Knitting and Ironing: Redistribution via Auctions*

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Abstract

This paper considers the design of an auction-induced wealth transfer mechanism to maximize social surplus. Two items, one good, the other bad, are to be assigned to bidders who value money differently, and the taker of the bad is compensated with proceeds from the good. Transfers that improve social welfare occur indirectly when bidders who value money less buy the good, and those who value money more are paid to take the bad. We obtain the solution for the optimal mechanism given general type-distributions. We introduce a new concept, two-part operator, to integrate a bidder's endogenously bifurcated information rent in buying the good versus taking the bad. To solve the optimal mechanism problem, the objective of which is nonlinear, we bisect it into two linear programmings, solve each via ironing, and knit the two into the solution for the original problem.

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1 Introduction

The mechanism design literature has been silent about how to improve social welfare through wealth transfers among individuals. That is because a main assumption that affords tractability in the literature is quasilinearity of individuals' payoff functions, which renders money transfers irrelevant to the sum of individual payoffs, or *social surplus*. To remove this restriction, we consider a situation where different players value money differently, say due to social, economic restrictions. Imagine that a social planner auctions off two items, one desirable (say a high-tech giant's headquarter location), the other undesirable (say the location of a nuclear plant), and uses the revenues collected from the former to compensate the "winner" of the latter.[1] Such a wealth transfer enlarges the social surplus when money is valued less for the winner of the good than it is for the taker of the bad. Both sides of the transfer being willing participants of the auctions, such welfare-improving redistributions, different from income taxes, do not rely on mandates and are robust to asymmetric information. The question is how to abstract such situations—where non-quasilinearity is the key to capture the societal gain from wealth transfers—into a model tractable for mechanism design.

This paper thus presents a new, tractable model to solve for the social-surplus maximizing mechanism in a particular non-quasilinear environment. Two items, one good, the other bad, are to be allocated among n asymmetrically distributed bidders. The good has a positive value, and the bad a negative value, both commonly known to all bidders. A bidder knows privately his per-dollar value of money so that high types value the same amount of money less than low types, with high types interpreted as being rich, and low being poor.

From an individual bidder's viewpoint, our model appears the same as an independent private value setup, so we can use auction design techniques to characterize the design constraints: incentive compatibility, individual rationality and budget balance. From a social planner's viewpoint, however, the design objective—social surplus—is not quasilinear, and she needs to balance the budget in wealth redistributions; thus, the optimization problem involves new challenges, each requiring a new technique to overcome.

First, in contrast to the literature on the possibility of efficient trades (Myerson and Satterthwaite [7]; Cramton, Gibbons and Klemperer [1]), the counterpart of the efficient allocation here, known a priori as giving the good to the highest type and the bad to the

[1] According to Herodotus [2], such auctions took place in ancient Babylon for marriage markets.

lowest one, is in general suboptimal in our model; thus the optimal allocation is unknown a priori and the social planner has to find it subject to the above-listed constraints. That poses the first challenge to the design problem: how to reduce the design objective to a tractable form. The challenge comes from the fact that a bidder's information rent switches from one form to another, depending on his realized type and whether the realized type is supposed to be assigned the good or to be assigned the bad. Such endogenous bifurcation of one's information rents would cause no complication had the design objective been linear, as in Myerson and Satterthwaite and Cramton, Gibbons and Klemperer. The objective in our model, by contrast, is the sum of the non-quasilinear payoffs across bidders, and hence a nonlinear outcome of the bifurcation.

To derive a tractable form of the objective, we introduce a new concept, two-part operators on the space of allocations. Albeit nonlinear, such an operator is concave and continuous. It keeps track of the different kinds of information rents that need to be deducted from a bidder's surplus depending on whether the bidder's type would give him the good item or the bad one. We then reduce the design objective to the outcome of a two-part operator through analyzing the social planner's choice of payment rules to implement an allocation—such payment rule decisions, unlike those in the optimal auction theory, are nontrivial because the payoff-equivalence theorem is upset by non-quasilinearity.

Next we face the second challenge: multiple kinds of binding constraints. In contrast to the optimal auction theory (Myerson [6]), where incentive compatibility is the only nontrivial binding constraint and hence can be handled by the ironing technique alone, our problem is subject to not only incentive compatibility but also budget balance constraints, both possibly binding. When the latter is also binding, it is impossible for the Lagrangian associated with the design problem to be a linear functional on the choice set, hence ironing alone does not work. This challenge stems from a crucial difference between optimal auction theory and this paper. While optimal auction theory is about extracting surplus from players through a one-way allocation (selling a good) and hence budget balancing is a nonissue, this paper is about transferring wealth among players through two-way allocations (selling a good and buying acceptance of a bad) and hence budget balancing is necessary in wealth transfers.

We overcome this challenge through first characterizing any optimal mechanism by the saddle point condition, which gives rise to a Lagrange problem whose objective is a nonlinear outcome of a two-part operator. We then bisect the Lagrange problem into two linear problems, each solved by ironing—incentive compatibility binding for each—and finally knit the two into a solution for the original problem. This bisection technique works because of a well-ordered property possessed by the family of two-part operators that we introduce.

We obtain the solution for the optimal mechanism given a general class of type-distributions (Theorems 1). The optimal mechanism is the concatenation of two auctions, one to allocate the good by a ranking criterion resulting from one part of the aforementioned two-part operator, and the other to allocate the bad by another ranking criterion resulting from the other part of the same operator. In contrast to the virtual utilities in the optimal auction literature, which are usually negative for low types and, given regular type-distributions, increasing in types, our counterpart is non-monotone, peaking up to positive levels at both ends of the type support, and negative at some types in between. Thus the ironing operation, often avoided in the literature, is mostly unavoidable here. In other words, while the good may sometimes be allocated to the richest bidder, the bad is often allocated to middle and poorer ones through an egalitarian lottery.

A policy implication of our solution is that any optimal mechanism given some type-distributions allocates the bad with a strictly positive probability (Theorem 2). Thus, given the extent to which existence of an item to be assigned need not be exogenous to the social planner (e.g., a soldier's post in an avoidable war), sometimes it is socially desirable to assign artificially made-up social hierarchies to society members: the threat of being assigned a low status makes the rich types willing to transfer more money to the poor.

Our modeling choice of having the set of items include an item of negative value is substantively different from merely normalizing the payoff of not winning the good to a negative level. First, the real world abounds with issues regarding assignment of undesirable items such as the location of Nimbys (fracking, housing development, homeless shelters, landfill sites, chemical plants, hydro poles, oil pipeline terminals, etc.). The cost of a Nimby, borne mainly by the neighborhood and not by others, cannot be captured by normalization. Second, from the viewpoint of wealth redistributions, the "badness" of not winning a good is shared by all except the winner, rendering the social planner's instrument too blunt to target various bidder-types differently. Third, and most importantly, introduction of a bad item leads to a new mathematical structure. Incentivizing a player to receive the bad item requires a different kind of information rents than incentivizing him to buy the good requires. Which kind of rents to apply is unknown unless his type is revealed, and the allocation chosen.

To keep track of the two kinds of rents systematically, a new construct, the aforementioned two-part operator, comes to be.

The kind of nonlinearity handled in this paper is different from those in the non-quasilinear optimal auction literature initiated by Maskin and Riley [4]. The main challenge for that literature is the analytical complexity of a bidder's incentive given non-quasilinearity (e.g., risk aversion) in his payoff function, while the designer, merely to extract profits from selling a good, does not have to deal with the budget balance constraint. The focus of this paper, by contrast, is to maximize the sum of players' surpluses through inducing wealth transfers among them with two items of opposite values. Such a duality gives rise to the endogenous bifurcation of information rents, and the wealth transfers need to satisfy the budget balance constraint, each absent in that literature. Nonlinearity of the objective in our design problem is just an outcome of such endogenous bifurcation and binding constraints combined with our minimum departure from the quasilinear model just to allow wealth transfers among players to have an effect on the sum of their payoffs.

The following Section 2 defines the model and the design problem. Section 3 introduces the concept of two-part operators and the general notions of information rents and virtual utilities, as well as summarizing the ironing procedure in the optimal auction theory. Section 4 solves the problem. Then Section 5 presents the "goodness of bad" implication and constructs an example where the budget balance constraint is binding. Section 5.4 extends our results to the case where the values of the two items are heterogeneous among bidders. Section 5.5 spells out three operations that constitute the precise description of our departure from the literature. Section 6 concludes.

2 The Preliminaries

2.1 The Good, the Bad, and n Bidders

Two items, named A and B, each indivisible, are to be allocated among n bidders ($n \geq 3$), each of whom can get one or both or none of the items. The value of item A is commonly known to be equal to one, and that of item B commonly known equal to $-c$, such that $c \geq 0$. Each bidder's type is independently drawn from a commonly known cumulative distribution function (CDF) F_i such that its support is $T_i := \left[\underline{t}_i, \bar{t}_i\right]$, its density f_i is continuous and positive on the support, and $\underline{t}_i > 0$. After the allocation mechanism is announced, each bidder, privately informed of his own type, can opt out of the mechanism thereby getting zero payoff. If $\mathbf{1}_i^j$ denotes the indicator function for bidder i to get item j ($j \in \{A, B\}$), and if p_i is the net money transfer from bidder i to others (if $p_i < 0$ then the transfer is from others to i), then the ex post payoff for bidder i, given realized type t_i, is equal to $\mathbf{1}_i^A - c\mathbf{1}_i^B - p_i/t_i$. Each bidder is assumed risk neutral in his payoff.

2.2 Allocations and Mechanisms

For each bidder i, let $T_{-i} := \prod_{j \neq i} T_j$ and F_{-i} be the product measure on T_{-i} generated by $(F_j)_{j \neq i}$. Let $\mathscr{Q}$ denote the set of all $(Q_i)_{i=1}^n$, each being a profile of functions $Q_i : T_i \to \mathbb{R}$ such that, for some ex post allocation $(q_{iA}, q_{iB})_{i=1}^n$ with $q_{iA}, q_{iB} : \prod_{j=1}^n T_j \to [0, 1]$ satisfying $\sum_i q_{iA}(\cdot) \leq 1$ and $\sum_i q_{iB}(\cdot) \leq 1$, we have for each $i = 1, \ldots, n$ and each $t_i \in T_i$,

$$Q_i(t_i) = \int_{T_{-i}} q_{iA}(t_i, t_{-i}) dF_{-i}(t_{-i}) - c \int_{T_{-i}} q_{iB}(t_i, t_{-i}) dF_{-i}(t_{-i}). \tag{1}$$

Let $\mathscr{Q}_+$ be the set of all $(Q_i)_{i=1}^n \in \mathscr{Q}$ such that $Q_i \geq 0$ for all i. Due to the coefficient c in (1), the set of all $(Q_i)_{i=1}^n \in \mathscr{Q}$ such that $Q_i \leq 0$ for all i is $-c\mathscr{Q}_+$.[2] Both $\mathscr{Q}_+$ and $-c\mathscr{Q}_+$ are obviously convex. Let $\mathscr{Q}_{\text{mon}}$ be the set of all $(Q_i)_{i=1}^n \in \mathscr{Q}$ such that Q_i is weakly increasing for every i. One can prove that $\mathscr{Q}_{\text{mon}}$ is convex (Appendix A.1). It is easy to verify that $\mathscr{Q}$ belongs to a normed vector space. Endow $\mathscr{Q}$ with this norm topology.[3]

By the revelation principle and risk neutrality of bidders, it is clear that any mechanism-equilibrium pair corresponds to some $(Q_i, P_i)_{i=1}^n$ such that $(Q_i)_{i=1}^n \in \mathscr{Q}$ and, for each i, $P_i : T_i \to \mathbb{R}$ and $P_i(\cdot) = \int_{T_{-i}} p_i(\cdot, t_{-i}) dF_{-i}(t_{-i})$ for some ex post payment rule $(p_i)_{i=1}^n$, a profile of functions $p_i : \prod_j T_j \to \mathbb{R}$ ($\forall i$). Call $(Q_i)_{i=1}^n$ (reduced form) allocation, and $(P_i)_{i=1}^n$ (reduced form) payment rule.

