Simultaneous Versus Sequential Disclosure[*]

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Abstract

We study an environment in which a decision maker has access to several expert advisers. The experts all have access to an identical set of facts. The decision maker's utility is increasing in the number of facts that the experts reveal. The experts have (potentially) different preferences. The game in which experts simultaneously disclose information typically has multiple equilibria. When multiple equilibria exist, the decision maker's favorite equilibrium fails to survive iterative deletion of weakly dominated strategies. We characterize the set of equilibria that survive iterative deletion of weakly dominated strategies. In a leading special case, only one outcome survives iterative deletion of weakly dominated strategies. It is the most preferred equilibrium from the perspective of the experts. We study the outcomes that can arise when the decision maker can consult the experts sequentially. We demonstrate that if the decision maker can select the order of consultations, can consult experts multiple times, and can commit to ending the consultation process, then in leading cases he can induce the same disclosure as with simultaneous disclosure. Sequential disclosure may perform worse than simultaneous disclosure from the perspective of the decision maker when it is not possible to consult experts multiple times or if commitment is not feasible.

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1   Introduction

Decision makers often take actions about things that they do not understand. When someone experiences pain, he may consult a doctor, who describes treatment options. When a car breaks down, the driver may ask a mechanic for advice. When a firm considers a product innovation, it may consult division managers before deciding a marketing strategy. Often, decision makers approach these problems by consulting just one expert. But there are situations in which it is common to request advice from more than one source. We are interested in these situations. We focus on two questions: what is the value of consulting more than one expert and how best to structure the consultations.

It is common in the literature to assume that there is an unknown state of the world. In such a model, disclosures provide information about that state. In contrast, we consider an abstract model in which experts have access to elements in a finite set. Experts choose which element to reveal. The decision maker can use everything revealed to make his decision. In our preferred interpretation, disclosures introduce new actions to the decision maker. In the case of a consultation with a doctor or a mechanic, the disclosure may be a treatment option. In the case of an executive, the disclosures may be marketing strategies or demonstrations of the valuable qualities of the new product. The disclosures may equip a department chair with arguments to persuade administrators to hire the job candidate. We view experts as individuals capable of permitting a decision maker to do something that would not be feasible without consultation. Once a doctor describes a surgical procedure, a mechanic suggests a repair, or a marketing advisor describes an ad campaign, the approach becomes available to the decision maker. Without advice, the decision maker would not have access to the idea. We assume that more disclosure benefits the decision maker (as he can ignore disclosures that he does not use), but experts may have different incentives. It is the conflict of interest between decision maker and expert that creates the possibility that the decision maker does not learn everything needed to make a decision from a single expert. Because of the possibility of incomplete disclosure, the decision maker might gain from consulting more than one expert.

Formally, we study a model in which a finite set of experts have access to an identical, partially ordered set of "facts." They play a game in which facts are disclosed. Their payoffs depend on the maximum (component wise when the set of facts is multidimensional) disclosure. We do not model the decision maker explicitly, but we assume that he makes a decision that depends on the maximum disclosure. The experts have preferences over the decision. We study two different ways of structuring the consultation process. In the first, the experts simultaneously make disclosures. In the second, the decision maker consults experts in a sequence.

An immediate result from this formulation is that competition between experts permits the decision maker to acquire the maximum amount of information when disclosure is simultaneous. The reason is straightforward: if one expert discloses everything, then it is a best response for other experts to do the same. While intuition suggests that full disclosure is possible with only two experts, we argue that the full disclosure result is too good to be true. More precisely, we show that an equilibrium refinement typically eliminates the full-disclosure outcome. The formal result is stronger than this. To get a flavor of the results, consider the case in which disclosure is one-dimensional (possi-

ble disclosures are completely ordered). In this setting, the simultaneous disclosure game typically has multiple Nash equilibrium disclosures. Moreover, the experts have the same preferences over equilibria: They prefer equilibria in which the maximum disclosure is smaller. (If an expert preferred disclosure x' to disclosure $x < x'$, then the maximum disclosure would never be x in equilibrium.) The decision maker's preference, of course, is the opposite: he prefers more disclosure to less. We show for generic payoffs that when disclosure is simultaneous, iterative deletion of weakly dominated strategies selects the experts' most preferred equilibrium. That is, the full-disclosure equilibrium is not robust if there exists another equilibrium. The paper extends this result to multi-dimensional disclosure and non-generic payoffs.

Because simultaneous disclosure need not lead to full disclosure, it is natural to ask if different ways of acquiring advice could lead to more disclosure. We study a family of sequential disclosure procedures. Here our main result is an equivalence theorem. Again, for simplicity, consider the one-dimensional case with generic preferences. We show that sequential disclosure can never do better for the decision maker than simultaneous disclosure, but that the decision maker is always able to design a sequential procedure that duplicates the outcome that survives iterative deletion of weakly dominated strategies with simultaneous disclosure. The sequential procedure requires that the decision maker have the ability to do three things: pick the order that he consults experts; commit to ending the consultation process; and have the power to return to an expert (giving her multiple opportunities to make a disclosure). This result suggests that our model does not provide a strong reason to favor simultaneous disclosure over sequential disclosure or vice versa. The paper again generalizes the finding to multi-dimensional disclosure and non-generic payoffs. The qualitative results are now less clear cut: we provide examples under which simultaneous consultation permits more disclosure than sequential consultation and others where sequential consultation allows more disclosure, but even when the set of disclosures is not linearly ordered we obtain similar bounds on the range of disclosures that can arise.

The analysis provides several other insights. Given a group of experts, we can identify when it is unnecessary to consult an expert. We can also establish a comparative-static result that confirms the intuition that adding experts must be weakly beneficial to the decision maker.

A very special case might provide useful intuition for the results. Suppose that experts have single-peaked preferences over a one-dimensional disclosure. That is, Expert i is characterized by an optimal disclosure x_i^*. Her preferences increase for $x < x_i^*$ and decrease thereafter. In this setting, it is intuitive that disclosures greater than x_i^* are weakly dominated and that the salient prediction for the simultaneous disclosure game is that the decision maker will learn the maximum of the x_i^*. It is also apparent that the decision maker need only consult a single expert – the one with the maximum x_i^*. When preferences are not single peaked, the analysis is more complicated. By constructing a sequential procedure with repeated consultations, the decision maker can secure a disclosure of one expert's global maximum and then use this knowledge to induce further disclosures from other experts.

We organize the remainder of the paper as follows. Section 2 describes some of the papers related to ours. Section 3 describes the basic model. Section 4 contains the

analysis of the simultaneous disclosure game. Section 5 contains the analysis of the sequential disclosure game. Section 6 compares simultaneous to sequential disclosure. Section 7 contains concluding remarks. The appendix contains proofs that are not in the main text.

2   Related Literature

There is an extensive literature that discusses whether competition between information sources leads to better decisions. Gentzkow and Kamenica (2017a) and Gentzkow and Kamenica (2017b) study a model in which experts simultaneously choose how much to communicate to a decision maker in a Bayesian Persuasion framework. In these models, the decision maker wants to know the value of the state of the world and the strategies of experts are arbitrary signals (joint probability distributions on the state and message received by the decision maker). Gentzkow and Kamenica (2017b) shows that adding an agent need not increase the amount of information revelation, but provides a condition under which increasing the number of experts increases the amount of information revealed. Although we do not interpret our model in this way, one can view our model as an example of the Bayesian Persuasion problems studied by Gentzkow and Kamenica with restrictions on the kind of signals available to the experts. Additional experts are always valuable in the sense that the minimal equilibrium disclosure is increasing in the number of experts. Gentzkow and Kamenica do not focus on equilibrium selection, but they note the existence of multiple equilibria and the tendency of experts to prefer less disclosure. Li and Norman (2018b) studies a sequential version of the Gentzkow and Kamenica model. They provide an existence and partial characterization result. They show that sequential persuasion results in no more informative equilibria than simultaneous persuasion. Li and Norman (2018a) also notes that the order of disclosure matters, pointing out it is possible that adding an expert into a sequence may decrease the amount of information disclosure. In addition to the different interpretation of the nature of the disclosure game, our analysis contributes an equilibrium refinement of the simultaneous game and studies the optimal order of consultation in the sequential game.

Krishna and Morgan (2001a) and Krishna and Morgan (2001b) study competition in disclosure in a cheap-talk setting. When experts report simultaneously, they construct a fully equilibrium, but they show that full disclosure need not be an equilibrium when experts move sequentially. Milgrom and Roberts (1986) study the issue in a model of verifiable information. In this setting, full disclosure is an equilibrium with only a single expert in leading cases (Grossman (1981) and Milgrom (1981)).

We know of three papers that compare simultaneous to sequential interactions in different contexts. Dekel and Piccione (2000) compare simultaneous to sequential voting institutions. There are a finite number of voters and two options. Voters can either vote for or against the status quo. Voters do not know their valuations, but receive private signals. Dekel and Piccone compare the equilibria of games in which voters cast votes simultaneously to those in which votes are sequential. They show that a symmetric informative equilibrium of the simultaneous game is an equilibrium to any sequential

game. Weaker results hold for asymmetric equilibria.[1] Although this paper researches a conclusion that is similar to ours, we do not see a formal connection between the analyses. Dekel and Piccone's model focuses on aggregation of preferences, while our setting involves disclosure. Our equivalence result requires an equilibrium refinement and commitment power in the design of sequential disclosure mechanisms. Schummer and Velez (2018) identify conditions under which social choice functions that can be implemented in truthful strategies when players move simultaneously cannot be truthfully implemented when players move sequentially. The context is quite different from our paper, but it suggests environments in which sequential procedures will perform less well than simultaneous ones.

