the probability of type $i$ is $\tau$ and the probability of type $j$ is $1 - \tau$. Let $f^{i,j}$ refer to such a market.

**Definition 4** *For a market $f$, type $i \neq 1$, and $\epsilon \geq 0$, let $\tilde{\mu}_{i,\epsilon}(f)$ be a segmentation of $f$ with two segments $f', f''$ and probabilities $\epsilon, 1 - \epsilon$, respectively, where $f' = f^{1,i}$ and $f'' = \frac{f - \epsilon f'}{1 - \epsilon}$.*

Note that $f''$ is defined such that $\tilde{\mu}_{i,\epsilon}(f)$ is indeed a segmentation of $f$, that is, $\epsilon f' + (1 - \epsilon) f'' = f$. For $f''$ to be a market (a distribution over types), $\epsilon$ must be small enough such that $f - \epsilon f' \geq 0$. For the remainder of the discussion we assume that this is indeed the case.

We show later that if type $i$ is a type that gets alternative $\bar{a}$ in an optimal mechanism of $f$, then $\tilde{\mu}_{i,\epsilon}(f)$ improves consumer surplus. The next lemma shows that such a type exists.

**Lemma 9** *In any optimal mechanism $(x, p)$ for a market $f$, there exists a type $i$ that gets alternative $\bar{a}$ with probability one, $x_{\bar{a}}(i) = 1$.*

**Proof.** Assume for contradiction that $x_{\bar{a}}(j) \neq 1$ for all $j$ in an optimal mechanism $(x, p)$. Consider offering alternative $\bar{a}$ for a high price $q$ and decreasing $q$ until some type $i$ is indifferent
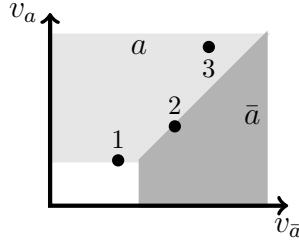
Figure 14: Mechanism $M$ is represented by a menu that offers alternative $a$ for a lower price than alternative $\bar{a}$. Type 3 is the highest type but does not get alternative $\bar{a}$.

between its allocation in $(x, p)$ and paying $q$ for $\bar{a}$. Notice that since $i$ has a higher valuation for $\bar{a}$ than $x(i)$, $q$ must be strictly higher than $p(i)$. Since the payment of other types are unaffected, the alternative mechanism obtains higher expected revenue than $(x, p)$, contradicting the assumption that $(x, p)$ is optimal for $f$. ∎

Note that the type that gets alternative $\bar{a}$ need not be the highest type $n$. See Figure 14.

We now define our notion of genericity. A set of markets is generic if its complement is a subset of a finite union of hyperplanes. A set of markets $H \subseteq \Delta(T)$ is a hyperplane if there exists $a_1, \ldots, a_n$, where $a_i \neq 0$ for at least one $i \in \{1, \ldots, n\}$, such that $H = \{f \in \Delta(T) \mid \sum_i a_i f_i = 0\}$.[34] Let $\mathcal{F}_{Tr} \subseteq \Delta(T)$ be the set of trivial markets (Definition 2). We have the following definition.

**Definition 5** *A set of markets $\mathcal{F} \subseteq \Delta(T)$ is generic if there exists a finite set of hyperplanes $\{H_\ell\}_\ell$ such that $\Delta(T) \setminus \mathcal{F} \subseteq \cup_\ell H_\ell$. A set of markets $\mathcal{F} \subseteq \Delta(T)$ is generic among non-trivial markets if $\mathcal{F} \cup \mathcal{F}_{Tr}$ is generic.*

The main result of this section states that the segmentation defined in Definition 4, parameterized appropriately, generically improves consumer surplus for non-trivial markets. To be precise, let $\mathcal{F}_{Im} \subseteq \Delta(T)$ be the set of markets $f$ for which $\tilde{\mu}_{i,\epsilon}(f)$ of Definition 4 strictly improves consumer surplus for some $i, \epsilon$. The proposition below shows that the set of markets for which consumer surplus can be improved with the segmentation in Definition 4 is generic among non-trivial markets.[35] Type $i$ is any type that receives $\bar{a}$ with probability one in $f$.

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[34] The standard definition of a hyperplane is that $\sum_i a_i f_i = b$ for some $b$. The definition with $b = 0$ suffices for our purposes.

[35] The set $\mathcal{F}_{IM}$ itself is not generic. The reason is that improving consumer surplus is impossible for the set of non-trivial markets, which is a generic set. In Figure 7, $\mathcal{F}_{IM} = (\tau_1, \tau_2) \cup (\tau_2, 1)$, which is not generic. However, $\mathcal{F}_{IM}$ is generic among non-trivial markets, $(\tau_1, 1)$.

**Theorem 3** *For any given set of types, the set of markets $\mathcal{F}_{Im}$ for which improving consumer surplus is possible is generic among non-trivial markets.*

The proof of Theorem 3 consists of three steps. First, we show that for any non-trivial market $f$, in any optimal mechanism $M$, any type that gets alternative $\bar{a}$ with probability one pays strictly more than $v_{\bar{a}}^1$. Second, we show that if $f$ has a unique optimal mechanism, the segmentation in Definition 4 strictly improves consumer surplus. Third, we show that the set of markets with a unique optimal mechanism is generic.

We start with the first step. Consider any optimal mechanism $M$ for market $f$. By Lemma 9, there exists a type $i$ that gets alternative $\bar{a}$ with probability one. The lemma below shows that the payment of $i$ is strictly higher than $v_{\bar{a}}^1$. Therefore, type $i$ has a strictly higher utility in mechanism $N^1$, the optimal mechanism in segment $f'$ of segmentation $\tilde{\mu}_{i,\epsilon}(f)$ (Definition 4) than it does in mechanism $M$.