[2] For any scaler c and any subset S of a vector space, $-cS := \{-cs : s \in S\}$.

[3] For example, for each bidder i let $L^2(T_i)$ be the L^2-space of measurable real functions defined on T_i, endowed with the measure F_i. Clearly $\mathscr{Q} \in \prod_i L^2(T_i)$. Define the norm for $\prod_i L^2(T_i)$ by $\|Q\| := \sum_i \|Q_i\|_2$ for any $Q := (Q_i)_{i=1}^n \in \prod_i L^2(T_i)$. Should one want to prove existence of optimal mechanisms based on Weierstrass's extreme value theorem, the norm topology is too strong to guarantee compactness of any closed and bounded choice set. However, we do not need Weierstrass's theorem to establish existence.

2.3 Constraints

Any type-t_i bidder i's decision in response to an equilibrium is:

$$U_i(t_i) := \max_{\hat{t}_i \in T_i} Q_i(\hat{t}_i) - P_i(\hat{t}_i)/t_i,$$

which, since $\underline{t}_i > 0$ by assumption, is equivalent to

$$\tilde{U}_i(t_i) := \max_{\hat{t}_i \in T_i} t_i Q_i(\hat{t}_i) - P_i(\hat{t}_i).$$

Thus, by auction design routines, *incentive compatibility* (IC) for bidder i is equivalent to simultaneous satisfaction of two conditions: (i) $Q \in \mathscr{Q}_{\text{mon}}$; (ii) for any t_i, t_i^0 of T_i,

$$P_i(t_i) - P_i(t_i^0) = \int_{t_i^0}^{t_i} s dQ_i(s). \tag{2}$$

Since each bidder can opt out of a mechanism before it operates, *individual rationality* (IR) means $U_i(t_i) \geq 0$ for all i and all $t_i \in T_i$. By the above definitions of U_i and $\tilde{U}_i$, $U_i(t_i) = \tilde{U}_i(t_i)/t_i$ for any $t_i \in T_i$, and $\tilde{U}_i$ is convex, with derivative almost everywhere equal to the weakly increasing Q_i. Thus, $\tilde{U}_i$ attains its minimum at

$$\tau(Q_i) := \inf \left\{ t_i \in T_i : Q_i(t_i) \geq 0 \text{ or } t_i = \bar{t}_i \right\}. \tag{3}$$

Consequently, $\tilde{U}_i(\tau(Q_i)) \geq 0$ iff "$\tilde{U}_i(t_i) \geq 0$ for all $t_i \in T_i$" iff "$U_i(t_i) \geq 0$ for all $t_i \in T_i$." Thus, IR is equivalent to $\tilde{U}_i(\tau(Q_i)) \geq 0$ for all bidders i.

Budget balance (BB) means that the society consisting of the n bidders never requires any outside subsidy to carry out the allocation. More precisely, a reduced-form payment rule $(P_i)_{i=1}^n$ satisfies BB if and only if it is generated by some ex post payment rule $(p_i)_{i=1}^n$ such that, for each i, $\sum_i p_i(t) \geq 0$ for all $t \in \prod_i T_i$.

2.4 The Problem

Social surplus given any mechanism-equilibrium pair means the sum of all bidders' payoffs in the equilibrium calculated from the ex ante standpoint before bidders' types are realized. Given any mechanism $(Q_i, P_i)_{i=1}^n$ and (1), it is clear that the social surplus is equal to

$$\sum_i \int_{T_i} Q_i(t_i) dF_i(t_i) - \sum_i \int_{T_i} \frac{P_i(t_i)}{t_i} dF_i(t_i). \tag{4}$$

The problem is to maximize (4) among all mechanisms $(Q_i, P_i)_{i=1}^n$ subject to IC, IR and BB.

This problem involves three difficulties. First, the design objective (4) is nonlinear in the payment rule $(P_i)_{i=1}^n$, so the routine technique of rewriting an objective into a linear form of $(Q_i)_{i=1}^n$, through eliminating $(P_i)_{i=1}^n$ via the envelope equation (2), is not directly available. Second, IC and BB may be binding simultaneously at an optimum—which is confirmed in a later section—hence the ironing technique, which suffices to handle binding IC when BB is absent, does not suffice here. And in general we cannot isolate only one binding constraint to handle. Third, since a bidder may be assigned the bad item, an allocation Q_i may be negative for some types (cf. (1)). Since a bidder i behaves like a buyer when $Q_i(t_i) > 0$, and like a seller when $Q_i(t_i) < 0$, his information rent depends on which case his type belongs to. That, as will be clear later, compounds the nonlinearity issue.

3 The Operator

To convert the design problem into a tractable form, we introduce a new concept of two-part operators on the space of allocations. It is motivated by the fact that a bidder's incentive when he is supposed to be assigned the good is qualitatively different from his incentive when supposed to be assigned the bad. The operator is to systematically keep track of his information rents between the two cases when his type varies.

3.1 Densities for Information Rents

More general than CDFs, a *distribution* on T_i, say μ_i, means a weakly increasing real function that is nonnegative, right-continuous, and supported by T_i, though $\mu_i(\bar{t}_i)$ need not be equal to one. For any i, any distribution μ_i on T_i and any $t_i \in T_i$, define

$$\rho_+(\mu_i)(t_i) \quad := \quad -\int_{T_i} d\mu_i + \int_{\underline{t}_i}^{t_i} d\mu_i, \tag{5}$$

$$\rho_-(\mu_i)(t_i) \quad := \quad \int_{\underline{t}_i}^{t_i} d\mu_i. \tag{6}$$

It will be clear that $\rho_+(\mu_i)$ reflects i's information rent density when i is a buyer, and $\rho_-(\mu_i)$, i's information rent density when i is a seller, had i's type been measured by μ_i (Eq, (15)).

3.2 Two-Part Profiles

A *two-part function* for bidder i means an ordered pair (φ_+, φ_-) of F_i-integrable functions $\varphi_+, \varphi_- : T_i \to \mathbb{R}$. A *two-part profile* means a profile $\varphi := (\varphi_i)_{i=1}^n$ such that, for each i, $\varphi_i = (\varphi_{i,+}, \varphi_{i,-})$ is a two-part function for i. For example, $(\rho_+(\mu_i), \rho_-(\mu_i))_{i=1}^n$, defined by (5) and (6), is a two-part profile. A two-part profile $(\varphi_i)_{i=1}^n$ is said *well-ordered* if and only if $\varphi_{i,+} \le \varphi_{i,-}$ for each bidder i.

Obviously, two-part profiles constitute a vector space. Furthermore, the set of well-ordered two-part profiles constitutes a positive cone, as "$\varphi_{i,+}^k \le \varphi_{i,-}^k$ and $\gamma^k \ge 0$ for all i and all k" implies $\sum_k \gamma^k \varphi_{i,+}^k \le \sum_k \gamma^k \varphi_{i,-}^k$.

3.3 Two-Part Operators

For any two functions ϕ and ψ on T_i, denote $\phi\psi$ for their point-wise product: $(\phi\psi)(t_i) := \phi(t_i)\psi(t_i)$ for all t_i; denote $\frac{\phi}{\psi} := \phi/\psi := \phi(1/\psi)$; and denote their inner product by

$$\langle \phi \, , \psi \rangle := \int_{T_i} \phi(t_i)\psi(t_i)dt_i,$$

provided that the (Lebesgue) integral exists. By contrast, for any $Q \in \mathscr{Q}$ and any two-part profile $\varphi := (\varphi_{i,+}, \varphi_{i,-})_{i=1}^n$, denote the two-part operator

$$\langle Q : \varphi| := \sum_{i=1}^n \left(\langle Q_i^+ \, , \varphi_{i,+} f_i \rangle - \langle Q_i^- \, , \varphi_{i,-} f_i \rangle \right), \tag{7}$$

where f_i is the density specified in Section 2.1, $\varphi_{i,+} f_i$ and $\varphi_{i,-} f_i$ are point-wise products, $Q_i^+(t_i) := \max\{0, Q_i(t_i)\}$, $Q_i^-(t_i) := \max\{0, -Q_i(t_i)\}$, and hence $Q_i = Q_i^+ - Q_i^-$.

The colon between Q and φ, and the asymmetric bracket, are to distinguish $\langle Q : \varphi|$ from an inner product: Obviously, $\langle Q : \varphi|$ is not linear in Q unless $\varphi_{i,+} = \varphi_{i,-}$ for all i. By contrast, $\langle Q : \varphi|$ is linear in φ: $\langle Q : \alpha\varphi + \beta\phi| = \alpha\langle Q : \varphi| + \beta\langle Q : \phi|$ for any two-part profiles φ and ϕ, and any scalers α and β. The next lemma is proved in Appendix A.2.

Lemma 1 *If a two-part profile φ is well-ordered, then $Q \mapsto \langle Q : \varphi|$ is concave on $\mathscr{Q}$.*

3.4 Virtual Utilities

Denote $\mathbb{I}$ for the identity mapping on $\mathbb{R}$, i.e., $\mathbb{I}(x) = x$ for all $x \in \mathbb{R}$. For any profile $\mu := (\mu_i)_{i=1}^n$ such that μ_i is a differentiable distribution on T_i, with density μ_i', for each i, define a two-part profile $V(\mu) := (V_{i,+}(\mu), V_{i,-}(\mu))_{i=1}^n$ by, for any i,

$$V_{i,+}(\mu) := \frac{\mu_i' \mathbb{I} + \rho_+(\mu_i)}{f_i}, \quad V_{i,-}(\mu) := \frac{\mu_i' \mathbb{I} + \rho_-(\mu_i)}{f_i}. \tag{8}$$

For example, if μ is the profile $F := (F_i)_{i=1}^n$ of distributions specified in Section 2.1, plug (5) and (6) into (8) to obtain

$$V_{i,+}(F)(t_i) = t_i - (1 - F_i(t_i))/f_i(t_i), \quad V_{i,-}(F)(t_i) = t_i + F_i(t_i)/f_i(t_i), \tag{9}$$

which are of course familiar. The generality of V, however, allows it to act on other distribution profiles. That is important because, as the proof of Lemma 3 will show, calculation of the design objective requires a linear combination of the actions by V not only on F but also on another distribution profile derived from F. The next lemma is proved in Appendix A.3.

