Supermodular correspondences and comparison of multi-prior beliefs*

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Abstract

Economic decisions often involve maximising an objective whose value is itself the outcome of another optimisation problem. This decision structure arises in multi-output production and choice under uncertainty with multi-prior beliefs. To analyse comparative statics in these models, we introduce a theory of supermodular correspondences. In particular, we employ this theory to generalise the notion of first order stochastic dominance to multi-prior beliefs, allowing us to characterise conditions under which greater optimism leads to higher action.

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1 Introduction

Consider a firm that uses ℓ factors to produce a single good sold at a fixed price. The factors of production are said to be *complements* if a fall in the price of one factor raises the demand for all factors, at least weakly. It is well-known that complementarity holds if the production function is supermodular; in this context, supermodularity says that the marginal productivity of a factor is increasing in the level of the other factors.[1]

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[1]See Topkis (1978), Milgrom and Roberts (1990), and Milgrom and Shannon (1994).

A natural follow up question is to ask what conditions on the production technology will guarantee factor complementarity when the firm is producing multiple output goods. In that case, the firm's production possibility can be represented by a correspondence Γ where set $\Gamma(x)$ consists of all the combinations of output goods that are producible using factors x. Assuming that there are m output goods priced at $q = (q_1, q_2, \ldots, q_m)$, factor complementarity holds if the maximum revenue

$$f(x) \;:=\; \max\Big\{q \cdot y : y \in \Gamma(x)\Big\}$$

is a supermodular function of x.[2] What conditions on Γ will guarantee this?

This issue is one of many in economic modelling that requires supermodularity of a value function after some optimisation procedure. For another example, consider an agent who has to take an action under uncertainty. Suppose that the agent's payoff is $g(x, s)$, where $x \in X \subseteq \mathbb{R}$ is the chosen action at the state $\tilde{s} \in S \subseteq \mathbb{R}$. The expected utility of action x is therefore $f(x, t) := \int g(x, \tilde{s})d\lambda(\tilde{s}, t)$, where $t \in T \subseteq \mathbb{R}$ parametrises the distribution function $\lambda(\cdot, t)$ over S. Suppose that g is such that the marginal payoff of a higher action increases with s, i.e., function g is supermodular. It seems reasonable that the *expected* marginal payoff of a higher action should be greater when higher states are more likely. This intuition is correct: if g is a supermodular function of (x, s), then f is supermodular in (x, t) if $\lambda(\cdot, t)$ first order stochastically increases with t. This in turn implies that the optimal policy $\operatorname{argmax}\big\{f(x, t) : x \in X\big\}$ increases with t.

As a simple application of this result, consider an agent who decides on his savings x in period 1, given uncertainty in his period 2 income, denoted by s. Then,

$$g(x, s) \;=\; u(w - x) + \delta u\big(x(1 + r) + s\big),$$

where u is the per-period utility, δ is the discount rate, and r is the interest. In this case, function g is submodular in (x, s), or $g_{xs} \leq 0$, so long as u is concave. Thus, we conclude that a first order shift in the distribution of period 2 income will *reduce* savings.

Suppose that instead of being an EU-maximiser, the agent is endowed with maxmin preferences as in Gilboa and Schmeidler (1989), so that the ex-ante utility of action x is

$$f(x, t) \;:=\; \min\left\{\int_S g(x, \tilde{s})d\lambda(\tilde{s}) : \lambda \in \Lambda(t)\right\}, \tag{1}$$

[2] We are assuming here that the firm is a price-taker in all markets. For an alternative interpretation of vector q and correspondence Γ see Section 4.

where $\Lambda(t)$ denotes a set of distributions over S parameterised by t. Note that f is the value function arising from Nature choosing $\lambda \in \Lambda(t)$. Assuming that g is supermodular, what conditions on the correspondence Λ will guarantee that f is supermodular (and hence that the optimal choice of x increases with t)? That is, how do we compare *sets* of distributions in a way that generalises first order stochastic dominance?

Our results. In Section 2 we formally introduce the generalised notion of supermodularity for a correspondence $\Gamma : X \to Y$, where X is a lattice and Y is an ordered vector space. Our main results are presented in Section 3. In Main Theorem we show that supermodularity of the correspondence is sufficient to guarantee that the function $f(x) := \max\{\phi(y) : y \in \Gamma(x)\}$ is supermodular, for any positive linear functional $\phi : Y \to \mathbb{R}$. Moreover, in Proposition 1 we argue that the condition is also necessary. Similarly, we develop a related notion of supermodularity of Γ which guarantees that the map $f(x) := \min\{\phi(y) : y \in \Gamma(x)\}$ is supermodular. The remainder of the paper is devoted to exploiting these results in different economic contexts.

In Section 4, we employ our main theorems to production analysis. We formulate a notion of input complementarity for multi-output technologies and provide examples of production correspondences Γ where the property holds. We also use our results to examine a related but distinct issue: the relationship between factor prices and marginal cost. It is natural to imagine that a firm's marginal cost would increase when the price of a factor increases, however, this is *not* generally true. We identify necessary and sufficient conditions under which this property holds; in particular, we show that it is satisfied when the production is (i) homothetic and quasiconcave, or (ii) supermodular and concave.

Section 5 deals with the comparative statics of decision-making with maxmin, variational, and multiplier preferences. For each of these models, we formulate what it means for "*beliefs to shift towards higher states.*" In the case of the maxmin model, we show that if function f, defined in (1), is supermodular in (x, t), for any supermodular g, if and only if the belief correspondence Λ shifts in the following sense: for any $t' \geq t$ and $\lambda \in \Lambda(t)$, $\lambda' \in \Lambda(t')$, there is some $\mu \in \Lambda(t)$ and $\mu' \in \Lambda(t')$ such that $\lambda' \succeq \mu$, $\mu' \succeq \lambda$, and

$$\tfrac{1}{2}\lambda + \tfrac{1}{2}\lambda' \;=\; \tfrac{1}{2}\mu + \tfrac{1}{2}\mu',$$

where $\succeq$ denotes first order stochastic dominance.[3] Returning to our example of the ambiguity averse saver, higher optimism about her period 2 income, captured by the shift in Λ in the sense defined, would lead to lower savings in period 1.

Our definition, which compares *sets* of distributions, is an extension of first order stochastic dominance. Indeed, if $\Lambda(t)$ and $\Lambda(t')$ are singletons then our definition is equivalent to the first order stochastic dominance. It is not difficult to show that the following correspondence Λ is an instance of our property:

$$\Lambda(t) \;:=\; \Big\{\lambda : \nu(\cdot,t) \succeq \lambda \succeq \mu(\cdot,t)\Big\},$$

where the probability distributions $\mu(\cdot,t)$ and $\nu(\cdot,t)$ are both increasing in t with respect to the first order stochastic dominance. That is, the agent's uncertainty is captured by an interval of distributions, with its upper and lower bounds increasing with t.

Unlike for comparisons over probability distributions, the notion of stochastic dominance over multi-prior beliefs that is required for monotonicity of the optimal policy is stronger from the ordering that implies higher value of the utility. That is, the above condition on the correspondence Λ is sufficient for the value $\min\big\{\int_S u(\tilde{s})d\lambda(\tilde{s}) : \lambda \in \Lambda(t)\big\}$ to be increasing in t, for any increasing function u, but not necessary. We specify the tight condition under which the latter property holds.

We consider applications to dynamic programming in Section 6; specifically, we show that the method in Hopenhayn and Prescott (1992) can be extended to the case where, instead of maximising expected discounted utility, the agent's preference over uncertain future utility streams conforms to the maxmin model.

2 Basic concepts

In this section we introduce the basic mathematical concepts that are crucial to our main analysis. We begin with a brief revision of the existing lattice theory.

[3] Note that, if $\lambda' \succeq \lambda$, then μ and μ' can be chosen to be λ and λ', respectively.

2.1 Orderings, lattices, and comparative statics

A *partial order* $\geq_X$ over a set X is a reflexive, transitive, and antisymmetric binary relation.[4] A partially ordered set, or a *poset*, is a pair $(X, \geq_X)$ consisting of a set X and a partial order $\geq_X$. Whenever it causes no confusion, we denote $(X, \geq_X)$ with X.

For any two elements x, x' of a poset X, their *meet*, or the greatest lower bound, is denoted by $(x \wedge x')$, and their *join*, or the least upper bound, by $(x \vee x')$, where both elements are defined with respect to the corresponding partial order $\geq_X$. A poset X is a *lattice* if for any x, $x' \in X$ both their meet $(x \wedge x')$ and their join $(x \vee x')$ belong to the set. For any subset Y of X, the poset $(Y, \geq_X)$ is a *sublattice* of X if it contains elements $(y \wedge y')$, $(y \vee y')$, for any y, $y' \in Y$, both defined with respect to $\geq_X$.

For the purposes of this paper, the most important lattice is the Euclidean space $\mathbb{R}^\ell$ endowed with the natural product order $\geq$. That is, for any vectors x, $x' \in \mathbb{R}^\ell$, we denote $x' \geq x$ if $x'_i \geq x_i$, for all $i = 1, \ldots, \ell$. In this case, the meet and the join are defined as $(x \wedge x')_i = \min\{x_i, x'_i\}$ and $(x \vee x')_i = \max\{x_i, x'_i\}$, for all $i = 1, \ldots, \ell$.

A function $f : X \to \mathbb{R}$ defined over a lattice X is *supermodular* if for any elements x, $x' \in X$, we have $f(x \wedge x') + f(x \vee x') \geq f(x) + f(x')$. Alternatively, we say that f is *submodular* if and only if $(-f)$ is supermodular.

Our generalisation of supermodularity applies to correspondences that map a lattice to an ordered vector space. A binary relation $\geq_Y$ over a set Y is a *preorder* if it is reflexive and transitive. An *ordered vector space* is a pair $(Y, \geq_Y)$ of a vector space Y and a preorder $\geq_Y$ that preserves vector space operations.[5] That is, for any y, $y' \in Y$, we have $y' \geq_Y y$ only if $(y' + z) \geq_Y (y + z)$ and $\alpha y' \geq \alpha y$, for any $z \in Y$ and $\alpha \geq 0$.

Clearly, the Euclidean space is an ordered vector space. Another important example is the space of signed finite measures defined over a measurable space $(S, \mathcal{S})$. This is a vector space that contains, crucial to our purposes, the set of probability measures. Whenever set S is partially ordered, the signed measures can be ranked with respect to the first order stochastic dominance, i.e., for any two signed measures λ, $\lambda' \in Y$, we have $\lambda' \geq_Y \lambda$ if $\int_S u(\tilde{s}) d\lambda'(\tilde{s}) \geq \int_S u(\tilde{s}) d\lambda(\tilde{s})$, for any measurable, bounded function $u : S \to \mathbb{R}$ that increases on S with respect to the corresponding partial order. For probability measures,

[4] See Topkis (1998) for a handbook treatment of the concepts covered here.

[5] Set Y is a vector space if for any y, $y' \in Y$ and α, $\beta \in \mathbb{R}$, element $(\alpha y + \beta y')$ belongs to Y.

this is equivalent to the standard notion of the first order stochastic dominance.

Lattice theory plays a particularly important role when analysing comparative statics in optimisation problems. For any two subsets Y, Y' of a lattice X, we say that Y' dominates Y in the *strong set order* induced by $\geq_X$, if for any $y \in Y$ and $y' \in Y'$, we have $(y \wedge y') \in Y$ and $(y \vee y') \in Y'$. Whenever Y and Y' both contain their greatest elements y and y', respectively, then Y' dominates Y in the strong set order only if $y' \geq_X y$.[6] While the strong set order is not complete, it is transitive over the subsets of X (see Topkis, 1978). The basic results outlined below provide conditions under which the set of maximisers of some objective function is increasing in the strong set order.

Let X be a lattice and T be a partially order set. A function $f : X \times T \to \mathbb{R}$ is said to have *increasing differences* if, for all $x' \geq_X x$, the difference $\delta(t) = \big[f(x', t) - f(x, t)\big]$ is increasing in t. This notion is closely related to supermodularity; indeed if T is totally ordered (and hence a lattice), then it is straightforward to check that function $f(x, t)$ is supermodular in (x, t), with respect to the product order on $X \times T$, if and only if it is supermodular in x and has increasing differences in (x, t).

Topkis (1978) shows that whenever function $f : X \times T \to \mathbb{R}$ is supermodular in x, then the set of maximisers $\Phi(t) := \arg\max \big\{f(x, t) : x \in X\big\}$ is a sublattice of X.[7] In addition, if the function f has increasing differences in (x, t), then the set increases in t with respect to the strong set order, i.e., set $\Phi(t')$ dominates $\Phi(t)$ in the strong set order, for any $t' \geq_T t$. In the remainder of the paper, we refer to this result as the *Monotone Comparative Statics* (MCS) theorem.[8] If $\Phi(t)$ is a *compact* sublattice of a Euclidean space, it admits the least and the greatest elements, that are also increasing in t.

2.2 Upper and lower supermodular correspondences

Suppose that $(X, \geq_X)$ is a lattice and $(Y, \geq_Y)$ is an ordered vector space. A correspondence $\Gamma : X \to Y$ is *upper supermodular* if for any two elements x, $x' \in X$ and $y \in \Gamma(x)$,

[6] Similarly, if Y and Y' contain their least elements y and y' respectively, then Y' dominates Y in the strong set order only if $y' \geq_X y$. Moreover, whenever $Y = \{y\}$ and $Y' = \{y'\}$, i.e., the sets are singletons, then $y' \geq_X y$ if and only if Y' dominates Y in the strong set order.

[7] The set of maximisers $\Phi(t)$ is the set of all arguments $x \in X$ such that $f(x, t) \geq f(y, t)$, for all $y \in X$.

[8] Milgrom and Shannon (1994) present a generalisation of the above results that employs an ordinal notion of complementarity, called *quasisupermodulrity*.

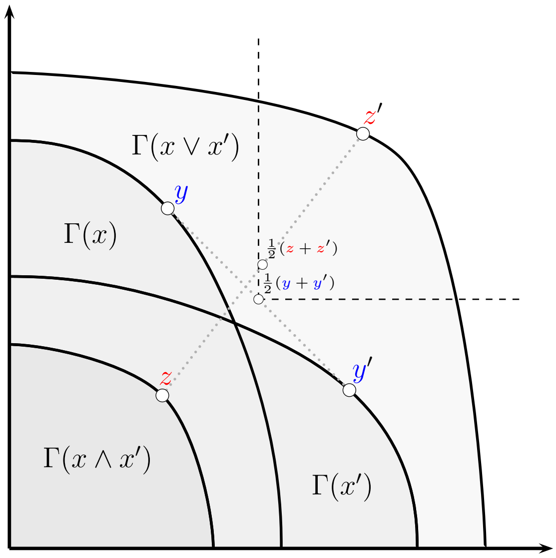

Figure 1: An upper supermodular correspondence for $Y = \mathbb{R}^2_+$.

$y' \in \Gamma(x')$, there is some $z \in \Gamma(x \wedge x')$, $z' \in \Gamma(x \vee x')$ such that

$$z + z' \;\geq_Y\; y + y'. \tag{2}$$

Equivalently, the condition can be expressed in terms of average vectors that satisfy $(1/2)z + (1/2)z' \geq_Y (1/2)y + (1/2)y'$. See Figure 1 for a graphical interpretation.

The correspondence Γ is *lower supermodular* if for any x, $x' \in X$ and $z \in \Gamma(x \wedge x')$, $z' \in \Gamma(x \vee x')$ there are some vectors $y \in \Gamma(x)$, $y' \in \Gamma(x')$ that satisfy (2).[9] Finally, the correspondence is *supermodular* if it is both upper and lower supermodular.

The above definitions can be restated in terms of ordering of sets $\Gamma(x) + \Gamma(x')$ and $\Gamma(x \wedge x') + \Gamma(x \vee x')$. Indeed, upper supermodularity requires that for any element v in $\Gamma(x)+\Gamma(x')$ there is some vector v' in $\Gamma(x \wedge x')+\Gamma(x \vee x')$ such that $v' \geq_Y v$. Analogously, lower supermodularity implies that for any element v' in the latter there is a vector v in the former for which $v' \geq_Y v$. Finally, for a correspondence to be supermodular, both conditions must hold. See Figure 2 for a graphical interpretation.

Submodularity of correspondences can be defined analogously. The correspondence Γ is *upper submodular* if for any x, $x' \in X$ and $y \in \Gamma(x)$, $y' \in \Gamma(x')$, there is some

[9]Notice that, the distinction between upper and lower supermodularity disappears if Γ is a function, i.e., Γ is a singleton-valued, rather than a set-valued correspondence.

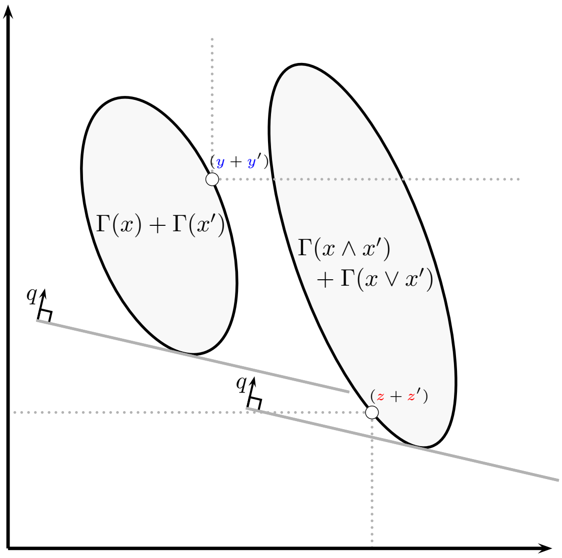

Figure 2: For any element v in $\Gamma(x)+\Gamma(x')$ there is some v' in $\Gamma(x\wedge x')+\Gamma(x\vee x')$ such that $v'\geq v$. Thus, the correspondence is upper supermodular. However, there is some $v'=(z+z')$ in $\Gamma(x\wedge x')+\Gamma(x\vee x')$ for which there is no v in $\Gamma(x)+\Gamma(x')$ satisfying $v'\geq v$. Therefore, it is not lower supermodular. In particular, there is a positive vector q for which $f(x)+f(x')>f(x\wedge x')+f(x\vee x')$, where $f(x):=\min\{q\cdot y: y\in\Gamma(x)\}$.

$z\in\Gamma(x\wedge x')$, $z'\in\Gamma(x\vee x')$ that satisfy

$$y+y' \ \geq_Y \ z+z';$$

equivalently, this means that correspondence $(-\Gamma)$ is upper supermodular. Analogously, one may define lower submodularity and submodularity.

Our definition of supermodular correspondences generalises the familiar notion of supermodularity applied to real-valued functions, presented at the beginning of this section. This notion also extends the concept of *stochastic supermodularity* introduced in Topkis (1968) to correspondences;[10] a function mapping a lattice to the set of probability measures on some measurable space is said to be stochastically supermodular if condition (2) holds with $\geq_Y$ representing the first order stochastic dominance.