**Lemma 10** *Consider any optimal mechanism $(x, p)$ for a non-trivial market $f$. For any type $i$, if $x_{\bar{a}}(i) = 1$, then $p(i) > v_{\bar{a}}^1$.*

**Proof.** Consider an optimal mechanism $(x, p)$ of a non-trivial market $f$, and assume for contradiction that $x_{\bar{a}}(i) = 1$ and $p(i) \leq v_{\bar{a}}^1$ for some type $i$. For a type $j$ to not benefit from mimicking type $i$, it must be that

$$v^j \cdot x(j) - p(j) \geq v^j \cdot x(i) - p(i) \geq v_{\bar{a}}^j - v_{\bar{a}}^1.$$

Thus, $p(j) \leq v^j \cdot x(j) - (v_{\bar{a}}^j - v_{\bar{a}}^1)$. Since $v^j \cdot x(j) \leq v_{\bar{a}}^j$, we must have $p(j) \leq v_{\bar{a}}^1$ for all $j$. Thus the mechanism $N^1$, in which all types pay $v_{\bar{a}}^1$, must also be optimal. Since $N^1$ is optimal and has an efficient allocation, $f$ is trivial, which is a contradiction. ∎

For the remainder of the proof, it is useful to consider the set of implementable payment functions. Say that a mechanism $(x, p)$ is $IR_0$ if it is IR and the individual rationality constraint for type 1 holds with equality, $v^1 \cdot x(1) - p(1) = 0$. Any optimal mechanism must be $IR_0$. Consider the set $P = \{p : T \rightarrow R \mid \exists x, \text{ s.t. } (x, p) \text{ is IC \& } IR_0\}$ of implementable payment functions. The set $P$ is convex with finitely many extreme points. To see this, note that the set of IC and $IR_0$ mechanisms is a bounded convex polytope. It is the intersection of a finite number of halfspaces specified by the IC constraints (1), IR constraints (2) (plus $v^1 \cdot x(1) - p(1) = 0$), and the constraints that $x_a(i) \geq 0$ and $\sum_a x_a(i) \leq 1$, defined over the finite set of variables $x$

and $p$. The set $P$ is a projection of the set of IC and $IR_0$ mechanisms $(x, p)$. Formally, we have the following lemma.

**Lemma 11** *There exists a finite set $\mathcal{E}_P \subseteq R^{|T|}$ such that $P$ is the convex hull of $\mathcal{E}_P$.*

We say that a market $f$ has a unique optimal payment function if $p = p'$ for any two optimal mechanisms $(x, p)$ and $(x', p')$ of $f$. We next show that if a non-trivial market $f$ has a unique optimal payment function, then the segmentation $\tilde{\mu}_{i,\epsilon}(f)$ with any type $i$ that receives $\bar{a}$ with probability one in $f$ and small enough $\epsilon > 0$ strictly improves consumer surplus. The key observation is that if a market $f$ has a unique optimal payment function, then any optimal mechanism for $f$ is also optimal for all markets that are close enough to $f$, and in particular segment $f''$ of $\tilde{\mu}_{i,\epsilon}(f)$. Therefore, all the types in segment $f''$ are unaffected by the segmentation. Since type $i$ prefers mechanism $N^1$ to the optimal mechanism for $f$, the segmentation strictly improves consumer surplus.

**Lemma 12** *It a non-trivial market $f$ has a unique optimal payment function, then $f \in \mathcal{F}_{Im}$.*

**Proof.** Consider a non-trivial market $f$ with a unique optimal payment function $p$, and its optimal mechanism $M = (x, p)$ such that $CS(f) = EU(f, M)$. By Lemma 9, there exists a type $i$ that receives alternative $\bar{a}$ with probability one in $M$.

Consider the segmentation $\tilde{\mu}_{i,\epsilon}(f)$ two with segments $f'$ and $f''$. For small enough $\epsilon > 0$, the mechanism $M$ is optimal for $f''$. To see this, note that since the set $\mathcal{E}_P$ is finite (Lemma 11) and $p$ is the unique optimal payment function, there exists $\delta > 0$ such that $E_{i \sim f}[p(i)] \geq E_{i \sim f}[p'(i)] + \delta$ for all $p' \in \mathcal{E}_P$. Thus, by continuity of the expected revenue in $f$, for small enough $\epsilon$, $E_{i \sim f''}[p(i)] \geq E_{i \sim f''}[p'(i)]$ for all $p' \in \mathcal{E}_P$. Since all payment functions are convex combinations of the payment functions in $\mathcal{E}_P$ (Lemma 11), we must have $E_{i \sim f''}[p(i)] \geq E_{i \sim f''}[p'(i)]$ for all payment functions $p' \in P$, that is, the mechanism $M$ is optimal for $f''$. Since $CS(f'')$ is the maximum consumer surplus across all optimal mechanisms of $f''$, $CS(f'') \geq EU(f'', M)$.

We now complete the proof by showing that $\tilde{\mu}_{i,\epsilon}(f)$ improves consumer surplus. By construction, the mechanism $N^1$ is optimal for $f'$. Type 1 has the same utility in $M$ and $N^1$ (which is zero), and type $i$ has strictly higher utility in $N^1$ than in $M$ by Lemma 10 and the fact that $i$ receives alternative $\bar{a}$ with probability one in $M$. Thus, $CS(f') \geq EU(f', N^i) > EU(f', M)$.

Now write the consumer surplus of the segmentation,

$$CS(\tilde{\mu}_{i,\epsilon}(f)) = \epsilon CS(f') + (1 - \epsilon)CS(f'') > \epsilon EU(f', M) + (1 - \epsilon)CS(f'')$$

$$\geq \epsilon EU(f', M) + (1 - \epsilon)EU(f'', M)$$

$$= EU(f, M) = CS(f).$$

Thus the segmentation $\tilde{\mu}_{i,\epsilon}(f)$ has higher consumer surplus than the market $f$. ■

The last step in the proof of Theorem 3, is to show that the set of markets that admit a unique optimal payment function is generic.

