

# Hyperadditive Games and Applications to Networks or Matching Problems\*

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## Abstract

For the class of cooperative games with transferable utility, we introduce and study the notion of hyperadditivity, a new cohesiveness property weaker than convexity and stronger than superadditivity. It is first established that every hyperadditive game is balanced: we propose a formula allowing to compute some core allocations; and this leads to the definition of a single-valued solution (for hyperadditive games) satisfying the axioms of symmetry, dummy and core selection. This solution coincides with the Shapley value on the subclass of convex games. Furthermore, we prove that the bargaining set of a hyperadditive game always coincides with its core. It is shown that many well-known economic applications satisfy hyperadditivity. Our work extends (and gives a unifying explanation for) various results found in the literature on network games, assignment games and convex games. Some new results are also derived for these classes of games.

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# 1 Introduction

Cooperative games with transferable utility (TU games, for short) allow to model a wide array of economic problems where side payments between agents are possible. Their solution concepts include the *core*, which is the set of allocations that no coalition of players can improve upon, and the Shapley value, a single-valued solution characterized by a few natural axioms (see Shapley, 1953). Among other applications, these solution concepts have been used to describe (i) the reallocation of endowments in an economy [see for instance Shapley and Shubik (1969) or Wilson (1978)], (ii) bankruptcy and bargaining problems between economic agents [Gul (1989) or Montez (2014)], (iii) matching between firms and workers [Crawford and Knoer (1981) or Kelso and Crawford (1982)].

A distinguished class of TU games is the family of *convex games*, whose characteristic functions are supermodular. These convex games, which were first studied by Shapley (1971), exhibit some remarkable properties. For example, every convex game has a nonempty core (a property not guaranteed as soon as one drops convexity). Moreover, the Shapley value (in addition to its other desirable properties) occupies a central position in the core of convex games: it obtains as the average of all extreme core allocations. For TU games that are not convex, a major drawback of the Shapley value is that it generally does not fall in the core.

Another well-known family of TU games, which contains all convex games, is the class of superadditive games —see for instance Young (1985) and Solymosi (1999). However, it turns out that the core of a superadditive game is not always nonempty and, even if the core is nonempty, the Shapley value typically does not produce a core allocation in superadditive games (see footnote 10).

Most economic applications of TU games are superadditive (bargaining, matching, networks, production economies, voting, etc.). However, as pointed out above, superadditive games do not have the nice properties found in convex games. The main contribution of the present paper is to bridge this gap by introducing and studying a new class that is contained in the family of superadditive games. The interest of this new class lies in the facts that (a) it encompasses most of the aforementioned applications of TU games and (b) it meets many of the desirable properties satisfied by convex games. These games, which we call *hyperadditive games*, are formally defined in Section 2; and we then proceed to prove the points (a) and (b) mentioned above.

More specifically, the notion of hyperadditivity is defined using the concepts of reduced game (Davis and Maschler, 1965) and marginal game (Núñez and Rafels, 1998). Given a superadditive TU game with player set  $N$  and characteristic func-

tion  $v$ , call *player  $i$ 's marginal contribution* the quantity  $m_i = v(N) - v(N \setminus \{i\})$ . One can then define the reduced game with player set  $N \setminus \{i\}$  and characteristic function  $v^i$  such that  $v^i(S) = \max(v(S), v(S \cup \{i\}) - m_i)$ , for every subset  $S$  of this reduced player set. Call this particular reduced game  $(N \setminus \{i\}, v^i)$  the *marginal game* associated with  $i$ . In turn, one can compute the reduced game  $(N \setminus \{i, j\}, v^{ij})$  obtained by using the marginal contribution of some player  $j$  to the game  $(N \setminus \{i\}, v^i)$ , and so forth. A game is then called *hyperadditive* if these successive marginal games are all superadditive.

Just like convexity, one can check hyperadditivity for any given TU game. It is important to point out that hyperadditivity has a natural and intuitive economic interpretation in terms of *cohesiveness*: if agents are compensated one after the other (in the amount of their marginal contribution) then the remaining agents will have incentives to act together (as opposed to splitting into multiple subgroups). We argue that this cohesiveness property is satisfied in many problems relating to voting, networks, bargaining and matching.

We prove in Theorem 2-(a) that every convex game is hyperadditive, thus establishing that convexity is a stronger requirement than hyperadditivity. Moreover, we show that every hyperadditive game has a nonempty core (Theorem 3). Precisely, we define a new single-valued solution concept (the *average marginal value*) which always falls in the core of hyperadditive games.<sup>1</sup> Interestingly, it is shown in Theorem 2-(b) that this new value coincides with the Shapley value on the set of convex games. Importantly, unlike the Shapley value, the average marginal value is a core selection on the set of hyperadditive games.

These findings mean that many properties exhibited by the Shapley value on the class of convex games carry through to the wider class of hyperadditive games if one extends the restriction of the Shapley value (to the set of convex games) by using our average marginal value. We also derive new results on the bargaining set of hyperadditive games. Shapley, Maschler and Peleg (1971) showed that the bargaining set and the core coincide in convex games. The same result has been shown for veto games (Bahel, 2016) and assignment games (Solymosi, 1999). By showing that the core and the bargaining set coincide in every hyperadditive game (see Theorem 4), the present paper provides a unifying explanation for these three seemingly unrelated results.

Proof of applicability of our results is given in Sections 4-5. We show in Section

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<sup>1</sup>In the context of bargaining, Vidal-Puga (2004) introduced a mechanism, called the selective value, which coincides with the average marginal value. The author characterized the selective value as the unique subgame perfect outcome of a bargaining game with commitment.

4 that veto games, shortest path games and minimum cost arborescence games (including minimum cost spanning tree problems) are all hyperadditive. In Section 5 we focus on assignment games and *quasi-hyperadditivity*, a slightly weaker requirement which turns out to be sufficient to guarantee many properties exhibited by hyperadditive games (see Theorem 9). As pointed by Núñez and Rafels (2003), a marginal game of an assignment game is typically not an assignment game (it is not even superadditive in general). Regardless, we show in the proof of Proposition 10 that it has the same core (up to a geometric translation) as some suitably constructed assignment game. Our analysis of assignment games through the lens of hyperadditivity allows to understand many results found in this literature.

In addition to providing a unifying explanation for many well-known results from different strands of literature, our work also highlights some new results as corollaries to the propositions of Sections 4-5. For instance, the average marginal value studied in this paper is a new, stable and symmetric solution for assignment games, shortest path and minimum cost arborescence (as well as minimum cost spanning tree) problems.<sup>2</sup> Moreover, the core of every minimum cost arborescence problem coincides with its bargaining set. All proofs are relegated to the appendix.

## 2 Hyperadditive games: definition

### 2.1 TU games and superadditivity

Recall that a *cooperative game with transferable utility* (or *TU game*, for short) is a pair  $G \equiv (N, v)$ , where  $N$  is the finite set of players (with  $n \equiv |N| \geq 1$ ) and  $v : 2^N \rightarrow \mathbb{R}_+$  is a function such that  $v(\emptyset) = 0$ . The mapping  $v$  is called the *characteristic function* of  $G$ ; and, for any coalition  $S \in 2^N$ , the quantity  $v(S)$  is the *worth* of  $S$ . For expositional convenience, we restrict attention to nonnegative characteristic functions, that is to say,  $v(S) \geq 0, \forall S \subseteq N$ . Here and throughout the paper, the symbol  $\subseteq (\subset)$  stands for weak (strict) set inclusion.

For any  $S \subseteq N$  and  $x \in \mathbb{R}^N$ , define  $x_S \equiv \sum_{i \in S} x_i$  (with the convention that  $x_\emptyset = 0$ ). To ease on notation, we often write  $i$  instead of  $\{i\}$ ,  $ij$  instead of  $\{i, j\}$ , and so forth. Finally, let  $\Pi(N)$  denote the set of permutations of  $N$ .

A TU game  $(N, v)$  is *superadditive* if we have  $v(S \cup T) \geq v(S) + v(T)$ , for any

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<sup>2</sup>Minimum cost arborescence problems were first modeled by Dutta and Mishra (2012) as cooperative games, and further studied by Bahel and Trudeau (2017). These problems generalize the standard minimum cost spanning tree problems —see for instance Bird (1976) or Bergantiños and Vidal-Puga (2007).

$S, T \in 2^N$  such that  $S \cap T = \emptyset$ . This property, which is naturally satisfied in most economic applications, means that it is beneficial for disjoint coalitions of players to merge. We denote by  $\mathcal{S}^N$  the class of superadditive TU games (with player set  $N$ ) satisfying  $v(S) \geq 0$ ,  $\forall S \subseteq N$ . It is easy to see that every  $G \in \mathcal{S}^N$  is *monotonic*, that is,  $v(S) \leq v(S')$  for all  $S, S' \in 2^N$  such that  $S \subseteq S'$ .

## 2.2 Marginal games and a new value for TU games

Given a TU game  $G = (N, v)$  and  $i \in N$ , call  $i$ 's *marginal contribution* the quantity  $m_i(G) \equiv v(N) - v(N \setminus i)$  and write  $m(G) = (m_i(G))_{i \in N}$ . Obviously, for all  $i \in N$ , we have  $m_i(G) \geq 0$ , whenever  $G$  is monotonic.

### 2.2.1 Marginal games of order $k$

We call  $i$ -*marginal game* of  $G = (N, v)$  the TU game  $G^i = (N \setminus i, v^i)$ , which is defined by  $v^i(S) = \max(v(S), v(S \cup i) - m_i(G))$ , for any  $S \subseteq N \setminus i$ . In words, if we assign to player  $i$  her marginal contribution,  $G^i$  is the reduced game obtained by giving to every subgroup of remaining agents the option to use  $i$  to their benefit in exchange for a compensation equal to  $m_i(G)$ .

For any (ordered) sequence of distinct players  $p = \{i_1, i_2, \dots, i_{k-1}, i_k\}$  such that  $2 \leq k \leq n-1$ , write  $p \setminus i_k \equiv \{i_1, i_2, \dots, i_{k-1}\}$ . Then one can recursively define the ( $k$ th-order)  $p$ -*marginal game* of  $G$  as the  $i_k$ -*marginal game* of  $G^{p \setminus i_k}$ . We use the notation  $G^p = (N \setminus p, v^p)$  to refer to the  $p$ -*marginal game* of  $G$  (with the convention  $G^\emptyset \equiv G$ ).

The lemma below follows from the definition of marginal games by using a simple induction argument (the proof is omitted).

**Lemma 1.** *Consider a TU game  $G = (N, v)$  and an ordered sequence  $p = \{i_1, \dots, i_k\} \subsetneq N$ , where  $k \geq 1$ . Then for all  $S \subseteq N \setminus p$ , we have*

$$v^p(S) = \max_{T \subseteq p} [v(S \cup T) - \sum_{i_l \in T} m_{i_l}(G^{\{i_1, \dots, i_l\} \setminus i_l})].$$

We now provide two examples illustrating the computation of marginal games.

**Example 1.** *Consider the TU game  $G = (N, v)$  such that  $N = \{1, 2, 3, 4\}$  and,*

$$\forall S \subseteq N, \quad v(S) = \begin{cases} 20, & \text{if } S = N; \\ 14, & \text{if } |S| = 3; \\ 6, & \text{if } |S| = 2; \\ 0, & \text{if } |S| = 1. \end{cases}$$

It is readily checked that  $G$  is superadditive. For every  $i \in N$ , we have  $m_i(G) = v(N) - v(N \setminus i) = 20 - 14 = 6$ ; and therefore the  $i$ -marginal game  $G^i = (N \setminus i, v^i)$

is defined as follows: for any  $S \subseteq N \setminus i$ ,  $v^i(S) = \begin{cases} 14, & \text{if } S = N \setminus i; \\ 8, & \text{if } |S| = 2; \\ 0, & \text{if } |S| = 1. \end{cases}$

Indeed, note for instance that we have:  $v^4(12) = \max(v(12), v(124) - m_4(G)) = \max(6, 14 - 6) = 14 - 6 = 8$ .