3   Underlying Strategic Environment

There is a finite set of players. I denotes the player set.[2] We assume that there is a common set of strategies X available to each player and that $X = X(1) \times \cdots \times X(n)$, where each $X(k)$ is a finite set of real numbers. For $x, x' \in \mathbb{R}^n$ let $x \vee x' \in \mathbb{R}^n$ be the component-wise maximum (the kth component of $x \vee x'$ is the maximum of the kth components of x and x'). For $x = (x(1), \ldots, x(n)) \in X$, we say $x > x'$ if $x(k) \geq x'(k)$ for $k = 1, 2, \ldots, n$ and at least one inequality is strict. For $y = (y_1, \ldots, y_l)$, $y_i \in \mathbb{R}^n$ let $M(y) = y_1 \vee \cdots \vee y_l$. Each player i has a utility function $\tilde{u}_i : X \to \mathbb{R}$. We assume that there exist $u_i : X \to \mathbb{R}$ such that $\tilde{u}_i(x) \equiv u_i(M(x))$. We call elements of X disclosures. We denote the minimal element of X by $\underline{x}$.

We assume each $u_i(\cdot)$ is quasi supermodular.[3] Quasi-supermodularity is a complementarity assumption that implies, roughly, that increasing the disclosure in one dimension makes disclosure in another dimension more attractive. It will hold if utility is separable across components, but it will fail if facts in one dimension substitute for facts in another dimension (and experts strictly prefer intermediate disclosures).

We view this setting as a reduced form for disclosure games. The player set consists of a group of experts. There are n issues. For each issue, there is an ordered list of "facts." Each of the agents has access to these facts. A strategy for a player describes the facts that the expert discloses to a decision maker. The decision maker aggregates the disclosures $y_1, \ldots, y_I$ and can make a decision based on all of the facts presented, $M(y_1, \ldots, y_I)$. We do not model the choices of the decision maker explicitly, but we have in mind a situation in which the decision maker selects an action given available facts and that the optimal action rule is a function of the join of all of the disclosures. Keeping with this interpretation, we assume that the decision maker's utility is increasing in what is disclosed.

One version of our interpretation is that there is an underlying state of the world and experts provide the decision maker with "experiments" – procedures that produce for each state of the world a probability distribution over a set of signals observable by

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[1]Dekel and Piccone (2014) studies a voting model in which the timing of votes is a strategic choice.

[2]In an abuse of notation, we also let I denote the cardinality of the player set.

[3]The function $f : \mathbb{R}^n \to \mathbb{R}$ is supermodular if $f(x \vee y) + f(x \wedge y) \geq f(x) + f(y)$ and quasi-supermodular if $f(x) \geq (>)f(x \wedge y)$ implies $f(x \vee y) \geq (>)f(y)$.

the decision maker. The decision maker then makes a decision based on the signals he observes (and knowledge of the experiments and the prior distribution on the state of the world). This interpretation is consistent with the model of competition in persuasion in Gentzkow and Kamenica (2017b). We prefer the interpretation that the experts provide tools for solving problems that the decision maker could not access without the experts' disclosures.

4 Simultaneous Disclosure

In this section, we study the game in which experts simultaneously select an element in X. If $x = (x_1, \dots, x_I)$ is the profile of disclosures, then Expert i's payoff is $\tilde{u}_i(x)$.

A disclosure $\pi = (\pi_1, \dots, \pi_n)$ such that $u_i(\pi) \geq u_i(\pi')$ for all $\pi' > \pi$ and all i is a Nash equilibrium disclosure. For any equilibrium disclosure π, a strategy profile x that satisfies $x_i \leq \pi$ and at least two $x_j = \pi$ is a Nash Equilibrium. Full disclosure arises when the maximum disclosure is equal to the maximum element in X. Full disclosure is always an equilibrium disclosure. It can be supported by a strategy profile in which at least two players reveal the maximal element of X. Typically there are other Nash Equilibria. The pure-strategy Nash Equilibria are Pareto ranked when information structures are linearly ordered (if x^* and x^{**} are equilibrium disclosures and $x^{**} > x^*$ then all experts prefer x^* to x^{**}). The equilibria are not necessarily Pareto ranked if information structures are partially ordered, but even in this case, if x^* and x^{**} are both Nash Equilibria and $M(x^*) \leq M(x^{**})$, then $\tilde{u}_i(x^*) \geq \tilde{u}_i(x^{**})$ for all i.

The possibility of multiple equilibria leads us to consider a more restrictive solution concept.

Definition 1. *Given subsets $X_i' \subset X$, with $X' = \prod_{i \in I} X_i'$, Player i's strategy $x_i^* \in X_i'$ is a best response to $x_{-i} \in X_{-i}'$ relative to X_i if $\tilde{u}_i(x_i^*, x_{-i}) \geq \tilde{u}_i(x_i, x_{-i})$ for all $x_i \in X_i$. Player i's strategy $x_i \in X_i'$ is weakly dominated relative to X' if there exists $z_i \in X_i'$ such that $\tilde{u}_i(x_i, x_{-i}) \leq \tilde{u}_i(z_i, x_{-i})$ for all $x_{-i} \in X_{-i}'$, with strict inequality for at least one $x_{-i} \in X_{-i}'$.*

Definition 2. *The set $S = S_1 \times \cdots \times S_I \subset X$ survives iterated deletion of weakly dominated strategies (IDWDS) if for $m = 0, 1, 2, \dots$ there are sets $S^m = S_1^m \times \cdots \times S_I^m$, such that $S^0 = X$, $S^m \subset S^{m-1}$ for $m > 0$; S_i^m is obtained by (possibly) removing strategies in S_i^{m-1} that are weakly dominated relative to S^{m-1}; $S^m = S^{m-1}$ if and only if for each i no strategy in S_i^{m-1} is weakly dominated relative to S^{m-1}; and $S_i = \bigcap_{m=1}^{\infty} S_i^m$ for each i.*[4]

For finite games, it must be the case that there exists an m such that $S^r = S^m$ for all $r > m$. There are typically many different procedures that are consistent with Definition 2. These procedures may lead to different sets that survive the process. We discuss properties that are common to all sets that survive and give conditions under which all sets that survive lead to the same maximum disclosure.

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[4] Our notation follows these rules: superscripts denote steps in an iterative process; subscripts denotes players; arguments are components.

IDWDS is a powerful concept that makes strong demands on the rationality of agents. It is also delicate – the order of deletion matters and it is sometimes poorly behaved in games with continuous strategy spaces.[5] Nevertheless, this concept appears appropriate in contexts such as ours where an individual agent's decision is relevant to her own payoff in a limited number of circumstances. Just as in voting models one wants to condition behavior on the event that a voter is pivotal, in the disclosure model, one wants to focus attention on circumstances when an expert's disclosure will alter the total amount of information available to the decision maker. Weak dominance arguments capture these strategic circumstances.

We analyze the implications of applying iterated deletion of weakly dominated strategies. Sobel (2019) introduces a class of supermodular games called WID-supermodular games and describes general properties of strategies that survive the process of iteratively deleting weakly dominated strategies in these games. He shows that if $X \subset \mathbb{R}$, then the simultaneous disclosure game is a WID-supermodular game. It is straightforward to show that when $u_i(\cdot)$ is supermodular, the simultaneous disclosure game is a WID-supermodular game even when X is not one dimensional.

Quasi-supermodularity imposes structure on the set of Nash equilibria. We state some useful properties. Topkis (2011, Theorem 2.72) reports Fact 1.

Fact 1. *For any sublattice $X' \subset X$, $\arg\max_{z \in X'} u_i(z)$ is a sublattice of X.*

Hence the set of best replies to any pure strategy forms a sublattice and the smallest best response exists.

Lemma 1. *If π' and π'' are equilibrium disclosures, then $\pi' \wedge \pi''$ is an equilibrium disclosure.*

Lemma 1 implies that there is a minimum equilibrium disclosure. We have observed that maximal disclosure is an equilibrium, but now we know that a minimal equilibrium disclosure exists. We next show that every disclosure that survives iterated deletion of weakly dominated strategies, whether it is an equilibrium or not, is greater than or equal to this disclosure. Before we describe the procedure, we first define two quantities π^* and $\tilde{\pi}^*$ that are an upper bound and a lower bound of the refined equilibria.

Definition 3. *The smallest strictly preferred equilibrium disclosure is*

$$\pi^* = \min\{\pi : u_i(\pi) > u_i(x_i) \text{ for all } x_i > \pi \text{ and all } i\}.$$

Definition 4. *The smallest equilibrium disclosure is*

$$\tilde{\pi}^* = \min\{\pi : u_i(\pi) \geq u_i(x_i) \text{ for all } x_i > \pi \text{ and all } i\}.$$

Note that these disclosures are Pareto efficient (from perspective of the experts) in the set of Nash equilibrium payoffs.

It is immediate that $\tilde{\pi}^*$ is well defined and the smallest Nash equilibrium disclosure. That π^* is well defined is a straightforward consequence of the definition. Clearly, $\pi^* \geq$

$\tilde{\pi}^*$. Equality will hold if $u_i(\cdot)$ is one-to-one for each player. Any strategy profile x that satisfies $x_i \leq \pi$ and at least two $x_j = \pi$ is a Nash Equilibrium for $\pi = \pi^*$ and $\tilde{\pi}^*$. We will show that $\tilde{\pi}^*$ is an lower bound of disclosures, i.e., all disclosures that survive IDWDS are greater than or equal to $\tilde{\pi}^*$.

We need a bit more notation to state our main result.

Definition 5. *Let $\bar{x}(k)$ be the largest (feasible) disclosure on dimension k. Let $\bar{x}(-k)$ be the largest (feasible) disclosure on dimensions other than k.*

Definition 6. *The bounding disclosure is*

$$\bar{\pi}^* = \min\{\pi : u_i(\pi(k), \bar{x}(-k)) > u_i(x_i(k), \bar{x}(-k)) \text{ for all } x_i(k) > \pi(k), \text{ all dimensions } k, \text{ and all } i\}.$$

Claim 1 in the appendix contains an argument (using quasi-supermodularity) that guarantees that $\bar{\pi}^*$ is well defined and proves all players get a strictly higher payoff from $\bar{\pi}^*$ than anything larger and $\pi^* \leq \bar{\pi}^*$. When X is one dimensional, $\pi^* = \bar{\pi}^*$.

We can now state the main result of this section.