**Lemma 13** *The set of markets that admit a unique optimal payment function is generic.*

**Proof.** Consider a market $f$ for which more than one payment function in $P$ maximizes revenue. Since $\mathcal{E}_P$ is the set of extreme points of $P$, there must exist $p, p' \in \mathcal{E}_P, p \neq p'$ that are optimal for $f$. Thus such a market is contained in a hyperplane $H_{p,p'}$ defined by the equation $\sum_i f_i(p(i) - p'(i)) = 0$. Since $\mathcal{E}_P$ is finite, the set of markets $f$ with more than one optimal payment function is contained in a finite union of hyperplanes, one for each pair of payment functions in $\mathcal{E}_P$, i.e., $\cup_{p,p' \in \mathcal{E}_P} H_{p,p'}$. Thus, by Definition 5, the set of markets with a unique revenue maximizing payment function is generic. ■

We now complete the proof of Theorem 3.

**Proof of Theorem 3.** Consider the set of markets $f$ that admit a unique optimal payment function, which is generic by Lemma 13. If $f$ is non-trivial, then by Lemma 12, $f \in \mathcal{F}_{Im}$. As a result, the set of markets that admit a unique optimal mechanism, which is a generic set, is contained in $\mathcal{F}_{Tr} \cup \mathcal{F}_{Im}$. Since a superset of a generic set is generic, the set $\mathcal{F}_{Tr} \cup \mathcal{F}_{Im}$ is generic. ■

# 7 Concluding Remarks

We studied the maximum consumer surplus across all possible segmentations of a given market. A key feature of our model is that the seller may find it profitable to screen consumers in a market by offering multiple bundle-price pairs. With two consumer types, we provided a complete characterization of consumer-optimal segmentations as well as a characterization of when improving consumer surplus is possible and first best consumer surplus is achievable. With more than two types, we provided a characterization of when first best consumer surplus

is achievable for all markets or no market with a given set of types. We also showed that there are markets for which improving consumer surplus is impossible, but we constructed a simple segmentation that improves consumer surplus for generic markets.

We wrap up by first discussing the consequences of relaxing the assumption that the sets of types and alternatives are finite, and then briefly discussing how our results extend to a setting with non-linear costs.

## 7.1   Infinitely Many Types Or Alternatives

The results of Section 5 regarding the achievability of first best consumer surplus extend straightforwardly when the set of types $T$ and the set of alternatives $A$ are compact but not necessarily finite. In particular, Proposition 3 and Lemma 8, which are the key steps in the proofs of Corollary 2 and Theorem 2, continue to hold with compact $T$ and $A$.

The result in Section 6 regarding the possibility of improving consumer surplus for generic markets, Theorem 3, may no longer hold if either $T$ or $A$ is infinite. Recall from Subsection 4.2 that consumer surplus consists of at most $k-1$ linear pieces, where $k$ is the number of alternatives (Figure 6). Thus the number of linear pieces, and the number of markets for which improving consumer surplus is impossible, grows with the number of alternatives. If the set $A$ is countable, the the set markets for which improving consumer surplus is impossible is also countable.[36] If $A$ is uncountable, then the set of markets for which improving consumer surplus is impossible may be uncountable, as the following example shows.

Consider the following extension of the example in Section 2. There are two types, 1 and 2, and a continuum of alternatives $A \subseteq R$. The valuations are $v_a^1 = 2a - a^2$, and $v_a^2 = 2a$. Consumer surplus $CS$ and the maximum consumer surplus $MCS$ (the concavification of $CS$) are illustrated in Figure 15 (a) and (b), for $A = [0, 1]$ and $A = [0, 0.5]$.[37] If $A = [0, 1]$, then improving consumer surplus is impossible for a generic set of non-trivial markets, namely $[0, 0.5]$.[38] If $A = [0, 0.5]$,

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[36]Therefore, the set of markets for which improving consumer surplus is possible is generic if the definition of genericity (Definition 5) is adjusted to account for a countable (instead of finite) union of hyperplanes.

[37]In the optimal mechanism for market $q$, type 1 gets an alternative $a$ that maximizes $v_a^1 - q v_a^2$. If $A = [0, 1]$, then type 1 gets an alternative $a = 1 - q$, and consumer surplus is $CS(q) = q(1-q)^2$. If $A = [0, 0.5]$, then type 1 gets an alternative $a = 0.5$ if $q \leq 0.5$ and $a = 1 - q$ if $q \geq 0.5$. Therefore, $CS(q) = q/4$ if $q \leq 0.5$ and $CS(q) = q(1-q)^2$ if $q \geq 0.5$.)
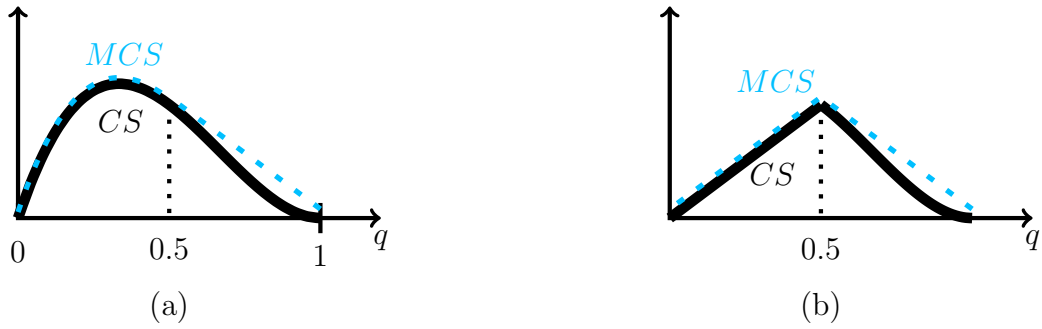
[38]Bergemann et al. (2015) develop a related example.