Lemma 2 *if μ_i' is uniformly bounded for any i, then $Q \mapsto \langle Q : V(\mu)|$ is continuous on $\mathscr{Q}$.*

3.5 Allocation by Ranks

For any profile $(\varphi_i)_{i=1}^n$ of functions $\varphi_i : T_i \to \mathbb{R}$ $(\forall i)$, a $(Q_i)_{i=1}^n \in \mathscr{Q}$ is call *allocation by the rank of* $(\varphi_i)_{i=1}^n$ if and only if, for some $(q_i)_{i=1}^n$ such that $q_i : \prod_j T_j \to [0, 1]$ is defined by

$$q_i\left((t_j)_{j=1}^n\right) = \frac{|\{i\} \cap \arg\max_j \varphi_j(t_j)|}{|\arg\max_j \varphi_j(t_j)|} \mathbf{1}_{\varphi_i(t_i)>0} \tag{10}$$

for any i, any $t_i \in T_i$ and any $(t_j)_{j=1}^n \in \prod_j T_j$,[4]

$$Q_i(\cdot) = \begin{cases} \int_{T_{-i}} q_i(\cdot, t_{-i}) dF_{-i}(t_{-i}) & \text{if } (Q_i)_{i=1}^n \in \mathscr{Q}_+ \\ -c \int_{T_{-i}} q_i(\cdot, t_{-i}) dF_{-i}(t_{-i}) & \text{if } (Q_i)_{i=1}^n \in -c\mathscr{Q}_+ \end{cases} \tag{11}$$

for any i. Note that (10) is independent of the item being allocated, be it the good or the bad. The reference to the specific item is recovered via the bifurcation in (11).

For any integrable function ψ_i on T_i, define the *ironed copy* $\overline{\psi}_i$ of ψ_i by

$$\overline{\psi}_i(t_i) \overset{\text{a.e.}}{=} \frac{d}{ds} \widehat{H}_i^{\psi_i}(s) \bigg|_{s=F_i(t_i)}, \tag{12}$$

where $\widehat{H}_i^{\psi_i}$ is the convex hull of the $H_i^{\psi_i} : [0, 1] \to \mathbb{R}$ defined by

$$H_i^{\psi_i}(s) := \int_0^s \psi_i\left(F_i^{-1}(r)\right) dr \tag{13}$$

[4] Denote $|X|$ for the cardinality of set X, and $\mathbf{1}_S$ for the indicator function for the truth of statement S.

for any $s \in [0,1]$. Following Myerson [6, Section 6], one can verify that one solution for

$$\max_{Q \in X \cap \mathscr{Q}_{\text{mon}}} \sum_i \langle Q_i \,, \psi_i f_i \rangle \tag{14}$$

is an allocation by the rank of $\left(\overline{\psi}_i\right)_{i=1}^n$ if $X = \mathscr{Q}_+$, and an allocation by the rank of $\left(-\overline{\psi}_i\right)_{i=1}^n$ if $X = -c\mathscr{Q}_+$. Furthermore, this solution is unique in the sense that it is identical to any other solution for (14) almost everywhere in $\prod_i \left\{t_i \in T_i : \overline{\psi}_i(t_i) \neq 0\right\}$.[5]

4 The Solution

4.1 Characterizing Budget Balance

For any bidder i and any differentiable distribution μ_i on T_i, if (Q_i, P_i) is IC for i then

$$\langle P_i \,, \mu_i' \rangle = \langle Q_i^+ \,, \mu_i' \mathbb{I} + \rho_+(\mu_i) \rangle - \langle Q_i^- \,, \mu_i' \mathbb{I} + \rho_-(\mu_i) \rangle - \tilde{U}_i \left(\tau(Q_i)\right) \int_{T_i} d\mu_i. \tag{15}$$

Proved in Appendix A.4, (15) is just the "integration-by-parts" routine in optimal auction theory for μ_i that need not be the primitive prior F_i. Summing (15) across all i to get

$$\sum_i \langle P_i \,, \mu_i' \rangle = \langle Q : V(\mu) | - \sum_i \tilde{U}_i \left(\tau(Q)\right) \int_{T_i} d\mu_i. \tag{16}$$

Apply (16) to the case $\mu_i = F_i$ (so $F_i' = f_i$) to obtain $\sum_i \langle P_i \,, f_i \rangle = \langle Q : V(F)| - \sum_i \tilde{U}_i \left(\tau(Q_i)\right)$. Thus, BB and linearity of the inner product together imply that

$$\langle Q : V(F)| \geq \sum_i \tilde{U}_i \left(\tau(Q_i)\right), \tag{17}$$

which coupled with IR implies

$$\langle Q : V(F)| \geq 0. \tag{18}$$

Thus, $Q \in \mathscr{Q}_{\text{mon}}$ constitutes an IC, IR and BB mechanism only if Q satisfies (18). The converse is also true: Appendix A.5, similar to Cramton, Gibbons and Klemperer [1, Lemma

[5] Any $Q \in \mathscr{Q}_+ \cap \mathscr{Q}_{\text{mon}}$ that differs from the allocation Q_* by the rank of $\left(\overline{\psi}_i\right)_{i=1}^n$ on a positive-measure subset of $\prod_i \left\{t_i \in T_i : \overline{\psi}_i(t_i) \neq 0\right\}$ yields a smaller value than Q_* does in terms of Myerson's (6.11), as (6.11) is maximization of a linear functional defined by $\left(\overline{\psi}_i\right)_{i=1}^n$, and never performs better than Q_* does in terms of Myerson's (6.12) due to the monotonicity requirement of $\mathscr{Q}_{\text{mon}}$. The case of $X = -c\mathscr{Q}_+$ is similar.

4],[6] constructs an ex post payment rule $(p_i)_{i=1}^n$ such that for any $t \in \prod_j T_j$,

$$\sum_i p_i(t) = \langle Q : V(F)|, \tag{19}$$

which is nonnegative if (18) holds.

4.2 Calculating the Objective

For each bidder i and any $t_i \in T_i$,, define

$$G_i(t_i) := \int_{\underline{t}_i}^{t_i} \frac{1}{s} dF_i(s). \tag{20}$$

Thus $G := (G_i)_{i=1}$ is a profile of differentiable distributions, with $G_i' = f_i/\mathbb{I}$ for all i. Let

$$\begin{aligned} \alpha_i &:= \int_{T_i} \frac{1}{s} dF_i(s), && (21) \\ \alpha_* &:= \max_i \alpha_i. && (22) \end{aligned}$$

Lemma 3 *Maximization of social surplus among all IC, IR and BB direct revelation mechanisms is equivalent to*

$$\begin{aligned} \max_{Q \in \mathscr{Q}_{\text{mon}}} \quad & \langle Q : \alpha_* V(F) - V(G) + 1| && (23) \\ s.t. \quad & \langle Q : V(F)| \geq 0. \end{aligned}$$

Proof Previous sections have shown that the constraints $\langle Q : V(F)| \geq 0$ and $Q \in \mathscr{Q}_{\text{mon}}$ together constitute the choice set for the problem. To verify the objective in (23), calculate the social surplus (4). Note that its term

$$\int_{T_i} \frac{P_i(t_i)}{t_i} dF_i(t_i) = \int_{T_i} P_i(t_i) \cdot \frac{f_i(t_i)}{t_i} dt_i = \langle P_i \,, f_i/\mathbb{I} \rangle = \langle P_i \,, G_i' \rangle. \tag{24}$$

[6] Different from its counterpart in Cramton, Gibbons and Klemperer, our construction allows for asymmetric allocations while they assume symmetric allocations. In addition, our budget balance condition is weaker than theirs in allowing surplus for the auctioneer, though we show later that the social optimum is to leave zero surplus to the auctioneer (Remark 1).

Apply (16) to $\mu_i = G_i$ (hence $\mu_i' = f_i/\mathbb{I}$) to obtain

$$\begin{aligned}\sum_i \int_{T_i} \frac{P_i(t_i)}{t_i} dF_i(t_i) &= \langle Q : V(G)| - \sum_i \alpha_i \tilde{U}_i\left(\tau(Q_i)\right) \\ &= \langle Q : 1 + V(G) - 1| - \sum_i \alpha_i \tilde{U}_i\left(\tau(Q_i)\right) \\ &= \langle Q : 1| + \langle Q : V(G) - 1| - \sum_i \alpha_i \tilde{U}_i\left(\tau(Q_i)\right) \\ &= \sum_i \langle Q_i \, , f_i\rangle + \langle Q : V(G) - 1| - \sum_i \alpha_i \tilde{U}_i\left(\tau(Q_i)\right),\end{aligned}$$

with the third equality due to linearity of $\varphi \mapsto \langle Q : \varphi|$, and the last equality due to the definition of two-part operators. Plug this into (4) to calculate the social surplus

$$\begin{aligned}&\sum_i \langle Q_i \, , f_i\rangle - \left(\sum_i \langle Q_i \, , f_i\rangle + \langle Q : V(G) - 1| - \sum_i \alpha_i \tilde{U}_i\left(\tau(Q_i)\right)\right) \\ = &\sum_i \alpha_i \tilde{U}_i\left(\tau(Q_i)\right) - \langle Q : V(G) - 1|.\end{aligned}$$

By (17) and (22), $\sum_i \alpha_i \tilde{U}_i\left(\tau(Q_i)\right) \le \alpha_* \langle Q : V(F)|$. Furthermore, the right-hand side of this inequality can be attained: pick a bidder i_* for whom $\alpha_{i_*} = \alpha_*$; for any realized type profile $t \in \prod_i T_i$ and any $i \neq i_*$, set the money transfer $p_i^*(t)$ from i to others to be $p_i(t)$, with p_i being the ex post payment rule in (19) (defined by (49), Appendix A.5); set the money transfer $p_{i_*}^*(t)$ from i_* to others as $p_{i_*}(t) - \langle Q : V(F)|$. Given $(p_i^*)_{i=1}^n$, BB follows from (19), and $\tilde{U}_i\left(\tau(Q_i)\right) = 0$ for all $i \neq i_*$, while $\tilde{U}_i\left(\tau(Q_{i_*})\right) = \langle Q : V(F)|$. Thus, an optimizing social planner would make $\sum_i \alpha_i \tilde{U}_i\left(\tau(Q_i)\right) = \alpha_* \langle Q : V(F)|$ and hence the social surplus $\alpha_* \langle Q : V(F)| - \langle Q : V(G) - 1| = \langle Q : \alpha_* V(F) - V(G) + 1|$, the objective in (23). ■

Remark 1 The ending part of the proof of Lemma 3 characterizes the optimal payment rule: The surplus left for the auctioneer is always zero; in any solution Q for (23), the equilibrium expected payoff for the type $\tau(Q_i)$ of bidder i is equal to zero if $i \neq i_*$, and equal to $\frac{1}{\tau(Q_{i*})} \langle Q : V(F)|$ if $i = i_*$, such that i_* is one of the bidders with the property $\alpha_{i_*} = \alpha_*$.

4.3 Saddle Point Characterization of the Solution

Since examples exist where the constraint $\langle Q : V(F)| \ge 0$ in (23) is binding (Remark 3), to solve (23) we start with characterizing any of its solutions by the saddle point condition.