Suppose that $\Gamma: X\to Y$ has *downward comprehensive* values, i.e., $y\in\Gamma(x)$ and $y\geq_Y z$ implies $z\in\Gamma(x)$, for any $z\in Y$ and $x\in X$. It is straightforward to show that

[10]Although Topkis (1968) refers to this property as *stochastic convexity*, the term *stochastic supermodularity* is more commonly used; see also Curtat (1996) or Balbus, Reffett, and Woźny (2014).

correspondence Γ is upper supermodular if and only if, for all x, $x' \in X$,

$$\Gamma(x \wedge x') + \Gamma(x \vee x') \;\supseteq\; \Gamma(x) + \Gamma(x'). \tag{3}$$

The fact that (3) implies upper supermodularity is clear and does not require for Γ to have downward comprehensive values. To show the converse, suppose that Γ is upper supermodular. Thus, for any $y \in \Gamma(x)$, $y' \in \Gamma(x')$ there is some $z \in \Gamma(x \wedge x')$, $z' \in \Gamma(x \vee x')$ such that $z + z' \geq_Y y + y'$. In particular, we have $z \geq_Y (y + y' - z')$. Since Γ is downward comprehensive, it must be that $(y + y' - z') \in \Gamma(x \wedge x')$. Consequently, this implies that $(y + y') = (y + y' - z') + z'$ belongs to $\Gamma(x \wedge x') + \Gamma(x \vee x')$.

A special case of property (3) appears in the study of cooperative games with non-transferable utility. In that context, set X is the collection of coalitions of a finite set N of players in a game, i.e., the power set of N, endowed with is the set inclusion order $\geq_X \,=\, \supseteq$. Thus, the pair $(X, \supseteq)$ is a lattice. For any coalition x, set $\Gamma(x) \subseteq \mathbb{R}^N$ consists of utility profiles (across all players in the game) that could result from the formation of that coalition. The game is said to be *cardinally convex* if (3) holds (see Sharkey, 1981, Section 2). That is, whenever the correspondence Γ is upper supermodular.

2.3 Examples of supermodular correspondences

An immediate implication of the definition is that upper supermodularity is preserved by downward comprehensive transformations. That is, if correspondence $\Gamma : X \to Y$ is upper supermodular, then so is $\bar{\Gamma}(x) := \big\{ y \in Y : y \leq_Y z, \text{ for some } z \in \Gamma(x) \big\}$. Analogously, lower supermodularity is preserved by upward comprehensive transformations.

It should be clear that upper and lower supermodularity are preserved by weighted sums; i.e., for any upper (lower) supermodular correspondences Γ, $\Lambda : X \to Y$, mapping $\Omega(x) := \alpha\Gamma(x) + \beta\Lambda(x)$ is an upper (lower) supermodular correspondence, for any positive scalars α and β. Below is a list of particular examples of supermodular correspondences.

Example 1. Function $g_i : X \to \mathbb{R}$ are supermodular over a lattice X, for all $i = 1, \ldots, \ell$, if and only if the map $G : X \to \mathbb{R}^\ell$, given by $G(x) := \big(g_1(x), \ldots, g_\ell(x)\big)$, is a supermodular function, i.e., we have $G(x \wedge x') + G(x \vee x') \geq G(x) + G(x')$, for all x, $x' \in X$, where $\geq$ denotes the natural product order on $\mathbb{R}^\ell$.

Example 2. Consider correspondence $\Gamma_i : X_i \to Y$, where $X_i \subseteq \mathbb{R}$ and Y is an ordered vector space, for $i = 1, 2$. The map $\Lambda : X_1 \times X_2 \to Y$, where $\Lambda(x_1, x_2) := \Gamma_1(x_1) + \Gamma_2(x_2)$, is a supermodular correspondence (in fact, it is also submodular).

Example 3. For any subset Z of an ordered vector space Y, a supermodular function $h : X \to Y$ over a lattice X, and positive scalars α and β, the mapping $\Gamma : X \to Y$, given by $\Gamma(x) := \big\{\alpha y + \beta h(x) : y \in Z\big\}$, is a supermodular correspondence.

Suppose that Y is the space of finite signed measures endowed with the first order stochastic dominance. If $Z \subseteq Y$ is a set of probability measures, while $h(x)$ is a probability measure for all $x \in X$, then for any positive scalars α and β such that $\alpha + \beta = 1$, set $\Gamma(x)$ is a subset of probability measures. This is an example of a supermodular correspondence that maps a lattice to the space of probability measures.

Example 4. Let Z be a convex subset of an ordered vector space Y, such that $z \geq 0$, for all $z \in Z$. For any positive, supermodular function $h : X \to \mathbb{R}_+$ over a lattice X, the correspondence $\Gamma : X \to Y$ given by $\Gamma(x) := \big\{h(x)z : z \in Z\big\}$ is supermodular.

This claim requires a short proof. Since Z is convex and non-negative, Lemma 5.27 in Aliprantis and Border (2006) guarantees that $\alpha Z + \beta Z = (\alpha + \beta)Z$, for any positive scalars α and β. To show that Γ is upper supermodular, take any $h(x)y \in \Gamma(x)$ and $h(x')y' \in \Gamma(x')$. Given the above property of set Z, there is some vector $v \in Z$ such that $h(x)y + h(x')y' = \big[h(x) + h(x')\big]v$. By supermodularity of function h,

$$\big[h(x) + h(x')\big]v \;\leq\; \big[h(x \wedge x') + h(x \vee x')\big]v.$$

Since $h(x \wedge x')v \in \Gamma(x \wedge x')$ and $h(x \vee x')v \in \Gamma(x \vee x')$, this concludes the proof. An analogous argument guarantees that Γ is also lower supermodular.

Example 5. Let X, T be lattices and Z be a sublattice of $X \times T$ (endowed with the product order). By X_Z we denote the set of elements in X for which there is some $t \in T$ such that $(x, t) \in Z$; it is straightforward to check that X_Z is a sublattice of X. Suppose that $h : Z \to Y$ is a supermodular function, where Y is an ordered real vector space. Then the correspondence $\Gamma : X_Z \to Y$, given by

$$\Gamma(x) \;:=\; \Big\{h(x, t) : (x, t) \in Z\Big\},$$

is upper supermodular. Indeed, take any $y \in \Gamma(x)$ and $y' \in \Gamma(x')$. By the definition of Γ, there is some t and t' in T such that $y = h(x,t)$ and $y' = h(x',t')$. Moreover, the supermodularity of function h implies that

$$h\big((x \wedge x'), (t \wedge t')\big) + h\big((x \vee x'), (t \vee t')\big) \;\geq\; h(x,t) + h(x',t').$$

Hence, the element $h\big((x,t) \wedge (x',t')\big)$ belongs to $\Gamma(x \wedge x')$ and $h\big((x,t) \vee (x',t')\big)$ is in $\Gamma(x \vee x')$, which concludes our argument.

3 Value functions of supermodular correspondences

In this section we present our main theorems on supermodular correspondences. While the proofs are simple, these results lead naturally to a wide range of applications.

Main Theorem. *Suppose that X is a lattice and Y is an ordered vector space. For any positive linear functional $\phi : Y \to \mathbb{R}$,*[11]

(i) *if correspondence $\Gamma : X \to Y$ is upper supermodular then the function $f : X \to \mathbb{R}$, given by $f(x) := \max\big\{\phi(y) : y \in \Gamma(x)\big\}$, is supermodular;*[12]

(ii) *if correspondence $\Gamma : X \to Y$ is lower supermodular then the function $f : X \to \mathbb{R}$, given by $f(x) := \min\big\{\phi(y) : y \in \Gamma(x)\big\}$, is supermodular.*

Proof. To show (i), take any x, $x' \in X$ and $y \in \Gamma(x)$, $y' \in \Gamma(x')$. By the upper supermodularity of Γ, there is some $z \in \Gamma(x \wedge x')$, $z' \in \Gamma(x \vee x')$ such that $z + z' \geq_Y y + y'$. Therefore, for any positive linear functional $\phi : Y \to \mathbb{R}$, we have

$$\begin{aligned}\phi(y) + \phi(y') \;&=\; \phi\big(y + y'\big) \;\leq\; \phi\big(z + z'\big) \;=\; \phi(z) + \phi(z') \\ &\leq\; \max\big\{\phi(v) : v \in \Gamma(x \wedge x')\big\} + \max\big\{\phi(v) : v \in \Gamma(x \vee x')\big\} \\ &=\; f(x \wedge x') + f(x \vee x'),\end{aligned}$$

where the first inequality follows from ϕ being positive and the second is implied by the definition of maximum. By taking the maximum over the left side of the inequality, we conclude that $f(x) + f(x') \leq f(x \wedge x') + f(x \vee x')$. Hence, f is supermodular.

[11] A linear functional $\phi : Y \to \mathbb{R}$ is *positive*, whenever $y \geq_Y z$ implies $\phi(y) \geq \phi(z)$, for all y, z in Y.

[12] We shall assume throughout this paper that a solution exists to any optimisation problem we consider, so that we could always speak of the maximum (minimum) rather than the supremum (infimum). That said, it is easy to check that both the Main Theorem and Main Theorem $(*)$ remain valid if the existence of an optimum is not guaranteed and we have to replace max (min) with sup (inf).

To prove (ii), take any $z \in \Gamma(x \wedge x')$, $z' \in \Gamma(x \vee x')$. By the lower supermodularity of Γ, there is some $y \in \Gamma(x)$, $y' \in \Gamma(x')$ such that $z + z' \geq_Y y + y'$. Therefore,

$$\begin{aligned}\phi(z) + \phi(z') &= \phi(z + z') \geq \phi(y + y') = \phi(y) + \phi(y') \\ &\geq \min\{\phi(v) : v \in \Gamma(x')\} + \min\{\phi(v) : v \in \Gamma(x)\} = f(x) + f(x'),\end{aligned}$$

where the first inequality follows from ϕ being positive and the second is implied by the definition of minimum. Once we take the minimum on the left of this inequality, we obtain $f(x \wedge x') + f(x \vee x') \geq f(x) + f(x')$, which concludes the proof. $\square$

In some applications one would like to investigate submodular properties of the value functions; in those instances the following analogue to the Main Theorem applies. We skip the proof since it is similar to the one supporting the previous result.

Main Theorem ($*$)**.** *Suppose that X is a lattice and Y is an ordered vector space. For any positive linear functional $\phi : Y \to \mathbb{R}$,*

(i) *if correspondence $\Gamma : X \to Y$ is upper submodular then function $f : X \to \mathbb{R}$, given by $f(x) := \min\{\phi(y) : y \in \Gamma(x)\}$, is submodular;*

(ii) *if correspondence $\Gamma : X \to Y$ is lower submodular then function $f : X \to \mathbb{R}$, given by $f(x) := \max\{\phi(y) : y \in \Gamma(x)\}$, is submodular.*

It is not hard to see that the assumptions in the Main Theorem are essentially tight. The following result gives a converse to the theorem in the case where Y is an Euclidean space. We postpone the proof until Appendix B.

Proposition 1. *Let X be a lattice, Y be an Euclidean space, and let correspondence $\Gamma : X \to Y$ have closed, convex values that are bounded from below (or above).*[13]

(i) *If function $f : X \to \mathbb{R}$, given by $f(x) := \max\{\phi(y) : y \in \Gamma(x)\}$, is supermodular for any positive linear functional $\phi : Y \to \mathbb{R}$, then Γ is upper supermodular.*

(ii) *If function $f : X \to \mathbb{R}$, given by $f(x) := \min\{\phi(y) : y \in \Gamma(x)\}$, is supermodular for any positive linear functional $\phi : Y \to \mathbb{R}$, then Γ is lower supermodular.*

[13] A subset $A \subseteq \mathbb{R}^\ell$ is *bounded from below* if there is some $z \in \mathbb{R}^\ell$ such that $z \leq y$, for all $y \in A$. Analogously, the set is bounded from *above* if there is some $z \in \mathbb{R}^\ell$ such that $z \geq y$, for all $y \in A$.

We prove the above result by contradiction and show that whenever a correspondence is not upper or lower supermodular, there always exists a positive linear functional ϕ for which the corresponding function f is not supermodular. Figure 2 summarises the principal idea of our proof. Since the argument employs the strong separating hyperplane theorem, it is crucial that values of Γ are closed, convex, and bounded. Nevertheless, correspondences satisfying these properties arise naturally in our economic applications.

4 Applications to production

The results developed in the last section lead to a wide range of applications that confirm the value of extending the concept of supermodularity to correspondences. We begin by applying our generalisation of supermodularity to comparative statics in production.

4.1 Complementarity in multi-output production

Consider a firm endowed with a technology that employs ℓ inputs to manufacture m output goods. We represent it by a production set $P \subseteq \mathbb{R}^\ell_+ \times \mathbb{R}^m_+$, where element (x, y) is a feasible production profile that uses inputs $x \in \mathbb{R}^\ell_+$ to produce an output $y \in \mathbb{R}^m_+$.[14]

Let X be the set of all input profiles under which a production is feasible. That is, it consists of all vectors x for which there is some y such that $(x, y) \in P$. The *production possibility correspondence* $\Gamma : X \to \mathbb{R}^m_+$ maps input vectors $x \in X$ to those combinations of output y that are feasible given the firm's technology, i.e.,

$$\Gamma(x) \;:=\; \Big\{ y \in \mathbb{R}^m_+ : (x, y) \in P \Big\}. \tag{4}$$

Conditional on strictly positive prices of inputs $p \in \mathbb{R}^\ell_{++}$ and outputs $q \in \mathbb{R}^m_{++}$, the problem of the firm is to choose input $x \in X$ in order to maximise

$$\pi(x, p) \;:=\; \max \Big\{ q \cdot y : y \in \Gamma(x) \Big\} - p \cdot x.$$

Even though we interpret vectors y as output profiles, there is a related but slightly different interpretation. Suppose that the firm is operating in a risky environment with

[14] A reader may notice that our definition of a production set is not the usual one, because we have not adopted the convention of writing inputs as negative entries in a production vector. The formulation we adopt is more convenient for our purposes.

m states of the world. Then, vector y determines all the contingent revenues that the firm may choose, when the input vector x is employed. If q is the probability distribution over different states, then $q \cdot y$ is the expected revenue under profile y.

We are interested in conditions on the technology P under which inputs x are *complements*. That is, all else constant, when it is optimal for the firm to increase demand for all factors of production if the price of at least one of them falls. Formally, we determine properties of the set P under which correspondence $\Phi : \mathbb{R}^{\ell}_{++} \to X$, given by

$$\Phi(p) \ := \ \arg\max \Big\{ \pi(x, p) : x \in X \Big\}.$$

for some $q \in \mathbb{R}^{m}_{++}$, decreases in p with respect to the strong set order. Complementarity of inputs can be guaranteed through a condition on Γ.

Proposition 2. *Inputs are complements if set X is a sublattice of $\mathbb{R}^{\ell}_{+}$ and correspondence $\Gamma : X \to \mathbb{R}^{m}_{+}$, defined above, is upper supermodular.*

This proposition follows from the Main Theorem and the MCS Theorem. First, for each $x \in X$, given output prices $q \in \mathbb{R}^{m}_{++}$, the firm determines the maximal revenue that is achievable under the available technology, i.e., $f(x) := \max \big\{ q \cdot y : y \in \Gamma(x) \big\}$. In the second step, the firm chooses input profile $x \in X$ to maximise its profit $\pi(x, p) = f(x) - p \cdot x$. From this observation and the MCS Theorem, we know that inputs are complements if function π is supermodular in x and has increasing differences in $(x, -p)$. The latter is always true given the formula for π, while the former is satisfied whenever f is supermodular. The Main Theorem guarantees that f is supermodular if the correspondence Γ is upper supermodular. This completes our argument.

The following examples show applications of this proposition.

Application 1. For a specific upper supermodular output correspondence, suppose there are three inputs and two outputs (or state contingent revenues), where

$$\Gamma(x_1, x_2, x_3) \ := \ \Big\{ (y_1, y_2) \in \mathbb{R}^2_{+} : y_1 \leq \sqrt[3]{x_1\, x_2\, t},\ y_2 \leq \sqrt{x_1} + \sqrt{x_3 - t},\ \text{for } t \in [0, x_3] \Big\}.$$

In the above example, input 1 is non-rivalrous since it can be used in its entirety to produce both outputs. On the other hand, input 3 has to be shared between the two productions, while input 2 is only used in the production of good 1.

To see that the above correspondence is upper supermodular, first notice that set

$$Z \;:=\; \Big\{(x_1, x_2, x_3, t) \in \mathbb{R}^4 : x_i \geq 0, \text{ for } i = 1, 2, 3, \text{ and } t \in [0, x_3]\Big\}$$

is a sublattice of $\mathbb{R}^4$. Moreover, $h : Z \to \mathbb{R}^2$, where $h(x, t) := \big(\sqrt[3]{x_1\, x_2\, t}, \sqrt{x_1} + \sqrt{x_3 - t}\big)$, is a supermodular function. Therefore, by the claim in Example 5, the correspondence $\tilde{\Gamma}(x) := \big\{h(x, t) : (x, t) \in Z\big\}$ is upper supermodular. Since $\Gamma(x)$ is a downward comprehensive hull of $\tilde{\Gamma}(x)$, correspondence Γ must also be upper supermodular.

Application 2. We are interested in conditions under which inputs are complements when the firm's production set is given by

$$P \;:=\; \Big\{(x, y) \in \mathbb{R}^\ell_+ \times \mathbb{R}^m_+ : g(x) \geq h(y)\Big\},$$

where $g : \mathbb{R}^\ell_+ \to \mathbb{R}_+$ and $h : \mathbb{R}^m_+ \to \mathbb{R}_+$ are strictly increasing functions.[15] In this case, for each input vector x in $X = \mathbb{R}^\ell_+$, we have

$$\Gamma(x) \;:=\; \Big\{y \in \mathbb{R}^m_+ : g(x) \geq h(y)\Big\}.$$

We claim that, whenever function g is supermodular and h is convex and homogeneous of degree 1, then correspondence Γ is supermodular, and thus upper supermodular, so that Proposition 2 applies. Indeed, define the set $Z := \big\{z \in \mathbb{R}^\ell_+ : 1 \geq h(z)\big\}$, which is positive and convex. By homogeneity of function h and our claim in Example 4, the correspondence $\Gamma(x) = g(x)Z$ is supermodular.

A straightforward modification of Proposition 2 allows us to discuss complementarities not only among inputs, but also across inputs and outputs. Below we discuss a specific example, however, this approach can be extended to general problems.

Application 3. Consider a firm producing outputs y_1 and y_2 using two inputs x_1 and x_2. The use of the first input is non-rivalrous but each unit of input 2 can be assigned to the production of either y_1 or y_2, but not both. Suppose that input (x_1, z_1) allows to produce up to $g_1(x_1, z_1)$ units of output y_1, while $g_2(x_1, z_2)$ is the output of good 2 when (x_1, z_2) is employed. Hence, the firm's production possibility set is given by

$$\begin{aligned} P \;:=\; \Big\{(x_1, x_2, y_1, y_2) \in \mathbb{R}^4_+ : y_1 \leq g_1(x_1, z_1),& \\ y_2 \leq g_2(x_1, z_2),\; \text{with } z_1 + z_2 = x_2,\; \text{and } z_1, z_2 \geq 0\Big\}&. \end{aligned}$$

[15] We could interpret $g(x)$ as the level of some intermediate good which can be produced with x; this intermediate good can then be transformed into different output goods via the function h.

We determine conditions on the primitives under which *input x_1 and output y_1 are complements*. That is, the firm finds it optimal to increase the demand for x_1 and the production of y_1 if either the price of input 1 decreases or the price of output 1 increases. We claim that the above property holds whenever: (i) function $g_2 : \mathbb{R}^2_+ \to \mathbb{R}_+$ is supermodular; (ii) $g_1 : \mathbb{R}^2_+ \to \mathbb{R}_+$ is continuous, strictly increasing, and supermodular, while $g_1(x_1, \cdot)$ is unbounded and concave, for all $x_1 > 0$; and (iii) both factors are essential, i.e., $g_1(x_1, 0) = g_1(0, z_1) = 0$ for all $x_1 \geq 0$ and $z_1 \geq 0$.