Proposition 1. *If x is a strategy profile that survives IDWDS, then $M(x) \in [\tilde{\pi}^*, \bar{\pi}^*]$. If x is a Nash equilibrium strategy profile that survives IDWDS, then $\tilde{u}_i(x) \geq u_i(\bar{\pi}^*)$ for all i.*

The proposition bounds the set of disclosures that survive iterative deletion of weakly dominates strategies. The lower bound is the lower bound of the set of Nash equilibria. The upper bound is typically lower than full disclosure. We provide examples to demonstrate that the bounds need not survive IDWDS, but we are able to describe conditions when the bounds are tight.

To prove the proposition, we first show that there is always a disclosure less than or equal to π^* that survives IDWDS. This observation follows because if $x_j \leq \pi^*$ for all $j \neq i$, then Expert i must have an undominated best reply to x_{-i} that is less than or equal to π^*. Next we show that disclosures strictly lower than $\tilde{\pi}^*$ must eventually be eliminated. This argument uses the definition of $\tilde{\pi}^*$ and in particular, the fact that for any $\tilde{\pi} \not\geq \tilde{\pi}^*$, there must exist an expert i such that $u_i(x_i) > u_i(\tilde{\pi})$. Finally, we show how to delete strategies than are not less or equal to $\bar{\pi}^*$. Proving this is more involved. The argument involves constructing a strategy that dominates the smallest remaining strategy that is not below $\bar{\pi}^*$. The appendix contains the details.

Sobel (2019, Proposition 3) proves this result when X is one dimensional.

Corollary 1. *If $\bar{\pi}^* = \tilde{\pi}^*$, then for all x that survives IDWDS, $M(x) = \pi^* = \bar{\pi}^* = \tilde{\pi}^*$.*

Corollary 1 follows directly from Proposition 1. It follows from the definition of $\bar{\pi}^*$ and $\bar{\pi}^* \geq \pi^* \geq \tilde{\pi}^*$ that $\bar{\pi}^* = \tilde{\pi}^*$ will hold provided that $u_i(\tilde{\pi}^*(k), \bar{x}(-k)) > u_i(x_i(k), \bar{x}(-k))$ for all $x_i(k) > \tilde{\pi}^*(k)$, all dimensions k, and all i.

Remark 1. *When $X_i \subset \mathbb{R}$, the disclosure that survives is unique if $u_i(\cdot)$ is one-to-one for all i (so that $\pi^* = \tilde{\pi}^*$). When $X_i \subset \mathbb{R}^n$ for $n \geq 2$, the disclosure that survives is unique if $u_i(\cdot)$ is one-to-one for all i and all players' preferences toward disclosures are separable across dimensions: $u_i(x(k), x(-k)) \geq u_i(x'(k), x(-k))$ implies $u_i(x(k), x'(-k)) \geq u_i(x'(k), x'(-k))$ and $u_i(x(k), x(-k)) > u_i(x'(k), x(-k))$ implies $u_i(x(k), x'(-k)) > u_i(x'(k), x'(-k))$ for all i.*

As the examples in the next section demonstrate, we cannot generally say more than that equilibrium disclosures must be in the interval $[\tilde{\pi}^*, \bar{\pi}^*]$. That is, the bounds are not attained in every game.

4.1 Examples

In this subsection we present examples that demonstrate that we cannot strengthen the conclusion of Proposition 1 and that, in general, the order of deleting strategies matter. In this case Proposition 1 identifies a unique payoff that survives IDWDS when payoff functions are generic. Hence the pathologies are all due to ties in payoff functions. We cannot guarantee that disclosure π^* or disclosure $\tilde{\pi}^*$ will survive IDWDS nor can we guarantee that all payoffs that survive IDWDS are greater than or equal to $u_i(\pi^*)$.

Example 1. *Consider the following game:*

	None	*Some*	*All*
None	$2, 0$	$1, 2$	$1, 1$
Some	$1, 2$	$1, 2$	$1, 1$
All	$1, 1$	$1, 1$	$1, 1$

We have $\pi^* = All$ and $\tilde{\pi}^* = Some$. Discarding the bottom two strategies of Expert 1 (the row player) initially leaves $(None, Some)$; discarding Expert 2's None and All, leaves $(any, Some)$ (so either Some or All is disclosed). You cannot delete the outcome $(None, Some)$. So the set of disclosures that survive IDWDS always includes $\tilde{\pi}^*$, but may or may not include π^*. Although Proposition 5 guarantees that π^* is a disclosure that survives IDWDS for some order of deletion, this example demonstrates that we cannot guarantee that π^* survives independent of the deletion order.

Example 2. *Consider the following game:*

	None	*Some*	*All*
None	$1, 1$	$-1, 0$	$1, 0$
Some	$-1, 0$	$-1, 0$	$1, 0$
All	$1, 0$	$1, 0$	$1, 0$

We have $\pi^* = All$ and $\tilde{\pi}^* = None$. Discarding Expert 2's Some and All, leads to $(None, None)$ and $(All, None)$. Discarding Expert 1's None and Some, leads to (All, Any). You cannot delete Expert 1's All strategy. Here the set of disclosures that survive IDWDS always includes π^*, but may or may not include $\tilde{\pi}^*$.

Taken together, the examples show that you need not select π^* or $\tilde{\pi}^*$. The examples are consistent with the observation that you will select one or the other and that there is a way of deleting weakly dominated strategies that will select both. The second claim is not true, however.

Example 3. *Consider the following game:*

	None	Some	All
None	$0,0$	$0,-1$	$-1,-1$
Some	$0,-1$	$0,-1$	$-1,-1$
All	$-1,-1$	$-1,-1$	$-1,-1$

We have $\pi^* = All$ and $\tilde{\pi}^* = None$. (*Some, None*) and (*None, None*) are equilibria that survive IDWDS, but it is weakly dominated to disclose *All*. Consequently $\tilde{\pi}^*$ is an equilibrium disclosure; π^* is not an equilibrium disclosure that survives IDWDS; and there is an equilibrium disclosure that survives IDWDS strictly between $\tilde{\pi}^*$ and π^*.

It is also possible to construct an example in which $\tilde{\pi}^*$ is not an equilibrium disclosure that survives IDWDS.

Example 4. *Consider the following game:*

	None	Some	All
None	$2,1$	$0,1$	$2,0$
Some	$0,1$	$0,1$	$2,0$
All	$2,0$	$2,0$	$2,0$

We have $\pi^* = All$ and $\tilde{\pi}^* = None$. The only equilibrium disclosure that survives IDWDS is π^*.

IDWDS need not bound the utility of the experts.

Example 5. *Consider the following game:*

	None	Some	All
None	$1,0$	$-1,0$	$0,0$
Some	$-1,0$	$-1,0$	$0,0$
All	$0,0$	$0,0$	$0,0$

In this example, $\pi^* = All$ and $\tilde{\pi}^* = None$. Expert 1's "Some" strategy is weakly dominated, but all other strategies survive. Consequently there exists an outcome that survives IDWDS in which Expert 1's payoff is less than π^*. This outcome, (*None, Some*), is not an equilibrium.

Now we turn to a situation in which the strategies are not completely ordered.

Example 6. *Consider the following game:*

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(0,0)$	$2,2$	$1,-1$	$-1,1$	$0,0$
$(1,0)$	$1,-1$	$1,-1$	$0,0$	$0,0$
$(0,1)$	$-1,1$	$0,0$	$-1,1$	$0,0$
$(1,1)$	$0,0$	$0,0$	$0,0$	$0,0$

The two players' preferences are: $\{0,0\} \succ_1 \{1,0\} \succ_1 \{1,1\} \succ_1 \{0,1\}$ and $\{0,0\} \succ_2 \{0,1\} \succ_2 \{1,1\} \succ_2 \{1,0\}$. There are 2 pure-strategy equilibrium joint disclosures: $(0,0), (1,1)$. It is dominated for Expert 1 to disclose 1 on the second dimension and for Expert 2 to disclose 1 on the first dimension. So discard $(1,1), (0,1)$ for Expert 1 and $(1,1), (1,0)$ for Expert 2. No other strategy can be deleted. Every disclosure survives IDWDS. In this example $\pi^* = \tilde{\pi}^* = (0,0)$, but $\bar{\pi}^* > \pi^*$ and there are disclosures that survive IDWDS that exceed π^*.

Example 6 illustrates how full disclosure may survive iterative deletion of weakly dominated strategies in the simultaneous game even when the strategy of full disclosure is weakly dominated, which cannot happen in the one-dimensional case. In the example, $(1,0)$ is an unattractive disclosure from the perspective of Expert 2. If this disclosure is a possibility, then Expert 2 will have a justification for making a disclosure in the second dimension. Similarly, Expert 1 would not delete her strategy $(1,0)$ because she prefers that the decision maker have full information than information only about the second dimension. Although no disclosure remains an equilibrium that survives IDWDS, the full disclosure outcome is more robust than it is in the one-dimensional case.

5 Sequential Disclosure

Full disclosure does not always survive iterative deletion of weakly dominated strategies. This leaves open the question of whether the decision maker could do better by consulting the experts in a different way.

What will be generated by an alternative arrangement where each expert sequentially discloses information and observes prior disclosures when making their choice? This section gives conditions under which the outcomes in these two environments are essentially the same.

5.1 Preliminaries

Let $H_0 = \emptyset$, $H_t = X^t$, where $X \subset \mathbb{R}^n$ is the (finite) set of possible disclosures.[6] Let $H = \bigcup_{t=0}^{\infty} H_t$ be the set of histories. If $h_t = (h_t^1, \ldots, h_t^t) \in H_t$ and $h_{t'} = (h_{t'}^1, \ldots, h_{t'}^{t'}) \in H_{t'}$, then $h_t h_{t'} \in H_{t+t'}$ is the history obtained by the natural concatenation: $h_t h_{t'} = (h^1, \ldots, h^t, h^{t+1}, \ldots, h^{t+t'})$ where

$$h^m = \begin{cases} h_t^m & \text{if } 1 \leq m \leq t \\ h_{t'}^{m-t} & \text{if } t < m \leq t + t'. \end{cases}$$

Definition 7. *A finite sequential disclosure protocol is a mapping $P : H \to \{0, 1, \ldots, N\}$ such that for all $h, h_t \in H$,*

$$P(h_t) = 0 \implies P(h_t h) = 0 \tag{1}$$

and there exists T such that $P(h_T) = 0$ for all $h_T \in H_T$.