Figure 15: (a) $A = [0,1]$ and $CS(q) = q(1-q)^2$. (b) $A = [0,.5]$, and $CS(q) = q/4$ if $q \leq 0.5$ and $CS(q) = q(1-q)^2$ if $q \geq 0.5$.

improving consumer surplus is possible for all non-trivial markets (markets in $[0, 0.5]$ are trivial).

The reason for the failure of Theorem 3 is that Lemma 11 no longer holds when $T$ or $A$ are not finite. In particular, the set of IC and IR mechanisms can no longer be written as a polytope defined by finitely many variables and over finitely many constraints. As a result, the set of implementable payments functions no longer has finitely many extreme points. Vincent and Manelli (2007) provide an analysis of the set of feasible mechanisms with infinitely many types.

## 7.2   A Mussa-Rosen Setting with Non-linear Costs

Our analysis examined the interplay between screening and market segmentation. Since screening is not profitable if valuations and costs are linear (Riley and Zeckhauser, 1983), we focused on a model with non-linear valuations and zero costs. In this section we briefly discuss the alternative model with linear valuations and non-linear costs, as in Mussa and Rosen (1978). As we will argue, the technical difficulties that we had to overcome persist, and our conclusions continue to hold, in such a setting.

Consider the following setting with a finite number of types $T = \{1, \ldots, n\}$ and a set of alternatives $A \subseteq [0, 1]$. The valuation of type $i \in T$ for alternative $a$ is $v^i a$, where $v^i \in R$ is increasing in $i$. Thus the valuation of type $i$ is linear in $a$. The cost of producing alternative $a$ is $C(a)$.

One difference between the two models is the application. Our general setting (Section 3) can capture a bundling application with multi unit demands and heterogeneous products since

we do not require all alternatives to be ranked. In contrast, the setting above (as in Mussa and Rosen, 1978 and Maskin and Riley, 1984) assumes that alternatives are ranked and allows for multi unit demands but not selling heterogeneous products.

Another point is the tractability of the models. Can the main technical difficulties throughout this paper be avoided in a setting with linear valuations and non-linear costs? The answer is negative. First, recall the problem with concavification. Concavification requires an identification of consumer surplus *for all* markets, which remains a difficult task. With linear valuations, optimal mechanisms can be identified via ironing for a given market. Nevertheless, for different markets, ironing needs to be performed over different regions of the type space.[39] Thus, ironing does not easily lend itself to a closed-form characterization of consumer surplus for all markets. We discuss this issue in more detail below. Second, even if concavified consumer surplus is identified in closed form, it remains unclear how it can be compared to the two bounds unless there are two types, in which case a geometric comparison can be made.

To be more precise, consider a market $f \in \Delta(T)$ in the setting with linear valuations discussed above. Let $F^i = \sum_{j>i} f^j$ and $\phi^i = v^i - \frac{(v^{i+1}-v^i)F^i}{f^i}$. Using standard tools (e.g., Myerson, 1981), the problem of maximizing profit becomes to choose $x^1, \ldots, x^n$ to maximize

$$E_{i \sim f}[x^i \phi^i - C(x^i)],$$

subject to monotonicity of $x$, that is, $x^i$ is non-decreasing in $i$. If the cost function $c$ is convex and $\phi^i$ is non-decreasing in $i$, then monotonicity of $x$ is satisfied at the solution to the unconstrained problem and $x^i$ solves $\phi^i = C'(x^i)$. If $\phi$ is not non-decreasing, then optimal mechanisms can be obtained via ironing. Nevertheless, ironing needs to be applied to different regions of the type space for different markets, complicating the task of obtaining a closed form expression for consumer surplus.

Despite these technical issues, the main insights of this paper can be obtained with linear valuations and non-linear costs. This can be seen with the example below that obtains two conclusions analogous to those of the example in Section 2. Namely, first best consumer surplus is achievable for a non-trivial market if and only if non-screening is optimal for all markets, and improving consumer surplus is possible for all non-trivial markets except for possibly one such market. Even though we expect these conclusions to extend beyond this example, we do not pursue such extensions formally in this paper.

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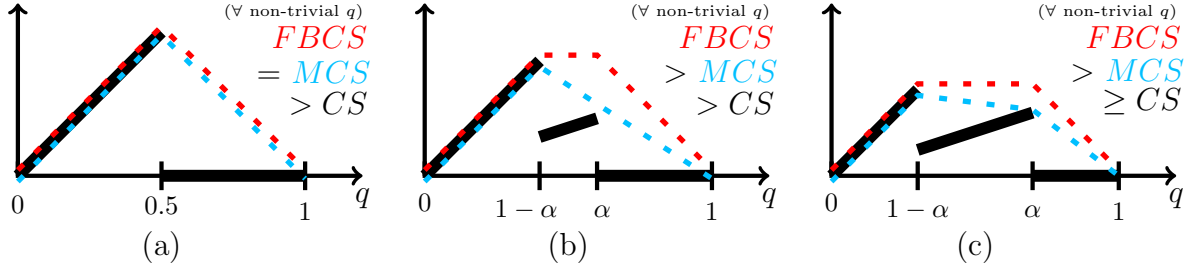[39]In other words, different incentive constraints bind in different markets.

Figure 16: The relationship between $CS$, $MCS$, and $FBCS$. (a) $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. (b) $\alpha \in (\frac{1}{2}, \frac{\sqrt{5}-1}{2})$. (c) $\alpha \in [\frac{\sqrt{5}-1}{2}, \frac{2}{3})$.