In addition, for any  $j \in N \setminus i$ , remark that  $m_j(G^i) = v^i(N \setminus i) - v^i(N \setminus ij) = 14 - 8 = 6$ ; and hence the second-order  $ij$ -marginal game of  $G$ ,  $G^{ij} = (N \setminus ij, v^{ij})$ ,

is characterized by:  $\forall S \subseteq N \setminus ij$ ,  $v^{ij}(S) = \begin{cases} 8, & \text{if } S = N \setminus ij; \\ 2, & \text{if } |S| = 1. \end{cases}$

Notice in particular that we have  $v^{43}(1) = v^{43}(2) = \max(0, 8 - 6) = 2$ .

Given that every  $G \in \mathcal{S}^N$  is monotonic, it is straightforward to see that all  $p$ -marginal games of  $G$  are also monotonic. However, the superadditivity of  $G$  does not necessarily carry through to its marginal games (as illustrated by the following example).

**Example 2.** Let  $G = (N, v)$  be the so-called “5-player majority game”, which is characterized by  $N = \{1, 2, 3, 4, 5\}$  and, for any  $S \subseteq N$ ,  $v(S) = \begin{cases} 1, & \text{if } |S| \geq 3; \\ 0, & \text{otherwise.} \end{cases}$

One can easily see that  $G \in \mathcal{S}^N$ . For every  $i \in N$ ,  $m_i(G) = v(N) - v(N \setminus i) = 0$ ; and therefore the  $i$ -marginal game  $G^i = (N \setminus i, v^i)$  is defined by: for any  $S \subseteq N \setminus i$ ,

$v^i(S) = \begin{cases} 1, & \text{if } |S| \geq 2; \\ 0, & \text{otherwise.} \end{cases}$

Obviously,  $G^i$  is not superadditive since  $v^i(S) + v^i(T) = 2 > 1 = v^i(S \cup T)$ , for any disjoint  $S, T \subseteq N \setminus i$  s.t.  $|S| = |T| = 2$ .

The following observation obtains directly from the definition of marginal contributions and marginal games.

**Remark 1.** Let  $G = (N, v)$  and  $p = \{i_1, i_2, \dots, i_k\}$ . Then notice from the definition of  $p$ -marginal games that

$$v^p(N \setminus p) = v^{p \setminus i_k}(N \setminus p) = v(N) - m_{i_1}(G) - m_{i_2}(G^{i_1}) - \dots - m_{i_k}(G^{p \setminus i_k}).$$

This remark will often be recalled in what follows.

### 2.2.2 The average marginal value

Recall that a *value* (for TU games) is a mapping  $\varphi$ , which associates with every TU game  $G = (N, v)$  a payoff profile  $\varphi(G) \in \mathbb{R}^N$  that satisfies the efficiency

condition  $\sum_{i \in N} \varphi_i(G) = v(N)$ . As a classic example, the Shapley value is defined by:  $\forall i \in N, \varphi_i^{Sh}(G) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} [v(p_\pi(i) \cup i) - v(p_\pi(i))]$ , where  $\Pi(N)$  is the set of permutations of  $N$  and  $p_\pi(i) \equiv \{j \in N : \pi(j) < \pi(i)\}$  stands for the ordered set of  $i$ 's predecessors under the permutation  $\pi$ . The following definition introduces a new value for TU games.

**Definition 1.** (a) Given a permutation  $\pi \in \Pi(N)$ , call  $\pi$ -**marginal value** the mapping which, to every TU game  $G = (N, v)$ , assigns the vector of payoffs  $\varphi^\pi(G)$  such that

$$\varphi_i^\pi(G) = m_i(G^{p_\pi(i)}), \forall i \in N. \quad (1)$$

(b) Define the **average marginal value**  $\varphi^{AM}$  by: for any  $G = (N, v)$ ,

$$\varphi_i^{AM}(G) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} m_i(G^{p_\pi(i)}), \forall i \in N. \quad (2)$$

Thus, under the  $\pi$ -marginal value, each player  $i$  receives a payoff equal to her marginal contribution to  $G^{p_\pi(i)}$ , the  $p_\pi(i)$ -marginal game of  $G$ . The value  $\varphi^{AM}$  then obtains as the average of all  $\pi$ -marginal values (for all permutations  $\pi \in \Pi(N)$ ). Note from Remark 1 that all  $\varphi^\pi(G)$  satisfy the efficiency condition  $\sum_{i \in N} \varphi_i(G) = v(N)$  [hence,  $\varphi^{AM}(G)$  also meets efficiency].

As an illustration, letting  $\pi = \overline{1234}$  be the natural ordering of the players in Example 1, one gets  $\varphi_i^{\overline{1234}}(G) = (6, 6, 6, 2)$ ; and it is also easy to see that  $\varphi_i^{AM}(G) = (5, 5, 5, 5)$ . Further examples are provided in Sections 4 and 5.

Let us now introduce some well-known properties that a value may exhibit. We say that a value  $\varphi$  for TU games is:

- *additive* if  $\varphi(N, v_1 + v_2) = \varphi(N, v_1) + \varphi(N, v_2)$  for all  $(N, v_1)$  and  $(N, v_2)$ ;
- *dummy* if  $\varphi_i(N, v) = 0$  whenever  $v(S \cup i) = v(S), \forall S \subseteq N \setminus i$ ;
- *symmetric* if  $\varphi_i(N, v) = \varphi_j(G)$  whenever  $v(S \cup i) = v(S \cup j), \forall S \subseteq N \setminus ij$ .

It is well known that the Shapley value is characterized by additivity, dummy and symmetry (see Shapley, 1953). The following result, which easily follows from the definition of  $\varphi^{AM}$ , states that the average marginal value satisfies two of these three properties (the proof is omitted).

**Theorem 1.** *The average marginal value  $\varphi^{AM}$  is dummy and symmetric.*

However, in contrast with the Shapley value,  $\varphi^{AM}$  does not satisfy additivity. Indeed, the marginal game of the sum of two TU games is generally not the sum of their marginal games. If  $G_1 = (N, v_1)$ ,  $G_2 = (N, v_2)$ , and  $G = (N, v_1 + v_2)$  then we

typically have  $G^i \neq G_1^i + G_2^i$  (given  $i \in N$ ). Using the notion of marginal games, we introduce a new class of TU games, which is a proper subset of  $\mathcal{S}^N$ .

**Definition 2.** Let  $G \in \mathcal{S}^N$  and suppose that  $n = |N| \geq 2$ . We say that  $G$  is *hyperadditive* if we have  $G^p \in \mathcal{S}^{N \setminus p}$ , for every  $p = \{i_1, \dots, i_k\} \subsetneq N$ .

In words, a TU game  $G$  is hyperadditive if every one of its marginal games (of order  $k = 1, 2, \dots, n-1$ ) is superadditive. We denote by  $\mathcal{H}^N$  the set containing all hyperadditive games associated with the player set  $N$ . As an illustration, the TU game  $G$  of Example 1 is hyperadditive: it is straightforward to see that all marginal games (of order  $k = 1, 2$ ) described are superadditive.

Remark that every hyperadditive game is superadditive by definition, but the reverse inclusion does not hold (that is,  $\mathcal{H}^N \subsetneq \mathcal{S}^N$ ). For instance, Example 2 shows that the 5-player majority game, which is superadditive, is not hyperadditive.

We now examine the relationship between  $\mathcal{H}^N$  and another well-known class of TU games. Recall that a TU game  $G = (N, v)$  is *convex* if:

$$v(S \cup i) - v(S) \leq v(T \cup i) - v(T), \quad \forall i \in N, \quad \forall S \subseteq T \subseteq N \setminus i. \quad (3)$$

Thus, in a convex game, the marginal contribution of every player increases (weakly) with the size of the coalition of players she joins. This convexity property in TU games is often referred to as “snowballing effect” –see for instance Shapley (1971). We show in the following theorem that all convex games are hyperadditive. Moreover, the average marginal value coincides with the Shapley value on the set of convex TU games. Denote by  $\mathcal{C}^N$  the set of convex TU games with player set  $N$ .

**Theorem 2.** Fix a player set  $N$  s.t.  $|N| \geq 3$ . Then we have:

- (a)  $\mathcal{C}^N \subsetneq \mathcal{H}^N$ ;
- (b)  $\varphi^{Sh}(G) = \varphi^{AM}(G), \quad \forall G \in \mathcal{C}^N$ .

Our average marginal value thus coincides with the Shapley value on the set of convex games. Note that Theorem 2 assumes  $|N| \geq 3$ . Indeed, in the case where  $|N| < 3$ , it is easy to see that  $\mathcal{C}^N = \mathcal{S}^N = \mathcal{H}^N$ . To see why the inclusion stated in Theorem 2-(a) is strict, observe that the TU game of Example 1 is hyperadditive (as argued earlier) but not convex:  $v(N) - v(N \setminus i) = 20 - 14 = 6 < 8 = 14 - 6 = v(ijk) - v(jk)$ , for any distinct  $i, j, k \in N$ .



### 3 Core, bargaining set and hyperadditivity

This section examines two widely used set-valued solution concepts, namely the core and the bargaining set, in the specific case of hyperadditive games. Throughout, we consider a fixed player set  $N$  such that  $|N| = n \geq 2$ .

#### 3.1 Core

Given a TU game  $(N, v)$ , an *imputation* is a tuple  $x \in \mathbb{R}^N$  satisfying  $x_N = v(N)$  (efficiency) and  $x_i \geq v(i)$  (individual rationality),<sup>3</sup> for any  $i \in N$ . An imputation  $x$  is called *stable* if  $x_S \geq v(S)$ , for any  $\emptyset \neq S \subsetneq N$ . The *core* of  $(N, v)$  is the set of stable imputations, that is to say,

$$\text{Core}(N, v) \equiv \{x \in \mathbb{R}^N \mid x_N = v(N) \text{ and } x_S \geq v(S), \text{ for all } \emptyset \neq S \subsetneq N\}.$$

A TU game is called *balanced* if its core is nonempty. The notion of the core can be traced back to Edgeworth (1881) and was formally introduced by Gillies (1953) as a solution concept for TU games. It has been established by Shapley (1971) that every convex game is balanced. On the other hand, it is well known that superadditive games are in general not balanced.<sup>4</sup>

**Definition 3.** We say that a value  $\varphi$  satisfies Core selection for hyperadditive games (CSH) if  $\varphi(G) \in \text{Core}(G)$ , for every  $G \in \mathcal{H}^N$ .

We show next that all hyperadditive games are balanced, thus extending to the class  $\mathcal{H}^N$  the well-known balancedness property of convex games.

**Theorem 3.** For all  $G \in \mathcal{H}^N$  we have:

- (a)  $\varphi^\pi(G)$  is an extreme point of  $\text{Core}(G)$ , for all  $\pi \in \Pi(N)$ ;
- (b)  $\varphi_i^{AM}(G) \in \text{Core}(G)$ .