Let $X := \big\{(x_1, y_1) \in \mathbb{R}^2 : \ (x_1, x_2, y_1, y_2) \in P \text{ for some } x_2,\, y_2\big\}$. Since g_1 is continuous, $g_1(x_1, \cdot)$ is unbounded, and $g_1(x_1, 0) = 0$, for any $x_1 > 0$ and $y_1 \geq 0$, there is a unique $\varphi(x_1, y_1) \geq 0$ such that $g_1\big(x_1, \varphi(x_1, y_1)\big) = y_1$. In other words, $\varphi(x_1, y_1)$ is the least amount of input 2 needed to produce y_1 when x_1 units of the first input are used. As φ is well-defined for all $x_1 > 0$ and $y_1 \geq 0$, while $g_1(x_1, 0) = g_1(0, z_1) = 0$, for all $x_1 \geq 0$ and $z_1 \geq 0$, this implies that $X = (\mathbb{R}_{++} \times \mathbb{R}_+) \cup \big\{(0, 0)\big\}$, which is a lattice. Let

$$\Gamma(x_1, y_1) \ := \ \Big\{(-x_2, y_2) \in \mathbb{R}^2 \ : \ -x_2 \leq -\varphi(x_1, y_1) - z_2 \qquad \text{and} \ \ y_2 \leq g_2(x_1, z_2), \ \text{for } z_2 \geq 0\Big\}.$$

Since function g_1 is monotone, supermodular, and concave in the second argument, the function φ is submodular.[16] This implies that $h(x_1, y_1, z_2) := \big(-\varphi(x_1, y_1) - z_2, g_2(x_1, z_2)\big)$ is a supermodular function over a sublattice of $\mathbb{R}^3$. Following Example 5, correspondence $\Gamma(x_1, y_1) := \big\{y \in \mathbb{R}^2 : y \leq h(x_1, y_1, z_2), \ \text{for } z_2 \geq 0\big\}$ is upper supermodular.

Given the above, the problem of the firm can be reduced to:

$$\text{maximise} \ \ f(x_1, y_1) + q_1 \cdot y_1 - p_1 x_1, \ \ \text{subject to} \ \ (x_1, y_1) \in X,$$

where $f(x_1, y_1) := \max\big\{q_2 y_2 - p_2 x_2 : (-x_2, y_2) \in \Gamma(x_1, y_1)\big\}$. Since correspondence Γ is upper supermodular, the Main Theorem guarantees that the function f is supermodular. By MCS Theorem, this suffices for the optimal level of input x_1 and output y_1 to increase with respect to $(-p_1, q_1)$. Hence, the two commodities are complements.

[16] A quick way of verifying this is to assume that g_1 is sufficiently smooth and show that $\partial^2 \varphi / \partial a \partial k \leq 0$, but it is not difficult to give a discrete proof that dispenses with differentiability.

4.2 Factor prices and output

Suppose a firm produces a single output using ℓ inputs and has the production function $F : \mathbb{R}^\ell_+ \to \mathbb{R}_+$. We assume that the firm derives some benefit from output q, which we denote by $B(q)$ in $\mathbb{R}$. The objective of the firm is to choose inputs x in order to maximise $B\big(F(x)\big) - p \cdot x$, where we denote the input prices by p in $\mathbb{R}^\ell_+$.

The benefit $B(q)$ may be interpreted as the revenue derived from selling q units of the good, which is generally non-linear in q if the firm has monopoly power. Alternatively, the firm could face a risky output price s, in which case

$$B(q) \ := \ \int_0^\infty u(\tilde{s}q) d\lambda(\tilde{s}),$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is the Bernoulli index that summarises the firm's attitude towards risk, while λ is the probability distribution over the output price $\tilde{s}$.

We know that the factors are complements if the map from x to $B\big(F(x)\big)$ is supermodular. However, unless we make further assumptions about B, we cannot obtain such a conclusion even if F is supermodular. Nonetheless, with suitable assumptions on F alone, we can guarantee the the firm will *raise its output as the price of a factor falls.*[17]

Given the production function F, the firm's cost function $C : \mathbb{R}^\ell_+ \times \mathbb{R}_+ \to \mathbb{R}$ is

$$C(p, q) \ := \ \min \Big\{ p \cdot y : F(y) \geq q \Big\}. \tag{5}$$

To keep the exposition short, suppose that C is well-defined. Hence, the firm's optimisation is equivalent to choosing an output $q \geq 0$ that maximises $B(q) - C(p, q)$.

We wish to find conditions on function F under which the firm's output increases when prices of input factors fall. By the MCS Theorem, it suffices for the cost function C to have increasing differences in (p_i, q), for any prices p_{-i} of the remaining inputs. It is clear from the argument in Quah (2007) that a sufficient condition to imply this property on C is supermodularity and $(-i)$-concavity of the production function F.[18] However, the argument in that paper is rather roundabout — it uses the Envelope Theorem and relies on the differentiability of C, as well as various (rather strong) ancillary assumptions. Moreover, the conditions are only sufficient and not necessary. We provide a direct proof of this result and identify tight conditions on F under which the property holds.

[17]We are grateful to Eddie Dekel, whose queries inspired us to look at this issue more closely.

[18]A function f is $(-i)$-*concave* if, for any fixed x_i, it is a concave function of x_{-i}. Quah (2007) referred to this notion as "i-concavity". However, we find this term to be confusing and simply idiotic.

Proposition 3. *Suppose that a production function $F : \mathbb{R}^\ell_+ \to \mathbb{R}_+$ is continuous, increasing, and quasiconcave. The following statements are equivalent.*

(i) For any $q' \geq q$ and z, z' such that $F(z) \geq q$, $F(z') \geq q'$, there is some y, y' such that $F(y) \geq q$, $F(y') \geq q'$, where $z' \geq y$, $y' \geq z$, and $z + z' = y + y'$.

(ii) Let X be a subset of $\mathbb{R}$. Correspondence $\Gamma : X \times \mathbb{R}_+ \to \mathbb{R}^\ell_+$,

$$\Gamma(x, q) \ := \ \Big\{ a \in \mathbb{R}^\ell_+ : a_i = h_i(x) y_i, \ \textit{for all } i = 1, \dots, \ell, \ \textit{where } F(y) \geq q \Big\},$$

is lower supermodular for any positive and increasing functions $h_i : X \to \mathbb{R}_+$.

(iii) The cost function C has increasing differences in prices p and output level q.

Proof. Since the argument is rather extensive, we show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and postpone the proof of the converse until Appendix B.

To prove that (i) implies (ii), take any $x' \geq x$, $q' \geq q$ and $a \in \Gamma(x, t)$, $a' \in \Gamma(x', t')$. By definition of the correspondence Γ, there is some z, z' satisfying $F(z) \geq q$, $F(z') \geq q'$, where $a_i = h_i(x) z_i$ and $a'_i = h_i(x') z'_i$, for all $i = 1, \dots, \ell$. Given property (i), we can always find some y, y' such that $F(y) \geq q$, $F(y') \geq q'$, where $z' \geq y$, $y' \geq z$ and $z + z' = y + y'$. By monotonicity of functions h_i, it must be that

$$h_i(x') \big[z'_i - y_i \big] \ \geq \ h_i(x) \big[z'_i - y_i \big] \ = \ h_i(x) \big[y'_i - z_i \big],$$

for all $i = 1, \dots, \ell$. Define vectors b, b' such that $b_i = h_i(x') z_i$ and $b'_i = h_i(x) y'_i$. for all i. Clearly, we have $b \in \Gamma(x', q)$ and $b' \in \Gamma(x, q')$, where $a + a' \geq b + b'$.

Implication (ii) $\Rightarrow$ (iii) follows from the Main Theorem. Let $X = \{0, 1\}$. For any prices $p' \geq p$, take some positive and increasing functions $h_i : X \to \mathbb{R}_+$ that satisfy $h_i(1) = p'_i \geq p_i = h_i(0)$, for all $i = 1, \dots, \ell$. By (ii), the mapping $\Gamma : X \times \mathbb{R}_+ \to \mathbb{R}^\ell$,

$$\Gamma(x, q) \ := \ \Big\{ a \in \mathbb{R}^\ell_+ : a_i = h_i(x) y_i, \ \text{for all } i = 1, \dots, \ell, \ \text{where } F(y) \geq q \Big\},$$

is lower supermodular. The Main Theorem implies that function $f : X \times \mathbb{R}_+ \to \mathbb{R}$,

$$f(x, q) \ := \ \min \Big\{ \mathbf{1} \cdot a : a \in \Gamma(x, q) \Big\},$$

where $\mathbf{1}$ is the unit vector, is supermodular. By construction of Γ, we have

$$C(p, q) + C(p', q') \ = \ f(0, q) + f(1, q') \ \geq \ f(1, q) + f(0, q') \ = \ C(p', q) + C(p, q').$$

Thus, the cost function has increasing differences in prices p and output level q. $\square$

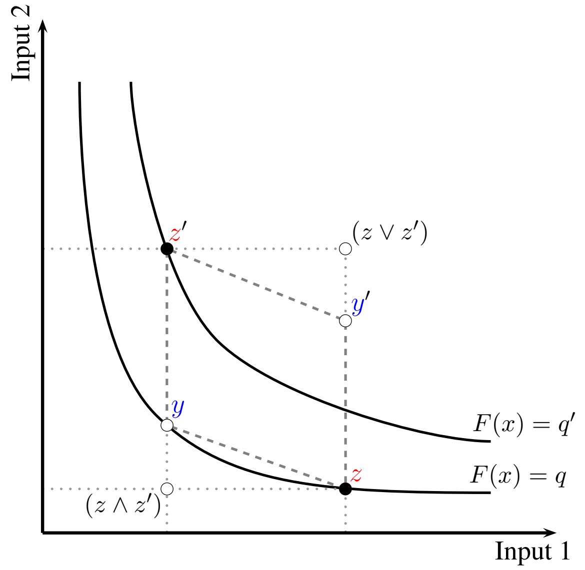

Figure 3: A graphical interpretation of the arguments presented in Example 6 (left) and 7 (right). In each case, the production F satisfies condition (i) in Proposition 3.

Remark 1. Implication (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *does not* require for F to be continuous, increasing, or quasiconcave. We employ the ancillary conditions only to prove the converse.

Proposition 3 characterises technologies for which the cost function has increasing differences with respect to input prices and output. The condition states that, for any input vectors z, z' under which the firm can produce q, q' units of the final good, respectively, it is always possible to determine inputs y, y' that produce at least the same output q, q' and satisfy $z + z' = y + y'$. Therefore, this technology allows the firm to maintain the same level of production with at most the same amount of each input.

We conclude this subsection by introducing two classes of production functions that satisfy condition (i) in Proposition 3.

Example 6. Let production $F : \mathbb{R}^\ell_+ \to \mathbb{R}$ be continuous, increasing, supermodular, and $(-i)$-concave, for all $i = 1, \ldots, \ell$.[19] Then, it satisfies condition (i) in Proposition 3.

Indeed, take any $q' \geq q$ and z, z' such that $F(z) \geq q$, $F(z') \geq q'$. We show that there is some y, y' such that $F(y) \geq q$, $F(y') \geq q'$, where $z' \geq y$, $y' \geq z$, and $z + z' = y + y'$. Whenever $F(z \wedge z') \geq q$, choose $y := (z \wedge z')$ and $y' := (z \vee z')$. Given that $z' \geq (z \wedge z')$, $(z \vee z') \geq z$ and $z + z' = (z \wedge z') + (z \vee z')$, the condition is satisfied.

[19]Note that, it is possible for a function to be $(-i)$-concave for all i without being concave. For example, $f(x_1, x_2) = x_1 x_2$ is $(-i)$-concave for $i = 1, 2$, but it is not concave.

Alternatively, suppose that $F(z \wedge z') < q$. As in Figure 3 (left), continuity of the function F guarantees that there is some $y := \alpha z' + (1-\alpha)(z \wedge z')$ such that $F(y) = q$, for $\alpha \in (0,1)$. In addition, it must be that $y_i = z_i = (z \wedge z)_i$, for some i. Since the production function F is supermodular and $(-i)$-concave, by Proposition 2 in Quah (2007), it is $\mathcal{C}_i$-supermodular. This suffices to show that $F(y') \geq q'$, where $y' = (1-\alpha)(z \vee z') + \alpha z$. Clearly, we have $z' \geq y$, $y' \geq z$, and $z + z' = y + y'$.

Example 7. Suppose that production function $F : \mathbb{R}^\ell_+ \to \mathbb{R}$ is homothetic and quasi-concave. Then, it satisfies condition (i) in Proposition 3.

Take any $q' \geq q$ and z, z' such that $F(z) \geq q$, $F(z') \geq q'$. As previously, we show that there is some y, y' such that $F(y) \geq q$, $F(y') \geq q'$, where $z' \geq y$, $y' \geq z$, and $z + z' = y + y'$. If $F(z) \geq F(z')$, choose $y := z'$ and $y' := z$. Since $F(z) \geq F(z') \geq q' \geq q$, the claim holds trivially. Whenever $F(z') > F(z)$, homotheticity of F implies that there is some $\alpha > 1$ such that $F(\alpha z) = F(z')$. As in Figure 3 (right), denote $\tilde{z} := \alpha z$ and choose $y := (1/\alpha)z'$, $y' := z' - (1/\alpha)(z' - \tilde{z})$. Clearly, we have $F(y) \geq q$. Moreover, since F is quasiconcave, it must be that $F(y') \geq q'$. Finally, by construction, we obtain $y' - z = z' - y \geq 0$, which concludes our proof.

5 Supermodular correspondences and uncertainty

In this subsection we apply the tools developed in Section 3 to the study of comparative statics in various models of choice under uncertainty.

Consider an agent who chooses an action $x \in X$ before the realisation of some state $s \in S$, where X and S are both subsets of $\mathbb{R}$. Given x, the agent's utility is $g(x,s)$ whenever state s is realised. Assuming that λ is a probability distribution over S, the agent chooses x to maximise the expected utility $\int_S g(x,\tilde{s})d\lambda(\tilde{s})$. If g is a supermodular function and the agent is allowed to choose her action *after* observing the state, then we know that her action will increase with the state. Therefore, it is intuitive that under the same condition on g, if the agent has to make a decision *before* the state is realised, then she will pick a higher action if she thinks that higher states are more likely. This turns out to be true; more precisely, it can be shown that a first order stochastic dominance shift in the distribution of s will indeed lead to a higher optimal action.

In this section we extend this result to the case where the agent has non-expected utility preferences. We introduce the notion of the first order stochastic dominance that is applicable to the study of multi-prior beliefs.

Throughout this subsection we denote the set of cumulative probability distributions over a state space $S \subseteq \mathbb{R}$ by $\triangle_S$. In addition, we order its elements with respect to the first order stochastic dominance $\succeq$. That is, for any distributions λ, λ' in $\triangle_S$, we have $\lambda' \succeq \lambda$ whenever $\int_S u(\tilde{s})d\lambda'(\tilde{s}) \geq \int_S u(\tilde{s})d\lambda(\tilde{s})$, for any increasing function $u : S \to \mathbb{R}$. Equivalently, this is to say that $\lambda'(s) \leq \lambda(s)$, for all $s \in S$.

An important feature of the space of distributions $\triangle_S$ ordered with respect to the first order stochastic dominance, is that it is a lattice. In particular, for any two probability distributions λ, λ' their meet and join are defined by $(\lambda \wedge \lambda')(s) = \max\big\{\lambda(s), \lambda'(s)\big\}$ and $(\lambda \vee \lambda')(s) = \min\big\{\lambda(s), \lambda'(s)\big\}$, for all $s \in S$, respectively.[20]

5.1 Shifts of beliefs under ambiguity aversion

Suppose that the agent does not know the probabilities over the states of the world or no such objective probabilities exist. Instead, beliefs of the decision maker are represented by a subset Λ of probability distributions over S. This is to say that, since the actual stochastic model of the world is ambiguous, the agent is endowed with multiple beliefs that she considers a possible description of the environment.

The *maxmin* model of Gilboa and Schmeidler (1989) investigates the case in which the agent is *ambiguity averse*; in this case, the agent's preference is represented by

$$\min\left\{\int_S g(x,\tilde{s})d\lambda(\tilde{s}) : \lambda \in \Lambda\right\},$$

for some function $g : X \times S \to \mathbb{R}$. In this subsection we are interested in shifts in beliefs under which the agent finds it optimal to increase the optimal action x, assuming that the function g is supermodular with respect to (x, s).

Before we proceed with the main question, we investigate a more fundamental issue. Suppose that an ambiguity averse agent evaluates each state $s \in S$ with a utility function $u : S \to \mathbb{R}$. Given beliefs Λ, the utility of the decision maker is

$$\min\left\{\int_S u(\tilde{s})d\lambda(\tilde{s}) : \lambda \in \Lambda\right\}.$$

[20]Note that, it is crucial that the state space S is a subset of $\mathbb{R}$.

Assuming that the function u is increasing, what shift in the beliefs Λ would make the agent better off? Consider the following proposition.

Proposition 4. *Let S be a subset of $\mathbb{R}$ and Λ, Λ' be subsets of $\triangle_S$. If for any $\lambda' \in \Lambda'$ there is some distribution $\lambda \in \Lambda$ such that $\lambda' \succeq \lambda$, then*

$$\min\left\{\int_S u(\tilde{s})d\lambda(\tilde{s}) : \lambda \in \Lambda'\right\} \;\geq\; \min\left\{\int_S u(\tilde{s})d\lambda(\tilde{s}) : \lambda \in \Lambda\right\},$$

for any increasing function $u : S \to \mathbb{R}$. Moreover, if set S is finite and the beliefs Λ, Λ' are compact and convex, then the converse is also true.

Indeed, take any $\lambda' \in \Lambda'$. By our assumption and the definition of the first order stochastic dominance $\succeq$, there is some $\lambda \in \Lambda$ such that $\int_S u(\tilde{s})d\lambda'(\tilde{s}) \geq \int_S u(\tilde{s})d\lambda(\tilde{s})$. In particular, it must be that $\int_S u(\tilde{s})d\lambda'(\tilde{s}) \geq \min\big\{\int_S u(\tilde{s})d\lambda(\tilde{s}) : \lambda \in \Lambda\big\}$. To conclude our argument, it suffices to take the minimum over the left hand side of the inequality. We delay the proof the second part of the result until Appendix B.

The above observation characterises shifts in beliefs that are preferable for any ambiguity averse agent with an increasing Bernoulli utility u. Set of beliefs Λ' is better that Λ, if for any distribution in the former set at least one belief in the latter is stochastically dominated. For singleton beliefs, this reduces to the first order stochastic dominance.

We return to our initial question. The notion of stochastic dominance specified in Proposition 4 is *not* sufficient for the agent to increase the optimal action, even if her utility function is supermodular with respect to the action and the state.

For example, suppose that $S = \{s_1, s_2, s_3\}$ and consider three distributions λ, λ', and μ in $\triangle_S$, given by $\lambda(s_1) = 1/4$, $\lambda(s_2) = 7/8$, $\lambda'(s_1) = \lambda'(s_2) = \mu(s_1) = 1/2$, and $\mu(s_2) = 3/4$, where $\lambda(s_3) = \lambda'(s_3) = \mu(s_3) = 1$. Moreover, suppose that $\Lambda := \{\lambda, \mu\}$ and $\Lambda' := \{\lambda'\}$. Since $\lambda' \succeq \mu$, the condition in Proposition 4 is satisfied. Nevertheless, it is possible to determine a supermodular function $g : \{0, 1\} \times S \to \mathbb{R}$ such that the agent strictly prefers 1 to 0 under beliefs Λ, and 0 to 1 under Λ'.[21] Thus, it is optimal to lower the action from 1 to 0 as beliefs increase in the aforementioned sense.