The example is parameterized by some $\alpha$, where $1/3 < \alpha < 2/3$. There are two types, 1 and 2, and three alternatives $A = \{0, 1-\alpha, 1\}$. The valuations are linear, $v_a^1 = 2a$ and $v_a^2 = 3a$. The costs are $C(0) = 0, C(1-\alpha) = 2 - 3\alpha$, and $C(1) = 1$.[40] A market $q$ consists of a fraction $1 - q$ of type 1, and a fraction $q$ of type 2. The relationship between consumer surplus $CS$, maximum consumer surplus $MCS$, and first best consumer surplus $FBCS$ is illustrated in Figure 16.[41] Notice the analogy to Figure 4.

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[40]The assumption that $1/3 < \alpha < 2/3$ is to ensure that costs are positive and $c(1-\alpha) < c(1)$. If costs may be negative and non-monotone, then this assumption is not necessary.

[41]Mechanism $N^1$ has profit 1. Mechanism $N^2$ has profit $2q$. Mechanism $S$, which offers alternative $1 - \alpha$ at price $2(1-\alpha)$ and alternative 1 at price $2 + \alpha$, and has profit $q + \alpha$. If $\alpha \le 0.5$, then mechanism $N^1$ is optimal for markets $[0, 0.5]$ and mechanism $N^2$ is optimal for markets $[0.5, 1]$. If $\alpha \ge 0.5$, then mechanism $N^1$ is optimal for markets $[0, 1-\alpha]$, mechanism $S$ is optimal for markets $[1-\alpha, \alpha]$, and mechanism $N^2$ is optimal for markets $[\alpha, 1]$. Consumer surplus is $CS(q) = q$ if mechanism $N^1$ is optimal, $CS(q) = (1-\alpha)q$ if mechanism $S$ is optimal, and $CS(q) = 0$ if mechanism $N^2$ is optimal. First best consumer surplus $FBCS(q)$ is $1 + q$ minus the optimal profit in market $q$.

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# A    Proofs of Section 3

## A.1    Proof of Lemma 1

**Proof of Lemma 1.**   We argued in the text that $CS(f) \leq MCS(f)$. To see that $MCS(f) \leq E_{i \sim f}[v_{\bar{a}}^i] - ER(f)$, consider any market $f'$ and its optimal mechanism $M' = (x', p')$ such that $CS(f') = EU(f', M')$. We have

$$CS(f') + ER(f') = E_{i \sim f'}[v^i \cdot x'(i) - p'(i)] + E_{i \sim f'}[p'(i)] = E_{i \sim f'}[v^i \cdot x'(i)] \leq E_{i \sim f'}[v_{\bar{a}}^i]. \quad (4)$$

Now consider a market $f$ and a segmentation $\mu$ of $f$. From (4) we have

$$CS(\mu) + E_{f' \sim \mu}[ER(f')] = E_{f' \sim \mu}[CS(f') + ER(f')] \leq E_{f' \sim \mu}[E_{i \sim f'}[v_{\bar{a}}^i]] = E_{i \sim f}[v_{\bar{a}}^i], \quad (5)$$

where the equality followed since $E_{f' \sim \mu}[f'] = f$. Now consider any optimal mechanism $(x, p)$ of $f$. We have

$$E_{f' \sim \mu}[ER(f')] \geq E_{f' \sim \mu}[\sum_i f'_i p(i)] = \sum f_i p(i) = ER(f). \quad (6)$$

Combining (5) and (6), we conclude that

$$CS(\mu) \leq E_{i \sim f}[v_{\bar{a}}^i] - ER(f).$$

∎

## A.2    Proof of Lemma 2

**Proof of Lemma 2.**   Consider any market $f'$ and its optimal mechanism $M' = (x', p')$ such that $CS(f') = EU(f', M')$. We have

$$CS(f') + ER(f') = E_{i \sim f'}[v^i \cdot x'(i) - p'(i)] + E_{i \sim f'}[p'(i)] = E_{i \sim f'}[v^i \cdot x'(i)] \leq E_{i \sim f'}[v_{\bar{a}}^i]. \quad (7)$$

Now consider a market $f$ and a segmentation $\mu$ of $f$. From (7) we have

$$CS(\mu) + E_{f' \sim \mu}[ER(f')] = E_{f' \sim \mu}[CS(f') + ER(f')] \leq E_{f' \sim \mu}[E_{i \sim f'}[v_{\bar{a}}^i]] = E_{i \sim f}[v_{\bar{a}}^i], \quad (8)$$

where the equality followed since $E_{f' \sim \mu}[f'] = f$. Now consider any optimal mechanism $(x, p)$ of $f$. We have

$$E_{f' \sim \mu}[ER(f')] \geq E_{f' \sim \mu}[\sum_i f'_i p(i)] = \sum f_i p(i) = ER(f). \quad (9)$$

Combining (8) and (9), we conclude that

$$CS(\mu) \le E_{i \sim f}[v_{\bar{a}}^i] - ER(f),$$

with equality if and only if both (8) and (9) hold with equality. For (8) to hold with equality, it is necessary and sufficient that $x_{\bar{a}}(i) = 1$ for all $i$ in the support of every segment of $\mu$. That is, every segment of $\mu$ is trivial (Definition 2). For (9) to hold with equality, it is necessary and sufficient that the mechanism $(x, p)$ is optimal for every segment of $\mu$. ∎

# B  Proofs of Section 4

## B.1  Optimal Mechanisms For Two Types

The following lemma provides a characterization of optimal mechanisms for two types. A mechanism is optimal if and only if the following conditions hold. First, the allocation of type 1 is a distribution over alternatives that maximize $v_a^1 - q v_a^2$, and the allocation of type 2 is alternative $\bar{a}$. Second, the payments are calculated via the binding IR constraint of type 1 and the binding IC constraint of type 2.