It thus comes from Theorem 3-(b) that the average marginal value  $\varphi_i^{AM}$  meets CSH—in addition to symmetry and dummy (from Theorem 1).<sup>5</sup> By contrast,

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<sup>3</sup>Observe that the set of imputations is always nonempty for any  $G \in \mathcal{S}^N$ .

<sup>4</sup>As an illustration, the 5-player majority game of Example 2 has an empty core. Indeed, a stable imputation  $x$  requires that  $x_S \geq 1$ , for any  $S$  s.t.  $|S| = 4$ . But given that each player  $i$  belongs to exactly 4 (out of a total of 5) coalitions of size  $|S| = 4$ , this implies that  $4 \sum_{i=1}^5 x_i \geq 5$  and thus contradicts efficiency ( $\sum_{i=1}^5 x_i = 1$ ).

<sup>5</sup>Tijs *et al.* (2011) show (from of a theoretical standpoint) the existence of “lexicographical allocations” in the core of every balanced game, without providing a formula or computation procedure. By contrast, our formula giving  $\varphi^\pi(G)$  allows to define the new value  $\varphi^{AM}$ , which can

the Shapley value does not satisfy CSH (see for instance footnote 10). Thus, in hyperadditive games, the average marginal value has the advantage (over the Shapley value) that it cannot be blocked by some coalition.

### 3.2 Bargaining set

Suppose that the imputation set is nonempty (as is the case whenever  $G \in \mathcal{S}^N$ ); and let  $x$  be an imputation of  $G$ . In addition, let  $i, j \in N$  be two distinct players of  $G$ . We say that a pair  $(S, y)$  is an *objection* of  $i$  against  $j$  at  $x$  if:

$$i \in S \subseteq N \setminus j; \quad (4)$$

$$y \in \mathbb{R}^S, \text{ with } y_S = v(S); \quad (5)$$

$$y_k > x_k, \forall k \in S. \quad (6)$$

Furthermore, a pair  $(T, z)$  will be called a *counter-objection* of  $j$  to the objection  $(S, y)$  of  $i$  against  $j$  at  $x$  if:

$$j \in T \subseteq N \setminus i; \quad (7)$$

$$z \in \mathbb{R}^T \text{ with } z_T = v(T); \quad (8)$$

$$z_k \geq y_k, \forall k \in S \cap T; \text{ and } z_k \geq x_k, \forall k \in T \setminus S. \quad (9)$$

The *bargaining set* of  $G$ ,  $\mathcal{M}(G)$ , is then defined as the set of all imputations at which no player  $i$  has an objection (against some  $j$ ) that is not met by a counter-objection (of  $j$ ).

Note from the combination of (5) and (6) that there is no possible objection (against any player  $i$ ) at a stable imputation  $x$ .<sup>6</sup> As a result, the core is always a subset of the bargaining set. This set-valued solution concept was introduced and studied by Aumann and Maschler (1964) and Davis and Maschler (1967).

It is known from Maschler, Peleg and Shapley (1971) that the core of every convex games coincides with its bargaining set. Similar results exist in the literature for veto games and some specific classes of network games (which are all hyperadditive, as will be shown in the next Section). Remarkably, we are able to obtain these respective results (and some new ones) as corollaries to the following theorem.

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be computed for any TU game and turns out to be a core selection on the set of hyperadditive games. Also note that  $\varphi^\pi(G)$  need not be (a lexicographic vertex) in the core if  $G$  is not hyperadditive.

<sup>6</sup> An objection  $(S, y)$  would imply that  $y_S = v(S) > x_S$ , which contradicts the stability of  $x$ .

**Theorem 4.** *For every hyperadditive game  $G \in \mathcal{H}^N$ , we have*

$$\text{Core}(G) = \mathcal{M}(G).$$

Theorem 4 states that the core coincides with the bargaining set in all hyperadditive games. It is therefore a robust set-valued solution concept; for any allocation not in the core, there exists a valid objection which cannot be met by a counter-objection. Note that the aforementioned result by Maschler, Peleg and Shapley (1971) now obtains as a corollary of Theorems 2 and 4: since convex games are hyperadditive, their core and bargaining set must coincide.

## 4 Veto and network games

As shown earlier, all convex games are hyperadditive. In this section we argue that many applications of TU games (which are typically not convex) also satisfy hyperadditivity.

### 4.1 Veto games

Many economic problems involve some distinguished players who are essential for any coalition to achieve a surplus. This is the case for instance in committee voting with veto power (e.g., the UN Security Council), bankruptcy problems with big claimants (Potters *et al.*, 1989) or distribution of profits with some monopolized production factors (Chetty *et al.*, 1976).

**Definition 4.**

*We say that a TU monotonic game  $G = (N, v)$  is a **veto game** if there exists a player  $i \in N$  such that  $v(S) = 0$ , for any  $i \notin S$ . Any such player  $i$  is called a **veto player** of  $G$ .*

Note from Definition 4 that a game  $G$  may exhibit several veto players. Denoting by  $T^*(G)$  the set of veto players for any game  $G$ , one can see that  $G$  is a veto game if and only if  $T^*(G) \neq \emptyset$ .

**Proposition 5.** *All veto games are hyperadditive, that is, if  $G = (N, v)$  is a veto game then  $G \in \mathcal{H}^N$ .*

Recalling Theorem 2-(b) and Theorem 4, one thus obtains the result below as an immediate corollary to Proposition 5.

**Corollary 1.** *Consider a veto game  $G = (N, v)$ . Then we have*

$$\varphi^{AM}(G) \in \text{Core}(G) = \mathcal{M}(G).$$

It is readily checked that big-boss games and clan games, introduced respectively by Muto *et al.* (1988) and Potters *et al.* (1989), are examples of veto games.<sup>7</sup> Therefore, our Proposition 5 and Corollary 1 also apply to big-boss games and clan games.

## 4.2 Network applications

It will be shown in this part that many well-known network games satisfy hyperadditivity. In order to do so, let us first extend the definition of hyperadditivity (given in Section 2 for value games) to cost games.

A *cooperative game with transferable cost* (or *TC game*, for short) is a pair  $G \equiv (N, c)$ , where  $N$  is the finite set of players and  $c : 2^N \rightarrow \mathbb{R}$  is a function such that  $c(\emptyset) = 0$ . The mapping  $c$  is called the *characteristic cost function* of  $G$ ; and, for any coalition  $S \in 2^N$ , the amount  $c(S)$  is the *cost* of  $S$ .

A TC game  $(N, c)$  is *subadditive* if we have  $c(S \cup T) \leq c(S) + c(T)$ , for any disjoint  $S, T \in 2^N$ . We denote by  $\underline{\mathcal{S}}^N$  the class of subadditive TC games with player set  $N$  satisfying the property  $c(S) \geq 0$ ,  $\forall S \subseteq N$ . Note that the core of a TC game  $(N, c)$  is defined by  $\text{Core}(N, c) \equiv \{x \in \mathbb{R}^N \mid x_N = c(N) \text{ and } x_S \leq c(S), \text{ for all } \emptyset \neq S \subsetneq N\}$ . Likewise, the notions of objection and counter-objection are easily defined by replacing in (4)-(9) the respective symbols  $v, >, \geq$  with  $c, <, \leq$  (everything else unchanged); and the bargaining set of  $\mathcal{M}(N, c)$  is then the set of cost imputations of  $(N, c)$  at which no player has an objection that is not met by a counter-objection.

For any  $(N, c) \in \underline{\mathcal{S}}^N$ , let us define the associated value game  $(N, v^c)$  by

$$v^c(S) \equiv \sum_{i \in S} c(i) - c(S) \geq 0, \quad \forall S \in 2^N. \quad (10)$$

That is to say,  $v^c(S)$  represents the total savings made by the members of  $S$  when using the whole coalition instead of their respective individual connections. It is straightforward to see that  $(N, v^c) \in \mathcal{S}^N$ , since  $(N, c) \in \underline{\mathcal{S}}^N$ .

**Definition 5.** *Call a TC game  $(N, c) \in \underline{\mathcal{S}}^N$  **cost-hyperadditive** if  $(N, v^c)$  is hyperadditive. In addition, let  $\underline{\mathcal{H}}^N$  be the set of cost-hyperadditive games with player set  $N$ .*

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<sup>7</sup>In essence, clan games are veto games such that, for every coalition  $S$  containing all veto players, the worth  $v(S)$  and the marginal contributions of outsiders sum up to a number less than  $v(N)$ . A big-boss game is then a clan game that exhibits a single veto player.

Using the same reasoning, one can define  $\underline{\varphi}^{AM}$ , the average marginal value for TC games, by:  $\underline{\varphi}_i^{AM}(N, c) \equiv c(i) - \varphi_i^{AM}(N, v^c)$ , for every TC game  $(N, c)$  and every  $i \in N$ . Likewise, we will have  $\underline{\varphi}_i^\pi(N, c) \equiv c(i) - \varphi_i^\pi(N, v^c)$ , for all  $\pi \in \Pi(N)$ . The following result then easily obtains as the counterpart to both Theorems 3 and 4 (which were stated for value games).

**Theorem 6.** *For all  $(N, c) \in \underline{\mathcal{H}}^N$ , we have:*

$$\underline{\varphi}^{AM}(N, c) \in \text{Core}(N, c) = \mathcal{M}(N, c).$$

The proof of Theorem 6 is omitted (it is a straightforward adaptation of the proofs of Theorem 3 and Theorem 4). To illustrate some interesting applications of this result, let us now give examples of standard network problems that are hyperadditive TC games.

#### 4.2.1 Shortest path problems

This subsection examines network problems where the cost on every arc is linear in the flow crossing it. The resulting cost sharing problem is studied in Rosenthal (2013) and Bahel and Trudeau (2014). Let us give a formal definition as follows.

Consider a fixed point  $s$  from which agents (residing at various locations) need to ship their respective demands of some homogeneous goods —  $s$  is called *the source*. A *Shortest Path Problem* (SPP) is a tuple  $P = (N, \alpha, x)$ , where (i)  $N$  is the set of agents (or nodes) that need to connect to the source  $s$ ; (ii)  $\alpha = \{\alpha(i, j) | i, j \in N \cup \{s\}, i \neq j\}$  is a collection (of nonnegative numbers) giving the **constant unit cost** of shipping demands through every arc  $(i, j)$  s.t.  $i \neq j$ ; (iii)  $x \in \mathbb{N}^N$  is the demand profile: each agent  $i$  has  $x_i$  units of demand to ship from the source to her location.

For any  $i \in N$ , we call *path* (of length  $K$ ) to  $i$  any ordered sequence  $p \equiv \{p_k\}_{k=0, \dots, K}$  such that: (i)  $p_k \in N \cup \{s\}$ , for  $k = 0, 1, \dots, K$ ; (ii)  $p_0 = s$  and  $p_K = i$ ; (iii)  $p_k \neq p_{k'}$  for any distinct  $k, k'$ . Given  $P = (N, \alpha, x)$ , one can extend the function  $\alpha$  to paths as follows: for any path  $p$  (of length  $K$ ) to  $i$ ,

$$\alpha(p) = \sum_{k=1}^K \alpha(p_{k-1}, p_k).$$

In words,  $\alpha(p)$  is the cost of shipping one unit from the source to agent  $i$  via the path  $p$ . Denoting by  $\mathcal{P}(i)$  the set of paths to  $i \in N$ , one can then define the TC game induced by  $P = (N, \alpha, x)$  as  $(N, c_P)$ , which is characterized by

$$c_P(S) \equiv \min \left\{ \sum_{i \in S} x_i \alpha(p^i) \mid p^i \in \mathcal{P}(i) \text{ and } p^i \subseteq S, \forall i \in S \right\}. \quad (11)$$

Equation (11) gives the lowest possible cost of shipping (from the source) the respective demands of the members of  $S$  *when using only the connections available in  $S$* . Given the linear cost structure of an SPP, the problem stated in (11) is solved by finding the cost-minimizing path (or *shortest path*) to every member of  $S$ . For every  $j \in N$ , denote by  $e^j \in \mathbb{R}^N$  the vector of demands characterized by  $e_j^j = 1$  and  $e_i^j = 0$ , if  $i \in N \setminus j$ . Let  $A, B \subseteq \mathbb{R}^N$  and  $\lambda \in \mathbb{R}$ . We use the following conventions:  $A + B \equiv \{a + b \mid a \in A \text{ and } b \in B\}$ ;  $\lambda \cdot A \equiv \{\lambda a \mid a \in A\}$ . Finally, call *elementary SPP* any problem  $P^j \equiv (N, \alpha, e^j)$  where  $j$  demands one unit and all other agents have null demands. One can then write the following result.