In order to guarantee that the agent finds it optimal to increase the action as beliefs improve, we need to impose a stronger condition.

[21]For example let $g(0, s_1) = g(0, s_2) = 5$, $g(0, s_3) = 21$, $g(1, s_1) = 0$, $g(1, s_2) = 8$, and $g(1, s_3) = 24$. This function is both supermodular on $\{0, 1\} \times S$ and increasing on S.

Proposition 5. *Let X, S, $T \subseteq \mathbb{R}$, where $S = \{s_1, s_2, \dots, s_{\ell+1}\}$ is finite, and $\Lambda : T \to \triangle_S$ be a correspondence with compact, convex values. The following are equivalent.*

(i) For any $t' \geq t$ and $\lambda \in \Lambda(t)$, $\lambda' \in \Lambda(t')$ there is some $\mu \in \Lambda(t)$, $\mu' \in \Lambda(t')$ such that

$$\lambda' \succeq \mu, \quad \mu' \succeq \lambda, \quad \textit{and} \quad \tfrac{1}{2}\lambda + \tfrac{1}{2}\lambda' = \tfrac{1}{2}\mu + \tfrac{1}{2}\mu'.$$

(ii) For any supermodular function $g : X \times S \to \mathbb{R}$, correspondence $\Gamma : X \times T \to \mathbb{R}^\ell$,

$$\Gamma(x,t) \ := \ \Big\{ a \in \mathbb{R}^\ell : a_i = -\delta_i(x)\lambda(s_i), \ \text{for all } i = 1, \dots, \ell, \ \text{where } \lambda \in \Lambda(t) \Big\}$$

is lower supermodular, where $\delta_i(x) = \big[g(x, s_{i+1}) - g(x, s_i)\big]$, for all $i = 1, \dots, \ell$.

(iii) Function $f : X \times T \to \mathbb{R}$, where

$$f(x,t) \ := \ \min \left\{ \int_S g(x, \tilde{s}) d\lambda(\tilde{s}) : \lambda \in \Lambda(t) \right\},$$

is supermodular for any supermodular function $g : X \times S \to \mathbb{R}$.

Proof. To show that (i) implies (ii), take any $x' \geq x$, $t \geq t$ and $a \in \Gamma(x,t)$, $a' \in \Gamma(x', t')$. By definition of the correspondence Γ, there are some distributions $\lambda \in \Lambda(t)$, $\lambda' \in \Lambda(t')$ such that $a_i = -\delta_i(x)\lambda(s_i)$ and $a'_i = -\delta_i(x')\lambda'(s_i)$, for all $i = 1, \dots, \ell$. Given property (i), there is some $\mu \in \Lambda(t)$, $\mu' \in \Lambda(t')$ such that $\lambda'(s_i) \leq \mu(s_i)$, $\mu'(s_i) \leq \lambda(s_i)$, and $\lambda(s_i) + \lambda'(s_i) = \mu(s_i) + \mu'(s_i)$, for all $i = 1, \dots, \ell$. Since function $g : X \times S \to \mathbb{R}$ is supermodular if and only if $\delta_i(x)$ is increasing, for all $i = 1, \dots, \ell$, we obtain

$$\delta_i(x')\big[\mu(s_i) - \lambda'(s_i)\big] \ \geq \ \delta_i(x)\big[\mu(s_i) - \lambda'(s_i)\big] \ = \ \delta_i(x)\big[\lambda(s_i) - \mu'(s_i)\big],$$

for all $i = 1, \dots, \ell$. Construct vectors b, b' where $b_i := -\delta_i(x')\mu(s_i)$ and $b'_i := -\delta_i(x)\mu'(s_i)$, for all i. Clearly, we have $b \in \Gamma(x', t)$, $b' \in \Gamma(x, t')$, and $a + a' \geq b + b'$.

To show (ii) $\Rightarrow$ (iii), note that for any function $g : X \times S \to \mathbb{R}$ and distribution λ,

$$\begin{aligned} \int_S g(x, \tilde{s}) d\lambda(\tilde{s}) \ &= \ g(x, s_1)\lambda(s_1) + \sum_{i=1}^{\ell} g(x, s_{i+1})\big[\lambda(s_{i+1}) - \lambda(s_i)\big] \\ &= \ g(x, s_{\ell+1})\lambda(s_{\ell+1}) + \sum_{i=1}^{\ell} \big[g(x, s_i) - g(x, s_{i+1})\big]\lambda(s_i) \qquad (6) \\ &= \ g(x, s_{\ell+1}) + \sum_{i=1}^{\ell} \big[-\delta_i(x)\lambda(s_i)\big], \end{aligned}$$

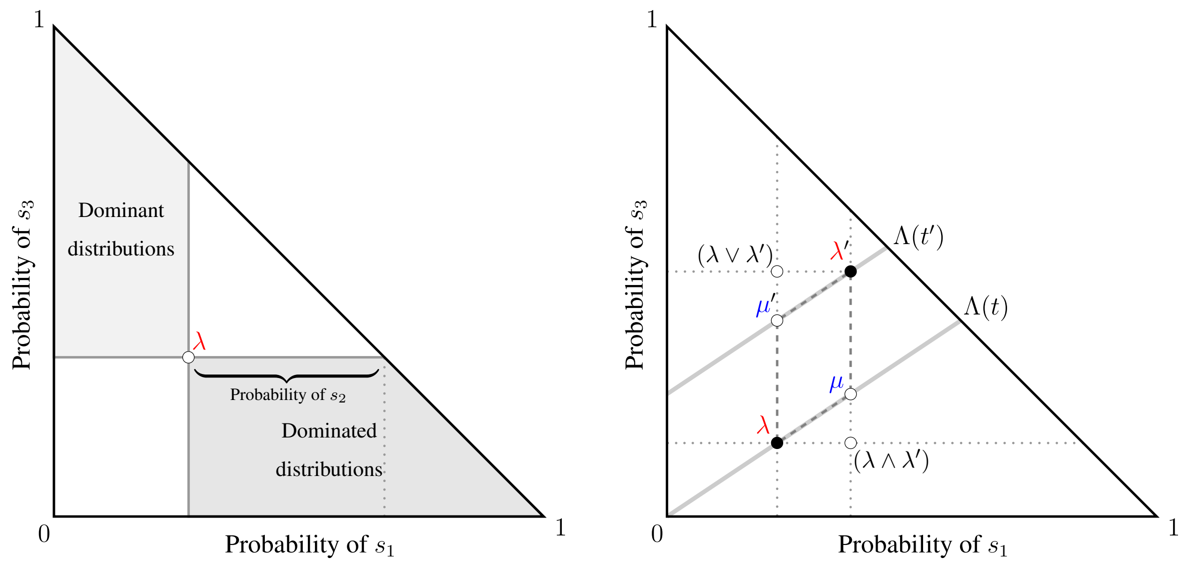

Figure 4: The points on the graphs denote probability measures corresponding to cumulative distributions in $\triangle_S$, for $S = \{s_1, s_2, s_3\}$. On the right, the thick straight lines represent values $\Lambda(t)$ and $\Lambda(t')$ from Example 9, for some function $h : S \to \mathbb{R}$.

since $\lambda(s_{\ell+1}) = 1$. Therefore, function f in part (iii) can be reformulated as

$$f(x,t) \;=\; g(x, s_{\ell+1}) + \min\Big\{\mathbf{1} \cdot a : a \in \Gamma(x,t)\Big\},$$

where $\mathbf{1}$ is the unit vector and Γ is defined as in (ii) for the function g. Since Γ is lower supermodular, by the Main Theorem function f is supermodular. The proof of implication (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) is extensive, hence, we postpone it until Appendix B. $\square$

Remark 2. Implication (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) does not require for the values Λ to be compact or convex. The additional assumptions are employed to prove the converse.

We prove the following remark formally in Appendix B.

Remark 3. Proposition 5 remains true if S is a compact interval of $\mathbb{R}$ and function $g(x, \cdot)$ is Riemann-Stieltjes integrable over S with respect to each $\lambda \in \Lambda(t)$, for all $x \in X$ and $t \in T$. In particular, this holds if at least one of the following conditions is satisfied: (a) function $g(x,s)$ is continuous in $s \in S$; (b) $g(x,s)$ is bounded on S and has only finitely many discontinuities in s, and all distributions in $\Lambda(t)$ are atomless; or (c) $g(x,s)$ is bounded on S and monotone, and all distributions in $\Lambda(t)$ are atomless.

Condition (i) in Proposition 5 requires that for any $t' \geq t$ and distribution $\lambda' \in \Lambda(t')$ there is some $\mu \in \Lambda(t)$ such that $\lambda' \succeq \mu$. Therefore, the belief correspondence has to satisfy the property introduced in Proposition 4. However, the condition imposes an additional form of consistency on how the beliefs are formed. Namely, it requires that any two distributions $\lambda \in \Lambda(t)$ and $\lambda' \in \Lambda(t')$ can be explained in terms of equivalent shifts of probabilities across the states of the world. That is, one can always find distributions $\mu \in \Lambda(t)$, $\mu' \in \Lambda(t')$ such that the stochastic shift $(\lambda' - \mu)$ is equal to $(\mu' - \lambda)$. Therefore, if λ' can be obtained via particular changes in probabilities across states from μ, then μ' can be constructed through the same changes starting from λ.

Below we present examples of correspondences satisfying condition (i) in Proposition 5.

Example 8 (Strong set order)**.** Suppose that correspondence $\Lambda : T \to \triangle_S$ increases in the *strong set order* induced by the first order stochastic dominance $\succeq$. Then it satisfies condition (i) from Proposition 5. Recall that Λ increases in that sense if for any $t' \geq t$ and $\lambda \in \Lambda(t)$, $\lambda' \in \Lambda(t')$, we have $(\lambda \wedge \lambda') \in \Lambda(t)$ and $(\lambda \vee \lambda') \in \Lambda(t')$, where the meet and the join are defined at the beginning of this section. Since $\lambda' \succeq (\lambda \wedge \lambda')$, $(\lambda \vee \lambda') \succeq \lambda$, and $(\lambda \wedge \lambda') + (\lambda \vee \lambda') = \lambda + \lambda'$, the condition is satisfied.

For example, suppose that $\mu(\cdot, t)$ and $\nu(\cdot, t)$ are two probability distributions in $\triangle_S$, parametrised by $t \in T$. In addition, assume that the two distributions increase in t. That is, if $t' \geq t$ then $\mu(\cdot, t') \succeq \mu(\cdot, t)$ and $\nu(\cdot, t') \succeq \nu(\cdot, t)$. It can be shown that, whenever $\nu(\cdot, t) \succeq \mu(\cdot, t)$, for all $t \in T$, then correspondence $\Lambda : T \to \triangle_S$, given by

$$\Lambda(t) \ := \ \Big\{\lambda \in \triangle_S : \nu(\cdot, t) \succeq \lambda \succeq \mu(\cdot, t)\Big\},$$

increases in the strong set order induced by the stochastic dominance $\succeq$.

Example 9 (Increasing mean)**.** Take any function $h : S \to \mathbb{R}$ and suppose that values $\Lambda(t)$ of correspondence $\Lambda : T \to \triangle_S$ consist of all distributions over S for which the expected value of this random variable is equal to t. Formally, let

$$\Lambda(t) \ := \ \left\{\lambda \in \triangle_S : \int_S h(\tilde{s}) d\lambda(\tilde{s}) = t\right\}.$$

We show in Appendix B that, in such a case, correspondence Λ satisfies condition (i) in Proposition 5. Moreover, unlike the correspondence in Example 8, the above mapping Λ does not increase with respect to the strong set order. See also Figure 4 (right).

Example 10 (Ambiguous vs non-ambiguous states)**.** In this example we consider a case in which the agent distinguishes between states that are non-ambiguous, i.e., the probability of these events is known; and ambiguous, for which the agent assigns a range of probabilities when formulating beliefs. Formally, let Ω denote a finite event space and $\mathcal{M}_\Omega$ be the set of all probability measures over that set. Suppose that non-ambiguous states are contained in set $\Omega' \subseteq \Omega$ and probabilities of these events are induced by measure $\nu^* \in \mathcal{M}_\Omega$. The space of beliefs consistent with ν^* is summarised by set

$$\mathcal{P}_{\Omega'} \;:=\; \Big\{\nu \in \mathcal{M}_\Omega : \nu(A) = \nu^*(A), \text{ for all } A \subseteq \Omega'\Big\}.$$

Consider an act $h : \Omega \times T \to S$ that assigns event $\omega \in \Omega$ and an exogenous parameter $t \in T$ to some outcomes in S, and let correspondence $\Lambda : T \to \triangle_S$ be given by

$$\Lambda(t) \;:=\; \Big\{\lambda \in \triangle_S : \lambda(s) = \nu\big(\{\omega \in \Omega : h(\omega, t) \leq s\}\big), \text{ for all } s \in S, \text{ where } \nu \in \mathcal{P}_{\Omega'}\Big\}.$$

Therefore, set $\Lambda(t)$ contains all cumulative probability distributions over values of the function h that are consistent with beliefs in $\mathcal{P}_{\Omega'}$.

Suppose that function h increases in t only for non-ambiguous states and remains constant for the ambiguous ones. Formally, for any $t' \geq t$, we have $h(\omega, t') \geq h(\omega, t)$, if $\omega \in \Omega'$, and $h(\omega, t') = h(\omega, t)$ otherwise. We show in the Appendix B that, in such a case, correspondence Λ satisfies condition (i) in Proposition 5.

By the MCS Theorem, Proposition 5 can be applied directly to determine how optimal decisions vary with respect to shifts in beliefs in the model of Gilboa and Schmeidler. Under the conditions imposed on the belief correspondence Λ, the utility $f(x, t)$ of an ambiguity averse agent is supermodular. Therefore, the corresponding set of optimal actions $\Phi(t) = \operatorname{argmax}\big\{f(x, t) : x \in X\big\}$ increases in the strong set order in t.

Remark 4. The claim in Proposition 5 remains true even if we leave part (i) unchanged; replace "lower supermodularity" in part (ii) with "upper supermodularity"; and replace the "min" operator in part (iii) with "max".[22]

Remark 4 allows us to go beyond ambiguity aversion. The α-maxmin model by Ghirardato, Maccheroni, and Marinacci (2004), which generalises the above, allows for both ambiguity averse and ambiguity loving behaviour, with the agent's utility

$$\alpha \;\min\left\{\int_S g(x, \tilde{s}) d\lambda(\tilde{s}) : \lambda \in \Lambda(t)\right\} \;+\; (1 - \alpha)\;\max\left\{\int_S g(x, \tilde{s}) d\lambda(\tilde{s}) : \lambda \in \Lambda(t)\right\}$$

[22]This can be shown analogously to Proposition 5. Following Proposition A.2 in Appendix B, condition (i) is necessary and sufficient for the correspondence Γ in (ii) to be upper supermodular. By the Main Theorem, the latter is necessary and sufficient for the function f in (iii) to be supermodular.

for some $\alpha \in [0, 1]$. By Proposition 5 and Remark 4, both elements of the sum are supermodular functions. Since supermodularity of functions is preserved under summation, an agent whose preference is described by the α-maxmin preferences will take a higher action when there is a shift in her beliefs towards higher states.

In the case of the maxmin model of choice under ambiguity, whenever function g is increasing with respect to $s \in S$, for all $x \in X$, one can assume that values of the belief correspondence Λ is upper comprehensive, without loss of generality.[23] That is, for all $t \in T$, if $\lambda \in \Lambda(t)$ and $\lambda' \succeq \lambda$ then $\lambda' \in \Lambda(t)$, for any $\lambda' \in \triangle_S$. In the following result we show that, even when function g is *increasing* over S and supermodular, condition (i) in Proposition 5 remains necessary and sufficient for function f to be supermodular.

Proposition 6. *Let X, S, $T \subseteq \mathbb{R}$, where S is finite, and $\Lambda : T \to \triangle_S$ be a correspondence with compact, convex, an upper comprehensive values. Correspondence Λ satisfies condition (i) in Proposition 5 if and only if, function $f : X \times T \to \mathbb{R}$,*

$$f(x,t) \;:=\; \min\left\{\int_S g(x,\tilde{s})d\lambda(\tilde{s}) : \lambda \in \Lambda(t)\right\},$$

is supermodular for all functions $g : X \times S \to \mathbb{R}$ that are increasing over S, for any fixed $x \in X$, and supermodular over $X \times S$.

We postpone the proof of this proposition until Appendix B. We conclude this subsection with two economic applications of the results discussed above.

Application 4 (Portfolio problem)**.** An investor divides her wealth $m > 0$ between a *safe asset*, that pays out $r > 0$ for sure, and a *risky asset* with an uncertain gross payout of s in $S \subseteq \mathbb{R}_+$. The investor's beliefs over the risky return is captured by the correspondence $\Lambda : T \to \triangle_S$, where $\triangle_S$ is the space of probability distributions over S.

The investor chooses to invest $x \in X \subset \mathbb{R}$ in the risky asset, with the rest of her wealth invested in the safe security. We allow the investor to go short on either asset but require her to be solvent, i.e., it must be that $xs + (m-x)r \geq 0$, for all $s \in S$ and $x \in X$. Assuming that her Bernoulli index is $u : \mathbb{R}_+ \to \mathbb{R}$ and the investor is ambiguity averse,

[23] Take any subset Λ of $\triangle_S$ and let $\bar{\Lambda} := \{\lambda \in \triangle_S : \lambda \succeq \lambda',\ \text{for}\ \lambda' \in \Lambda\}$ be its upper comprehensive hull. Since $\Lambda \subseteq \bar{\Lambda}$, we have $\min\{\int_S u(\tilde{s})d\lambda(\tilde{s}) : \lambda \in \Lambda\} \geq \min\{\int_S u(\tilde{s})d\lambda(\tilde{s}) : \lambda \in \bar{\Lambda}\}$, for any function $u : S \to \mathbb{R}$. Moreover, for any $\lambda' \in \bar{\Lambda}$ there is some $\lambda \in \Lambda$ such that $\lambda' \succeq \lambda$. By Proposition 4, this implies that the reveres inequality also holds for any increasing u. Hence, the two values are equal.

the investor's utility at $x \in X$ is given by

$$f(x,t) \ := \ \min\left\{\int_S u\big(x\tilde{s} + (m-x)r\big)d\lambda(\tilde{s}) : \lambda \in \Lambda(t)\right\}. \quad (7)$$

To capture the idea that a higher t represents greater optimism, we assume that correspondence Λ satisfies the property (i) in Proposition 5. In particular, this implies that the function f is supermodular if $g(x,s) := u\big(xs + (m-x)r\big)$ is supermodular. Assuming that u is strictly increasing, concave, and twice continuously differentiable, it is straightforward to check that g is supermodular if the coefficient of relative risk aversion of u is less than 1.[24] Therefore, with this condition on u, we can apply the MCS Theorem to guarantee that the investor's holding in the risky asset increases with t. This conclusion holds even if the investor's preference has the α-maxmin form.[25]

The next example has a different flavour from Example 4: it has both x and t as choice variables and exploits the fact that supermodularity is preserved by the sum.

Application 5. A firm operating in uncertain market conditions must decide on how much to produce and how much to spend on promoting its product via advertising. In period 1, the marginal cost of production is $c > 0$ and the marginal cost of advertising is $a > 0$. If the firm chooses t units of advertising, its belief on the demand for its output is given by a multi-prior set $\Lambda(t)$ of probability distributions over the set $S \subseteq \mathbb{R}_+$. We assume throughout that the price of the good is normalised to 1.