The Lagrangian corresponding to (23) is

$$\begin{aligned}\mathscr{L}(Q,\lambda) &:= \langle Q : \alpha_* V(F) - V(G) + 1| + \lambda\langle Q : V(F)| \\ &= \langle Q : (\alpha_* + \lambda)V(F) - V(G) + 1|,\end{aligned} \tag{25}$$

with the second line due to linearity of $\varphi \mapsto \langle Q : \varphi|$. We observe that Q^* is a solution for (23) if and only if there exists a $\lambda^* \in \mathbb{R}_+$ such that (Q^*, λ^*) is a saddle point in the sense that, for all $Q \in \mathscr{Q}_{\text{mon}}$ and all $\lambda \in \mathbb{R}_+$,

$$\mathscr{L}(Q^*, \lambda) \geq \mathscr{L}(Q^*, \lambda^*) \geq \mathscr{L}(Q, \lambda^*).$$

The "if" part of the observation is trivial. To prove the "only if" part, we need only to verify all the conditions corresponding to those in Luenberger [3, Corollary 1, p219]. The verification is based on the next lemma, proved in Appendix A.6.

Lemma 4 *The two-part profiles $V(F)$ and $\alpha_* V(F) - V(G) + 1$ are each well-ordered. And*

$$\langle Q : V(F)| \geq \sum_i \tau(Q_i) \int_{T_i} Q_i(t_i) dF_i(t_i). \tag{26}$$

Now that $V(F)$ and $\alpha_* V(F) - V(G) + 1$ are each well-ordered, Lemma 1 implies that $\langle Q : V(F)|$ and $\langle Q : \alpha_* V(F) - V(G) + 1|$ are each a concave function of Q. Thus, the objective in (23) is concave in the choice variable, and the constraint $\langle Q : V(F)| \geq 0$ defines a convex set of Q's. This, coupled with convexity of $\mathscr{Q}_{\text{mon}}$ (Appendix A.1), means that the proof is complete if there exists a $Q \in \mathscr{Q}_*$ such that $\langle Q : V(F)| > 0$. Such Q exists: Let Q be the allocation that always assigns the good to bidder 1 and never assigns the bad at all. That is, $Q_1 = 1$, hence $\tau(Q_1) = \underline{t}_1$, and $Q_i = 0$ for all $i \neq 1$. Note $Q \in \mathscr{Q}_{\text{mon}}$. By (26),

$$\langle Q : V(F)| \geq \underline{t}_1 \int_{T_1} Q_1(t_1) dF_1(t_1) = \underline{t}_1 > 0,$$

with the last ">" due to an assumption. Now that all conditions are verified, the saddle point characterization follows.

4.4 Solution through Bisection

For any $\lambda \geq 0$, denote

$$Z^\lambda := (\alpha_* + \lambda)V(F) - V(G) + 1. \tag{27}$$

Hence $\mathscr{L}(Q,\lambda) = \left\langle Q : Z^{\lambda}\right|$ and the saddle point condition for any solution requires

$$\max_{Q\in\mathscr{Q}_{\text{mon}}} \left\langle Q : Z^{\lambda}\right| = \max_{Q\in\mathscr{Q}_{\text{mon}}} \sum_i \left(\left\langle Q_i^+ \,, Z_{i,+}^{\lambda} f_i\right\rangle - \left\langle Q_i^- \,, Z_{i,-}^{\lambda} f_i\right\rangle\right), \tag{28}$$

with the equality due to (7). To solve the problem in (28), observe that the two-part profile Z^{λ} is well-ordered, because $(\alpha_* + \lambda)V(F) - V(G) + 1$ belongs to the positive cone generated by $\alpha_* V(F) - V(G) + 1$ and $V(F)$, each well-ordered (Section 3.2 and Lemma 4). With this property, we bisect the problem in (28) into two independent linear programmings.

For any $x := (x_i)_{i=1}^n \in \prod_i T_i$, define

$$\begin{aligned}\mathscr{Q}_x &:= \left\{Q \in \mathscr{Q} : (\forall i)\,[t_i < x_i \Rightarrow Q_i(t_i) = 0]\right\},\\ \mathscr{Q}^x &:= \left\{Q \in \mathscr{Q} : (\forall i)\,[t_i > x_i \Rightarrow Q_i(t_i) = 0]\right\}.\end{aligned}$$

Then $\mathscr{Q}_x \subseteq \mathscr{Q}$ and $\mathscr{Q}^x \subseteq \mathscr{Q}$. Note: If $\overline{Q} \in \mathscr{Q}_x \cap \mathscr{Q}_+ \cap \mathscr{Q}_{\text{mon}}$ and $\tilde{Q} \in \mathscr{Q}^x \cap (-c\mathscr{Q}_+) \cap \mathscr{Q}_{\text{mon}}$, then Q defined by

$$\forall i \;\forall t_i \in T_i: \;\; Q_i(t_i) := \overline{Q}_i(t_i)\mathbf{1}_{t_i \geq x_i} + \tilde{Q}_i(t_i)\mathbf{1}_{t_i \leq x_i}$$

belongs to $\mathscr{Q}_{\text{mon}}$, and $Q^+ = \overline{Q}$ and $Q^- = -\tilde{Q}$. Conversely, for any $Q \in \mathscr{Q}_{\text{mon}}$ there exists an $x := (x_i)_{i=1}^n \in \prod_i T_i$ for which $(Q_i^+)_{i=1}^n \in \mathscr{Q}_x \cap \mathscr{Q}_+ \cap \mathscr{Q}_{\text{mon}}$, $(-Q_i^-)_{i=1}^n \in \mathscr{Q}^x \cap (-c\mathscr{Q}_+) \cap \mathscr{Q}_{\text{mon}}$. Thus, for any $\lambda \geq 0$, the problem in (28) is equivalent to

$$\begin{aligned}\max_{(x,Q^+,Q^-)} \quad & \sum_i \left(\left\langle Q_i^+ \,, Z_{i,+}^{\lambda} f_i\right\rangle + \left\langle -Q_i^- \,, Z_{i,-}^{\lambda} f_i\right\rangle\right)\\ \text{s.t.} \quad & x \in \prod_i T_i\\ & Q^+ \in \mathscr{Q}_x \cap \mathscr{Q}_+ \cap \mathscr{Q}_{\text{mon}}\\ & -Q^- \in \mathscr{Q}^x \cap (-c\mathscr{Q}_+) \cap \mathscr{Q}_{\text{mon}}.\end{aligned}$$

This in turn is equivalent to

$$\max_{x\in\prod_i T_i} \quad R(x|\lambda) + C(x|\lambda), \tag{29}$$

where, for any $x := (x_i)_{i=1}^n \in \prod_i T_i$,

$$R(x|\lambda) \;:=\; \max_{Q\in\mathscr{Q}_x\cap\mathscr{Q}_+\cap\mathscr{Q}_{\text{mon}}} \sum_i \left\langle Q_i \,, Z_{i,+}^{\lambda} f_i\right\rangle, \tag{30}$$

$$C(x|\lambda) \;:=\; \max_{Q\in\mathscr{Q}^x\cap(-c\mathscr{Q}_+)\cap\mathscr{Q}_{\text{mon}}} \sum_i \left\langle Q_i \,, Z_{i,-}^{\lambda} f_i\right\rangle. \tag{31}$$

Since $\mathscr{Q}_x \subseteq \mathscr{Q}$ and $\mathscr{Q}^x \subseteq \mathscr{Q}$, we have $\mathscr{Q}_x \cap \mathscr{Q}_+ \cap \mathscr{Q}_{\text{mon}} \subseteq \mathscr{Q}_+ \cap \mathscr{Q}_{\text{mon}}$ and $\mathscr{Q}^x \cap (-c\mathscr{Q}_+) \cap \mathscr{Q}_{\text{mon}} \subseteq (-c\mathscr{Q}_+) \cap \mathscr{Q}_{\text{mon}}$. Thus, for any $x \in \prod_i T_i$,

$$R(x|\lambda) \leq \max_{Q \in \mathscr{Q}_+ \cap \mathscr{Q}_{\text{mon}}} \sum_i \left\langle Q_i \, , Z^\lambda_{i,+} f_i \right\rangle , \tag{32}$$

$$C(x|\lambda) \leq \max_{Q \in (-c\mathscr{Q}_+) \cap \mathscr{Q}_{\text{mon}}} \sum_i \left\langle Q_i \, , Z^\lambda_{i,-} f_i \right\rangle . \tag{33}$$

The problems on the right-hand side of (32) and (33) are each in the form of (14), with the role of X there played by $\mathscr{Q}_+$ for (32), and $-c\mathscr{Q}_+$ for (33). Thus, with $\overline{Z^\lambda_{i,\cdot}}$ signifying the ironed copy of $Z^\lambda_{i,\cdot}$ (Eqs. (12) and (13)), denote:

A. $Q(\cdot|\lambda, +)$: the allocation that belongs to $\mathscr{Q}_+$ and is by the rank of $\left(\overline{Z^\lambda_{i,+}}\right)_{i=1}^n$, and

B. $Q(\cdot|\lambda, -)$: the allocation that belongs to $-c\mathscr{Q}_+$ and is by the rank of $\left(-\overline{Z^\lambda_{i,-}}\right)_{i=1}^n$;

then $Q(\cdot|\lambda, +)$ is a solution for the problem in (32), and $Q(\cdot|\lambda, -)$ a solution for the problem in (33). Define for each bidder i

$$\underline{r}^\lambda_i := \sup \left\{ t_i \in T_i : \overline{Z^\lambda_{i,-}}(t_i) < 0 \right\} ,$$
$$\overline{r}^\lambda_i := \inf \left\{ t_i \in T_i : \overline{Z^\lambda_{i,+}}(t_i) > 0 \right\} .$$

Since $Q(\cdot|\lambda, +)$ accords with the rank of $\left(\overline{Z^\lambda_{i,+}}\right)_{i=1}^n$, it obeys (10) and the upper branch of (11), with the role ψ_i there played by $\overline{Z^\lambda_{i,+}}$ here. Analogously $Q(\cdot|\lambda, -)$ obeys (10) and the lower branch of (11), with the role ψ_i there played by $-\overline{Z^\lambda_{i,-}}$ here. Thus, for each i,

$$\text{support}\, Q_i(\cdot|\lambda, +) \subseteq \left[\overline{r}^\lambda_i, \overline{t}_i\right] ,$$
$$\text{support}\, Q_i(\cdot|\lambda, -) \subseteq \left[\underline{t}_i, \underline{r}^\lambda_i\right] .$$

Recall that Z^λ is well-ordered, i.e., $Z^\lambda_{i,+} \leq Z^\lambda_{i,-}$ for all i. Thus one can prove (Appendix A.7), as a consequence of the ironing operation ((12) and (13)), that

$$\underline{r}^\lambda_i \leq \overline{r}^\lambda_i \tag{34}$$

for each i. Then at any $x \in \prod_i \left[\underline{r}^\lambda_i, \overline{r}^\lambda_i\right]$, the objectives in (30) and (31) simultaneously attain their upper bounds (32) and (33) . Thus (29) is solved by any $x \in \prod_i \left[\underline{r}^\lambda_i, \overline{r}^\lambda_i\right]$. In other words, the allocation $Q(\cdot|\lambda) := (Q_i(\cdot|\lambda))_{i=1}^n$ defined by

$$\forall i \, \forall t_i \in T_i : \; Q_i(t_i|\lambda) := Q_i(t_i|\lambda, +)\mathbf{1}_{t_i \geq \overline{r}^\lambda_i} + Q_i(t_i|\lambda, -)\mathbf{1}_{t_i \leq \underline{r}^\lambda_i} , \tag{35}$$

which belongs to $\mathscr{Q}_{\text{mon}}$ and has the property that $Q^+(\cdot|\lambda) = Q_i(\cdot|\lambda,+)$ and $Q^-(\cdot|\lambda) = -Q_i(\cdot|\lambda,-)$, solves the problem in (28) for any $\lambda \in \mathbb{R}_+$. Thus the main result is at hand.