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[6]Histories should also include the identity of the expert who reveals, but not just what is revealed. The extra generality does not affect the analysis so we use a simpler definition of history.

The decision maker observes a history of disclosures, h_t. He then decides whether to stop the process ($P(h_t) = 0$) or to ask Expert i to disclose ($P(h_t) = i$). Condition (1) means that once the decision maker stops the process, he cannot restart it. We limit attention (without loss of generality) to protocols that end after a finite number of periods.

A sequential disclosure protocol induces a game in which the players are the experts. Player i's strategy specifies a disclosure as a function of h_t for each h_t such that $P(h_t) = i$. Given a history of length t, $h_t = (h_t^1, \ldots, h_t^t)$, let $\mu(h_t) = \max\{h_t^1, \ldots, h_t^t\}$, where the maximum is taken component wise. A strategy profile $s = (s_1, \ldots, s_n)$ determines disclosures $d_t^*(s)$ and histories $h_t^*(s)$ for $t = 1, \ldots, T$ where $h_1^*(s) = d_1^*(s) = s_{P(\emptyset)}(\emptyset)$, $d_2^*(s) = s_{P(h_1^*(s))}(h_1^*(s))$, $h_2^*(s) = h_1^*(s)d_2^*(s)$, and, in general, $d_k^*(s) = s_{P(h_{k-1}^*(s))}(h_{k-1}^*(s))$, $h_k^*(s) = h_{k-1}^*(s)d_k^*(s)$ with the convention that $s_0(h) = \mu(h)$. Expert i's payoff as a function of the strategy profile is $\tilde{u}_i(s) = u_i(\mu(h_T^*(s)))$. We say that a disclosure π is generated by a sequential disclosure protocol if the induced game has a strategy profile that survives IDWDS in which π is disclosed. A disclosure π is uniquely generated by a sequential disclosure protocol if π is the only disclosure generated by the protocol.

5.2 Example

The definition of sequential protocol permits the decision maker to do three things: vary the order in which he consults experts; return to experts more than once; and commit to ending the consultation process. The following example demonstrates why these three features are important and gives some insight into the general construction.

Example 7. *There are five information structures, $1, 2, 3, 4, 5$ with higher numbers representing more information, and two experts. Expert 1 has strict preferences: $2 \succ 4 \succ 1 \succ 5 \succ 3$. Expert 2 has strict preferences: $1 \succ 3 \succ 2 \succ 5 \succ 4$. Here $\bar{\pi}^* = \pi^* = \tilde{\pi}^* = 5$. So the unique outcome that survives IDWDS in the simultaneous game is full disclosure. There are four possible disclosure sequences without returning to an expert: consulting exactly one expert, or consulting both in either order. Without commitment, the possible disclosures are:*

Sequence	Disclosure
Expert 1	2
Expert 2	1
Expert 1, then 2	1
Expert 2, then 1	2

It is straightforward to confirm that returning to experts will not lead to more disclosure. Hence, without commitment, sequential disclosure need not lead to disclosure π^*.

If the decision maker can commit to stopping, but can only consult each expert (at most) once, then commitment ability cannot improve upon the protocols in which only one expert is consulted (in these cases, the decision maker commits to stop after the first disclosure independent of the disclosure itself). If he starts with Expert 1 and sometimes asks Expert 2, then more disclosure is possible with commitment. If the decision maker

stops after 1, 2, *or* 3, *Expert 1 will disclose* 2, *which will be the final disclosure. If the decision maker stops after* 4 *or* 5, *the disclosure will be* 4. *This means that if the decision maker consults Expert 1 first, then he would do best by committing not to consult Expert 2 if Expert 1 discloses at least* 4. *Alternatively, the decision maker can consult Expert 2 first. The disclosure generated will be* 1 *if the decision maker stops after disclosure* 1; 2 *if the decision maker stops after* 2, 4 *or* 5, *and* 3 *if the decision maker stops at* 3. *This means that the decision maker can obtain disclosure* 3 *by consulting first Expert 2, then Expert 1 (with commitment). We conclude that the best the decision maker can do with commitment but without returning to experts is disclosure* 4. *Hence commitment increases the disclosure, but does not generate full disclosure.*

Proposition 2 guarantees that there is a sequential disclosure protocol that generates disclosure $\tilde{\pi}^*(= \pi^* = \bar{\pi}^*)$. The protocol involves asking Expert 1, then Expert 2, and then going back to Expert 1, with the commitment to stop if Expert 2 discloses 3.

Adding an additional expert is quite beneficial to the decision maker in the example. Full disclosure is possible even though it is ranked fourth out of five disclosures for both experts.

The example illustrates the importance of the three features of sequential protocols. The order of consultation matters. Consulting Expert 2 is better for the decision maker than consulting Expert 1 first because Expert 1 is certain to disclose more than 1 if she moves last. Commitment is important because it motivates an expert to make a partial disclosure without fear that a future expert will disclose more. Finally, returning to an expert may be valuable because an expert's favorite disclosure depends on what has already been disclosed. In the example, full disclosure is unattractive to Expert 2 ex ante, but once 4 has been disclosed, it is her favorite option. What complicates the construction is the possibility that an expert will want to disclose more in response to a partial disclosure from another expert. Well designed sequential protocols permit the decision maker to leverage conflicts of interest between experts to obtain greater disclosure.

5.3   Bounds on Disclosures from Sequential Consultation

This subsection describes what is possible for the decision maker using general sequential protocols.

Imagine that the decision maker commits to consulting only Expert i. Expert i would disclose an x that maximizes $u_i(x)$. Provided that $\underline{x} < \tilde{\pi}^*$, this disclosure is greater than $\underline{x}$ for at least one i. Consequently, the decision maker can guarantee a positive disclosure. Call this disclosure π^1. Now imagine the decision maker reaches a stage in which $x < \tilde{\pi}^*$ has been disclosed. By the definition of $\tilde{\pi}^*$ this means that some expert can gain strictly by disclosing (if she were guaranteed that the decision maker would consult no further experts). Hence if the decision maker can guarantee the disclosure x, then by picking the correct expert to consult next, he would be able to guarantee further disclosure.

We construct a protocol that builds on this insight. We define a finite, increasing sequence of disclosures that satisfies

$$\underline{x} = \pi^0 < \pi^1 < \cdots < \pi^{\bar{m}} = \tilde{\pi}^* \tag{2}$$

such that the decision maker can guarantee a disclosure of at least π^m by consulting m times. The basic idea is that as long as $\pi < \tilde{\pi}^*$, the decision maker can leverage the ability to guarantee disclosure π to induce an expert who prefers to disclose more than π to make a further disclosure. We observe that this task would be straightforward if experts had single-peaked preferences. In that case, the decision maker need only consult a single expert (the one with the greatest peak).

Describing the protocol requires introducing some notation.

Let $\bar{v}_i(x) = \min \arg \max\{u_i(v) : v \geq x\}$. The quantity $\bar{v}_i(x)$ is the smallest disclosure that maximizes Expert i's utility given that x has already been disclosed. (If there is more than one disclosure that maximizes Sender i's utility, then $\bar{v}_i(x)$ is the smallest one.) Let $\pi^0 = \underline{x}$; and for $m > 0$,

$$\underline{i}(m) = \min\{i : \bar{v}_i(\pi^{m-1}) = \max_j\{\bar{v}_j(\pi^{m-1})\}\}$$

and $\pi^m = \bar{v}_{\underline{i}(m)}(\pi^{m-1})$. Given that $\bar{v}_i(\cdot)$ is well defined, so are $\underline{i}(m)$ and π^m for all m. $\underline{i}(m)$ is the expert whose smallest maximizing disclosure is the greatest given that π^{m-1} has already been disclosed. In case of ties we select the expert with the smallest index among those with the largest $\bar{v}_j(\pi^{m-1})$.

Note that $\underline{i}(m) \neq \underline{i}(m-1)$, but it is possible to have $\underline{i}(m) = \underline{i}(m')$ for $m > m'+1$. That is, this disclosure mechanism may require asking the same expert to disclose more than once. Finally we define π^m to be the smallest disclosure of Expert $\underline{i}(m)$ that maximizes her payoff given that π^{m-1} has already been disclosed.

It follows from the definition of $\tilde{\pi}^*$ that if $x \not\geq \tilde{\pi}^*$, then $\bar{v}_{\underline{i}}(x) > x$ for all x. Hence, there is a finite $\bar{m}$ such that $\pi^m \not\geq \tilde{\pi}^*$ for $m < \bar{m}$ and $\pi^{\bar{m}} \geq \tilde{\pi}^*$. The next result refines this observation.

Lemma 2. $\pi^m < \tilde{\pi}^*$ for $m < \bar{m}$ and $\pi^{\bar{m}} = \tilde{\pi}^*$.

Lemma 2 implies that (2) holds.

We now can define the disclosure protocol.

Definition 8 (Canonical Disclosure Protocol). *In the Canonical Disclosure Protocol, for any $h_t \in H_t$,*

$$P(h_t) = \begin{cases} 0 & \text{if } \mu(h_t) \geq \pi^{\bar{m}-t} \\ \underline{i}(\bar{m} - t) & \text{otherwise.} \end{cases}$$

In the canonical disclosure protocol the decision maker first asks Expert $\bar{m}$; if this expert discloses at least $\pi^{\bar{m}}$, then the decision maker stops. Otherwise, he moves to Expert $\bar{m} - 1$. The process continues. If the disclosure after the rth report is at least $\pi(\bar{m} - r)$, then the decision maker stops (and requests no further reports). Otherwise, the decision maker requests a report from Expert $\underline{i}(\bar{m} - r)$.

Proposition 2. *Every disclosure that survives IDWDS in the game determined by the canonical disclosure protocol is at least $\tilde{\pi}^*$.*

Proposition 2 provides a lower bound to what the decision maker can obtain from a sequential disclosure protocol.