In period 2, the firm's actual demand s is realised and the firm has to meet this demand even if it exceeds its period 1 output; the profit in period 2 is

$$\pi(x,s) \ := \ s - \kappa\big(\max\{s-x, 0\}\big).$$

Function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ should be interpreted as the cost of producing the additional units to meet demand in period 2. At the same time, goods for which there is no demand can be freely disposed. Also, notice that $\pi(x,s)$ need not be increasing in s.

[24]Note that, since x can take negative values, function g does not increase in s.

[25]We are not the first to discuss comparative statics of the portfolio choice model under ambiguity. For example, Gollier (2011) examines how the demand for the risky asset changes with the level of ambiguity aversion, in the context of the smooth ambiguity model. Cherbonnier and Gollier (2015) study both the smooth ambiguity model and the α-maxmin model; the authors provide conditions under which the demand for the risky asset increases with respect to initial wealth.

The firm chooses $x \geq 0$ and $t \geq 0$ in period 1 to maximise

$$\Pi(x,t,c,a) \;:=\; \min\left\{\int_S \pi(x,\tilde{s})d\lambda(\tilde{s}) : \lambda \in \Lambda(t)\right\} - cx - at.$$

It is straightforward to check that the function π is supermodular if κ is increasing, convex, and $\kappa(0) = 0$.[26] Given this, Proposition 5 guarantees that

$$f(x,t) \;=\; \min\left\{\int_S \pi(x,\tilde{s})d\lambda(\tilde{s}) : \lambda \in \Lambda(t)\right\}$$

is a supermodular function of (x,t) and therefore Π is supermodular in (x,t). Furthermore, Π has increasing differences in $\big((x,t),(-c,-a)\big)$. Applying the MCS Theorem, we conclude that more advertising and higher output will ensue from either a fall in the cost of advertising a or a fall in the cost of period 1 production c.

5.2 Variational and multiplier preferences

It is possible to extend Proposition 5 to cover a broader class of choice problems. Maccheroni, Marinacci, and Rustichini (2006) introduce and axiomatise a generalisation of the Gilboa and Schmeidler maxmin model, called *variational preferences*. In this case, the utility of some action x is $f(x) = \min\big\{\int_S g(x,\tilde{s})d\lambda(\tilde{s}) + c(\lambda) : \lambda \in \triangle_S\big\}$. Loosely speaking, the agent's utility from the action x is obtained by minimising her expected utility over a set of all probability distributions. However, unlike the maxmin model where the agent is restricted to a subset of $\triangle_S$, any distribution in $\triangle_S$ could be 'picked' in the variational preferences model, though each element λ is associated with a different cost $c(\lambda)$. For a detailed discussion see Maccheroni, Marinacci, and Rustichini (2006) or the survey in Epstein and Schneider (2010).

In the following result, we parametrise the cost function c by $t \in T \subseteq \mathbb{R}$ and identify conditions under which the agent's utility is supermodular in (x,t).

Proposition 7. *Let X, S, $T \subseteq \mathbb{R}$, where $S = \{s_1, s_2, \ldots, s_{\ell+1}\}$ is finite, and function $c : \triangle_S \times T \to \mathbb{R}_+$ is continuous and convex on $\triangle_S$, for all $t \in T$. These are equivalent:*

(i) For any $t' \geq t$ in T and λ, λ' in $\triangle_S$ there is some μ, μ' in $\triangle_S$ such that $\lambda' \succeq \mu$, $\mu' \succeq \lambda$, $\frac{1}{2}\lambda + \frac{1}{2}\lambda' = \frac{1}{2}\mu + \frac{1}{2}\mu'$, and $c(\lambda,t) + c(\lambda',t') \geq c(\mu,t) + c(\mu',t')$.

[26] Take any $x' \geq x$ and consider three cases. If (i) $s \leq x$, then $\delta(s) := \big[\pi(x',s) - \pi(x,s)\big] = 0$; whenever (ii) $x < s \leq x'$, then $\delta(s) = \kappa(s-x)$; and finally (iii) $s > x'$ implies $\delta(s) = \kappa(s-x) - \kappa(s-x')$. In either case, under the assumptions imposed on κ, the function δ is increasing in s.

(ii) For any supermodular function $g : X \times S \to \mathbb{R}$, correspondence $\Gamma : X \times T \to \mathbb{R}^{\ell+1}$,

$$\Gamma(x,t) := \Big\{a \in \mathbb{R}^{\ell+1} : a_i = -\delta_i(x)\lambda(s_i), \\ \text{if } i = 1, \dots, \ell, \text{ and } a_{\ell+1} \geq c(\lambda, t), \text{ for } \lambda \in \triangle_S\Big\}$$

is lower supermodular, where $\delta_i(x) = \big[g(x, s_{i+1}) - g(x, s_i)\big]$, for $i = 1, \dots, \ell$.

(iii) Function $f : X \times T \to \mathbb{R}$, where

$$f(x,t) := \min\left\{\int_S g(x,\tilde{s})d\lambda(\tilde{s}) + c(\lambda,t) : \lambda \in \triangle_S\right\},$$

is supermodular for any supermodular function $g : X \times S \to \mathbb{R}$.

We postpone the proof until Appendix B. Analogously to Proposition 5, implication (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) in Proposition 7 does not require for the cost function c to be convex or continuous. We employ the additional assumption to prove the converse. Below, we introduce two particular examples of cost functions that satisfy property (i) above.

Example 11 (Submodular cost)**.** Suppose that function $c : \triangle_S \times T \to \mathbb{R}_+$ is submodular.[27] Then it satisfies condition (i) in Proposition 7. Indeed, take any $t' \geq t$ and λ, λ'. Since $S \subseteq \mathbb{R}$, both $(\lambda \wedge \lambda')$ and $(\lambda \vee \lambda')$ belong to $\triangle_S$. Clearly, we have

$$\lambda' \succeq (\lambda \wedge \lambda'), \ (\lambda \wedge \lambda') \succeq \lambda, \ \text{and} \ \tfrac{1}{2}\lambda + \tfrac{1}{2}\lambda' = \tfrac{1}{2}(\lambda \wedge \lambda') + \tfrac{1}{2}(\lambda \vee \lambda').$$

Finally, by submodularity of c, we obtain $c(\lambda, t') + c(\lambda', t) \geq c(\lambda \wedge \lambda', t) + c(\lambda \vee \lambda', t')$.

This above result is quite natural. When c is submodular, the marginal cost of choosing a higher λ, with respect to first order stochastic dominance, falls as t increases. This guarantees that the set of distributions that solve the minimisation problem

$$\min\left\{\int_S g(x,\tilde{s})d\lambda(\tilde{s}) + c(\lambda,t) : \lambda \in \triangle_S\right\} \tag{8}$$

increases with t in the strong set order.[28] In other words, when evaluating the ex-ante utility of a given action x, a higher distribution is used when t is higher. When g is supermodular, higher actions are favoured at higher states, so it is intuitive that the ex-ante utility f will favour higher actions when t is higher.

[27] This is with respect to the product order of $\succeq$ on $\triangle_S$ and the natural order on $T \subseteq \mathbb{R}$.

[28] It is easy to check that the objective function is submodular in (λ, t) when c is submodular. By the MCS Theorem, the set of minimisers increases in the strong set order as t increases.

Example 12. Suppose that $\tilde{c} : \mathbb{R} \times T \to \mathbb{R}$ is a submodular function and the cost function $c : \triangle_S \times T \to \mathbb{R}$ is evaluated by

$$c(\lambda, t) \;:=\; \tilde{c}\left(\int_S h(\tilde{s}) d\lambda(\tilde{s}), t\right),$$

for some function $h : S \to \mathbb{R}$. That is, the cost function depends only on the mean of the random variable h with respect to the distribution λ, and the parameter t. Such a function c satisfies condition (i) in Proposition 7.[29]

Indeed, take any λ, λ' in $\triangle_S$ and denote the mean of function h corresponding to each distribution by m, m', respectively. Suppose that $m' \geq m$. Following the argument from Example 9, there are some distributions μ, μ' with means m, m', respectively, such that $\lambda' \succeq \mu$, $\mu' \succeq \lambda$, and $(1/2)\lambda + (1/2)\lambda' = (1/2)\mu + (1/2)\mu'$. Since $c(\lambda, t) = c(\mu, t)$ and $c(\lambda', t') = c(\mu', t')$, the condition (i) is trivially satisfied. Whenever $m' < m$, choose $\mu := \lambda'$ and $\mu' := \lambda$. By submodularity of the function $\tilde{c}$, we obtain

$$c(\lambda, t) + c(\lambda', t') \;=\; \tilde{c}(m, t) + \tilde{c}(m', t') \;\geq\; \tilde{c}(m', t) + \tilde{c}(m, t') \;=\; c(\mu, t) + c(\mu', t').$$

This suffices for condition (i) in Proposition 7 to hold.

Proposition 5 in Section 5.1 can be thought of as a special case of Proposition 7. Indeed, given a correspondence $\Lambda : T \to \triangle_S$, we can define the function c by

$$c(\lambda, t) \;:=\; \begin{cases} 0 & \text{if } \lambda \in \Lambda(t); \\ \infty & \text{otherwise.} \end{cases}$$

Then, function f specified in part (iii) of Proposition 7 takes the maxmin form given in part (iii) of Proposition 5. Furthermore, the function c satisfies the property (i) in the former result if and only if correspondence Λ satisfies property (i) in the latter. In fact, there is an analogous equivalence between Examples 8 and 11, i.e., function c is submodular if and only if the correspondence Λ increases in the strong set order.

Another prominent example of this class of models are *multiplier preferences*, introduced in Sargent and Hansen (2001) and axiomatised by Strzalecki (2011a). In this case, the cost c is given by $c(\lambda, t) := \theta R\big(\lambda \| \lambda^*(\cdot, t)\big)$, for some $\theta \geq 0$ and $\lambda^*(\cdot, t) \in \triangle_S$, where

$$R\big(\lambda \| \lambda^*(\cdot, t)\big) \;:=\; \int_S \log\left(\frac{d\lambda(\tilde{s})}{d\lambda^*(\tilde{s}, t)}\right) d\lambda(\tilde{s})$$

[29] Submodularity of function $\tilde{c}$ over $\mathbb{R} \times T$ does *not* imply submodularity of c over $\triangle_S \times T$.

is the *relative entropy*.[30] Note that $d\lambda(s)$, $d\lambda^*(s,t)$ denote the probability of state s in the distribution λ, $\lambda^*(\cdot,t)$, respectively. This representation can be interpreted in the following manner. The decision maker has a belief over the states of the world given by a *reference* or *benchmark* $\lambda^*(\cdot,t)$, but she is not completely confident that she is exactly correct. To accommodate this concern, the decision maker takes all distributions in $\triangle_S$ into account when evaluating her utility from a given action, though distributions further away from $\lambda^*(\cdot,t)$ cost more and are thus less likely to be the distribution that solves the minimisation problem in (8). The minimisation over all possible distributions reflects aversion to misspecification of the model or ambiguity.

Proposition 8. *Let X, S, $T \subseteq \mathbb{R}$, where S is finite. For any distribution $\lambda^*(\cdot,t)$ in $\triangle_S$, function $c : \triangle_S \times T \to \mathbb{R}$, given by $c(\lambda,t) := \theta R\big(\lambda \| \lambda^*(\cdot,t)\big)$ is submodular on $\triangle_S$, for all $t \in T$ and positive θ. Furthermore, if $\lambda^*(\cdot,t)$ is increasing in t with respect to the monotone likelihood ratio, then function c is submodular in (λ,t).*

Remark 5. Monotone likelihood ratio implies that the ratio $d\lambda^*(s,t')/d\lambda^*(s,t)$ is increasing in s, for all $t' \geq t$. It is straightforward to check that this condition implies that $\lambda(\cdot,t')$ dominates $\lambda(\cdot,t)$ with respect to the first order stochastic dominance.

By combining Propositions 7 and 8, we conclude that $f(x,t)$ is supermodular if function $g(x,s)$ is supermodular and the agent has multiplier preferences, with the benchmark distribution increasing with t in the sense of the monotone likelihood ratio order. In other words, the marginal utility of a higher action becomes greater when the benchmark distribution shifts in favour of higher states.

In Applications 4 and 5 we assume that the agent has maxmin preferences; it is clear that, by appealing to Proposition 7 (instead of Proposition 5), the conclusions in those examples will continue to hold, mutatis mutandi, if the agent has variational or, more specifically, multiplier preferences.

[30] See Strzalecki (2011b) for a detailed discussion on the relation between variational preferences, multiplier preference, and subjective expected utility.

6 Dynamic programming under ambiguity aversion

In an influential paper, Hopenhayn and Prescott (1992) used the tools of monotone comparative statics to analyse stationary dynamic optimisation problems. In this section, we show how the results we developed could also be fruitful in that context.

We consider an agent who faces a stochastic control problem where X and S are the sets of endogenous and exogenous state variables, respectively. To keep the exposition simple, we shall assume that X is sublattice of a Euclidean space and S is a subset of another Euclidean space. The evolution of s over time follows a Markov process with the transition function λ. The agent's problem can be formulated in the following way (see Stokey, Lucas, and Prescott, 1989). At each period τ, given the current state $(x_\tau, s_\tau) \in X \times S$, the agent chooses the endogenous variable $x_{\tau+1}$ for the following period; $x_{\tau+1}$ is chosen from a non-empty feasible set which may depend on the current state, which we denote by $B(x_\tau, s_\tau) \subseteq X$. The single-period return is given by the function $F : X \times S \times X \to \mathbb{R}$; $F(x, s, y)$ is the payoff when (x, s) is the state variable in period τ and y is the endogenous state variable in period $\tau + 1$ chosen in period τ. We assume that the payoffs are discounted by a constant factor $\beta \in (0, 1)$.

The agent's objective is to maximise her expected discounted payoffs over an infinite horizon, given the initial condition (x, s). We denote the value of this optimisation problem by $v^*(x, s)$. Under standard assumptions — in particular, the continuity and boundedness of F and the continuity of B — this problem admits a recursive representation, where $v = v^*$ is the unique solution to the Bellman equation

$$v(x, s) \;=\; \max \left\{ F(x, s, y) + \beta \int_S v(y, \tilde{s}) d\lambda(\tilde{s}, s) : y \in B(x, s) \right\},$$

where $\lambda(\cdot, s)$ is a cumulative probability distribution over states of the world in the following period, conditional on the current state s.[31] Furthermore, the set

$$\Phi(x, s) \;:=\; \arg\max \left\{ F(x, s, y) + \beta \int_S v^*(y, \tilde{s}) d\lambda(\tilde{s}, s) : y \in B(x, s) \right\}$$

is non-empty and compact, for all $(x, s) \in X \times S$, and the correspondence $\Phi : X \times S \to X$ is upper hemi-continuous. We refer to any optimal control problem in which v^* and Φ have the properties listed in this paragraph as a *well-behaved* problem.

[31]See Theorem 9.6 in Stokey, Lucas, and Prescott (1989) for details.

Given a well-behaved problem, Hopenhayn and Prescott (1992) (henceforth HP) apply Theorem 4.3 in Topkis (1978) to show that the value $v^*(x,s)$ *is supermodular in* x *and has increasing differences in* (x,s) under the following assumptions: (i) $F(x,s,y)$ is supermodular in (x,y) and has increasing differences in $\big((x,y),s\big)$; (ii) the graph of B is a sublattice of $X \times S \times X$; (iii) $\lambda(\cdot,s)$ is increasing in s with respect to the first order stochastic dominance. The properties of v^* in turn guarantee that the function

$$f(x,s,y) \ := \ F(x,s,y) + \beta \int_S v^*(y,\tilde{s}) d\lambda(\tilde{s},s)$$

is supermodular in y and has increasing differences in $\big(y,(x,s)\big)$. By the MCS Theorem, $\Phi(x,s)$ is a compact sublattice of X and is increasing in (x,s).[32] This in turn guarantees the existence of the greatest optimal selection

$$g(x,s) \ := \ \Big\{ y \in \Phi(x,s) : y \geq_X z, \text{ for all } z \in \Phi(x,s) \Big\}.$$[33]

In addition, function g is increasing and Borel measurable. Lastly, the policy function g induces a Markov process on $X \times S$, where, for measurable sets $Y \subseteq X$ and $T \subseteq S$, the probability of $Y \times T$ conditional on (x,s) is the probability of T conditional on s if $g(x,s) \in Y$, and it is zero otherwise. HP make use of the monotonicity of g to guarantee that this Markov process has a stationary distribution.[34]

We now apply our results to discuss comparative statics of a dynamic model under ambiguity aversion. We consider a stochastic control problem identical to the one described at the beginning of this section. Since at each period τ the exogenous variable is drawn from the set S, the set of all possible realisations of the exogenous variable over time is given by S^∞. An expected utility maximiser behaves as though she is guided by a distribution over S^∞; to obtain the utility of a given plan of action, the agent evaluates the discounted utility on every possible path, i.e., over every element in S^∞ and takes the average across paths, weighing each path with its probability.

[32]Condition (ii) on B guarantees that $B(x,s)$ is sublattice of X and that it increases with (x,s) in the strong set order. Given with the properties on f, we know that $\Phi(x,s)$ is a sublattice and that it increases with (x,s); this follows from a more general version of the MCS Theorem (than the one stated in Section 2) that allows for increasing constraint sets. See Topkis (1978).

[33]Function is well-defined because Φ is compact-valued and a sublattice.

[34]The focus in this section is on primitive conditions guaranteeing the monotonicity of the policy function. Readers who are interested in how the distribution over (x,s) evolves over time (under monotonicity or weaker assumptions) should consult Huggett (2003). HP and, more recently, Stachurski and Kamihigashi (2014) also discuss uniqueness and other issues relating to the stationary distribution.

When the agent has a maxmin preference, her behaviour can be modelled by a *set* of distributions $\mathcal{M}$ over S^∞. The utility of a plan is then given by the minimum of the expected discounted utility for every distribution in $\mathcal{M}$. In contrast to expected discounted utility, it is known that in this case the agent's utility will not generally have the recursive representation. However, there is a condition on $\mathcal{M}$ called *rectangularity* which is sufficient (and effectively necessary) for this to hold (see Epstein and Schneider, 2003). Furthermore, it is known that a time-invariant version of rectangularity is also sufficient to guarantee that the agent's problem can be solved through the Bellman equation, in a way analogous to that for expected discounted utility (see Iyengar, 2005). This condition says that the agent's belief over the possible value of the exogenous variable in the following period, after observing s in the current period, is given by a *set* of distribution functions $\Lambda(s)$; this set depends on the current realisation of the exogenous variable and is time-invariant. The set $\mathcal{M}$, given an initial value s_0, is then obtained by concatenating the transition probabilities. Therefore, the probability associated with a path $(s_1, s_2, s_3, \ldots)$ is given by $\prod_{i=1}^{\infty} p_i$, where p_1 is the probability of s_1 for some distribution in $\Lambda(s_0)$, p_2 is the probability of s_2 for some distribution in $\Lambda(s_2)$, etc.

With this assumption on $\mathcal{M}$ in place, and some regularity standard conditions, one could guarantee that the value $v^*(x, s)$ of the control problem with the initial state (x, s), is the unique solution to the Bellman equation

$$v(x,s) \;=\; \max\Big\{F(x,s,y) + \beta(Av)(y,s) : y \in B(x,s)\Big\}$$

where $(Av)(y, s) := \min\big\{\int_S v(y, \tilde{s}) d\lambda(\tilde{s}) : \lambda \in \Lambda(s)\big\}$ (see Iyengar, 2005). Furthermore, the problem is *well-behaved* in the sense defined at the beginning of this section.