**Lemma 14** *With two types, a mechanism $(x, p)$ is optimal if and only if*

1. *$x(1) \in \Delta(\arg\max_a v_a^1 - q v_a^2)$, $x_{\bar{a}}(2) = 1$.*

2. *$p(1) = v^1 \cdot x(1)$, $p(2) = p(1) + v^2 \cdot (x(2) - x(1))$.*

**Proof.** Consider any IC and IR mechanism $(x', p')$. The IR constraint for type 1 is

$$v^1 \cdot x'(1) - p'(1) \ge 0, \tag{10}$$

Similarly, the IC constraint for type 2 is

$$v^2 \cdot x'(2) - p'(2) \ge v^2 \cdot x'(1) - p'(1), \tag{11}$$

Substituting (10) and (11), the expected revenue is

$$
\begin{aligned}
(1-q)p'(1) + q p'(2) &= p'(1) + q(p'(2) - p'(1)) \\
&\le v^1 \cdot x'(1) + q v^2 \cdot (x'(2) - x'(1))] \\
&= x'(1) \cdot (v^1 - q v^2) + q x'(2) \cdot v^2
\end{aligned} \tag{12}
$$

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Thus, maximum revenue across all mechanisms is at most the maximum of (12) across all $x'(1)$ and $x'(2)$, which is obtained by setting $x(1) \in \Delta(\arg\max_a v_a^1 - qv_a^2)$ and $x_{\bar{a}}(2) = 1$. Furthermore, note that the expected revenue of mechanism $(x, p)$ equals $x(1) \cdot (v^1 - qv^2) + qx(2) \cdot v^2$ if the two constraints (10) and (11) hold with equality, which is obtained by setting $p(1) = v^1 \cdot x(1)$ and $p(2) = p(1) + v^2 \cdot (x(2) - x(1))$. Thus, if mechanism $(x, p)$ is IC and IR, it is optimal. Furthermore, if such a mechanism is IC and IR, then all optimal mechanisms must satisfy the conditions specified by the lemma.

To verify that the mechanism $(x, p)$ is IC and IR, note that (10) and (11) imply that the IR constraint for type 2 is satisfied. The only remaining constraint is the incentive constraint for type 1. The utility of type 1 for deviating to type 2 is

$$
\begin{aligned}
v^1 \cdot x(2) - p(2) &= v^1 \cdot x(2) - p(1) - v^2 \cdot (x(2) - x(1)) \\
&= v^1 \cdot x(2) - v^1 \cdot x(1) - v^2 \cdot (x(2) - x(1)) \\
&= (v^2 - v^1) \cdot x(1) - (v^2 - v^1) \cdot x(2) \\
&= \sum_a x_a(1)\big((v_a^2 - v_a^1) - (v^2 - v^1) \cdot x(2)\big)
\end{aligned}
\tag{13}
$$

where the last equality followed since $\sum_a x_a(1) = 1$. Now consider any alternative $a$ such that $x_a(1) > 0$. By definition, it must be that $a \in \arg\max_a v_a^1 - qv_a^2$. Thus, in particular, $v_a^1 - qv_a^2 \geq v_{\bar{a}}^1 - qv_{\bar{a}}^2$. Since $q \leq 1$, we have $v_{\bar{a}}^2 - v_a^2 \geq v_{\bar{a}}^1 - v_a^1$. Therefore, for any $a$ where $x_a(1) > 0$,

$$
(v_a^2 - v_a^1) - (v^2 - v^1) \cdot x(2) \leq 0.
$$

Now note that for any $a$, either $x_a(1) = 0$, or $x_a(1) > 0$ in which case the above inequality holds. In either case, we have

$$
x_a(1)\big((v_a^2 - v_a^1) - (v^2 - v^1) \cdot x(2)\big) \leq 0.
$$

Summing over all $a$, and given (13), we have

$$
v^1 \cdot x(2) - p(2) = \sum_a x_a(1)\big((v_a^2 - v_a^1) - (v^2 - v^1) \cdot x(2)\big) \leq 0.
$$

Thus, the incentive constraint for type 1 is satisfied. ∎

The following lemma reveals the structural properties of $\arg\max_a v_a^1 - qv_a^2$, which identifies the allocation of type 1 by Lemma 14. The problem is one of maximizing over functions $v_a^1 - qv_a^2$
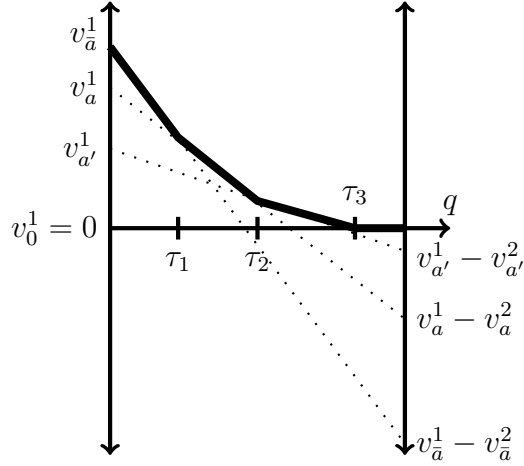
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Figure 17: An example with $k = 4$, $m = 3$, $\tau_1 = a$, and $\tau_2 = a'$.

that are linear in $q$. Thus, the set of markets $[0, 1]$ can be divided into intervals, each assigned a unique alternative, such that the alternative is the unique maximizer of $v_a^1 - qv_a^2$ in the interior of the interval, and is an optimizer of $v_a^1 - qv_a^2$ at the endpoints of the interval. Additionally, $\max_a v_a^1 - qv_a^2$ is convex in $q$. See Figure 17.