Lemma 3. *If $\tilde{\pi}^* \leq \pi$ and $\pi \not\leq \pi^*$, then $u_i(\pi) < u_i(\tilde{\pi}^*)$.*

The next result provides an upper bound for this sequential mechanism.

Proposition 3. *If π is generated by the canonical protocol, then $\pi \leq \pi^*$.*

Corollary 2. *If $\tilde{\pi}^* = \pi^*$, then there exists a sequential disclosure protocol that uniquely generates the disclosure π^*.*

In particular, if the experts all have strict preferences, then the canonical sequence uniquely generates the disclosure π^*. Corollary 2 is an immediate consequence of Propositions 2 and 3.

We next consider situations in which $\tilde{\pi}^* < \pi^*$. Proposition 2 guarantees that at least $\tilde{\pi}^*$ is disclosed. There may be no protocol that guarantees disclosure π^*. A simple example demonstrates this fact. Assume there is a single expert and two disclosures (high and low), and the expert is indifferent between the two. Every protocol will have an outcome with high disclosure and an outcome with low disclosure.

5.4   Comparative Statics

In this section we make a few observations about the value of adding experts.

Adding an expert cannot harm the decision maker in the sense that if π is a disclosure for the original players that survives IDWDS in the simultaneous game a disclosure of at least π will survive if additional players are added and a new player need not be consulted in a sequential protocol. Li and Norman (2018b) show that adding an expert may hurt the decision maker if the expert must be inserted in a particular place.

In situations in which our bounds on disclosure are tight ($\bar{\pi}^* = \tilde{\pi}^* = \pi^*$) adding an additional expert can only be beneficial if doing so increases one of these quantities. An expert that does not increase one of these quantities is **redundant**. It is clear that if preferences are single-peaked, all experts except the one with the greatest peak is redundant. More generally, if there are a pair of experts i and j such that $\bar{v}_i(x) \geq \bar{v}_j(x)$ for all x, then Expert j is redundant.

6   Comparing Disclosures in Simultaneous and Sequential Games

We have described the set of disclosures that can survive IDWDS in the simultaneous games and can be outcomes of a sequential disclosure protocol. When $\tilde{\pi}^* = \pi^* = \bar{\pi}^*$, we can assert that the decision maker does equally well with either form of consultation. This means that for generic games with one-dimensional strategy spaces there is no obvious reason to favor sequential versus simultaneous disclosure. In general, the different forms of consultation may perform differently. Sometimes simultaneous disclosure games enable larger disclosures than sequential games, sometimes sequential disclosure will be superior from the perspective of the decision maker.

When $\pi^* < \bar{\pi}^*$, the simultaneous game may admit equilibria surviving IDWDS that disclose strictly more that π^* (Example 6). That is, it is possible for simultaneous disclosure to strictly exceed the upper bound for disclosure in any sequential protocol.

When $\tilde{\pi}^* < \pi^* = \bar{\pi}^*$, the following results demonstrate that if there is an expert who is indifferent between $\tilde{\pi}^*$ and π^*, then π^* will survive in some equilibrium that survives IDWDS.

Proposition 4. *If there is an expert who is indifferent between $\tilde{\pi}^*$ and π^*, there is a sequential disclosure protocol that generates the disclosure π^*.*

If there is an expert who is indifferent between $\tilde{\pi}^*$ and π^*, Example 1 demonstrates that π^* need not survive all procedures of IDWDS in the simultaneous game. One can check that in this game there is no sequential protocol that guarantees that π^* is the unique equilibrium disclosure that survives IDWDS.

Proposition 5. *If there is an expert who is indifferent between $\tilde{\pi}^*$ and π^*, there is a procedure of IDWDS in the simultaneous game so that the disclosure π^* survives.*

By Proposition 4, there is a sequential disclosure protocol that generates π^* if an expert is indifferent between $\tilde{\pi}^*$ and π^*. By Proposition 5, π^* also survives IDWDS in the simultaneous game under this condition. If all experts, however, strictly prefer $\tilde{\pi}^*$ to π^*, π^* may not survive IDWDS in either the simultaneous or the sequential game.

Even when there is an expert who is indifferent between $\tilde{\pi}^*$ and π^*, sequential and simultaneous consultation still need not be equivalent, because π^* may be the unique disclosure that survives IDWDS in the simultaneous game, but not the sequential game or in the sequential game, but not simultaneous game.

Finally, if there is no expert who is indifferent between $\tilde{\pi}^*$ and π^*, the next example illustrates that it is possible to construct a sequential protocol that has equilibrium disclosure π^*, but that the disclosures surviving IDWDS are strictly smaller. This means that there are situations in which a sequential protocol may be more attractive than simultaneous disclosure.

Example 8. *Consider the following game:*

	None	Some	More	All
None	3, 2	0, 2	2, 1	1, 1
Some	0, 2	0, 2	2, 1	1, 1
More	2, 1	2, 1	2, 1	1, 1
All	1, 1	1, 1	1, 1	1, 1

It is straightforward to confirm that $\tilde{\pi}^* = None$ and $\pi^* = All$ and that any order of deletion of weakly dominated strategies leaves $\{None, More\}$ for Expert 1 and $\{None, Some\}$ for Expert 2. Consequently, the disclosure in the simultaneous game is always less than All. In contrast, when Expert 1 discloses first and Expert 2 follows, there is an equilibrium in undominated strategies in which Expert 1 discloses More and Expert 2 discloses All when Expert 1 discloses More, Some when Expert 1 discloses None, and otherwise

does not disclose more than Expert 1. Expert 2's strategy is not weakly dominated because it plays a best response at each information set. Expert 1's only best replies to Expert 2's strategies involve disclosing More or All, so More cannot be weakly dominated. Hence there is a sequential protocol that generates a greater disclosure than what is possible in the simultaneous game.

7   Discussion

One of our goals in this study was to identify features that favored simultaneous versus sequential disclosure. Our results suggest that the choice of organization does not matter, at least in a leading case. Nevertheless, even in our setting, there are differences between the procedures. The sequential protocol that we identify requires that the decision maker know a lot about the preferences of the experts. The fine details of the optimal protocol depend on these preferences. In contrast, the simultaneous protocol uses the same game to induce disclosure independent of the preferences of the experts. This observation makes the simultaneous protocol more appealing. Both organizations require that the experts know a lot about the preferences of other experts. It is not clear whether uncertainty about preferences would have a differential impact on performance. In one way, sequential disclosure may be superior to simultaneous disclosure. Simultaneous disclosure requires that all experts (or, at least, an essential subset of experts) participate. Under the optimal sequential protocol, however, it is an equilibrium for the first expert to make the equilibrium disclosure (under the threat that, if not, the decision maker will consult further experts). Consequently, if it is costly to consult experts, a sequential protocol might be more attractive than a simultaneous one.

There appear to be natural settings in which simultaneous disclosure is preferred and other settings in which sequential procedures are the norm. On one hand, editors typically consult several reviewers simultaneously to obtain reports on a submission. Committee deliberations, anonymous voting, and obtaining multiple bids for a construction contract have features of simultaneous disclosure. On the other hand, it is frequent to consult medical experts in sequence. A full understanding of the relative merits of simultaneous and sequential procedures therefore requires modifying our model. Two directions seem promising. Certainly adding costs (whether direct payment or waiting times) to consultation will change the analysis, presumably in the direction of favoring sequential procedures. Somewhat less obvious, in our model experts cannot learn from each other. It would be interesting to study a variation of the model in which experts had access to different facts or where the disclosure of one expert influences the set of disclosures available to subsequent agents.

We have studied two ways to organize consultations. It is natural to ask whether there is an organization that is superior to either the simultaneous or sequential procedures that we examined. We lack a complete answer to this question, which differs from standard implementation questions in two ways. First, we must study something more general than implementation in Nash equilibrium (because otherwise it is straightforward to obtain full disclosure). Second, in the standard implementation problem, the decision maker's set of action is fixed. We assume that the what the decision maker can do depends on

the actual disclosures of the expert (the decision maker cannot take an action unless an expert identifies it). Although we lack a general formulation, given that all agents disclose no more than π^*, there is never an incentive for a particular agent to disclose more than π^*, at least in the one-dimensional case. This observation suggests that any procedure that eliminates weakly dominated strategies cannot induce a disclosure greater than π^*.

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Appendix

Lemma 1. *If π' and π'' are equilibrium disclosures, then $\pi' \wedge \pi''$ is an equilibrium disclosure.*

Proof. If π' and π'' are equilibrium disclosures, then $u_i(\pi'') \geq u_i((\pi' \wedge \hat{\pi}) \vee \pi'')$ and $u_i(\pi') \geq u_i(\pi' \vee \hat{\pi})$ for any $\hat{\pi}$. It follows from quasi-supermodularity that

$$u_i((\pi' \wedge \hat{\pi}) \wedge \pi'') \geq u_i(\pi' \wedge \hat{\pi}) \tag{3}$$

and

$$u_i(\pi' \wedge \hat{\pi}) \geq u_i(\hat{\pi}). \tag{4}$$

Now assume that $\hat{\pi} > \pi' \wedge \pi''$. Consequently, $(\pi' \wedge \hat{\pi}) \wedge \pi'' = \pi' \wedge \pi''$ so that inequality (3) implies

$$u_i(\pi' \wedge \pi'') \geq u_i(\pi' \wedge \hat{\pi}). \tag{5}$$

It follows from (4) and (5) that $u_i(\pi' \wedge \pi'') \geq u_i(\hat{\pi})$ so that $\pi' \wedge \pi''$ is an equilibrium disclosure. $\qquad\square$

Claim 1. *Let π be a disclosure that satisfies $u_i(\pi(k), \bar{x}(-k)) > u_i(x_i(k), \bar{x}(-k))$ for all $x_i(k) > \pi(k)$, all dimensions k, and all i. Then π is an equilibrium disclosure such that $u_i(\pi) > u_i(x_i)$ for all $x_i > \pi$ and all i.*

Proof. Fix a disclosure $\pi' > \pi$ and an arbitrary expert i. Then $\pi'(k) \geq \pi(k)$ on all dimensions and the inequality is strict on at least one dimension. We will show that $u_i(\pi) > u_i(\pi')$. Without loss of generality, suppose that $\pi'(1) > \pi(1)$, where $\pi(1)$ is the disclosure of π on dimension 1 and $\pi(-1)$ is the disclosure of π on dimensions other than 1. We will show by induction on r that

$$u_i(\pi) > u_i(\pi'(1), \ldots, \pi'(r), \pi(r+1), \ldots, \pi(n)) \tag{6}$$

for $r = 1, \ldots, n$. The claim follows from inequality (6) for $r = n$.