With this basic set-up, we are almost in a position to recover a monotone result of the HP type: all that is needed is a condition guaranteeing that $(Av)(y, s)$ is a supermodular function of (y, s), whenever v is supermodular. When X and S are one-dimensional, Proposition 5 tells us that this holds if the beliefs correspondence Λ satisfies property (i) therein. The proof of the next proposition is supplied in the Appendix B.

Proposition 9. *Consider a well-behaved optimal control problem where X and S are subsets of $\mathbb{R}$, with X compact and S finite. Suppose that $F(x, s, y)$ is supermodular in all three arguments, $\Lambda : S \to \triangle_S$ satisfies property (i) in Proposition 5, and the graph of*

$B : X \times S \to X$ is a sublattice. Then, the value function $v^*(x, s)$ is supermodular; the correspondence $\Phi : X \times S \to \mathbb{R}$, where

$$\Phi(x, s) \;:=\; \arg\max \Big\{ F(x, s, y) + \beta(Av^*)(y, s) : y \in B(x, s) \Big\}$$

is sublattice-valued and increasing in the strong set order; and the greatest selection $g : X \times S \to \mathbb{R}$ of Φ is well-defined, increasing, and Borel measurable.

Below we discuss a specific application of this result.

Application 6. Consider the following dynamic optimisation problem of a firm. In each period, the firm collects revenue $\pi(x, s)$, where $s \in S$ denotes the realised exogenous state of the world and $x \in \mathbb{R}_+$ is the level of capital stock currently available to the firm. Once s is revealed to the firm and the revenue collected, the firm may invest $a \in [0, K]$ at a cost $c(a)$, K being a finite positive number. With this investment, capital stock in the next period is $y = \delta x + a$, where $\delta \in [0, 1]$ denotes the fraction of non-depreciated capital. Therefore, the dividend in each period is

$$F(x, s, y) \;:=\; \pi(x, s) - c(y - \delta x),$$

where the firm chooses y from the interval $\Phi(x, s) = [\delta x, \delta x + K]$. We know from HP that if the firm is an expected utility maximiser and the optimal control problem is well-behaved, then the firm has a policy function that is increasing in (x, s) under the following additional conditions: the transition *function* $\Lambda : S \to \triangle_S$ is increasing with respect to first order stochastic dominance and F is supermodular; the latter property is guaranteed if π is supermodular and c is concave. Proposition 9 goes further by saying that this conclusion remains true if the firm has a maxmin preference, so long as the transition *correspondence* Λ satisfies property (i) in Proposition 5.

A Auxiliary results

We devote this section to three auxiliary results.

A.1 Proposition A.1

The following proposition plays a fundamental role in our main applications.

Proposition A.1. *Let K be a subset of $\{1, 2, \ldots, \ell\}$. Moreover, suppose that X, T are subsets of $\mathbb{R}$ and mapping $\Lambda : T \to \mathbb{R}^\ell$ is a correspondence. If for any $t' \geq t$ and $z \in \Lambda(t)$, $z' \in \Lambda(t')$ there is some $y \in \Lambda(t)$, $y' \in \Lambda(t')$ such that $z'_k \geq y_k$, for all $k \in K$, and $z + z' \geq y + y'$, then correspondence $\Gamma : X \times T \to \mathbb{R}^\ell$, given by*

$$\Gamma(x,t) \;:=\; \Big\{a \in \mathbb{R}^\ell : a_k = h_k(x)y_k,\;\; \textit{if } k \in K,\;\; \textit{and } a_k = y_k \;\textit{otherwise},\;\; \textit{for } y \in \Lambda(t)\Big\},$$

is lower supermodular for any positive, increasing functions $h_k : X \to \mathbb{R}_+$, for $k \in K$. If values of Λ are closed, convex, and bounded from below, then the converse is also true.

We devote the reminder of this subsection to the proof. To show the first part, define correspondence Γ as above. Take any $x' \geq x$, $t' \geq t$, and $a \in \Gamma(x,t)$, $a' \in \Gamma(x',t')$. We need to find some $b \in \Gamma(x',t)$ and $b' \in \Gamma(x,t')$ such that $a + a' \geq b + b'$.

By definition of Γ, there is some $z \in \Lambda(t)$ and $z' \in \Lambda(t')$ such that $a_k = h_k(x)z_k$, $a'_k = h_k(x')z'_k$, for $k \in K$, and $a_k = z_k$, $a'_k = z'_k$ otherwise. In addition, by the assumption imposed on Λ, there is some $y \in \Lambda(t)$, $y' \in \Lambda(t')$ such that $z'_k \geq y_k$, for all $k \in K$, and $z'_i - y_i \geq y'_i - z_i$, for all $i = 1, \ldots, \ell$. Given that h_k is positive and increasing,

$$h_k(x')\big[z'_k - y_k\big] \;\geq\; h_k(x)\big[z'_k - y_k\big] \;\geq\; h_k(x)\big[y'_k - z_k\big],$$

for all $k \in K$. Construct vectors b, b' by $b_k = h_k(x')y_k$, $b'_k = h_k(x)y'_k$, for all $k \in K$, and $b_k = y_k$, $b'_k = y'_k$ otherwise. Clearly, we have $b \in \Gamma(x',t)$, $b' \in \Gamma(x,t')$, and $a + a' \geq b + b'$.

Before proving the second part of the proposition, we need to introduce additional notation and two auxiliary results. Let ϵ_i denote the i'th versor, i.e., the ℓ-dimensional vector with all entries equal to 0 apart from entry i, which is equal to 1. Given the subset of indices K, define a closed, convex cone

$$\begin{aligned} D \;:=\; \Big\{(a,a') \in \mathbb{R}^\ell \times \mathbb{R}^\ell \;:\; (a,a') = {\textstyle\sum_{i=1}^{\ell}} \theta_i(\epsilon_i, 0) + {\textstyle\sum_{i=1}^{\ell}} \theta'_i(0, \epsilon_i) \\ + {\textstyle\sum_{i=1}^{\ell}} \vartheta_i(-\epsilon_i, \epsilon_i),\;\text{for some } \theta_i, \theta'_i \geq 0,\;\text{for all } i = 1, \ldots, \ell\;, \\ \vartheta_k \geq 0,\;\text{for all } k \in K,\;\text{and any } \vartheta_k,\;\text{for } k \notin K\Big\}. \end{aligned} \tag{A1}$$

Lemma A.1. *Let B be a subset of $\mathbb{R}^\ell \times \mathbb{R}^\ell$ that is closed, convex, and bounded from below. Then, the sum $(D + B)$ is closed, where the set D is defined in (A1).*

Proof. Let AB and AD denote the asymptotic cones of sets B and D, respectively.[35] By Proposition 2.38 in Border (1985) it suffices to show that AB and AD are *positively semi-independent*. That is, whenever $(y, y') + (z, z') = (0, 0)$, for $(y, y') \in AB$ and $(z, z') \in AD$, then $(y, y') = (z, z') = (0, 0)$. Given that set D is a closed cone, while B is closed and convex, by Theorem 8.2 in Rockafellar (1970), we have $AD = D$ and

$$AB \;=\; \Big\{(z, z') \in \mathbb{R}^\ell \times \mathbb{R}^\ell : \big[(a, a') + \alpha(z, z')\big] \in B, \text{ for all } (a, a') \in B \text{ and } \alpha \geq 0\Big\}.$$

That is, the asymptotic cone of B is equal to its recession cone.

Suppose that $(y, y') + (z, z') = 0$, or equivalently $(z, z') = -(y, y')$, for some $(y, y') \in D$ and $(z, z') \in AB$. Therefore, there are some positive θ_i, θ'_i, and some ϑ_i such that

$$(a, a') - \alpha \sum\nolimits_{i=1}^{\ell} \theta_i(\epsilon_i, 0) - \alpha \sum\nolimits_{i=1}^{\ell} \theta'_i(0, \epsilon_i) - \alpha \sum\nolimits_{i=1}^{\ell} \vartheta_i(-\epsilon_i, \epsilon_i)$$

belongs to B, for all $(a, a') \in B$ and $\alpha \geq 0$. This holds only if $\theta_i = \theta'_i = \vartheta_i = 0$, for all $i = 1, \ldots, \ell$. Otherwise, if $\vartheta_i > 0$, for some i, then for any $(a, a') \in B$ and number c there would be a large enough $\alpha \geq 0$ such that $a'_i - \alpha\theta'_i - \alpha\vartheta_i < c$, contradicting that B is bounded from below. Analogously, if $\vartheta_i < 0$, for some i, we would have $a_i - \alpha\theta_i + \alpha\vartheta_i < c$, for a sufficiently large α. For the same reason, it must be that $\theta_i = \theta'_i = 0$. $\square$

We proceed with the proof of the second part of Proposition A.1. Suppose that values of correspondence Λ are closed, convex, and bounded from below. We show that whenever the condition stated in the proposition is violated, there exist positive, increasing functions h_i for which the correspondence Γ is not lower supermodular.

Take any $z \in \Lambda(t)$, $z' \in \Lambda(t')$ and define a subset of $\mathbb{R}^\ell \times \mathbb{R}^\ell$ by

$$C \;:=\; \Big\{(z - y', z' - y) \in \mathbb{R}^\ell \times \mathbb{R}^\ell \;:\; y \in \Lambda(t) \text{ and } y' \in \Lambda(t')\Big\}. \tag{A2}$$

Given the assumptions imposed on $\Lambda(t)$ and $\Lambda(t')$, set C is closed, convex, and bounded from above. This makes $(-C)$ closed, convex and bounded from below.

Lemma A.2. *Take any $t' \geq t$ and $z \in \Lambda(t)$, $z \in \Lambda(t')$. There is some $y \in \Lambda(t)$ and $y' \in \Lambda(t')$ such that $z'_k \geq y_k$, for all $k \in K$, and $z + z' \geq y + y'$ if and only if $C \cap D \neq \emptyset$, where sets D and C are defined in (A1) and (A2), respectively.*

[35]The *asymptotic* cone of E is the set of limits of sequences $\{\lambda_n x_n\}$, where $x_n \in E$ and $0 < \lambda_n \to 0$.

Proof. If $C \cap D \neq \emptyset$, there must be some $y \in \Lambda(t)$, $y' \in \Lambda(t')$ such that

$$(z - y', z' - y) \;=\; \sum_{i=1}^{\ell} \theta_i(\epsilon_i, 0) + \sum_{i=1}^{\ell} \theta_i'(0, \epsilon_i) + \sum_{i=1}^{\ell} \vartheta_i(-\epsilon_i, \epsilon_i),$$

for some positive θ_i, θ_i', for all $i = 1, \ldots, \ell$, positive ϑ_k, for $k \in K$, and some ϑ_k, for $k \notin K$. This implies that $z_k' - y_k = \theta_k' + \vartheta_k \geq 0$, for all $k \in K$. Moreover, for all i,

$$z_i - y_i' + z_i' - y_i \;=\; \theta_i - \vartheta_i + \theta_i' + \vartheta_i \;=\; \theta_i + \theta_i' \;\geq\; 0.$$

To show the converse, take any $y \in \Lambda(t)$, $y' \in \Lambda(t')$ such that $z_k' \geq y_k$, for all $k \in K$, and $z + z' \geq y + y'$. For each dimension $i = 1, \ldots, \ell$, define weights $\theta_i = (z_i' - y_i + z_i - y_i')$, $\theta_i' = 0$, and $\vartheta_i = (z_i' - y_i)$. Notice that weights θ_i, θ_i' are positive, for all $i = 1, \ldots, \ell$, while $\vartheta_k \geq 0$ for all $k \in K$. By construction, we obtain

$$(z - y', z' - y) \;=\; \sum_{i=1}^{\ell} \theta_i(\epsilon_i, 0) + \sum_{i=1}^{\ell} \theta_i'(0, \epsilon_i) + \sum_{i=1}^{\ell} \vartheta_i(-\epsilon_i, \epsilon_i).$$

Therefore, we have $(z - y', z' - y) \in D$, which implies that $C \cap D \neq \emptyset$. $\square$

Given the above observation, we prove the result by contradiction. Suppose that correspondence Λ violates the condition stated in the proposition for some $t' \geq t$ and $z \in \Lambda(t)$, $z' \in \Lambda(t')$. By Lemma A.2, this is equivalent to $C \cap D = \emptyset$, with C and D defined above. Since both sets are convex, while $(D - C)$ is closed (recall Lemma A.1), by the strong separating hyperplane theorem, there is some $(\hat{p}, \hat{p}')$ in $\mathbb{R}^\ell \times \mathbb{R}^\ell$ such that

$$\hat{p} \cdot (z - y') + \hat{p}' \cdot (z' - y) \;<\; 0 \;\leq\; \hat{p} \cdot a + \hat{p}' \cdot a',$$

for all $y \in \Lambda(t)$, $y' \in \Lambda(t')$, and $(a, a') \in D$ — since $(0, 0) \in D$. Given that $(\epsilon_i, 0)$, $(0, \epsilon_i)$ belong to D, for all $i = 1, \ldots, \ell$, vectors $\hat{p}$ and $\hat{p}'$ are positive. Moreover, since $(-\epsilon_k, \epsilon_k)$ is an element of D, for all $k \in K$, we have $\hat{p}_k' \geq \hat{p}_k$, for $k \in K$. Finally, since both $(-\epsilon_i, \epsilon_i)$ and $(\epsilon_i, -\epsilon_i)$ are in D, for all $k \notin K$, we also have $\hat{p}_k' = \hat{p}_k$, for all $k \notin K$.

The above inequality implies that, for any $y \in \Lambda(t)$ and $y' \in \Lambda(t')$ either

$$\hat{p}_k y_k' + \hat{p}_k' y_k \;>\; \hat{p}_k z_k + \hat{p}_k' z_k',$$

for some $k \in K$, or $y_k + y_k' > z_k + z_k'$, for some $k \notin K$ (since $\hat{p}_k' = \hat{p}_k$).

Take any $x' > x$ in X. Define function $h_k : X \to \mathbb{R}_+$,

$$h_k(z) \;:=\; \begin{cases} \hat{p}_k & \text{for } z \leq x, \\ \hat{p}_k' & \text{otherwise;} \end{cases}$$

for all $k \in K$. Clearly, it is positive and increasing.

Define correspondence Γ as in the lemma for the functions h_k above. Take vectors a, a' such that $a_k = h_k(x)z_k$, $a'_k = h_k(x')z'_k$, for all $k \in K$, and $a_k = z_k$, $a'_k = z'_k$ otherwise. Clearly, we have $a \in \Gamma(x,t)$ and $a' \in \Gamma(x',t')$. However, our previous observation implies that there are no vectors $b \in \Gamma(x',t)$, $b' \in \Gamma(x,t')$ that satisfy $a + a' \geq b + b'$. Therefore, whenever Λ violates the condition stated in the proposition, there are positive, increasing functions h_k, $k \in K$, for which Γ is not lower supermodular.

A.2 Proposition A.2

In the following lemma we introduce a slightly stronger condition on the correspondence Λ under which it is supermodular, i.e., both upper and lower supermodular, for any increasing (but not necessarily positive) functions h_k, for $k \in K$.

Proposition A.2. *Let K be a subset of $\{1, 2, \ldots, \ell\}$. Moreover, suppose that X, T are subsets of $\mathbb{R}$ and mapping $\Lambda : T \to \mathbb{R}^\ell$ is a correspondence. If for any $t' \geq t$ and $z \in \Lambda(t)$, $z' \in \Lambda(t')$ there is some $y \in \Lambda(t)$, $y' \in \Lambda(t')$ such that $z'_k \geq y_k$, $y'_k \geq z_k$, for all $k \in K$, and $z + z' = y + y'$, then correspondence $\Gamma : X \times T \to \mathbb{R}^\ell$, given by*

$$\Gamma(x,t) \;:=\; \Big\{a \in \mathbb{R}^\ell : a_k = h_k(x)y_k, \;\; \textit{if } k \in K, \;\; \textit{and } a_k = y_k \;\; \textit{otherwise}, \;\; \textit{for } y \in \Lambda(t)\Big\},$$

is supermodular for any increasing functions $h_k : X \to \mathbb{R}_+$, for $k \in K$. If values of Λ are closed, convex, and bounded from below (or above), then the converse is also true.

The condition on Λ is stronger than in Proposition A.1. However, it implies not only that Γ is both upper and lower supermodular, but also that the properties hold for increasing functions that need not be positive. Moreover, as we show in the proof below, if values of the correspondence Λ are closed, convex, and bounded from below (or above), then the condition on Λ is necessary for the correspondence Γ to be upper supermodular, and for the correspondence to be lower supermodular, for any increasing functions h_k. Therefore, it is stronger than showing that the property is necessary for Γ to be supermodular. We devote the remainder of this subsection to our argument.

The first part of the lemma can be shown by applying an analogous argument to the one in the proof of Proposition A.1. Before proving the necessity part, we introduce additional notation and two auxiliary result. As in the previous subsection, let ϵ_i denote the i'th versor in $\mathbb{R}^\ell$. Given the subset of indices K, define a closed, convex cone

$$D \;:=\; \Big\{(a,a') \in \mathbb{R}^\ell \times \mathbb{R}^\ell \;:\; (a,a') = \sum\nolimits_{i=1}^{\ell} \vartheta_i(-\epsilon_i, \epsilon_i), \text{ where } \vartheta_k \geq 0, \text{ if } k \in K\Big\}. \quad \text{(A3)}$$

Lemma A.3. *Let B be a subset of $\mathbb{R}^\ell \times \mathbb{R}^\ell$ that is closed, convex, and bounded from below (or above). Then, the sum $(D + B)$ is closed, where the set D is defined in (A3).*

We skip the proof as it is analogous to the one supporting Lemma A.1. Take any $z \in \Lambda(t)$, $z' \in \Lambda(t')$ and define subset C as in (A2). Given the assumptions on $\Lambda(t)$ and $\Lambda(t')$, set C is closed, convex, and bounded from below (or above).

Lemma A.4. *Take any $t' \geq t$ and $z \in \Lambda(t)$, $z \in \Lambda(t')$. There is some $y \in \Lambda(t)$ and $y' \in \Lambda(t')$ such that $z'_k \geq y_k$, $y'_k \geq z_k$, for all $k \in K$, and $z + z' = y + y'$ if and only if $C \cap D \neq \emptyset$, where sets C and D are defined in (A2) and (A3), respectively.*

The proof of the above lemma is analogous to the argument supporting Lemma **??**. Given the above observation, we prove the result by contradiction. Suppose that correspondence Λ violates the condition stated in the proposition for some $t' \geq t$ and $z \in \Lambda(t)$, $z' \in \Lambda(t')$. By Lemma A.4, this is equivalent to $C \cap D = \emptyset$, with C and D defined above. Since set $(D - C)$ is convex and closed (recall Lemma A.3), by the strong separating hyperplane theorem, there is a pair of vectors $(\hat{p}, \hat{p}')$ in $\mathbb{R}^\ell \times \mathbb{R}^\ell$ such that

$$\hat{p} \cdot (z - y') + \hat{p}' \cdot (z' - y) \;<\; 0 \;\leq\; \hat{p} \cdot a + \hat{p}' \cdot a',$$

for all $y \in \Lambda(t)$, $y' \in \Lambda(t')$, and $(a, a') \in D$ — since $(0,0) \in D$. Given that $(-\epsilon_k, \epsilon_k)$ belongs to D, for all $k \in K$, we have $\hat{p}'_k \geq \hat{p}_k$, for $k \in K$. Moreover, since both $(-\epsilon_k, \epsilon_k)$ and $(\epsilon_k, -\epsilon_k)$ are elements of D, for all $k \notin K$, we have $\hat{p}'_k = \hat{p}_k$, for all $k \notin K$.

Following the argument from the proof of Proposition A.1, there are increasing functions h_k, for $k \in K$, such that correspondence Γ is not lower supermodular. Analogously, we show that the same condition is necessary for Γ to be upper supermodular.