**Lemma 15** *For some $m \leq k$, there exist thresholds $\tau_0 < \ldots < \tau_{m+1}$, $\tau_0 = 0$, $\tau_{m+1} = 1$ and an injective function $g : \{0, \ldots, m\} \to A$, $g(0) = \bar{a}$, $g(m) = 0$, such that for all $j \leq m$, $g(j) \in \arg\max_a v_a^1 - qv_a^2$ if and only if $q = [\tau_j, \tau_{j+1}]$. Additionally, $v_{g(j)}^2 - v_{g(j)}^1$ is strictly decreasing in $j$.*

**Proof.** Consider the problem of maximizing $v_a^1 - qv_a^2$. For each $q$, the problem is to maximize over $k$ linear functions. The graph is drawn in Figure 17. Note the properties of the solution. First, the set of markets $[0, 1]$ can be divided into intervals with thresholds $\tau_0 < \ldots < \tau_{m+1}$, with each $\tau_j$ assigned a unique alternative $g(j)$, such that for all $j \leq m$, $g(j) \in \arg\max_a v_a^1 - qv_a^2$ if and only if $q = [\tau_j, \tau_{j+1}]$. Since $v_{\bar{a}}^1 > v_a^1$ for all $a \neq \bar{a}$, $g(0) = \bar{a}$. Similarly, since $v_a^1 - v_a^2 < 0$ for all $a \neq 0$, $g(m) = 0$. Additionally, consider $j$ such that $g_{j-1} = a$ and $g_j = a'$. We must have $v_a^1 - qv_a^2 > v_{a'}^1 - qv_{a'}^2$ for all $q < \tau_j$ and $v_a^1 - qv_a^2 < v_{a'}^1 - qv_{a'}^2$ for all $q > \tau_j$. Thus, $v_a^2 - v_a^1 > v_{a'}^2 - v_{a'}^1$. ∎

## B.2 Proof of Lemma 4

**Proof of Lemma 4.** By Lemma 14, the mechanism $N^1$ is optimal if and only if $\bar{a} \in \arg\max_a v_a^1 - qv_a^2$. By Lemma 15, the set of such markets is $[0, q_1]$ for $q_1 = \tau_1 > 0$. Furthermore,

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$N^1$ is the unique optimal mechanism for $[0, q_1)$.

Similarly, by Lemma 14, the mechanism $N^2$ is optimal if and only if $0 \in \arg\max_a v_a^1 - q v_a^2$. By Lemma 15, the set of such markets is $[q_2, 1]$ for $q_2 = \tau_m < 1$. Furthermore, $N^2$ is the unique optimal mechanism for $(q_2, 1]$.

Finally, consider a mechanism $M$ that is optimal for $(q_1, q_2)$. Since neither $N^1$ nor $N^2$ is optimal for any market $(q_1, q_2)$, $M \neq N^1, N^2$, and thus $M$ cannot be optimal for $[0, q_1)$ or $(q_2, 1]$. ∎

## B.3 Proof of Lemma 5

**Proof of Lemma 5.** By Lemma 14 and Lemma 15, there exists thresholds $0 < \tau_1 < \ldots < \tau_{m+1}$, $\tau_0 = 0$, $\tau_{m+1} = 1$, and a function $g$ such that a mechanism $(x, p)$ is optimal for a market $q$ if and only if the IR constrain for type 1 and the IC constraint for type 2 bind, $x_{\bar{a}}(2) = 1$, and $x_a(1) > 0$ only if $q \in [\tau_j, \tau_{j+1}]$ and $a = g(j)$.

Define $\alpha_j = v_{g(j)}^2 - v_{g(j)}^1$. The consumer surplus in a market $q \in (\tau_j, \tau_{j+1})$ is $CS(q) = q\alpha_j = q(v_{g(j)}^2 - v_{g(j)}^1)$. Consider a threshold market $q = \tau_j$, and two alternatives $a = g(j-1)$ and $a = g(j)$. Recall from Lemma 15 that $v_{g(j-1)}^2 - v^1(g(j-1)) > v_{g(j)}^2 - v^1(g(j))$. Thus, $x_{g(j-1)} = 1$ maximizes consumer surplus across all optimal mechanisms for a threshold market $\tau_j$. Therefore, $CS(q) = q\alpha_j$ for all $q \in (\tau_j, \tau_{j+1}]$ and $\alpha_j$ is decreasing in $j$. ∎

## B.4 Impossibility Of Improving Consumer Surplus for $K - 1$ Non-trivial Markets

We provide an example with $K$ alternatives such that improving consumer surplus is impossible for $K - 1$ non-trivial markets.

There are two types 1 and 2, and alternatives are 0 to $K$. For $a \geq 1$, the valuations are $v_a^1 = 2v(a) - v(a)^2$, and $v_a^2 = 2v(a)$, where $v(a) = \frac{1}{2} + \frac{a}{2K}$. By Lemma 14, in an optimal mechanism type 1 gets an alternative that maximizes $2v(a) - v(a)^2 - 2qv(a)$. For alternatives $0 < a < a'$, we have

$$2v(a) - v(a)^2 - 2qv(a) \geq 2v(a') - v(a')^2 - 2qv(a')$$

if and only if

$$q \geq 1 - \frac{v(a) + v(a')}{2}.$$

As a result, alternative $a > 1$ maximizes $2v(a) - v(a)^2 - 2qv(a)$ if and only if $q \in [\frac{1}{2} - \frac{2a+1}{4K}, \frac{1}{2} - \frac{2a-1}{4K}]$, and alternative $a = 1$ maximizes $2v(a) - v(a)^2 - 2qv(a)$ if and only if $q \in [\frac{1}{2} - \frac{3}{4K}, \frac{3}{4} - \frac{1}{4K}]$. The consumer surplus at a threshold market $q = \frac{1}{2} - \frac{2a+1}{4K}$ is $CS(q) = qv(a)^2 = q(1 - q - \frac{1}{2K})^2$. Thus, consumer surplus at all threshold markets is convex.