**Lemma 2.** *Given the problem  $P = (N, c, x)$ , we have*

$$\sum_{j \in N} x_j \cdot \text{Core}(P^j) \subseteq \text{Core}(P).$$

Lemma 2 says that, in order to find a core allocation in any problem  $P$ , it suffices to find core allocation in all elementary problems  $P^j$ .

**Example 3.** *Consider the SPP given by  $P = (N, \alpha, x)$ , where  $N = \{1, 2, 3\}$ ,  $x = (2, 1, 1)$  and the cost structure  $\alpha$  is depicted by Figure 1, with  $\alpha_{ij} = \alpha_{ji}$ , for  $i, j \in N \cup s$ . For example, we have  $\alpha(s, 1) = 120$ ,  $\alpha(3, 1) = \alpha(1, 3) = 60$  and  $\alpha(2, 1) = \alpha(1, 2) = 0$ .*

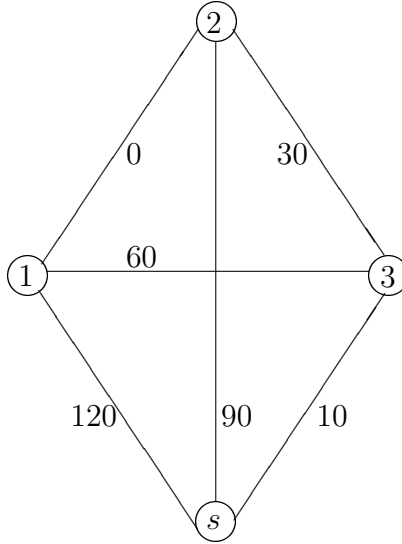


Figure 1: A three-agent SPP

*In particular, the shortest path to agent 1 is  $p^1 = \overline{s, 3, 2, 1}$ , with cost  $\alpha(p^1) = 10 + 30 + 0 = 40$ . Considering the elementary problem  $P^1$ , one easily gets the associated TC game  $(N, c_P)$ , defined by  $c_{P^1}(1) = 120$ ,  $c_{P^1}(2) = 0$ ,  $c_{P^1}(3) = 0$ ;*

$c_{P^1}(12) = 90$ ;  $c_{P^1}(13) = 70$ ,  $c_{P^1}(23) = 0$ ;  $c_{P^1}(N) = 40$ . Using (10), one then obtains the associated TU game:  $v^{c_{P^1}}(1) = v^{c_{P^1}}(2) = v^{c_{P^1}}(3) = v^{c_{P^1}}(23) = 0$ ;  $v^{c_{P^1}}(12) = 30$ ,  $v^{c_{P^1}}(13) = 50$ , and  $v^{c_{P^1}}(N) = 80$ .

In Example 3, it is straightforward to see that  $c_{P^1}$  is cost-hyperadditive, since  $v^{c_{P^1}}$  is hyperadditive. The following result shows that this is not a coincidence: cost-hyperadditivity holds for all elementary SPP.

**Proposition 7.** *Let  $P = (N, \alpha, x)$  be an SPP. Then for all  $j \in N$ , the TC game  $(N, c_{P^j})$  associated with  $P^j$  is cost-hyperadditive, that is to say,  $(N, c_{P^j}) \in \underline{\mathcal{H}}^N$ .*

Interestingly, Proposition 7 means that our results of Sections 2 and 3 apply to elementary shortest path problems. By exploiting Lemma 2, one can then define a value which picks a core allocation in every SPP, as stated by the following result.

**Corollary 2.** *Let  $P = (N, \alpha, x)$  be an SPP. Then we have*

$$\underline{\varphi}^*(N, c_{P^j}) \equiv \sum_{j \in N} x_j \underline{\varphi}^{AM}(N, c_{P^j}) \in \text{Core}(N, c_P) \neq \emptyset.$$

Therefore, the notion of cost-hyperadditivity explains the known result that the core of an SPP is always nonempty —see Rosenthal (2013) and Bahel and Trudeau (2014). In Example 3, the imputations  $\underline{\varphi}^\pi$  in the respective elementary problems  $P^j$  are given by the following table. Recalling that  $x = (2, 1, 1)$  and using the formula

Order $\pi$	$\underline{\varphi}^\pi(P^1)$	$\underline{\varphi}^\pi(P^2)$	$\underline{\varphi}^\pi(P^3)$
$\overline{123}$	(40,0,0)	(0,40,0)	(0,0,10)
$\overline{132}$	(40,0,0)	(0,90,-50)	(0,0,10)
$\overline{213}$	(70,-30,0)	(0,40,0)	(0,0,10)
$\overline{231}$	(120,-30,-50)	(0,40,0)	(0,0,10)
$\overline{312}$	(90,0,-50)	(0,90,-50)	(0,0,10)
$\overline{321}$	(120,-30,-50)	(0,90,-50)	(0,0,10)

Table 1: Computation of  $\underline{\varphi}^\pi$  in  $P^j$

given in Corollary 2, it thus follows from Table 1 that  $\underline{\varphi}^*(N, c_{P^j}) = (160, 35, -65)$ .<sup>8</sup> The reader can easily check that  $\underline{\varphi}^*(N, c_{P^j})$  above is a stable cost imputation.

<sup>8</sup>Note the negative cost share of player 3, that is,  $\underline{\varphi}_3^*(N, c_{P^j}) = -65$ . Indeed in network problems, it makes sense (and is sometimes necessary for stability) to award a subsidy to some players (who help others connect to the source at a lower cost).

### 4.2.2 Minimum cost arborescence problems

In a minimum cost arborescence problem, the cost of using an edge does not vary with the number of agents who connect to the source  $s$  through that edge. Let  $N = \{1, \dots, n\}$  be the set of agents needing to connect to the source  $s$ ; and define  $N_s \equiv N \cup \{s\}$ . Denote by  $(i, j)$  the directed edge from  $i \in N_0$  to  $j \in N$ , and by  $\gamma_{ij} \geq 0$  its cost. Let  $\mathbb{E} = \{(i, j) \in N_s \times N \mid i \neq j\}$  be the set of edges. We call *cost matrix* any matrix of the form  $\gamma = (\gamma_e)_{e \in \mathbb{E}}$ , that is to say,  $\gamma \in \mathbb{R}_+^{\mathbb{E}}$ . Let  $\Gamma$  be the set of cost matrices. A matrix  $\gamma \in \Gamma$  is *undirected* if  $\gamma_{ij} = \gamma_{ji}$  for all  $i, j \in N$ . We use the symbol  $\Gamma^u$  to denote the set of undirected cost matrices (note that  $\Gamma^u \subsetneq \Gamma$ ).

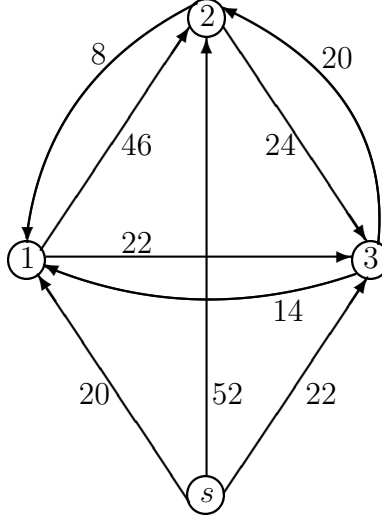


Figure 2: A three-agent mca problem

**Example 4.** Figure 2 describes an example of mca problem with 3 agents; the nodes are identified in the circles. The number beside each directed edge represents its cost. Note for instance that  $\gamma_{12} = 46 \neq \gamma_{21} = 8$ .

Given a subset of agents  $S \subseteq N$ , an  $S$ -arborescence is a directed graph that contains a path (from  $s$ ) to any  $i \in S$ . As a graph, an arborescence  $\mathbf{A}$  is completely characterized by the set of edges it contains; and its associated cost is given by  $\bar{\gamma}(\mathbf{A}) \equiv \sum_{e \in \mathbf{A}} \gamma_e$ . An  $N$ -arborescence that achieves the minimum possible cost is called *minimum cost arborescence* (mca).

An *mca problem* is a triple  $(s, N, \gamma)$ , with  $\gamma \in \Gamma$ . Furthermore, call *minimum cost spanning tree (mcst) problem* any triple  $(s, N, \gamma)$  such that  $\gamma \in \Gamma^u$ , that is, an mcst problem is an mca problem where the cost on each edge is independent of the direction of the flow.

For any coalition  $S \subseteq N$ , let  $c^\gamma(S) \equiv \min\{\bar{\gamma}(\mathbf{A}) \mid \mathbf{A} \text{ is an } S\text{-arborescence}\}$  denote the minimum cost to connect the members of  $S$  using an  $S$ -arborescence.



We refer to  $(N, c^\gamma)$  as the *TC game* associated with the mca problem  $(s, N, \gamma)$ . As an illustration, for the mca problem described in Figure 2, note that the unique mca is  $\{(s, 1); (1, 2); (2, 3)\}$  with cost  $c^\gamma(N) = 22 + 20 + 8 = 50$ . We are now ready to state the following result.

**Proposition 8.** *Let  $P = (s, N, \gamma)$  be an mca problem. Then the associated TC game  $(N, c^\gamma)$  is cost-hyperadditive, that is,  $(N, c^\gamma) \in \underline{\mathcal{H}}^N$ .*

The result above says that all TC games associated with mca problems are cost-hyperadditive. It thus follows from Theorem 3 and Definition 5 that, given any  $\pi \in \Pi(N)$ , the value  $\underline{\varphi}^\pi$  gives an extreme core imputation in all mca problems. In the particular case of mcs problems, it is known from Trudeau and Vidal-Puga (2017) that **all** extreme core imputations are of the type  $\underline{\varphi}^\pi(N, c^\gamma)$ , for some  $\pi \in \Pi(N)$ . Let us now state the main implication of Proposition 8.

**Corollary 3.** *Let  $P = (s, N, \gamma)$  be an mca problem. Then we have*

$$\underline{\varphi}^{AM}(N, c^\gamma) \in \text{Core}(N, c^\gamma) = \mathcal{M}(N, c^\gamma).$$

Thus, in an mca problem, every allocation not in the core can be ruled out by allowing agents to formulate objections against any of their peers (who have the option of making counter-objections). Only core imputations survive this bargaining process. Furthermore, the average marginal value always produces a stable cost allocation. As an illustration, in Example 4 we find  $\underline{\varphi}^{\overline{321}}(N, c^\gamma) = (20, 40, -10)$  and  $\underline{\varphi}^{AM}(N, c^\gamma) = (13, 32, 5)$ . It is readily checked that these imputations are stable.