A.3 Lemma A.5

Below we present a lemma that was applied in the proof of Proposition 6. Suppose that S is a finite subset of $\mathbb{R}$ and $\triangle_S$ consists of cumulative probability distributions over S. As previously, we rank elements of $\triangle_S$ with the first order stochastic dominance $\succeq$.

Lemma A.5. *For any distributions λ, λ' and μ, μ' in $\triangle_S$ such that*

$$\lambda' \succeq \mu \quad \textit{and} \quad \tfrac{1}{2}\lambda + \tfrac{1}{2}\lambda' \succeq \tfrac{1}{2}\mu + \tfrac{1}{2}\mu',$$

there are some distributions ν and ν' such that

$$\lambda' \succeq \nu \succeq \mu, \quad \nu' \succeq \mu', \quad \textit{and} \quad \tfrac{1}{2}\lambda + \tfrac{1}{2}\lambda' = \tfrac{1}{2}\nu + \tfrac{1}{2}\nu'.$$

Proof. Since the state space is finite, denote $S = \{s_i\}_{i=1}^{\ell}$, where $s_1 < s_2 < \ldots < s_\ell$. By assumption, we have $\mu(s_i) \geq \lambda'(s_i)$ and $\mu(s_i) + \mu'(s_i) \geq \lambda(s_i) + \lambda'(s_i)$, for all $i = 1, \ldots, \ell$. We construct distributions ν and ν' recursively, with respect to i.

First, take the lowest state s_1 and set $\nu(s_1) := \max\left\{\lambda'(s_1),\ \lambda(s_1)+\lambda'(s_1)-\mu'(s_1)\right\}$ and $\nu'(s_1) := \lambda(s_1) + \lambda'(s_1) - \nu(s_1)$. By construction, we have $\nu(s_1) + \nu'(s_1) = \lambda(s_1) + \lambda'(s_1)$ as well as $\nu(s_1) \geq \lambda'(s_1)$. Therefore, $\nu(s_1)$ must be positive and $\nu'(s_1) \leq \lambda(s_1)$. We claim that $\nu'(s_1)$ is positive, while $\nu(s_1) \leq \mu(s_1)$ and $\nu'(s_1) \leq \mu'(s_1)$.

Consider two cases. If (i) $\nu(s_1) = \lambda'(s_1)$, then $\nu'(s_1) = \lambda(s_1)$, which is positive. Since $\lambda'(s_i) \leq \mu(s_i)$, for all $i = 1, \ldots, \ell$, in particular, we have $\nu(s_1) = \lambda'(s_1) \leq \mu(s_1)$. Given that case (i) holds only if $\lambda'(s_1) \geq \lambda(s_1)+\lambda'(s_1)-\mu'(s_1)$, we have $\nu'(s_1) = \lambda(s_1) \leq \mu'(s_1)$.

Whenever (ii) $\nu(s_1) = \lambda(s_1) + \lambda'(s_1) - \mu'(s_1)$, then $\nu'(s_1) = \mu'(s_1)$, which is positive. Given that $\mu(s_i) + \mu'(s_i) \geq \lambda(s_i) + \lambda'(s_i)$, for all $i = 1, \ldots, \ell$, in particular

$$\mu(s_1) + \mu'(s_1) \ \geq \ \lambda(s_1) + \lambda'(s_1) \ = \ \nu(s_1) + \nu'(s_1) \ = \ \nu(s_1) + \mu'(s_1),$$

which implies $\mu(s_1) \geq \nu(s_1)$. This concludes the base step.

For the recursive step, take any $j = 1, \ldots, (\ell-1)$ and suppose there are some positive $\nu(s_j)$, $\nu'(s_j)$ that satisfy $\nu(s_j) + \nu'(s_j) = \lambda(s_j) + \lambda'(s_j)$, with $\mu(s_j) \geq \nu(s_j) \geq \lambda'(s_j)$, and $\nu'(s_j) \leq \mu'(s_j)$. Clearly, this implies that $\nu'(s_j) \leq \lambda(s_j)$.

Define the numbers $\nu(s_{j+1}) := \max\left\{\lambda'(s_{j+1}), \lambda(s_{j+1})+\lambda'(s_{j+1})-\mu'(s_{j+1}), \nu(s_j)\right\}$ and $\nu'(s_{j+1}) := \lambda(s_{j+1}) + \lambda'(s_{j+1}) - \nu(s_{j+1})$. By construction, we have

$$\nu(s_{j+1}) + \nu'(s_{j+1}) \ = \ \lambda(s_{j+1}) + \lambda'(s_{j+1}),$$

as well as $\nu(s_{j+1}) \geq \lambda'(s_{j+1})$ and $\nu(s_{j+1}) \geq \nu(s_j)$. Thus, we have $\nu'(s_{j+1}) \leq \lambda(s_{j+1})$. We claim that $\nu'(s_{j+1}) \geq \nu'(s_j)$, while $\nu(s_{j+1}) \leq \mu(s_{j+1})$ and $\nu'(s_{j+1}) \leq \mu'(s_{j+1})$.

Consider three cases. If (i) $\nu(s_{j+1}) = \lambda'(s_{j+1})$, then $\nu'(s_{j+1}) = \lambda(s_{j+1})$, which implies $\nu'(s_{j+1}) = \lambda(s_{j+1}) \geq \lambda(s_j) \geq \nu'(s_j)$. Since $\lambda'(s_i) \leq \mu(s_i)$, for all $i = 1, \ldots, \ell$, in particular, we have $\nu(s_{j+1}) = \lambda'(s_{j+1}) \leq \mu(s_{j+1})$. Given that the case under consideration holds only if $\lambda'(s_{j+1}) \geq \lambda(s_{j+1}) + \lambda'(s_{j+1}) - \mu'(s_{j+1})$, we obtain $\nu'(s_{j+1}) = \lambda(s_{j+1}) \leq \mu'(s_{j+1})$.

Whenever (ii) $\nu(s_{j+1}) = \lambda(s_{j+1}) + \lambda'(s_{j+1}) - \mu'(s_{j+1})$, then $\nu'(s_{j+1}) = \mu'(s_{j+1})$. Moreover, $\nu'(s_j) \leq \mu'(s_j)$ implies $\nu'(s_{j+1}) = \mu'(s_{j+1}) \geq \mu(s_j) \geq \nu'(s_j)$. Additionally, given that $\mu(s_i) + \mu'(s_i) \geq \lambda(s_i) + \lambda'(s_i)$, for all $i = 1, \ldots, \ell$, we obtain

$$\mu(s_{j+1}) + \mu'(s_{j+1}) \;\geq\; \lambda(s_{j+1}) + \lambda'(s_{j+1}) \;=\; \nu(s_{j+1}) + \nu'(s_{j+1}) \;=\; \nu(s_{j+1}) + \mu'(s_{j+1}),$$

which implies that $\nu(s_{j+1}) \leq \mu(s_{j+1})$.

Finally, suppose that (iii) $\nu(s_{j+1}) = \nu(s_j)$. In particular, this implies

$$\nu'(s_{j+1}) \;=\; \lambda(s_{j+1}) + \lambda'(s_{j+1}) - \nu(s_j) \;\geq\; \lambda(s_j) + \lambda'(s_j) - \nu(s_j) \;=\; \nu'(s_j).$$

Moreover, since $\nu(s_j) \leq \mu(s_j)$, we have $\nu(s_{j+1}) = \nu(s_j) \leq \mu(s_j) \leq \mu(s_{j+1})$. Given that the case under consideration is satisfied only if

$$\nu(s_{j+1}) \;=\; \nu(s_j) \;\geq\; \lambda(s_{j+1}) + \lambda'(s_{j+1}) - \mu'(s_{j+1}),$$

we have $\mu'(s_{j+1}) \geq \lambda(s_{j+1}) + \lambda'(s_{j+1}) - \nu(s_{j+1}) = \nu'(s_{j+1})$.

The above argument guarantees that it is always possible to find positive numbers $\nu(s_i)$, $\nu'(s_i)$ such that $\nu(s_i) + \nu'(s_i) = \lambda(s_i) + \lambda'(s_i)$, where $\mu(s_i) \geq \nu(s_i) \geq \lambda'(s_i)$, and $\nu'(s_i) \leq \mu'(s_i)$, for all $i = 1, \ldots, \ell$. What is more, we have $\nu(s_{i+1}) \geq \nu(s_i)$ and $\nu'(s_{i+1}) \geq \nu'(s_i)$, for all $i = 1, \ldots, (\ell - 1)$. This concludes our argument. $\square$

B Proofs

We devote this subsection to the proofs that were omitted in the main body of the paper.

Proof of Proposition 1

Before we state the proof, note that, whenever sets B, $C \subseteq \mathbb{R}^\ell$ are closed, convex, and bounded from below, then their sum $(B + C)$ is closed. This follows directly from Proposition 2.38 in Border (1985). Indeed, since both sets are closed and convex, their asymptotic cones are equal to their recession cones. Since B and C are bounded from below, their asymptotic cones must positively semi-independent. The result is also true whenever sets B and C are bounded from above.

We only prove part (i) of Proposition 1; the proof of (ii) is analogous. Suppose that Γ is *not* upper supermodular. There exist some x, x' in X and $y \in \Gamma(x)$, $y' \in \Gamma(x')$ such that for any $z \in \Gamma(x \wedge x')$, $z' \in \Gamma(x \vee x')$, we have $z + z' \not\geq y + y'$. Define set

$$U \ := \ \Big\{u \in Y : u \leq v, \text{ for some } v \in \Gamma(x \wedge x') + \Gamma(x \vee x')\Big\},$$

which is convex, downward comprehensive, and $(y + y') \notin U$. Given the assumptions imposed on values of Γ, set U is closed and convex.[36] By the strong separating hyperplane theorem, there is a non-zero, linear functional ϕ^* such that $\phi^*(y + y') > \phi^*(u)$, for all $u \in U$. As U is downward comprehensive, the functional ϕ^* must be positive.

We claim that function $f(x) := \max\{\phi^*(u) : u \in \Gamma(x)\}$ is *not* supermodular. Indeed,

$$\begin{aligned} f(x \wedge x') + f(x \vee x') \ &= \ \max\{\phi^*(u) : u \in \Gamma(x \wedge x')\} + \max\{\phi^*(u) : u \in \Gamma(x \vee x')\} \\ = \ \max\{\phi^*(u) : u \in \Gamma(x \wedge x') + \Gamma(x \vee x')\} \ &< \ \phi^*(y + y') \ = \ \phi^*(y) + \phi^*(y') \\ &\leq \ \max\{\phi^*(u) : u \in \Gamma(x)\} + \max\{\phi^*(u) : u \in \Gamma(x')\} \ = \ f(x) + f(x'), \end{aligned}$$

which contradicts supermodularity of function f.

Continuation of the proof of Proposition 3

We show (iii) $\Rightarrow$ (ii) by contradiction. By assumptions imposed on function F, correspondence Γ is closed, convex, and bounded from below. Suppose it is not lower supermodular for some positive, increasing functions h_i. By Proposition 1, there is some positive vector p and $x' \geq x$, $q' \geq q$ such that function $f : X \times \mathbb{R}_+ \to \mathbb{R}$, where

$$f(x, q) \ := \ \min\{p \cdot a \ : \ a \in \Gamma(x, q)\},$$

satisfies $f(x, q) + f(x', q') < f(x', q) + f(x, q')$. However, this implies that there are some prices $\hat{p}'_i = p_i h_i(x') \geq p_i h_i(x) = \hat{p}_i$, for all $i = 1, \ldots, \ell$, such that

$$C(\hat{p}, q) + C(\hat{p}', q') \ = \ f(x, q) + f(x', q') \ < \ f(x', q) + f(x, q') \ = \ C(\hat{p}', q) + C(\hat{p}, q').$$

This contradicts that C has increasing differences in prices p and output level q.

Finally, define correspondence Λ as in Section 4.2 To show (ii) $\Rightarrow$ (i), recall from Proposition A.1 and the definition of correspondence Λ that statement (ii) holds only if

[36] This is the only instance where we use the assumption that Γ has bounded values. In fact, we only require that Γ satisfies the following property: for any x, x' in X, set $\Gamma(x) + \Gamma(x')$ is closed.

for any $q' \geq q$ and z, z' satisfying $F(z) \geq q$ and $F(z') \geq q'$, there is some y and y' such that $F(y) \geq q$, $F(y') \geq q'$, while $z' \geq y$ and $z+z' \geq y+y'$. Since function F is increasing, there is some $\tilde{y}'$ such that $F(\tilde{y}') \geq q'$ and $z + z' = y + \tilde{y}'$.

Continuation of the proof of Proposition 4

In this subsection we discuss the second part of Proposition 4. Since S is finite, we can denote $S = \{s_i\}_{i=1}^{\ell+1}$, where $s_1 < s_2 < \ldots < s_{\ell+1}$. Suppose that the condition stated in the proposition is violated. Thus, there is some $\lambda' \in \Lambda'$ such that $\lambda' \not\geq \lambda$, for all $\lambda \in \Lambda$.

Define set $V := \big\{y \in \mathbb{R}^\ell : y_i \geq \lambda'(s_i)$, for all $i = 1, \ldots, \ell\big\}$. Clearly, $V \cap \Lambda = \emptyset$. Since both sets are convex, set V is closed, and Λ is compact, by the strong separating hyperplane theorem, there is some $\hat{p} \in \mathbb{R}^\ell$ such that

$$\min \left\{ \sum_{i=1}^{\ell} \hat{p}_i y_i : y \in V \right\} \; > \; \max \left\{ \sum_{i=1}^{\ell} \hat{p}_i \lambda(s_i) : \lambda \in \Lambda \right\}.$$

Given that V is downward comprehensive, vector $\hat{p}$ must be positive. In particular, we have $\sum_{i=1}^{\ell} \hat{p}_i \lambda'(s_i) = \min \big\{\hat{p} \cdot y : y \in V\big\}$. Define function $u : S \to \mathbb{R}$ by $u(s_1) := \hat{p}_1$ and $u(s_{i+1}) := \big[u(s_i) + \hat{p}_{i+1}\big]$, for all $i = 1, \ldots, \ell$, which is increasing. Since

$$\int_S u(\tilde{s}) d\mu(\tilde{s}) \; = \; u(s_{\ell+1}) - \sum_{i=1}^{\ell} \hat{p}_i \mu(s_i),$$

for any distribution $\mu \in \triangle_S$ (recall (6) in Section 5.1), we have

$$\begin{aligned}
\min \left\{ \int_S u(\tilde{s}) d\lambda(\tilde{s}) : \lambda \in \Lambda \right\} \; &= \; u(s_{\ell+1}) - \max \left\{ \sum_{i=1}^{\ell} \hat{p}_i \lambda(s_i) : \lambda \in \Lambda \right\} \\
&> \; u(s_{i+1}) - \sum_{i=1}^{\ell} \hat{p}_i \lambda'(s_i) \; \geq \; u(s_{i+1}) - \max \left\{ \sum_{i=1}^{\ell} \hat{p}_i \lambda(s_i) : \lambda \in \Lambda' \right\} \\
&= \; \min \left\{ \int_S u(\tilde{s}) d\lambda(\tilde{s}) : \lambda \in \Lambda' \right\},
\end{aligned}$$

which contradicts the claim stated in the proposition.

Continuation of the proof of Proposition 5

We show (iii) $\Rightarrow$ (ii) by contradiction. Suppose there is a supermodular function g for which Γ is not lower supermodular. Since values of Γ are compact and convex, by

Proposition 1, there is a positive vector $p \in \mathbb{R}$ for which function

$$f(x,t) \;=\; p_\ell g(x, s_{\ell+1}) + \min\Big\{p \cdot a : a \in \Gamma(x,t)\Big\},$$

is not supermodular. Recall that $\delta_i(x) := \big[g(x, s_{i+1}) - g(x, s_i)\big]$, for all $i = 1, \ldots, \ell$, and let $\hat{g} : X \times S \to \mathbb{R}$ be given by $\hat{g}(x, s_1) = p_1 g(x, s_1)$ and $\hat{g}(x, s_{i+1}) = \hat{g}(x, s_i) + p_i \delta_i(x)$, for all $x \in X$ and $i = 1, \ldots, \ell$. Clearly, function $\hat{g}$ is supermodular. However, since

$$f(x,t) \;=\; \min\left\{\int_S \hat{g}(x, \tilde{s}) d\lambda(\tilde{s}) : \lambda \in \Lambda(t)\right\},$$

this contradicts that f is a supermodular function.

Implication (ii) $\Rightarrow$ (i) follows directly from Proposition A.2. Function $g : X \times S \to \mathbb{R}$ is supermodular if and only if function $\delta_i(x)$ is increasing, for all $i = 1, \ldots, \ell$. Since values of the correspondence Λ are compact and convex, condition (ii) implies that for any $t' \geq t$ and $\lambda \in \Lambda(t)$, $\lambda' \in \Lambda(t')$, there are some $\mu \in \Lambda(t)$, $\mu' \in \Lambda(t')$ such that

$$-\lambda'(s) \geq -\mu(s), \;\; -\mu'(s) \geq -\lambda(s), \;\; \text{and} \quad -\lambda(s) - \lambda'(s) = -\mu(s) - \mu'(s),$$

for all $s \in S$. This is equivalent to property (i).

Proof of Remark 3

Suppose that $S = [a, b]$. For each natural number n, choose a sequence $\{s_i^n\}_{i=1}^n$ such that $a = s_0^n < s_1^n < \ldots < s_{n-1}^n < s_n^n = b$, with the mesh approaching 0 as $n \to \infty$. Since at each (x,t), function $g(x, \cdot)$ is Riemann-Stieltjes integrable with respect to $\lambda \in \Lambda(t)$,

$$\int_S g(x, \tilde{s}) d\lambda(\tilde{s}) \;=\; \lim_{n \to \infty} \sum_{i=0}^{n-1} g(x, s_{i+1}) \big[\lambda(s_{i+1}) - \lambda(s_i)\big]$$

for all $\lambda \in \Lambda(t)$. This guarantees that $\lim_{n\to\infty} f_n(x,t) = f(x,t)$ for all (x,t), where

$$f_n(x,t) \;:=\; \min\left\{\sum_{i=0}^{n-1} g(x, s_{i+1}) \big[\lambda(s_{i+1}) - \lambda(s_i)\big] : \lambda \in \Lambda(t)\right\}.$$

We know, from the case where S is finite, that $f_n : X \times T \to \mathbb{R}$ is a supermodular function. Since supermodularity is preserved by pointwise convergence, f is supermodular.