Because $u_i(\pi(1), \bar{x}(-1)) > u_i(\pi'(1), \bar{x}(-1))$, it must be that $u_i(\pi) > u_i(\pi'(1), \pi(-1))$ by quasi-supermodularity. Hence inequality (6) holds for $r = 1$. Assume inequality (6) for $r \leq k$. It suffices to show that it holds for $r = k + 1$.

Because $u_i(\pi(k+1), \bar{x}(-(k+1))) \geq u_i(\pi'(k+1), \bar{x}(-(k+1)))$,

$$u_i(\bar{x}(1), \ldots, \bar{x}(k), \pi(k+1), \ldots, \pi(n)) \geq u_i(\bar{x}(1), \ldots, \bar{x}(k), \pi'(k+1), \pi(k+2), \ldots, \pi(n)) \tag{7}$$

by quasi-supermodularity. Inequality (7) and quasi-supermodularity imply that

$$u_i(\pi'(1), \ldots, \pi'(k), \pi(k+1), \ldots, \pi(n)) \geq u_i(\pi'(1), \ldots, \pi'(k+1), \pi(k+2), \ldots, \pi(n)). \tag{8}$$

It follows from inequality (6) for $r = k$ and inequality (8) that inequality (6) holds for $r = k + 1$, which completes the proof. $\qquad\square$

Proposition 1. *If x is a strategy profile that survives IDWDS, then $M(x) \in [\tilde{\pi}^*, \bar{\pi}^*]$. If x is a Nash equilibrium strategy profile that survives IDWDS, then $\tilde{u}_i(x) \geq u_i(\bar{\pi}^*)$ for all i.*

Denote the set of strategies that survive IDWDS by S. We prove the proposition in a series of steps, which we state and prove as claims.

Claim 2. *For all $x \in S$ and every i, there exists $x_i \in S_i$ that is a best response to x relative to X.*

Proof. The result is clear if the best response to x has not yet been deleted. If the best response to x has been deleted, then it was deleted by a strategy that weakly dominates it. This strategy must be a best reply to x. $\square$

Claim 3. *There exists a strategy profile $x \in S$ such that $M(x_1, \ldots, x_I) \leq \pi^*$.*

Proof. Suppose that after r iterations, there exists a strategy profile x satisfying the condition in the claim. In the next iteration, every agent must have a strategy that is a best response to x by Claim 2. Suppose that the best response for Expert i to x is some $y_i \not\leq \pi^*$. We will argue to a contraction. Because $M(y_i, x_{-i}) = y_i \vee M(x_{-i}) = M(y_i \vee M(x_{-i}), x_{-i})$,

$$\tilde{u}_i(y_i, x_{-i}) = u_i(M(y_i, x_{-i})) = u_i(M(y_i \vee M(x_{-i}), x_{-i})) = \tilde{u}_i(y_i \vee M(x_{-i}), x_{-i}). \quad (9)$$

It follows from (9) that $y_i' \equiv y_i \vee M(x_{-i})$ is also a best response to x. Note that because $y_i' > M(x_{-i})$,

$$M(y_i', x_{-i}) = y_i'. \quad (10)$$

In addition, note that

$$M(y_i' \wedge \pi^*, x_{-i}) = (y_i' \wedge \pi^*) \vee M(x_{-i}) = y_i' \wedge \pi^*, \quad (11)$$

where second equality holds because $M(x_{-i}) < y_i'$ and $M(x_{-i}) \leq \pi^*$ imply that $M(x_{-i}) \leq y_i' \wedge \pi^*$.

By definition of π^*, $u_i(\pi^*) > u_i(x_i)$ for all $x_i > \pi^*$. Because $y_i \not\leq \pi^*$ and $y_i' > y_i$, $y_i' \not\leq \pi^*$ and therefore $y_i' \vee \pi^* > \pi^*$. It follows that

$$u_i(\pi^*) > u_i(y_i' \vee \pi^*). \quad (12)$$

Hence it must be that

$$\tilde{u}_i(y_i' \wedge \pi^*, x_{-i}) = u_i(M(y_i' \wedge \pi^*, x_{-i})) = u_i(y_i' \wedge \pi^*) > u_i(y_i') = u_i(M(y_i', x_{-i})) = \tilde{u}_i(y_i', x_{-i}) \quad (13)$$

where the second equation follows from (11), the inequality follows by (12) and quasi-supermodularity, and the third equation follows from (10). Inequalities (9) and (13) combine to show that y_i is not a best response to x_{-i}, which is a contradiction. We conclude that no strategy $y_i' \not\leq \pi^*$ can do at least as well as $y_i' \wedge \pi^*$ against x. Therefore, all best responses to x are less than or equal to π^* and a strategy less than or equal to π^* must remain. $\square$

Then we show that all disclosures that survive IDWDS are greater than or equal to the smallest equilibrium disclosure.

Claim 4. *If $x \in S$, then $M(x) \geq \tilde{\pi}^*$.*

Proof. Let $\tilde{\pi} \equiv \wedge_{x \in S} M(x)$ be the meet of all disclosures that survive IDWDS. We wish to show that $\tilde{\pi} \geq \tilde{\pi}^*$. In order to reach a contradiction, assume that $\tilde{\pi} \not\geq \tilde{\pi}^*$. By the definition of $\tilde{\pi}^*$, it must be the case that for some i,

$$\text{there exists } x_i \text{ with } x_i > \tilde{\pi} \text{ such that } u_i(x_i) > u_i(\tilde{\pi}), \tag{14}$$

because otherwise $\tilde{\pi}^*$ would not be the smallest equilibrium disclosure. Let

$$y_i = \min\{\arg\max_{x_i \geq \tilde{\pi}} u_i(x_i)\}$$

be the smallest best response of Player i to disclosure $\tilde{\pi}$, which is well defined by Fact 1. It is apparent that $y_i > \tilde{\pi}$ and so $y_i(k) > \tilde{\pi}(k)$ for some k. Select $\tilde{x} \in S$ such that $M(\tilde{x})(k) = \tilde{\pi}(k)$. This is possible by the definition of $\tilde{\pi}$. We assert that $y_i \vee \tilde{x}_i$ weakly dominates $\tilde{x}_i$. The assertion is sufficient to prove the claim because it means that if (14) holds, then we can show an element in S is weakly dominated. This contradicts the definition of S. It remains to show that $y_i \vee \tilde{x}_i$ weakly dominates $\tilde{x}_i$. For any x_{-i} such that $M(x_{-i}) \geq y_i \vee \tilde{x}_i$,

$$\tilde{u}_i(y_i \vee \tilde{x}_i, x_{-i}) = u_i(M(x_{-i})) = \tilde{u}_i(\tilde{x}_i, x_{-i}).$$

For any x_{-i} such that $M(x_{-i}) \not\geq y_i \vee \tilde{x}_i$, i's utility from playing $y_i \vee \tilde{x}_i$ and $\tilde{x}_i$ are $u_i((y_i \vee \tilde{x}_i) \vee M(x_{-i})) = u_i(y_i \vee (\tilde{x}_i \vee M(x_{-i})))$ and $u_i(\tilde{x}_i \vee M(x_{-i}))$, respectively. We now claim that

$$\tilde{u}_i(y_i \vee \tilde{x}_i, x_{-i}) = u_i(y_i \vee (\tilde{x}_i \vee M(x_{-i}))) \geq u_i(\tilde{x}_i \vee M(x_{-i})) = \tilde{u}_i(\tilde{x}_i, x_{-i}). \tag{15}$$

Hence $y_i \vee \tilde{x}_i$ is weakly better than $\tilde{x}_i$. The equations in (15) follow from the definition of $\tilde{u}_i$. The inequality holds by quasi-supermodularity because

$$u_i(y_i) \geq u_i(y_i \wedge (\tilde{x}_i \vee M(x_{-i}))). \tag{16}$$

Note that $\tilde{x}_i \vee M(x_{-i}) \geq \tilde{\pi}$ by the definition of $\tilde{\pi}$ and $y_i > \tilde{\pi}$, imply that $y_i \wedge (\tilde{x}_i \vee M(x_{-i})) \geq \tilde{\pi}$. Inequality (16) now follows from the definition of y_i.

It follows from (14) and the definition of y_i that $y_i > \tilde{\pi}$. Also $y_i > y_i \wedge M(\tilde{x}) \geq \tilde{\pi}$, $y_i \wedge M(\tilde{x}) = y_i$ and so the definition of y_i also implies that $u_i(y_i) > u_i(y_i \wedge M(\tilde{x}))$. Consequently, $u_i(y_i \vee M(\tilde{x})) > u_i(M(\tilde{x}))$ by quasi-supermodularity. It follows that

$$\tilde{u}_i(y_i \vee \tilde{x}_i, \tilde{x}_{-i}) = u_i(y_i \vee M(\tilde{x})) > u_i(M(\tilde{x})) = \tilde{u}_i(\tilde{x}) = \tilde{u}_i(\tilde{x}_i, \tilde{x}_{-i}). \tag{17}$$

Inequality (17) guarantees that $y_i \vee \tilde{x}_i$ is strictly better than $\tilde{x}_i$ when $x_{-i} = \tilde{x}_{-i}$. Establishing this completes the proof that $y_i \vee \tilde{x}_i$ weakly dominates $\tilde{x}_i$. $\square$

All disclosures that survive IDWDS are greater than or equal to $\tilde{\pi}^*$. Next, we show that all disclosures that survive IDWDS are less than or equal to the bounding disclosure.