## 5 Assignment games

Shapley and Shubik (1971) introduced the assignment model as a two-sided TU game describing the matching of buyers and sellers in a given market. More precisely, let  $N = M \cup M'$  be the player set, where  $M$  is the set of buyers and  $M'$  is the set of sellers (with  $M \cap M' = \emptyset$ ). The surplus generated by any pair  $(i, j) \in M \times M'$  is given by  $a_{ij} \geq 0$ ; and the respective values of all such pairs are given by a matrix of the form  $A = (a_{ij})_{i \in M, j \in M'}$ . We will use the notation  $(M \cup M', A)$  to describe an assignment problem.

Given any  $S \subseteq M$  and  $T \subseteq M'$ , a subset  $\mu \subseteq S \times T$  will be called an *assignment* (or *matching*) between  $S$  and  $T$  if each player of  $S \cup T$  belongs to **at most** one pair in  $\mu$ . Since there is no possible ambiguity, we will write  $\mu(i) = j$  and  $\mu(j) = i$  whenever  $(i, j) \in \mu$ . Denote by  $\mathcal{A}(S, T)$  the set containing all assignments between  $S$  and  $T$ .

Every assignment problem  $(M \cup M', A)$  generates a cooperative game  $G_A = (M \cup M', v_A)$  defined as follows: for all  $S \subseteq M$  and  $T \subseteq M'$ ,

$$v_A(S \cup T) = \max_{\mu \in \mathcal{A}(S, T)} \sum_{(i, j) \in \mu} a_{ij}. \quad (12)$$

Note in particular that any coalition containing only buyers (sellers) is worth zero. We call  $G_A = (M \cup M', v_A)$  the assignment game associated with the matrix  $A$ . Moreover, we say that an assignment  $\mu \in \mathcal{A}(S, T)$  is  $S \times T$ -*optimal* for  $A$  if  $v_A(S \cup T) = \sum_{(i, j) \in \mu} a_{ij}$ . It is known from Núñez and Rafels (2003) that marginal games of assignment games are in general not superadditive. The following assignment problem illustrates this fact.

**Example 5.** Let  $M = \{1, 3, 5, 7\}$ ,  $M' = \{2, 4, 6, 8\}$ , and consider the assignment

$$\text{problem } (M \cup M', A), \text{ where } A = \begin{pmatrix} a_{12} & a_{14} & a_{16} & a_{18} \\ a_{32} & a_{34} & a_{36} & a_{38} \\ a_{52} & a_{54} & a_{56} & a_{58} \\ a_{72} & a_{74} & a_{76} & a_{78} \end{pmatrix} = \begin{pmatrix} \underline{20} & 22 & 22 & 0 \\ 0 & \underline{16} & 0 & 0 \\ 0 & 0 & \underline{14} & 0 \\ 38 & 0 & 0 & \underline{30} \end{pmatrix}.$$

The underlined figures depict the unique optimal matching for the matrix  $A$ , which

is  $\mu^* = \{(1, 2); (3, 4); (5, 6); (7, 8)\}$ , with a worth  $v_A(\overbrace{M \cup M'}^N) = 20 + 16 + 14 + 30 = 80$ . Also note that  $v_A(N \setminus 1) = 16 + 14 + 38 = 68$ , and hence  $m_1(v_A) = 80 - 68 = 12$ . Consider now the marginal game  $G_A^1 = (N \setminus 1, v_A^1)$ , which is defined by  $v_A^1(P) = \max(v_A(P), v_A(P \cup 1) - m_1(v_A))$ , for any  $P \subseteq N \setminus 1$ . Observe that  $G_A^1$  is **not** superadditive since  $v_A^1(\{4\}) + v_A^1(\{6\}) = 10 + 10 > 10 = v_A^1(\{4, 6\})$ .

Hence, assignment games are not necessarily hyperadditive. Yet, we show in the following lines that they satisfy the remarkable features of hyperadditive games identified in Section 3.

**Definition 6.** We say that a TU game  $G = (N, v) \in S^N$  is **quasi-hyperadditive** if, for any sequence  $p = \{i_1, \dots, i_k\} \subsetneq N$ , there exists a superadditive game  $\bar{G}_p = (N \setminus p, \bar{v}_p) \in S^{N \setminus p}$  such that  $\text{Core}(G^p) = \text{Core}(\bar{G}_p)$  and  $v^p(j) = \bar{v}_p(j), \forall j \in N \setminus p$ .

From the definition above, a superadditive game is quasi-hyperadditive if each of its  $p$ -marginal games has the same core (and single-player worths) as some superadditive game with player set  $N \setminus p$ . It is straightforward to see that every hyperadditive game  $G$  is also quasi-hyperadditive (by taking  $\bar{G}_p = G^p$ ). The example below illustrates the case where the two games  $\bar{G}_p$  and  $G^p$  are different, and yet exhibit the same core and individual worths.

Call a game  $(N, v)$  *additive* if there exists  $b \in \mathbb{R}^N$  such that  $v(S) = \sum_{i \in S} b_i \equiv b_S$ , for all  $S \subseteq N$ . Moreover, write  $\hat{G}_b = (N, \hat{v}_b)$  to denote the (unique) additive game associated with every  $b \in \mathbb{R}^N$ . As usual, given two TU games  $G = (N, v)$  and  $G' = (N, v')$ , the sum of  $G$  and  $G'$  is the TU game  $G + G' \equiv (N, v + v')$ .

**Example 6.** Recall the assignment game  $(N, v_A)$  introduced in Example 5. It has been shown that the marginal game  $G_A^1 = (N \setminus 1, v_A^1)$  is not superadditive. Regardless, we note here that  $\text{Core}(G_A^1) = \text{Core}(\hat{G}_b + G_{\bar{A}^1})$ , where  $b \in \mathbb{R}^{N \setminus 1}$  satisfies  $b_2 = 8, b_4 = b_6 = 10, b_3 = b_5 = b_7 = b_8 = 0$ , and  $\bar{A}^1 \in \mathbb{R}^{(M \setminus 1) \times M'}$  is given by  $\bar{A}^1 = \begin{pmatrix} 0 & \underline{6} & 0 & 0 \\ 0 & 0 & \underline{4} & 0 \\ \underline{38} & 0 & 0 & 30 \end{pmatrix}$ . Indeed, it is not difficult to see that, for either  $G_A^1$  or  $\bar{G}_1 = \hat{G}_b + G_{\bar{A}^1}$ , the core is the set of imputations  $x \in \mathbb{R}^{N \setminus 1}$  that satisfy:  $x_2 = 8, x_4 \in [10, 16], x_3 = 16 - x_4, x_6 \in [10, 14], x_5 = 14 - x_6, x_7 = 30$  and  $x_8 = 0$ . Obviously,  $\bar{G}_1$  is superadditive (as the sum of two superadditive TU games). Moreover, one can easily check that the single-player worths are the same:  $v^1(j) = \bar{v}_1(j), \forall j \in N \setminus 1$ . For instance, we have  $v^1(4) = \bar{v}_1(4) = 10$ .

The observations made in Example 6 can be generalized; we show below that these properties hold for all  $p$ -marginal games of every assignment game, that is to say, assignment games are quasi-hyperadditive. Interestingly, the property of quasi-hyperadditivity confers to any assignment games the remarkable properties exhibited by hyperadditive games.

**Theorem 9.** If  $G \in \mathcal{S}^N$  is quasi-hyperadditive then the following statements hold:

- (a)  $\varphi^\pi(G)$  is an extreme point of  $\text{Core}(G)$ , for all  $\pi \in \Pi(N)$ ;
- (b)  $\varphi^{AM}(G) \in \text{Core}(G)$ ;
- (c)  $\text{Core}(G) = \mathcal{M}(G)$ .

The proof of Theorem 9 is omitted: it is an easy adaptation of the arguments found in the proofs of Theorem 3 and Theorem 4 (which are available in the Appendix). More importantly, one can now state the following result on assignment games.

**Proposition 10.** Let  $(M \cup M', A)$  be an assignment problem. Then the associated assignment game  $G_A = (M \cup M', v_A)$  is quasi-hyperadditive; and therefore  $\varphi_i^{AM}(G_A) \in \text{Core}(G_A) = \mathcal{M}(G_A)$ .

An immediate corollary of Proposition 10 and Theorem 9 is the known result that each player  $i$  achieves her marginal contribution  $m_i(G_A)$  in at least one core

allocation.<sup>9</sup> See for instance Núñez and Rafels (2003), who also show that every vertex of the core of an assignment game is of the form  $\varphi^\pi(G_A)$ , for some  $\pi \in \Pi(N)$ . Another corollary of our Proposition 10 and Theorem 9 is the fact there is always (at least) one player who receives a payoff of zero in each of these extreme core imputations  $\varphi^\pi(G_A)$ , in line with the findings of Balinski and Gale (1987).

As an illustration of these results, recall the assignment game of Example 5. Picking for instance the permutations  $\pi' = \overline{13572468}$  and  $\pi'' = \overline{46281357}$ , we get  $\varphi^{\pi'}(G_A) = (12, 8, 6, 10, 4, 10, 30, 0)$  and  $\varphi^{\pi''}(G_A) = (8, 12, 0, 16, 0, 14, 26, 4)$ . In addition, averaging over all possible permutations  $\pi$  of  $N$ , one obtains

$$\varphi^{AM}(G_A) = \frac{109}{168}(12, 8, 3, 13, 2, 12, 30, 0) + \frac{59}{168}(8, 12, 1, 15, 0, 14, 26, 4),$$

One can easily check that the three imputations described above are stable. As stated in Proposition 10, the average marginal value always picks a stable imputation in assignment games; and this is an advantage over the Shapley value.<sup>10</sup>

## 6 Conclusion

We defined and studied the property of hyperadditivity for TU games, thereby extending the family of convex games. It has been shown that every hyperadditive game is balanced. In addition, we have defined a new value which satisfies core selection on the class of hyperadditive games and coincides with the Shapley value on the subclass of convex games. We argue that this new value is a sensible and fair solution concept, since it is also dummy and symmetric (as stated in Theorem 1). Finally, we established that the core of a hyperadditive game always coincides with its bargaining set. This result illustrates the robustness of the core as a set-valued solution concept for hyperadditive games.

Hyperadditivity is verifiable in many different economic contexts. The practicality of our results has been shown by proving that several classic applications (veto games, shortest path and minimum cost arborescence problems, assignment games) can be modeled as (quasi-)hyperadditive games. This exercise has allowed

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<sup>9</sup>In fact, we have a more general corollary from Theorem 3: each player of a hyperadditive game achieves her marginal contribution in at least one core allocation.

<sup>10</sup>Consider the assignment game  $G_A = (M \cup M', v_A)$ , with two buyers  $M = \{1, 3\}$ , one seller  $M' = \{2\}$ , and match values  $a_{12} = 36$ ,  $a_{32} = 30$ . Note that (i)  $G_A$  is hyperadditive; (ii) the core is the set of imputations  $x \in \mathbb{R}_+^3$  such that  $x_1 \in [0, 6]$ ,  $x_2 = 36 - x_1$  and  $x_3 = 0$ ; (iii) the Shapley value does not fall in the core:  $\varphi^{Sh}(G_A) = (8, 23, 5) \notin \text{Core}(G_A)$ . On the other hand, we have  $\varphi^{AM}(G_A) = (3, 33, 0) \in \text{Core}(G_A)$ . This example illustrates the fact that the average marginal value is more sensible than the Shapley value on the class of hyperadditive games.

not only to have a broader understanding of various results existing in the literature, but also to state some new results as corollaries to our main findings.