Theorem 1 *On the problem of maximizing social surplus subject to IC, IR and BB:*

a. for any solution Q_* there exists $\lambda \geq 0$ such that:

i. $Q_* = Q(\cdot|\lambda)$ a.e. on $\prod_i \left(\underline{t}_i, \underline{r}_i^\lambda\right] \cup \left[\overline{r}_i^\lambda, \overline{t}_i\right]$, with $Q(\cdot|\lambda)$ defined by (35), and

ii. $\lambda \left\langle Q(\cdot|\lambda) : V(F)\right| = 0$;

b. a solution exists.

Proof Claim (a): By Section 4.3, any solution for (23) corresponds to a saddle point (Q_*, λ). Then (a.ii) is immediate. The saddle point condition also implies that Q_* solves $\max_{Q\in\mathscr{Q}_{\text{mon}}} \mathscr{L}(Q,\lambda)$. By the reasoning preceding the statement of this theorem,

$$\max_{Q^+-Q^-\in\mathscr{Q}_{\text{mon}}} \mathscr{L}(Q,\lambda) = \max_{Q^+\in\mathscr{Q}_+\cap\mathscr{Q}_{\text{mon}}} \sum_i \left\langle Q_i^+, Z_{i,+}^\lambda f_i\right\rangle + \max_{-Q^-\in(-c\mathscr{Q}_+)\cap\mathscr{Q}_{\text{mon}}} \sum_i \left\langle -Q_i^-, Z_{i,-}^\lambda f_i\right\rangle.$$

Thus, $Q^+ - Q^-$ solves the problem on the left-hand side (LHS) if and only if Q^+ and $-Q^-$ solve their corresponding problems on the right-hand side (RHS). Since $Q(\cdot|\lambda,+)$ and $Q(\cdot|\lambda,-)$ are each the unique solution (unique modulo measure zero, cf. Footnote 5) for the corresponding RHS problems, $Q(\cdot|\lambda,+)+Q(\cdot|\lambda,-)$ is the unique solution for the LHS problem modulo measure zero. By (34), the supports of $Q(\cdot|\lambda,+)$ and $Q(\cdot|\lambda,-)$ are non-overlapping, hence $Q(\cdot|\lambda,+)+Q(\cdot|\lambda,-)$ is the $Q(\cdot|\lambda)$ defined in (35). Thus Claim (a.i) follows.

Claim (b): Consider two possibilities: either (i) $\langle Q(\cdot|0) : V(F)| \geq 0$ or (ii) $\langle Q(\cdot|0) : V(F)| < 0$. Since $Q(\cdot|0)$ solves $\max_{Q\in\mathscr{Q}_{\text{mon}}} \mathscr{L}(Q,0)$, Case (i) means that $Q(\cdot|0)$ solves (23). Hence consider Case (ii). It suffices to show existence of $\lambda > 0$ such that $\langle Q(\cdot|\lambda) : V(F)| = 0$. We show that by the intermediate-value theorem: Since $Q \mapsto \langle Q : V(F)|$ is continuous (Lemma 2 coupled with continuity of f_i on the compact T_i); by the theorem of maximum applied to (28)—applicable because of Lemma 2 and the fact that $\langle \cdot : V(F)|$ and $\langle \cdot : V(G)|$ are both continuous, with the latter due to $G_i' \leq \sup_{T_i} f_i/\underline{t}_i$—we know that $\lambda \mapsto Q(\cdot|\lambda)$ is continuous. Thus, $\langle Q(\cdot|\lambda) : V(F)|$ is continuous in λ. Within Case (ii), $\langle Q(\cdot|0) : V(F)| < 0$. On the other hand, $\langle Q(\cdot|\lambda) : V(F)| \geq 0$ for any sufficiently large λ: For all $t_i \in T_i$,

$$\frac{\partial}{\partial\lambda} Z_{i,-}^\lambda(t_i) = V_{i,-}(F)(t_i) \overset{(9)}{=} t_i + F_i(t_i)/f_i(t_i) \geq \underline{t}_i,$$

a constant bounded away from zero. Thus, for any sufficiently large λ, $Z^{\lambda}_{i,-} > 0$, hence $\overline{Z^{\lambda}_{i,-}} > 0$ by (12) and (13). Consequently, any solution for (28) would set $Q^{-}_{i}(\cdot|\lambda) = 0$ for all i. Hence $Q(\cdot|\lambda) \geq 0$ and, by (26), $\langle Q(\cdot|\lambda) : V(F)| \geq 0$. Thus, there exists $\lambda > 0$ for which $\langle Q(\cdot|\lambda) : V(F)| = 0$, as desired. ■

Interpretation Any optimum in its reduced form is almost everywhere identical to the concatenation of two allocations, defined by the above Provisions A and B. One is to allocate the good by the rank of $(\overline{Z^{\lambda}_{i,+}})^{n}_{i=1}$, and the other, to allocate the bad by the rank of $(-\overline{Z^{\lambda}_{i,+}})^{n}_{i=1}$. That implies two features of the optimal mechanism. First, according to the definition of allocations by ranks, for each bidder i there is a price floor (or minimum type $\overline{r}^{\lambda}_{i}$) for the good, and a price ceiling (or maximum type $\underline{r}^{\lambda}_{i}$) for the bad. Second, except special cases, for each bidder i there are types, constituting a set of positive measure, that are treated indiscriminately by the mechanism. That is because, as we will see, $Z^{\lambda}_{i,+}$ and $Z^{\lambda}_{i,-}$ are in general non-monotone even with regular type-distributions and hence there are nondegenerate intervals on which $Z^{\lambda}_{i,x} > \overline{Z^{\lambda}_{i,x}}$ for some $x \in \{+, -\}$. An implication of the two features is that the counterpart of the efficient allocation in the literature, allocating the good to a bidder with the highest realized type and the bad to the lowest one, is in general suboptimal.

5 Implications and Remarks

5.1 The Goodness of Bad

Theorem 1 implies that it is sometimes socially desirable to assign the bad to someone even when assigning it to no one is an option. Furthermore, since the model allows the value of the bad to cancel out that of the good ($c = 1$), sometimes the social planner should artificially make up a service for one society member to pay tribute to another society member.

Theorem 2 *If, for some bidder i,*

$$H_{i}^{Z^{0}_{i,-}}\left(F_{i}\left(\arg\min_{T_i} Z^{0}_{i,-}\right)\right) < 0, \tag{36}$$

then, for any solution $(Q_j)^{n}_{j=1}$ for (23), $Q_i < 0$ on a positive-measure subset of T_i.

Proof Suppose, to the contrary, that there exists a solution Q^* for (23) such that $Q^*_i \geq 0$ almost everywhere for any bidder i. That implies, by monotonicity of Q^*_i ($Q^* \in \mathscr{Q}_{\text{mon}}$),

$\tau(Q_i^*) = \underline{t}_i$ for all i. Then (26) implies

$$\langle Q^* : V(F)| \geq \left(\min_i \underline{t}_i\right) \sum_i \int_{T_i} Q_i^*(t_i) dF_i(t_i). \tag{37}$$

Note that $\sum_i \int_{T_i} Q_i^*(t_i) dF_i(t_i) > 0$. Otherwise, the supposition $Q_i^* \geq 0$ for all i implies that $Q_i^* = 0$ for all i. Consequently, the social surplus generated by Q^*, $\langle Q^* : \varphi|$ for some two-part profile (the objective in (23)), is equal to zero because $\varphi \mapsto \langle Q^* : \varphi|$ is a linear operator. But then Q is suboptimal because assigning the good to someone for free generates a positive surplus. Thus $\sum_i \int_{T_i} Q_i^*(t_i) dF_i(t_i) > 0$, hence $\langle Q^* : V(F)| > 0$. Then Theorem 1.a implies $\lambda = 0$ and Q^* maximizes $\mathscr{L}(Q, 0)$ among all $Q \in \mathscr{Q}_{\text{mon}}$. Hence Q^* is almost everywhere identical to the $Q(\cdot|0)$ in (35), which requires $Q_i^-(t_i|0) > 0$ whenever $\overline{Z_{i,-}^0}(t_i) < 0$. By (13) and (36), $H_i^{Z_{i,-}^0}(0) = 0 > H_i^{Z_{i,-}^0}\left(F_i\left(\arg\min Z_{i,-}^0\right)\right)$ and hence the convex hull of $H_i^{Z_{i,-}^0}$ is negatively sloped on $\left[0, F_i\left(\arg\min Z_{i,-}^0\right)\right]$. Then (12) implies $\overline{Z_{i,-}^0} < 0$, and hence $Q_i^* < 0$, on the nondegnerate interval $\left[\underline{t}_i, \arg\min Z_{i,-}^0\right]$, contradiction. ■

Remark 2 Condition (36) for Theorem 2 is not vacuous. The function $Z_{i,-}^0$ in the condition can be derived from (27) as

$$Z_{i,-}^0(t_i) = \alpha_* t_i + \frac{\alpha_* F_i(t_i) - G_i(t_i)}{f_i(t_i)}$$

for any i and any $t_i \in T_i$ ((51), Appendix A.6). An example is the uniform distribution $F(t) := (t - \underline{t})/(\bar{t} - \underline{t})$ for all $t \in [\underline{t}, \bar{t}]$. Denote $\Delta := \bar{t} - \underline{t}$ and $\delta := \ln \bar{t} - \ln \underline{t}$. Then

$$Z_{i,-}^0(t) = \frac{2\delta}{\Delta} t - \ln t - \frac{\delta}{\Delta}\underline{t} + \ln \underline{t}$$

for all $t \in [\underline{t}, \bar{t}]$, hence $\arg\min Z_{i,-}^0 = \Delta/(2\delta)$. Thus, by (13), (36) is equivalent to

$$\int_{\underline{t}}^{\Delta/(2\delta)} \left(\frac{2\delta}{\Delta} t - \ln t - \frac{\delta}{\Delta}\underline{t} + \ln \underline{t}\right) \frac{1}{\Delta} dt < 0,$$

which in turn is equivalent to $\frac{3}{2} - \frac{3\delta}{\Delta}\underline{t} + \ln \underline{t} < \ln \frac{\Delta}{2\delta}$, which is true when $\underline{t} \leq 1$ and $\left(\bar{t} - \underline{t}\right) / \left(\ln \bar{t} - \ln \underline{t}\right) \geq 2e^{3/2}$. The forthcoming Remark 3 gives another example.