Continuation of Example 9

We show that correspondence Λ introduced in Example 9 satisfies condition (i) in Proposition 5. Take any $t' \geq t$ and $\lambda \in \Lambda(t)$, $\lambda' \in \Lambda(t')$. Given that

$$\int_S h(\tilde{s})d(\lambda \wedge \lambda')(\tilde{s}) \;\leq\; \int_S h(\tilde{s})d\lambda'(\tilde{s}) \;=\; t \;\leq\; t' \;=\; \int_S h(\tilde{s})d\lambda'(\tilde{s}),$$

there is a number $\alpha \in [0,1]$ such that

$$\alpha \int_S h(\tilde{s})d\lambda'(\tilde{s}) + (1-\alpha)\int_S h(\tilde{s})d(\lambda \wedge \lambda')(\tilde{s}) \;=\; t.$$

Denote $\mu := \alpha\lambda' + (1-\alpha)(\lambda \wedge \lambda')$ and $\mu' := \alpha\lambda + (1-\alpha)(\lambda \vee \lambda')$. Clearly, we have $\mu \in \Lambda(t)$ as well as $\lambda' \succeq \mu$. By construction, we obtain

$$\begin{aligned}
\int_S h(\tilde{s})d\mu'(\tilde{s}) \;&=\; \alpha\int_S h(\tilde{s})d\lambda(\tilde{s}) + (1-\alpha)\int_S h(\tilde{s})d(\lambda \vee \lambda')(\tilde{s}) \\
&=\; \alpha\int_S h(\tilde{s})d\lambda(\tilde{s}) + (1-\alpha)\left[\int_S h(\tilde{s})d\lambda(\tilde{s}) + \int_S h(\tilde{s})d\lambda'(\tilde{s}) - \int_S h(\tilde{s})d(\lambda \wedge \lambda')(\tilde{s})\right] \\
&=\; \int_S h(\tilde{s})d\lambda(\tilde{s}) + \int_S h(\tilde{s})d\lambda'(\tilde{s}) - \int_S h(\tilde{s})d\mu(\tilde{s}) \;=\; t + t' - t \;=\; t'.
\end{aligned}$$

Hence, we have $\mu' \in \Lambda(t')$. Finally, it must be that

$$\tfrac{1}{2}\lambda + \tfrac{1}{2}\lambda' \;=\; \tfrac{1}{2}\big[\alpha\lambda' + (1-\alpha)(\lambda \wedge \lambda')\big] + \tfrac{1}{2}\big[\alpha\lambda + (1-\alpha)(\lambda \vee \lambda')\big] \;=\; \tfrac{1}{2}\mu + \tfrac{1}{2}\mu',$$

which implies that correspondence Λ satisfies condition (i) in Proposition 5.

Continuation of Example 10

We prove that correspondence Λ introduced in Example 10 satisfies condition (i) in Proposition 5. Take any $t' \geq t$ and $\lambda \in \Lambda(t)$, $\lambda' \in \Lambda(t')$. There are measures ν, $\nu' \in \mathcal{P}_{\Omega'}$ such that $\lambda(s) = \nu\big(\{\omega \in \Omega : f(\omega, t) \leq s\}\big)$ and $\lambda'(s) = \nu'\big(\{\omega \in \Omega : f(\omega, t') \leq s\}\big)$, for all $s \in S$. Define μ, μ' by $\mu(s) := \nu'\big(\{\omega \in \Omega : f(\omega, t) \leq s\}\big)$ and $\mu'(s) := \nu\big(\{\omega \in \Omega : f(\omega, t') \leq s\}\big)$, for $s \in S$. Clearly, we have $\mu \in \Lambda(t)$ and $\mu' \in \Lambda(t')$, while $\lambda' \succeq \mu$. Therefore, it suffices to show that $(1/2)\lambda + (1/2)\lambda' = (1/2)\mu + (1/2)\mu'$.

Since $\nu \in \mathcal{P}_{\Omega'}$ and $f(\omega, t') = f(\omega, t)$, for all $\omega \in \Omega \setminus \Omega'$, we obtain

$$\begin{aligned}
\lambda'(s) - \mu(s) &= \nu\big(\{\omega \in \Omega : f(\omega, t') \leq s\}\big) - \nu\big(\{\omega \in \Omega : f(\omega, t) \leq s\}\big) \\
&= \nu\big(\{\omega \in \Omega' : f(\omega, t') \leq s\}\big) + \nu\big(\{\omega \in \Omega \setminus \Omega' : f(\omega, t') \leq s\}\big) \\
&\qquad - \nu\big(\{\omega \in \Omega' : f(\omega, t) \leq s\}\big) - \nu\big(\{\omega \in \Omega \setminus \Omega' : f(\omega, t) \leq s\}\big) \\
&= \nu^*\big(\{\omega \in \Omega' : f(\omega, t') \leq s\}\big) + \nu\big(\{\omega \in \Omega \setminus \Omega' : f(\omega, t') \leq s\}\big) \\
&\qquad - \nu^*\big(\{\omega \in \Omega' : f(\omega, t) \leq s\}\big) - \nu\big(\{\omega \in \Omega \setminus \Omega' : f(\omega, t) \leq s\}\big) \\
&= \nu^*\big(\{\omega \in \Omega' : f(\omega, t') \leq s\}\big) - \nu^*\big(\{\omega \in \Omega' : f(\omega, t) \leq s\}\big),
\end{aligned}$$

for all $s \in S$. Analogously, we can show that

$$\mu'(s) - \lambda(s) = \nu^*\big(\{\omega \in \Omega' : f(\omega, t') \leq s\}\big) - \nu^*\big(\{\omega \in \Omega' : f(\omega, t) \leq s\}\big),$$

which completes our argument.

Proof of Proposition 6

Implication ($\Rightarrow$) follows directly from Proposition 5. To show the converse, recall from the proof of Proposition 5 that the function f satisfies

$$f(x, t) = g(x, s_{\ell+1}) + \min\Big\{\mathbf{1} \cdot a : a_i = -\delta_i(x)\lambda(s_i), \text{ for } i = 1, \ldots, \ell, \text{ and } \lambda \in \Lambda(t)\Big\},$$

where $\delta_i(x) = \big[g(x, s_{i+1}) - g(x, s_i)\big]$, for all $i = 1, \ldots, \ell$. Notice that function g is increasing and supermodular if and only if functions δ_i are positive and increasing. Therefore, whenever function f is supermodular for any increasing and supermodular function g, by Propositions 1 and A.1, it must be that, for any $t' \geq t$ and $\lambda \in \Lambda(t)$, $\lambda' \in \Lambda(t')$ there is some $\mu \in \Lambda(t)$, $\mu' \in \Lambda(t')$ such that $\lambda'(s) \leq \mu(s)$ and $\lambda(s) + \lambda'(s) \leq \mu(s) + \mu'(s)$, for all $s \in S$. Equivalently, this is to say that

$$\lambda' \succeq \mu \quad \text{and} \quad \tfrac{1}{2}\lambda + \tfrac{1}{2}\lambda' \succeq \tfrac{1}{2}\mu + \tfrac{1}{2}\mu'.$$

In Lemma A.5 in Appendix A, we show that it is always possible to find probability distributions ν, ν' in $\triangle_S$ such that

$$\lambda' \succeq \nu \succeq \mu, \quad \nu' \succeq \mu', \quad \text{and} \quad \tfrac{1}{2}\lambda + \tfrac{1}{2}\lambda' = \tfrac{1}{2}\nu + \tfrac{1}{2}\nu'.$$

Since values of correspondence Λ are upper comprehensive, $\nu \succeq \mu$ implies $\nu \in \Lambda(t)$ and $\nu' \succeq \mu'$ implies $\nu' \in \Lambda(t')$. This concludes the proof.

Proof of Proposition 7

We proceed with the proof of implication (i) $\Rightarrow$ (ii). Take some $x' \geq x$, $t' \geq t$ and any $a \in \Gamma(x,t)$, $a' \in \Gamma(x',t')$. By definition, there are distributions λ, λ' such that $a_i = -\delta_i(x)\lambda(s_i)$, $a'_i = -\delta_i(x')\lambda'(s_i)$, for $i = 1, \ldots, \ell$, and $a_{\ell+1} \geq c(\lambda, t)$, $a'_{\ell+1} \geq c(\lambda', t')$. By assumption, there are distributions μ, μ' such that $\mu(s_i) - \lambda'(s_i) = \lambda(s_i) - \mu'(s_i) \geq 0$, for all $i = 1, \ldots, \ell$, and $c(\lambda, t) + c(\lambda', t') \geq c(\mu, t) + c(\mu', t')$. Since function g is supermodular if and only if δ_i is increasing, for all $i = 1, \ldots, \ell$, it must be that

$$\delta_i(x')\big[\mu(s_i) - \lambda'(s_i)\big] \;\geq\; \delta_i(x)\big[\mu(s_i) - \lambda'(s_i)\big] \;=\; \delta_i(x)\big[\lambda(s_i) - \mu'(s_i)\big],$$

for all $i = 1, \ldots, \ell$. Moreover, it must be that

$$a_{\ell+1} + a'_{\ell+1} \;\geq\; c(\lambda, t) + c(\lambda', t') \;\geq\; c(\mu, t) + c(\mu', t').$$

Define vectors b, b' so that $b_i = -\delta_i(x')\mu(s_i)$, $b'_i = -\delta_i(x)\mu'(s_i)$, for $i = 1, \ldots, \ell$, and $b_{\ell+1} = c(\mu, t)$, $b'_{\ell+1} = c(\mu', t')$. Clearly, we have $b \in \Gamma(x', t)$, $b' \in \Gamma(x, t')$ and $a + a' \geq b + b'$.

In order to prove (ii) $\Rightarrow$ (iii), as in (6), we can show that

$$f(x,t) \;=\; g(x, s_{\ell+1}) + \min\Big\{\mathbf{1} \cdot a : a \in \Gamma(x,t)\Big\},$$

for any function $g : X \times S \to \mathbb{R}$. Therefore, by the Main Theorem, (ii) implies (iii).

We show (iii) $\Rightarrow$ (ii) by contradiction. Suppose there is some supermodular function g for which correspondence Γ is not lower supermodular. Clearly, values of Γ are bounded from below. Moreover, since c is continuous and convex, the values also are closed and convex. By Proposition 1, there is some $p \in \mathbb{R}^{\ell+1}_{+}$ such that function

$$f(x,t) \;=\; p_\ell g(x, s_{\ell+1}) + \min\Big\{p \cdot a : a \in \Gamma(x,t)\Big\}$$

is not supermodular. Clearly, it must be that $p_{\ell+1} > 0$. Otherwise f would be constant over T and trivially supermodular. Take the vector $\hat{p} = (1/p_{\ell+1})p$ and define a function $\hat{g} : X \times S \to \mathbb{R}$ by $\hat{g}(x, s_1) = \hat{p}_1 g(x, s_1)$ and $\hat{g}(x, s_{i+1}) = \hat{g}(x, s_i) + \hat{p}_i \delta_i(x)$, for $i = 2, \ldots, \ell$, where $\delta_i(x) := \big[g(x, s_{i+1}) - g(x, s_i)\big]$, for all $x \in X$. Clearly, $\hat{g}$ is supermodular and

$$p_{\ell+1} f(x,t) \;=\; \min\left\{\int_S \hat{g}(x, \tilde{s})d\lambda(\tilde{s}) + c(\lambda, t) : \lambda \in \triangle_S\right\},$$

which contradicts that f is a supermodular function.

Implication (ii) $\Rightarrow$ (i) follows directly from Proposition A.2. Function $g : X \times S \to \mathbb{R}$ is supermodular if and only if function $\delta_i(x)$ is increasing, for all $i = 1, \ldots, \ell$. Since values of the correspondence Λ are compact and convex, condition (ii) implies that for any $t' \geq t$ and $z \in \Lambda(t)$, $z' \in \Lambda(t')$, there are some $y \in \Lambda(t)$, $y' \in \Lambda(t')$ such that

$$\begin{aligned} z'_i &= -\lambda'(s_i) &\geq& -\mu(s_i) &=& y_i; \\ z_i + z'_i &= -\lambda(s_i) - \lambda'(s_i) &=& -\mu(s_i) - \mu'(s_i) &=& y_i + y'_i, \end{aligned}$$

for all $i = 1, \ldots, \ell$, and $c(\lambda, t) + c(\lambda', t') \leq z_{\ell+1} + z'_{\ell+1} = y_{\ell+1} + y_{\ell+1}$. Since $y_{\ell+1} \geq c(\mu, t)$ and $y'_{\ell+1} \geq c(\mu', t')$, the latter is satisfied only if $c(\lambda, t) + c(\lambda', t') \geq c(\mu, t) + c(\mu', t')$. Clearly, this is equivalent to the property (i).

Proof of Proposition 8

To show that $c(\lambda, t) := R\big(\lambda \| \lambda^*(\cdot, t)\big)$ is submodular in (λ, t) it suffices to show that it is submodular in λ, for all $t \in T$, and has decreasing differences in (λ, t). Recall that by $d\lambda(s)$ we denote the probability of state s that is induced by distribution λ. That is, let $d\lambda(s_1) = \lambda(s_1)$ and $d\lambda(s_i) = \big[\lambda(s_i) - \lambda(s_{i-1})\big]$, for all $i = 2, \ldots, \ell$.

We start with proving the former. Without loss of generality, let $S = \{s_1, s_2, \ldots, s_\ell\}$, where $s_1 < s_2 < \ldots < s_\ell$. In order to prove the result, it suffices to show that, for any measures λ, λ' and any state s_i, we have

$$\begin{aligned} & d\lambda(s_i) \log d\lambda(s_i) + d\lambda'(s_i) \log d\lambda'(s_i) - \big[d\lambda(s_i) + d\lambda'(s_i)\big] \log d\lambda^*(s_i, t) \\ & \qquad \geq \ d(\lambda \wedge \lambda')(s_i) \log d(\lambda \wedge \lambda')(s_i) + d(\lambda \vee \lambda')(s_i) \log d(\lambda \vee \lambda')(s_i) \\ & \qquad\qquad - \ \big[d(\lambda \wedge \lambda')(s_i) + d(\lambda \vee \lambda')(s_i)\big] \log d\lambda^*(s_i, t). \end{aligned}$$

We prove this claim by induction. Clearly, this condition holds trivially for $i = 1$. To show the inductive step, suppose that the above inequality holds for some $i = 1, \ldots, (\ell-1)$. We prove that it is true for $i + 1$. With no loss of generality, let $(\lambda \wedge \lambda')(s_i) = \lambda(s_i)$ and $(\lambda \vee \lambda')(s_i) = \lambda'(s_i)$. Consider two cases. First, assume that

$$d\lambda'(s_{i+1}) + \lambda'(s_i) \ \leq \ d\lambda(s_{i+1}) + \lambda(s_i)$$

so that $(\lambda \wedge \lambda')(s_{i+1}) = \lambda(s_{i+1})$ and $(\lambda \vee \lambda')(s_{i+1}) = \lambda'(s_{i+1})$. In such a case, we have $d(\lambda \wedge \lambda')(s_{i+1}) = d\lambda(s_{i+1})$ and $d(\lambda \vee \lambda')(s_{i+1}) = d\lambda'(s_{i+1})$, hence, the above inequality holds trivially. Alternatively, suppose that

$$d\lambda'(s_{i+1}) + \lambda'(s_i) \;>\; d\lambda(s_{i+1}) + \lambda(s_i),$$

which implies $(\lambda \wedge \lambda')(s_{i+1}) = \lambda'(s_{i+1})$ and $(\lambda \vee \lambda')(s_{i+1}) = \lambda(s_{i+1})$. Define number $\delta := \big[\lambda(s_i) - \lambda'(s_i)\big]$ and notice that, we have $0 \leq \delta < \big[d\lambda'(s_{i+1}) - d\lambda(s_{i+1})\big]$. In addition, given that $d(\lambda \wedge \lambda')(s_{i+1}) = d\lambda'(s_{i+1}) - \delta$ and $d(\lambda \vee \lambda')(s_{i+1}) = d\lambda(s_{i+1}) + \delta$, we obtain

$$\begin{aligned}
& d\lambda(s_{i+1}) \log d\lambda(s_{i+1}) + d\lambda'(s_{i+1}) \log d\lambda'(s_{i+1}) \\
& \qquad\qquad - \big[d\lambda(s_{i+1}) + d\lambda'(s_{i+1})\big] \log d\lambda^*(s_{i+1}, t) \\
\geq\; & \big[d\lambda'(s_{i+1}) - \delta\big] \log \big[d\lambda'(s_{i+1}) - \delta\big] + \big[d\lambda(s_{i+1}) + \delta\big] \log \big[d\lambda(s_{i+1}) + \delta\big] \\
& \qquad\qquad - \big[d\lambda(s_{i+1}) + d\lambda'(s_{i+1})\big] \log d\lambda^*(s_{i+1}, t) \\
=\; & d(\lambda \wedge \lambda')(s_{i+1}) \log d(\lambda \wedge \lambda')(s_{i+1}) + d(\lambda \vee \lambda')(s_{i+1}) \log d(\lambda \vee \lambda')(s_{i+1}) \\
& \qquad\qquad - \big[d(\lambda \wedge \lambda')(s_{i+1}) + d(\lambda \vee \lambda')(s_{i+1})\big] \log d\lambda^*(s_{i+1}, t),
\end{aligned}$$

where the inequality follows from convexity of function $z \to z \log z$. Therefore, function c is submodular on $\triangle_S$, for all $t \in T$.

In order to show that function $c(\lambda, t) = R\big(\lambda \| \lambda^*(\cdot, t)\big)$ has decreasing differences in (λ, t), take any distribution $\lambda' \succeq \lambda$ and notice that

$$\begin{aligned}
R\big(\lambda' \| \lambda^*(\cdot, t)\big) - R\big(\lambda \| \lambda^*(\cdot, t)\big) \;=\; & \sum_{i=1}^{\ell} d\lambda'(s_i) \log d\lambda'(s_i) - \sum_{i=1}^{\ell} d\lambda(s_i) \log d\lambda(s_i) \\
& + \sum_{i=1}^{\ell} \big[d\lambda(s_i) - d\lambda'(s_i)\big] \log d\lambda^*(s_i, t),
\end{aligned}$$

for any $t \in T$. Therefore, for any $t' \geq t$ in T, we obtain

$$\begin{aligned}
& \Big[R\big(\lambda' \| \lambda^*(\cdot, t')\big) - R\big(\lambda \| \lambda^*(\cdot, t')\big)\Big] - \Big[R\big(\lambda' \| \lambda^*(\cdot, t)\big) - R\big(\lambda \| \lambda^*(\cdot, t)\big)\Big] \;= \\
& \qquad \sum_{i=1}^{\ell} \big[\log d\lambda^*(s_i, t') - \log d\lambda^*(s_i, t)\big] \big[d\lambda(s_i) - d\lambda'(s_i)\big].
\end{aligned}$$

We claim that the above expression is negative. Indeed, recall that $\lambda^*(\cdot, t)$ increases in the monotone likelihood ratio, hence, function $g(s) := \big[\log d\lambda^*(s, t') - \log d\lambda^*(s, t)\big]$ is increasing. Given that $\lambda' \succeq \lambda$, the above expression must be negative.

Proof of Proposition 9

Let $v : X \times S \to \mathbb{R}$ be a continuous and bounded function. Since the problem is well-behaved we know that the function $(\mathscr{T} v)$, given by

$$\big(\mathscr{T} v\big)(x,s) \;=\; \max\Big\{F(x,s,y) + \beta(Av)(y,s) : y \in B(x,s)\Big\},$$

is a continuous function on $X \times S$ and $(\mathscr{T}^n v)$ converges uniformly to v^* as $n \to \infty$.

By Proposition 5, whenever function v is supermodular, then so is (Av). Clearly, this implies that $F(x,s,y) + \beta(Av)(y,s)$ is supermodular over $X \times S \times X$. Given that the graph of correspondence B is a sublattice, by Theorem 4.3 in Topkis (1978), function $(\mathscr{T} v)$ is supermodular. Since supermodularity is preserved under uniform convergence, we conclude that $v^* = (\mathscr{T} v^*)$ is a supermodular.

In order to show (ii), notice that set $\Phi(x,s)$ consists of elements y that maximise function $F(x,s,y) + \beta(Av^*)(x,s)$ over $B(x,s)$. Since the function is supermodular, while values of correspondence B are complete sub-lattices of X, by the MCS Theorem, set $\Phi(s,x)$ is a complete sub-lattices of X. Furthermore, since B increases over $X \times S$ in the strong set order, so does Φ. As the problem is well-behaved, the latter admits the greatest selection $g(x,s)$ that is increasing and measurable. This follows from HP.

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