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Claim 5. *If $x \in S$, then $M(x) \leq \bar{\pi}^*$.*

Proof. Let S_i^r be the set of strategies remaining for i after r rounds of deleting strategies. For each i let $P_i^r = \{s_i^r \in S_i^r : s_i^r \not\leq \bar{\pi}^*\}$. If there exists r such that $\bigcup_i P_i^r = \emptyset$, then the proof is complete. Otherwise, there is at least one dimension k such that $s_i^r(k) > \bar{\pi}^*(k)$ for some $s_i^r \in P_i^r$ and some i. Let $Q^r(k) = \bigcup_i \{s_i^r \in P_i^r : s_i^r(k) > \bar{\pi}^*(k)\}$. Let $z^r \in \arg\min\{s^r(k) : s^r \in Q^r(k)\}$. If $\bigcup_i P_i^r \neq \emptyset$, then z^r exists and $z^r \in P_j^r$ for some j; we write $z^r = z_j^r$ to indicate that $z_j^r \in P_j^r$. We claim that z_j^r is weakly dominated by $(\bar{\pi}^*(k), z_j^r(-k))$.[7]

For any x such that $M(x_{-j}(k)) \geq z_j^r(k)$,

$$\tilde{u}_j((\bar{\pi}^*(k), z_j^r(-k)), x_{-j}) = u_j(M(x_{-j}(k)), M(z_j^r(-k), x_{-j}(-k))) = \tilde{u}_j(z_j^r, x_{-j}).$$

This follows because $z_j^r(k) > \bar{\pi}^*(k)$.

For any x such that $M(x_{-j}(k)) < z_j^r(k)$, $M(x_{-j}(k)) \leq \bar{\pi}^*(k)$ by the definition of z_j^r. Hence, Expert j's utility from using z_j^r is $u_j(z_j^r(k), M(z_j^r(-k), x_{-j}(-k)))$, while j's utility from using $(\bar{\pi}^*(k), z_j^r(-k))$ is $u_j(\bar{\pi}^*(k), M(z_j^r(-k), x_{-j}(-k)))$. We claim that

$$u_j(\bar{\pi}^*(k), M(z_j^r(-k), x_{-j}(-k))) > u_j(z_j^r(k), M(z_j^r(-k), x_{-j}(-k))). \tag{18}$$

It follows from the definition of $\bar{\pi}^*$ that $u_j(\bar{\pi}^*(k), \bar{x}(-k)) > u_j(z_j^r(k), \bar{x}(-k))$, therefore Inequality (18) follows from quasi-supermodularity. That is, j does strictly better using $(\bar{\pi}^*(k), z_j^l(-k))$ than z_j^r whenever the kth dimension of $M(x_{-j})$ is less than $z_j^r(k)$. Because there always exists a strategy in which $M(x_{-j}(k))$ is less than $z_j^r(k)$ by Claim 3 and $\pi^* \leq \bar{\pi}^*$, $(\bar{\pi}^*(k), z_j^r(-k))$ must be strictly better than z_j^r against one strategy profile that survives IDWDS. Consequently, $(\bar{\pi}^*(k), z_j^r(-k))$ weakly dominates z_j^r. Therefore, z_j^r must eventually be deleted. We conclude that there must exist a r^* such that $P_i^{r^*} = \emptyset$ for all i, which establishes the result. $\square$

Proof of Proposition 1. Claim 4 guarantees that all $x \in S$ satisfy $M(x) \geq \tilde{\pi}^*$. Claim 5 guarantees that all $x \in S$ satisfy $M(x) \leq \bar{\pi}^*$. This establishes the first part of the Proposition. Given any $x \in S$, it follows from Claim 2 that each player has a surviving strategy that is a best response to x_{-i} relative to the full strategy set. Because $x_i = \bar{\pi}^*$ leads to payoff $u_i(\bar{\pi}^*)$ for Expert i against any surviving strategy by Claim 5, the second part of the proposition follows. $\square$

Lemma 2. $\pi^m < \tilde{\pi}^*$ *for $m < \bar{m}$ and $\pi^{\bar{m}} = \tilde{\pi}^*$.*

Proof. It is sufficient to show that $\pi^m \leq \tilde{\pi}^*$ for all m. Suppose that $\pi^m \not\leq \tilde{\pi}^*$ for some m. Let $\pi^{m'}$ be the disclosure such that $\pi^{m'} \not\leq \tilde{\pi}^*$ and $\pi^{m'-1} \leq \tilde{\pi}^*$. Because $\pi^0 = \underline{x} \leq \tilde{\pi}^*$, m' is well defined. The fact that $\pi^{m'} \not\leq \tilde{\pi}^*$, implies that $\pi^{m'} \wedge \tilde{\pi}^* < \pi^{m'}$. By quasi-supermodularity of $\underline{i}(m')$'s utility function, $u_{\underline{i}(m')}(\tilde{\pi}^*) \geq u_{\underline{i}(m')}(\pi^{m'} \vee \tilde{\pi}^*)$ implies that $u_{\underline{i}(m')}(\pi^{m'} \wedge \tilde{\pi}^*) \geq u_{\underline{i}(m')}(\pi^{m'})$. Further, note that $\pi^{m'-1} \leq \tilde{\pi}^*$ implies that $\pi^{m'-1} \leq \pi^{m'} \wedge \tilde{\pi}^*$. Hence $\pi^{m'}$ cannot be the smallest disclosure of Expert $\underline{i}(m')$ that maximizes her payoff given that $\pi^{m'-1}$ has already been disclosed. This contradicts the definition of $\pi^{m'}$, which means that $\pi^m \leq \tilde{\pi}^*$ for $m < \bar{m}$ and $\pi^{\bar{m}} = \tilde{\pi}^*$. $\square$

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[7]If $X \subset \mathbb{R}$, $(\bar{\pi}^*(k), z_j^r(-k)) = \bar{\pi}^* = \pi^*$.

Proposition 2. *Every disclosure that survives IDWDS in the game determined by the canonical disclosure protocol is at least $\tilde{\pi}^*$.*

Proof. Consider a disclosure $\hat{\pi} \geq \pi^{m-1}$ such that $\hat{\pi} \not\geq \pi^m$ and suppose that there exists a strategy profile $\hat{x}$ such that $\mu(h_{\bar{m}}(\hat{x})) = \hat{\pi}$. We will show that there exists an expert i such that $\hat{x}_i$ is weakly dominated.

Because $\hat{\pi} \not\geq \pi^m$, the decision maker asks Expert $\underline{i}(m)$ to make a disclosure at stage $\bar{m}-m+1$ (that is, the protocol will not stop before stage $\bar{m}-m+1$) and also $\pi^m \wedge \hat{\pi} < \pi^m$. Because $\pi^m \geq \pi^{m-1}$ and $\hat{\pi} \geq \pi^{m-1}$,

$$\pi^m \wedge \hat{\pi} \geq \pi^{m-1}. \tag{19}$$

Inequality (19) implies for all $\pi \not\geq \pi^m$, $u_{\underline{i}(m)}(\pi^m \wedge \hat{\pi}) < u_{\underline{i}(m)}(\pi^m)$ by definition of π^m, and

$$u_{\underline{i}(m)}(\pi^m \vee \hat{\pi}) > u_{\underline{i}(m)}(\hat{\pi}) \tag{20}$$

by quasi-supermodularity. Now let $H_m = \{h = (h_1, \ldots, h_{\bar{m}-m} : \mu(h) = \hat{\pi}\}$ be the set of histories that generate disclosure $\hat{\pi}$ prior to the start of stage $\bar{m} - m + 1$

$$x'_{\underline{i}(m)}(h) = \begin{cases} \pi^m \vee \hat{\pi} & \text{if } h \in H_m \\ \hat{x}_{\underline{i}(m)}(h) & \text{if } h \notin H_m \end{cases}.$$

It follows that $x'_{\underline{i}(m)}$ weakly dominates $\hat{x}_{\underline{i}(m)}$ because it does exactly as well as $\hat{x}_{\underline{i}(m)}$ for $h \notin H_m$ and strictly better otherwise. That $x'_{\underline{i}(m)}$ is strictly better follows from (20) because $\pi^m \vee \hat{\pi} > \pi^m$ guarantees that for $h \in H_m$ the decision maker stops asking experts after Expert $\underline{i}(m)$ discloses π^m in stage $\bar{m} - m + 1$.

We conclude that a disclosure $\hat{x}$ can only be generated if a player uses a weakly dominated strategy. This establishes the proposition. $\square$

Lemma 3. *If $\tilde{\pi}^* \leq \pi$ and $\pi \not\leq \pi^*$, then $u_i(\pi) < u_i(\tilde{\pi}^*)$.*

Proof. It follows from the assumptions that $\tilde{\pi}^* \leq \pi \wedge \pi^* < \pi$ and $\pi \vee \pi^* > \pi^*$. Consequently, $u_i(\pi^*) > u_i(\pi^* \vee \pi)$ by the definition of π^* and $u_i(\pi^* \wedge \pi) > u_i(\pi)$ by quasi-supermodularity. The lemma follows because $u_i(\tilde{\pi}^*) \geq u_i(\pi^* \wedge \pi)$ by the definition of $\tilde{\pi}^*$. $\square$

Proposition 3. *If π is generated by the canonical protocol, then $\pi \leq \pi^*$.*

Proof. By Lemma 3, Expert $\underline{i}(\bar{m})$ can do strictly better than any disclosure π such that $\tilde{\pi}^* \leq \pi$ and $\pi \not\leq \pi^*$ by disclosing $\tilde{\pi}^*$. This disclosure ends the protocol. By Proposition 2, all disclosures generated by the canonical sequence are greater than or equal to $\tilde{\pi}^*$. This establishes the proposition. $\square$

Proposition 4. *If there is an expert who is indifferent between $\tilde{\pi}^*$ and π^*, there is a sequential disclosure protocol that generates the disclosure π^*.*

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Proof. Proposition 2 guarantees the existence of a sequential disclosure protocol that generates at least the disclosure $\tilde{\pi}^*$. Find the expert i^* who is indifferent between $\tilde{\pi}^*$ and π^*. The decision maker asks this expert to report first. If the expert discloses $x \geq \tilde{\pi}^*$, then the decision maker stops. Otherwise, the decision maker follows the canonical sequential protocol. Subsequent experts extend their strategies from the canonical protocol – following their strategy in the protocol that generates $\tilde{\pi}^*$ – by making decisions independent of the first disclosure.[8] Because the final disclosure must be at least $\tilde{\pi}^*$, one can construct extended strategies that survive IDWDS. $\qquad\square$

Proposition 5. *If there is an expert who is indifferent between $\tilde{\pi}^*$ and π^*, there is a procedure of IDWDS in the simultaneous game so that the disclosure π^* survives.*

To prove the proposition we need a preliminary result. We describe a process that deletes weakly dominated strategies that establishes the proposition. To do this, we describe a specific procedure of IDWDS that uses some of the structure we created to study the sequential problem.