5.2 Possibility of Binding Budget Balance Constraint

Were there no loss of generality to relax the constraint $\langle Q : V(F)| \geq 0$ in (23), one could avoid nonlinearity of the objective in (28) through a restrictive assumption of the primitive:

$$\forall i : \alpha_i = \alpha_*. \tag{38}$$

Then the two-part operator $\langle \cdot : \alpha_* V(F) - V(G) + 1|$ for the objective in (23) becomes a linear operator, as one can show (comparing (50) with (51), Appendix A.6) from (38) that

$$\langle Q : \alpha_* V(F) - V(G) + 1| = \sum_i \langle Q_i, Y_i f_i \rangle, \tag{39}$$

where, for any bidder i and any $t_i \in T_i$,

$$Y_i(t_i) := \alpha_* t_i + \frac{\alpha_* F_i(t_i) - G_i(t_i)}{f_i(t_i)}. \tag{40}$$

Thus, had $\langle Q : V(F)| \geq 0$ been relaxed, the problem (28) required by the saddle point condition would become a linear programming $\max_{Q \in \mathscr{Q}_{\text{mon}}} \sum_i \langle Q_i, Y_i f_i \rangle$, and any solution Q_* for this problem, and hence for (23), would be the allocation Q_*^+ by the ranks of $\left(\overline{Y}_i\right)_{i=1}^n$ coupled with the allocation Q_*^- by the ranks of $\left(-\overline{Y}_i\right)_{i=1}^n$. However, the next remark shows existence of parametric configurations given which the constraint $\langle Q : V(F)| \geq 0$ is binding in every solution for (23).

Remark 3 Let $0 < L < H$ such that $\ln(H/L) \geq 11/6$. Pick any $m = 1, 2, \ldots$ large enough for $L < H - 1/m < H$. For any $t \in [L, H]$, let

$$\phi(t) := t^3/3 - (H - 1/m)t^2 + (H - 1/m)^2 t + t/m^4.$$

Consider a symmetric-bidder case where the common distribution F is defined by

$$F(t) = \frac{\phi(t) - \phi(L)}{\phi(H) - \phi(L)}$$

for all t in its support $[L, H]$. Then (38) is satisfied, and hence the objective in (23) becomes the linear form (39). By (21), (40), and the parametric condition $\ln(H/L) \geq 11/6$, one can show (Appendix A.8):

$$Y_i(H - 1/m) \to -\infty \quad \text{as} \quad m \to \infty. \tag{41}$$

Should the constraint $\langle Q : V(F)| \geq 0$ be non-binding at a solution Q^* for (23), Q^* would be the concatenation of the allocation by the rank of $\left(\overline{Y}_i\right)_{i=1}^n$ and the allocation by the rank of $\left(-\overline{Y}_i\right)_{i=1}^n$. By (41), for all sufficiently large m, $\overline{Y}_i < 0$ on $[L, H - 1/m)$ and hence $Q_i^* < 0$ on $[L, H - 1/m)$ for all i. This, coupled with the fact that $V_{i,+}(F) \leq \bar{t}_i = H$ and $V_{i,-}(F) \geq \underline{t}_i = L$ for all i (due to (9)), implies that

$$\begin{aligned} \langle Q^* : V(F)| &\leq H \sum_i \int_{T_i} \max\{0, Q_i^*(t_i)\} dF(t_i) - L \sum_i \int_{T_i} \max\{0, -Q_i^*(t_i)\} dF(t_i) \\ &\leq H \left(1 - F(H - 1/m)^n\right) - Lc \left(1 - (1 - F(H - 1/m))^n\right), \end{aligned}$$

which is negative for all sufficiently large m, contradiction.

5.3 Non-monotonicity of Virtual Utilities

In the literature, a virtual utility function is typically negative for low types and, given regular type-distributions, is increasing throughout. By contrast, our counterpart Z^{λ} of the virtual utility functions is non-monotone even given regular type-distributions. To highlight the contrast, suppose that all bidders' types are drawn from a type-distribution F (density f) that is *regular* in the sense that $t_i - (1 - F(t_i))/f(t_i)$ and $t_i + F(t_i)/f(t_i)$ are each strictly increasing in t_i. The optimal auction in the literature applied to the allocation of the good would sell the good to a highest-type bidder and, if applied to the allocation of the bad, would procure acceptance of the bad from a lowest-type one, subject to a price floor for the good and a price ceiling for the bad. In our model, by contrast, one can show (incorporating λ into (50) and (51), Appendix A.6) that

$$\begin{aligned} Z^{\lambda}_{+}(t_i) &= \alpha_* t_i + \frac{\alpha_* F(t_i) - G(t_i)}{f(t_i)} + \lambda \left(t_i - \frac{1 - F(t_i)}{f(t_i)} \right), \\ Z^{\lambda}_{-}(t_i) &= \alpha_* t_i + \frac{\alpha_* F(t_i) - G(t_i)}{f(t_i)} + \lambda \left(t_i + \frac{F(t_i)}{f(t_i)} \right), \end{aligned} \tag{42}$$

(subscripts i dropped from Z_i and G_i by symmetry) for any i and any $t_i \in T_i$.

One can see that the graph of Z^{λ}_{+} and Z^{λ}_{-} are each roughly U-shape, peaking at $\underline{t}_i$ and $\bar{t}_i$, with possibly multiple local extremums in between: First,[7]

$$0 < Z^{\lambda}_{-}(\underline{t}_i) = (\alpha_* + \lambda)\underline{t}_i < \max_{T_i} Z^{\lambda}_{-} = Z^{\lambda}_{-}(\bar{t}_i). \tag{43}$$

Second, Z^{λ}_{-} is strictly increasing on a neighborhood of $\bar{t}_i$ and, if $\underline{t}_i < 1/(2\alpha_*)$, Z^{λ}_{-} is strictly decreasing on a neighborhood of $\underline{t}_i$. The shape of Z^{λ}_{+} is similar. Thus, the ironing operation is mostly unavoidable in our model even when type-distributions are regular. If someone's realized type belongs to the neighborhood of $\bar{t}_i$ where Z^{λ}_{+} is strictly increasing, the good would still go to the highest-type bidder; otherwise, each item is allocated through lotteries. In particular, the set of types to which the bad may be allocated, being the lower truncation $\left[\underline{t}, \underline{r}^{\lambda}\right]$ of the type-support, are likely to be treated equally by the ironed copy of Z^{λ}_{-}; consequently, the bad is likely to be assigned via an egalitarian lottery.

The difference is driven by the fact that, while virtual utilities in the literature are measured in monetary payoffs, our counterparts are measured in monetary payoffs divided

[7] By (20) and (42), $Z^{\lambda}_{-}(\underline{t}_i) = (\alpha_* + \lambda)\underline{t}_i > 0$ and hence, by continuity, $Z^{\lambda}_{-} > 0$ on a neighborhood of $\underline{t}_i$. By (20) and (21), $G_i \geq \alpha_* F_i$, with "=" attained if and only if $t_i \in \left\{\underline{t}_i, \bar{t}_i\right\}$. Thus (43) follows from (42).

by a bidder's type. Thus, among relatively low types, a bidder's virtual utility here becomes higher when his type gets lower, so that the social planner would be less eager to pay him for accepting the bad, whereas in the literature's standard model she would be more willing to procure the service from lower types, which means in that model lower virtual costs.

5.4 Extension to Heterogeneous Values

All results of this paper can be extended to the case where the values of the items being assigned are heterogeneous across bidders. For each bidder i, let v_i be i's payoff from having the good, and $-c_i$ his payoff from having the bad, such that $v_i \geq 0 \geq c_i$ and $(v_i, c_i)_{i=1}^n$ is commonly known. Replace (1) by

$$Q_i(t_i) = v_i \int_{T_{-i}} q_{iA}(t_i, t_{-i}) dF_{-i}(t_{-i}) - c_i \int_{T_{-i}} q_{iB}(t_i, t_{-i}) dF_{-i}(t_{-i}),$$

and replace (11) by

$$Q_i(\cdot) = \begin{cases} v_i \int_{T_{-i}} q_i(\cdot, t_{-i}) dF_{-i}(t_{-i}) & \text{if } (Q_i)_{i=1}^n \in \overline{v}\mathscr{Q}_+ \\ -c_i \int_{T_{-i}} q_i(\cdot, t_{-i}) dF_{-i}(t_{-i}) & \text{if } (Q_i)_{i=1}^n \in -\overline{c}\mathscr{Q}_+, \end{cases}$$

where $\overline{v} := \max_i v_i$ and $\overline{c} := \max_i c_i$. Then, when the set X in (14) is $\overline{v}\mathscr{Q}_+$, the objective in (14) is equal to

$$\sum_i \left\langle v_i \int_{T_{-i}} q_i(\cdot, t_{-i}) dF_{-i}(t_{-i}), \psi_i f_i \right\rangle = \sum_i \left\langle \int_{T_{-i}} q_i(\cdot, t_{-i}) dF_{-i}(t_{-i}), v_i \psi_i f_i \right\rangle$$

and hence a solution for (14) is the allocation by the rank of $\left(v_i \overline{\psi}_i\right)_{i=1}^n$. Analogously, when the set X in (14) is $-\overline{c}\mathscr{Q}_+$, the objective in (14) is equal to

$$\sum_i \left\langle \int_{T_{-i}} q_i(\cdot, t_{-i}) dF_{-i}(t_{-i}), -c_i \psi_i f_i \right\rangle$$

and hence a solution for (14) is the allocation by the rank of $\left(-c_i \overline{\psi}_i\right)_{i=1}^n$. It follows that the optimal mechanism becomes the concatenation between $Q(\cdot|\lambda, +)$ and $Q(\cdot|\lambda, -)$ such that $Q(\cdot|\lambda, +)$ is the allocation by the rank of $\left(v_i \overline{Z_{i,+}^\lambda}\right)_{i=1}^n$, and $Q(\cdot|\lambda, -)$ the allocation by the rank of $\left(-c_i \overline{Z_{i,-}^\lambda}\right)_{i=1}^n$.

5.5 Operations That Set Our Model apart from the Literature

First, we set $c > 0$ (having an item of negative value $-c$). That means an allocation in reduced form may be negative: $Q_i = Q_i^+ - Q_i^-$, with $Q_i^- \neq 0$. Consequently, a bidder's information rent density, depending on his realized type t_i, switches between two functions ρ_+ and ρ_-, one for $Q_i(t_i) > 0$, the other for $Q_i(t_i) < 0$ (Eq. (15)). To systematically keep track of such endogenous switching between the information rents, the concept of two-part operators comes to be. Thus our first point of departure is represented by the transformation

$$[\rho : \mu_i \mapsto \rho(\mu_i)] \quad \longmapsto \quad [(\rho_+, \rho_-) : \mu_i \mapsto (\rho_+(\mu_i), \rho_-(\mu_i))]\,. \tag{44}$$

From (44) we also see that introducing an item of negative value is substantively different from merely normalizing anyone's payoff from not winning the good to a negative level. The former, as (44) does, necessitates an endogenous bifurcation of information rent densities, one to assign the good, the other to assign the bad. Normalization of payoffs, by contrast, does not require such bifurcation and hence leaves no room for (44).