Another class containing all convex games is the family of *totally balanced games* (TU games whose subgames all exhibit a nonempty core). It may be worth noting that there is no inclusion relationship between our class of hyperadditive games and that of totally balanced games. Indeed, not all hyperadditive games are totally balanced. Consider for instance the 5-player game where  $v(N) = 40$ ,  $v(ijkl) = 30$ ,  $v(ijk) = 16$ ,  $v(ij) = 10$  and  $v(i) = 0$ , for all distinct  $i, j, k, l \in N$ . One can easily check that this game is hyperadditive (and hence balanced). However, it is not totally balanced: all 3-player subgames have an empty core. Likewise, one can construct a totally balanced game that is not hyperadditive.

The study of the average marginal value beyond the class of hyperadditive games could prove instructive. Indeed, it would be interesting to investigate the question of characterizing the set of TU games for which this value is a core selection. Our results on quasi-hyperadditive games and assignment games (in Theorem 9 and Proposition 10) provide a first step in this direction. Another interesting topic would be the search for additional properties (other than core selection for hyperadditive games, symmetry and dummy) satisfied by the average marginal value. Finally, one could focus on the design of mechanisms allowing to implement this solution concept in matching markets or auction problems. These questions lay ground for future work.

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# Appendix (Proofs)

## A Theorem 2

(a) Let  $G = (N, v) \in \mathcal{C}^N$  be a convex game; and suppose that  $N = \{1, 2, \dots, n\}$  where  $n \geq 3$ . Without loss of generality, we will prove the result for the natural ordering  $\bar{\pi} \in \Pi(N)$ , which is defined by:  $\bar{\pi}(i) = i$ , for all  $i \in N$ . Define the shorthands  $[i] \equiv \{1, \dots, i\}$  and  $[i] \equiv \{i + 1, \dots, n\}$ , for any  $i = 1, \dots, n$ . We will use the convention  $[0] = \emptyset$ . It is shown in what follows that  $G^{[i]} \in \mathcal{C}^{[i]}$ , for all  $i = 1, \dots, n$ .

First, recall that  $G^{[1]} = (N^{[1]}, v^{[1]})$  is defined by

$$v^{[1]}(S) = \max(v(S), v(S \cup i) - m_1(G)), \quad \forall S \subseteq N \setminus 1. \quad (13)$$

Since  $(N, v) \in \mathcal{C}^N$ , we have  $m_1(G) = v(N) - v(N \setminus 1) \geq v(S \cup i) - v(S)$ , for all  $S \subseteq N \setminus 1$ . In other words,  $v(S) \geq v(S \cup i) - m_1(G)$ . Substituting this in (13) gives  $v^{[1]}(S) = v(S) \quad \forall S \subseteq N \setminus 1$ . Thus,  $G^{[1]}$  coincides with the subgame of  $G$  defined on  $N \setminus 1$ . It is well known that every subgame of a convex game is also convex, that is to say,  $G^{[1]} \in \mathcal{C}^{[1]}$ .

By induction, suppose now that  $G^{[i-1]} = (v, [i-1]) \in \mathcal{C}^{[i-1]}$  for  $i = 2, \dots, n$ . Then we have  $v^{[i]}(S) = \max(v(S), v(S \cup i) - m_i(G^{[i-1]}))$ ,  $\forall S \subseteq [i] = N \setminus [i]$ . Once again, since  $G^{[i-1]}$  is convex, we have  $m_i(G^{[i-1]}) = v([i-1]) - v([i]) \geq v(S \cup i) - v(S)$ ; and hence  $v^{[i]}(S) = v(S)$ , for all  $S \subseteq [i]$ . That is to say,  $G^{[i]} = (v, [i]) \in \mathcal{C}^{[i]}$ . Since every convex game is superadditive, we have thus shown that all marginal games of  $G$  are superadditive. Therefore, we have  $G \in \mathcal{H}^N$ , for all  $G \in \mathcal{C}^N$  (i.e.,  $\mathcal{C}^N \subseteq \mathcal{H}^N$ ). To see why the inclusion is strict, recall that the game of Example 1 is hyperadditive, but not convex.

(b) As shown in the proof of (a) above, if  $G = (N, v)$  is convex and  $\pi \in \Pi(N)$ , then we have  $G^{p_\pi(i)} = (v, N \setminus \pi(i))$ , where  $\pi(i)$  denotes the set of  $i$ 's predecessors under  $\pi$ . Therefore, for any  $G \in \mathcal{C}^N$ , it holds that  $\varphi_i^\pi(G) = m_i(G^{p_\pi(i)}) = v(N \setminus p_\pi(i)) - v(N \setminus (i \cup p_\pi(i)))$ . It thus follows from (2) that

$$\varphi_i^{AM}(G) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} m_i(G^{p_\pi(i)}) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} [v(N \setminus p_\pi(i)) - v(N \setminus (i \cup p_\pi(i)))] = \varphi^{Sh}(G).$$

That is, the average marginal value coincides with the Shapley value for convex games.  $\square$



## B Theorem 3

Consider a hyperadditive game  $G \in \mathcal{H}^N$ .

(a) Without loss of generality, we will prove the claim for  $\bar{\pi}$ , the natural ordering of  $N = \{1, \dots, n\}$ , which is defined by  $\bar{\pi}(i) = i, \forall i \in N$ . Let us fix  $S \subseteq N$ . We prove below that  $\varphi_S^{\bar{\pi}}(G) \geq v(S)$ . Define  $k \equiv \max_{i \in S} i$ .

Note first from Remark 1 that we have: for all  $i = 1, \dots, n$

$$v^{[i-1]}(\lfloor i \rfloor) = v(N) - \varphi_1^{\bar{\pi}}(G) - \dots - \varphi_i^{\bar{\pi}}(G). \quad (14)$$

We discuss two cases below:  $k < n$  or  $k = n$ .

Case 1. Suppose that  $k \equiv \max_{i \in S} i < n$  and write from Definition 1-(a) that  $\varphi_k^{\bar{\pi}}(G) = m_k(G^{[k-1]}) = v^{[k-1]}(\lfloor k-1 \rfloor) - v^{[k-1]}(\lfloor k \rfloor)$ , where the marginal game  $G^{[k-1]}$  is defined by the player set  $\lfloor k-1 \rfloor = \{k, \dots, n\}$  and the characteristic function  $v^{[k-1]}$ . Since  $G \in \mathcal{H}^N$ , the marginal game  $G^{[k-1]}$  is superadditive;<sup>11</sup> and one can hence write  $v^{[k-1]}(\lfloor k \rfloor) + v^{[k-1]}(k) \leq v^{[k-1]}(\lfloor k-1 \rfloor)$ . That is to say,

$$v^{[k-1]}(k) \leq v^{[k-1]}(\lfloor k-1 \rfloor) - v^{[k-1]}(\lfloor k \rfloor) = \varphi_k^{\bar{\pi}}(G). \quad (15)$$

Recalling Lemma 1, one may write

$$\begin{aligned} v^{[k-1]}(k) &= \max_{T \subseteq \lfloor k-1 \rfloor} \left( v(T \cup k) - \sum_{i \in T} m_i(G^{[i-1]}) \right) \\ &\geq v(\overbrace{(S \setminus k) \cup k}^T) - \varphi_{S \setminus k}^{\bar{\pi}}(G) = v(S) - \varphi_{S \setminus k}^{\bar{\pi}}(G) \end{aligned} \quad (16)$$

Combining (15)-(16), we thus get  $v(S) - \varphi_{S \setminus k}^{\bar{\pi}}(G) \leq \varphi_k^{\bar{\pi}}(G)$ , *i.e.*,  $v(S) \leq \varphi_S^{\bar{\pi}}(G)$ .

Case 2. Suppose now that  $k \equiv \max_{i \in S} i = n$ . Then since  $S \neq N$ , note that there must exist  $j = 1, \dots, n-1$  such that  $j \notin S$  and  $\lfloor j \rfloor \subseteq S$ . By contradiction, suppose that  $v(S) > \varphi_S^{\bar{\pi}}(G) = \varphi_{S \cap [j-1]}^{\bar{\pi}}(G) + \varphi_{[j]}^{\bar{\pi}}(G)$ , that is to say,

$$v(S) - \varphi_{S \cap [j-1]}^{\bar{\pi}}(G) > \varphi_{[j]}^{\bar{\pi}}(G) = v(N) - \varphi_1^{\bar{\pi}}(G) - \dots - \varphi_j^{\bar{\pi}}(G). \quad (17)$$

Next, recall from Remark 1 that  $v(N) - \varphi_1^{\bar{\pi}}(G) - \dots - \varphi_j^{\bar{\pi}}(G) = v^{[j-1]}(\lfloor j \rfloor)$ . Substituting the last equality in (17), we get  $v(S) - \varphi_{S \cap [j-1]}^{\bar{\pi}}(G) > v^{[j-1]}(\lfloor j \rfloor)$ . But note that this is a contradiction, given that Lemma 1 (applied with  $p = \lfloor j-1 \rfloor$ ) requires that

$$v^{[j-1]}(\lfloor j \rfloor) = \max_{T \subseteq \lfloor j-1 \rfloor} (v(\lfloor j \rfloor \cup T) - \varphi_T^{\bar{\pi}}(G)) \geq v(\underbrace{\lfloor j \rfloor \cup (S \cap [j-1])}_T) - \varphi_{S \cap [j-1]}^{\bar{\pi}}(G).$$

<sup>11</sup>Recall that, if  $k = 1$ , we have  $G^{[k-1]} = G^\emptyset = G$ .

We have thus shown that  $\varphi^{\bar{\pi}}(G) \in \text{Core}(G) \neq \emptyset$ . It remains to argue that  $\varphi^{\bar{\pi}}(G)$  is an extreme point in  $\text{Core}(G)$ . Consider the lexicographic ordering  $\succ^{\bar{\pi}}$ , defined on  $\text{Core}(G)$  by:

$$x \succ^{\bar{\pi}} y \text{ iff } \begin{cases} x_1 > y_1 \text{ or} \\ (\exists k \in [n-1] \text{ s.t. } x_i = y_i \ \forall i \in [k] \text{ and } x_{k+1} > y_{k+1}). \end{cases}$$

It is not difficult to check that  $\succ^{\bar{\pi}}$  is a linear order over  $\text{Core}(G)$ . Moreover, one can see that, by construction,  $\varphi^{\bar{\pi}}(G)$  maximizes  $\succ^{\bar{\pi}}$  on  $\text{Core}(G)$ . Therefore, writing  $\varphi^{\bar{\pi}}(G) = \alpha x + (1 - \alpha)y$ —for some  $\alpha \in (0, 1)$  and distinct  $x, y \in \text{Core}(G)$ —would lead to a contradiction, since it would imply that either  $x$  or  $y$  is  $\succ^{\bar{\pi}}$ -preferred to  $\varphi^{\bar{\pi}}(G)$ .