Let S_i^m be the set of remaining strategies of i after m rounds and $S^m = \prod_{i \in I} S_i^m$ be the set of strategy profiles. In the mth stage ($1 \leq m \leq \bar{m}$), we delete all strategies $x_{\underline{i}(m)}$ of Expert $\underline{i}(m)$ such that $x_{\underline{i}(m)} \vee \pi^{m-1} \not\geq \pi^m$.

The next result shows that strategies of this kind are weakly dominated and shows that we can eliminate all strategy profiles x such that $M(x) \not\geq \tilde{\pi}^*$.

In the next claim, $\pi^{-1} = \underline{x}$.

Claim 6. *For all r,*

$$\text{if } x_i^r \in S_i^r \text{ for all } i, \text{ then } M(x_1^r, \dots, x_I^r) \geq \pi^r \tag{21}$$

and

$$x_{\underline{i}(l)}^r \vee \pi^{r-1} \geq \pi^r. \tag{22}$$

Proof. The claim holds for $r = 0$. We present a proof by induction. Suppose that for $r = m - 1$, Conditions (21) and (22) hold. We will show that they hold for $r = m$. Let $x \in S^{r-1}$. We want to show that if $x_{\underline{i}(m)} \vee \pi^{m-1} \not\geq \pi^m$, then $x_{\underline{i}(m)}$ is weakly dominated by $\pi^m \vee x_{\underline{i}(m)}$. Expert $\underline{i}(m)$'s utility from playing $\pi^m \vee x_{\underline{i}(m)}$ and $x_{\underline{i}(m)}$ are

$$u_{\underline{i}(m)}((\pi^m \vee x_{\underline{i}(m)}) \vee M(x_{-\underline{i}(m)})) = u_{\underline{i}(m)}(\pi^m \vee (x_{\underline{i}(m)} \vee M(x_{-\underline{i}(m)})))$$

and $u_{\underline{i}(m)}(x_{\underline{i}(m)} \vee M(x_{-\underline{i}(m)}))$, respectively. By the definition of π^m, $u_{\underline{i}(m)}(\pi^m) \geq u_{\underline{i}(m)}(y)$ for $\pi^m \geq y \geq \pi^{m-1}$. By the induction hypothesis for $r = m - 1$,

$$\pi^m \wedge (x_{\underline{i}(m)} \vee M(x_{-\underline{i}(m)})) \geq \pi^{m-1}.$$

It follows that

$$u_{\underline{i}(m)}(\pi^m) \geq u_{\underline{i}(m)}(\pi^m \wedge (x_{\underline{i}(m)} \vee M(x_{-\underline{i}(m)}))).$$

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[8] Formally, if $\boldsymbol{s}$ is the strategy from the protocol and $\boldsymbol{s}'$ is the strategy in the augmented game, then $\boldsymbol{s}'_{i^*}(\emptyset) = \pi^*$ and if $h'_{t+1} = h_0 h_t$, then (denoting by h_0 a history generated by Expert i^*'s initial report, $\boldsymbol{s}'_i(h'_{t+1}) = \boldsymbol{s}_i(h_t)$.

So $u_{\underline{i}(m)}(\pi^m \vee (x_{\underline{i}(m)} \vee M(x_{-\underline{i}(m)}))) \geq u_{\underline{i}(m)}(x_{\underline{i}(m)} \vee M(x_{-\underline{i}(m)}))$ by quasi-supermodularity. It follows that $\pi^m \vee x_{\underline{i}(m)}$ is weakly better than $x_{\underline{i}(m)}$. Furthermore, there is some $x_{-\underline{i}(m)}$ such that $M(x_{-\underline{i}(m)}) = \pi^{m-1}$ because $\underline{i}(m) \neq \underline{i}(m-1)$.

Because $x_{\underline{i}(m)} \vee \pi^{m-1} \not\geq \pi^m$, $u_{\underline{i}(m)}(\pi^m) > u_{\underline{i}(m)}(\pi^m \wedge (x_{\underline{i}(m)} \vee \pi^{m-1}))$ by the definition of π^m. Quasi-supermodularity implies that $u_{\underline{i}(m)}(\pi^m \vee (x_{\underline{i}(m)} \vee \pi^{m-1})) > u_{\underline{i}(m)}(x_{\underline{i}(m)} \vee \pi^{m-1})$. We conclude that $u_{\underline{i}(m)}(\pi^m \vee x_{\underline{i}(m)}) > u_{\underline{i}(m)}(x_{\underline{i}(m)})$ when $M(x_{-\underline{i}(m)}) = \pi^{m-1}$. Hence $\pi^m \vee x_{\underline{i}(m)}$ weakly dominates $x_{\underline{i}(m)}$ as claimed. We are therefore able to delete all $x_{\underline{i}(m)}$ such that $x_{\underline{i}(m)} \vee \pi^{m-1} \not\geq \pi^m$ in stage m. This establishes (22) for $l = m$. By the induction hypothesis,

$$x^m_{\underline{i}(m)} \vee M(x^{m-1}_{-\underline{i}(m)}) = x^m_{\underline{i}(m)} \vee (x^m_{\underline{i}(m)} \vee M(x^{m-1}_{-\underline{i}(m)})) \geq x^m_{\underline{i}(m)} \vee \pi^{m-1} \geq \pi^m$$

for all $x^m_{\underline{i}(m)} \in S^m_{\underline{i}(m)}$ and $x^{m-1}_{-\underline{i}(m)} \in S^{m-1}_{-\underline{i}(m)}$.

Because we only delete strategies of Expert $\underline{i}(m)$ in stage m, $S^m_i = S^{m-1}_i$ for all $i \neq \underline{i}(m)$. It follows that

$$\max\{x^m_1, \ldots, x^m_I\} = x^m_{\underline{i}(m)} \vee M(x^m_{-\underline{i}(m)}) \geq \pi^m$$

for all $x^m \in S^m$ by the induction hypothesis (22). Hence all disclosures $M(x^m)$ for $x^m \in S^m$ are greater than or equal to π^m, which establishes (21) for $r = m$. $\qquad \square$

Proof of Proposition 5. Let j be the expert who is indifferent between $\tilde{\pi}^*$ and π^*. We show that j's strategy π^* survives IDWDS, i.e. $\pi^* \in S_j$. Consider a different strategy $x_j \in S^{\bar{m}}_j$ of Expert j and a strategy $x_{-j} \in S^{\bar{m}}_{-j}$ of other players that remain after stage $\bar{m}$. Expert j's payoffs from playing π^* and x_j are $u_j(\pi^* \vee M(x_{-j}))$ and $u_j(x_j \vee M(x_{-j}))$, respectively.

First assume that $x_j \leq \pi^*$, then $\pi^* \vee M(x_{-j}) = (\pi^* \vee x_j) \vee M(x_{-j}) = \pi^* \vee (x_j \vee M(x_{-j}))$. Furthermore, the disclosure $x_j \vee M(x_{-j}) \geq \tilde{\pi}^*$ by Claim 6. So $\pi^* \wedge (x_j \vee M(x_{-j})) \geq \tilde{\pi}^*$. It follows that $u_j(\pi^*) = u_j(\tilde{\pi}^*) \geq u_j(\pi^* \wedge (x_j \vee M(x_{-j})))$ by the definition of $\tilde{\pi}^*$. This implies $u_j(\pi^* \vee M(x_{-j})) = u_j(\pi^* \vee (x_j \vee M(x_{-j}))) \geq u_j(x_j \vee M(x_{-j}))$ by quasi-supermodularity. So Expert j can never get a lower payoff from π^* than any $x_j \leq \pi^*$.

Now assume that $x_j \not\leq \pi^*$. By Lemma 3, there is a strategy profile x that survives IDWDS such that $M(x) \leq \pi^*$. Then there exists some $x'_{-j} \in S_{-j}$ such that $M(x'_{-j}) \leq \pi^*$. It follows that $\pi^* \vee M(x'_{-j}) = \pi^*$ and $x_j \vee M(x'_{-j}) \not\leq \pi^*$, which implies that $\pi^* \vee (x_j \vee M(x'_{-j})) > \pi^*$. It follows that $u_j(\pi^*) > u_j(\pi^* \vee (x_j \vee M(x'_{-j})))$ by the definition of π^*. This implies $u_j(\pi^* \wedge (x_j \vee M(x'_{-j}))) > u_j(x_j \vee M(x'_{-j}))$ by quasi-supermodularity. By Claim 6, $x_j \vee M(x'_{-j}) \geq \tilde{\pi}^*$. It follows that

$$u_j(\pi^* \vee M(x'_{-j})) = u_j(\pi^*) = u_j(\tilde{\pi}^*) \geq u_j(\pi^* \wedge (x_j \vee M(x'_{-j}))) > u_j(x_j \vee M(x'_{-j}))$$

by the definition of $\tilde{\pi}^*$. So there is no $x_j \not\leq \pi^*$ that dominates the strategy π^*.

Hence there is no strategy that could weakly dominate π^*, regardless of the order of deletion after stage $\bar{m}$. So $\pi^* \in S_j$. Because $M(x'_{-j}) \leq \pi^*$, the disclosure $\pi^* = \pi^* \vee M(x'_{-j})$ survives IDWDS and the result follows. $\qquad \square$