# C Proofs of Section 5

## C.1 Proof of Lemma 7

**Proof of Lemma 7.** Consider two markets $f', f''$ for which a mechanism $(x', p')$ is optimal. Let $f = \alpha f' + (1 - \alpha) f''$. For any mechanism $(x, p)$, we have

$$E_{i \sim f}[p(i)] = \alpha E_{i \sim f'}[p(i)] + (1 - \alpha) E_{i \sim f''}[p(i)] \leq \alpha E_{i \sim f'}[p'(i)] + (1 - \alpha) E_{i \sim f''}[p'(i)] = E_{i \sim f}[p'(i)].$$

That is, for market $f$, the revenue of any mechanism $(x, p)$ is at most the revenue of $(x', p')$. Thus, $(x', p')$ is optimal for $f$. ∎

## C.2 Proof of Theorem 1

We start with the following lemma from Bergemann et al. (2015).

**Lemma 16** *For any market $f$ and $i \in \arg\max_j v_{\bar{a}}^j(\sum_{i' \geq j} f_{i'})$, there exists a segmentation $\mu$ of $f$ such that every segment $f'$ of $\mu$ includes $i$ in its support and for all $i'$ in the support of $f'$, $i' \in \arg\max_j v_{\bar{a}}^j(\sum_{i' \geq j} f_{i'})$.*

We now complete the proof of Theorem 1.

**Proof of Theorem 1.** To complete the proof, we show that (2) implies (1). Consider a market $f$. By assumption, the mechanism $N^i$ is optimal for some $i$. In particular, $N^i$ is optimal for any $i \in \arg\max_j v_{\bar{a}}^j(\sum_{i' \geq j} f_{i'})$. By Lemma 16, there exists a segmentation $\mu$ such that every segment $f'$ of $\mu$ includes $i$ in its support and for all $i'$ in the support of $f'$, $i' \in \arg\max_j v_{\bar{a}}^j(\sum_{i' \geq j} f_{i'})$. Since $N^j$ is optimal for $f'$ for some $j$, $N^{i'}$ is optimal for all $i'$ in the support of $f'$. That is, $f'$ is an equal-revenue market. Thus, by Corollary 1, $\mu$ achieves first best consumer surplus. ∎
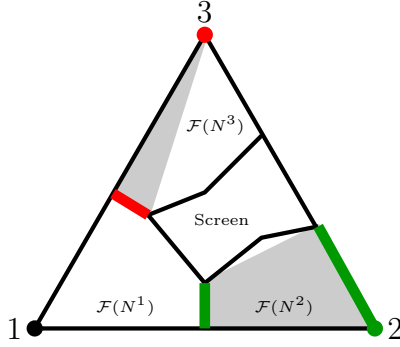
Figure 18: The set of non-trivial markets for which first best consumer surplus is achievable is shaded.

## C.3  Corollary 3: An Example

Corollary 3 identifies conditions under which first best consumer surplus is achievable for some, but not all, non-trivial markets. In this section we identify with an example the set of markets for which first best consumer surplus is achievable.

Consider an example with 3 types and its set of optimal mechanisms as shown in Figure 18. The set of non-trivial markets for which first best consumer surplus is achievable is shaded in Figure 18. Markets in $\mathcal{F}(N^1)$ are trivial. First best consumer surplus is unachievable for any market for which screening is optimal by Proposition 3. Thus, if first best consumer surplus is achievable for a non-trivial market, then the market must be in $\mathcal{F}(N^2) \cup \mathcal{F}(N^3)$. Suppose that a segmentation $\mu$ of a market $f$ in $\mathcal{F}(N^2)$ achieves first best consumer surplus. By Lemma 2, two conditions must hold for every segment $f'$ of $\mu$. First. $f'$ must be in $\mathcal{F}(N^2)$. Second, $N^{\underline{i}(f')}$ must be optimal for $f'$. The set of such markets $f'$ is shown in green in Figure 18. Market $f$ must be in the convex hull of such market, which is shaded in Figure 18. Similarly, if first best consumer surplus is achievable for a market $f \in \mathcal{F}(N^3)$, then $f$ must be in the convex hull of the set of markets that are shown in red in Figure 18.

# D  Proofs of Section 6

## D.1  Proof of Lemma 11

**Proof of Lemma 11.**  As argued in the text, the set of IC and IR$_0$ mechanisms is convex. The set is also bounded, since $0 \leq x_a \leq 1$, $p(i) \leq v_{\bar{a}}^i$, and $p(i) \geq 0$ (if $p(i) < 0$, then the incentive constraint of type 1 implies that $v^1 \cdot x(1) - p(1) > 0$, violating IR$_0$). Thus there exist a finite

set $\mathcal{E}_M$ such that the set of IC and $\mathrm{IR}_0$ mechanisms is equal to the convex hull of $\mathcal{E}_M$. Let $\mathcal{E}_P = \{p \mid \exists x, (x, p) \in \mathcal{E}_M\}$.

If $p \in P$, then there exists $x$ such that $(x, p)$ is IC and $\mathrm{IR}_0$, implying that $(x, p)$ is in the convex hull of $\mathcal{E}_M$, which in turn implies that $p$ is in the convex hull of $\mathcal{E}_P$. Conversely, if $p$ is in the convex hull of $\mathcal{E}_P$, then there exists a distribution $\nu$ over IC and $\mathrm{IR}_0$ mechanisms $(x', p')$ such that $p = E_\nu[p']$. Define $x = E_\nu[x']$. The mechanism $(x, p)$ is IC and $\mathrm{IR}_0$, and thus $p \in P$.

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