Second, we replace the independent-private-value payoff function $t_i Q_i(\hat{t}_i) - P_i(\hat{t}_i)$ by a common-value one, $Q_i(\hat{t}_i) - P_i(\hat{t}_i)/t_i$. That, as explained in (24), amounts to a transform of the distribution, $f_i(t_i) \mapsto f_i(t_i)/t_i$ density-wise, on the domain of the payment function. This transform amounts to a change of the expected-value operator of the payment function:

$$[P_i \mapsto \langle P_i, f_i \rangle] \quad \longmapsto \quad [P_i \mapsto \langle P_i, f_i/\mathbb{I} \rangle]\,. \tag{45}$$

This, combined with (44) and the envelope equation, implies a transform of the design objective—social surplus—from a linear form to a nonlinear one (Lemma 3):

$$\langle Q\,, 1 \rangle \quad \longmapsto \quad \langle Q : \alpha_* V(F) - V(G) + 1|. \tag{46}$$

Note that the right-hand side of (46) would have collapsed to $\langle Q\,, 1\rangle$ had (45) been absent, which would make $\alpha_* = 1$ and $G = F$. Had (44) been absent, two-part operators would have collapsed to linear operators and so the right-hand side of (46) would have been an inner product, a linear form. Thus transforms (44) and (45) together set our model apart from the quasilinearity-based optimal auction models.

Third, the budget balance constraint may bind in our model. That means the Lagrangian associated with the design problem is transformed:

$$\langle Q : \alpha_* V(F) - V(G) + 1| \quad \longmapsto \quad \langle Q : (\alpha_* + \lambda)(V(F) - V(G) + 1|. \tag{47}$$

Had (47) been absent, it would still be possible to collapse the two-part operator $\langle Q : \alpha_* V(F) - V(G) + 1|$ to a linear operator, as has been explained at (38)–(40), Section 5.2.

In sum, (44), (45) and (47) combined, nonlinearity of the design objective and the associated Lagrangian are unavoidable, and our bisection method indispensable.

6 Conclusion

This paper makes a methodology contribution to the mechanism design literature. We introduce a new concept, a two-part operator, to systematically keep track of each player's bifurcated incentive of playing the role of a buyer sometimes and the role of a seller other times. We devise a bisection technique to solve an optimal mechanism problem the objective of which is a nonlinear outcome of the two-part operator. Our method proves successful because we solve the design problem given a general class of parameter configurations.

The paper also makes an application contribution. We have extended the mechanism design theory from the design of selling mechanisms to the design of wealth redistributions. Furthermore, our solution of the optimal mechanism points out an important role that auctions can play in wealth redistribution programs: Instead of mandating wealth transfers from one individual to another, whose idiosyncrasies are uncertain to the regulator, a social planner could have used auctions to induce the right amount of wealth transfers voluntarily conducted between the right types of individuals.

Our model can be applied to matching theory in the case where one side of the matching market has both desirable and undesirable items (e.g., toxic assets that need to be absorbed by other financial institutions; enrollment of schools in bad neighborhoods; thankless tasks to be carried out by some team members). While much of the matching theory literature assumes that money transfers are banned by the regulator, an implication of our result is that it is suboptimal of the regulator to ban money transfers from matching markets.

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A Proof Details

A.1 Convexity of $\mathscr{Q}_{\text{mon}}$

Let $\gamma \in [0,1]$ and $Q, \hat{Q} \in \mathscr{Q}_{\text{mon}}$. Now that $Q \in \mathscr{Q}_{\text{mon}}$, there is a $(q_{iA}, q_{iB})_{i=1}^n$ with $\sum_i q_{iA}(\cdot) \leq 1$ and $\sum_i q_{iB}(\cdot) \leq 1$, such that Q_i satisfies (1) and is weakly increasing for all i. And $\hat{Q} = (\hat{Q}_i)_{i=1}^n$ is likewise an analogous $(\hat{q}_{iA}, \hat{q}_{iB})_{i=1}^n$. Then $\sum_i (\gamma q_{iA} + (1-\gamma)\hat{q}_{iA}) \leq 1$ and $\sum_i (\gamma q_{iB} + (1-\gamma)\hat{q}_{iB}) \leq 1$; furthermore, for each i, $\gamma Q_i + (1-\gamma)\hat{Q}_i$ satisfies (1) with respect to $(\gamma q_{iA} + (1-\gamma)\hat{q}_{iA}, \gamma q_{iB} + (1-\gamma)\hat{q}_{iB})$, and is weakly increasing because both Q_i and $\hat{Q}_i$ are so. Thus $\left(\gamma Q_i + (1-\gamma)\hat{Q}_i\right)_{i=1}^n \in \mathscr{Q}_{\text{mon}}$.

A.2 Proof of Lemma 1

For any $Q \in \mathscr{Q}$ and any well-ordered two-part profile $\varphi := (\varphi_{i,+}, \varphi_{i,-})_{i=1}^n$, use the definition of two-part operators and the fact $Q_i = Q_i^+ - Q_i^-$ to obtain

$$\begin{aligned}
\langle Q : \varphi| &= \sum_i \int_{T_i} Q_i^+(t_i)\varphi_{i,+}(t_i)dF_i(t_i) - \sum_i \int_{T_i} Q_i^-(t_i)\varphi_{i,-}(t_i)dF_i(t_i) \\
&= \sum_i \int_{T_i} Q_i(t_i)\varphi_{i,-}(t_i)dF_i(t_i) + \sum_i \int_{T_i} Q_i^+(t_i)\left(\varphi_{i,+}(t_i) - \varphi_{i,-}(t_i)\right)dF_i(t_i).
\end{aligned}$$

On the second line, the first sum on the second line is linear in Q, and the second sum concave in Q because Q_i^+ is convex in Q_i and, because φ is well-ordered, $\varphi_{i,+} - \varphi_{i,-} \leq 0$ for all i. Thus $\langle Q : \varphi|$ is concave in Q.

A.3 Proof of Lemma 2

By (7), $\langle Q : V(\mu)| = \sum_i \left(\langle Q_i^+, V_{i,+}(\mu)f_i\rangle - \langle Q_i^-, V_{i,-}(\mu)f_i\rangle\right)$, and Q^+ and Q^- are each continuous in Q. Thus it suffices to show continuity of $\langle \cdot, V_{i,+}(\mu)f_i\rangle$ and $\langle \cdot, V_{i,-}(\mu)f_i\rangle$. As the two are both linear operators, we need only to show that they are also bounded. (That suffices continuity because $\mathscr{Q}$ belongs to a normed vector space, cf. Footnote 3.) To show that, note from (5), (6) and (8) that

$$\begin{aligned}
V_{i,+}(\mu)f_i &= \mu_i'\mathbb{I} + \rho_+(\mu_i) \leq \left(\sup_{T_i} \mu_i'\right)\bar{t}_i, \\
V_{i,-}(\mu)f_i &= \mu_i'\mathbb{I} + \rho_-(\mu_i) \leq \left(\sup_{T_i} \mu_i'\right)\bar{t}_i + \mu_i\left(\bar{t}_i\right).
\end{aligned}$$

Hence each is uniformly bounded, as desired.

A.4 Proof of (15)

Denote $t_i^0 := \tau(Q_i)$. Since (Q_i, P_i) is IC, (2) implies

$$\begin{aligned}\langle P_i\,, \mu_i'\rangle &= \int_{T_i}\left(t_iQ_i(t_i) - \int_{t_i^0}^{t_i} Q_i(s)ds - \tilde{U}_i(t_i^0)\right)\mu_i'(t_i)dt_i \\ &= \langle Q_i\,, \mu_i'\mathbb{I}\rangle - \tilde{U}_i(t_i^0)\int_{T_i} d\mu_i - \int_{T_i}\int_{t_i^0}^{t_i} Q_i(s)ds\mu_i'(t_i)dt_i.\end{aligned}$$

Decompose the last double integral to obtain

$$\begin{aligned}\int_{T_i}\int_{t_i^0}^{t_i} Q_i(s)ds\mu_i'(t_i)dt_i &= \int_{\underline{t}_i}^{t_i^0}\int_{t_i^0}^{t_i} Q_i(s)ds\mu_i'(t_i)dt_i + \int_{t_i^0}^{\bar{t}_i}\int_{t_i^0}^{t_i} Q_i(s)ds\mu_i'(t_i)dt_i \\ &= -\int_{\underline{t}_i}^{t_i^0}\int_{t_i}^{t_i^0} Q_i(s)\mu_i'(t_i)dsdt_i + \int_{t_i^0}^{\bar{t}_i}\int_{t_i^0}^{t_i} Q_i(s)\mu_i'(t_i)dsdt_i \\ &= -\int_{\underline{t}_i}^{t_i^0}\int_{\underline{t}_i}^{s} Q_i(s)\mu_i'dt_ids + \int_{t_i^0}^{\bar{t}_i}\int_{s}^{\bar{t}_i} Q_i(s)\mu_i'dt_ids \\ &= -\int_{\underline{t}_i}^{t_i^0} Q_i(s)\int_{\underline{t}_i}^{s} \mu_i'dt_ids + \int_{t_i^0}^{\bar{t}_i} Q_i(s)\int_{s}^{\bar{t}_i} \mu_i'dt_ids \\ &= \int_{\underline{t}_i}^{t_i^0} Q_i^-(s)\int_{\underline{t}_i}^{s} \mu_i'dt_ids + \int_{t_i^0}^{\bar{t}_i} Q_i^+(s)\left(\int_{\underline{t}_i}^{\bar{t}_i} \mu_i'dt_i - \int_{\underline{t}_i}^{s} \mu_i'dt_i\right)ds \\ &= \int_{\underline{t}_i}^{t_i^0} Q_i^-(s)\rho_-(\mu_i)(s)ds - \int_{t_i^0}^{\bar{t}_i} Q_i^+(s)\rho_+(\mu_i)(s)ds \\ &= \langle Q_i^-\,, \rho_-(\mu_i)\rangle - \langle Q_i^+\,, \rho_+(\mu_i)\rangle,\end{aligned}$$

with the third equality due to Fubini's theorem, and the second last equality due to (5) and (6). Combining the two multiline formulas displayed above, we have

$$\begin{aligned}\langle P_i\,, \mu_i'\rangle &= \langle Q_i\,, \mu_i'\mathbb{I}\rangle + \langle Q_i^+\,, \rho_+(\mu_i)\rangle - \langle Q_i^-\,, \rho_-(\mu_i)\rangle - \tilde{U}_i(t_i^0)\int_{T_i} d\mu_i \\ &= \langle Q_i^+\,, \mu_i'\mathbb{I}\rangle - \langle Q_i^-\,, \mu_i'\mathbb{I}\rangle + \langle Q_i^+\,, \rho_+(\mu_i)\rangle - \langle Q_i^-\,, \rho_-(\mu_i)\rangle - \tilde{U}_i(t_i^0)\int_{T_i} d\mu_i \\ &= \langle Q_i^+\,, \mu_i'\mathbb{I} + \rho_+(\mu_i)\rangle - \langle Q_i^-\,, \mu_i'\mathbb{I} + \rho_-(\mu_i)\rangle - \tilde{U}_i(t_i^0)\int_{T_i} d\mu_i,\end{aligned}$$

with the second equality due to $Q_i = Q_i^+ - Q_i^-$.