(b) It has been shown in (a) above that  $\varphi_i^{\pi}(G) \in \text{Core}(G)$ , for every  $\pi \in \Pi(N)$ . Since the core of a TU game is always a convex set, it easily follows that  $\varphi_i^{AM}(G) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \varphi_i^{\pi}(G) \in \text{Core}(G)$ .  $\square$

## C Theorem 4

Consider  $G = (N, v) \in \mathcal{H}^N$ . Recall  $\bar{\pi}$ , the natural ordering of  $N = \{1, \dots, n\}$ , that is,  $\pi(i) = i$  for all  $i \in N$ . To ease on notation, we will write  $m_i$  instead of  $m_i(G)$  throughout this proof (since  $G$  and  $\bar{\pi}$  are fixed). We know from Theorem 3 that there exists  $\bar{x} \in \text{Core}(G)$  such that  $\bar{x}_1 = m_1(G)$ . Thus, defining  $l_1 \equiv \min_{x \in \text{Core}(G)} x_1$ , one can write:<sup>12</sup>  $l_1 \leq x_1 \leq m_1$ , for all  $x \in \text{Core}(G)$ . Since the core is a convex set, it comes that

$$[l_1, m_1] = \{t \in \mathbb{R} \mid \exists x \in \text{Core}(G) \text{ s.t. } x_1 = t\}. \quad (18)$$

To avoid triviality, assume in what follows that  $n \geq 3$ . Next, by induction, for any  $k = 1, \dots, n-2$  and  $z \in \mathbb{R}^k$  s.t.  $X^z \equiv \{x \in \text{Core}(G) \mid x_i = z_i, \forall i \in [k]\} \neq \emptyset$ , let us define

$$l_{k+1}^z \equiv \max_{T \subseteq [k]} [v(T \cup \{k+1\}) - z_T]; \quad (19)$$

$$m_{k+1}^z \equiv v(N) - x_{[k]} - \max_{T \subseteq [k]} [v(T \cup [k+1]) - z_T]. \quad (20)$$

Also, if  $k = n-1$  and  $z \in \mathbb{R}^{n-1}$ , we will write

$$l_n^z = \max_{T \subseteq [n-2]} [v(T \cup n) - z_T] = m_n^z. \quad (21)$$

One can then make the following statement.

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<sup>12</sup>Recall from the proof of Theorem 3 that  $x_1 \leq m_1$ , for all  $x \in \text{Core}(G)$ .

**Claim 1.** Let  $z \in \mathbb{R}^k$  be such that  $X^z \neq \emptyset$ . Then we have:

$$\begin{aligned} (z, l_{k+1}^z, m_{k+2}^{(z, l_{k+1}^z)}, \dots, m_n^{(z, l_{k+1}^z, m_{k+2}^{(z, l_{k+1}^z)}, \dots, m_{n-1}^{(z, l_{k+1}^z, \dots)} )}) &\in \text{Core}(G); \\ (z, m_{k+1}^z, m_{k+2}^{(z, m_{k+1}^z)}, \dots, m_n^{(z, m_{k+1}^z, m_{k+2}^{(z, m_{k+1}^z)}, \dots, m_{n-1}^{(z, m_{k+1}^z, \dots)} )}) &\in \text{Core}(G). \end{aligned}$$

The proof of Claim 1 is similar to that of Theorem 3 and will not be written explicitly. Note that Claim 1 generalizes (18): for all  $k = 1, \dots, n-1$ ,

$$[z \in \mathbb{R}^k \text{ and } X^z \neq \emptyset] \Rightarrow [l_{k+1}^z, m_{k+1}^z] = \{x_{k+1}, x \in X^z\}.$$

This means that we have recursive (and implicit) expression of the core of a hyperadditive as follows:<sup>13</sup>

$$X \in \text{Core}(G) \Leftrightarrow \begin{cases} l_1 \leq x_1 \leq m_1 \\ l_{k+1}^{(x_1, \dots, x_k)} \leq x_{k+1} \leq m_{k+1}^{(x_1, \dots, x_k)}, \text{ for } k = 1, \dots, n-1, \end{cases} \quad (22)$$

**Claim 2.** We must have

$$l_1 = \min\{t \in \mathbb{R}_+ \mid (t, m_{k+1}^t, m_{k+2}^{(t, m_{k+1}^t)}, \dots, m_n^{(t, m_{k+1}^t, \dots, m_{n-1}^{(t, m_{k+1}^t, \dots)} )}) \in \text{Core}(G)\}. \quad (23)$$

Note that Claim 2 follows from the combination of Claim 1 and the fact that  $m_1 \in \{t \in \mathbb{R}_+ \mid (t, m_{k+1}^t, m_{k+2}^{(t, m_{k+1}^t)}, \dots, m_n^{(t, m_{k+1}^t, \dots, m_{n-1}^{(t, m_{k+1}^t, \dots)} )}) \in \text{Core}(G)\} \neq \emptyset$  (by Theorem 3).

We are now ready to show that  $\mathcal{M}(G) = \text{Core}(G)$ . Recalling from footnote 6 that  $\text{Core}(G) \subseteq \mathcal{M}(G)$  always holds, it suffices to show that, at any imputation  $x \notin \text{Core}(G)$ , there exists an objection (against some player) for which there is no possible counter-objection.

Let us then fix an imputation  $x \notin \text{Core}(G)$ . From (22) above, either  $x_1 \notin [l_1, m_1]$  or there exists  $k = 1, \dots, n-1$  such that  $x_{k+1} \notin [l_{k+1}^{(x_1, \dots, x_k)}, m_{k+1}^{(x_1, \dots, x_k)}]$ . Assume that  $x_{k+1} \notin [l_{k+1}^{(x_1, \dots, x_k)}, m_{k+1}^{(x_1, \dots, x_k)}]$  for some  $k = 1, \dots, n-1$  (similar argument if  $x_1 \notin [l_1, m_1]$ ). We discuss below the two possible cases.

(a)- If  $x_{k+1} < l_{k+1}^{(x_1, \dots, x_k)} = \max_{T \subseteq [k]} \left[ v(T \cup \{k+1\}) - \sum_{i \in T} x_i \right]$  then  $\exists T \subseteq [k]$  such that  $x_{k+1} < v(T \cup \{k+1\}) - \sum_{i \in T} x_i$ , that is,  $\varepsilon \equiv v(T \cup \{k+1\}) - \sum_{i \in T \cup \{k+1\}} x_i > 0$ .

Note that this inequality is possible only if  $k+1 < n$  (because  $\varepsilon = 0$  for  $k = n-1$ ). Thus, letting  $i = k+1$ ,  $j = n$ ,  $S = T \cup \{k+1\}$  and  $y = (x_t + \frac{\varepsilon}{|S|})_{t \in S}$ , it follows that  $(S, y)$  is an objection of  $i$  against  $j$  at  $x$ —in the sense of (4)-(6). Moreover,

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<sup>13</sup>Remark that the expression of the core in (22) is implicit because  $l_1 \equiv \min_{x \in \text{Core}(G)} x_1$ . Regardless, we will see that this expression yields the equality between core and bargaining set.

since  $x_n \geq l_n^z = \max_{T \subseteq [n-2]} [v(T \cup n) - x_T]$ , where the equality comes from (21), one can see that  $j = n$  has no possible counter-objection against  $(S, y)$ .

(b)- If instead  $x_{k+1} > m_{k+1}^{(x_1, \dots, x_k)} = v(N) - x_{[k]} - \max_{T \subseteq [k]} [v(T \cup [k+1]) - x_T]$  then  $\exists T \subseteq [k]$  such that  $v(T \cup [k+1]) - x_T > v(N) - x_{[k+1]} = x_{[k+1]}$ . Thus, observe that player  $i = n \in S = T \cup [k+1]$  has an objection  $(S, y)$  against player  $j = k+1$  at  $x$ , where  $y = (x_t + \frac{\varepsilon}{|S|})_{t \in S}$  with  $\varepsilon = v(T \cup [k+1]) - x_{T \cup [k+1]} > 0$ . As was done in the previous case, one can check that  $j = k+1$  has no possible counter-objection against  $(S, y)$ .  $\square$

## D Proposition 5

Let  $G = (N, v)$  be a veto game with veto set  $T^*(G)$  and  $|N| \geq 2$ . First, note that every veto game is superadditive. Indeed, given two disjoint subsets  $S, S' \subsetneq N$ , the veto set  $T^*(G)$  is a subset of at most one of them, and hence we must have either  $v(S) = 0$  or  $v(S') = 0$ . This means that  $v(S \cup S') \geq v(S) + v(S') = \max(v(S), v(S'))$  (by monotonicity of a veto game). Second, note that every marginal game of a veto game is also a veto game (and is hence superadditive). It thus follows that  $G = (N, v) \in \mathcal{H}^N$ .  $\square$

## E Proposition 7

Let  $P = (N, \alpha, x)$  be a shortest path problem (with  $|N| \geq 2$ ) and fix  $j \in N$ . Then it is easy to see that the elementary SPP  $(P^j = N, \alpha, e^j)$  is associated with a value game  $v^{c_{P^j}}$  that is a big-boss game (with big boss  $j$ ). Indeed, for any coalition  $S \in 2^N$  such that  $j \notin S$ , one can see from (10)-(11) that  $v^{c_{P^j}}(S) = 0$ . Since big-boss games are veto games, it then follows from Proposition 5 that  $(N, v^{c_{P^j}}) \in \mathcal{H}^N$  and therefore  $(N, c_{P^j})$  is cost-hyperadditive.  $\square$

## F Proposition 8

We will proceed in two steps. First, we extend the family of mca problems as follows. Fix a set of agents  $M$  (with  $|M| \geq 2$ ) and a set of Steiner points  $T$  (with  $|T| \geq 1$ ). Suppose that we have a matrix  $\gamma$  giving the cost of every edge  $e = (i, j) \in (M \cup T \cup s) \times (M \cup T)$  such that  $i \neq j$ . Also assume that the cost of using any Steiner point  $k \in T$  is given by  $\theta_k \geq 0$ . For any collection

of edges  $\mathbf{A} \subseteq (M \cup T \cup s) \times (M \cup T)$ , we will write  $[\mathbf{A}] \equiv \{i \in M \cup T \mid \exists j \in M \cup T \text{ s.t. either } (i, j) \in \mathbf{A} \text{ or } (j, i) \in \mathbf{A}\}$ .

The problem of connecting agents in  $M$  to the source  $s$  (while possibly using Steiner points) can then be described as the TC game  $(M, c_T^\gamma)$  such that  $c_T^\gamma(S) \equiv \min\{\bar{\gamma}(\mathbf{A}) + \sum_{k \in T \cap [\mathbf{A}]} \theta_k \mid \mathbf{A} \text{ is an } (M \cup T) - \text{arborescence}\}$ . It is not difficult to see that the TC game  $(M, c_T^\gamma)$  is *subadditive*, i.e.,  $c_T^\gamma(S' \cup S'') \leq c_T^\gamma(S') + c_T^\gamma(S'')$ , for all disjoint  $S', S'' \subseteq M$ . Hence, letting  $v^{c_T^\gamma}(S) \equiv \sum_{i \in S} c_T^\gamma(i) - c_T^\gamma(S)$  for all  $S \subseteq M$ , one can easily see that  $v^{c_T^\gamma}(S' \cup S'') \geq v^{c_T^\gamma}(S') + v^{c_T^\gamma}(S'')$ , that is,  $v^{c_T^\gamma}$  is superadditive. We call every such  $(M, c_T^\gamma)$  an *arborescence game with Steiner points*.

The second step consists in noticing from Lemma 1 that, given an mca problem  $(s, N, \gamma)$ , every  $p$ -marginal game of the associated TC game  $G = (N, c^\gamma)$  is an arborescence game with Steiner points where  $M = N \setminus p$ ,  $T = p = \{i_1, \dots, i_L\}$ , and  $\theta_{i_l} = m_{i_l}(G^{i_1 \dots i_{l-1}})$ , for all  $l = 1, \dots, L$ . Using the result of the above paragraph, it thus follows that every marginal game of  $(N, v^{c^\gamma})$  is superadditive. Therefore, we have  $G = (N, c^\gamma) \in \underline{\mathcal{H}}^N$ , that is to say,  $G = (N, c^\gamma)$  is cost-hyperadditive.  $\square$

## G Proposition 10

Consider an assignment problem  $(M \cup M', A)$  and let  $G_A = (M \cup M', v_A)$  be the associated assignment game (with  $N = M \cup M'$ ). Moreover, fix an ordered sequence  $p_k = \{i_1, \dots, i_{k-1}, i_k\} \subsetneq N$ . Write  $p_l = \{i_1, \dots, i_l\}$ , for all  $l = 1, \dots, k$ ; and  $p_0 = \emptyset$ . We will show below that the marginal game  $G_A^{p_k}$  satisfies  $\text{Core}(G_A^{p_k}) = \text{Core}(\bar{G})$ , where  $\bar{G} = (N \setminus p_k, \bar{v}) \in \mathcal{S}^{N \setminus p_k}$  is some suitably chosen superadditive game.

Recall that  $v^\emptyset = v$  and  $m_{i_l}(v_A^{p_{l-1}}) = v_A^{p_{l-1}}(N \setminus p_{l-1}) - v_A^{p_{l-1}}(N \setminus p_l) \geq 0$ , for all  $l = 0, \dots, k$ . **By induction over  $k$** , we will assume that all  $v^{p_{l-1}}$  (with  $l = 1, \dots, k$ ) satisfy:  $v^{p_{l-1}}(N \setminus p_{l-1}) \geq v^{p_{l-1}}(j) + v^{p_{l-1}}(N \setminus (p_{l-1} \cup j))$ , for all  $j \in N \setminus p_{l-1}$ . Notice that this condition is trivially satisfied when  $k = 1$ , since then the game  $v^{p_{k-1}} = v^{p_0} = v^\emptyset = v$  is superadditive.

Next, define the vector  $b \in \mathbb{R}_+^{N \setminus p_k}$  by

$$b_j = \begin{cases} \max(0, \max_{i_l \in M' \cap p_k} \overbrace{(a_{ji_l} - m_{i_l}(v_A^{p_{l-1}}))}^{r_{ji_l}}), & \text{if } j \in M; \\ \max(0, \max_{i_l \in M \cap p_k} \overbrace{(a_{i_lj} - m_{i_l}(v_A^{p_{l-1}}))}^{r_{i_lj}}), & \text{if } j \in M'. \end{cases} \quad (24)$$

Also note from Lemma 1 that

$$v_A^{p_k}(S \cup T) = \max_{Q \subseteq p_k} \left[ v_A(S \cup T \cup Q) - \sum_{i_l \in Q} m_{i_l}(v_A^{p_l-1}) \right], \forall S \subseteq M \setminus p_k, T \subseteq M' \setminus p_k. \quad (25)$$

Because  $(N, v_A)$  is an assignment game, value obtains by matching pairs of agents (from opposing sides). Hence it comes from (25) that the total worth  $v_A^{p_k}(S \cup T)$  can be accounted for by combining pairs of the following three types: (a) matching pairs  $(i, j)$  where  $i \in S$  and  $j \in T$ ; (b) matching pairs  $(i_l, j)$  where either  $j \in S$  and  $i_l \in p_k \cap M'$  or  $j \in T$  and  $i_l \in p_k \cap M$ , by paying the fee  $m_{i_l}(v_A^{p_l-1})$ ; (c) matching pairs  $(i_l, i_{l'})$  where  $i_l \in p_k \cap M$  and  $i_{l'} \in p_k \cap M'$ , by paying the fee  $m_{i_l}(v_A^{p_l-1}) + m_{i_{l'}}(v_A^{p_{l'}-1})$ .

Although these three types of pairings are available, notice that a coalition  $S \cup T$  will never use the type (c) because it results in a negative added-value. Indeed, observe that  $a_{i_l i_{l'}} - m_{i_l}(v_A^{p_l-1}) + m_{i_{l'}}(v_A^{p_{l'}-1}) \leq 0$ . To see why this holds, assume (without loss of generality) that  $l < l'$ . Then it comes from our induction hypothesis that  $v_A^{p_{l'}-1}(i_{l'}) + v_A^{p_{l'}-1}(N \setminus p_{l'}) \leq v_A^{p_{l'}-1}(N \setminus p_{l'-1})$ . That is to say,

$$v_A^{p_{l'}-1}(i_{l'}) \leq v_A^{p_{l'}-1}(N \setminus p_{l'-1}) - v_A^{p_{l'}-1}(N \setminus p_{l'}) = m_{i_{l'}}(v_A^{p_{l'}-1}). \quad (26)$$

From Lemma 1 we also get

$$v_A^{p_{l'}-1}(i_{l'}) = \max_{Q \subseteq p_{l'-1}} [v_A(i_{l'} \cup Q) - \sum_{i_l \in Q} m_{i_l}(v_A^{p_l-1})] \geq a_{i_l i_{l'}} - m_{i_l}(v_A^{p_l-1});$$

and substituting this in (26) gives:  $a_{i_l i_{l'}} - m_{i_l}(v_A^{p_l-1}) + m_{i_{l'}}(v_A^{p_{l'}-1}) \leq 0$ .

The previous paragraph says that  $v_A^{p_k}(S \cup T)$  should be computed using only pairs of the types (a) and (b); and it thus comes from (25) and (24) that, for all  $S \subseteq M \setminus p_k, T \subseteq M' \setminus p_k$ ,

$$v_A^{p_k}(S \cup T) = \max_{\mu \in \mathcal{A}(S \cup p_k^M, T \cup p_k^{M'})} \left[ \overbrace{\sum_{\substack{(i,j) \in \mu \\ i,j \in S \cup T}} a_{ij}}^{(a)} + \overbrace{\sum_{\substack{(i_l,j) \in \mu \\ i_l \in p_k^M \\ j \in T}} r_{i_l j} + \sum_{\substack{(j,i_l) \in \mu \\ j \in S \\ i_l \in p_k^{M'}}} r_{j i_l}}^{(b)} \right], \quad (27)$$

where  $p_k^M = p_k \cap M$  and  $p_k^{M'} = p_k \cap M'$ .

**Step 1.** Definition and superadditivity of  $\bar{v}$ .

Next, we define the game  $\bar{G} = (N \setminus p_k, \bar{v})$ : for all  $S \subseteq M \setminus p_k, T \subseteq M' \setminus p_k$ ,

$$\bar{v}(S \cup T) = \max_{\substack{S' \subseteq S \\ T' \subseteq T}} [v_A(S' \cup T') + \sum_{j \in S \setminus S'} b_j + \sum_{j \in T \setminus T'} b_j] \quad (28)$$

It is easy to see from (24) and (27)-(28) that we have

$$v_A^{p_k}(S \cup T) \leq \bar{v}(S \cup T), \quad \forall S \subseteq M \setminus p_k, T \subseteq M' \setminus p_k. \quad (29)$$

Moreover, notice from (28) that (i)  $\bar{G} = (N \setminus p_k, \bar{v})$  is superadditive and (ii)  $v_A^{p_k}(j) = \bar{v}(j)$ , for all  $j \in N \setminus p_k$ .

To conclude the proof, we follow the two steps below.

**Step 2.** We show next that  $v_A^{p_k}(N \setminus p_k) = \bar{v}(N \setminus p_k) = v_A(N \setminus p_k)$ .

Note that we already have  $v_A(N \setminus p_k) \leq v_A^{p_k}(N \setminus p_k) \leq \bar{v}(N \setminus p_k)$  from (27) and (29); and it thus remains to show that  $v_A(N \setminus p_k) = \bar{v}(N \setminus p_k)$ . **Let then**  $\tilde{\mu}$  be an  $(M \setminus p_k) \times (M' \setminus p_k)$ -**optimal matching** for  $A$ , that is to say,  $v_A(N \setminus p_k) = \sum_{(i,j) \in \tilde{\mu}} a_{ij}$ .

**Claim 1:** We have  $a_{ij} \geq b_i + b_j$  for all  $(i, j) \in \tilde{\mu}$ .

By contradiction, suppose there exists  $(i', j') \in \mu$  such that  $a_{i'j'} < b_{i'} + b_{j'}$ . That is,  $a_{i'j'} < \underbrace{a_{i'i_l} - m_{i_l}(v_A^{p_l-1})}_{r_{i'i_l}} + \underbrace{a_{i_tj'} - m_{i_t}(v_A^{p_t-1})}_{r_{i_tj'}}$ , for some  $i_l \in p_k^{M'}$  and  $i_t \in p_k^M$ .

$$v_A(N \setminus p_k) = \sum_{(i,j) \in \tilde{\mu}} a_{ij} = \sum_{(i,j) \in \tilde{\mu} \setminus (i',j')} a_{ij} + a_{i'j'} < \sum_{(i,j) \in \hat{\mu} \setminus (i',j')} a_{ij} + r_{i'i_l} + r_{i_tj'},$$

where  $\hat{\mu} \in \mathcal{A}(M, M')$  is defined by  $\hat{\mu}(j) = \tilde{\mu}(j)$  for  $j \in N \setminus (p_k \cup i'j')$ ,  $\hat{\mu}(i') = i_l$ ,  $\hat{\mu}(j') = i_t$ , and  $\hat{\mu}(i) = i$  for  $i \in p_k \setminus i_l i_t$ . But remark that the above inequality is a contradiction: taking  $S \cup T = N \setminus p_k$  in (27), we should rather have  $v_A(N \setminus p_k) \geq$

$$\sum_{(i,j) \in \hat{\mu} \setminus (i',j')} a_{ij} + r_{i'i_l} + r_{i_tj'}.$$

We leave it to the reader to check [using Claim 1 above and (28)] that

$$\bar{v}(N \setminus p_k) = \max_{S' \subseteq N \setminus p_k} [v_A(S') + \sum_{j \in N \setminus (S' \cup p_k)} b_j] \leq \sum_{(i,j) \in \tilde{\mu}} a_{ij} = v_A(N \setminus p_k).$$

This concludes Step 2.

**Step 3.**  $\text{Core}(N \setminus p_k, \bar{v}) = \text{Core}(N \setminus p_k, v_A^{p_k})$ .

Note from (29) and Step 2 above that  $\text{Core}(N \setminus p_k, v_A^{p_k}) \subseteq \text{Core}(N \setminus p_k, \bar{v})$ . We prove the reverse inclusion. Suppose that  $x \in \text{Core}(N \setminus p_k, v_A^{p_k})$  and fix  $S \subseteq M \setminus p_k, T \subseteq M' \setminus p_k$ . Then, for all  $S' \subset S \cup T$ , coalitional rationality gives:

$$\begin{aligned} x_{S \cup T} &= x_{S'} + x_{(S \cup T) \setminus S'} \\ &\geq \underbrace{v_A^{p_k}(S')}_{\geq v_A(S')} + \sum_{i \in (S \cup T) \setminus S'} \underbrace{v_A^{p_k}(i)}_{\geq b_i} \\ &\geq v_A(S') + \sum_{j \in (S \cup T) \setminus S'} b_j, \quad \forall S' \subset S \cup T \\ &\geq \max_{S' \subseteq S \cup T} [v_A(S') + \sum_{j \in (S \cup T) \setminus S'} b_j] \equiv \bar{v}(S \cup T). \end{aligned}$$

Therefore, we have  $Core(N \setminus p_k, \bar{v}) \subseteq Core(N \setminus p_k, v_A^{p_k})$ ; and this concludes the proof of Proposition 10.  $\square$