A.5 Proof of the Sufficiency of (18)

For each bidder i, denote $t_i^0 := \tau(Q_i)$ (τ defined in (3)). For each bidder i, define

$$c_i := t_i^0 Q_i(t_i^0) - \int_{\underline{t}_i}^{t_i^0} s dQ_i(s) + \frac{1}{n-1} \sum_{j \neq i} \int_{\underline{t}_j}^{\bar{t}_j} s \left(1 - F_j(s)\right) dQ_j(s) \tag{48}$$

and, for any $(t_i, t_{-i}) \in T_i \times T_{-i}$, let the money transfer from i to others be equal to

$$p_i(t_i, t_{-i}) := c_i + \int_{\underline{t}_i}^{t_i} s dQ_i(s) - \frac{1}{n-1} \sum_{j \neq i} \int_{\underline{t}_j}^{t_j} s dQ_j(s). \tag{49}$$

Integrating $p_i(t_i, t_{-i})$ across t_{-i} gives the envelope equation (2), which coupled with the monotonicity hypothesis of Q_i implies IC. The integration also implies $\tilde{U}_i(t_i^0) = 0$, hence IR follows. To complete the proof, we prove BB: It suffices to prove (19), $\sum_i p_i(t) = \langle Q : V \rangle$ for all $t \in \prod_i T_i$, for then BB follows from (18). Hence pick any $t := (t_i)_{i=1}^n \in \prod_i T_i$. By (49),

$$\sum_i p_i(t) = \sum_i c_i + \sum_i \int_{\underline{t}_i}^{t_i} s dQ_i(s) - \frac{1}{n-1} \sum_i \sum_{j \neq i} \int_{\underline{t}_j}^{t_j} s dQ_j(s) = \sum_i c_i.$$

Thus, by (48),

$$\begin{aligned}
\sum_i p_i(t) &= \sum_i t_i^0 Q_i(t_i^0) - \sum_i \int_{\underline{t}_i}^{t_i^0} s dQ_i(s) + \frac{1}{n-1} \sum_i \sum_{j \neq i} \int_{\underline{t}_j}^{\bar{t}_j} s \left(1 - F_j(s)\right) dQ_j(s) \\
&= \sum_i t_i^0 Q_i(t_i^0) - \sum_i \int_{\underline{t}_i}^{t_i^0} s dQ_i(s) + \sum_i \int_{\underline{t}_i}^{\bar{t}_i} s \left(1 - F_i(s)\right) dQ_i(s) \\
&= \sum_i \left(t_i^0 Q_i(t_i^0) - \int_{\underline{t}_i}^{t_i^0} s dQ_i(s) + \int_{\underline{t}_i}^{\bar{t}_i} s \left(1 - F_i(s)\right) dQ_i(s) \right).
\end{aligned}$$

Calculate the two integrals in the last line through integration by parts and then combine terms to obtain

$$\begin{aligned}
\sum_i p_i(t) &= \sum_i \left(\int_{\underline{t}_i}^{t_i^0} Q_i(s) ds - \int_{\underline{t}_i}^{\bar{t}_i} Q_i(s) \left(1 - F_i(s) - s f_i(s)\right) ds \right) \\
&= \sum_i \left(\int_{\underline{t}_i}^{t_i^0} Q_i(s) \left(1 - \left(1 - F_i(s) - s f_i(s)\right)\right) ds - \int_{t_i^0}^{\bar{t}_i} Q_i(s) \left(1 - F_i(s) - s f_i(s)\right) ds \right) \\
&= \sum_i \left(\int_{\underline{t}_i}^{t_i^0} Q_i(s) \left(s + \frac{F_i(s)}{f_i(s)} \right) f_i(s) ds + \int_{t_i^0}^{\bar{t}_i} Q_i(s) \left(s - \frac{1 - F_i(s)}{f_i(s)} \right) f_i(s) ds \right) \\
&= \langle Q : V(F) | ,
\end{aligned}$$

with the last line due to $t_i^0 = \tau(Q_i)$, (3), (7) and (9). That proves (19) and hence BB.

A.6 Proof of Lemma 4

$V(F)$ is well-ordered This follows from (8) coupled with the fact $\rho_+(F_i) \le \rho_-(F_i)$ due to (5) and (6).

$\alpha_* V(F) - V(G) + 1$ is well-ordered Plug (5), (6) and (20) into (8) to calculate

$$\begin{aligned}
& \alpha_* V_{i,+}(F))(t_i) - V_{i,+}(G)(t_i) + 1 \\
= \;& \alpha_* \cdot \frac{f_i(t_i)t_i - 1 + F_i(t_i)}{f_i(t_i)} - \frac{(f_i(t_i)/t_i)t_i - \alpha_i + G_i(t_i)}{f_i(t_i)} + 1 \\
= \;& \alpha_* t_i + \frac{\alpha_* F_i(t_i) - G_i(t_i)}{f_i(t_i)} - \frac{\alpha_* - \alpha_i}{f_i(t_i)},
\end{aligned} \tag{50}$$

with the middle line using the fact $G_i'(t_i) = f_i(t_i)/t_i$. Likewise,

$$\begin{aligned}
& \alpha_* V_{i,-}(F))(t_i) - V_{i,-}(G)(t_i) + 1 \\
= \;& \alpha_* \cdot \frac{f_i(t_i)t_i + F_i(t_i)}{f_i(t_i)} - \frac{(f_i(t_i)/t_i)t_i + G_i(t_i)}{f_i(t_i)} + 1 \\
= \;& \alpha_* t_i + \frac{\alpha_* F_i(t_i) - G_i(t_i)}{f_i(t_i)}.
\end{aligned} \tag{51}$$

By (22), $\alpha_* \ge \alpha_i$ for all i, thus $\alpha_* V(F) - V(G) + 1$ is well-ordered.

Proof of (26) Let $Q := (Q_i)_{i=1}^n \in \mathscr{Q}_{\text{mon}}$. Denote $t_i^0 := \tau(Q_i)$ for any i. By (3), (7) and (9),

$$\begin{aligned}
\langle Q : V(F)| &= \sum_i \left(\int_{t_i^0}^{\bar{t}_i} Q_i(t_i) \left(t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} \right) dF_i(t_i) + \int_{\underline{t}_i}^{t_i^0} Q_i(t_i)(t_i + \frac{F_i(t_i)}{f_i(t_i)}) dF_i(t_i) \right) \\
&= \sum_i \left(- \int_{t_i^0}^{\bar{t}_i} Q_i(t_i) d\left(t_i(1 - F_i(t_i))\right) + \int_{\underline{t}_i}^{t_i^0} Q_i(t_i) d\left(t_i F_i(t_i)\right) \right) \\
&= \sum_i \left(Q(t_i^0)t_i^0 + \int_{t_i^0}^{\bar{t}_i} t_i(1 - F_i(t_i)) dQ_i(t_i) - \int_{\underline{t}_i}^{t_i^0} t_i F_i(t_i) dQ_i(t_i) \right) \\
&\ge \sum_i \left(Q(t_i^0)t_i^0 + t_i^0 \left[\int_{t_i^0}^{\bar{t}_i} (1 - F_i(t_i)) dQ_i(t_i) - \int_{\underline{t}_i}^{t_i^0} F_i(t_i) dQ_i(t_i) \right] \right) \\
&= \sum_i \left(Q(t_i^0)t_i^0 + t_i^0 \left[\int_{\underline{t}_i}^{\bar{t}_i} Q_i(t_i) dF_i(t_i) - Q(t_i^0) \right] \right) \\
&= \sum_i t_i^0 \int_{T_i} Q_i(t_i) dF_i(t_i),
\end{aligned}$$

with the third and fourth equalities due to integration by parts, and the inequality due to Q_i being weakly increasing.

A.7 Proof of (34)

The following general observation on ironing (defined in (12) and (13)) implies (34).

Lemma 5 *For any two integrable functions φ and ϕ defined on T_i, if $\varphi \geq \phi$ on T_i then*

$$\sup \{t \in T_i : \overline{\varphi}(t) < 0\} \leq \inf \left\{t \in T_i : \overline{\phi}(t) > 0\right\}. \tag{52}$$

Proof Note from (12) and (13) that the left-hand side of (52) is equal to

$$\inf \left(\arg\min_{t \in T_i} H^{\varphi}\left(F_i(t)\right)\right),$$

and the right-hand side of (52) equal to

$$\sup \left(\arg\min_{t \in T_i} H^{\phi}\left(F_i(t)\right)\right).$$

By (13), for any $t' > t$ the difference $H^{\varphi}\left(F_i(t')\right) - H^{\varphi}\left(F_i(t)\right) = \int_t^{t'} \varphi(s) dF_i(s)$ increases when φ increases pointwise. Thus, with $\varphi \geq \phi$ on T_i, $\arg\min_{t \in T_i} H^{\varphi}\left(F_i(t)\right)$ is less than $\arg\min_{t \in T_i} H^{\phi}\left(F_i(t)\right)$ in strong-set order (Milgrom and Shannon [5]). Thus (52) follows. ■

A.8 Proof of (41)

Denote $\Delta := \phi(H) - \phi(L)$. Note that the density function is, for all $t \in [L, H]$,

$$f(t) = \frac{1}{\Delta}\left((t - (H - 1/m))^2 + 1/m^4\right).$$

By definition of α_*,

$$\begin{aligned}
\alpha_* &= \frac{1}{\Delta}\left(\frac{1}{2}(H^2 - L^2) - 2(H - 1/m)(H - L) + ((H - 1/m)^2 + 1/m^4)\ln(H/L)\right) \\
&= \frac{1}{\Delta}\left(H^2 \ln(H/L) - (H - L)\left(\frac{3}{2}H - \frac{1}{2}L\right)\right) + O(1/m) \\
&> \frac{1}{\Delta}H^2\left(\ln(H/L) - 3/2\right) + O(1/m),
\end{aligned}$$

with the inequality due to $H > L$. Plug into this the definitions of Δ to obtain

$$\begin{aligned}
\alpha_* - \frac{1}{H - 1/m} &> \frac{H^2\left(\ln(H/L) - 3/2\right)}{(H^3 - L^3)/3 - H(H^2 - L^2) + H^2(H - L)} - \frac{1}{H} + O(1/m) \\
&= \frac{H^2\left(\ln(H/L) - 3/2\right)}{(H - L)^3/3} - \frac{1}{H} + O(1/m) \\
&> \frac{1}{H}\left(3\left(\ln(H/L) - 3/2\right) - 1\right) + O(1/m) \\
&\geq O(1/m),
\end{aligned} \tag{53}$$

with the last line due to $\ln(H/L) \geq 11/6$. By definitions of G and α_*,

$$\begin{aligned}
G(H-1/m) - \alpha_* F(H-1/m) &= \int_L^{H-1/m} \frac{1}{s} dF(s) - F(H-1/m) \int_L^H \frac{1}{s} dF(s) \\
&= (F(H) - F(H-1/m)) \int_L^H \frac{1}{s} dF(s) - \int_{H-1/m}^H \frac{1}{s} dF(s) \\
&> (F(H) - F(H-1/m)) \left(\alpha_* - 1/(H-1/m)\right) \\
&> O(1/m^2) O(1/m) \\
&= O(1/m^3),
\end{aligned}$$

with the second last line due to Taylor's formula and (53). Plug the above-derived inequality and the fact $f(H-1/m) = 1/(\Delta m^4) = o(1/m^3)$ into the definition of Y_i to obtain

$$Y_i(H-1/m) = \alpha_*(H-1/m) - \left| \frac{O(1/m^3)}{o(1/m^3)} \right| = - \left| \frac{1}{O(1/m)} \right| \longrightarrow_m -\infty.$$